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A STUDY OF BOUNDEDNESS RELATED TO SOME MAXIMAL OPERATORS OF FEJÉR MEANS ON MARTINGALE HARDY SPACES

Abstract. The aim of this monograph is to discuss, develop and apply the newest developments of this fascinating theory connected with modern harmonic analysis. In particular, we investigate Fejér kernels and study two-sided estimates for them, then we prove the almost everywhere convergence of Fejér means. Moreover, we define the Lebesgue and Walsh–Lebesgue points of integrable functions and prove the convergence of Fejér means of integrable functions with respect to the Walsh system at Walsh–Lebesgue points, which are the almost everywhere points for any integrable function. We also study the convergence of Fejér means in L_p norm. Moreover, we investigate some important upper and lower estimates of Fejér kernels that are very crucial in proving the important results in the theory of martingale Hardy spaces. In particular, we consider the maximal operator, restricted maximal operators and weighted maximal operators of Fejér means with respect to the Walsh system and prove the (H_p, L_p) and $(H_p, \textit{weak-}L_p)$ type inequalities for them. Next, we apply these results to find the necessary and sufficient conditions for the modulus of continuity under which the convergence of norm Fejér means occurs. We also study analogous problems for the subsequences of Fejér means in terms of modulus of continuity. Finally, we prove some new Hardy type inequalities for Fejér means, which are called strong convergence results of Fejér means. We also prove the sharpness of all our main results in the present monograph.

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Key words and phrases. Dyadic group, Walsh system, L_p space, *weak*-L_p space, modulus of continuity, Fejér means, dyadic martingale, martingale Hardy spaces, maximal operators, restricted maximal operators, weighted maximal operators, strong convergence.

რეზიუმე. მონოგრაფიის მიზანია თანამედროვე ჰარმონიულ ანალიზთან დაკავშირებული ამ საინტერესო თეორიის უახლესი მიღწევების განხილვა, განვითარება და გამოყენება. კერძოდ, ჩვენ ვიკვლევთ ფეიერის საშუალოებს და ვსწავლობთ მათთვის ორმხრივ შეფასებებს, შემდეგ კი ვამტკიცებთ ფეიერის საშუალოების თითქმის ყველგან კრებადობას. გარდა ამისა, განვსაზღვრავთ ინტეგრებადი ფუნქციების ლებეგის და უოლშ-ლებეგის წერტილებს და ვამტკიცებთ, რომ ინტეგრებადი ფუნქციების ფეიერის საშუალოები უოლშის სისტემის მიმართ კრებადია უოლშ-ლებეგის წერტილებში, რომლებიც წარმოადგენს თითქმის ყველა წერტილს ნებისმიერი ინტეგრებადი ფუნქციისთვის. ასევე შევისწავლით ფეიერის საშუალოების კრებადობას L_p ნორმით. ამასთან, გამოვიკვლევთ ფეიერის გულების ზოგიერთ მნიშვნელოვან ზედა და ქვედა შეფასებას, რომლებიც არსებით როლს ასრულებს მარტინგალური ჰარდის სივრცეების თეორიაში მნიშვნელოვანი შედეგების დამტკიცებისას. კერძოდ, განვიხილავთ ფეიერის საშუალოების მაქსიმალურ ოპერატორს, შეზღუდულ მაქსიმალურ ოპერატორებსა და წონიან მაქსიმალურ ოპერატორებს უოლშის სისტემის მიმართ და მათთვის დავამტკიცებთ (H_p, L_p) და $(H_p, \text{weak-}L_p)$ ტიპის უტოლობებს. შემდეგ ამ შედეგებს გამოვიყენებთ უწყვეტობის მოდულისათვის აუცილებელი და საკმარისი პირობების საპოვნელად, რომელთა შესრულების პირობებში ადგილი აქვს ფეიერის საშუალოების ნორმით კრებადობას. უწყვეტობის მოდულის ტერმინებში ასევე ვსწავლობთ ანალოგიურ ამოცანებს ფეიერის საშუალოების ქვემიმდევრობებისთვის. საბოლოოდ, დავამტკიცებთ ჰარდის ტიპის რამდენიმე ახალ უტოლობას ფეიერის საშუალოებისთვის, რომლებიც ცნობილია, როგორც ფეიერის საშუალოების ძლიერად კრებადობის თეორემები. მონოგრაფიაში ასევე დავამტკიცებთ წარმოდგენილი ყველა ძირითადი შედეგის გაუძლიერებადობას.

1 Dyadic group and Fourier analysis on it

By $\mathbb{N}_+$ we denote the set of the positive integers, by $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ the set of non-negative integers and by $\mathbb{R}$ the set of the real numbers.

Denote by Z_2 the additive group of integers modulo-2, which contains only two elements: $Z_2 := \{0, 1\}$, the group operation is modulo-2 sum and all subsets are open.

Define the group G as the complete direct product of the groups Z_2 with the product of the discrete topologies of Z_2.

The elements of G are represented by the sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$, $x_k = 0, 1$. This group is called the Walsh group or dyadic group.

The dyadic group is metrizable with the following metric:

$$\rho(x, y) =: \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^{k+1}}, \quad x, y \in G. \tag{1.1}$$

The direct product μ of the measures $\mu_n(\{j\}) := 1/2$, $j \in Z_2$ is the Haar measure on G with $\mu(G) = 1$.

Translation of a subset $I_n(x) \in G$ by y is defined by $\tau_y(I_n(x)) = \{I_n(x) + y\}$. Since μ is a product measure, we get

$$\mu(I_n(x)) = \prod_{k=0}^{n-1} \left(\mu_k\{x_k\}\right) \prod_{k=n}^{\infty} \left(\mu_k\{0, \ldots, 1\}\right) = \prod_{k=0}^{n-1} \frac{1}{2} \prod_{k=n}^{\infty} 1 = \frac{1}{2^n}.$$

On the other hand, $\mu(\tau_y(I_n(x))) = \mu(I_n(x+y)) = 1/2^n$ for all $y \in G$. Hence, $\mu(\tau_y(I_n(x))) = \mu(I_n(x))$.

In particular, G is a locally compact Abelian group and it can be equipped with the Haar measure (see Pontryagin [80]), which coincides with the product measure μ. We will not construct the Haar measure here, since, for the groups we consider, the construction is trivial.

It is easy to give a base for the neighborhoods of G:

$$I_0(x) := G,$$
$$I_n(x) := \left\{y \in G \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\right\}, \quad x \in G, \quad n \in \mathbb{N}.$$

Set $I_n := I_n(0)$ for any $n \in \mathbb{N}$ and $\overline{I}_n := G \setminus I_n$.

Define

$$I_M^{k,l} =: \begin{cases} I_{l+1}(e_k + e_l) & \text{for } \ k < l < M, \\ I_M(e_k) & \text{for } \ l = M. \end{cases}$$

It is evident that

$$\begin{aligned}\overline{I}_M &= \Big(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_M^{k,l}\Big) \cup \Big(\bigcup_{k=1}^{M-1} I_M^{k,M}\Big) \\ &= \Big(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l)\Big) \cup \Big(\bigcup_{k=0}^{M-1} I_M(e_k)\Big) = \bigcup_{k=0}^{M-1} I_k \setminus I_{k+1}.\end{aligned} \tag{1.2}$$

The norm (quasi-norm) of the space $L_p(G)$ for $0 < p < \infty$ is defined as

$$\|f\|_p^p := \Big(\int_G |f(x)|^p \, d\mu(x)\Big).$$

Denote by $L_\infty(G)$ the space of all $f \in L_0(G)$ for which $\|f\|_\infty =: \inf\{C > 0, \ \mu\{x \in G, |f| > C\} = 0\} < \infty$.

Let $L_0(G)$ represent the collection of functions that are almost everywhere limits of step functions with respect to the measure μ of sequences in $\mathcal{P}$.

The space $C(G)$ consists of all continuous functions for which $\|f\|_C =: \sup_{x \in G} |f(x)| < c < \infty$. It is obvious that if $f \in C(G)$, then $\|f\|_C = \|f\|_\infty$ and $C(G) \subset L_\infty(G)$. Moreover, if $1 < p_1 < p_2 \leq \infty$, then $L_{p_2}(G) \subset L_{p_1}(G)$.

It is well-known that if $f \in L_p(G)$ for all $p > 0$, then $f \in L_\infty(G)$ and $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

Using Minkowski's inequality, we easily obtain

$$\|f\|_p = 0 \Longleftrightarrow f = 0 \text{ a.e.},$$
$$\|cf\|_p = |c|\,\|f\|_p, \quad c \in \mathbb{C},$$
$$\|f+g\|_p \le C_p\big(\|f\|_p + \|g\|_p\big),$$

where $C_p = 1$ for $p \ge 1$ and $C_p \le 2^{1/p}$ for $p < 1$. In view of the last property, $\|\cdot\|_p$ is a norm for $p \ge 1$ and quasi-norm for $p < 1$.

Proposition 1.1 (Generalized Minkowski inequality)**.** *Let $p \ge 1$. Then*

$$\Big\| \int_G f(\,\cdot\,,t)\,dt \Big\|_p \le \int_G \|f(\,\cdot\,,t)\|_p\,dt.$$

The convolution of two functions $f, g \in L_1(G)$ is defined by

$$(f * g)(x) =: \int_G f(x+t)g(t)\,dt, \quad x \in G.$$

It is easy to see that

$$(f * g)(x) = \int_G f(t)g(x+t)\,dt, \quad x \in G.$$

Proposition 1.2. *Let $f \in L_r(G)$, $g \in L_1(G)$ and $1 \le r \le \infty$. Then $f * g \in L_r(G)$ and*

$$\|f * g\|_r \le \|f\|_r \|g\|_1.$$

The modulus of continuity of functions in the Lebesgue spaces is defined by

$$\omega_p\Big(\frac{1}{2^n}, f\Big) =: \sup_{h \in I_n} \big\|f(\,\cdot\,+h) - f(\,\cdot\,)\big\|_p.$$

The space *weak-*$L_p(G)$ consists of all measurable functions f for which

$$\|f\|^p_{weak\text{-}L_p(G)} := \sup_{\lambda>0} \lambda^p \mu\big(x \in G : \ |f| > \lambda\big).$$

The following properties of the *weak-*$L_p(G)$ spaces can be proved easily:

$$\|f\|_{weak\text{-}L_p} = 0 \Longleftrightarrow f = 0 \text{ a.e.},$$
$$\|cf\|_{weak\text{-}L_p} = |c|\|f\|_{weak\text{-}L_p} \quad (c \in \mathbb{R}),$$
$$\|f+g\|_{weak\text{-}L_p} \le C_p\big(\|f\|_{weak\text{-}L_p} + \|g\|_{weak\text{-}L_p}\big),$$

where $C_p = \max(2, 2^{1/p})$. Due to the last property $\|\cdot\|_{weak\text{-}L_p}$ is a quasi-norm.

Proposition 1.3 (Tsebisev's inequality)**.** *If $0 < p \le \infty$, then*

$$y^p \mu(|f| > y) \le \int_G |f(x)|^p\,dx.$$

Proof. It is easy to see that

$$\int_G |f(x)|^p\,dx \ge \int_{\{x:\,|f(x)|>y\}} |f(x)|^p\,dx \ge y^p \mu(|f| > y),$$

which proves the proposition. □

We will show that the *weak-*$L_p(G)$ spaces are larger than the $L_p(G)$ spaces. In particular, Chebyshev's inequality can be rewritten in the following form.

Proposition 1.4. *If* $0 < p \leq \infty$*, then* $L_p(G) \subset$ *weak-*$L_p(G)$ *and* $\|f\|_{weak\text{-}L_p} \leq \|f\|_p$.

Note that the inclusion $L_p(G) \subset$ *weak-*$L_p(G)$ is proper for any $0 < p < \infty$. Indeed, let $h(x) =: |x|^{-1/p}$. Then, obviously, $h \notin L_p(G)$, but $h \in$ *weak-*$L_p(G)$ because of $y^p \mu(\{x : \ |x|^{-1/p} > y\}) = y^p y^{-p} = 1$. Recall that the *weak-*$L_p(G)$ space is also complete for each $p \geq 1$.

We define the distribution function λ_f as follows: $\lambda_f(y) =: \mu\{x : \ |f(x)| > y\}$. We now establish a useful formula which allows us to express the $L_p(G)$ norm of f in terms of the function $\lambda_f(y)$, namely, we have the following

Proposition 1.5. *We have*

$$\int\limits_G |f(x)|^p \, d\mu(x) = p \int\limits_0^\infty y^{p-1} \lambda_f(y) \, dy, \quad p \geq 1. \tag{1.3}$$

An operator T which maps a linear space of measurable functions on G in the collection of measurable functions on G is called sublinear if

$$|T(f+g)| \leq |T(f)| + |T(g)| \ \text{ a.e. on } G, \quad |T(\alpha f)| = |\alpha| \, |T(f)|$$

for all scalars α and all f in the domain of T. An operator T is a σ-sublinear operator if, for any $\alpha \in \mathbb{R}$,

$$\Big| T\Big(\sum_{k=1}^\infty f_k\Big)\Big| \leq \sum_{k=1}^\infty |T(f_k)| \ \text{ and } \ |T(\alpha f)| = |\alpha| \, |T(f)|.$$

An operator T from $L_p(G)$ into $L_0(G)$ is said to be of strong type (p,q) (or of (p,q) type) for some $1 \leq p \leq \infty$ and $1 < q < \infty$ if there is a constant $A > 0$ such that $\|Tf\|_q \leq A \|f\|_p$ for all $f \in L_p(G)$. An operator T from $L_p(G)$ into $L_0(G)$ is said to be of weak type (p,q) for some $1 \leq p \leq \infty$ and $1 < q < \infty$ if there is a constant $A > 0$ such that $\|Tf\|_{weak\text{-}L_q} \leq A \|f\|_p$ for all $f \in L_p(G)$. Hence, any operator of type (L_p, L_q) is necessarily of weak type (p,q).

