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EXISTENCE RESULTS FOR NONLINEAR
WEIGHTED ANISOTROPIC ELLIPTIC SYSTEMS
WITH VARIABLE EXPONENTS

Abstract. This paper establishes the existence of distributional solutions for a class of anisotropic elliptic $\vec{p}(\cdot)$ -systems. These systems are weighted by a positive function that belongs to the $\vec{p}(\cdot)$ -Sobolev space and involves $L^1(\Omega; \mathbb{R}^m)$ -data associated with the $L^1(\Omega)$ -coefficient of the zero order term.

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რეზიუმე. ნაშრომში ანიზოტროპული ელფისური $\vec{p}(\cdot)$ -სისტემების ერთი კლასისათვის დადგენილია განზოგადებული ამონახსნების არსებობა. ეს სისტემები შეწონილია დადებითი ფუნქციით, რომელიც ეკუთვნის $\vec{p}(\cdot)$ -სობოლევის სივრცეს და შეიცავს $L^1(\Omega; \mathbb{R}^m)$ -მონაცემებს, რომლებიც ასოცირებულია ნულოვანი რიგის წევრის $L^1(\Omega)$ -კოეფიციენტთან.

1 Introduction

This study focuses on establishing the existence of at least one distributional solution $u = (u_1, \dots, u_m)^\top$ for a particular class of weighted elliptic $\vec{p}(\cdot)$ -systems expressed in the form

$$-\sum_{i=1}^N \partial_i(v(x)\sigma_i(x, \partial_i u)) + A(x)g(x, u) = f \text{ in } \Omega, \quad (1.1)$$

$$u = 0 \text{ on } \partial\Omega.$$

Here, $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is an open bounded Lipschitz domain, $\partial_i u = \frac{\partial u}{\partial x_i}$, $i = 1, \dots, N$, $f \in L^1(\Omega; \mathbb{R}^m)$, $A(\cdot) \in L^1(\Omega)$, and $v(\cdot)$ is in $\overset{\circ}{W}^{1, \vec{p}(\cdot)}(\Omega)$ such that there exists $\alpha, \beta > 0$:

$$|f(x)| \leq \alpha A(x), \quad (1.2)$$

$$v(x) \geq \beta. \quad (1.3)$$

$\sigma_i : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $i = 1, \dots, N$, are the Carathéodory functions such that, for almost every $x \in \Omega$ and every $s, s' \in \mathbb{R}^m$ ($s, s' \neq (0, 0)$), we have

$$\sigma_i(x, s) \cdot s \geq c_1 |s|^{p_i(x)}, \quad (1.4)$$

$$|\sigma_i(x, s)| \leq c_2 (|s|^{p_i(x)} + |h|)^{1 - \frac{1}{p_i(x)}}, \quad h \in L^1(\Omega) \quad (1.5)$$

$$(\sigma_i(x, s) - \sigma_i(x, s')) \cdot (s - s') \geq \begin{cases} c_3 |s - s'|^{p_i(x)}, & \text{if } p_i(x) \geq 2, \\ c_4 \frac{|s - s'|^2}{(|s| + |s'|)^{2 - p_i(x)}}, & \text{if } 1 < p_i(x) < 2, \end{cases} \quad (1.6)$$

where c_1, c_2, c_3, c_4 are the positive constants.

The function $g : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a Carathéodory function and meets the following conditions almost everywhere for $x \in \Omega$:

$$g(x, s) \cdot (s - s') \geq 0, \quad \forall s, s' \in \mathbb{R}^m, \quad |s| = |s'|, \quad (1.7)$$

$$\sup_{|s| \leq t} |g(x, s)| \in L^1(\Omega; \mathbb{R}^m), \quad \forall s \in \mathbb{R}^m, \text{ and } \forall t > 0, \quad (1.8)$$

$$g(x, s) \cdot s \geq \sum_{i=1}^N |\rho|^{p_i(x)+1}, \quad \forall s \in \mathbb{R}^m. \quad (1.9)$$

As a typical example, consider the following:

$$-\sum_{i=1}^N \partial_i(v(x)|\partial_i u|^{p_i(x)-2} \partial_i u) + A(x)u \sum_{i=1}^N |u|^{p_i(x)-1} = f \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

where $f, A(\cdot), v(\cdot)$, and $p_i(\cdot)$, $i = 1, \dots, N$, are restricted as in Theorem 3.1.

