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NONLOCAL BOUNDARY VALUE PROBLEMS FOR SECOND ORDER LINEAR HYPERBOLIC SYSTEMS WITH TWO INDEPENDENT VARIABLES

In memory of Professor N. A. Perestyuk

Abstract. For second order linear hyperbolic systems nonlocal boundary value problems in a characteristic rectangle are investigated. Necessary and sufficient conditions of solvability, unique solvability, and well-posedness of problems under consideration are established.

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რეზიუმე. მეორე რიგის წრფივი ჰიპერბოლური სისტემებისთვის შესწავლილია არალოკალური სასაზღვრო ამოცანები მახასიათებელ სწორკუთხედში. დადგენილია ამ ამოცანათა ამოხსნადობის, ცალსახად ამოხსნადობისა და კორექტულობის აუცილებელი და საკმარისი პირობები.

Introduction

In the present paper, we study the linear hyperbolic system

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x) \tag{0.1}$$

in the characteristic rectangle $\Omega = [0,\omega_1] \times [0,\omega_2]$ subject to the nonlocal boundary conditions

$$\ell u(\,\cdot\,,y)) = \varphi(y), \quad h(u_x(x,\,\cdot\,)) = \psi(x), \tag{0.2}$$

where $P_j : C(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$, and $\ell : C([0,\omega_1];\mathbb{R}^n) \to \mathbb{R}^n$ and $h : C([0,\omega_2];\mathbb{R}^n) \to \mathbb{R}^n$ are bounded linear operators. Goursat initial-characteristic conditions

$$u(0,y) = \varphi(y), \quad u_x(x,0) = \psi(x) \tag{0.3}$$

and initial-boundary conditions

$$u(0,y) = \varphi(y), \quad h(u_x(x,\,\cdot\,)) = \psi(x) \tag{0.4}$$

are particular cases of (0.2). Problem (0.1), (0.2), except for the Goursat problem (0.1), (0.3), does not belong to the classical boundary value problems of mathematical physics.

Initial-periodic and doubly periodic problems are one of the most important particular cases of problem (0.1), (0.2). In fact, problems on periodic solutions were the original problems that have been intensively studied for partial differential equations of hyperbolic type (see [4–17, 37–49]). These problems naturally led to the initial-boundary value problems in a rectangle with general boundary conditions

$$u(0,y) = \varphi(y), \quad \mathcal{F}(u_x(x,\,\cdot\,)) = \psi(x), \tag{0.5}$$

where $\mathcal{F} : C([0,\omega_2];\mathbb{R}^n) \to C([0,\omega_1];\mathbb{R}^n)$ is a bounded linear operator. Notice that (0.4) is a particular case of (0.5). However, (0.5) is not a particular case of (0.2).

Initial-boundary value problems with integral boundary conditions for quasi-linear systems were studied in [1–3].

Initial-periodic boundary value problems for quasi-linear and nonlinear hyperbolic equations and systems were studied in [22, 23, 32].

Initial-boundary value problems for higher order linear hyperbolic equations were studied in [30] and [31].

Initial-boundary value problems for higher order nonlinear hyperbolic equations were studied in [29, 35].

Problems on doubly periodic solutions for linear hyperbolic systems were studied in [24].

Problems on doubly periodic solutions for nonlinear hyperbolic equations were studied in [25].

Nonlocal boundary value problems, in particular the Dirichlet problem and problems on doubly periodic solutions, for higher order hyperbolic equations were studied in [26, 27, 31, 34, 35].

A comprehensive theory of initial-boundary value problems (0.1), (0.5) was constructed in [21].

As for problem (0.1), (0.2), in its general form it has remained virtually unexplored.

This is the reason why in the present paper we do not consider initial-boundary value problem (0.1), (0.4) and focus our attention to fully nonlocal boundary value problem (0.1), (0.2).

The paper is organized as follows:

In Section 1, the necessary and sufficient conditions for the solvability and well-posedness of the general boundary value problem (0.1), (0.2), as well as several of its special cases, are formulated.

In Section 2, the problem of doubly periodic solutions in the plane is considered. This approach seems more appropriate than studying (equivalent) doubly-periodic problem in a characteristic rectangle Ω. One of the main reasons for studying the periodic problem separately is that it is not well-posed in the sense of Definition 1.2. The definition of well-posedness of the problem on doubly-periodic solutions is given in Definition 2.1 and differs from Definition 1.2.

Auxiliary lemmas are formulated and proved in Section 3.

The main results are proved in Section 4.

Throughout the paper, the following notations will be used:

- $\mathbb{R}^{m\times n}$ is the space of $m\times n$ matrices $X=(x_{kl})$ with real components x_{kl} ($k=1,\dots,m$; $l=1,\dots,n$) and the norm
$$\|X\|=\sum_{k=1}^{m}\sum_{l=1}^{n}|x_{kl}|.$$
- I is the unit matrix, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$.
- By the absolute value of the matrix $X=(x_{ij})\in\mathbb{R}^{m\times n}$ we understand the matrix $|X|=(|x_{ij}|)\in\mathbb{R}^{m\times n}$ with components $|x_{ij}|$ ($i=1,\dots,m$; $j=1,\dots,n$).
- A matrix $X=(x_{ij})\in\mathbb{R}^{m\times n}$ is called non-negative if $x_{ij}\geq 0$ ($i=1,\dots,m$; $j=1,\dots,n$).
- The inequalities between the matrices $X=(x_{ij})$ and $Y=(y_{ij})\in\mathbb{R}^{m\times n}$ are understood componentwise, i.e.,
$$X\leq Y\iff x_{ij}\leq y_{ij}\quad (i=1,\dots,m;\ \ j=1,\dots,n).$$
- If $X_k=(x_{ijk})\in\mathbb{R}^{m\times n}$ ($k=1,\dots,k_0$), then
$$\max_{1\leq k\leq k_0}X_k=\Big(\max_{1\leq k\leq k_0}x_{ijk}\Big).$$
- $\Omega=[0,\omega_1]\times[0,\omega_2]$, $\Omega_x=[0,x]\times[0,\omega_2]$.
$$u^{(j,k)}=\frac{\partial^{j+k}u}{\partial x^j\partial y^k}\,.$$
- $\langle x,y\rangle:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ ($\langle x,y\rangle:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$) is the standard inner product in $\mathbb{R}^n$ (in $\mathbb{C}^n$)
$$\langle x,y\rangle=\sum_{i=1}^{n}x_iy_i\quad\Big(\langle x,y\rangle=\sum_{i=1}^{n}x_i\overline{y}_i\Big).$$
- $C^k([0,\omega];\mathbb{R}^{m\times n})$ is the Banach space of functions $u:[0,\omega]\to\mathbb{R}^{m\times n}$, having continuous partial derivatives $u^{(i)}$ ($i=0,\dots,k$), endowed with the norm
$$\|u\|_{C^k([0,\omega])}=\sum_{i=0}^{k}\|u^{(i)}\|_{C([0,\omega])}.$$
- $C^k(\Omega;\mathbb{R}^{m\times n})$ is the Banach space of functions $u:\Omega\to\mathbb{R}^{m\times n}$, having continuous partial derivatives $u^{(i,j)}$ ($i+j=0,\dots,k$), endowed with the norm
$$\|u\|_{C^k(\Omega)}=\sum_{i+j=0}^{k}\|u^{(i,j)}\|_{C(\Omega)}.$$
- $C^{k,l}(\Omega;\mathbb{R}^{m\times n})$ is the Banach space of functions $u:\Omega\to\mathbb{R}^{m\times n}$, having continuous partial derivatives $u^{(i,j)}$ ($i=0,\dots,k$; $j=0,\dots,l$), endowed with the norm
$$\|u\|_{C^{k,l}(\Omega)}=\sum_{i=0}^{k}\sum_{j=0}^{l}\|u^{(i,j)}\|_{C(\Omega)}.$$

– $C^k_\omega(\mathbb{R};\mathbb{R}^{m\times n})$ is the space of k-times continuously differentiable ω-periodic functions, endowed with the norm

$$\|u\|_{C^k_\omega}=\sum_{i=0}^{k}\|u^{(i)}\|_{C([0,\omega])}.$$

– $C^{k,l}_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$ is the Banach space of functions that are ω_1-periodic with respect to the first variable and ω_2-periodic with respect to the second variable, having continuous partial derivatives $u^{(i,j)}$ ($i=0,\dots,k$; $j=0,\dots,l$), endowed with the norm

$$\|u\|_{C^{k,l}_{\omega_1\,\omega_2}}=\sum_{i=0}^{k}\sum_{j=0}^{l}\|u^{(i,j)}\|_{C(\Omega)}.$$

1 Nonlocal boundary value problems

In this section, for the system

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y+q(x,y), \tag{1.1}$$

we consider the fully nonlocal boundary conditions

$$\ell(u(\,\cdot\,,y))=\varphi(y),\quad h(u_x(x,\,\cdot\,))=\psi(x), \tag{1.2}$$

where $\ell:C([0,\omega_1];\mathbb{R}^n)\to\mathbb{R}^n$ and $h:C([0,\omega_2];\mathbb{R}^n)\to\mathbb{R}^n$ are *admissible* bounded linear operators.

Definition 1.1. A bounded linear operator $\mathcal{F}:C([a,b];\mathbb{R}^n)\to\mathbb{R}^n$ is called *admissible* if there exists $P\in C([a,b];\mathbb{R}^{n\times n})$ such that the problem

$$z'=P(t)z,\quad \mathcal{F}(z)=0$$

has only the trivial solution.

One may think that the boundary conditions

$$\ell(u(\,\cdot\,,y))=\varphi(y),\quad h(u(x,\,\cdot\,))=\Psi(x) \tag{$\widetilde{2.2}$}$$

are more natural than conditions (1.2). All the more so, conditions ($\widetilde{2.2}$) obviously imply conditions (1.2) with $\psi(x)=\Psi'(x)$. The main reason for studying problem (1.1), (1.2) instead of problem (1.1), ($\widetilde{2.2}$) is that problem (1.1), ($\widetilde{2.2}$) is inherently ill-posed, since the functions φ and Ψ should satisfy certain compatibility condition. In general, it is virtually impossible to establish that condition explicitly. However, if the operators ℓ and h commute, i.e., if

$$\ell\circ h=h\circ\ell, \tag{1.3}$$

then the functions φ and Ψ must satisfy the following compatibility condition $\ell(\Psi)=h(\varphi)$. Indeed, if $u\in C(\Omega)$ is an arbitrary function satisfying conditions ($\widetilde{2.2}$), then, in view of (1.3), we get $\ell(\Psi)=\ell\circ h\,(u)=h\circ\ell\,(u)=h(\varphi)$.

By a solution of problem (1.1), (1.2) we understand a *classical* solution, i.e., a function $u\in C^{1,1}(\Omega)$ satisfying system (1.1) and boundary conditions (1.2) everywhere in Ω.

Along with problem (1.1), (1.2), consider its corresponding homogeneous problem

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y, \tag{1.1_0}$$
$$\ell(u(\,\cdot\,,y))=0,\quad h(u_x(x,\,\cdot\,))=0, \tag{1.2_0}$$

as well as the problems

$$v'=P_2(x,y^*)v, \tag{1.1_1}$$
$$\ell(v)=0 \tag{1.2_1}$$

and

$$v' = P_1(x^*, y)v, \tag{1.1$_2$}$$
$$h(v) = 0. \tag{1.2$_2$}$$

Problems $(1.1_1), (1.2_1)$ and $(1.1_2), (1.2_2)$ are called the ***associated problems*** of problem (1.1), (1.2). Notice that problem $(1.1_1), (1.2_1)$ (problem $(1.1_1), (1.2_1)$) is a boundary value problem for a linear ordinary differential equation depending on the parameter $y^* \in [0, \omega_2]$ (on the parameter $x^* \in [0, \omega_1]$).

The concept of $\boldsymbol{\sigma}$-associated problems for n-dimensional periodic problems was introduced in [38], and for two-dimensional Dirichlet type problems in [46].

1.1 Necessary conditions of solvability

Theorem 1.1. *Let condition (1.3) hold, let h be an admissible operator, and let problem (1.1), (1.2) be solvable for arbitrary $\varphi \in C^1([0, \omega_2]; \mathbb{R}^n)$ and $\psi \in C([0, \omega_1]; \mathbb{R}^n)$. Then the problem*

$$z' = 0, \quad \ell(z) = 0 \tag{1.4}$$

has only the trivial solution.

Remark 1.1. If problem (1.4) has only the trivial solution, then problem $(1.1_0), (1.2_0)$ is equivalent to the homogeneous problem

$$u_{xy} = P_0(x, y)\, u + P_1(x, y)\, u_x + P_2(x, y)\, u_y, \tag{1.1$_0$}$$
$$\ell(u(\,\cdot\,, y)) = 0, \quad h(u(x, \,\cdot\,)) = 0. \tag{$\widetilde{1.2}_0$}$$

Theorem 1.2. *Let P_j $(j = 1, 2)$ be constant matrices, let problem $(1.1_1), (1.2_1)$ have a nontrivial solution, and let the following conditions hold:*

$$h\big(P_0(x, \,\cdot\,)z(\,\cdot\,)\big) = h(P_0(x, \,\cdot\,))\, h(z(\,\cdot\,)) \;\text{ for every }\; z \in C([0, \omega_2]; \mathbb{R}^n), \tag{1.5}$$
$$h(P_1 z(\,\cdot\,)) = P_1\, h(z(\,\cdot\,)) \;\text{ for every }\; z \in C([0, \omega_2]; \mathbb{R}^n), \tag{1.6}$$
$$h(P_2 z(\,\cdot\,)) = P_2\, h(z(\,\cdot\,)) \;\text{ for every }\; z \in C([0, \omega_2]; \mathbb{R}^n).$$

Then, for the solvability of problem (1.1), (1.2), it is necessary that the problem

$$v' = P_2 v + \big(h(P_0(x, \,\cdot\,)) + P_2 P_1\big)\Psi(x) + h(q(x, \,\cdot\,)), \tag{1.7}$$
$$\ell(v) = h(\varphi') - \ell(P_1\, \Psi), \tag{1.8}$$

be solvable, where Ψ is a solution of the problem

$$z' = \psi(x), \quad \ell(z) = h(\varphi).$$

Theorem 1.3. *Let P_j $(j = 1, 2)$ be constant matrices, let problem $(1.1_2), (1.2_2)$ have a nontrivial solution, and let along with (1.4) the following conditions hold:*

$$\ell\big(P_0(\,\cdot\,, y)z(\,\cdot\,)\big) = \ell(P_0(\,\cdot\,, y))\, \ell(z(\,\cdot\,)) \;\text{ for every }\; z \in C([0, \omega_1]; \mathbb{R}^n),$$
$$\ell(P_1 z(\,\cdot\,)) = P_1\, \ell(z(\,\cdot\,)) \;\text{ for every }\; z \in C([0, \omega_1]; \mathbb{R}^n),$$
$$\ell\big(P_2 z(\,\cdot\,)\big) = P_2\, \ell(z(\,\cdot\,)) \;\text{ for every }\; z \in C([0, \omega_1]; \mathbb{R}^n).$$

Then, for the solvability of problem (1.1), (1.2), it is necessary that the problem

$$v' = P_1 v + \big(\ell(P_0(\,\cdot\,, y)) + P_1 P_2\big)\varphi(y) + \ell(q(\,\cdot\,, y)), \tag{1.9}$$
$$h(v) = \ell(\psi) - h(P_2 \varphi) \tag{1.10}$$

be solvable.