Now, we state three important theorems of Fourier analysis.

Theorem 1.1 (Marcinkiewicz interpolation theorem)**.** *Let* $1 \leq p_1 < p_2$ *and suppose that the sublinear operator* T *satisfies the condition*

$$y^{p_i} \lambda_{Tf}(y) \leq C_i^{p_i} \int\limits_G |f(y)|^{p_i} \, d\mu(y)$$

for all $y > 0$, $f \in L_{p_i}(G)$ *and* $i = 1, 2$*, where* C_i*'s are the absolute constants independent of* y *and* f*. In case* $p_2 = \infty$*, suppose that* T *is of strong type, i.e.,* $\|Tf\|_\infty \leq C_2 \|f\|_\infty$*. Then*

$$\|Tf\|_p^p \leq p 2^p C_1^{p_1 \frac{p_2 - p}{p_2 - p_1}} C_2^{p_2 \frac{p - p_1}{p_2 - p_1}} \Big(\frac{1}{p - p_1} + \frac{1}{p_2 - p} \Big) \|f\|_p^p$$

for all $f \in L_p(G)$, $p_1 < p < p_2$.

In particular, if $p_1 < p < p_2$ *and the operator* T *is of weak* (p_1, p_1) *type and weak* (p_2, p_2) *type, then it will be of* (p,p) *type.*

Theorem 1.2. *Suppose* X_0 *is dense in* $L_p(G)$ *and let* $T, T_n : X \to L_p(G)$ *be sublinear operators, for some* $1 \leq p < \infty$ *with* T *bounded and* $T_n f \to Tf$ *a.e. on* G*, as* $n \to \infty$*, for each* $f \in X_0$*. Set*

$$T^* f =: \sup_{n \in \mathbb{N}} |T_n f|, \quad f \in L_p.$$

If there is a constant $C > 0$*, independent of* f *and* n *such that*

$$y^p \mu\big(\{|Tf| > y\}\big) \leq C \, \|f\|_p^p, \quad y^p \mu\big(\{T^* f > y\}\big) \leq C \, \|f\|_p^p$$

for all $y > 0$ *and* $f \in L_p$*, then* $Tf = \lim_{n \to \infty} T_n f$ *a.e. on* G*, for every* $f \in L_p$.

Theorem 1.3. *Suppose that the σ-sublinear operator V is bounded from L_{p_1} to L_{p_1} for some $1 < p_1 \leq \infty$ and*

$$\int\limits_{\overline{I}} |Vf| \, d\mu \leq C \, \|f\|_1$$

for $f \in L_1$ and the dyadic interval I satisfies

$$\operatorname{supp} f \subset I, \quad \int\limits_{G} f \, d\mu = 0. \tag{1.4}$$

Then the operator V is of weak-type $(1,1)$, i.e., $\sup\limits_{y>0} y\mu(\{Vf > y\}) \leq \|f\|_1$.

2 The numerical characterization of natural numbers

It is easy to see that every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum\limits_{j=0}^{\infty} n_j \cdot 2^j$, where $n_j \in Z_2$, $j \in \mathbb{N}$, and only a finite number of n_js differs from zero.

Let us denote $n^{(k)} := 2^{n_{k+1}} + 2^{n_{k+2}} + \cdots + 2^{n_r}$, $n^{(0)} := n$.

For any natural number $n \in \mathbb{N}$, we need the following expression: $n = \sum\limits_{i=1}^{s} 2^{n_i}$, $n_1 < n_2 < \cdots < n_s$.

Set $n^{(i)} := 2^{n_1} + \cdots + 2^{n_{i-1}}$, $i = 2, \ldots, s$, and

$$\mathbb{A}_{0,2} := \Big\{ n \in \mathbb{N} : \quad n = 2^0 + 2^2 + \sum_{i=3}^{s_n} 2^{n_i} \Big\}.$$

In the study of our problems, the central role is played by their important characteristics $[n]$, $|n|$, $d(n)$ and $V(n)$, which are defined by

$$[n] := \min\big\{ j \in \mathbb{N}, \, n_j \neq 0 \big\}, \quad |n| := \max\big\{ j \in \mathbb{N}, \, n_j \neq 0 \big\},$$
$$d(n) = |n| - [n] \tag{2.1}$$

and

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}| \ \text{ for all } \ n \in \mathbb{N}. \tag{2.2}$$

For any natural number $n \in \mathbb{N}$, there exist the numbers $0 \leq l_1 \leq m_1 \leq l_2 - 2 < l_2 \leq m_2 \leq \cdots \leq l_s - 2 < l_s \leq m_s$ such that $n = \sum\limits_{i=1}^{s} \sum\limits_{k=l_i}^{m_i} 2^k$, where s depends on n.

It is evident that $s \leq V(n) \leq 2s + 1$.

Moreover, for any natural numbers $\{n_{s_j}\}$, $j = 1, 2, \ldots, r$, satisfying $2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} < 2^{s+1}$, $s \in \mathbb{N}$, we define the numbers

$$s_- := \min\big\{[n_{s_j}]\big\}, \quad s_+ := \max\big\{|n_{s_j}|\big\} = s, \quad d_s(n_{s_j}) := s_+ - s_- \,. \tag{2.3}$$

Furthermore, if $2^s \leq n_{s_j} \leq 2^{s+1}$, $j = 1, \ldots, r$, $s \in \mathbb{N}$, which can be written as $n_{s_j} = \sum\limits_{i=1}^{r_{s_j}} \sum\limits_{k=l_i^{s_j}}^{m_i^{s_j}} 2^k$, where $0 \leq l_1^{s_j} \leq m_1^{s_j} \leq l_2^{s_j} - 2 < l_2^{s_j} \leq m_2^{s_j} \leq \cdots \leq l_{r_j}^{s_j} - 2 < l_{r_{s_j}}^{s_j} \leq m_{r_{s_j}}^{s_j}$, we define

$$A_s := \{l_1^s, l_2^s, \ldots, l_{r_s^1}^s\} \cup \{m_1^s, m_2^s, \ldots, m_{r_s^2}^s\} = \{u_1^s, u_2^s, \ldots, u_{r_s^3}^s\}, \tag{2.4}$$

where $u_1^s < u_2^s < \cdots < u_{r_s^3}^s$ and $m_{r_{s_j}}^{s_j} = s \in A_s$ for $j = 1, 2, \ldots, r$.

We also denote the cardinality of the set A_s by $|A_s|$, that is, $\operatorname{card}(A_s) := |A_s|$. By this definition, we can conclude that $|A_s| = r_s^3 \leq r_s^1 + r_s^2$.

It is evident that $\sup\limits_{s\in\mathbb{N}} |A_s| < \infty$ if and only if the sets $\{n_{s_1}, n_{s_2}, \ldots, n_{s_r}\}$ are uniformly finite for all $s \in \mathbb{N}_+$, and each n_{s_j} has bounded variation $V(n_{s_j})$, i.e., $V(n_{s_j}) < c < \infty$ for each $j = 1, 2, \ldots, r$.

3 Walsh system and introduction to Fourier analysis on dyadic group

As we mentioned above, if $n \in \mathbb{N}$, then it can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j \cdot 2^j$, where $n_j \in Z_2$, $j \in \mathbb{N}$, and only a finite number of n_js differs from zero.

Define the k-th Rademacher functions by $r_k(x) := (-1)^{x_k}$, $x \in G$, $k \in \mathbb{N}$.

Using the Rademacher functions, we define the Walsh system $w := (w_n : \ n \in \mathbb{N})$ G as

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_k}, \quad n \in \mathbb{N}.$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [83]).

For any two natural numbers $n, m \in \mathbb{N}$, let

$$n \oplus m := \sum_{k=0}^{\infty} |n_k - m_k| 2^k.$$

Then it is easy to see that $w_n(x) w_m(x) = w_{n \oplus m}(x)$.

Let us prove that the set of characteristic functions $\widehat{G}$ is an orthonormal system.

Proposition 3.1. *Let $n, m \in \mathbb{N}$. Then*

$$\int_G w_n(x) w_m(x)\, d\mu(x) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Proof. Since $w_n(x) w_m(x) = w_{n \oplus m}(x)$, we have only to prove that $\int_G f(x)\, d\mu(x) = 0$ for $f \in \widehat{G} \setminus \{w_0\}$, where $\widehat{G}$ is the set of Walsh functions. For a fixed f, choose $y \in G$ such that $f(y) = -1$. Then it is not difficult to see that

$$\begin{aligned} \int_G f(x)\, d\mu(x) &= \int_G f(x+y)\, d\mu(x+y) \\ &= \int_G f(x+y)\, d\mu(x) = f(y) \int_G f(x)\, d\mu(x) = -\int_G f(x)\, d\mu(x), \end{aligned}$$

from which the proof of the theorem follows. □

For any $f \in L_1(G)$, the numbers

$$\widehat{f}(n) := \int_G f(x) w_n(x)\, d\mu(x)$$

are called the n-th Walsh–Fourier coefficients of f.

The n-th partial sum is denoted by

$$S_n f(x) := \sum_{i=0}^{n-1} \widehat{f}(i) w_i(x).$$

The Dirichlet kernels are defined by

$$D_n(x) := \sum_{i=0}^{n-1} w_i(x).$$

For the one-dimensional case, the Fejér means with respect to the one-dimensional Walsh–Fourier series σ_n are defined by

$$\sigma_n f(x) := \frac{1}{n}\sum_{k=1}^{n} S_k f(x) \ \ (n \in \mathbb{N}_+).$$

Since

$$|\sigma_n f - A| \leq \frac{1}{n}\sum_{k=1}^{n} |S_k f - A|,$$

it is easy to see that if a sequence is convergent, then it is also $(C,1)$-convergent, but the converse is not true in general.

The following equality is true (for details see [26] and [83]):

$$\sigma_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1}(D_k * f)(x) = (f * K_n)(x) = \int_G f(t)K_n(x+t)\,d\mu(t),$$

where

$$K_n(x) := \frac{1}{n}\sum_{k=1}^{n} D_k(x), \ \ n \in \mathbb{N}_+.$$

In the literature K_n is called n-th Fejér kernel.

We also define the following maximal operators:

$$S^* f = \sup_{n\in\mathbb{N}} |S_n f|,$$

$$\widetilde{S}^*_{\#} f = \sup_{n\in\mathbb{N}} |S_{2^n} f|, \tag{3.1}$$

$$\sigma^* f = \sup_{n\in\mathbb{N}} |\sigma_n f|, \tag{3.2}$$

$$\widetilde{\sigma}^*_{\#} f = \sup_{n\in\mathbb{N}} |\sigma_{2^n} f|.$$

4 Background and state of the art about the partial sums and Fejér means with respect to Walsh system in Lebesgue spaces

It is well-known (see, e.g., the book [83]) that if $f \in L_1(G)$ and the Walsh series

$$T(x) = \sum_{j=0}^{\infty} C_j w_j(x)$$

converges to f in L_1-norm, then $C_j = \int_G f w_j\, d\mu =: \widehat{f}(j)$, $j = 0, 1, 2, \dots$, where C_j is called the j-th Walsh–Fourier coefficient and μ is the Haar measure on locally compact Abelian groups G. This means that in this case the approximation series must be a Walsh–Fourier series:

$$f \sim \sum_{j=0}^{\infty} \widehat{f}(j) w_j.$$

According to the Riemann–Lebesgue lemma (see Proposition 6.3) for the Walsh system (for details see, e.g., the book [83]), it is well-known that the Fourier coefficients of integrable functions vanish, that is, $\widehat{f}(k) \to 0$ as $k \to \infty$, for each $f \in L_1$. Moreover, for $f \in L_1(G)$ and $2^N \leq n \leq 2^{N+1}$,

$$|\widehat{f}(n)| \leq c\,\omega_1\Big(\frac{1}{2^N}, f\Big),$$

where $\omega_1(\delta, f)$ is a modulus of continuity in the $L_1(G)$ space.

The classical theory of Hilbert spaces (for details see, e.g., the books [93, 121]) says that if we consider the partial sums

$$S_n f =: \sum_{k=0}^{n-1} \widehat{f}(k) w_k$$

with respect to any orthonormal systems and among them to Walsh system, then $\|S_n f\|_2 \le \|f\|_2$. It follows that for every $f \in L_2$, $\|S_n f - f\|_2 \to 0$ as $n \to \infty$. Moreover, the so-called Bessel's identity (Parseval's identity) asserts that for every $L_2(G)$,

$$\sum_{i=1}^{n} |\widehat{f}(i)|^2 = \|f\|_2^2.$$

It is well-known (see, e.g., [3, 26, 54, 83] and [135]) that the Walsh system does not form bases in the spaces L_1 and L_∞. It follows that the boundedness does not hold for $p = 1$ and $p = \infty$. In these cases, the Dirichlet kernel plays a central role in studying the boundedness of partial sums from the space $L_1(G)$ to the space $L_1(G)$ and from the space $L_\infty(G)$ to the space $L_\infty(G)$.