In this work, we focus on the $\vec{p}(x)$ -anisotropic differential operator, which has a wide range of applications in applied sciences. For example, such operators are frequently employed in the study of electro-rheological fluids and image processing, as highlighted in references [10, 17, 27]. Various existence results for systems involving these operators under different conditions and data have been established, as detailed in [5, 19, 24–26]. For the anisotropic scalar case, the related results can be found in [1–4, 13]. Furthermore, the existence of solutions for variable exponent anisotropic nonlinear weighted elliptic equations has been demonstrated in [20–23], with the corresponding isotropic scalar case discussed in [6].

In the present paper, we establish existence results for distributional solutions to a class of anisotropic nonlinear elliptic systems with variable exponents described by (1.1). These systems

are weighted by a positive function $v(\cdot) \in \overset{\circ}{W}^{1,\vec{p}(\cdot)}(\Omega)$, and the given datum satisfies $f \in L^1(\Omega; \mathbb{R}^m)$. A key assumption is the interaction described in (1.2) between the datum and the $L^1(\Omega)$ coefficient $A(\cdot)$ of the zero-order term inducing a regularizing effect on (1.1). Our approach to proving the existence of solutions depends critically on the requirement that the weight function belongs to the $\vec{p}(\cdot)$ -Sobolev space.

We developed our proof based on a sequence of approximate solutions (u_n) , which requires demonstrating their existence through the main theorem on pseudo-monotone operators and the findings presented in [25]. Subsequently, we employ a priori estimates to establish the boundedness of (u_n) and the almost everywhere convergence of their partial derivatives $\partial_i u_n$ for $i = 1, \dots, N$, which can be interpreted as a strong L^1 -convergence. With this convergence established, we take the limit in the strong L^1 sense for $v_n(x)\sigma_i(x, \partial_i u_n)$ for $i = 1, \dots, N$ and in $A_n(x)g(x, u_n)$. Ultimately, we conclude that the approximate solutions u_n converge to the solution of (1.1).

Our work is organized as follows. Section 2 focuses on the mathematical preliminaries, covering isotropic and anisotropic variable exponent Lebesgue–Sobolev spaces, along with several embedding theorems. The primary theorem, along with its proof, is presented in Section 3.

2 Preliminaries

This section aims to introduce fundamental definitions and properties related to isotropic and anisotropic variable exponent Lebesgue–Sobolev spaces (refer to [11, 12, 15]).

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded open subset. We denote

$$\mathcal{C}_+(\bar{\Omega}) = \left\{ \text{continuous function } p(\cdot) : \bar{\Omega} \mapsto \mathbb{R}, \ 1 < p^- \left(= p^- = \min_{x \in \bar{\Omega}} p(x) \right) \right\}.$$

Let $p(\cdot) \in \mathcal{C}_+(\bar{\Omega})$. Variable exponent Lebesgue space with $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ \text{measurable functions } u : \Omega \mapsto \mathbb{R}; \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

where $u \mapsto \varrho_{p(\cdot)}(u) := \int_{\Omega} |u(x)|^{p(x)} dx$ is called the convex modular.

It forms a reflexive Banach space when equipped with the Luxemburg norm

$$u \mapsto \|u\|_{p(\cdot)} := \inf \left\{ s > 0 \mid \varrho_{p(\cdot)}\left(\frac{u}{s}\right) \leq 1 \right\}.$$

The Hölder type inequality

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}$$

holds true.

The subsequent results are presented in [11, 12].

If $u \in L^{p(\cdot)}(\Omega)$, then we obtain

$$\begin{aligned} \min &\leq (\varrho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \varrho_{p(\cdot)}(u)^{\frac{1}{p^-}}) \leq \|u\|_{p(\cdot)} \leq \max (\varrho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \varrho_{p(\cdot)}(u)^{\frac{1}{p^-}}), \\ \min &(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}) \leq \varrho_{p(\cdot)}(u) \leq \max (\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}). \end{aligned} \quad (2.1)$$

We now proceed to introduce the $\vec{p}(\cdot)$ -Sobolev spaces $W^{1,\vec{p}(\cdot)}(\Omega)$.