Remark 1.2. Solvability of the ill-posed nonhomogenous problem (1.9), (1.10) means additional compatibility conditions between the boundary values φ and ψ, matrices P_0, P_1 and P_2, and the vector function q. Indeed, consider the problem

$$u_{xy} = P_0(x,y)u + q(x,y), \tag{1.11}$$

$$u(0,y) = \varphi(y), \quad u_x(x,0) = u_x(x,\omega_2). \tag{1.12}$$

Let u be a solution of problem (1.11), (1.12). Set $v(y) = u_x(0,y)$. Then v is a solution of the problem

$$v' = P_0(0,y)\,\varphi(y) + q(0,y), \tag{1.13}$$

$$v(0) = v(\omega_2). \tag{1.14}$$

In other words, the solvability of (1.13), (1.14) is necessary for the solvability of problem (1.11), (1.12). Problem (1.13), (1.14) itself is ill-posed. It is solvable if and only if the following equality holds:

$$\int_0^{\omega_2} \big(P_0(0,t)\,\varphi(t) + q(0,t)\big)\,dt = 0.$$

Remark 1.3. Solvability of the ill-posed nonhomogenous problem (1.9), (1.10) is necessary for the solvability of problem (1.1), (1.2), but is by no means sufficient. Indeed, consider the two-dimensional problem

$$\begin{aligned} v_{xy} &= w - q_1(x),\\ w_{xy} &= -v + q_2(x), \end{aligned} \tag{1.15}$$

$$\begin{gathered} v(0,y) = 0, \quad v(\omega_1,y) = 0,\\ v_x(x,0) = v_x(x,\omega_2), \quad w_x(x,0) = w_x(x,\omega_2). \end{gathered} \tag{1.16}$$

Let us show that the corresponding homogeneous problem has only the trivial solution. Let $\begin{pmatrix} v(x,y)\\ w(x,y)\end{pmatrix}$ be an arbitrary solution of the homogeneous system

$$v_{xy} = w, \tag{1.17}$$

$$w_{xy} = -v, \tag{1.18}$$

satisfying conditions (1.16). Multiply (1.17) by w and integrate over Ω. After integrating by parts and taking into account conditions (1.16), we arrive at the equality

$$-\int_0^{\omega_1}\int_0^{\omega_2} v_x(x,y)w_y(x,y)\,dy\,dx = \int_0^{\omega_1}\int_0^{\omega_2} w^2(x,y)\,dy\,dx. \tag{1.19}$$

Similarly, after multiplying (1.18) by v and integrating over Ω, we get

$$-\int_0^{\omega_1}\int_0^{\omega_2} w_y(x,y)v_x(x,y)\,dy\,dx = -\int_0^{\omega_1}\int_0^{\omega_2} v^2(x,y)\,dy\,dx. \tag{1.20}$$

After subtracting (1.20) from (1.19), we arrive at the equality

$$\int_0^{\omega_1}\int_0^{\omega_2} \big(v^2(x,y) + w^2(x,y)\big)\,dy\,dx = 0.$$

Consequently, the homogeneous problem (1.17), (1.18) has only the trivial solution. Therefore, problem (1.15), (1.16) has at most one solution. Hence, the only possible (ω_2-*periodic with respect to the second variable*) solution of problem (1.15), (1.16) should be independent of y. Consequently,

$$\begin{pmatrix} v(x,y)\\ w(x,y)\end{pmatrix} = \begin{pmatrix} q_1(x)\\ q_2(x)\end{pmatrix}$$

is the only possible solution of problem (1.15), (1.16). It is clear that u is a weak solution, but not a classical one, if q_1 and q_2 are *nowhere differentiable* continuous functions.

1.2 Necessary and sufficient conditions of well-posedness

Theorem 1.4. *Let the following conditions hold:*

(A_0) *problem* (1.4) *have only the trivial solution;*

(A_1) *problem* $(1.1_1), (1.2_1)$ *have only the trivial solution for every* $y^* \in [0, \omega_2]$*;*

(A_1) *problem* $(1.1_2), (1.2_2)$ *have only the trivial solution for every* $x^* \in [0, \omega_1]$*.*

Then problem (1.1), (1.2) *has the Fredholm property, i.e., the following assertions hold:*

(i) *problem* $(1.1_0), (1.2_0)$ *has a finite dimensional space of solutions;*

(ii) *if problem* $(1.1_0), (1.2_0)$ *has only the trivial solution, then problem* (1.1), (1.2) *is uniquely solvable, and its solution* u *admits the estimate*

$$\|u\|_{C^{1,1}(\Omega)} \le M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])}\Big), \tag{1.21}$$

where M *is a positive constant independent of* φ*,* ψ *and* q*.*

Definition 1.2. Problem (1.1), (1.2) is called *well-posed* if it is uniquely solvable for arbitrary $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$ and $q \in C(\Omega)$, and its solution u admits the estimate (1.21) where M is a positive constant independent of φ, ψ and q.

Theorem 1.5. *Let problem* (1.1), (1.2) *be well-posed. Then conditions* (A_1) *and* (A_2) *of Theorem 1.4 hold. Moreover, if equality* (1.3) *holds, then problem* (1.4) *has only the trivial solution.*

Remark 1.4. Consider the problem

$$u_{xy} = p(x)\,u_x + p(x)\,u_y - p^2(x)\,u + q(x,y), \tag{1.22}$$

$$u(0,y) = 2u(\omega_1,y), \quad u_x(x,0) = u_x(x,\omega_2), \tag{1.23}$$

where $p \in C^\infty([0,\omega_1])$ is a *nonnegative* function and $q \in C^\infty(\Omega)$. Let $q(x,y) = p(x)\,\widetilde{q}(x,y)$. Set: $J_p = \{x \in [0,\omega_1] :\ p(x) = 0\}$. Then:

(i) problem (1.22), (1.23) is well-posed if and only if $J_p = \varnothing$. Moreover, if $J_p = \varnothing$, then the unique solution

$$u(x,y) = \int\limits_0^{\omega_1}\int\limits_0^{\omega_2} g_1(x,s)g_2(y,t;x)q(s,t)\,dt\,ds$$

of problem (1.22), (1.23), where

$$g_1(x,s) = \begin{cases} \dfrac{2\exp\big(\int_s^{\omega_1} p(\xi)\,d\xi\big)}{1 - 2\exp\big(\int_0^{\omega_1} p(\xi)\,d\xi\big)} + \exp\bigg(\displaystyle\int_s^x p(\xi)\,d\xi\bigg) & \text{for } 0 \le s < x \le \omega_1, \\ \dfrac{2\exp\big(\int_s^{\omega_1} p(\xi)\,d\xi\big)}{1 - 2\exp\big(\int_0^{\omega_1} p(\xi)\,d\xi\big)} & \text{for } 0 \le x \le s \le \omega_1 \end{cases}$$

and

$$g_2(y,t;x) = \begin{cases} \dfrac{\exp(\omega_2 p(x))}{1 - \exp(\omega_2 p(x))} + \exp((y-t)p(x)) & \text{for } 0 \le s < x \le \omega_1, \\ \dfrac{\exp(\omega_2 p(x))}{1 - \exp(\omega_2 p(x))} & \text{for } 0 \le x \le s \le \omega_1, \end{cases}$$

belongs to $C^\infty(\Omega)$;

(ii) if $\widetilde{q} \in L^\infty([0,\omega_1])$, then

$$u(x,y) = \int_0^{\omega_1}\int_0^{\omega_2} g_1(x,s)\widetilde{g}_2(y,t;x)\widetilde{q}(s,t)\,dt\,ds, \tag{1.24}$$

where

$$\widetilde{g}_2(y,t;x) = \begin{cases} p(x)g_2(y,t;x) & \text{for } x \in [0,\omega_1]\setminus J_p, \\ 0 & \text{for } x \in J_p \end{cases}$$

is a unique weak solution of problem (1.22), (1.23) *if and only if* $\operatorname{mes} J_p = 0$. Moreover, if $\operatorname{mes} J_p > 0$, then problem (1.22), (1.23) has *infinite dimensional* set of *nonclassical* weak solutions. If $\widetilde{q} \in C([0,\omega_1])$ and $\operatorname{mes} I_p = 0$, then the unique weak solution is a classical solution;

(iii) If $\widetilde{q} \in C([0,\omega_1])$, then $u(x,y)$, given by (1.24), is a unique *classical* solution of problem (1.22), (1.23) if and only if J_p is nowhere dense in $[0,\omega_1]$, i.e., $\operatorname{supp} p = [0,\omega_1]$. Moreover, if $\operatorname{supp} p \neq [0,\omega_1]$, then problem (1.22), (1.23) has an *infinite dimensional set* of classical solutions;

(iv) problem (1.22), (1.23) has a *unique classical* solution and *infinite dimensional set* of *weak* solutions if J_p is a nowhere dense set of a positive measure;

(v) if $q(x,y) = 1$ and $J_p \neq \varnothing$, then problem (1.22), (1.23) has *no classical solution* despite the fact that the coefficients of equation (1.22) belong to $C^\infty(\Omega)$.

Theorem 1.6. *Let conditions $(A_0), (A_1), (A_2)$ of Theorem 1.4 hold, and let $P_2 \in C^{0,1}(\Omega)$ be such that*

$$h(v) = 0 \implies h(P_2(\,\cdot\,,y)v(\,\cdot\,)) = 0 \ \text{ for } \ y \in [0,\omega_2], \ \ v \in C([0,\omega_2]). \tag{1.25}$$

Then there exists $\varepsilon > 0$ such that if

$$\big\|P(x,y) + P_1(x,y)\,P_2(x,y) - P_{2y}(x,y)\big\| \le \varepsilon \ \text{ for } \ (x,y) \in \Omega,$$

then problem (1.1), (1.2) is well-posed. In particular, if

$$P(x,y) + P_1(x,y)\,P_2(x,y) - P_{2y}(x,y) = \mathrm{O}, \tag{1.26}$$

then the solution of problem (1.1), (1.2_0) admits the representation

$$u(x,y) = \int_0^{\omega_1}\int_0^{\omega_2} G_1(x,s,y)\,G_2(y,t,s)\,q(s,t)\,dt\,ds, \tag{1.27}$$

where G_j is Green's matrix of problem $(1.1_j), (1.2_j)$ $(j = 1,2)$.

1.3 Two-point boundary value problems

For system (1.1), consider the boundary conditions

$$u(a,y) = Au(b,y) + \varphi(y), \quad u_x(x,c) = Bu_x(x,d) + \psi(x), \tag{1.28}$$

and

$$u(a,y) = Au(b,y) + \varphi(y), \quad u_x(x,c) = u_x(x,d) + \psi(x), \tag{1.29}$$

where $0 \le a < b \le \omega_1$, $0 \le c < d \le \omega_2$, and $A, B \in \mathbb{R}^{n\times n}$.

By $G_1(x,s;y^*)$ and $G_2(y,t;x^*)$, respectively, denote Green's matrices of the problems

$$\frac{dz}{dx} = P_2(x,y^*)z; \quad z(a) = Az(b), \tag{1.30}$$

and

$$\frac{dz}{dy} = P_1(x^*,y)z; \quad z(c) = Bz(d). \tag{1.31}$$

Theorem 1.7. *Let $P_1(x,y) \equiv P_1(x)$, $P_2(x,y) \equiv P_2(y)$, and let*

$$\det(I - A) \neq 0, \tag{1.32}$$

$$\det\big(I - A\exp((b-a)P_2(y))\big) \neq 0 \ \ \textit{for} \ \ y \in [0,\omega_2], \tag{1.33}$$

$$\det\big(I - B\exp((d-c)P_1(x))\big) \neq 0 \ \ \textit{for} \ \ x \in [0,\omega_1]. \tag{1.34}$$

Then problem (1.1), (1.28) has the Fredholm property.

Theorem 1.8. *Let inequality (1.32) hold, and let there exist $\sigma_i \in \{-1,1\}$ $(i=1,2)$ such that either*

$$\sigma_1(B^TB - I) \ \ \textit{is positive semi-definite},$$
$$\sigma_1\widehat{P}_1(x,y) \ \ \textit{is positive definite for} \ \ (x,y) \in [0,\omega_1]\times[c,d],$$

or

$$\sigma_1(B^TB - I) \ \ \textit{is positive definite},$$
$$\sigma_1\widehat{P}_1(x,y) \ \ \textit{is positive semi-definite for} \ \ (x,y) \in [0,\omega_1]\times[c,d],$$

and either

$$\sigma_2(A^TA - I) \ \ \textit{is positive semi-definite},$$
$$\sigma_2\widehat{P}_2(x,y) \ \ \textit{is positive definite for} \ \ (x,y) \in [a,b]\times[0,\omega_2],$$

or

$$\sigma_2(A^TA - I) \ \ \textit{is positive definite},$$
$$\sigma_2\widehat{P}_2(x,y) \ \ \textit{is positive semi-definite for} \ \ (x,y) \in [a,b]\times[0,\omega_2].$$

Then problem (1.1), (1.28) has the Fredholm property.

Theorem 1.9. *Let inequality (1.32) hold, let problem (1.30) have only the trivial solution for every $y^* \in [0,\omega_2]$, and let problem (1.31) have only the trivial solution for every $x^* \in [0,\omega_1]$. Furthermore, let*

$$P_2 \in C^{0,1}(\Omega;\mathbb{R}^{n\times n}), \quad P_2(x,c) = P_2(x,d), \quad P_2(x,d)B = BP_2(x,d),$$

and

$$\int_a^b\int_c^d \Big|G_1(x,s;y)G_2(y,t;s)\big(P_0(s,t) + P_1(s,t)P_2(s,t) - P_{2t}(s,t)\big)\Big|\,dt\,ds \le \Lambda,$$

where $\Lambda \in \mathbb{R}_+^{n\times n}$ is a nonnegative matrix with the spectral radius less than 1. Then problem (1.1), (1.28) is well-posed.

Theorem 1.10. *Let inequality (1.32) hold, let $P_j(x,y) \equiv P_j(x)$ $(j=0,1,2)$ be symmetric matrix functions such that*

$$P_j(x)P_k(x) = P_k(x)P_j(x) \ \ \textit{for} \ \ x \in [0,\omega_1] \ \ (j,k=0,1,2), \tag{1.35}$$

and let there exist $\sigma_i \in \{-1,1\}$ $(i=0,1,2)$ such that

$$\sigma_0\,\sigma_1\,\sigma_2 = -1, \tag{1.36}$$

$$\sigma_2(A^TA - I) \ \ \textit{is positive semi-definite}, \tag{1.37}$$

$$\sigma_iP_i(x) \ \ \textit{is positive definite for every} \ \ x \in [0,\omega_1], \ \ (i=0,1,2). \tag{1.38}$$

Then problem (1.1), (1.29) is well-posed.

Consider the one-dimensional case of problem (1.1), (1.28):

$$u_{xy} = p_0(x)u + p_1(x)u_x + p_2(x)u_y + q(x,y), \tag{1.39}$$
$$u(a,y) = \alpha u(b,y) + \varphi(y), \quad u_x(x,c) = u_x(x,d) + \psi(x). \tag{1.40}$$

Corollary 1.1. *Let the following inequalities hold:*

$$p_0(x)\,p_1(x)\,p_2(x) < 0 \ \ \textit{for} \ \ x \in [0,\omega_1]$$

and

$$(1-\alpha)\,p_2(x) < 0 \ \ \textit{for} \ \ x \in [0,\omega_1].$$

Then problem (1.39), (1.40) *is well-posed.*

Theorem 1.11. *Let inequality* (1.32) *hold, let $P_j(x,y) \equiv P_j$ $(j = 0,1,2)$, and let the inequalities*

$$\det\big(I - A\exp((b-a)P_2)\big) \neq 0, \tag{1.41}$$
$$\det\big(I - \exp((d-c)P_1)\big) \neq 0 \tag{1.42}$$

hold. Then problem (1.1), (1.29) *is well-posed if and only if*

$$\det\big(I - A\exp((b-a)\Lambda_k)\big) \neq 0 \ \ \textit{for} \ \ k \in \mathbb{Z},$$

where

$$\Lambda_k = \Big(i\,\frac{2\pi}{d-c}\,kI - P_1\Big)^{-1}\Big(P_0 + i\,\frac{2\pi}{d-c}\,kP_2\Big).$$

Let $n = 2m$, $u = (v,w)$, and $v,w \in \mathbb{R}^m$. For the system

$$v_{xy} = Q_1(y)v_x + P_{11}(y)w_x + Q_2(x)v_y + P_{21}(x)w_y + Q_0(x,y)v + P_{01}(x,y)w + q_1(x,y), \tag{1.43}$$
$$w_{xy} = P_{12}(y)v_x - Q_1(y)w_x + P_{22}(x)v_y - Q_2(x)w_y - P_{02}(x,y)v + Q_0(x,y)w + q_2(x,y), \tag{1.44}$$

consider the following boundary conditions:

$$\begin{aligned} v(a,y) &= Av(b,y) + \varphi_1(y), \quad w(b,y) = Aw(a,y) + \varphi_2(y),\\ v_x(x,c) &= Bv_x(x,d) + \psi_1(x), \quad w_x(x,d) = Bw_x(x,a) + \psi_2(x). \end{aligned} \tag{1.45}$$

Periodic type, antiperiodic type, and Nicoletti type boundary conditions

$$\begin{aligned} v(a,y) &= \alpha\, v(b,y) + \varphi_1(y), \quad w(b,y) = \alpha\, w(a,y) + \varphi_2(y),\\ v_x(x,c) &= \beta\, v_x(x,d) + \psi_1(x), \quad w_x(x,d) = \beta\, w_x(x,a) + \psi_2(x); \end{aligned} \tag{1.46}$$

$$\begin{aligned} v(a,y) &= \alpha v(b,y) + \varphi_1(y), \quad w(b,y) = \alpha w(a,y) + \varphi_2(y),\\ v_x(x,c) &= v_x(x,d) + \psi_1(x), \quad w_x(x,d) = w_x(x,c) + \psi_2(x); \end{aligned} \tag{1.47}$$

$$\begin{aligned} v(a,y) &= -v(b,y) + \varphi_1(y), \quad w(b,y) = -w(a,y) + \varphi_2(y),\\ v_x(x,c) &= -v_x(x,d) + \psi_1(x), \quad w_x(x,d) = -w_x(x,c) + \psi_2(x) \end{aligned} \tag{1.48}$$

and

$$v(a,y) = \varphi_1(y), \quad w(b,y) = \varphi_2(y), \quad v_x(x,c) = \psi_1(x), \quad w_x(x,d) = \psi_2(x) \tag{1.49}$$

are particular cases of (1.45). Here, $0 \le a < b \le \omega_1$, $0 \le c < d \le \omega_2$, α, $\beta \in \mathbb{R}$, $\varphi_i \in C([0,\omega_2];\mathbb{R}^m)$, $\psi_i \in C([0,\omega_1];\mathbb{R}^m)$, $q_i \in C(\Omega;\mathbb{R}^m)$ $(i = 1,2)$, $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times m}$ are *symmetric* matrices, and $Q_1 \in C([0,\omega_2];\mathbb{R}^{m\times m})$, $Q_2 \in C([0,\omega_1];\mathbb{R}^{m\times m})$, $Q_0 \in C(\Omega;\mathbb{R}^{m\times m})$, $P_{1i} \in C([0,\omega_2];\mathbb{R}^{m\times m})$, $P_{2i} \in C([0,\omega_1];\mathbb{R}^{m\times m})$ and $P_{0i} \in C(\Omega;\mathbb{R}^{m\times m})$ $(i = 1,2)$ are *symmetric* matrix functions.