It is known that for every Walsh system, the Lebesgue constant L_n, defined by $L_n =: \|D_n\|_1$ satisfies the inequality $L_n \le c \log n$.

For every natural number $n = \sum_{j=0}^{\infty} n_j \cdot 2^j$, we have the following two-sided estimates of the Lebesgue constants (see [33, 60, 83]):

$$\frac{1}{8} V(n) \le L_n \le V(n), \tag{4.1}$$

$$\frac{m}{2^m} \sum_{k=1}^{2^m - 1} V(k) \ge \frac{1}{2}, \tag{4.2}$$

where $V(n)$ are defined by (2.2). Hence, for any $f \in L_1(G)$, the subsequence S_{n_k} is bounded from $L_1(G)$ to $L_1(G)$ and from $L_\infty(G)$ to $L_\infty(G)$ if and only if n_k has uniformly bounded variation, which means that

$$\sup_{n \in \mathbb{N}} V(n) < C < \infty.$$

As a corollary of the above-said, the subsequence S_{2^n} of partial sums is bounded from the Lebesgue space $L_1(G)$ to the Lebesgue space $L_1(G)$ and from the Lebesgue space $L_\infty(G)$ to the Lebesgue space $L_\infty(G)$. Moreover, for any $n \in \mathbb{N}$, $1 \le p \le \infty$ and $f \in L_p(G)$, we have $\|S_{2^n} f\|_p \le \|f\|_p$, and for any $1 \le p < \infty$ and $f \in L_p(G)$, $\|S_{2^n} f - f\|_p \to 0$ as $n \to \infty$. It follows that the Walsh polynomials are dense in the Lebesgue spaces $L_p(G)$ for all $1 \le p \le \infty$, which is very important to obtain almost everywhere convergence of sublinear operators of weak-(1,1) type.

Onneweer [70] showed that if the modulus of continuity of $f \in L_1$ satisfies the condition

$$\omega_1\Big(\frac{1}{2^n}, f\Big) = o\Big(\frac{1}{n}\Big) \text{ as } n \to \infty, \tag{4.3}$$

then its Walsh–Fourier series converges in the L_1-norm. He also proved that condition (4.3) cannot be improved.

Localization principle plays a central role in studying pointwise convergence of partial sums with respect to the Walsh system. In particular, this result says that the convergence of partial sums of the Fourier series of a function f at the point x_0 depends only on the neighborhood of x_0, since it is proved that for $f \in L_1(G)$, any $x_0 \in G$ and for some $N \in \mathbb{N}$,

$$\lim_{n \to \infty} \int\limits_{G \setminus I_N(x_0)} f(t) D_n(x_0 + t)\, dt = 0.$$

This implies that if $f, g \in L_1$ and $f = g$ for some $I_N(x)$, then $S_n f(x)$ and $S_n g(x)$ converge or diverge simultaneously for every $x \in I_N(x)$.

Moreover, Dini's test holds for any $x\in G$ and $f\in L_1(G)$. In particular, if we define the function g by

$$g(t) =: \frac{f(t)-f(x)}{|x+t|}, \quad x\in G, \quad t\in G\setminus\{x\}, \quad f\in L_1(G),$$

and have $g\in L_1(G)$, then $S_nf(x)\to f(x)$, $n\to\infty$.

It follows that if $f\in L_1(G)$ and

$$|f(x+t)-f(x)| = O\Big(\log\Big(\frac{1}{|t|}\Big)^{-1-\varepsilon}\Big), \quad \varepsilon>0, \quad t\to 0,$$

then $S_nf(x)\to f(x)$, $n\to\infty$.

In [83], it is proved that there exists an absolute constant C_p, depending only on p such that

$$\|S_nf\|_p \le C_p\|f\|_p \ \text{ for } \ 1<p<\infty.$$

The boundedness does not hold for $p=1$ or $p=\infty$, but Watari [126] (see also Gosselin [55] and Young [134]) proved that there exists an absolute constant C such that, for $n=1,2,\dots$, the weak type estimate $y\mu\{|S_nf(x)|>y\}\le C\,\|f\|_1$, $f\in L_1(G)$, $y>0$, holds.

For an integrable function $f\in L_1(G)$ and for any $x\in G$, we define the maximal function $M(f)$ by

$$M(f)(x) =: \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))}\Big|\int\limits_{I_n(x)} f(t)\,d\mu(t)\Big|. \tag{4.4}$$

Then it can be proved that this function is bounded from $L_1(G)$ to the *weak*-$L_1(G)$ space, that is, $y\mu\{|M(f)(x)|>y\}\le c\,\|f\|_1$, $f\in L_1(G)$, $y>0$. Moreover, if we consider a restricted maximal operator $\widetilde{S}^*_{\#}$ defined by (3.1), according to the fact that $\sup_{n\in\mathbb{N}}|S_{2^n}f(x)|=M(f)(x)$, we also get $y\mu\{\widetilde{S}^*_{\#}f>y\}\le c\,\|f\|_1$, $f\in L_1(G)$, $y>0$. Hence, if $f\in L_1(G)$, then $S_{2^n}f\to f$ a.e. on G. The almost everywhere convergence of subsequences of the Walsh–Fourier series was considered in [20], where some methods of the martingale theory were used.

The uniform and pointwise convergence and some approximation properties of the partial sums in $L_1(G)$ norms were investigated by Avdispahić and Memić [11], Gát, Goginava and Tkebuchava [43,50], Nagy [65,69], Onneweer [70] and Persson, Schipp, Tephnadze and Weisz [73,74]. Fine [38] obtained the sufficient conditions for the uniform convergence that are completely analogous to the Dini–Lipschits condition. Gulićev [56] estimated the rate of uniform convergence of a Walsh–Fourier series by using the Lebesgue constants and modulus of continuity.

Fejér's theorem shows (see, e.g., [3] and [83]) that if one replaces an ordinary summation by Fejér means, then σ_nf of any integrable function f converges almost everywhere on G to the function. Moreover, for the maximal operator of Fejér means σ^* defined by (3.2) we have $y\mu(\sigma^*f>y)\le c\,\|f\|_1$, $f\in L_1(G)$, $y>0$. This result can be found in Zygmund [135] for trigonometric series, in Schipp [82] for Walsh series and in Pál, Simon [71] for bounded Vilenkin series. The boundedness does not hold from the Lebesgue space $L_1(G)$ to the space $L_1(G)$.

It follows from the weak-type inequality that if $f\in L_1(G)$, then $\lim_{n\to\infty}\sigma_nf(x)=f(x)$ a.e. on G.

Weisz [127] (see also [63]) defined the Walsh–Lebesgue points and proved that a.e. points of $f\in L_1(G)$ are the Walsh–Lebesgue points. Moreover, he proved the convergence of Walsh–Fejér means of integrable functions at these points.

In the one-dimensional case, Yano [133] proved that $\|\sigma_nf-f\|_p\to 0$ as $n\to\infty$, $f\in L_p(G)$, $1\le p\le\infty$. However (see [57,83]), the rate of convergence cannot be better than $O(n^{-1})$, $n\to\infty$ for non-constant functions, i.e., if $f\in L_p$, $1\le p\le\infty$ and

$$\|\sigma_{2^n}f-f\|_p = o\Big(\frac{1}{2^n}\Big) \ \text{ as } \ n\to\infty,$$

then f is a constant function.

It is also known that (see, e.g., the books [3,83] and the paper [4]) for any $1 \leq p \leq \infty$ and $n \in \mathbb{N}$, we have the following estimate:

$$\|\sigma_n f - f\|_p \leq C_p \omega_p\Big(\frac{1}{2^N}, f\Big) + C_p \sum_{s=0}^{N-1} \frac{2^s}{2^N} \omega_p\Big(\frac{1}{2^s}, f\Big),$$

where $\omega_p(\frac{1}{2^n}, f)$ is the modulus of continuity of the function $f \in L_p$. Applying this estimate, we immediately find that if $f \in \operatorname{lip}(\alpha, p)$, i.e.,

$$\omega_p\Big(\frac{1}{2^n}, f\Big) = O\Big(\frac{1}{2^{n\alpha}}\Big), \quad n \to \infty,$$

then

$$\|\sigma_n f - f\|_p = \begin{cases} O\Big(\dfrac{1}{2^N}\Big) & \text{if } \alpha > 1, \\ O\Big(\dfrac{N}{2^N}\Big) & \text{if } \alpha = 1, \\ O\Big(\dfrac{1}{2^{\alpha N}}\Big) & \text{if } \alpha < 1. \end{cases}$$

Convergence and summability for some general summability methods in the Lebesgue spaces were investigated by Abesadze, Anakidze, Nasibov and Tephnadze [1], Anakidze, Areshidze, Persson and Tephnadze [5,6], Anakidze, Persson, Tephnadze and Yimer [7], Kopaliani, Tephnadze and Weisz [58]. The best constants in some classical inequalities and the summability methods were investigated in [72,81,120].

The convergence and summability of partial sums and their subsequences in variable Lebesgue spaces follow from results obtained by Adamadze and Kopaliani [2] and by Shalukhina [84,85].

5 Important equalities and estimates for partial sums with respect to Walsh–Fourier series

First we prove the so-called Paley's lemma.

Lemma 5.1. *Let $n \in \mathbb{N}$. Then*

$$D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n(0), \\ 0, & x \in G \setminus I_n(0). \end{cases}$$

Proof. It is easy to see that $w_k(x) = 1$ when $x \in I_n(0)$ and $k < 2^n$. It follows that

$$D_{2^n}(x) = \sum_{k=0}^{2^n-1} w_k(x) = \sum_{k=0}^{2^n-1} 1 = 2^n, \quad x \in I_n(0)$$

and we get the first equality.

We also have

$$\int_{I_n(0)} |D_{2^n}(x)|^2 \, d\mu(x) = 2^{2n} \mu(I_n(0)) = 2^n$$

On the other hand, if we take into account the fact that $\{w_k\}$ is orthonormal system, we find that

$$\int_G |D_{2^n}(x)|^2 \, d\mu(x) = \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^n-1} \int_G w_i(x) w_j(x) \, d\mu(x) = 2^n.$$

Hence,

$$2^n = \int\limits_G |D_{2^n}(x)|^2 \, d\mu(x)$$
$$= \int\limits_{I_n(0)} |D_{2^n}(x)|^2 \, d\mu(x) + \int\limits_{G\setminus I_n(0)} |D_{2^n}(x)|^2 \, d\mu(x) = 2^n + \int\limits_{G\setminus I_n(0)} |D_{2^n}(x)|^2 \, d\mu(x)$$

and

$$\int\limits_{G\setminus I_n(0)} |D_{2^n}(x)|^2 \, d\mu(x) = 0.$$

It follow that $D_{2^n}(x) = 0$ as $x \in G \setminus I_n(0)$. $\square$

Lemma 5.2. *Let $n \in \mathbb{N}$. Then*

$$\sup_{n\in\mathbb{N}} \int\limits_G D_n(x) \, d\mu(x) = 1 \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \int\limits_G |D_{2^n}(x)| \, d\mu(x) = 1.$$

Proof. Since D_{2^n} is non-negative functions, the proof of both equalities follows from the orthonormality. $\square$

Lemma 5.3. *Let $0 \le m \le 2^n$. Then $D_{2^n+m}(x) = D_{2^n}(x) + w_{2^n}(x)D_m(x)$.*

Proof. Indeed,

$$D_{2^n+m}(x) := \sum_{k=0}^{2^n+m-1} w_k(x) = \sum_{k=0}^{2^n-1} w_k(x) + \sum_{k=2^n}^{2^n+m-1} w_k(x) = D_{2^n}(x) + \sum_{k=0}^{m-1} w_{k+2^n}(x).$$

Since $k \le m-1 \le 2^n - 1$, we can conclude that $2^n + m = 2^n \oplus m$ and $w_{2^n+m}(x) = w_{2^n\oplus m}(x) = w_{2^n}(x)w_m(x)$. Hence,

$$D_{2^n+m}(x) = D_{2^n}(x) + w_{2^n}(x) \sum_{k=0}^{m-1} w_k(x) = D_{2^n}(x) + w_{2^n}(x)D_m(x). \qquad \square$$

The following theorem is true.