Let $\vec{p}(x) = (p_1(x), \dots, p_N(x)) \in (C(\bar{\Omega}, [1, +\infty)))^N$, and for every x in $\bar{\Omega}$ we set

$$\begin{aligned} p_+(x) &= \max_{1 \leq i \leq N} p_i(x), \quad p_-(x) = \min_{1 \leq i \leq N} p_i(x), \\ p_-^- &= \min_{x \in \bar{\Omega}} p_-(x), \quad p_+^+ = \max_{x \in \bar{\Omega}} p_+(x), \end{aligned}$$

$$\bar{p}(x) = \frac{N}{\sum_{i=1}^N \frac{1}{p_i(x)}}, \quad \bar{p}^*(x) = \begin{cases} \frac{N\bar{p}(x)}{N - \bar{p}(x)} & \text{for } \bar{p}(x) < N, \\ +\infty & \text{for } \bar{p}(x) \geq N. \end{cases}$$

The Banach space $W^{1,\bar{p}(\cdot)}(\Omega)$ is defined by $W^{1,\bar{p}(\cdot)}(\Omega) = \{u \in L^{p_+(\cdot)}(\Omega), \partial_i u \in L^{p_i(\cdot)}(\Omega), i = 1, \dots, N\}$ under the norm

$$u \mapsto \|u\|_{\bar{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^N \|\partial_i u\|_{p_i(\cdot)}. \quad (2.2)$$

The Banach space $\mathring{W}^{1,\bar{p}(\cdot)}(\Omega)$ is defined as follows: $\mathring{W}^{1,\bar{p}(\cdot)}(\Omega) = W^{1,\bar{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$, when equipped with the norm (2.2).

The following results can be found in [14, 15].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\bar{p}(\cdot) \in (\mathcal{C}_+(\bar{\Omega}))^N$.

(*) If $r \in \mathcal{C}_+(\bar{\Omega})$ and $\forall x \in \bar{\Omega}$, $r(x) < \max(p_+(x), \bar{p}^*(x))$, then the embedding $\mathring{W}^{1,\bar{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ is compact.

(*) If we have

$$\forall x \in \bar{\Omega}, \quad p_+(x) < \bar{p}^*(x), \quad (2.3)$$

then the following inequality holds:

$$\|u\|_{p_+(\cdot)} \leq C \sum_{i=1}^N \|\partial_i u\|_{p_i(\cdot)}, \quad \forall u \in \mathring{W}^{1,\bar{p}(\cdot)}(\Omega),$$

where $C > 0$ is independent of u . Thus

$$u \mapsto \sum_{i=1}^N \|\partial_i u\|_{p_i(\cdot)} \text{ is an equivalent norm to (2.2) on } \mathring{W}^{1,\bar{p}(\cdot)}(\Omega). \quad (2.4)$$

In our paper, we denote by $L^{p(\cdot)}(\Omega, \mathbb{R}^m)$ and $\mathring{W}^{1,\bar{p}(\cdot)}(\Omega, \mathbb{R}^m)$ ($m \geq 1$) the $\mathbb{R}^m$ -valued version of $L^{p(\cdot)}(\Omega)$ and $\mathring{W}^{1,\bar{p}(\cdot)}(\Omega)$, respectively.

The space $X = \mathring{W}^{1,\bar{p}(\cdot)}(\Omega, \mathbb{R}^m)$ becomes a Banach space, when equipped with the norm $\|\cdot\|_X = \|\cdot\|_{\bar{p}(\cdot)}$. Also $Y = L^{p(\cdot)}(\Omega, \mathbb{R}^m)$, when equipped with the norm $\|\cdot\|_Y = \|\cdot\|_{p(\cdot)}$.

For any $t > 0$, the scalar truncation function T_t on $[0, \infty)$ is defined as $T_t(s) := \begin{cases} s & \text{if } s \leq t, \\ t & \text{if } s > t. \end{cases}$ For any $t > 0$, define the spherical truncation function $T_t : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$T_t(s) := \begin{cases} s & \text{if } |s| \leq t, \\ \frac{s}{|s|} t & \text{if } |s| > t. \end{cases} \quad (2.5)$$

3 Statement of results and proof

Definition. The vector-valued function $u = (u_1, \dots, u_m)^\top : \Omega \rightarrow \mathbb{R}^m$ is a solution of system (1.1) in the sense of distributions if and only if $u \in W_0^{1,1}(\Omega; \mathbb{R}^m)$, and for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$,

$$\int_{\Omega} \sum_{i=1}^N v(x) \sigma_i(x, \partial_i u) \cdot \partial_i \varphi \, dx + \int_{\Omega} \sum_{i=1}^N A(x) g(x, u) \cdot \varphi \, dx = \int_{\Omega} f(x) \cdot \varphi \, dx.$$

Our main Theorem is the following.