Theorem 1.12. *Let*

$$\det(I - A) \neq 0, \tag{1.50}$$
$$AQ_1(y) = Q_1(y)A \ \ \textit{and} \ \ BQ_2(x) = Q_2(x)B, \tag{1.51}$$
$$AP_{1i}(y) = P_{1i}(y)A \ \ \textit{and} \ \ BP_{2i}(x) = P_{2i}(x)B \ \ (i = 1,2),$$

and let there exist $\sigma_j \in \{-1,1\}$ $(j = 1,2,3)$ such that the following conditions hold:

(A_1) *$\sigma_1 P_{11}(y)$ and $\sigma_1 P_{12}(y)$ are positive semi-definite symmetric matrix functions on $[c,d]$ such that $\sigma_1 P_{11}(y^*)$ and $\sigma_1 P_{12}(y^*)$ are positive definite symmetric matrices for some $y^* \in [c,d]$;*

(A_2) *$\sigma_2 P_{21}(x)$ and $\sigma_2 P_{22}(x)$ are positive semi-definite symmetric matrix functions on $[a,b]$ such that $\sigma_2 P_{21}(x^*)$ and $\sigma_2 P_{22}(x^*)$ are positive definite symmetric matrices for some $x^* \in [a,b]$;*

(A_3) *$\sigma_3 P_{1i}(y)(A^2 - I)$ and $\sigma_3 P_{2i}(y)(B^2 - I)$ $(i = 1,2)$ are positive semi-definite matrix functions on $[c,d]$ and $[a,b]$, respectively;*

(A_4) *$\sigma_3 P_{01}(x,y)$ and $\sigma_3 P_{02}(x,y)$ are positive semi-definite symmetric matrix functions on $[a,b] \times [c,d]$.*

Furthermore, let one of the following conditions hold:

(B_1) *$\sigma_3 P_{11}(y)(A^2 - I)$ and $\sigma_3 P_{12}(y)(A^2 - I)$ are positive definite matrix functions on $[c,d]$;*

(B_2) *$\sigma_3 P_{21}(x)(B^2 - I)$ and $\sigma_3 P_{22}(x)(B^2 - I)$ are positive definite matrix functions on $[a,b]$;*

(B_3) *$\sigma_3 P_{01}(x_0,y)$ and $\sigma_3 P_{02}(x_0,y)$ are positive definite matrix functions on $[c,d]$ for some $x_0 \in [a,b]$;*

(B_4) *$\sigma_3 P_{01}(x,y_0)$ and $\sigma_3 P_{02}(x,y_0)$ are positive definite matrix functions on $[a,b]$ for some $y_0 \in [c,d]$.*

Then problem (1.43), (1.44), (1.45) is well-posed.

Corollary 1.2. *Let there exist $\sigma \in \{-1,1\}$ such that the following conditions hold:*

(A_1) *$\sigma(\alpha^2 - 1)P_{11}(y)$ and $\sigma(\alpha^2 - 1)P_{12}(y)$ are positive semi-definite symmetric matrix functions on $[c,d]$ such that $\sigma(\alpha^2 - 1)P_{11}(y^*)$ and $\sigma(\alpha^2 - 1)P_{12}(y^*)$ are positive definite symmetric matrices for some $y^* \in [c,d]$;*

(A_2) *$\sigma(\beta^2 - 1)P_{21}(x)$ and $\sigma(\beta^2 - 1)P_{22}(x)$ are positive semi-definite symmetric matrix functions on $[a,b]$ such that $\sigma(\beta^2 - 1)P_{21}(x^*)$ and $\sigma(\beta^2 - 1)P_{22}(x^*)$ are positive definite symmetric matrices for some $x^* \in [a,b]$;*

(A_3) *$\sigma P_{01}(x,y)$ and $\sigma P_{02}(x,y)$ are positive semi-definite symmetric matrix functions on $[a,b] \times [c,d]$.*

Furthermore, let one of the following conditions hold:

(B_1) *$\sigma(\alpha^2 - 1)P_{11}(y)$ and $\sigma(\alpha^2 - 1)P_{12}(y)$ are positive definite matrix functions on $[c,d]$;*

(B_2) *$\sigma(\beta^2 - 1)P_{21}(x)$ and $\sigma(\beta^2 - 1)P_{21}(x)$ are positive definite matrix functions on $[a,b]$;*

(B_3) *$\sigma P_{01}(x_0,y)$ and $\sigma P_{02}(x_0,y)$ are positive definite matrix functions on $[c,d]$ for some $x_0 \in [a,b]$;*

(B_4) *$\sigma P_{01}(x,y_0)$ and $\sigma P_{02}(x,y_0)$ are positive definite matrix functions on $[a,b]$for some $y_0 \in [c,d]$.*

Then problem (1.43), (1.44), (1.46) is well-posed.

Corollary 1.3. *Let there exist $\sigma_j \in \{-1,1\}$ $(j = 1,2)$ such that the following conditions hold:*

(A_1) *$\sigma_1(\alpha^2 - 1)P_{11}(y)$ and $\sigma_1(\alpha^2 - 1)P_{12}(y)$ are positive semi-definite symmetric matrix functions on $[c,d]$ such that $\sigma(\alpha^2 - 1)P_{11}(y^*)$ and $\sigma(\alpha^2 - 1)P_{12}(y^*)$ are positive definite symmetric matrices for some $y^* \in [c,d]$;*

(A_2) *$\sigma_2 P_{21}(x)$ and $\sigma_2 P_{22}(x)$ are positive semi-definite symmetric matrix functions on $[a,b]$ such that $\sigma_2 P_{21}(x^*)$ and $\sigma_2 P_{22}(x^*)$ are positive definite symmetric matrices for some $x^* \in [a,b]$;*

(A_3) *$\sigma_1 P_{01}(x,y)$ and $\sigma_1 P_{02}(x,y)$ are positive semi-definite symmetric matrix functions on $[a,b] \times [c,d]$.*

Furthermore, let one of the following conditions hold:

(B_1) *$\sigma_1(\alpha^2 - 1)P_{11}(y)$ and $\sigma_1(\alpha^2 - 1)P_{12}(y)$ are positive definite matrix functions on $[c,d]$;*

(B_2) *$\sigma_1 P_{01}(x_0,y)$ and $\sigma_1 P_{02}(x_0,y)$ are positive definite matrix functions on $[c,d]$ for some $x_0 \in [a,b]$;*

(B_3) *$\sigma_1 P_{01}(x,y_0)$ and $\sigma_1 P_{02}(x,y_0)$ are positive definite matrix functions on $[a,b]$for some $y_0 \in [c,d]$.*

Then problem (1.43), (1.44), (1.47) is well-posed.

Corollary 1.4. *Let there exist $\sigma_j \in \{-1,1\}$ $(j = 1,2,3)$ such that the following conditions hold:*

(A_1) *$\sigma_1 P_{11}(y)$ and $\sigma_1 P_{12}(y)$ are positive semi-definite symmetric matrix functions on $[c,d]$ such that $\sigma_1 P_{11}(y^*)$ and $\sigma_1 P_{12}(y^*)$ are positive definite symmetric matrices for some $y^* \in [c,d]$;*

(A_2) *$\sigma_2 P_{21}(x)$ and $\sigma_2 P_{22}(x)$ are positive semi-definite symmetric matrix functions on $[a,b]$ such that $\sigma_2 P_{21}(x^*)$ and $\sigma_2 P_{22}(x^*)$ are positive definite symmetric matrices for some $x^* \in [a,b]$;*

(A_3) *$\sigma_3 P_{01}(x,y)$ and $\sigma_3 P_{02}(x,y)$ are positive semi-definite symmetric matrix functions on $[a,b]\times[c,d]$.*

Furthermore, let one of the following conditions hold:

(B_1) *$\sigma_3 P_{01}(x_0,y)$ and $\sigma_3 P_{02}(x_0,y)$ are positive definite matrix functions on $[c,d]$ for some $x_0 \in [a,b]$;*

(B_2) *$\sigma_3 P_{01}(x,y_0)$ and $\sigma_3 P_{02}(x,y_0)$ are positive definite matrix functions on $[a,b]$ for some $y_0 \in [c,d]$.*

Then problem (1.43), (1.44), (1.48) is well-posed.

Corollary 1.5. *Let there exist $\sigma \in \{-1,1\}$ such that the following conditions hold:*

(A_1) *$\sigma P_{11}(y)$ and $\sigma P_{12}(y)$ are positive semi-definite symmetric matrix functions on $[c,d]$ such that $\sigma_1 P_{11}(y^*)$ and $\sigma_1 P_{12}(y^*)$ are positive definite symmetric matrices for some $y^* \in [c,d]$;*

(A_2) *$\sigma P_{21}(x)$ and $\sigma P_{22}(x)$ are positive semi-definite symmetric matrix functions on $[a,b]$ such that $\sigma_2 P_{21}(x^*)$ and $\sigma_2 P_{22}(x^*)$ are positive definite symmetric matrices for some $x^* \in [a,b]$;*

(A_3) *$\sigma P_{01}(x,y)$ and $\sigma P_{02}(x,y)$ are negative semi-definite symmetric matrix functions on $[a,b]\times[c,d]$.*

Furthermore, let one of the following conditions hold:

(B_1) *$\sigma P_{11}(y)$ and $\sigma P_{12}(y)$ are positive definite matrix functions on $[c,d]$;*

(B_2) *$\sigma P_{21}(x)$ and $\sigma P_{21}(x)$ are positive definite matrix functions on $[a,b]$;*

(B_3) *$\sigma P_{01}(x_0,y)$ and $\sigma P_{02}(x_0,y)$ are negative definite matrix functions on $[c,d]$ for some $x_0 \in [a,b]$;*

(B_4) *$\sigma P_{01}(x,y_0)$ and $\sigma P_{02}(x,y_0)$ are negative definite matrix functions on $[a,b]$for some $y_0 \in [c,d]$.*

Then problem (1.43), (1.44), (1.49) is well-posed.

Remark 1.5. Notice that problems (1.1), (1.28) and (1.43), (1.44), (1.45) can be localized in the rectangle $[a,b]\times[c,d] \subset \Omega$. In other words, they can be considered as boundary value problems in the rectangle Ω, as well as boundary value problems in the rectangle $[a,b]\times[c,d] \subset \Omega$.

Moreover, if inequality (1.32) holds and the associated problems (1.30) and (1.31) are uniquely solvable for every $y^* \in [0,\omega_2]$ and $x^* \in [0,\omega_1]$, respectively, then problem (1.1), (1.28) is well-posed in Ω if and only if it is well-posed in $[a,b]\times[c,d] \subset \Omega$. In particular, the homogeneous problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y, \tag{1.1$_0$}$$
$$u(a,y) = Au(b,y), \quad u_x(x,c) = Bu_x(x,d), \tag{1.36$_0$}$$

has only the trivial solution in the rectangle Ω if and only if it has only the trivial solution in the rectangle $[a,b]\times[c,d] \subset \Omega$.

Indeed, by Theorem 1.4, problem (1.1), (1.28) has the Fredholm property in the rectangle Ω, as well as in the rectangle $[a,b]\times[c,d] \subset \Omega$.

Let problem (1.1$_0$), (1.36$_0$) have only the trivial solution in $[a,b]\times[c,d]$, and let u be a solution of problem (1.1$_0$), (1.36$_0$) in Ω. Then it is clear that the restriction of $u(x,y) \equiv 0$ on $[a,b]\times[c,d]$. Consequently, in the rectangle $[0,\omega_1]\times[c,d]$, u satisfies the initial-boundary conditions

$$u(a,y) = 0, \quad u_x(x,c) = Bu_x(x,d), \tag{1.52}$$

and, in the rectangle $[a,b]\times[0,\omega_2]$, u satisfies the initial-boundary conditions

$$u(a,y)=Au(b,y),\quad u(x,c)=0. \tag{1.53}$$

Conditions (1.53) imply the initial-boundary conditions

$$u_y(a,y)=Au_y(b,y),\quad u(x,c)=0. \tag{1.54}$$

Thus, in the rectangle $[0,\omega_1]\times[c,d]$, u is a solution of the homogeneous initial-boundary value problem $(1.1_0),(1.52)$, and, in the rectangle $[a,b]\times[0,\omega_2]$, u is a solution of the homogeneous initial-boundary value problem $(1.1_0),(1.54)$. By Lemma 3.3 (also Theorems 4.1 and 4.1′ from [24]), both problems have only the trivial solutions, i.e., $u(x,y)\equiv 0$ on $[0,\omega_1]\times[c,d]$ and $[a,b]\times[0,\omega_2]$. In particular, in the rectangle Ω, u satisfies the initial conditions

$$u(a,y)=0,\quad u(x,c)=0. \tag{1.55}$$

Thus, in the rectangle Ω u, is a solution of the homogeneous Goursat problem $(1.1_0),(1.55)$, which has only the trivial solution. Consequently, well-posedness of problem (1.1), (1.28) in $[a,b]\times[c,d]\in\Omega$ implies its well-posendess in Ω.

Now, let problem $(1.1_0),(1.36_0)$ have only the trivial solution in Ω. Let $u_0(x,y)$ be a solution of $(1.1_0),(1.36_0)$ in $[a,b]\times[c,d]\in\Omega$. By Lemma 3.3, the problem

$$\begin{gathered}u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y,\\ u(a,y)=u_0(a,y),\quad u_x(x,c)=Bu_x(x,d),\end{gathered}$$

has a unique solution $u_1(x,y)$ in the rectangle $[0,\omega_1]\times[c,d]$. Again, by Lemma 3.3 and Remark 1.1, in the rectangle Ω the problem

$$\begin{gathered}u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y,\\ u(a,y)=Au(b,y),\quad u(x,c)=u_1(x,c),\end{gathered}$$

has a unique solution $u(x,y)$. Thus, $u(x,y)$ is a solution of problem $(1.1_0),(1.36_0)$, whose restriction on $[a,b]\times[c,d]$ is $u_0(x,y)$. Therefore $u_0(x,y)\equiv 0$ on $[a,b]\times[c,d]$. Consequently, well-posedness of problem (1.1), (1.28) in Ω implies its well-posendess $[a,b]\times[c,d]\subset\Omega$.

Similarly, one can show that if inequality (1.50) holds and the associated problems

$$\begin{gathered}\frac{dv}{dx}=Q_1(x)v+P_{21}(x)w,\\ \frac{dw}{dx}=P_{22}(x)v-Q_2(x)w,\\ v(a)=Av(b),\quad w(b)=Aw(a)\end{gathered}$$

and

$$\begin{gathered}\frac{dv}{dy}=Q_1(y)v+P_{11}(y)w,\\ \frac{dv}{dy}=P_{12}(y)v-Q_1(y)w,\\ v(c)=Av(d),\quad w(d)=Aw(c)\end{gathered}$$

are uniquely solvable, then problem (1.43), (1.44), (1.45) is well-posed in Ω if and only if it is well-posed in $[a,b]\times[c,d]\subset\Omega$.

2 Periodic problem

Consider the problem on doubly periodic solutions

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y+q(x,y), \tag{2.1}$$

$$u(x+\omega_1,y)=u(x,y),\quad u(x,y+\omega_2)=u(x,y), \tag{2.2}$$

where

$$P_i \in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n}) \ \ (i=0,1,2), \qquad q \in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^n). \tag{2.3}$$

It is quite obvious that if (2.3) holds, then conditions (2.2) are equivalent to the conditions

$$u(0,y)=u(\omega_1,y) \ \text{ for } \ y\in[0,\omega_2], \quad u(x,0)=u(x,\omega_2) \ \text{ for } \ x\in[0,\omega_1]. \tag{2.4}$$

More precisely, the restriction of an arbitrary solution u of problem (2.1), (2.2) on the domain Ω is a solution of problem (2.1), (2.4) and vice versa, every solution of problem (2.1), (2.4) admits a doubly periodic continuation, which is a solution of problem (2.1), (2.2).