Theorem 5.1. *Let $n := 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, $n_1 > n_2 > \cdots > n_r \ge 0$. Then*

$$D_n(x) = w_n(x)(w_{2^{n_1}}(x)D_{2^{n_1}}(x) + w_{2^{n_2}}(x)D_{2^{n_2}}(x) + w_{2^{n_3}}(x)D_{2^{n_3}}(x) + \cdots + w_{2^{n_r}}(x)D_{2^{n_r}}(x))$$
$$= w_n(x) \sum_{k=1}^{r} w_{2^{n_k}}(x)D_{2^{n_k}}(x).$$

Proof. Using Lemma 5.3, we obtain

$$\begin{aligned} D_n(x) &= D_{2^{n_1}+n^{(1)}}(x) = D_{2^{n_1}}(x) + w_{2^{n_1}}(x)D_{n^{(1)}}(x) \\ &= D_{2^{n_1}}(x) + w_{2^{n_1}}(x)D_{2^{n_2}+n^{(2)}}(x) = D_{2^{n_1}}(x) + w_{2^{n_1}}(x)\big(D_{2^{n_2}}(x) + w_{2^{n_2}}(x)D_{n^{(2)}}(x)\big) \\ &= D_{2^{n_1}}(x) + w_{2^{n_1}}(x)D_{2^{n_2}}(x) + w_{2^{n_2}}(x)w_{2^{n_1}}(x)D_{n^{(2)}}(x) = \cdots \\ &= D_{2^{n_1}}(x)+w_{2^{n_1}}(x)D_{2^{n_2}}(x)+w_{2^{n_1}}(x)w_{2^{n_2}}(x)D_{2^{n_3}}(x)+\cdots+w_{2^{n_1}}(x)w_{2^{n_2}}(x)\cdots w_{2^{n_{r-1}}}(x)D_{2^{n_r}}(x), \end{aligned}$$

and using $w_n = \prod_{k=1}^{r} w_{2^{n_k}}$, we get

$$\begin{aligned} w_{2^{n_1}}(x) &= w_n(x)w_{2^{n_2}}(x)w_{2^{n_3}}(x)\cdots w_{2^{n_r}}(x) \\ w_{2^{n_1}}(x)w_{2^{n_2}}(x) &= w_n(x)w_{2^{n_3}}(x)\cdots w_{2^{n_r}}(x) \\ &= \cdots \\ \prod_{k=1}^{s} w_{2^{n_k}} &= w_n(x) \prod_{k=s+1}^{r} w_{2^{n_k}}, \quad s = 1, 2, \dots, r-1 \end{aligned}$$

This implies that

$$\begin{aligned}D_n(x) = w_n(x)\big(&w_{2^{n_1}}(x)\cdots w_{2^{n_r}}(x)D_{2^{n_1}}(x) + w_{2^{n_2}}(x)\cdots w_{2^{n_r}}(x)D_{2^{n_2}}(x)\\ &+ w_{2^{n_3}}(x)\cdots w_{2^{n_r}}(x)D_{2^{n_3}}(x) + \cdots + w_{2^{n_r}}(x)D_{2^{n_r}}(x)\big).\end{aligned}$$

It is easy to see that if $n_k < n_s$ then $w_{2^{n_k}} = 1$ as $x \in I_{n_s}(0)$. According to $D_{2^{n_s}} = 0$ as $x \notin I_{n_s}(0)$, we find that $w_{2^{n_s}}(x)D_{2^{n_s}}(x) = D_{2^{n_s}}(x)$ as $x \in I_{n_s}(0)$ and

$$\begin{aligned}w_n(x)D_{2^{n_1}}(x) &= w_{2^{n_1}}(x)D_{2^{n_1}}(x)\\ w_{2^{n_2}}(x)\cdots w_{2^{n_r}}(x)D_{2^{n_2}}(x) &= w_{2^{n_2}}(x)D_{2^{n_2}}(x)\\ &\cdots\\ w_{2^{n_k}}(x)\cdots w_{2^{n_r}}(x)D_{2^{n_k}}(x) &= w_{2^{n_k}}(x)D_{2^{n_k}}(x).\end{aligned}$$

Therefore

$$\begin{aligned}D_n(x) &= w_n(x)\big(w_{2^{n_1}}(x)D_{2^{n_1}}(x) + w_{2^{n_2}}(x)D_{2^{n_2}}(x) + w_{2^{n_3}}(x)D_{2^{n_3}}(x) + \cdots + w_{2^{n_r}}(x)D_{2^{n_r}}(x)\big)\\ &= w_n\sum_{k=1}^{r} w_{2^{n_k}}(x)D_{2^{n_k}}(x).\end{aligned}$$

The proof is complete. □

From this theorem also follows

Lemma 5.4. *Let $x \in G$ and $n := \sum_{j=0}^{|n|} n_j 2^j$, $n_j = 0, 1$. Then*

$$D_n(x) = w_n(x)\sum_{j=0}^{|n|} n_j w_{2^j}(x)D_{2^j}(x) \tag{5.1}$$

and

$$D_n(x) = w_n(x)\sum_{j=0}^{|n|} n_j(D_{2^{j+1}}(x) - D_{2^j}(x)). \tag{5.2}$$

Proof. The proof of the first equality is obvious. To prove the second equality, it suffices to use Lemma 5.3 when $n' = 2^N$. Then, by moving the first sum to the left-hand side, using this formula, we get

$$D_n(x) = w_n(x)\sum_{j=0}^{|n|} n_j w_{2^j}(x)D_{2^j}(x) = w_n(x)\sum_{j=0}^{|n|} n_j(D_{2^{j+1}}(x) - D_{2^j}(x)).$$ □

Lemma 5.5. *Let $n \in \mathbb{N}$ and $x \in G$. Then*

$$|D_n(x)| \le \sum_{j=0}^{|n|} n_j D_{2^j}(x) \le \sum_{j=0}^{|n|} D_{2^j}(x) \tag{5.3}$$

and

$$|D_n(x)| < \frac{C}{|x|}, \quad x \neq 0,$$

where C is an absolute constant.

Proof. The first inequality is a direct consequence of (5.1) in Lemma 5.4.

Let $x \in I_s \setminus I_{s+1}$ for some $s = 0, \ldots, N-1$. Then $x = (0, \ldots, 0, x_s \neq 0, x_{s+1}, x_{s+2}, \ldots)$ and

$$\frac{1}{2^s} \le |x| \le \sum_{i=s}^{\infty}\frac{1}{2^i} = \frac{1}{2^{s-1}} \le \frac{2}{2^s}.$$

Combining Lemma 5.1 and (5.2) from Lemma 5.4, we obtain that

$$|D_n(x)| \leq \sum_{j=0}^{s} n_j D_{M_j}(x) = \sum_{j=0}^{s} n_j M_j \leq \sum_{j=0}^{s} 2^j = 2^{s+1} - 1 \leq 2^{s+1} \leq c \cdot 2^s \leq \frac{C}{|x|}. \qquad \square$$

Lemma 5.6. *Let $x \in G$ and $n \in \mathbb{N}$. Then*

$$\|D_{2^n}\|_1 = 1 \tag{5.4}$$

and

$$\|D_n\|_1 \leq c \log(n). \tag{5.5}$$

Proof. The first equality follows from Lemma 5.1. The second inequality follows by (5.3) from Lemma 5.5. □

6 Fourier coefficients and partial sums with respect to Walsh–Fourier series

By a Walsh polynomial we mean a finite linear combination of Walsh functions. We denote the collection of Walsh polynomials by $\mathcal{P}$. It is easy to prove that $L_0(G)$ represents the collection of functions which are almost everywhere limits with respect to a measure μ of sequences in $\mathcal{P}$.

Remark 6.1. Since the Walsh function w_m is constant on $I_n(x)$ for every $x \in G$ and $0 \leq m < 2^n$, it is clear that each Walsh function is a real-valued step function, that is, it is a finite linear combination of characteristic functions. By Paley's Lemma, the equality

$$\chi_{I_n(t)}(x) = \frac{1}{2^n} \sum_{j=0}^{2^n-1} w_j(x+t), \quad x \in I_n(t),$$

holds for each $x, t \in G$ and $n \in \mathbb{N}$. Thus each step function is a Walsh polynomial. Consequently, we find that the collection of step functions coincides with the collection of Walsh polynomials $\mathcal{P}$.

Since the Lebesgue measure is regular, it follows that given $f \in L_1$, there exist the Walsh polynomials $P_1, P_2, \ldots$ such that $P_n \to f$ a.e., as $n \to \infty$. Moreover, any $f \in L_p(G)$ can be written in the form $f = g - h$, where the functions g and h are almost everywhere limits of increasing sequences of the non-negative Walsh polynomials. In particular, $\mathcal{P}$ is dense in the space L_p for all $p \geq 1$. That is, for any real numbers $1 \leq p \leq \infty$, $\varepsilon > 0$ and for the function $f \in L_p(G)$ there exists a Walsh polynomial $W_n(x) := \sum_{i=0}^{n-1} \alpha_i w_i(x)$ such that $\|f - W_n\|_p < \varepsilon$.

Using the density of Walsh polynomials in $L^p(G_m)$, we can derive the following information for the partial sums S_N.

Lemma 6.1. *Let $f \in L_p(G)$ or $f \in C(G)$ if $p = \infty$. Then the following statements are equivalent for any $1 \leq p \leq \infty$:*

(i) $\|S_N f - f\|_p \to 0$ *as* $N \to \infty$*;*

(ii) $\sup_N \|S_N f\|_p < C_p \|f\|_p$.

Proof. If (i) holds, then $\|S_N f - f\|_p < C_p$ and

$$\|S_N f\|_p = \|S_N f - f + f\|_p \leq \|S_N f - f\|_p + \|f\|_p < C_p + \|f\|_p,$$

so (ii) holds.

On the other hand, assume that (ii) holds. Then, by using the density of the Walsh polynomials (see Remark 6.1) we get that, for any $\varepsilon > 0$,

$$\|f - T\|_p < \frac{\varepsilon}{2} C_p.$$

Then, for every sufficiently large $N > \deg T$, we get

$$\begin{aligned}\|S_N f - f\|_p = \|S_N f - S_N T + S_N T - f\|_p &\le \|S_N(f-T)\|_p + \|S_N T - f\|_p \\ &= \|S_N(f-T)\|_p + \|T - f\|_p < C_p\|T-f\|_p + \|T-f\|_p < \varepsilon,\end{aligned}$$

so (i) holds and the proof is complete. □

Proposition 6.1. *Let $n \in \mathbb{N}$, $1 \le p < \infty$ and $f \in L_p(G)$. Then $\|S_{2^n} f\|_p \le \|f\|_p$ and $\|S_{2^n} f - f\|_p \to 0$ as $n \to \infty$.*

Proof. Using (5.4) in Lemma 5.6 and Proposition 1.2, we immediately get $\|S_{2^n} f\|_p \le \|f\|_p \|D_{2^n}\|_1 = \|f\|_p$. The convergence follows just by using Lemma 6.1. □

Proposition 6.2. *Let $N \in \mathbb{N}$ and $f \in L_p(G)$, where $1 \le p < \infty$. Then*

$$\lim_{N\to\infty} \sup_{t\in I_N} \|f(x+t) - f(x)\|_p = 0.$$

Proof. Using Corollary 6.1, for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $S_{2^k} \in \mathcal{P}$ and $\|f - S_{2^k} f\|_p < \varepsilon/2$.

Let $N > k$. To prove the first inequality, we observe that if $t \in I_N(0)$, then

$$S_{2^k} f(x+t) = \sum_{k=0}^{2^n-1} w_k(x+t) = \sum_{k=0}^{2^n-1} w_k(x) w_k(t) = \sum_{k=0}^{2^n-1} w_k(x) = S_{2^k} f(x).$$

Hence,

$$\begin{aligned}\sup_{t\in I_N} \|f(x+t) - f(x)\|_p &= \sup_{t\in I_N} \left\|f(x+t) - S_{2^k} f(x+t) + S_{2^k} f(x+t) - f(x)\right\|_p \\ &= \sup_{t\in I_N} \left\|f(x+t) - S_{2^k} f(x+t) + S_{2^k} f(x) - f(x)\right\|_p \\ &\le \sup_{t\in I_N} \|f(x+t) - S_{2^k} f(x+t)\|_p + \sup_{t\in I_N} \|S_{2^k} f(x) - f(x)\|_p = 2 \sup_{t\in I_N} \|S_{2^k} f - f\|_p < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

The Proposition 6.2 does not hold when $p = \infty$.

Corollary 6.1. *Let $N \in \mathbb{N}$ and $f \in L_\infty(G)$. Then*

$$\lim_{N\to\infty} \sup_{t\in I_N} \|f(x+t) - f(x)\|_\infty > c > 0.$$

Proof. Let $f(x) = \begin{cases} 1, & x \in I_1(0), \\ 0, & x \in G \setminus I_1(0). \end{cases}$ Then $\|f\|_\infty = 1$. On the other hand, $f(e_0 + t) - f(e_0) = f(t)$ for any $t \in I_N$, $N > 2$ and $\|f(e_0+t) - f(e_0)\|_\infty = 1$, which completes the proof. □

Proposition 6.3 (Riemann–Lebesgue lemma)**.** *Let $f \in L_1(G)$. Then $\widehat{f}(n) \to 0$ as $n \to \infty$.*

Proof. Since the set of Walsh polynomials is dense in $L_1(G)$, for any $\varepsilon > 0$ there exists a Walsh polynomial $p(x) := \sum_{k=0}^{2^N-1} \alpha_k w_k(x)$ such that $\|f-p\|_1 < \varepsilon$. On the other hand, using the orthonormality of the Walsh system, we get $\widehat{p}(n) = 0$, when $n \ge 2^N$. Therefore, when $n \ge 2^N$, we have

$$\widehat{f}(n) = \int_G f w_n \, d\mu = \int_G (f-p) w_n \, d\mu.$$

Hence, $|\widehat{f}(n)| \le \|f-p\|_1 < \varepsilon$. □

Lemma 6.2. *Let $f \in L_1$, $y \in G$ and $N \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \int\limits_{G\setminus I_N(y)} f(t)D_n(x+t)\,dt = 0$$

for all $x \in I_N(y)$.