Theorem 3.1. Let $p_i(\cdot)$, $i = 1, \dots, N$ be in $\mathcal{C}_+(\overline{\Omega})$ such that $\overline{p}(\cdot) < N$ in $\overline{\Omega}$, and let (2.3) hold. Assume f is in $L^1(\Omega; \mathbb{R}^m)$, $A(\cdot)$ is in $L^1(\Omega)$, and $v(\cdot)$ is in $\mathring{W}^{1, \overline{p}(\cdot)}(\Omega)$ such that (1.2), (1.3) and (2.3) hold.

Let σ_i , $i = 1, \dots, N$, and g be the Carathéodory functions, where σ_i satisfy (1.4)–(1.6), and g satisfies (1.7)–(1.9). Then problem (1.1) has at least one solution $u \in \mathring{W}^{1, \overline{p}(\cdot)}(\Omega; \mathbb{R}^m)$ in the sense of distributions.

3.1 Existence of approximate solutions

We consider the function $\theta(\cdot)$ defined as follows:

$$\begin{aligned} \theta(x) &= \frac{nx}{x+n}, \quad x \geq 0, \\ f_n(x) &= \theta(f(x)), \quad A_n(x) = \frac{1}{\alpha} \theta(A(x)), \quad v_n(x) = \theta(v(x)), \quad n \in \mathbb{N}^*. \end{aligned} \quad (3.1)$$

Noticing the increase of θ , (1.2) and (1.3), we can obtain $|f_n(x)| \leq \theta(\alpha A(x)) = \alpha A_n(x)$ and $\theta(\beta) \leq v_n(x) \leq \theta(n)$. Then we conclude that for all $x \in \overline{\Omega}$,

$$|f_n(x)| \leq \alpha A_n(x), \quad (3.2)$$

$$\frac{\beta}{1+\beta} \leq v_n(x) \leq n. \quad (3.3)$$

Lemma 3.1. Let $p_i(\cdot)$, $i = 1, \dots, N$, be in $\mathcal{C}_+(\overline{\Omega})$ such that $\overline{p}(\cdot) < N$ in $\overline{\Omega}$, and let (2.3) hold. Assume f is in $L^1(\Omega; \mathbb{R}^m)$, $A(\cdot)$ is in $L^1(\Omega)$, and $v(\cdot)$ is in $\mathring{W}^{1, \overline{p}(\cdot)}(\Omega)$ such that (1.2), (1.3) and (2.3) hold.

Let σ_i , $i = 1, \dots, N$, and g be the Carathéodory functions, where σ_i satisfy (1.4)–(1.6), and g satisfies (1.7)–(1.9).

Then there exists at least one weak solution $u_n \in \mathring{W}^{1, \overline{p}(\cdot)}(\Omega; \mathbb{R}^m)$ to the approximated problems

$$\begin{aligned} - \sum_{i=1}^N \partial_i (v_n(x) \sigma_i(x, \partial_i u_n)) + A_n(x) g(x, u_n) &= f_n \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (3.4)$$

in the following sense:

For every $\varphi \in \mathring{W}^{1, \overline{p}(\cdot)}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$,

$$\sum_{i=1}^N \int_{\Omega} v_n(x) \sigma_i(x, \partial_i u_n) \cdot \partial_i \varphi \, dx + \int_{\Omega} A_n(x) g(x, u_n) \cdot \varphi \, dx = \int_{\Omega} f_n \cdot \varphi \, dx. \quad (3.5)$$

Proof. Consider the system

$$\begin{aligned} - \sum_{i=1}^N \partial_i (v_n(x) \sigma_i(x, \partial_i u_{n_k})) + A_n(x) T_k(g(x, u_{n_k})) &= f_n \quad \text{in } \Omega, \\ u_{n_k} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (3.6)$$

In a similar manner to the results obtained in [25], applying the main Theorem on the pseudo-monotone operators ((Theorem 27.A in [28], see also [8, 9, 18])), we conclude that there exists a solution $u_{n_k} \in \mathring{W}^{1, \overline{p}(\cdot)}(\Omega; \mathbb{R}^m)$ to system (3.6), which satisfies

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} v_n(x) \sigma_i(x, \partial_i u_{n_k}) \cdot \partial_i \varphi \, dx \\ + \int_{\Omega} A_n(x) T_k(g(x, u_{n_k})) \cdot \varphi \, dx = \int_{\Omega} f_n \cdot \varphi \, dx, \quad \forall \varphi \in \mathring{W}^{1, \vec{p}(\cdot)}(\Omega; \mathbb{R}^m). \end{aligned} \quad (3.7)$$