Along with system (2.1), consider the homogeneous system

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y, \tag{2.1$_0$}$$

as well as the problems

$$v'=P_2(x,y^*)v, \tag{2.1$_1$}$$
$$v(x+\omega_1)=v(x) \tag{2.2$_1$}$$

and

$$v'=P_1(x^*,y)v, \tag{2.1$_2$}$$
$$v(y+\omega_2)=v(y). \tag{2.2$_2$}$$

Problems (2.1$_1$), (2.2$_1$) and (2.1$_2$), (2.2$_2$) are called ***associated problems*** of problem (2.1), (2.2).

Theorem 2.1. *Let the following conditions hold:*

(A_1) *problem* (2.1$_1$), (2.2$_1$) *has only the trivial solution for every for* $y^*\in[0,\omega_2]$*;*

(A_2) *problem* (2.1$_2$), (2.2$_2$) *has only the trivial solution for every for* $x^*\in[0,\omega_1]$.

Then problem (2.1), (2.2) *has the Fredholm property, i.e., the following assertions hold:*

(i) *problem* (2.1$_0$), (2.2) *has a finite dimensional space of solutions;*

(ii) *if problem* (2.1$_0$), (2.2) *has only the trivial solution, then problem* (2.1), (2.2) *is uniquely solvable, and its solution* u *admits the estimate*

$$\|u\|_{C^{1,1}(\Omega)} \le M\|q\|_{C(\Omega)}, \tag{2.5}$$

where M *is a positive number independent of* $q\in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^n)$.

Remark 2.1. Theorem 2.1 is an improvement of Theorem 1.1 from [24], since the latter, in addition to conditions (1.5) and (1.6), requires the fulfillment of an additional unnecessary inequality.

Definition 2.1. Problem (2.1), (2.2) is called well-posed if for every $q\in C(\Omega;\mathbb{R}^n)$, it has a unique solution u admitting the estimate (2.5), where M is a positive number independent of q.

Theorem 2.2. *Problem* (2.1), (2.2) *is well-posed if and only if conditions* (A_1) *and* (A_2) *hold, and problem* (2.1$_0$), (2.2) *has only the trivial solution.*

Theorem 2.3. *Let* $P_1(x,y)\equiv P_1(x)$, $P_2(x,y)\equiv P_2(y)$, *and let*

$$\det\big(I-\exp(\omega_1P_2(y))\big)\neq 0 \ \textit{ for } \ y\in[0,\omega_2],$$
$$\det\big(I-\exp(\omega_2P_1(x))\big)\neq 0 \ \textit{ for } \ x\in[0,\omega_1].$$

Then problem (2.1), (2.2) *has the Fredholm property.*

Theorem 2.4. *Let there exist $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $\sigma_1 \widehat{P}_1(x, y)$ and $\sigma_2 \widehat{P}_2(x, y)$ are positive semi-definite for every $(x, y) \in \Omega$. Furthermore, let for every $x \in [0, \omega_1]$ there exist $\eta(x) \in [0, \omega_2]$, and let for every $y \in [0, \omega_2]$ there exist $\zeta(y) \in [0, \omega_1]$ such that $\sigma_1 \widehat{P}_1(x, \eta(x))$ and $\sigma_2 \widehat{P}_2(\zeta(y), y)$ are positive definite. Then problem (2.1), (2.2) has the Fredholm property.*

Theorem 2.5. *Let there exist $\sigma_{1i}, \sigma_{2j} \in \{-1, 1\}$ $(i = 1, \dots, n)$ such that*

$$\sigma_{1i} \int\limits_0^{\omega_2} p_{1ii}(x, t)\, dt = -\alpha_i(x) < 0 \ \text{ for } \ x \in [0, \omega_1] \ \ (i = 1, \dots, n)$$

and

$$\sigma_{2i} \int\limits_0^{\omega_1} p_{2ii}(s, y)\, ds = -\beta_i(y) < 0 \ \text{ for } \ y \in [0, \omega_2] \ \ (i = 1, \dots, n),$$

and let there exist ***nonnegative*** *continuous functions $a_{ik} \in C([0, \omega_1])$ $(b_{ik} \in C([0, \omega_2]))$ $(i \neq k;$ $i, k = 1, \dots, n)$ such that the spectral radius of the matrix $A(x) = (a_{ik}(x))_{i,k=1}^n$ $(B(y) = (b_{ik}(y))_{i,k=1}^n)$, where $a_{ii}(x) \equiv 0$ $(b_{ii}(y) \equiv 0)$ $(i = 1, \dots, n)$, is less than 1 for every $x \in [0, \omega_1]$ $(y \in [0, \omega_2])$. Furthermore, let the inequalities*

$$\bigg| \int\limits_y^{y - \sigma_{1i}\omega_2} e^{\int\limits_t^y p_{1ii}(x, \tau)\, d\tau} |p_{1ik}(x, t)|\, dt \bigg| \le a_{ik}(x)\big(1 - e^{-\alpha_i(x)}\big) \ \ (i \neq k; \ \ i, k = 1, \dots, n)$$

and

$$\bigg| \int\limits_x^{x - \sigma_{2i}\omega_1} e^{\int\limits_s^x p_{2ii}(\xi, y)\, d\xi} |p_{2ik}(s, y)|\, ds \bigg| \le b_{ik}(y)\big(1 - e^{-\beta_i(y)}\big) \ \ (i \neq k; \ \ i, k = 1, \dots, n)$$

hold on Ω. Then problem (2.1), (2.2) has the Fredholm property.

Theorem 2.6. *Let the inequalities*

$$\sigma_{1i} p_{1ii}(x, y) \le a_{ii}(x), \ \ \sigma_{2i} p_{2ii}(x, y) \le b_{ii}(y) \ \text{ for } \ (x, y) \in \Omega \ \ (i = 1, \dots, n),$$
$$|p_{1ik}(x, y)| \le a_{ik}(x), \ \ |p_{2ik}(x, y)| \le b_{ik}(y) \ \text{ for } \ (x, y) \in \Omega \ \ (i \neq k; \ \ i, k = 1, \dots, n)$$

hold, where $\sigma_{ij} \in \{-1, 1\}$ $(i = 1, 2;\ j = 1, \dots, n)$, $a_{ik} \in C([0, \omega_1])$, $b_{ik} \in C([0, \omega_2])$ $(i, k = 1, \dots)$, and the real parts of eigenvalues of the matrix functions $(a_{ik}(x))_{i,k=1}^n$ and $(b_{ik}(y))_{i,k=1}^n$ are negative for every $x \in [0, \omega_1]$ and $y \in [0, \omega_2]$, respectively. Then problem (2.1), (2.2) has the Fredholm property.

Theorem 2.7. *Let conditions (A_1) and (A_2) of Theorem (2.1) hold, let $\Lambda \in \mathbb{R}_+^{n \times n}$ be a nonnegative matrix with the spectral radius less than 1, and let either $P_1 \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2; \mathbb{R}^{n \times n})$ and*

$$\int\limits_0^{\omega_2} \int\limits_0^{\omega_1} \Big| G_2(y, t; x) G_1(x, s; t) \big(P_0(s, t) + P_2(s, t) P_1(s, t) - P_{1s}(s, t)\big) \Big|\, ds\, dt \le \Lambda,$$

or $P_2 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2; \mathbb{R}^{n \times n})$ and

$$\int\limits_0^{\omega_1} \int\limits_0^{\omega_2} \Big| G_1(x, s; y) G_2(y, t; s) \big(P_0(s, t) + P_1(s, t) P_2(s, t) - P_{2t}(s, t)\big) \Big|\, dt\, ds \le \Lambda.$$

Then problem (2.1), (2.2) is well-posed.

Theorem 2.8. *Let $n = 2m$, let*

$$P_1(x, y) = \begin{pmatrix} Q_1(y) & P_{11}(y) \\ P_{12}(y) & -Q_1(y) \end{pmatrix}, \quad P_2(x, y) = \begin{pmatrix} Q_2(x) & P_{21}(x) \\ P_{22}(x) & -Q_2(x) \end{pmatrix},$$
$$P_0(x, y) = \begin{pmatrix} Q_0(x, y) & P_{01}(x, y) \\ -P_{02}(x, y) & Q_0(x, y) \end{pmatrix},$$

where $Q_1(y)$, $Q_2(x)$ and $Q_0(x,y)$ are symmetric matrix functions, and let there exist $\sigma_j \in \{-1,1\}$ $(j=1,2,3)$ such that the following conditions hold:

(A_1) *$\sigma_1 P_{11}(y)$ and $\sigma_1 P_{12}(y)$ are positive semi-definite symmetric matrix functions on $[0,\omega_2]$ such that $\sigma_1 P_{11}(y^*)$ and $\sigma_1 P_{12}(y^*)$ are positive definite symmetric matrices for some $y^* \in [0,\omega_2]$;*

(A_2) *$\sigma_2 P_{21}(x)$ and $\sigma_2 P_{22}(x)$ are positive semi-definite symmetric matrix functions on $[0,\omega_1]$ such that $\sigma_2 P_{21}(x^*)$ and $\sigma_2 P_{22}(x^*)$ are positive definite symmetric matrices for some $x^* \in [0,\omega_1]$;*

(A_3) *$\sigma_3 P_{01}(x,y)$ and $\sigma_3 P_{02}(x,y)$ are positive semi-definite symmetric matrix functions on Ω.*

Furthermore, let one of the following conditions hold:

(B_1) *$\sigma_3 P_{01}(x_0,y)$ and $\sigma_3 P_{02}(x_0,y)$ are positive definite matrix functions on $[0,\omega_2]$ for some $x_0 \in [0,\omega_1]$;*

(B_2) *$\sigma_3 P_{01}(x,y_0)$ and $\sigma_3 P_{02}(x,y_0)$ are positive definite matrix functions on $[0,\omega_1]$ for some $y_0 \in [0,\omega_2]$.*

Then problem (2.1), (2.2) is well-posed.

Consider the system

$$u_{xy} = P_0(x,y)u + u_x + u_y + q(x,y). \tag{2.6}$$

Theorem 2.9. *Let P_0 be a symmetric matrix function, and let one of the following three conditions hold:*

(i) $P_0 \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$,

$$P_0(x,y) + \frac{1}{2}\,\frac{\partial P_0(x,y)}{\partial x} \ \text{ is negative semi-definite for } \ (x,y)\in\Omega,$$
$$\int\limits_0^{\omega_1} P_0(s,y)\,ds \ \text{ is negative semi-definite for } \ y\in[0,\omega_2],$$
$$\int\limits_0^{\omega_1} P_0(s,y^*)\,ds \ \text{ is negative definite for some } \ y^*\in[0,\omega_2];$$

(ii) $P_0 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$,

$$P_0(x,y) + \frac{1}{2}\,\frac{\partial P_0(x,y)}{\partial y} \ \text{ is negative semi-definite for } \ (x,y)\in\Omega,$$
$$\int\limits_0^{\omega_2} P_0(x,t)\,dt \ \text{ is negative semi-definite for } \ x\in[0,\omega_1],$$
$$\int\limits_0^{\omega_2} P_0(x^*,t)\,dt \ \text{ is negative definite for some } \ x^*\in[0,\omega_1];$$

(iii) $P_0 \in C^{1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$,

$$P_0(x,y) + \frac{1}{4}\Big(\frac{\partial P_0(x,y)}{\partial x} + \frac{\partial P_0(x,y)}{\partial y}\Big) \ \text{ is negative semi-definite for } \ (x,y)\in\Omega,$$
$$\int\limits_0^{\omega_1}\int\limits_0^{\omega_2} P_0(s,t)\,dt\,ds \ \text{ is negative definite.}$$

Then problem (2.6), (2.2) is well-posed.

Finally, consider the system

$$u_{xy} = P_0(y)u + P_1(y)u_x + P_2(y)u_y + q(x,y) \tag{2.7}$$

and its particular case, the equation

$$u_{xy} = p_0(y)u + p_1(y)u_x + p_2(y)u_y + q(x,y).$$

Theorem 2.10. *Let P_j $(j=0,1,2)$ be symmetric matrix functions such that*

$$P_j(y)P_k(y) = P_k(y)P_j(y) \ \text{ for } \ y \in [0,\omega_1], \ \ (j,k=0,1,2), \tag{2.8}$$

and let there exist $\sigma_j \in \{-1,1\}$ $(j=0,1,2)$ such that

$$\sigma_0\,\sigma_1\,\sigma_2 = -1, \tag{2.9}$$
$$\sigma_j P_j(y) \ \text{ is positive definite for every } \ y \in [0,\omega_2].$$

Then problem (2.7), (2.2) is well-posed.

Corollary 2.1. *Let*

$$p_0(y)\,p_1(y)\,p_2(y) < 0 \ \text{ for } \ y \in [0,\omega_2].$$

Then problem (2.8), (2.2) is well-posed.

Theorem 2.11. *Let $P_j(y) \equiv P_j$ $(j=0,1,2)$ be constant matrices. Then problem (2.7), (2.2) is well-posed if and only if*

$$\det\big(I - \exp(\omega_1 P_2)\big) \neq 0, \tag{2.10}$$
$$\det\big(I - \exp(\omega_2 P_1)\big) \neq 0, \tag{2.11}$$

and

$$\det\Big(P_0 + i\,\frac{2\pi}{\omega_1}\,mP_1 + i\,\frac{2\pi}{\omega_2}\,kP_2 + \frac{4\pi^2}{\omega_1\omega_2}\,mk\,I\Big) \neq 0 \ \text{ for } \ m,k \in \mathbb{Z}.$$

3 Auxiliary statements

Consider the boundary value problem

$$z' = P(t)z + q(t), \tag{3.1}$$
$$h(z) = c \tag{3.2}$$

and its corresponding homogeneous problem

$$z' = P(t)z, \tag{3.1$_0$}$$
$$h(z) = 0, \tag{3.2$_0$}$$

where $P \in C([0,\omega];\mathbb{R}^{n\times n})$, $q \in C([0,\omega];\mathbb{R}^n)$, $c \in \mathbb{R}^n$, and $h : C([0,\omega];\mathbb{R}^n) \to \mathbb{R}^n$ is bounded linear operator.

Lemma 3.1. *The following statements are equivalent:*

(i) *problem (3.1), (3.2) is solvable for arbitrary $q \in C([0,\omega];\mathbb{R}^n)$ and $c \in \mathbb{R}^n$;*

(ii) *problem (3.1), (3.2$_0$) is solvable for arbitrary $q \in C([0,\omega];\mathbb{R}^n)$;*

(ii) *problem (3.1$_0$), (3.2) is solvable for arbitrary $c \in \mathbb{R}^n$;*

(iv) *problem (3.1$_0$), (3.2$_0$) has only the trivial solution;*

(v) $\det h(Z) \neq 0$, *where $Z(t)$ is an arbitrary fundamental matrix of the differential system (3.1$_0$).*

Lemma 3.1 is a well-known fact in the theory of boundary value problems for ordinary differential equations (see, e.g., Theorem 1.1 from [18]. If problem $(3.1_0), (3.2_0)$ has only the trivial solution, then a solution of problem $(3.1), (3.2)$ admits the representation $z(t) = \Gamma[c](t) + \mathcal{G}[q](t)$, where $\Gamma : \mathbb{R}^n \to C^1([0,\omega];\mathbb{R}^n)$ and $\mathcal{G} : C([0,\omega];\mathbb{R}^n) \to C^1([0,\omega];\mathbb{R}^n)$ are bounded linear operators. Moreover, the operator $\mathcal{G}$ admits the representation

$$\mathcal{G}[q](t) = \int\limits_0^\omega G(t,\tau)q(\tau)\,d\tau,$$

where $G : [0,\omega] \times [0,\omega] \to \mathbb{R}$ is called the **Green's matrix of problem** $(3.1_0), (3.2_0)$ (for more information about Green's matrix, see [18]).

Lemma 3.2. *Let problem $(3.1_0), (3.2_0)$ have a nontrivial solution. Then for an arbitrary $\varepsilon > 0$ there exist a bounded linear operator $\widetilde{h} : C([0,\omega];\mathbb{R}^n) \to \mathbb{R}^n$ such that $\|h - \widetilde{h}\| < \varepsilon$ and the problem*

$$\begin{aligned} z &= P(t)z,\\ \widetilde{h}(z) &= 0 \end{aligned} \tag{$\widetilde{3.2_0}$}$$

has only the trivial solution.

Proof. By Theorem 1.1 from [18], problem $(3.1_0), (\widetilde{3.2_0})$ has a nontrivial solution if and only if $\det \widetilde{h}(Z) = 0$, where $Z(t)$ is an arbitrary fundamental matrix of the differential system (3.1_0). Set $\widetilde{h}(z) = (1-\lambda)h(z) + \lambda z(0)$. Then $D(\lambda) = \det \widetilde{h}(Z)$ is a polynomial (of degree not greater than n) with respect to λ. Moreover, it is a non-identically zero polynomial. Indeed, $D(1) = \det \widetilde{h}(Z) \neq 0$ for $\lambda = 1$, since the initial value problem

$$z' = P(t)z, \quad z(0) = 0$$

has only the trivial solution. Hence, $D(\lambda)$ has at most n zeros.