Proof. Since $x+t \notin I_N(0)$, using (5.1) in Lemma 5.4, for $n = \sum_{i=0}^{\infty} n_i \cdot 2^i$, we get

$$\int\limits_{G\setminus I_N(y)} f(t)D_n(x+t)\,dt = \sum_{j=0}^{N-1} \int\limits_{G\setminus I_N(y)} f(t)w_n(x+t)D_{2^j}(x+t)w_{2^j}(x+t)\,dt = w_n(x)\sum_{j=0}^{N-1} \widehat{g}_j(n),$$

where $g_j(t) =: f(t)(D_{2^j}(x+t)w_{2^j}(x+t))$. According to the fact that $g_j \in L_1$, applying the Riemann–Lebesgue Lemma (see Proposition 6.3), we can conclude that $\widehat{g}_j(n) \to 0$, $n \to \infty$. Hence,

$$\lim_{n\to\infty} \int\limits_{G\setminus I_N(y)} f(t)D_n(x+t)\,dt = 0. \qquad \square$$

Theorem 6.1 (Localization principle)**.** *Let $y \in G$, $N \in \mathbb{N}$, $f, g \in L_1(G)$ and $f = g$ on $I_N(y)$. Then $S_n f(x)$ and $S_n g(x)$ simultaneously converge or diverge for every $x \in I_N(y)$.*

Proof. By Lemma 6.2, we obtain

$$S_n f(x) - S_n g(x) = S_n(f-g) = \int\limits_{G\setminus I_N(y)} (f(t)-g(t))D_n(x-t)\,dt \longrightarrow 0. \qquad \square$$

Theorem 6.2 (Dini's test)**.** *Let $x \in G$, $f \in L_1(G)$ and define the function g by*

$$g(t) =: \frac{f(t)-f(x)}{|x-t|}, \quad t \in G \setminus \{x\}.$$

If $g \in L_1(G)$, then $S_n f(x) \to f(x)$, $n \to \infty$.

Proof. Using Lemma 5.5 we find that

$$|S_n f(x) - f(x)| \le \left| \int\limits_{G\setminus I_N(x)} (f(t)-f(x))D_n(x+t)\,dt \right| + C \int\limits_{I_N(x)} |g(t)|\,dt.$$

Moreover, by Theorem 6.1,

$$\int\limits_{G\setminus I_N(x)} (f(t)-f(x))D_n(x+t)\,dt \longrightarrow 0, \quad n \to \infty.$$

Since $g \in L_1(G)$, by using the absolute continuity of Lebesgue integral, we find that, for every $\varepsilon > 0$, $\int_{I_N(x)} |g(t)|\,dt \le \varepsilon$ when N is large enough. $\square$

Dini's test immediately implies the following

Corollary 6.2. *Let $f \in L_1(G)$ and*

$$|f(x+t) - f(x)| = O\Big(\Big(\log\frac{1}{|t|}\Big)^{-1-\varepsilon}\Big), \quad \varepsilon > 0, \quad t \to 0.$$

Then $S_n f(x) \to f(x)$, $n \to \infty$.

7 The modulus of continuity of functions in the Lebesgue spaces

As mentioned above, the modulus of continuity of functions in the Lebesgue spaces is defined by

$$\omega_p\Big(\frac{1}{2^n}, f\Big) =: \sup_{h\in I_n} \big\|f(\,\cdot\,+h)-f(\,\cdot\,)\big\|_p.$$

Using an absolute continuity of Lebesgue integral, we obtain

$$\omega_p\Big(f, \frac{1}{2^n}\Big) \longrightarrow 0 \ \text{ as } \ n\to\infty.$$

We say that $f\in \operatorname{lip}\alpha$, where $\alpha>0$, if

$$\omega_p\Big(f, \frac{1}{2^n}\Big) := O\Big(\frac{1}{2^{n\alpha}}\Big) \ \text{ as } \ n\to\infty.$$

The best approximation $\mathcal{E}_n(f,L_p)$ of the function $f\in L_p(G)$, $1\le p\le\infty$ is defined by

$$\inf_{w\in\mathcal{P}_n}\|f-w\|_p = \|f-\mathcal{E}_n(f,L_p)\|_p$$

if it exists, where $\mathcal{P}_n$ is the set of all Walsh polynomials of order less than $n\in\mathbb{N}$. Set

$$E_n(f,L_p) =: \inf_{w\in\mathcal{P}_n}\|f-w\|_p.$$

Lemma 7.1. *For any function $f\in L_2(G)$, we have $\min\limits_{P\in\mathcal{P}_n}\|f-P\|_2 = \|f-S_nf\|_2$, i.e., $\mathcal{E}_n(f,L_p)=S_nf$ in the space $L_2(G)$.*

Proof. It is evident that

$$\Big\|f-\sum_{i=1}^n \alpha_i w_i\Big\|_2^2 = \|f\|_2^2 - 2\sum_{i=1}^n \widehat{f}(i)\alpha_i + \sum_{i=1}^n|\alpha_i|^2 = \|f\|_2^2 - \sum_{i=1}^n|\widehat{f}(i)|^2 + \sum_{i=1}^n|\widehat{f}(i)-\alpha_i|^2.$$

This identity implies the proof. □

Proposition 7.1. *Let $n\in\mathbb{N}$. Then*

$$\frac12\,\omega_p\Big(f,\frac{1}{2^n}\Big) \le E_{2^n}(f,L_p) \le \|f-S_{2^n}f\|_p \le \omega_p\Big(f,\frac{1}{2^n}\Big).$$

Proof. First, we note that

$$\begin{aligned}
E_{2^n}(f,L_p) &= \inf_{P\in\mathcal{P}_{2^n}}\|f-P\|_p \le \|f-S_{2^n}f\|_p = \bigg\|\int_G (f(x+t)-f(x))D_{2^n}(t)\,d\mu(t)\bigg\|_p \\
&\le \int_G \|f(x+t)-f(x)\|_p D_{2^n}(t)\,d\mu(t) = 2^n\int_{I_n(0)}\|f(x+t)-f(x)\|_p\,d\mu(t) \le \omega_p\Big(f,\frac{1}{2^n}\Big).
\end{aligned}$$

To prove the first inequality, we observe that if $P\in\mathcal{P}_{M_n}$ and $y\in I_n(0)$, then

$$P(x+y) = \sum_{k=0}^{2^n-1}\alpha_k w_k(x+y) = \sum_{k=0}^{2^n-1}\alpha_k w_k(x)w_k(y) = \sum_{k=0}^{2^n-1}\alpha_k w_k(x) = P(x). \tag{7.1}$$

Hence, $f(x+y)-f(x) = f(x+y)-P(x+y)+P(x)-f(x)$ and

$$\omega_p\Big(f,\frac{1}{2^n}\Big) = \sup_{y\in I_n(0)}\|f(x+y)-f(x)\|_p \le 2\|f-P\|_p$$

for every $P\in\mathcal{P}_{2^n}$. It follows that

$$\frac12\,\omega_p\Big(f,\frac{1}{2^n}\Big) \le E_{2^n}(f,L_p),$$

thus the first inequality is proved and so, the proof is complete. □

8 Important equalities and estimates for Fejér kernels with respect to Walsh–Fourier series

For $n>t$ and $t,n\in\mathbb{N}$, we have the following expression for the 2^n-th Fejér kernels with respect to the one-dimensional Walsh–Fourier series (see also [83] and [40, 41]).

Lemma 8.1. *Let $n\in\mathbb{N}$. Then*

$$\begin{aligned}2^nK_{2^n}(t)&=\sum_{j=0}^{n-2}2^jD_{2^j}(t)\Big(\prod_{k=j+1}^{n-1}(1+w_{2^k}(t))\Big)+2^{n-1}D_{2^{n-1}}(t)\\&=\sum_{j=0}^{n-1}2^jD_{2^j}(t)\Big(\prod_{k=j+1}^{n-1}(1+w_{2^k}(t))\Big).\end{aligned}$$

In the last equality, we mean that $\prod\limits_{k=n}^{n-1}(1+w_{2^k}(t))=1$. *Moreover, the following equalities hold true:*

$$K_{2^n}(t)=\begin{cases}2^{n-1}+\dfrac{1}{2}\,, & t\in I_n(0),\\ 2^{s-1}, & t\in I_n(e_s),\quad s=0,1,\dots,n-1,\\ 0, & t\in I_n(e_s+e_l),\quad s\neq l,\quad s,l=0,1,\dots,n-1.\end{cases}\tag{8.1}$$

Proof. By a simple calculation, we get

$$\begin{aligned}2^nK_{2^n}(t)&=\sum_{j=1}^{2^n}D_j(t)=\sum_{j=1}^{2^{n-1}}D_j(t)+\sum_{j=2^{n-1}+1}^{2^n}D_j(t)\\&=\sum_{j=1}^{2^{n-1}}D_j(t)+\sum_{j=1}^{2^{n-1}}D_{j+2^{n-1}}(t)=\sum_{j=1}^{2^{n-1}}D_j(t)+\sum_{j=1}^{2^{n-1}}\big(D_{2^{n-1}}(t)+w_{2^{n-1}}(t)D_j(t)\big)\\&=\sum_{j=1}^{2^{n-1}}D_j(t)+2^{n-1}D_{2^{n-1}}(t)+w_{2^{n-1}}(t)\sum_{j=1}^{2^{n-1}}D_j(t)=2^{n-1}D_{2^{n-1}}(t)+(1+w_{2^{n-1}}(t))\sum_{j=1}^{2^{n-1}}D_j(t)\\&=2^{n-1}D_{2^{n-1}}(t)+(1+w_{2^{n-1}}(t))\cdot2^{n-2}D_{2^{n-2}}(t)+(1+w_{2^{n-2}}(t))(1+w_{2^{n-1}}(t))\sum_{j=1}^{2^{n-2}}D_j(t)=\cdots\\&=2^{n-1}D_{2^{n-1}}(t)+(1+w_{2^{n-1}}(t))\cdot2^{n-2}D_{2^{n-2}}(t)+\cdots+(1+w_{2^{n-1}}(t))(1+w_{2^{n-2}}(t))\cdots(1+w_{2^1}(t))\\&=\sum_{j=0}^{n-1}\prod_{k=j+1}^{n-1}(1+w_{2^k}(t))\cdot2^jD_{2^j}(t),\end{aligned}$$

which proves the first equality.

Let $t\in I_n(0)$. Then $D_j=j$, when $j\le 2^n$, and

$$K_{2^n}(t)=\frac{1}{2^n}\sum_{j=1}^{2^n}j=\frac{2^n(2^n+1)}{2^{n+1}}=2^{n-1}+\frac{1}{2}\,.$$

Let $t_s=1$, $s=0,\dots,n-1$. Then, using Lemma 5.1, we get $D_{2^j}(t)=0$, when $j>s$, and

$$\sum_{j=s+1}^{n-1}2^jD_{2^j}(t)\Big(\prod_{k=j+1}^{n-1}(1+w_{2^k}(t))\Big)=0,\quad t_s=1.\tag{8.2}$$

On the other hand, the product $\prod_{k=j+1}^{n-1}(1+w_{2^k}(t))$, $j=0,\dots,s-1$, contains $1+w_{2^s}(t)$ s-th multiplier and $1+w_{2^s}(t)=1+(-1)^{t_s}=1-1=0$, $t_s=1$, and

$$\sum_{j=0}^{s-1}2^jD_{2^j}(t)\Big(\prod_{k=j+1}^{n-1}(1+w_{2^k}(t))\Big)=0,\quad t_s=1. \tag{8.3}$$

It follows that if $t\in I_n(e_s)$, $s=0,\dots,n-1$, then

$$\sum_{j=0}^{n-1}2^jD_{2^j}(t)\prod_{k=j+1}^{n-1}(1+w_{2^k}(t))\neq 0$$

if and only if $j=s$ and

$$2^nK_{2^n}(t)=2^sD_{2^s}(t)\prod_{k=s+1}^{n-1}(1+w_{2^k}(t)).$$

Since $t\in I_n(e_s)$, we get $t_{s+1}=t_{s+2}=\cdots=t_{n-1}=0$, which implies that $1+w_{2^k}(t)=1+(-1)^{t_k}=1+1=2$, $k=s+1,\dots,n-1$. On the other hand, for $t\in I_n(e_s)$, where $t_0=t_1=\cdots=t_{s-1}=0$, we have $D_{2^s}(t)=2^s$. Therefore,

$$2^nK_{2^n}(t)=2^sD_{2^s}(t)\prod_{k=s+1}^{n-1}(1+w_{2^k}(t))=2^{2s}\cdot 2^{n-s-1}=2^{n+s-1}.$$

Let $t\in I_n(t_0,\dots,t_{n-1})$ and $\sum_{j=0}^{n-1}t_j\geq 2$. Without loss of generality, we can assume that $t_s=t_l=1$, where $s<l\leq n-1$.

According to $t_s=1$, applying (8.2) and (8.3), we obtain

$$2^nK_{2^n}(t)=2^sD_{2^s}(t)\prod_{k=s+1}^{n-1}(1+w_{2^k}(t)).$$

The difference in this case is that $t_l=1$, where $n-1\geq l\geq s+1$, and this term contains an l-th multiplier, which equals to 0. Indeed, $1+w_{2^l}(t)=1+(-1)^{t_l}=1-1=0$, whence, finally, we find that if $t\in I_n(e_s+e_l)$, $s\neq l$, $s,l=0,1,\dots,n-1$, then $K_{2^n}(t)=0$. □

Corollary 8.1. *Let $t,n\in\mathbb{N}$. Then*

$$K_{2^n}(x)=\frac{1}{2}(2^{-n}+1)D_{2^n}(x)+\sum_{j=0}^{n-1}2^{j-n-1}D_{2^n}(x+e_j).$$

Proof. Using Lemma 5.1 and Theorem 8.1. we immediately get the proof (we leave out the details). □

The proof of the following lemma follows directly from Theorem 8.1.