Now, choosing $\varphi = u_{n_k}$ as a test function in (3.7) and using (3.3), (1.4), after dropping the nonnegative term (since $A_n(x) T_k(g(x, u_{n_k})) \cdot u_{n_k} \geq 0$, due to (1.9) and the fact that $A_n(x) \geq 0$), we get

$$\frac{c_1 \beta}{(1 + \beta)} \sum_{i=1}^N \int_{\Omega} |\partial_i u_{n_k}|^{p_i(x)} \, dx \leq n \int_{\Omega} |u_{n_k}| \, dx.$$

Using Young's inequality, for all $\varepsilon > 0$, we get

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} |\partial_i u_{n_k}|^{p_i(x)} \, dx &\leq \frac{n(1 + \beta)}{c_1 \beta} \int_{\Omega} |u_{n_k}| \, dx \\ &\leq \frac{n(1 + \beta)}{c_1 \beta} \left(\varepsilon \int_{\Omega} |u_{n_k}|^{p^-} \, dx + C(\varepsilon) \right) \leq \frac{n(1 + \beta)}{c_1 \beta} \left(\varepsilon c \int_{\Omega} |\partial_i u_{n_k}|^{p^-} \, dx + C(\varepsilon) \right) \\ &\leq \frac{n(1 + \beta)}{c_1 \beta} \left(\varepsilon c \left(1 + \int_{\Omega} |\partial_i u_{n_k}|^{p_i(x)} \, dx \right) + C(\varepsilon) \right) \\ &\leq \frac{n(1 + \beta)}{c_1 \beta} \left(\varepsilon c \left(1 + \sum_{i=1}^N \int_{\Omega} |\partial_i u_{n_k}|^{p_i(x)} \, dx \right) + C(\varepsilon) \right). \end{aligned}$$

Choosing $\varepsilon = \frac{c_1 \beta}{2nc(1 + \beta)}$, we obtain

$$\sum_{i=1}^N \int_{\Omega} |\partial_i u_{n_k}|^{p_i(x)} \, dx \leq c(n). \quad (3.8)$$

On the other hand, from (2.1) for all $i = 1, \dots, N$ we have

$$1 + \int_{\Omega} |\partial_i u_{n_k}|^{p_i(x)} \, dx \geq \| |\partial_i u_{n_k}| \|_{p_i(x)}^{p_i^-}, \quad (3.9)$$

$$1 + \| |\partial_i u_{n_k}| \|_{p_i(x)}^{p_i^-} \geq \| |\partial_i u_{n_k}| \|_{p_i(x)}^{p_i^-}. \quad (3.10)$$

Combining (3.9) and (3.10), we obtain

$$\sum_{i=1}^N \int_{\Omega} |\partial_i u_{n_k}|^{p_i(x)} \, dx \geq \sum_{i=1}^N \| |\partial_i u_{n_k}| \|_{p_i(x)}^{p_i^-} - 2N. \quad (3.11)$$

By (3.11) and (2.4) (due (2.3)), we deduce that

$$\sum_{i=1}^N \int_{\Omega} |\partial_i u_{n_k}|^{p_i(x)} \, dx \geq \left(\frac{1}{N} \| |u_{n_k}| \|_{\vec{p}(\cdot)} \right)^{p^-} - 2N. \quad (3.12)$$

From (3.8) and (3.12), we conclude

$$\| |u_{n_k}| \|_{\vec{p}(\cdot)} \leq C(n). \quad (3.13)$$

Through (3.13) we can conclude that there is a subsequence (still denoted by u_{n_k}) $u_n \in \mathring{W}^{1, \vec{p}(\cdot)}(\Omega; \mathbb{R}^m)$ such that $u_{n_k} \rightharpoonup u_n$ weakly in $\mathring{W}^{1, \vec{p}(\cdot)}(\Omega; \mathbb{R}^m)$ and a.e in Ω . In a similar manner to the results obtained

in [25], thanks to (1.4)–(1.6) and (1.7)–(1.9), we can obtain $\partial_i u_{n_k} \rightarrow \partial_i u_n$ strongly in $L^{p_i(\cdot)}(\Omega; \mathbb{R}^m)$ and a.e. in Ω . So, $v_n(x)\sigma_i(x, \partial_i u_{n_k}) \rightarrow v_n(x)\sigma_i(x, \partial_i u_n)$ in $L^{p_i(\cdot)}(\Omega; \mathbb{R}^m)$. Taking $T_t(u_{n_k})$ as a test function in (3.7), by (1.8), (2.5), and the fact that