Consequently, there exists $\delta > 0$ such that

$$D(\lambda) \neq 0 \ \text{ for } \ \lambda \in (0,\delta). \tag{3.3}$$

Notice that

$$\|h - \widetilde{h}\| \le \lambda\big(1 + \|h\|\big). \tag{3.4}$$

The latter inequality with

$$\lambda < \min\Big\{\delta, \frac{\varepsilon}{(1 + \|h\|)}\Big\}$$

implies $\|h - \widetilde{h}\| < \varepsilon$. □

Definition 3.1. $\mathcal{G} : C([0,\omega];\mathbb{R}^n) \to C^1([0,\omega];\mathbb{R}^n)$ is called the **Green's operator** of problem $(3.1_0), (3.2_0)$.

Definition 3.2. $\Gamma : \mathbb{R}^n \to C^1([0,\omega];\mathbb{R}^n)$ is called the **Green's boundary operator** of problem $(3.1_0), (3.2_0)$.

Lemma 3.3. *Let $x_0 \in [0,\omega_1]$, and let the problem*

$$\begin{aligned} v' &= P_1(x^*, y)v,\\ h(v) &= 0 \end{aligned}$$

have only the trivial solution for every $x^* \in [0,\omega_1]$. Then the linear problem

$$\begin{gathered} u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x,y),\\ u(x_0,y) = \varphi(y), \quad h(u_x(x,\cdot)) = \psi(x) \end{gathered} \tag{3.5}$$

has a unique solution u admitting the representation

$$u(x,y) = \mathcal{A}_0(\varphi,\psi)(x,y) + \mathcal{A}(q)(x,y),$$

where $\mathcal{A}_0 : C([0,\omega_2];\mathbb{R}^n) \times C([0,\omega_1];\mathbb{R}^n) \to C^{1,1}(\Omega;\mathbb{R}^n)$ *and* $A : C(\Omega;\mathbb{R}^n) \to C^{1,1}(\Omega;\mathbb{R}^n)$ *are bounded linear operators. Furthermore, there exists a positive constant* M_0*, independent of* q*, such that*

$$\Big\| \frac{\partial}{\partial x}\,\mathcal{A}(q) \Big\|_{C(\Omega_{x_0 x})} \le M_0 \|q\|_{C(\Omega_{x_0 x})} \ \ \textit{for} \ \ x \in [0,\omega_1],$$

$$\|\mathcal{A}(q)\|_{C(\Omega_{x_0 x})} + \Big\| \frac{\partial}{\partial y}\,\mathcal{A}(q) \Big\|_{C(\Omega_{x_0 x})} \le M_0 \bigg| \int\limits_{x_0}^{x} \|q\|_{C(\Omega_{x_0 s})}\, ds \bigg| \ \ \textit{for} \ \ x \in [0,\omega_1],$$

where $\Omega_{x_0\, x} = [x_0, x] \times [0,\omega_2]$ *if* $x_0 < x$*, and* $\Omega_{x_0\, x} = [x, x_0] \times [0,\omega_2]$*, if* $x_0 > x$*.*

Lemma 3.3′. *Let* $x_0 \in [0,\omega_1]$*, and let the problem*

$$v' = P_1(x^*, y)v,$$
$$h(v)(x^*) = 0$$

have only the trivial solution for every $x^* \in [0,\omega_1]$*. Then problem* (3.3), (3.4) *has a unique solution* u *admitting representation* (3.5)*, where* $\mathcal{A}_0 : AC([0,\omega_2];\mathbb{R}^n) \times L([0,\omega_1];\mathbb{R}^n) \to AC(\Omega;\mathbb{R}^n)$ *and* $A : L(\Omega;\mathbb{R}^n) \to AC(\Omega;\mathbb{R}^n)$ *are bounded linear operators. Furthermore, there exists a positive constant* M_0*, independent of* q*, such that*

$$\|\mathcal{A}(q)\|_{C(\Omega_x)} \le M_0 \|q\|_{L(\Omega_x)} \ \ \textit{for} \ \ x \in [0,\omega_1].$$

Lemmas 3.3 and 3.3′ immediately follow from Theorems 4.1 and 4.1′ from [24].

Lemma 3.4. *Let conditions* (A_0) *and* (A_2) *of Theorem 1.4 hold. Then there exists* $u \in C^{1,1}(\Omega;\mathbb{R}^{n\times n})$ *satisfying conditions* (1.2)*.*

Proof. Consider the problem

$$u_{xy} = P_2(x,0)u_x,$$
$$\ell(u(\,\cdot\,,y)) = \varphi(y), \quad h(u_x(x,\,\cdot\,)) = \psi(x).$$

The latter problem has a unique solution

$$u(x,y) = \Gamma_0[\varphi(y)](x) + \int\limits_{0}^{\omega_1} G_0(x,s)\,\Gamma_2[\psi(s)](y)\,ds,$$

where Γ_0 and $G_0(x,s)$, respectively, are Green's boundary operator and Green's matrix of problem (1.4), and Γ_2 is Green's boundary operator of the problem

$$v' = P_1(0,y)v,$$
$$h(v) = 0. \qquad \square$$

Consider the problem

$$v' = \widetilde{P}_2(x, y^*)v, \tag{3.6}$$
$$\ell(v) = 0, \tag{3.7}$$

where $\widetilde{P}_2 \in C^{0,1}(\Omega;\mathbb{R}^{n\times n})$.

Lemma 3.5. *Let conditions (A_0), (A_1), (A_2) of Theorem 1.4 hold, and let problem (3.6), (3.7) have only the trivial solution for every $x_2 \in [0, \omega_2]$. Then an arbitrary solution u of problem (1.1), (1.2) admits the following representations:*

$$
\begin{aligned}
u_x(x,y) &= \widetilde{P}_2(x,y)u + \int\limits_0^{\omega_2} G_2(y,t;x)\Big(\big(P_0(x,t) + P_1(x,t)\widetilde{P}_2(x,t) - \widetilde{P}_{2t}(x,t)\big)u(x,t) \\
&\qquad + \big(P_2(x,t) - \widetilde{P}_2(x,t)\big)u_t(x,t) + q(x,t)\Big)\,dt + \Gamma_2\big[\psi(x) - h(\widetilde{P}_2(x,\,\cdot\,)u(x,\,\cdot\,))\big](y;x);\\
u_y(x,y) &= \int\limits_0^{\omega_1} G_1(x,s;y)\Big(P_0(s,y)u(s,y) + P_1(s,y)u_s(s,y) + q(s,y)\Big)\,ds + \Gamma_1[\varphi'(y)](x;y);\\
u(x,y) &= \int\limits_0^{\omega_1}\int\limits_0^{\omega_2} \widetilde{G}_1(x,s;y)\,G_2(y,t;s)\Big(\Big(P_0(s,t) + P_1(s,t)\widetilde{P}_2(s,t) - \frac{\partial}{\partial t}\widetilde{P}_2(s,t)\Big)u(s,t)\\
&\qquad + \big(P_2(s,t) - \widetilde{P}_2(s,t)\big)u_t(s,t) + q(s,t)\Big)\,ds + \mathcal{P}[u;\psi](x,y) + \widetilde{\Gamma}_1[\varphi(y)](x;y),
\end{aligned}
$$

where

$$
\mathcal{P}\big[u;\psi\big](x,y) = \int\limits_0^{\omega_1} \widetilde{G}_1(x,s;y)\,\Gamma_2\big[\psi(s) - h(\widetilde{P}_2(s,\,\cdot\,)u(s,\,\cdot\,))\big](y;s)\,ds,
$$

G_j and Γ_j, respectively, are Green's matrix and Green's boundary operator of problem (1.1_j), (1.2_j) *($j = 1,2$), and $\widetilde{G}_1$ and $\widetilde{\Gamma}_1$, respectively, are Green's matrix and Green's boundary operator of problem* (3.6), (3.7).

Proof. Let u be a solution of problem (1.1), (1.2). Set

$$
v(x,y) = u_x(x,y); \quad w(x,y) = u_y(x,y); \quad \widetilde{v}(x,y) = u_x(x,y) - \widetilde{P}_2(x,y)u(x,y).
$$

To prove Lemma 3.4, one needs to note that v, w and $\widetilde{v}$, respectively, are solutions of the following boundary value problems:

$$
\begin{gathered}
v_y = P_1(x,y)v + P_0(x,y)u(x,y) + P_2(x,y)u_y(x,y) + q(x,y),\\
h(v(x,\,\cdot\,)) = \psi(x);\\
w_x = P_2(x,y)w + P_0(x,y)u(x,y) + P_1(x,y)u_x(x,y) + q(x,y),\\
\ell(w(\,\cdot\,,y)) = \varphi'(y);\\
\widetilde{v}_y = P_1(x,y)v + \Big(P_0(x,y) + P_1(x,y)\widetilde{P}_2(x,y) - \frac{\partial}{\partial y}\,\widetilde{P}_2(x,y)\Big)u(x,y)\\
+\big(P_2(x,y) - \widetilde{P}_2(x,y)\big)u_y(x,y) + q(x,y),\\
h(\widetilde{v}(x,\,\cdot\,)) = \psi(x) - h\big(\widetilde{P}_2(x,\,\cdot\,)u(x,\,\cdot\,)\big).
\end{gathered}
$$

□

Lemma 3.6. *Let conditions (A_0), (A_1), (A_2) of Theorem 1.4 hold. Then problem (1.1), (1.2) has the Fredholm property.*

Proof. First, note that, by Lemma 3.4, there exists $u \in C^{1,1}(\Omega;\mathbb{R}^{n\times n})$ satisfying conditions (1.2). Let us show that problem (1.1), (1.2) is equivalent to the following system of integral equations:

$$v(x,y) = \mathcal{F}_1(u,w)(x,y); \tag{3.8}$$

$$w(x,y) = \mathcal{F}_2(u,v)(x,y); \tag{3.9}$$

$$u(x,y) = \mathcal{F}_0(u,w)(x,y), \tag{3.10}$$

where

$$\mathcal{F}_1(u,w)(x_1,x_2)=\int_0^{\omega_2} G_2(y,t;x)\Big(P_0(x,t)u(x,t)+P_2(x,t)w(x,t)+q(x,t)\Big)\,dt+\Gamma_2[\psi(x)](y;x);$$

$$\mathcal{F}_2(u,v)(x,y)=\int_0^{\omega_1} G_1(x,s;y)\Big(P_0(s,y)u(s,y)+P_1(s,y)v(s,y)+q(s,y)\Big)\,ds+\Gamma_1[\varphi'(y)](x;y); \tag{3.11}$$

$$\mathcal{F}_0(u,w)(x,y)=\int_0^{\omega_1}\int_0^{\omega_2} G_0(x,s)\,G_2(y,t;s)\Big(P_0(s,t)u(s,t)+P_2(s,t)w(s,t)+q(s,t)\Big)\,ds \\ +\mathcal{P}_0[\psi](x,y)+\Gamma_0[\varphi(y)](x),$$

where

$$\mathcal{P}_0[\psi](x,y)=\int_0^{\omega_1} G_0(x,s)\,\Gamma_2[\psi(s)](y;s)\,ds, \tag{3.12}$$

G_j and Γ_j, respectively, are Green's matrix and Green's boundary operator of problem $(2.1_j),(2.2_j)$ $(j=1,2)$, and G_0 and Γ_0, respectively, are Green's matrix and Green's boundary operator of problem (1.4).

Indeed, if u is a solution to problem (1.1), (1.2), then according to Lemma 3.4, u admits the representations

$$u_x(x,y)=\int_0^{\omega_2} G_2(y,t;x)\Big(P_0(x,t)u(x,t)+P_2(x,t)u_t(x,t)+q(x,t)\Big)\,dt+\Gamma_2[\psi(x)](y;x);$$

$$u_y(x,y)=\int_0^{\omega_1} G_1(x,s;y)\Big(P_0(s,y)u(s,y)+P_1(s,y)u_s(s,y)+q(s,y)\Big)\,ds+\Gamma_1[\varphi'(y)](x;y);$$

$$u(x,y)=\int_0^{\omega_1}\int_0^{\omega_2} G_0(x,s;y)\,G_2(y,t;s)\Big(P_0(s,t)u(s,t)+P_2(s,t)u_t(s,t)+q(s,t)\Big)\,ds \\ +\mathcal{P}_0[\psi](x,y)+\Gamma_0[\varphi(y)](x).$$

Now, let (v,w,u) be a solution of system (3.8)–(3.10). Then from (3.10) we get

$$u_x(x,y)=\int_0^{\omega_2} G_2(y,t;x)\Big(P_0(x,t)u(x,t)+P_2(x,t)w(x,t)+q(x,t)\Big)\,dt+\Gamma_2[\psi(x)](y;x), \tag{3.13}$$

$$u_x(x,y)=v(x,y), \tag{3.14}$$

and

$$\ell(u(\cdot\,,y))=\varphi(y). \tag{3.15}$$

Consequently, in view of (3.9) and (3.11), we have

$$w(x,y)=\int_0^{\omega_1} G_1(x,s;y)\Big(P_0(s,y)u(s,y)+P_1(s,y)u_s(s,y)+q(s,y)\Big)\,ds+\Gamma_1[\varphi'(y)](x;y).$$

The latter equality immediately implies

$$w_x(x,y)=P_2(x,y)w(x,y)+P_0(x,y)u(x,y)+P_1(x,y)u_x(x,y)+q(x,y)$$

and

$$\ell(w(\cdot\,,y))=\varphi'(y). \tag{3.16}$$

From (3.13) and (3.15) we get

$$
\begin{gathered}
u_{xy}(x,y) = P_1(x,y)u_x(x,y) + P_0(x,y)u(x,y) + P_2(x,y)w(x,y) + q(x,y), \\
\ell(u_y(\,\cdot\,,y)) = \varphi'(y).
\end{gathered} \tag{3.17}
$$

Set

$$
z(x,y) = u_y(x,y) - w(x,y).
$$

Then

$$
\begin{gathered}
z_x(x,y) = P_2(x,y)z(x,y), \\
\ell(z(\,\cdot\,,y)) = 0.
\end{gathered} \tag{3.18}
$$

Consequently, in view of condition (A_1), $z(x,y) \equiv 0$, i.e., $u_y(x,y) = w(x,y)$.

Thus, problem (1.1), (1.2) is equivalent to system (3.8)–(3.10).

Let $\mathcal{F}_1^0(u,w)$, $\mathcal{F}_2^0(u,v)$ and $\mathcal{F}_0^0(u,w)$ be the homogeneous parts of the operators $\mathcal{F}_1(u,w)$, $\mathcal{F}_2(u,v)$ and $\mathcal{F}_0(u,w)$, respectively, and set

$$
\mathcal{K}(u,v,w) = \big(\mathcal{F}_1^0(u,w), \mathcal{F}_2^0(u,v), \mathcal{F}_0^0(u,w)\big).
$$

It is clear that $\mathcal{K}$ is a bounded linear operator from $C(\Omega;\mathbb{R}^n)\times C(\Omega;\mathbb{R}^n)\times C(\Omega;\mathbb{R}^n)$ into $C^{0,1}(\Omega;\mathbb{R}^n)\times C^{1,0}(\Omega;\mathbb{R}^n)\times C^{1,1}(\Omega;\mathbb{R}^n)$.

Notice that $\mathcal{K}^2 = \mathcal{K}\circ\mathcal{K}$ is a compact operator from $C(\Omega;\mathbb{R}^n)\times C(\Omega;\mathbb{R}^n)\times C(\Omega;\mathbb{R}^n)$ into $C(\Omega;\mathbb{R}^n)\times C(\Omega;\mathbb{R}^n)\times C(\Omega;\mathbb{R}^n)$. The latter fact implies that the system of operator equations (3.8)–(3.10) and, consequently, problem (1.1), (1.2) has the Fredholm property. $\square$

Lemmas 3.5 and 3.6 immediately imply

Lemma 3.7. *Let conditions (A_0), (A_1), (A_2) of Theorem 1.4 hold, and let $P_2 \in C^{0,1}(\Omega;\mathbb{R}^{n\times n})$. Then problem (1.1), (1.2) is equivalent to the integral equation*

$$
\begin{aligned}
\mathcal{F}(u)(x,y) = \int_0^{\omega_1}\!\!\int_0^{\omega_2} G_1(x,s;y)\,G_2(y,t;s)\Big(\big(P_0(s,y)+P_1(s,t)P_2(s,t)-P_{2t}(s,t)\big)u(s,t)+q(s,t)\Big)\,dt\,ds \\
+ \mathcal{P}[u;\psi](x,y) + \Gamma_1[\varphi(y)](x;y),
\end{aligned}
$$

where

$$
\mathcal{P}[u;\psi](x,y) = \int_0^{\omega_1} G_1(x,s;y)\,\Gamma_2\big[\psi(s) - h(P_2(s,\,\cdot\,)u(s,\,\cdot\,))\big](y;s)\,ds
$$

and G_j and Γ_j, respectively, are Green's matrix and Green's boundary operator of problem (1.1_j), (1.2_j) $(j=1,2)$.