Lemma 8.2. *Let $n\in\mathbb{N}$ and $x\in I_N^{k,l}$, where $k<l$. Then*

$$K_{2^n}(x)=0,\quad \textit{if}\ \ n>l, \tag{8.4}$$

$$|K_{2^n}(x)|\leq c\cdot 2^k, \tag{8.5}$$

and

$$|K_{2^n}(x)|\leq c\sum_{s=0}^{n}2^s\chi_{I_n(x+e_s)}, \tag{8.6}$$

where c is an absolute constant.

Theorem 8.1. *Let $n \in \mathbb{N}$. Then* $\sup_{n\in\mathbb{N}} \int_G |K_{2^n}(t)|\,dt = 1$.

Proof. Since $\int_G K_j(t)\,dt = 1$ and K_{2^n} are non-negative functions for any $n \in \mathbb{N}$, then

$$\int_G |K_{2^n}(t)|\,dt = \int_G K_{2^n}(t)\,dt = 1,$$

which completes the proof.

This equality can also be obtained from Lemma 8.1 proven above. Indeed,

$$\begin{aligned}\int_G |K_{2^n}(t)|\,dt &= \int_{I_n(0)} |K_{2^n}(t)|\,dt + \sum_{s=0}^{n-1} \int_{I_n(e_s)} |K_{2^n}(t)|\,dt = \frac{2^{n-1}+1/2}{2^n} + \sum_{s=0}^{n-1} 2^{s-n-1}\\ &= \frac{1}{2} + \frac{1}{2^{n+1}} + \sum_{k=2}^{n+1} 2^{-k} = \frac{1}{2} + \frac{1}{2^{n+1}} + \sum_{k=2}^{\infty} 2^{-k} - \sum_{k=n+2}^{\infty} 2^{-k}\\ &= \frac{1}{2} + \frac{1}{2^{n+1}} + \frac{1}{2} - \frac{1}{2^{n+1}} = 1.\end{aligned}$$

□

Theorem 8.2. *Let $n := 2^{n_1}+2^{n_2}+\cdots+2^{n_r}$, $n_1 > n_2 > \cdots > n_r$ and $n^{(k)} := 2^{n_{k+1}}+2^{n_{k+2}}+\cdots+2^{n_r}$. Then*

$$nK_n(t) = \sum_{j=1}^{r} \Big(\prod_{k=1}^{j-1} w_{2^{n_k}}(t)\Big) \cdot 2^{n_j} K_{2^{n_j}}(t) + \sum_{j=1}^{r-1} \Big(\prod_{k=1}^{j-1} w_{2^{n_k}}(t)\Big) n^{(j)} D_{2^{n_j}}(t).$$

Proof. It is easy to see that

$$\begin{aligned}nK_n(t) &= \sum_{j=1}^{n} D_j(t) = \sum_{j=1}^{2^{n_1}} D_j(t) + \sum_{j=2^{n_1}+1}^{n^{(1)}+2^{n_1}} D_j(t) = \sum_{j=1}^{2^{n_1}} D_j(t) + \sum_{j=1}^{n^{(1)}} D_{j+2^{n_1}}(t)\\ &= \sum_{j=1}^{2^{n_1}} D_j(t) + \sum_{j=1}^{n^{(1)}} \big(D_{2^{n_1}}(t) + w_{2^{n_1}}(t) D_j(t)\big) = \sum_{j=1}^{2^{n_1}} D_j(t) + n^{(1)} D_{2^{n_1}}(t) + w_{2^{n_1}}(t) \sum_{j=1}^{n^{(1)}} D_j(t)\\ &= 2^{n_1} K_{2^{n_1}}(t) + n^{(1)} D_{2^{n_1}}(t) + w_{2^{n_1}}(t) n^{(1)} K_{n^{(1)}}(t)\\ &= 2^{n_1} K_{2^{n_1}}(t) + n^{(1)} D_{2^{n_1}}(t) + w_{2^{n_1}}(t)\big(2^{n_2} K_{2^{n_2}}(t) + n^{(2)} D_{2^{n_2}}(t) + w_{2^{n_2}}(t) n^{(2)} K_{n^{(2)}}(t)\big) = \cdots\\ &= 2^{n_1} K_{2^{n_1}}(t) + n^{(1)} D_{2^{n_1}}(t) + w_{2^{n_1}}(t) \cdot 2^{n_2} K_{2^{n_2}}(t) + n^{(2)} w_{2^{n_1}}(t) D_{2^{n_2}}(t) + \cdots\\ &+ \Big(\prod_{k=1}^{r-2} w_{2^{n_k}}(t)\Big) \cdot 2^{n_{r-1}} K_{2^{n_{r-1}}}(t) + \Big(\prod_{k=1}^{r-2} w_{2^{n_k}}(t)\Big) n^{(r-1)} D_{2^{n_{r-1}}}(t) + \Big(\prod_{k=1}^{r-1} w_{2^{n_k}}(t)\Big) n^{(r-1)} K_{n^{(r-1)}}(t)\\ &= \sum_{j=1}^{r-1} \Big(\prod_{k=1}^{j-1} w_{2^{n_k}}(t)\Big) \cdot 2^{n_j} K_{2^{n_j}}(t) + \sum_{j=1}^{r-1} \Big(\prod_{k=1}^{j-1} w_{2^{n_k}}(t)\Big) n^{(j)} D_{2^{n_j}}(t) + \Big(\prod_{k=1}^{r-1} w_{2^{n_k}}(t)\Big) n^{r-1} K_{n^{r-1}}(t)\\ &= \sum_{j=1}^{r} \Big(\prod_{k=1}^{j-1} w_{2^{n_k}}(t)\Big) \cdot 2^{n_j} K_{2^{n_j}}(t) + \sum_{j=1}^{r-1} \Big(\prod_{k=1}^{j-1} w_{2^{n_k}}(t)\Big) n^{(j)} D_{2^{n_j}}(t).\end{aligned}$$

The proof is complete. □

Corollary 8.2. *Let $n \in \mathbb{N}$. Then there exists an absolute constant c such that*

$$n|K_n| \le 3 \sum_{l=0}^{|n|} 2^l K_{2^l}. \tag{8.7}$$

Proof. According to $n^{(j)} \leq 2^{n_j}$, using Lemma 5.1 and Theorem 8.2, for any $j \in \mathbb{N}$, we get

$$n^{(j)} D_{2^{n_j}}(t) \leq 2 \cdot 2^{n_j} K_{2^{n_j}}. \tag{8.8}$$

Hence, by Lemma 8.2, we obtain

$$n|K_n(x)| \leq \sum_{j=1}^{r} 2^{n_j} K_{2^{n_j}}(t) + \sum_{j=1}^{r-1} n^{(j)} D_{2^{n_j}}(t) \leq 3 \sum_{j=1}^{r} 2^{n_j} K_{2^{n_j}}(t) \leq 3 \sum_{l=0}^{|n|} 2^l K_{2^l}(x). \qquad \square$$

Corollary 8.3. *Let $n \in \mathbb{N}$. Then, for any $n, N \in \mathbb{N}_+$, we have*

$$\int_G K_n(x)\, d\mu(x) = 1, \tag{8.9}$$

$$\sup_{n\in\mathbb{N}} \int_G |K_n(x)|\, d\mu(x) \leq 6, \tag{8.10}$$

$$\int_{G\setminus I_N} |K_n(x)|\, d\mu(x) \longrightarrow 0 \ \text{ as } \ n \to \infty, \quad \forall\, N \in \mathbb{N}_+. \tag{8.11}$$

Proof. Using the orthonormality theorem of Walsh system, the proof of (8.9) is immediately obtained.
Combining Corollary 8.2 and Lemma 8.1, we get

$$\int_G |K_n(x)|\, d\mu(x) \leq \frac{3}{n} \sum_{l=0}^{|n|} 2^l \int_G |K_{2^l}(x)|\, d\mu(x) \leq \frac{3}{n} \sum_{l=0}^{|n|} 2^l \leq 3 \cdot \frac{2^{|n|+1}}{n} \leq 3 \cdot \frac{2^{|n|+1}}{2^{|n|}} \leq 6,$$

which follows from the proof of (8.8).

To prove (8.11), we use the equations in (1.2).

Let $x \in I_N^{k,l}$, $k = 0, \ldots, N-2$, $l = k+1, \ldots, N-1$. By theorem (8.1) and Corollary 8.2, we arrive at

$$|K_n(x)| \leq \frac{3}{n} \sum_{s=0}^{l} 2^s |K_{2^s}(x)| \leq \frac{3}{n} \sum_{s=0}^{l} 2^s \cdot 2^k \leq \frac{c \cdot 2^l \cdot 2^k}{n}. \tag{8.12}$$

Let $x \in I_{q+1}^{k,q} \subset I_N^{k,N}$, where $N \leq q \leq |n|-1$, that is,

$$x = \big(x_0 = 0, \ldots, x_{k-1} = 0, x_k = 1, \ldots, x_{N-1} = 0, \ldots, x_{q-1} = 0, x_q = 1, \ldots, x_{|n|-1}, \ldots\big).$$

Then

$$|K_n(x)| \leq \frac{3}{n} \sum_{i=0}^{q} 2^i \cdot 2^k \leq \frac{c \cdot 2^k \cdot 2^q}{n}. \tag{8.13}$$

Let $x \in I_{|n|}^{k,|n|} \subset I_N^{k,N}$, that is $x = (x_0 = 0, \ldots, x_{k-1} = 0, x_k = 1, x_{k+1} = 0, \ldots, x_{N-1} = 0, \ldots, x_{|n|-1} = 0, \ldots)$. Then

$$|K_n(x)| \leq \frac{3}{n} \sum_{s=0}^{|n|} 2^s |K_{2^s}(x)| \leq \frac{3}{n} \sum_{i=0}^{|n|} 2^i \cdot 2^k \leq \frac{c \cdot 2^k \cdot 2^{|n|}}{n}. \tag{8.14}$$

Since

$$I_N^{k,N} = \Big(\bigcup_{q=N}^{|n|-1} I_{q+1}^{k,q} \Big) \cup I_{|n|}^{k,|n|},$$

combining (8.13) and (8.14), we get

$$\int_{I_N^{k,N}} |K_n|\, d\mu = \sum_{q=N}^{|n|-1} \int_{I_{q+1}^{k,q}} |K_n|\, d\mu + \int_{I_{|n|}^{k,|n|}} |K_n|\, d\mu \leq \sum_{q=N}^{|n|-1} \frac{c \cdot 2^k}{n} + \frac{c \cdot 2^k}{n} \leq \frac{c\,(|n| - N) \cdot 2^k}{2^{|n|}}. \tag{8.15}$$

Hence, using (8.13), we can conclude that

$$\int\limits_{G\setminus I_N} |K_n|\,d\mu = \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{\substack{x_j=0,\\ j\in\{l+1,\dots,N-1\}}}^{1}\int\limits_{I_N^{k,l}} |K_n|\,d\mu + \sum_{k=0}^{N-1}\int\limits_{I_N^{k,N}} |K_n|\,d\mu$$
$$\leq c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{2^{N-l}}{2^N}\frac{c\cdot 2^l\cdot 2^k}{n} + c\sum_{k=0}^{N-1}\frac{|n|-N}{2^{|n|-k}} := I+II.$$

It is evident that the following estimate holds:

$$I \leq c\sum_{k=0}^{N-2}\sum l=k+1^{N-1}\frac{2^k}{2^{|n|}} \leq c\sum_{k=0}^{N-2}\frac{(N-k)\cdot 2^k}{2^{|n|}} \leq c\sum_{k=0}^{N-2}\frac{|n|-k}{2^{|n|-k}}$$
$$= c\sum_{k=0}^{N-2}\frac{|n|-k}{2^{(|n|-k)/2}}\frac{1}{2^{(|n|-k)/2}} \leq c\sum_{k=0}^{N-2}\frac{1}{2^{(|n|-k)/2}} \leq \frac{c}{2^{(|n|-N)/2}} \longrightarrow 0 \text{ as } n\to\infty.$$

Analogously, we can show that $II \leq \frac{c(|n|-N)}{2^{|n|-N}} \to 0$ as $n\to\infty$. Thus estimate (8.11) is fulfilled and theorem is proved. □

The following theorem is very important in proving the almost everywhere convergence of the Fejer means in the following sections.

Lemma 8.3. *Let $n\in\mathbb{N}$. Then $\int\limits_{G\setminus I_N}\sup\limits_{n>2^N}|K_n|\,d\mu \leq C<\infty$, where C is an absolute constant.*

Proof. Let $n>2^N$ and $x\in I_N^{k,l}$, $k=0,\dots,N-2$, $l=k+1,\dots,N-1$. Due to (8.12), we get

$$\sup_{n>2^N}|K_n(x)| \leq c\cdot 2^{l+k-N}. \tag{8.16}$$

Let $n>2^N$ and $x\in I_N^{k,N}$. Then, combining (8.13) and (8.14), we find that $|K_n(x)|\leq c\cdot 2^k$. Hence,

$$\sup_{n>2^N}|K_n(x)| \leq c\cdot 2^k. \tag{8.17}$$

If we use the estimates used in the previous theorem and follow similar steps as in this theorem, we finally get

$$\int\limits_{G\setminus I_N}\sup_{n>2^N}|K_n|\,d\mu = \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{\substack{x_j=0,\\ j\in\{l+1,\dots,N-1\}}}^{1}\int\limits_{I_N^{k,l}}\sup_{n>2^N}|K_n|\,d\mu + \sum_{k=0}^{N-1}\int\limits_{I_N^{k,N}}\sup_{n>2^N}|K_n|\,d\mu$$
$$\leq c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{2^{N-l}}{2^N}\frac{2^l\cdot 2^k}{2^N} + c\sum_{k=0}^{N-1}\frac{2^k}{2^N} = c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{2^k}{2^N} + c\sum_{k=0}^{N-1}\frac{2^k}{2^N}$$
$$\leq c\sum_{k=0}^{N-2}\frac{(N-k)}{2^{N-k}} + c \leq c\sum_{k=0}^{N-2}\frac{1}{2^{(N-k)/2}} + c < C < \infty. \tag{8.18}$$

The proof is complete. □

The following estimation is proved by Goginava [48].