$$|T_t(s)| \leq M + t1_{\{|s|>M\}}, \quad \forall s \in \mathbb{R}^m, \quad 0 < M < t,$$

for all $0 < M < t$ we get

$$\int_{\{|u_{n_k}|>t\}} A_n(x)|T_k(g(x, u_{n_k}))| dx \leq \frac{M}{t} \|f\|_{L^1(\Omega; \mathbb{R}^m)} + \int_{\{|u_{n_k}|>M\}} |f_n|. \quad (3.14)$$

Let $E \subset \Omega$ be any measurable set, we write

$$\int_E A_n(x)|T_k(g(x, u_{n_k}))| dx = \int_{E \cap \{|u_{n_k}| \leq t\}} A_n(x)|g(x, u_{n_k})| dx + \int_{E \cap \{|u_{n_k}| > t\}} A_n(x)|T_k(g(x, u_{n_k}))| dx.$$

Then, by (1.8) and the decomposition (3.14), we deduce that the sequence $\{A_n(x)T_k(g(x, u_{n_k}))\}$ is equi-integrable in $L^1(\Omega; \mathbb{R}^m)$, and since $T_k(g(x, u_{n_k})) \rightarrow g(x, u_n)$ a.e. in Ω , Vitali's theorem implies that $A_n(x)T_k(g(x, u_{n_k})) \rightarrow A_n(x)g(x, u_n)$ in $L^1(\Omega; \mathbb{R}^m)$. Therefore, we can obtain (3.5) by passing to the limit in (3.7). $\square$

3.1.1 A priori estimates

Lemma 3.2. *Let f , A , v , g and p_i , σ_i , $i = 1, \dots, N$, be restricted as in Theorem 3.1. Then*

$$|g(x, u_n)| \leq \alpha, \quad (3.15)$$

$$|u_n| \leq \left(\frac{\alpha}{N} + 1 \right)^{\frac{1}{p^-}}, \quad (3.16)$$

where u_n is the weak solution to problem (3.4).

Proof. After choosing $\varphi = u_n$ in the weak formulation (3.5), and dropping the nonnegative term (since $v_n(x)\sigma_i(x, \partial_i u_n) \cdot \partial_i u_n \geq 0$, $i = 1, \dots, N$, due (1.4), and (3.3)), we obtain

$$\int_{\Omega} A_n(x)g(x, u_n) \cdot u_n dx \leq \int_{\Omega} |f_n||u_n| dx.$$

Using (3.2) and the fact that $g(x, u_n) \cdot u_n \geq |g(x, u_n)||u_n|$ (it is produced through the following: by (1.7), we get

$$\frac{u_n}{|u_n|} \cdot g(x, u_n) - |g(x, u_n)| = \frac{1}{|u_n|} g(x, u_n) \cdot \left(u_n - |u_n| \frac{g(x, u_n)}{|g(x, u_n)|} \right) \geq 0,$$

we obtain

$$\int_{\Omega} A_n(x)|g(x, u_n)||u_n| dx \leq \alpha \int_{\Omega} A_n(x)|u_n| dx.$$

whence

$$\int_{\Omega} A_n(x)(|g(x, u_n)| - \alpha)|u_n| dx \leq 0. \quad (3.17)$$

Then (3.17) implies (3.15).

Also, by the fact that $1 + |u_n|^{p_i(x)} \geq |u_n|^{p^-}$, $i = 1, \dots, N$, due to (1.9) and (3.15), we get (3.16). $\square$

Remark 3.1. (3.15) and (3.16) imply that

$$\begin{aligned} (g(x, u_n)) & \text{ is bounded in } L^\infty(\Omega, \mathbb{R}^m), \\ (u_n) & \text{ is bounded in } L^\infty(\Omega, \mathbb{R}^m). \end{aligned} \quad (3.18)$$

Lemma 3.3.

$$(A_n(x)g(x, u_n)) \text{ is bounded in } L^1(\Omega, \mathbb{R}^m). \quad (3.19)$$

Proof. Through (3.15) and (3.1), we get

$$\int_{\Omega} |A_n(x)g(x, u_n)| dx \leq \alpha \int_{\Omega} |A_n(x)| dx \leq \|A\|_{L^1(\Omega)}. \quad (3.20)$$