Consider the problems

$$
v' = \widetilde{P}_2(x,y^*)v, \tag{3.19}
$$

$$
v(x+\omega_1) = v(x), \tag{3.20}
$$

and

$$
v' = \widetilde{P}_1(x,y^*)v, \tag{3.21}
$$

$$
v(y+\omega_2) = v(y), \tag{3.22}
$$

where $\widetilde{P}_1 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$ and $\widetilde{P}_2 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$.

Lemma 3.8. *Let conditions* (A_1) *and* (A_2) *of Theorem 2.1 hold. Then an arbitrary solution* u *of problem* (2.1), (2.2) *admits the following representations:*

$$u_x(x,y) = \widetilde{P}_2(x,y)u(x,y) + \int_x^{x+\omega_2} G_2(y,t;x)\Big(\big(P_0(x,t) + P_1(x,t)\widetilde{P}_2(x,t) - \widetilde{P}_{2t}(x,t)\big)u(x,t) + \big(P_2(x,t) - \widetilde{P}_2(x,t)\big)u_t(x,t) + q(s,t)\Big)\,ds;$$

$$u_y(x,y) = \int_x^{x+\omega_1} G_1(x,s;y)\big(P_0(s,y)u(s,y) + P_1(s,y)u_s(s,y) + q(s,y)\big)\,ds;$$

$$u(x,y) = \int_x^{x+\omega_1}\int_y^{y+\omega_2} \widetilde{G}_1(x,s;y)\,G_2(y,t;s)\Big(\big(P_0(s,t) + P_1(s,t)\widetilde{P}_2(s,t) - \widetilde{P}_{2t}(s,t)\big)u(s,t) + \big(P_2(s,t) - \widetilde{P}_2(s,t)\big)u_t(s,t) + q(s,t)\Big)\,dt\,ds,$$

where G_j *is Green's matrix of problem* $(2.1_j), (2.2_j)$ $(j = 1,2)$ *and* $\widetilde{G}_1$ *is Green's matrix of problem* (3.19), (3.20).

Proof. The proof of Lemma 3.8 is similar to that of Lemma 3.5. □

Lemma 3.9. *Let conditions* (A_1) *and* (A_2) *of Theorem 2.1 hold. Then an arbitrary solution* u *of problem* (2.1), (2.2) *admits the following representations:*

$$u_x(x,y) = \int_x^{x+\omega_2} G_2(y,t;x)\big(P_0(x,t)u(x,t) + P_2(x,t)u_t(x,t) + q(s,t)\big)\,ds;$$

$$u_y(x,y) = \widetilde{P}_1(x,y)u(x,y) + \int_x^{x+\omega_1} G_1(x,s;y)\Big(\big(P_0(s,y) + P_2(s,y)\widetilde{P}_1(s,y) - \widetilde{P}_{1s}(x,t)\big)u(s,y) + \big(P_1(s,y) - \widetilde{P}_1(s,y)\big)u_s(s,y) + q(s,y)\Big)\,ds;$$

$$u(x,y) = \int_y^{y+\omega_2}\int_x^{x+\omega_1} \widetilde{G}_2(y,t;x)\,G_1(x,s;t)\Big(\big(P_0(s,t) + P_2(s,t)\widetilde{P}_1(s,t) - \widetilde{P}_{1s}(s,t)\big)u(s,t) + \big(P_1(s,t) - \widetilde{P}_1(s,t)\big)u_s(s,t) + q(s,t)\Big)\,ds\,dt,$$

where G_j *is Green's matrix of problem* $(2.1_j), (2.2_j)$ $(j = 1,2)$ *and* $\widetilde{G}_2$ *is Green's matrix of problem* (3.21), (3.22).

Proof. The proof of Lemma 3.9 is similar to that of Lemma 3.5. □

Lemma 3.10. *Let conditions* (A_1) *and* (A_2) *of Theorem 2.1 hold. Then problem* (2.1), (2.2) *is equivalent to the system of integral equations*

$$v(x,y) = \int_x^{x+\omega_1}\int_y^{y+\omega_2} G(x,s)\,G_2(y,t;s)\big(P_0(s,t)u(s,t) + (P_2(s,t) - I)w(s,t) + q(s,t)\big)\,dt\,ds + \int_y^{y+\omega_2} G_2(y,t;x)\big(P_0(x,t)u(x,t) + (P_2(x,t) - I)w(x,t) + q(x,t)\big)\,dt;$$

$$w(x,y) = \int_x^{x+\omega_1} G_1(x,s;y)\big(P_0(s,y)u(s,y) + P_1(s,y)v(s,y) + q(s,y)\big)\,ds;$$

$$u(x,y)=\int\limits_{x}^{x+\omega_1}\int\limits_{y}^{y+\omega_2} G(x,s)\,G_2(y,t;s)\big(P_0(s,t)u(s,t)+(P_2(s,t)-I)w(s,t)+q(s,t)\big)\,dt\,ds,$$

where $G(x,s)=\frac{e^{x-s}}{e^{-\omega_1}-1}I$ *is Green's matrix of the problem*

$$z'=z,\quad z(x+\omega_1)=z(x).$$

Proof. The proof of Lemma 3.10 is similar to that of Lemma 3.6. □

Lemma 3.11. *Let conditions* (A_1) *and* (A_2) *of Theorem 2.1 hold. Then problem* (2.1), (2.2) *is equivalent to the system of integral equations*

$$v(x,y)=\int\limits_{y}^{y+\omega_1} G_2(y,t;x)\big(P_0(x,t)u(x,t)+P_2(x,t)w(x,t)+q(x,t)\big)\,dt;$$

$$w(x,y)=\int\limits_{y}^{y+\omega_2}\int\limits_{x}^{x+\omega_1} G(y,t)\,G_1(x,s;t)\big(P_0(s,t)u(s,t)+(P_1(s,t)-I)v(s,t)+q(s,t)\big)\,ds\,dt$$
$$+\int\limits_{x}^{x+\omega_1} G_1(x,s;y)\big(P_0(s,y)u(s,y)+(P_1(s,y)-I)v(s,y)+q(s,y)\big)\,ds;$$

$$u(x,y)=\int\limits_{y}^{y+\omega_2}\int\limits_{x}^{x+\omega_1} G(y,t)\,G_1(x,s;t)\big(P_0(s,t)u(s,t)+(P_1(s,t)-I)v(s,t)+q(s,t)\big)\,ds\,dt$$

where $G(y,t)=\frac{e^{y-t}}{e^{-\omega_2}-1}I$ *is Green's matrix of the problem*

$$z'=z,\quad z(y+\omega_2)=z(y).$$

Proof. The proof of Lemma 3.11 is similar to that of Lemma 3.6. □

Lemma 3.12. *Let conditions* (A_1) *and* (A_2) *of Theorem 2.1 hold, and let* $P_1\in C^{1,0}_{\omega_1\omega_2}(\Omega;\mathbb{R}^{n\times n})$. *Then problem* (2.1), (2.2) *is equivalent to the operator equation*

$$u(x,y)=\int\limits_{y}^{y+\omega_2}\int\limits_{x}^{x+\omega_1} G_2(y,t;x)\,G_1(x,s;t)\Big(\big(P_0(s,y)+P_1(s,t)P_2(s,t)-P_{1s}(s,t)\big)u(s,t)+q(s,t)\Big)\,ds\,dt,$$

where G_j *is Green's matrix of problem* $(2.1_j),(2.2_j)$ $(j=1,2)$.

Proof. Lemma 3.12 is an immediate consequence of Lemmas 3.8 and 3.10. □

Lemma 3.13. *Let conditions* (A_1) *and* (A_2) *of Theorem 2.1 hold, and let* $P_2\in C^{0,1}_{\omega_1\omega_2}(\Omega;\mathbb{R}^{n\times n})$. *Then problem* (2.1), (2.2) *is equivalent to the operator equation*

$$u(x,y)=\int\limits_{x}^{x+\omega_1}\int\limits_{y}^{y+\omega_2} G_1(x,s;y)\,G_2(y,t;s)\Big(\big(P_0(s,y)+P_1(s,t)P_2(s,t)-P_{2t}(s,t)\big)u(s,t)+q(s,t)\Big)\,dt\,ds,$$

where G_j *is Green's matrix of problem* $(2.1_j),(2.2_j)$ $(j=1,2)$.

Proof. Lemma 3.13 is an immediate consequence of Lemmas 3.9 and 3.11. □

In the interval $[0,\omega]$, consider the problem

$$z' = P(t)z, \tag{3.23}$$
$$z(a) = Az(b), \tag{3.24}$$

where $P \in C([0,\omega];\mathbb{R}^{n\times n})$, $A \in \mathbb{R}^{n\times n}$ and $0 \le a < b \le \omega$. Set $\widehat{P} = \frac{1}{2}\,(P + P^T)$, where P^T is the transpose of the matrix P.

Lemma 3.14. *Let there exist $\sigma \in \{-1,1\}$ such that $\sigma\widehat{P}(t)$ is positive semi-definite on $[a,b]$, and let either*

$$\sigma(A^TA - I) \ \text{ is positive semi-definite}, \tag{3.25}$$
$$\sigma\widehat{P}(t^*) \ \text{ is positive definite for some } \ t^* \in [a,b], \tag{3.26}$$

or

$$\sigma(A^TA - I) \ \text{ is positive definite}, \tag{3.27}$$

Then problem (3.23), (3.24) has only the trivial solution.

Proof. Let z be a solution of problem (3.23), (3.24). Then

$$\frac{1}{2}\,\langle z(t), z(t)\rangle\Big|_a^b = \int_a^b \langle z'(t), z(t)\rangle\,dt = \int_a^b \big\langle P(t)z(t), z(t)\big\rangle\,dt = \int_a^b \big\langle \widehat{P}(t)z(t), z(t)\big\rangle\,dt,$$

whence, in view of (3.13), we get

$$\frac{1}{2}\,\Big(\langle z(b), z(b)\rangle - \langle z(a), z(a)\rangle\Big) = \frac{1}{2}\,\Big(\langle z(b), z(b)\rangle - \big\langle Az(b), Az(b)\big\rangle\Big)$$

and

$$\frac{1}{2}\,\big\langle (A^TA - I)z(b), z(b)\big\rangle + \int_a^b \big\langle \widehat{P}(t)z(t), z(t)\big\rangle\,dt = 0. \tag{3.28}$$

The matrix function $\sigma\widehat{P}(t)$ is positive semi-definite on $[a,b]$. Notice that if (3.29) holds, then the matrix function $\sigma\widehat{P}(t)$ is positive definite in some neighborhood of t^*. Therefore, (3.25), (3.26) and (3.28) imply $z(t^*) = 0$, while (3.27) and (3.28) imply $z(b) = 0$. In either case, by the Existence and Uniqueness Theorem, $z(t) \equiv 0$. $\square$

Let $n = 2m$. In the interval $[0,\omega]$, for the linear system

$$\begin{aligned} z_1' &= Q(t)z_1 + P_1(t)z_2, \\ z_2' &= P_2(t)z_1 - Q(t)z_2, \end{aligned} \tag{3.29}$$

consider the boundary conditions

$$z_1(a) = Az_1(b), \quad z_2(b) = Az_2(a), \tag{3.30}$$

where $z = (z_1, z_2)$, $z_1, z_2 \in \mathbb{R}^m$, $P_i \in C([0,\omega];\mathbb{R}^{m\times m})$ $(i = 1,2)$, $Q \in C([0,\omega];\mathbb{R}^{m\times m})$, and $A \in \mathbb{R}^{m\times m}$).

Lemma 3.15. *Let $A \in \mathbb{R}^{m\times m}$ be symmetric matrix, let $Q \in C([0,\omega];\mathbb{R}^{m\times m})$ be a symmetric matrix function on $[a,b]$, let $\sigma P_1 \in C([0,\omega];\mathbb{R}^{m\times m})$ and $\sigma P_2 \in C([0,\omega];\mathbb{R}^{m\times m})$ be positive semi-definite symmetric matrix functions on $[a,b]$ for some $\sigma \in \{-1,1\}$, and let $\sigma P_1(t^*)$ and $\sigma P_2(t^*)$ be positive definite for some $t^* \in [a,b]$. Then problem (3.29), (3.30) has only the trivial solution.*

Proof. Let $(z_1(t), z_2(t))$ be a solution of problem (3.29), (3.30). Then

$$\int_a^b \langle z_1'(t), z_2(t)\rangle\, dt = \int_a^b \Big(\big\langle Q(t)z_1(t), z_2(t)\big\rangle + \big\langle P_1(t)z_2(t), z_2(t)\big\rangle\Big)\, dt \tag{3.31}$$

and

$$\int_a^b \langle z_2'(t), z_1(t)\rangle\, dt = \int_a^b \Big(\big\langle P_2(t)z_1(t), z_1(t)\big\rangle - Q(t)z_2(t), z_1(t)\big\rangle\Big)\, dt. \tag{3.32}$$

Adding (3.31) and (3.32), we get

$$\begin{aligned}\langle z_1(b), z_2(b)\rangle &- \langle z_1(a), z_2(a)\rangle \\ &= \int_a^b \frac{d}{dt}\langle z_1(t), z_2(t)\rangle\, dt = \int_a^b \Big(\big\langle P_1(t)z_2(t), z_2(t)\big\rangle + \big\langle P_2(t)z_1(t), z_1(t)\big\rangle\Big)\, dt.\end{aligned} \tag{3.33}$$

In view of (3.30) and symmetry of A, from (3.33) we get

$$\langle z_1(b), z_2(b)\rangle - \langle z_1(a), z_2(a)\rangle = \langle z_1(b), Az_2(a)\rangle - \langle Az_1(b), z_2(a)\rangle = 0 \tag{3.34}$$

and

$$\int_a^b \Big(\big\langle P_1(t)z_2(t), z_2(t)\big\rangle + \big\langle P_2(t)z_1(t), z_1(t)\big\rangle\Big)\, dt = 0. \tag{3.35}$$

The matrix functions $\sigma P_1(t)$ and $\sigma P_2(t)$ are positive semi-definite on $[a, b]$, and $\sigma P_1(t^*)$ and $\sigma P_2(t^*)$ are positive definite for some $t^* \in [a, b]$. Consequently, the matrix functions $\sigma P_1(t)$ and $\sigma P_2(t)$ are positive definite in some neighborhood of t^*. Therefore, from (3.35) we immediately get $(z_1(t^*), z_2(t^*)) = 0$. By the Existence and Uniqueness Theorem, $(z_1(t, z_2(t)) \equiv 0$. □

4 Proofs of the main results

Proof of Theorem 1.1. Let $\psi(x) \equiv 0$, let $c \in \mathbb{R}^n$ be an arbitrary vector, and let $\varphi \in C^1([0, \omega_2]; \mathbb{R}^n)$ be such that

$$h(\varphi) = c.$$

Existence of such φ follows from the admissibility of h. Indeed, let $P \in C([0, \omega_2]; \mathbb{R}^{n\times n})$ be such that the problem

$$z' = P(y)z, \quad h(z) = 0$$

has only the trivial solution. Then, in the role of φ one can choose the (unique) solution of the problem

$$z' = P(y)z, \quad h(z) = c.$$

Let u be an arbitrary solution of problem (1.1), (1.2). Set $z = h(u(x, \cdot))$. Then, in view of (1.3), z is a solution of the problem

$$z' = 0, \quad \ell(z) = c. \tag{4.1}$$

Thus, problem (4.1) is solvable for arbitrary $c \in \mathbb{R}^n$. By Lemma 3.1, the homogeneous problem (1.4) has only the trivial solution. □

Proof of Theorem 1.2. Let u be a solution of problem (1.1), (1.2). Set

$$w(x, y) = u_y(x, y) - P_1 u(x, y), \quad v(x) = h(w(x, \cdot)).$$

In view of (1.2) and (1.6), w is a solution of the problem

$$w^{(1,0)} = P_2 w + (P_0(x,y) + P_2 P_1) u(x,y) + q(x,y), \tag{4.2}$$
$$\ell(w(\,\cdot\,,y)) = \ell\big(u_y(\,\cdot\,,y) - P_1 u(\,\cdot\,,y)\big) = \varphi'(y) - P_1\varphi(y). \tag{4.3}$$

After applying the operator h to (4.2) and (4.3) and utilizing (1.3), we get

$$\begin{aligned} h(w'(x,\,\cdot\,)) = v'(x) = P_2 v + \big(h(P_0(x,\,\cdot\,)) + P_2 P_1\big)\Psi(x) + h(q(x,\,\cdot\,)) \\ = h\big(P_2 w(x,\,\cdot\,) + (P_0(x,\,\cdot\,) + P_2 P_1) u(x,\,\cdot\,) + q(x,\,\cdot\,)\big) \end{aligned}$$

and

$$h(\ell(w)) = \ell(h(w)) = \ell(v) = h(\varphi' - P_1\Psi).$$

Consequently, $v(x)$ is a solution of problem(1.7), (1.8). □

Proof of Theorem 1.3. Let u be a solution of problem (1.1), (1.2). Set

$$w(x,y) = u_x(x,y) - P_2 u(x,y), \quad v(y) = \ell(w(\,\cdot\,,y)).$$

In view of (1.2) and (1.6), w is a solution of the problem

$$w^{(0,1)} = P_1 w + (P_0(x,y) + P_1 P_2) u(x,y) + q(x,y), \tag{4.4}$$
$$h(w(x,\,\cdot\,) = h\big(u_x(x,\,\cdot\,) - P_2 u(x,\,\cdot\,)\big) = \psi(x) - P_2\Psi(x). \tag{4.5}$$

After applying the operator h to (4.4) and (4.5) and utilizing (1.3), we get

$$\begin{aligned} \ell(w'(\,\cdot\,,y)) = v'(y) = P_1 v + \big(\ell(P_0(\,\cdot\,,y)) + P_1 P_2\big)\varphi(y) + \ell(q(\,\cdot\,,y)) \\ = \ell\big(P_1 w(\,\cdot\,,y) + (P_0(\,\cdot\,,y) + P_1 P_2) u(\,\cdot\,,y) + q(\,\cdot\,,y)\big) \end{aligned}$$

and

$$\ell(h(w)) = h(\ell(w)) = h(v) = \ell(\psi - P_2\varphi).$$

Consequently, $v(y)$ is a solution of problem (1.9), (1.10). □

Proof of Theorem 1.4. Theorem 1.4 follows from Lemmas 3.4 and 3.5. □

Proof of Theorem 1.5. Let problem (1.1), (1.2) be well-posed. Assume, on the contrary, that either condition (A_1) or (A_2) is not satisfied.