Lemma 8.4. *Let $x\in I_M^{k,l}$, $k=0,\dots,M-1$, $l=0,\dots,M$. Then*

$$\int\limits_{I_M}|K_n(x+t)|\,d\mu(t) \leq \frac{c\cdot 2^{l+k}}{n\cdot 2^M}\,,\quad n>2^M. \tag{8.19}$$

where c is an absolute constant.

Proof. Let $n \in \mathbb{N}$ and $x \in I_M^{k,l}$ for $0 \le k < l \le M-1$ and $t \in I_M$. Since $x+t \in I_M^{k,l}$, combining (8.1) in Lemma 8.1 and (8.7) in Corollary 8.2, we obtain

$$\int_{I_N} n|K_n(x+t)|\,d\mu(t) \le C\sum_{i=0}^{l} 2^i \int_{I_N} |K_{2^i}(x+t)|\,d\mu(t) \le C\mu(I_N)\sum_{i=0}^{l} 2^{k+l} \le \frac{C\cdot 2^{k+l}}{2^N},$$

so

$$\int_{I_N} |K_n(x+t)|\,d\mu(t) \le \frac{C\cdot 2^{k+l}}{n\cdot 2^N},$$

and thus in this case (8.19) is proved.

Let $x \in I_M^{k,M}$. Then, applying (8.1) in Lemma 8.1 and (8.7) in Corollary 8.2, we have

$$\int_{I_N} n|K_n(x+t)|\,d\mu(t) \le \sum_{i=0}^{|n|} 2^i \int_{I_N} |K_{2^i}(x+t)|\,d\mu(t). \tag{8.20}$$

Let

$$\begin{cases} x = \big(0,\dots,0,x_k \neq 0,0,\dots,0,x_M,x_{M+1},x_q,\dots,x_{|n|-1},\dots\big),\\ t = \big(0,\dots,0,x_M,\dots,x_{q-1},t_q \neq x_q,t_{q+1},\dots,t_{|n|-1},\dots\big), \end{cases}$$

for some $q = M,\dots,|n|-1$. Using (8.1) in Lemma 8.1 and (8.7) in Corollary 8.2, it is easy to see that

$$\int_{I_M} |K_n(x+t)|\,d\mu(t) \le \frac{C}{n}\sum_{i=0}^{q-1} 2^i \int_{I_M} 2^k\,d\mu(t) \le \frac{C\cdot 2^{k+q}}{n\cdot 2^M} \le \frac{C\cdot 2^k}{2^M}. \tag{8.21}$$

Let

$$\begin{cases} x = \big(0,\dots,0,x_k \neq 0,0,\dots,0,x_M,x_{M+1},x_q,\dots,x_{|n|-1},\dots\big),\\ t = \big(0,\dots,0,x_M,x_{M+1},\dots,x_{|n|-1},\dots\big). \end{cases}$$

If we apply again (8.1) in Lemma 8.1 and (8.7) in Corollary 8.2, we obtain

$$\int_{I_N} |K_n(x+t)|\,d\mu(t) \le \frac{C}{n}\sum_{i=0}^{|n|-1} 2^i \int_{I_M} 2^k\,d\mu(t) \le \frac{C\cdot 2^k}{2^M}. \tag{8.22}$$

Combining (8.21) and (8.22), we conclude that (8.19) holds for $l = M$ as well, which completes the proof. □

Corollary 8.4. *Let $x \in I_M^{k,l}$, $k = 0,\dots,M-1$, $l = k+1,\dots,M$. Then*

$$\int_{I_N} |K_n(x+t)|\,d\mu(t) \le \frac{C\cdot 2^{l+k}}{2^{2M}} \quad \textit{for } n \ge 2^M.$$

The following estimations of Fejér kernels with respect to the one-dimensional Walsh–Fourier series is proved in [112].

Lemma 8.5. *Let $n = \sum_{i=1}^{r}\sum_{k=l_i}^{m_i} 2^k$, where $m_1 \ge l_1 > l_1 - 2 \ge m_2 \ge l_2 > l_2 - 2 > \cdots > m_s \ge l_s \ge 0$. Then*

$$|nK_n| \le c\sum_{A=1}^{r}\Big(2^{l_A}|K_{2^{l_A}}| + 2^{m_A}|K_{2^{m_A}}| + 2^{l_A}\sum_{k=l_A}^{m_A} D_{2^k}\Big) + c\,V(n),$$

where c is an absolute constant.

Proof. Let $n=\sum_{i=1}^{r}2^{n_i}$, $n_1>n_2>\cdots>n_r\geq 0$. Using Lemma 8.2 for the n-th Fejér kernels, we can conclude that

$$nK_n=\sum_{A=1}^{r}\Big(\prod_{j=1}^{A-1}w_{2^{n_j}}\Big)\big(2^{n_A}K_{2^{n_A}}+(2^{n_A}-1)D_{2^{n_A}}\big)$$
$$-\sum_{A=1}^{r}\Big(\prod_{j=1}^{A-1}w_{2^{n_j}}\Big)(2^{n_A}-1-n^{(A)})D_{2^{n_A}}=I_1-I_2.$$

For I_1, we have the following equality:

$$\begin{aligned}I_1&=\sum_{v=1}^{r}\Big(\prod_{j=1}^{v-1}\prod_{i=l_j}^{m_j}w_{2^i}\Big)\Big(\sum_{k=l_v}^{m_v}\Big(\prod_{j=k+1}^{m_v}w_{2^j}\Big)\big(2^kK_{2^k}-(2^k-1)D_{2^k}\big)\Big)\\&=\sum_{v=1}^{r}\Big(\prod_{j=1}^{v-1}\prod_{i=l_j}^{m_j}w_{2^i}\Big)\Big(\sum_{k=0}^{m_v}-\sum_{k=0}^{l_v-1}\Big)\Big(\prod_{j=k+1}^{m_v}w_{2^j}\Big)\big(2^kK_{2^k}-(2^k-1)D_{2^k}\big)\\&=\sum_{v=1}^{r}\Big(\prod_{j=1}^{v-1}\prod_{i=l_j}^{m_j}w_{2^i}\Big)\Big(\sum_{k=0}^{m_v}\Big(\prod_{j=k+1}^{m_v}w_{2^j}\Big)\big(2^kK_{2^k}-(2^k-1)D_{2^k}\big)\Big)\\&\qquad-\sum_{v=1}^{r}\Big(\prod_{j=1}^{v}\prod_{i=l_j}^{m_j}w_{2^i}\Big)\Big(\sum_{k=0}^{l_v-1}\Big(\prod_{j=k+1}^{l_v-1}w_{2^j}\Big)\big(2^kK_{2^k}-(2^k-1)D_{2^k}\big)\Big).\end{aligned}$$

Since $2^n-1=\sum_{k=0}^{n-1}2^k$ and

$$(2^n-1)K_{2^n-1}=\sum_{k=0}^{n-1}\Big(\prod_{j=k+1}^{n-1}w_{2^j}\Big)\big(2^kK_{2^k}-(2^k-1)D_{2^k}\big),$$

we obtain

$$I_1=\sum_{v=1}^{r}\Big(\prod_{j=1}^{v-1}\prod_{i=l_j}^{m_j}w_{2^i}\Big)(2^{m_v+1}-1)K_{2^{m_v+1}-1}-\sum_{v=1}^{r}\Big(\prod_{j=1}^{v}\prod_{i=l_j}^{m_j}w_{2^i}\Big)(2^{l_v}-1)K_{2^{l_v}-1}.$$

Applying the estimations $|K_{2^n}|\leq c|K_{2^{n-1}}|$ and $|K_{2^n-1}|\leq c|K_{2^n}|+c$, we get

$$|I_1|\leq c\sum_{v=1}^{r}\big(2^{l_v}|K_{2^{l_v}}|+2^{m_v}|K_{2^{m_v}}|+c\,r\big). \tag{8.23}$$

Let $l_j<n_A\leq m_j$, where $j=1,\dots,s$. Then $n^{(A)}\geq\sum_{v=l_j}^{n_A-1}2^v\geq 2^{n_A}-2^{l_j}$ and $2^{n_A}-1-n^{(A)}\leq 2^{l_j}$.

If $l_j=n_A$, where $j=1,\dots,s$, then $n^{(A)}\leq 2^{m_{j-1}+1}<2^{l_j}$. Using these estimations, we can conclude that

$$|I_2|\leq c\sum_{v=1}^{r}2^{l_v}\sum_{k=l_v}^{m_v}D_{2^k}. \tag{8.24}$$

Combining (8.23), (8.24), we get the proof of Lemma 8.5. □

The following estimations of Fejér kernels with respect to the one-dimensional Walsh–Fourier series are proved in [112].

Lemma 8.6. *Let* $n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k$*, where* $0 \le l_1 \le m_1 \le l_2 - 2 < l_2 \le m_2 \le \cdots \le l_s - 2 < l_s \le m_s$*. Then*

$$n|K_n(x)| \ge \frac{2^{2l_i}}{16} \quad \textit{for} \;\; x \in I_{l_i+1}(e_{l_i-1} + e_{l_i}).$$

Proof. If we apply Lemma 8.2 for $n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k$, we can write that

$$\begin{aligned} nK_n = & \sum_{r=1}^{s} \sum_{k=l_r}^{m_r} \Big(\prod_{j=r+1}^{s} \prod_{q=l_j}^{m_j} w_{2^q} \prod_{j=k+1}^{m_r} w_{2^j} \Big) \cdot 2^k K_{2^k} \\ & + \sum_{r=1}^{s} \sum_{k=l_r}^{m_r} \Big(\prod_{j=r+1}^{s} \prod_{q=l_j}^{m_j} w_{2^q} \prod_{j=k+1}^{m_r} w_{2^j} \Big) \Big(\sum_{t=1}^{r-1} \sum_{q=l_t}^{m_t} 2^q + \sum_{q=l_r}^{k-1} 2^q \Big) D_{2^k}. \end{aligned}$$

Let $x \in I_{l_i+1}(e_{l_i-1} + e_{l_i})$. Then

$$n|K_n| \ge |2^{l_i} K_{2^{l_i}}| - \sum_{r=1}^{i-1} \sum_{k=l_r}^{m_r} |2^k K_{2^k}| - \sum_{r=1}^{i-1} \sum_{k=l_r}^{m_r} |2^k D_{2^k}| = I - II - III.$$

From the equalities in (8.1) in Lemma 8.1, it follows that

$$I = |2^{l_i} K_{2^{l_i}}(x)| = \frac{2^{2l_i}}{4}. \tag{8.25}$$

Since $m_{i-1} \le l_i - 2$, we can easily obtain that the following estimation is true:

$$II \le \sum_{n=0}^{l_i-2} |2^n K_{2^n}(x)| \le \sum_{n=0}^{l_i-2} 2^n \frac{(2^n+1)}{2} \le \frac{2^{2l_i}}{24} + \frac{2^{l_i}}{4} - \frac{2}{3}. \tag{8.26}$$

For III, we get

$$III \le \sum_{k=0}^{l_i-2} |2^k D_{2^k}(x)| \le \sum_{k=0}^{l_i-2} 4^k = \frac{2^{2l_i}}{12} - \frac{1}{3}. \tag{8.27}$$

Combining (8.25)–(8.27), we can conclude that

$$n|K_n(x)| \ge I - II - III \ge \frac{2^{2l_i}}{8} - \frac{2^{l_i}}{4} + 1. \tag{8.28}$$

Suppose that $l_i \ge 2$. Then

$$n|K_n(x)| \ge \frac{2^{2l_i}}{8} - \frac{2^{2l_i}}{16} \ge \frac{2^{2l_i}}{16}.$$

If $l_i = 0$ or $l_i = 1$, then, applying (8.28), we get $n|K_n(x)| \ge \frac{7}{8} \ge \frac{2^{2l_i}}{16}$. □

The following estimations of Fejér kernels with respect to the one-dimensional Walsh–Fourier series is proved in [15].

Lemma 8.7. *Let* $n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k$*, where* $m_1 \ge l_1 > l_1 - 2 \ge m_2 \ge l_2 > l_2 - 2 > \cdots > m_s \ge l_s \ge 0$*. Then, for any* $i = 1, 2, \ldots, s$*,* $n|K_n(x)| \ge 2^{2m_i - 2}$ *for* $x \in E_{t_i} := I_{t_i+3}(e_{t_i+1} + e_{t_i+2})$*.*

Proof. It is evident that we always have $m_i + 2 \le l_{i-1}$. If $m_i + 2 = l_{i-1}$, then

$$E_{m_i} = I_{m_i+3}(e_{m_i+1} + e_{m_i+2}) = I_{l_{i-1}+1}(e_{l_{i-1}-1} + e_{l_{i-1}}) = E_{l_{i-1}}$$

and if we apply Lemma 8.6, we find that $n|K_n(x)| \ge 2^{2l_{i-1}-4} = 2^{2t_i}$ for $x \in E_{l_{i-1}} = E_{t_i}$.