So, (3.20) implies (3.19). $\square$

Lemma 3.4.

$$v_n \text{ is bounded in } \mathring{W}^{1, \vec{p}(\cdot)}(\Omega), \quad (3.21)$$

$$v_n \rightarrow v \text{ strongly in } \mathring{W}^{1, \vec{p}(\cdot)}(\Omega). \quad (3.22)$$

Proof. Since, for all $x \in \overline{\Omega}$,

$$\partial_i v_n(x) = \frac{\partial_i v(x)}{(1 + \frac{v(x)}{n})^2}, \quad i = 1, \dots, N,$$

we have $|\partial_i v_n(x)| \leq |\partial_i v(x)|$ and, therefore, $v_n(\cdot) \in \mathring{W}^{1, \vec{p}(\cdot)}(\Omega)$. From this and the fact that $0 \leq v_n(x) \leq v(x)$, we get (3.21) and (3.22). $\square$

Lemma 3.5. Let f , A , v , g and p_i , σ_i , $i = 1, \dots, N$, be restricted as in Theorem 3.1. Then

$$u_n \text{ is bounded in } \mathring{W}^{1, \vec{p}(\cdot)}(\Omega; \mathbb{R}^m), \quad (3.23)$$

where u_n is the weak solution to problem (3.4).

Proof. After choosing $\varphi = u_n$ in the weak formulation (3.5), and dropping the nonnegative term (since $A_n(x)g(x, u_n) \cdot u_n \geq 0$, due to (1.9) and the fact that $A_n(x) \geq 0$), and using (3.16), (1.4), (3.1) and (3.3), we can get

$$\frac{c_1 \beta}{(1 + \beta)} \sum_{i=1}^N \int_{\Omega} |\partial_i u_n|^{p_i(x)} dx \leq \left(\frac{\alpha}{N} + 1 \right)^{\frac{1}{p^-}} \|f\|_{L^1(\Omega, \mathbb{R}^m)}.$$

Then we have

$$\sum_{i=1}^N \int_{\Omega} |\partial_i u_n|^{p_i(x)} dx \leq c. \quad (3.24)$$

By a proof similar to the that of (3.12), we can get

$$\sum_{i=1}^N \int_{\Omega} |\partial_i u_n|^{p_i(x)} dx \geq \left(\frac{1}{N} \|u_n\|_{\vec{p}(\cdot)} \right)^{p^-} - 2N. \quad (3.25)$$

Combining (3.24) and (3.25), we obtain

$$\|u_n\|_{\vec{p}(\cdot)} \leq C, \quad (3.26)$$

where $C > 0$ is independent of n .

Then (3.26) implies (3.23). $\square$

Remark 3.2. It follows from (3.23) that there exist a function $u \in \mathring{W}^{1, \vec{p}(\cdot)}(\Omega; \mathbb{R}^m)$ and a subsequence (still denoted by (u_n)) such that

$$u_n \rightharpoonup u \text{ weakly in } \mathring{W}^{1, \vec{p}(\cdot)}(\Omega; \mathbb{R}^m) \text{ and a.e in } \Omega. \quad (3.27)$$

Lemma 3.6. For all $i = 1, \dots, N$,

$$\partial_i u_n \rightarrow \partial_i u \text{ a.e. in } \bar{\Omega}, \quad (3.28)$$

where u is the weak limit of the sequence (u_n) in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega; \mathbb{R}^m)$.

Proof. By (3.3), we obtain

$$\frac{1}{v_n(x)} \varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega, \mathbb{R}^m) \text{ for all } \varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m).$$

Therefore, we can choose it as a test function in (3.5) and get

$$\sum_{i=1}^N \int_{\Omega} \sigma_i(x, \partial_i u_n) \cdot \partial_i \varphi \, dx = \int_{\Omega} \Phi_n(x) \varphi \, dx,$$

where Φ_n is defined by

$$\Phi_n(x) = \frac{1}{v_n(x)} \left(f_n(x) - A_n(x)g(x, u_n) + \sum_{i=1}^N \sigma_i(x, \partial_i u_n) \partial_i v_n(x) \right).$$