If condition (A_1) is not satisfied, then problem $(1.1_1), (1.2_1)$ has a nontrivial solution $\xi_0(x)$ for some $y^* \in [0,\omega_2]$. Due to well-posedness of problem (1.1), (1.2) there exist $\delta > 0$ and $\widetilde{P}_2 \in C^{(0,1)}(\Omega;\mathbb{R}^{n\times n})$ such that $\widetilde{P}_2(x,y) = P_2(x,y^*)$ for $y \in [y^*-\delta, y^*+\delta] \cap [0,\omega_2]$, and the problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + \widetilde{P}_2(x,y)u_y + q(x,y), \tag{4.6}$$
$$\ell(u(\,\cdot\,,y)) = 0, \quad h(u_x(x,\,\cdot\,)) = \psi(x) \tag{4.7}$$

is well-posed. In other words, problem (4.6), (4.7) has a unique solution

$$u(x,y) = \mathcal{A}(\psi,q)(x,y),$$

where $\mathcal{A} : C([0,\omega_1];\mathbb{R}^n) \times C(\Omega;\mathbb{R}^n) \to C^{1,1}(\Omega;\mathbb{R}^n)$ is a bounded linear operator.

Consider the problem

$$u_{xy} = \big(-P_1(x,y)\widetilde{P}_2(x,y) + \widetilde{P}_{2y}(x,y)\big)u + P_1(x,y)u_x + \widetilde{P}_2(x,y)u_y, \tag{4.8}$$
$$\ell(u(\,\cdot\,,y)) = 0, \quad h\big(u_x(x,\,\cdot\,) - \widetilde{P}_2(x,\,\cdot\,)u(x,\,\cdot\,)\big) = 0. \tag{4.9}$$

Every solution u of problem (4.8), (4.9) is also a solution of the operator equation

$$u(x,y) = \widetilde{\mathcal{A}}(u)(x,y), \tag{4.10}$$

where

$$\widetilde{\mathcal{A}}(u)(x,y)=\mathcal{A}\big(\boldsymbol{f}(u),Q(u)\big)(x,y),\quad \boldsymbol{f}(u)(x)=h\big(\widetilde{P}_2(x,\cdot\,)u(x,\cdot\,)\big),$$
$$Q(u)(x,y)=\big(P_0(x,y)+P_1(x,y)\widetilde{P}_2(x,y)-\widetilde{P}_{2y}(x,y)\big)u(x,y).$$

It is clear that $\widetilde{\mathcal{A}}:C(\Omega;\mathbb{R}^n)\to C^{1,1}(\Omega;\mathbb{R}^n)$ is a bounded linear operator. Consequently, $\widetilde{\mathcal{A}}:C(\Omega;\mathbb{R}^n)\to C(\Omega;\mathbb{R}^n)$ is a compact operator. Therefore, (4.10) has a finite dimensional space of solutions in $C(\Omega;\mathbb{R}^n)$. But then problem (4.8), (4.9) also has a finite dimensional space of solutions.

On the other hand, system (4.8) is equivalent to the system

$$\big(u_x-\widetilde{P}_2(x,y)u\big)_y=P_1(x,y)\big(u_x-\widetilde{P}_2(x,y)u\big). \tag{4.11}$$

Hence, every solution $u\in C^{1,1}(\Omega;\mathbb{R}^n)$ of the problem

$$u_x=\widetilde{P}_2(x,y)u, \tag{4.12}$$
$$\ell(u(\,\cdot\,,y))=0 \tag{4.13}$$

is a solution of problem (4.11), (4.9) and, consequently, of problem (4.8), (4.9).

Let $\gamma\in C^\infty([0,\omega_2])$ be an arbitrary function such that $\operatorname{supp}\gamma\subset[y^*-\delta,y^*+\delta]\cap[0,\omega_2]$. Then

$$u(x,y)=\xi_0(x)\gamma(y)$$

is a solution of problem (4.12), (4.13) and, consequently, of problem (4.8), (4.9). Thus problem (4.8), (4.9) has an infinite dimensional space of solutions, which contradicts to the fact that equation (4.10) has a finite-dimensional space of solutions.

Now, let us assume that problem (1.1_2), (1.2_2) has a nontrivial solution $\eta_0(y)$ for some $x^*\in[0,\omega_1]$. In view of well-posedness of problem (1.1), (1.2) and Lemma 3.2, there exist $\delta>0$, $\widetilde{P}_1\in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ and a bounded linear operator $\widetilde{h}:C([0,\omega_1];\mathbb{R}^n)\to\mathbb{R}^n$ such that

$$\widetilde{P}_1(x,y)=P_1(x^*,y)\ \text{ for }\ x\in[x^*-\delta,x^*+\delta]\cap[0,\omega_1],$$

the problem

$$z'=0,\quad \widetilde{h}(z)=0 \tag{4.14}$$

has only the trivial solution, and the problem

$$u_{xy}=P_0(x,y)u+\widetilde{P}_1(x,y)u_x+P_2(x,y)u_y+q(x,y), \tag{4.15}$$
$$\ell(u(\,\cdot\,,y))=\varphi(y),\quad \widetilde{h}(u_x(x,\cdot\,))=0 \tag{4.16}$$

is well-posed. In other words, problem (4.15), (4.16) has a unique solution

$$u(x,y)=\mathcal{B}(\varphi,q)(x,y),$$

where $\mathcal{B}:C([0,\omega_2];\mathbb{R}^n)\times C(\Omega;\mathbb{R}^n)\to C^{1,1}(\Omega;\mathbb{R}^n)$ is a bounded linear operator.

Consider the problem

$$u_{xy}=\big(-P_2(x,y)\widetilde{P}_1(x,y)+\widetilde{P}_{1\,x}(x,y)\big)u+\widetilde{P}_1(x,y)u_x+P_2(x,y)u_y, \tag{4.17}$$
$$\ell\bigg(u(\,\cdot\,,y)-\int\limits_0^{\omega_2}G_0(y,t)\widetilde{P}_1(\,\cdot\,,t)u(\,\cdot\,,t)\,dt\bigg)=0,\quad \widetilde{h}(u_x(x,\cdot\,))=0, \tag{4.18}$$

where and G_0 is Green's matrix of problem (4.14).

Every solution u of problem (4.17), (4.18) is also a solution of the operator equation

$$u(x,y)=\widetilde{\mathcal{B}}(u)(x,y), \tag{4.19}$$

where

$$\widetilde{\mathcal{B}}(u)(x,y) = \mathcal{B}(\mathcal{P}(u), Q(u))(x,y), \quad \mathcal{P}(u)(y) = \ell\bigg(\int_0^{\omega_2} G_0(y,t)\,\widetilde{P}_1(\,\cdot\,,t)u(\,\cdot\,,t)\,dt\bigg),$$

$$Q(u)(\mathbf{x}) = \big(P_0(x,y) + P_2(x,y)\widetilde{P}_1(x,y) - \widetilde{P}_{1x}(x,y)\big)u(x,y).$$

It is clear that $\widetilde{\mathcal{B}} : C(\Omega;\mathbb{R}^n) \to C^{1,1}(\Omega;\mathbb{R}^n)$ is a bounded linear operator. Consequently, $\widetilde{\mathcal{B}} : C(\Omega;\mathbb{R}^n) \to C(\Omega;\mathbb{R}^n)$ is a compact operator. Therefore, (4.19) has a finite dimensional space of solutions in $C(\Omega;\mathbb{R}^n)$. But then problem (4.17), (4.18) also has a finite dimensional space of solutions.

On the other hand, system (4.17) is equivalent to the system

$$(u_y - \widetilde{P}_1(x,y)u)_x = P_2(x,y)(u_y - \widetilde{P}_1(x,y)u). \tag{4.20}$$

Hence, every solution $u \in C^{1,1}(\Omega;\mathbb{R}^n)$ of the problem

$$u_y = \widetilde{P}_1(x,y)u, \tag{4.21}$$

$$\widetilde{h}(u(x,\,\cdot\,)) = 0 \tag{4.22}$$

is a solution of problem (4.20), (4.18) and, consequently, of problem (4.17), (4.18).

Let $\gamma \in C^\infty([0,\omega_1])$ be an arbitrary function such that $\operatorname{supp}\gamma \subset [x^* - \delta, x^* + \delta] \cap [0,\omega_1]$. Then $u(x,y) = \eta_0(y)\gamma(x)$ is a solution of problem (4.21), (4.22) and, consequently, of problem (4.17), (4.18). Thus problem (4.17), (4.18) has an infinite dimensional space of solutions, which contradicts to the fact that equation (4.19) has a finite-dimensional space of solutions. The obtained contradiction completes the proof of the theorem. □

Proof of Theorem 1.6. In view of Lemma 3.7, problem (1.1), (1.2_0) is equivalent to the operator equation

$$u(x,y) = \mathcal{F}(u)(x,y) \tag{4.23}$$

in the space $C(\Omega;\mathbb{R}^n)$, where

$$\mathcal{F}(u)(x,y) = \int_0^{\omega_1}\int_0^{\omega_2} G_1(x,s;y)\,G_2(y,t;s)$$
$$\times \Big(\big(P_0(s,t) + P_1(s,t)P_2(s,t) - P_{2t}(s,t)\big)u(s,t) + q(s,t)\Big)\,dt\,ds.$$

Hence, it is obvious that if conditions (1.25) hold, then for $\varepsilon > 0$ sufficiently small, $\mathcal{F}$ is an operator of contraction. Therefore, equation (4.23) is uniquely solvable and, thus, problem (1.1), (1.2) is well-posed. Moreover, if equalities (1.26) hold, then the unique solution u of equation (2.9), as well as of problem (1.1), (1.2_0), admits representation (1.27). □

Proof of Theorem 1.7. Theorem 1.7 is a particular case of Theorem 1.4, since

$$X(x,y) = \exp\big((x-a)P_2(y)\big) \text{ and } Y(y,x) = \exp\big((y-c)P_1(x)\big),$$

respectively, are fundamental matrices of the differential systems

$$\frac{dz}{dx} = P_2(y^*)z \text{ and } \frac{dz}{dy} = P_1(x^*).$$

Consequently, by statement (v) of Lemma 3.1, the problems

$$\frac{dz}{dx} = P_2(y^*)z, \quad z(a) = Az(b)$$

and

$$\frac{dz}{dy} = P_1(x^*)z, \quad z(c) = Bz(d)$$

are uniquely solvable if and only if inequalities (1.33) and (1.34) hold. □

Proof of Theorem 1.8. Theorem 1.8 follows from Theorem 1.4 and Lemma 3.14. $\square$

Proof of Theorem 1.9. By Theorem 1.9, problem (1.1), (1.27) has the Fredholm property. According to Remark 1.5, well-posedness of problem (1.1), (1.27) in Ω is equivalent to the well-posedness of problem (1.1), (1.27) in $[a,b]\times[c,d]\subset\Omega$. It remains to show that in the rectangle $[a,b]\times[c,d]$ the homogeneous problem

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y, \tag{4.24}$$

$$u(a,y)=Au(b,y),\quad u_x(x,c)=Bu_x(x,d) \tag{4.25}$$

has only the trivial solution. Let u be a solution of problem (4.24), (4.25). By Lemma 3.7, u admits the integral representation

$$u(x,y)=\int\limits_a^b\int\limits_c^d G_1(x,s;y)\,G_2(y,t;s)\big(P_0(s,t)+P_1(s,t)P_2(s,t)-P_{2t}(s,t)\big)u(s,t)\,dt\,ds,$$

whence we get

$$|u|_{C(\Omega)}\le\int\limits_a^b\int\limits_c^d\Big|G_1(x,s;y)\,G_2(y,t;s)\big(P_0(s,t)+P_1(s,t)P_2(s,t)-P_{2t}(s,t)\big)\Big|\,dt\,ds\,|u|_{C(\Omega)}$$

and $|u|_{C(\Omega)}\le\Lambda|u|_{C(\Omega)}$. The latter inequality immediately implies $u(x,y)\equiv0$ in $[a,b]\times[c,d]$, since the spectral radius of Λ is less than 1. $\square$

Proof of Theorem 1.10. In view of (1.37) and (1.38), by Lemma 3.14 and Theorem 1.4, problem (1.1), (1.29) has the Fredholm property. According to Remark 1.5, problem (1.1), (1.29) is well-posed in Ω if and only if it is the well-posed in $[a,b]\times[c,d]\subset\Omega$. It remains to show that in the rectangle $[a,b]\times[c,d]$ the homogeneous problem

$$u_{xy}=P_0(x)u+P_1(x)u_x+P_2(x)u_y, \tag{4.26}$$

$$u(a,y)=Au(b,y),\quad u_x(x,c)=u_x(x,d) \tag{4.27}$$

has only the trivial solution. Let u be a solution of problem (4.26), (4.27). Then u admits the Fourier series representation

$$u(x,y)=\sum_{k=-\infty}^{+\infty}u_k(x)e^{\imath k\,\frac{2\pi}{d-c}\,y}. \tag{4.28}$$

After substituting (4.28) in (4.26), we get

$$u_k'(x)=\Big(-P_1(x)+\imath\,\frac{2\pi}{d-c}\,kI\Big)^{-1}\Big(P_0(x)+\imath\,\frac{2\pi}{d-c}\,kP_2(x)\Big)u_k(x),\quad u_k(a)=Au_k(b). \tag{4.29}$$

In view of (1.35) and (1.38), for every $j,k=0,1,2$, $\sigma_j\sigma_kP_j(x)P_k(x)$ is a positive definite symmetric matrix function, and

$$\begin{aligned}\Big(-P_1(x)+\imath\,\frac{2\pi}{d-c}\,kI\Big)^{-1}&\Big(P_0(x)+\imath\,\frac{2\pi}{d-c}\,kP_2(x)\Big)\\&=-\Big(P_0(x)P_1(x)-\frac{4\pi^2}{(d-c)^2}\,k^2P_2(x)\Big)\Big(P_1^2(x)+\frac{4\pi^2}{(d-c)^2}\,k^2I\Big)^{-1}\\&\qquad-\imath\,\frac{2\pi}{d-c}\,k\big(P_0(x)+P_1(x)P_2(x)\big)^{-1}\Big(P_1^2(x)+\frac{4\pi^2}{(d-c)^2}\,k^2I\Big)^{-1}.\end{aligned} \tag{4.30}$$

Moreover,

$$\sigma_2\Big(-P_0(x)P_1(x)+\frac{4\pi^2}{(d-c)^2}\,k^2P_2(x)\Big)\Big(P_1^2(x)+\frac{4\pi^2}{(d-c)^2}\,k^2I\Big)^{-1}$$

and

$$\sigma_0\big(P_0(x)+P_1(x)P_2(x)\big)^{-1}\Big(P_1^2(x)+\frac{4\pi^2}{(d-c)^2}\,k^2I\Big)^{-1}$$

are positive definite symmetric matrix functions. Hence, taking into account (4.29) and (4.30), we get

$$\begin{aligned}\big\langle (I-AA^T)u_k(b),u_k(b)\big\rangle &= \int\limits_a^b \frac{d}{dx}\,\langle u_k(x),u_k(x)\rangle\,dx=\int\limits_a^b\Big(\langle u_k'(x),u_k(x)\rangle+\langle u_k(x),u_k'(x)\rangle\Big)\,dx\\ &=2\int\limits_a^b\Big\langle\Big(-P_0(x)P_1(x)+\frac{4\pi^2}{(d-c)^2}\,k^2P_2(x)\Big)\Big(P_1^2(x)+\frac{4\pi^2}{(d-c)^2}\,k^2I\Big)^{-1}u_k(x),u_k(x)\Big\rangle\,dx. \qquad (4.31)\end{aligned}$$

(1.36)–(1.38) and(4.31) immediately imply $u_k(x)\equiv 0$ on $[a,b]$ $(k\in\mathbb{Z})$. Consequently, $u(x,y)\equiv 0$ on $[a,b]\times[c,d]$. ☐

Proof of Corollary 1.1. Corollary 1.1 is an immediate consequence of Theorem 1.10. ☐

Proof of Theorem 1.11. In view of (1.41) and (1.42), by Lemma 3.14 and Theorem 1.4, problem (1.1), (1.29) has the Fredholm property. According to Remark 1.5, problem (1.1), (1.29) is well-posed in Ω if and only if it is the well-posed in $[a,b]\times[c,d]\subset\Omega$. It remains to show that in the rectangle $[a,b]\times[c,d]$ the homogeneous system

$$u_{xy}=P_0u+P_1u_x+P_2u_y \qquad (4.32)$$

has only the trivial solution satisfying condition (4.27). Let u be a solution of problem (4.32), (4.27). Then u admits the Fourier series representation (4.28). After substituting (4.28) in (4.26), we get

$$u_k'(x)=\Lambda_k u_k(x), \quad u_k(a)=Au_k(b), \qquad (4.33)$$

where

$$\Lambda_k=\Big(i\,\frac{2\pi}{d-c}\,kI-P_1\Big)^{-1}\Big(P_0+i\,\frac{2\pi}{d-c}\,kP_2\Big).$$

By statement (v) of Lemma (3.1), problem (4.33) has a trivial solution if and only if

$$\det\big(I-A\exp((b-a)\Lambda_k)\big)\neq 0.$$ ☐

Proof of Theorem 1.12. By Lemma 3.15 and Theorem 2.1, problem (1.43), (1.44), (1.45) has the Fredholm property. According to Remark 1.5, well-posedness of problem (1.43), (1.44), (1.45) in Ω is equivalent to the well-posedness of problem (1.43), (1.44), (1.45) in $[a,b]\times[c,d]\subset\Omega$. It remains to show that in the rectangle $[a,b]\times[c,d]$ the homogeneous problem

$$v_{xy}=Q_1(y)v_x+P_{11}(y)w_x+Q_2(x)v_y+P_{21}(x)w_y+Q_0(x,y)v+P_{01}(x,y)w, \qquad (4.34)$$

$$w_{xy}=P_{12}(y)v_x-Q_1(y)w_x+P_{22}(x)v_y-Q_2(x)w_y-P_{02}(x,y)v+Q_0(x,y)w, \qquad (4.35)$$

$$\begin{aligned}v(a,y)&=Av(b,y), \quad w(b,y)=Aw(a,y),\\ v_x(x,c)&=Bv_x(x,d), \quad w_x(x,d)=Bw_x(x,a)\end{aligned} \qquad (4.36)$$

has only the trivial solution. Let (v,w) be an arbitrary solution of problem (4.34), (4.35), (4.36). Then we have

$$\begin{aligned}&\int\limits_a^b\int\limits_c^d\big\langle v_{xy}(x,y),w(x,y)\big\rangle\,dy\,dx\\ &\quad=\int\limits_a^b\int\limits_c^d\Big(\big\langle Q_1(y)v_x(x,y),w(x,y)\big\rangle+\big\langle P_{11}(y)w_x(x,y),w(x,y)\big\rangle+\big\langle Q_2(x)v_y(x,y),w(x,y)\big\rangle\\ &\quad+\big\langle P_{21}(x)w_y(x,y),w(x,y)\big\rangle+\big\langle Q_0(x,y)v(x,y),w(x,y)\big\rangle+\big\langle P_{01}(x,y)w(x,y),w(x,y)\big\rangle\Big)\,dy\,dx\end{aligned}$$

and

$$\int_a^b\int_c^d \big\langle w_{xy}(x,y),v(x,y)\big\rangle\,dy\,dx$$
$$= \int_a^b\int_c^d \Big(\big\langle P_{12}(y)v_x(x,y),v(x,y)\big\rangle - \big\langle Q_1(y)w_x(x,y),v(x,y)\big\rangle + \big\langle P_{22}(x)v_y(x,y),v(x,y)\big\rangle$$
$$- \big\langle Q_2(x)w_y(x,y),v(x,y)\big\rangle - \big\langle P_{02}(x,y)v(x,y),v(x,y)\big\rangle + \big\langle Q_0(x,y)w(x,y),v(x,y)\big\rangle\Big)\,dy\,dx.$$

Taking into account conditions (1.50), (1.51) and (4.36), and symmetry of the matrices and matrix functions A, B, Q_i, P_{ij} $(i=0,1,2;\ j=1,2)$, we get

$$\int_a^b\int_c^d \big\langle v_{xy}(x,y),w(x,y)\big\rangle\,dy\,dx = \int_a^b\int_c^d \big\langle w_{xy}(x,y),v(x,y)\big\rangle\,dy\,dx,$$
$$\int_a^b\int_c^d \big\langle Q_1(y)v_x(x,y),w(x,y)\big\rangle\,dy\,dx = -\int_a^b\int_c^d \big\langle Q_1(y)w_x(x,y),v(x,y)\big\rangle\,dy\,dx,$$
$$\int_a^b\int_c^d \big\langle Q_2(x)v_y(x,y),w(x,y)\big\rangle\,dy\,dx = -\int_a^b\int_c^d \big\langle Q_2(x)w_y(x,y),v(x,y)\big\rangle\,dy\,dx,$$
$$\int_a^b\int_c^d \big\langle P_{11}(y)w_x(x,y),w(x,y)\big\rangle\,dy\,dx = \frac{1}{2}\int_c^d \big\langle P_{11}(y)(A^2-I)w(a,y),w(a,y)\big\rangle\,dy,$$
$$\int_a^b\int_c^d \big\langle P_{12}(y)v_x(x,y),v(x,y)\big\rangle\,dy\,dx = \frac{1}{2}\int_c^d \big\langle P_{12}(y)(I-A^2)v(b,y),v(b,y)\big\rangle\,dy,$$
$$\int_a^b\int_c^d \big\langle P_{21}(x)w_y(x,y),w(x,y)\big\rangle\,dy\,dx = \frac{1}{2}\int_a^b \big\langle P_{21}(x)(B^2-I)w(x,c),w(x,c)\big\rangle\,dx,$$
$$\int_a^b\int_c^d \big\langle P_{22}(x)v_y(x,y),v(x,y)\big\rangle\,dy\,dx = \frac{1}{2}\int_a^b \big\langle P_{22}(x)(I-A^2)v(x,d),v(x,d)\big\rangle\,dx.$$

Therefore, after subtracting (3.17) from (3.16), we arrive at the equality

$$\frac{1}{2}\int_c^d \Big(\big\langle P_{11}(y)(A^2-I)w(a,y),w(a,y)\big\rangle + \big\langle P_{12}(y)(A^2-I)v(b,y),v(b,y)\big\rangle\Big)\,dy$$
$$+\frac{1}{2}\int_a^b \Big(\big\langle P_{21}(x)(B^2-I)w(x,c),w(x,c)\big\rangle + \big\langle P_{22}(x)(B^2-I)v(x,d),v(x,d)\big\rangle\Big)\,dx$$
$$+\int_a^b\int_c^d \Big(\big\langle P_{01}(x,y)v(x,y),v(x,y)\big\rangle + \big\langle P_{0,2}(x,y)w(x,y),v(x,y)\big\rangle\Big)\,dy\,dx = 0.$$

If (B_1) holds, then $v(a,y)=0$ and $w(a,y)=0$ (also, $v(b,y)=0$ and $w(b,y)=0$).

If (B_3) holds, then $v(x_0,y)=0$ and $w(x_0,y)=0$.

Therefore, $(v(x,y),w(x,y))$ is a solution of system (4.34), (4.35) satisfying the initial-boundary conditions

$$\begin{gathered} v(x_1,y)=0, \quad w(x_1,y)=0, \\ v_x(x,c)=Bv_x(x,d), \quad w_x(x,d)=Bw_x(x,a), \end{gathered} \tag{4.37}$$

where x_1 is either a, or x_0. By Lemma 3.3, the homogeneous problem (4.34), (4.35), (4.37) has only the trivial solution in $[a,b]\times[c,d]$.

Similarly, one can show that if either (B_2) or (B_4)holds, then $u(x,y)\equiv 0$ in $[a,b]\times[c,d]$. □

Corollaries 1.2–1.5 are particular cases of Theorem 1.12.

Proof of Theorem 2.1. The proof of Theorem 2.1 is similar to that of Theorem 1.4. □

Proof of Theorem 2.2. The proof of Theorem 2.2 is similar to that of Theorem 1.5. □

Proof of Theorem 2.3. The proof of Theorem 2.3 is similar to that of Theorem 1.7. □

Proof of Theorem 2.4. The proof of Theorem 2.4 is similar to that of Theorem 1.8. □

Proof of Theorem 2.5. By Lemma 9.3 and Theorem 9.2 from [18], the conditions of Theorem 2.5 guarantee unique solvability of the associated problems $(2.1_1), (2.2_1)$ and $(2.1_2), (2.2_2)$ for every $y^*\in[0,\omega_2]$ and $x^*\in[0,\omega_1]$, respectively. Consequently, by Theorem 2.1, problem (2.1), (2.2) has the Fredholm property. □

Proof of Theorem 2.6. By Lemma 9.4 and Theorem 9.2 from [18], the conditions of Theorem 2.6 guarantee unique solvability of the associated problems $(2.1_1), (2.2_1)$ and $(2.1_2), (2.2_2)$ for every $y^*\in[0,\omega_2]$ and $x^*\in[0,\omega_1]$, respectively. Consequently, by Theorem 2.1, problem (2.1), (2.2) has the Fredholm property. □

Proof of Theorem 2.7. The proof of Theorem 2.7 is similar to that of Theorem 1.9. □

Proof of Theorem 2.8. The proof of Theorem 2.8 is similar to that of Theorem 1.12. □

Proof of Theorem 2.9. By Lemma 3.14 and Theorem 2.1, problem (2.6), (2.2) has the Fredholm property. It remains to show that the corresponding homogeneous problem has only the trivial solution. Let u be a solution of the homogeneous system

$$u_{xy}=P_0(x,y)u+u_x+u_y$$

satisfying conditions (2.2). Then we have

$$\begin{aligned}\int_0^{\omega_1}\int_0^{\omega_2}\big\langle u_{xy}(x,y),u(x,y)\big\rangle\,dy\,dx=\int_0^{\omega_1}\int_0^{\omega_2}\Big(\big\langle P_0(x,y)u(x,y),u(x,y)\big\rangle\\ +\big\langle u_x(x,y),u(x,y)\big\rangle+\big\langle u_y(x,y),u(x,y)\big\rangle\Big)\,dy\,dx,\end{aligned}\tag{4.38}$$

$$\begin{aligned}\int_0^{\omega_1}\int_0^{\omega_2}\big\langle u_{xy}(x,y),u_x(x,y)\big\rangle\,dy\,dx=\int_0^{\omega_1}\int_0^{\omega_2}\Big(\big\langle P_0(x,y)u(x,y),u_x(x,y)\big\rangle\\ +\big\langle u_x(x,y),u_x(x,y)\big\rangle+\big\langle u_y(x,y),u_x(x,y)\big\rangle\Big)\,dy\,dx,\end{aligned}\tag{4.39}$$

$$\begin{aligned}\int_0^{\omega_1}\int_0^{\omega_2}\big\langle u_{xy}(x,y),u_y(x,y)\big\rangle\,dy\,dx=\int_0^{\omega_1}\int_0^{\omega_2}\Big(\big\langle P_0(x,y)u(x,y),u_y(x,y)\big\rangle\\ +\big\langle u_x(x,y),u_y(x,y)\big\rangle+\big\langle u_y(x,y),u_y(x,y)\big\rangle\Big)\,dy\,dx.\end{aligned}\tag{4.40}$$

In view of condition (2.2), from (4.38)–(4.40) we get

$$-\int_0^{\omega_1}\int_0^{\omega_2}\big\langle u_x(x,y),u_y(x,y)\big\rangle\,dy\,dx=\int_0^{\omega_1}\int_0^{\omega_2}\big\langle P_0(x,y)u(x,y),u(x,y)\big\rangle\,dy\,dx,\tag{4.41}$$

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big(\big\langle P_0(x,y)u(x,y),u_x(x,y)\big\rangle+\|u_x(x,y)\|^2+\big\langle u_y(x,y),u_x(x,y)\big\rangle\Big)\,dy\,dx=0, \tag{4.42}$$

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big(\big\langle P_0(x,y)u(x,y),u_y(x,y)\big\rangle+\big\langle u_x(x,y),u_y(x,y)\big\rangle+\big\|u_y(x,y)\big\|^2\Big)\,dy\,dx=0. \tag{4.43}$$

If $P_0\in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$, then (4.41) and (4.42) imply

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big(\|u_x(x,y)\|^2-\Big\langle\Big(P_0(x,y)+\frac{1}{2}\,P_{0x}(x,y)\Big)u(x,y),u(x,y)\Big\rangle\Big)\,dy\,dx=0. \tag{4.44}$$

If $P_0\in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$, then (4.41) and (4.43) imply

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big(\|u_y(x,y)\|^2-\Big\langle\Big(P_0(x,y)+\frac{1}{2}\,P_{0y}(x,y)\Big)u(x,y),u(x,y)\Big\rangle\Big)\,dy\,dx=0. \tag{4.45}$$

If $P_0\in C^{1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$, then (4.44) and (4.45) imply

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big(\|u_x(x,y)\|^2+\|u_y(x,y)\|^2 \\ -2\Big\langle\Big(P_0(x,y)+\frac{1}{4}\,P_{0x}(x,y)+\frac{1}{4}\,P_{0y}(x,y)\Big)u(x,y),u(x,y)\Big\rangle\Big)\,dy\,dx=0. \tag{4.46}$$

Set

$$Q_1(y)=\int_0^{\omega_1}P_0(x,y)\,dx;\quad Q_2(x)=\int_0^{\omega_2}P_0(x,y)\,dy;\quad Q_0=\int_0^{\omega_1}\int_0^{\omega_2}P_0(x,y)\,dy\,dx.$$

If condition (i) holds, then (4.43) implies $u(x,y)\equiv u(y)$,

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big\langle\Big(P_0(x,y)+\frac{1}{2}\,P_{0x}(x,y)\Big)u(y),u(y)\Big\rangle\,dy\,dx=\int_0^{\omega_2}\langle Q_1(y)u(y),u(y)\rangle\,dy=0$$

and $u(y^*)\equiv 0$ due to the negative semi-definiteness of $Q_1(y)$ on $[0,\omega_2]$ and negative definiteness of $Q_1(y^*)$. Hence, u is a solution of the homogeneous initial-periodic problem

$$u_{xy}=P_0(x,y)u+u_x+u_y,$$
$$u(x+\omega_1)=u(x),\quad u(x,y^ast)=0.$$

By Remark 1.1 and Lemma 3.3, $u(x,y)\equiv 0$.

Similarly, one can show that if condition (ii) holds, then $u(x,y)\equiv 0$.

If condition (iii) holds, then (4.46) implies $u(x,y)\equiv u=const$,

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big\langle\Big(P_0(x,y)+\frac{1}{4}\,P_{0x}(x,y)+\frac{1}{4}\,P_{0y}(x,y)\Big)u,u\Big\rangle\,dy\,dx=\langle Q_0u,u\rangle=0,$$

and $u=0$ due to the negative definiteness of Q_0. □

Proof of Theorem 2.10. The proof of Theorem 2.10 is similar to that of Theorem 1.10. □

Proof of Corollary 2.1. Corollary 2.1 is an immediate consequence of Theorem 2.6. □

Proof of Theorem 2.11. In view of (2.10) and (2.11), by Lemma 3.14 and Theorem 2.1, problem (2.7), (2.2) has the Fredholm property. It remains to show that the homogeneous problem

$$u_{xy} = P_0 u + P_1 u_x + P_2 u_y, \tag{4.47}$$

$$u(x+\omega_1, y) = u(x,y), \quad u(x, y+\omega_2) = u(x,y) \tag{4.48}$$

has only the trivial solution. Let u be a solution of problem (4.47), (4.48). Then u admits the double Fourier series representation

$$u(x,y) = \sum_{m,k=-\infty}^{+\infty} u_{mk}(x) e^{\imath m \frac{2\pi}{\omega_1} x + \imath k \frac{2\pi}{\omega_2} y}. \tag{4.49}$$

After substituting (4.49) in (4.47), we get

$$\det\Big(P_0 + i\,\frac{2\pi}{\omega_1}\, mP_1 + i\frac{2\pi}{\omega_2} kP_2 + \frac{4\pi^2}{\omega_1\omega_2}\, mk\, I\Big) u_{mk} = 0.$$

Hence, $u_{mk} = 0$ in and only if

$$\det\Big(P_0 + i\,\frac{2\pi}{\omega_1}\, mP_1 + i\,\frac{2\pi}{\omega_2}\, kP_2 + \frac{4\pi^2}{\omega_1\omega_2}\, mk\, I\Big) \neq 0. \qquad \square$$

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