By Young's inequality, (1.5), and since $\partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, for all $\varepsilon > 0$ we get

$$\begin{aligned} \int_{\Omega} |\sigma_i(x, \partial_i u_n)| \, dx &= c' + \int_{\Omega} |\partial_i u_n|^{p_i(x)-1} \, dx \\ &\leq c' + C(\varepsilon) + \varepsilon \int_{\Omega} |\partial_i u_n|^{p_i(x)} \, dx \leq c' + C(\varepsilon) + \varepsilon c = C'(\varepsilon). \end{aligned}$$

Then, for any fixed choice for $\varepsilon > 0$, we conclude that for all $i = 1, \dots, N$,

$$(\sigma_i(x, \partial_i u_n)) \text{ is bounded in } L^1(\Omega, \mathbb{R}^m). \quad (3.29)$$

From (3.1) (implying that $f_n \in L^1(\Omega, \mathbb{R}^m)$), (3.29) and (3.19), we conclude that

$$\left(f_n(x) - A_n(x)g(x, u_n) + \sum_{i=1}^N \sigma_i(x, \partial_i u_n) \partial_i v_n(x) \right) \text{ is bounded in } L^1(\Omega, \mathbb{R}^m). \quad (3.30)$$

Through (3.30) and the boundedness of the sequence $(\frac{1}{v_n(x)})$ (due to (3.3)), we obtain (Φ_n) is bounded in $L^1(\Omega, \mathbb{R}^m)$. So, applying the results obtained in [7] to the sequence (u_n) , we can simply obtain (3.28). $\square$

3.2 Proof of Theorem 3.1

For all $i = 1, \dots, N$, we put

$$\begin{aligned} \sigma_i(x, \partial_i u_n) &= (\sigma_i^{(1)}(x, \partial_i u_n), \dots, \sigma_i^{(m)}(x, \partial_i u_n)), \\ \sigma_i(x, \partial_i u) &= (\sigma_i^{(1)}(x, \partial_i u), \dots, \sigma_i^{(m)}(x, \partial_i u)). \end{aligned}$$

By (3.28), for all $i = 1, \dots, N$ we have $\sigma_i(x, \partial_i u_n) \rightharpoonup \sigma_i(x, \partial_i u)$ weakly in $L^{p'_i(\cdot)}(\Omega; \mathbb{R}^m)$. Then we conclude that, for all $i = 1, \dots, N$ and all $j = 1, \dots, m$,

$$\sigma_i^{(j)}(x, \partial_i u_n) \rightharpoonup \sigma_i^{(j)}(x, \partial_i u) \text{ weakly in } L^{p'_i(\cdot)}(\Omega), \quad (3.31)$$

where $p'_i(\cdot)$ denotes the Hölder conjugate of $p_i(\cdot)$ in $\bar{\Omega}$.

By (3.22), we conclude that for all $i = 1, \dots, N$,

$$v_n(\cdot) \rightarrow v(\cdot) \text{ strongly in } L^{p_i(\cdot)}(\Omega). \quad (3.32)$$

Then, from (3.31) and (3.32), for all $i = 1, \dots, N$ and all $j = 1, \dots, m$, we obtain

$$v_n(x)\sigma_i^{(j)}(x, \partial_i u_n) \rightarrow v(x)\sigma_i^{(j)}(x, \partial_i u) \text{ strongly in } L^1(\Omega). \quad (3.33)$$

So, (3.33) implies that

$$v_n(x)\sigma_i(x, \partial_i u_n) \rightarrow v(x)\sigma_i(x, \partial_i u) \text{ strongly in } L^1(\Omega; \mathbb{R}^m). \quad (3.34)$$

Now, we put $g(x, u_n) = (g_1(x, u_n), \dots, g_m(x, u_n))$ and $g(x, u) = (g_1(x, u), \dots, g_m(x, u))$. Through (3.18) and the fact that $|g_j(x, u_n)| \leq |g(x, u_n)|$, $j = 1, \dots, m$, we conclude that

$$(g_j(x, u_n)) \text{ is bounded in } L^\infty(\Omega). \quad (3.35)$$

Then, as $A_n \in L^1(\Omega)$, from (3.35) and (3.27), for all $j = 1, \dots, m$, we obtain

$$A_n(x)g_j(x, u_n) \rightarrow A(x)g_j(x, u) \text{ strongly in } L^1(\Omega). \quad (3.36)$$

So, (3.36) implies that

$$A_n(x)g(x, u_n) \rightarrow A(x)g(x, u) \text{ strongly in } L^1(\Omega; \mathbb{R}^m). \quad (3.37)$$

Then, through (3.34) and (3.37), we can pass to the limit in (3.5). Thus Theorem 3.1 is proved.

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