Let $m_i+2<l_{i-1}$. Combining Lemma 5.1 and (8.1) in Lemma 8.1, for any $n\geq m_i+3$, we get $D_{2^n}(x)=K_{2^n}(x)=0$ for $x\in E_{t_i}$.

From Lemma 8.2, for $x\in E_{m_i}$ we can conclude that

$$\begin{aligned} nK_n &= \sum_{r=1}^{s}\sum_{k=l_r}^{m_r}\Big(\prod_{j=i+1}^{s}\prod_{q=l_j}^{m_j}w_{2^q}\prod_{j=k+1}^{m_i}w_{2^j}\Big)\Big(2^kK_{2^k}+\Big(\sum_{j=i+1}^{s}\sum_{q=l_j}^{m_j}2^q+\sum_{q=l_i}^{k-1}2^q\Big)D_{2^k}\Big)\\ &= \sum_{r=i}^{s}\sum_{k=l_r}^{m_r}\Big(\prod_{j=i+1}^{s}\prod_{q=l_j}^{t_j}w_{2^q}\prod_{j=k+1}^{m_i}w_{2^j}\Big)\Big(2^kK_{2^k}+\Big(\sum_{j=i+1}^{s}\sum_{q=l_j}^{m_j}2^q+\sum_{q=l_i}^{k-1}2^q\Big)D_{2^k}\Big). \end{aligned} \tag{8.29}$$

Suppose that $l_i<m_i$. Since

$$\sum_{j=i+1}^{s}\sum_{q=l_j}^{m_j}2^q+\sum_{q=l_i}^{m_i-1}2^q\geq 2^{m_i-1}$$

for $x\in E_{m_i}$, we find that

$$n|K_n|\geq |2^{m_i}K_{2^{m_i}}+2^{m_i-1}D_{2^{m_i}}|-\sum_{k=0}^{m_i-1}|2^kK_{2^k}|-\sum_{k=0}^{m_i-1}|2^kD_{2^k}|:=I-II-III. \tag{8.30}$$

Moreover, Combining Lemma 5.1 and (8.1) in Lemma 8.1, we obtain

$$I=2^{m_i}K_{2^{m_i}}(x)+2^{m_i-1}D_{2^{m_i}}=\frac{2^{2m_i}}{2}+2^{m_i-1}+\frac{2^{2m_i}}{2}=2^{2m_i}+2^{m_i-1}. \tag{8.31}$$

For II, we have

$$II\leq\sum_{k=0}^{m_i-1}2^k\cdot\frac{(2^k+1)}{2}=\frac{1}{2}\cdot\frac{2^{2m_i}-1}{4-1}+\frac{1}{2}\cdot\frac{2^{m_i}-1}{2-1}\leq\frac{2^{2m_i}}{6}+2^{m_i-1}. \tag{8.32}$$

Moreover, III can be estimated as follows:

$$III\leq\sum_{k=0}^{l_i-1}4^k=\frac{2^{2m_i}}{3}\,. \tag{8.33}$$

Combining (8.31)–(8.33) and substituting them into (8.30), we obtain

$$n|K_n(x)|\geq I-II-III\geq\frac{2^{2l_i}}{2}\,. \tag{8.34}$$

If $m_i=l_i$, we get $m_{i+1}\leq l_i-2=m_i-2$. Hence, using (8.29), we find that

$$n|K_n|\geq|2^{m_i}K_{2^{m_i}}|-\sum_{k=0}^{m_i-2}|2^kK_{2^k}|-\sum_{k=0}^{m_i-2}|2^kD_{2^k}|:=I-II-III. \tag{8.35}$$

By simple calculations, we get

$$I\geq 2^{m_i}K_{2^{m_i}}(x)=\frac{2^{2m_i}}{2}+\frac{2^{m_i}}{2}\,, \tag{8.36}$$

$$II\leq\sum_{k=0}^{m_i-2}2^k\cdot\frac{(2^k+1)}{2}\leq\frac{1}{2}\cdot\frac{2^{2m_i-2}-1}{4-1}+\frac{1}{2}\cdot\frac{2^{t_i-1}-1}{2-1}\leq\frac{2^{2m_i}}{24}+2^{m_i-2}\,, \tag{8.37}$$

$$III\leq\sum_{k=0}^{l_i-2}4^k=\frac{2^{2m_i}}{12}\,. \tag{8.38}$$

We substitute (8.36)–(8.38) into (8.35) and find that

$$n|K_n(x)|\geq I-II-III\geq\frac{2^{2l_i}}{4}\,. \tag{8.39}$$

The proof is complete by simply combining the estimates (8.34) and (8.39). □

Corollary 8.5. *Let* $n=\sum_{i=1}^{s}\sum_{k=l_i}^{m_i}2^k$, *where* $m_1\geq l_1>l_1-2\geq m_2\geq l_2>l_2-2>\cdots>m_s\geq l_s\geq 0$. *Then, for any* $i=1,2,\ldots,s$, $n|K_n(x)|\geq 2^{2m_i-5}$ *for* $x\in E_{m_i}:=I_{t_i+1}(e_{m_i+1}+e_{m_i+2})$ *and* $n|K_n(x)|\geq 2^{2l_i-5}$ *for* $x\in E_{l_i}:=I_{l_i+1}(e_{l_i-1}+e_{l_i})$, *where* $I_1(e_{-1}+e_0)=I_2(e_0+e_1)$.

Corollary 8.6. *Let* $2<n\in\mathbb{N}_+$ *and* $q_n=2^{2n}+2^{2n-2}+\cdots+2^2+2^0$. *Then*

$$q_{n-1}|K_{q_{n-1}}(x)|\geq\frac{2^{4k}}{128}\quad for\ \ x\in I_{2k+1}(e_{2k-1}+e_{l_{2k}}),\quad k=0,1,\ldots,n.$$

Corollary 8.7. *Let* $n\in\mathbb{N}$, $\langle n\rangle\neq|n|$ *and* $x\in I_{\langle n\rangle+1}(e_{\langle n\rangle-1}+e_{\langle n\rangle})$. *Then*

$$|nK_n(x)|=\left|(n-2^{|n|})K_{n-2^{|n|}}(x)\right|\geq\frac{2^{2\langle n\rangle}}{16}\,.$$

9 Norm and almost everywhere convergences of Fejér means with respect to Walsh system in Lebesgue spaces

We study the rate of convergence of Fejér means of the Walsh–Fourier series in the Lebesgue spaces.

Theorem 9.1. *Suppose that* $f\in L_p(G)$ *for some* $1\leq p<\infty$. *Then* $\|\sigma_nf-f\|_p\to 0$ *as* $n\to\infty$. *The same holds for* $p=\infty$ *if* f *is uniformly continuous.*

Proof. Let $\varepsilon>0$. Using an absolute continuity of the Lebesgue integral combining with (8.10) in Corollary 8.3, we get

$$\sup_{t\in I_N}\|f(\cdot+t)-f(\cdot)\|_p\sup_{n\in\mathbb{N}}\|K_n\|_1<\frac{\varepsilon}{2}\,,$$

whenever N is large enough. Applying Minkowski's integral inequality as well as (8.9) and (8.11) in Corollary 8.3, we find that

$$\begin{aligned}\|\sigma_nf-f\|_p&=\left\|\int_G K_n(t)(f(\cdot+t)-f(\cdot))\,d\mu(t)\right\|_p\leq\int_G|K_n(t)|\,\|f(\cdot+t)-f(\cdot)\|_p\,d\mu(t)\\&=\int_{I_N}|K_n(t)|\,\|f(\cdot+t)-f(\cdot)\|_p\,d\mu(t)+\int_{G\setminus I_N}|K_n(t)|\,\|f(\cdot+t)-f(\cdot)\|_p\,d\mu(t)\\&\leq\sup_{t\in I_N}\|f(\cdot+t)-f(\cdot)\|_p\sup_{n\in\mathbb{N}}\|K_n\|_1+\sup_{t\in G}\|f(\cdot+t)-f(\cdot)\|_p\int_{G\setminus I_N}|K_n(t)|\,d\mu(t)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon\end{aligned}$$

if n is large enough. □

The following theorem is true.

Theorem 9.2. *Let* $1\leq p<\infty$, $f\in L_p(G)$ *and* $n\in\mathbb{N}$. *Then*

$$\|\sigma_nf-f\|_p\leq 8\,\omega_p(f,2^{-N})+\sum_{k=0}^{N-1}2^{k-N-1}\omega_p(f,2^{-k}),\quad 2^N\leq n<2^{N+1}.$$

Proof. Let $P:=S_{2^N}f$. Then $\sigma_nf-f=\sigma_n(f-P)+(P-f)+(\sigma_nP-P)$ and

$$\|\sigma_nf-f\|_p\leq\|\sigma_n(f-P)\|_p+\|P-f\|_p+\|\sigma_nP-P\|_p\,.$$

Using Proposition 7.1, we find that $\|P-f\|_p\leq\omega_p(f,2^{-N})$. On the other hand,

$$\|\sigma_n(f-P)\|_p\leq\left\|\int_G(f-P)(t)K_n(x+t)\,dt\right\|_p\leq 6\,\|f-P\|_p\leq 6\,\omega_p(f,2^{-N}).$$

Using the identities

$$\begin{aligned}\sigma_n S_{2^N} f - S_{2^N} f &= \frac{1}{n}\sum_{k=0}^{2^N} S_k S_{2^N} f + \frac{1}{n}\sum_{k=2^N+1}^{n} S_k S_{2^N} f - S_{2^N} f \\ &= \frac{1}{n}\sum_{k=0}^{2^N} S_k f + \frac{1}{n}\sum_{k=2^N+1}^{n} S_{2^N} f - S_{2^N} f = \frac{1}{n}\sum_{k=0}^{2^N} S_k f + \frac{n-2^N}{n} S_{2^N} f - S_{2^N} f \\ &= \frac{2^N}{n}\sigma_{2^N} f - \frac{2^N}{n} S_{2^N} f = \frac{2^N}{n}(S_{2^N}\sigma_{2^N} f - S_{2^N} f) = \frac{2^N}{n} S_{2^N}(\sigma_{2^N} f - f), \qquad (9.1)\end{aligned}$$

we can conclude that $\|\sigma_n(S_{2^N} f) - S_{2^N} f\|_p \le \|S_{2^N}(\sigma_{2^N} f - f)\|_p \le \|\sigma_{2^N} f - f\|_p$. It follows that $\|\sigma_n f - f\|_p \le 7\omega_p(f, 2^{-N}) + \|\sigma_{2^N} f - f\|_p$. We have only to estimate $\|\sigma_{2^N} f - f\|_p$. Since

$$\sigma_{2^N} f(x) - f(x) = \int_G (f(x+t) - f(x)) K_{2^N}(t)\, dt,$$

applying Theorem 8.1, we get

$$\sigma_{2^N} f(x) - f(x) = \left(2^{N-1} + \frac{1}{2}\right)\int_{I_N} (f(x+t) - f(x))\, dt + \sum_{l=0}^{N-1} 2^{l-1} \int_{I_N(e_l)} (f(x+t) - f(x))\, dt$$

and

$$\|\sigma_{2^N} f - f\|_p \le \omega_p(f, 2^{-N}) + \sum_{l=0}^{N-1} 2^{l-N-1}\omega_p(f, 2^{-l}). \qquad \square$$

Remark 9.1. Let $f \in L_p(G)$. Then $\lim_{N\to\infty} \sum_{k=0}^{N-1} 2^{k-N-1}\omega_p(f, 2^{-k}) = 0$.

Proof. Since $f \in L_p(G)$, we obtain $\lim_{n\to\infty} \omega(f, 2^{-n}) = 0$ and, therefore, there exists N_0 such that when $k > N_0$, the following inequality holds: $\omega_p(f, 2^{-k}) < \varepsilon/2$.

Let $N > N_0$. Then

$$\sum_{k=0}^{N-1} 2^{k-N-1}\omega_p(f, 2^{-k}) = \sum_{k=0}^{N_0} 2^{k-N-1}\omega_p(f, 2^{-k}) + \sum_{k=N_0+1}^{N-1} 2^{k-N-1}\omega_p(f, 2^{-k}) = I + II.$$

For II, we have

$$II \le \frac{\varepsilon}{2}\sum_{k=N_0+1}^{N-1} 2^{k-N-1} \le \frac{\varepsilon}{2}\sum_{n=0}^{\infty} 2^{-n} < \varepsilon.$$

It is also easy to check that

$$\begin{aligned} I &\le \sum_{k=0}^{N_0} 2^{k-N-1}\omega_p(f, 2^{-k}) \le 2^{N_0-N} \max_{0\le k\le N_0} \omega(f, 2^{-k}) \\ &\le 2^{N_0-N} \max_{0\le k\le \infty} \omega_p(f, 2^{-k}) \le c\cdot 2^{N_0-N} \longrightarrow 0 \ \text{ as } \ N\to\infty. \qquad \square \end{aligned}$$

From the theorem proven above, we find that the following is true.

Theorem 9.3. *Let $f \in L_p(G)$. Then $\|\sigma_n f - f\|_p \to 0$ as $n \to \infty$. Moreover, if $f \in \mathrm{lip}(\alpha, p)$, that is,*

$$\omega_p\left(\frac{1}{2^n}, f\right) = O\left(\frac{1}{2^{n\alpha}}\right), \quad n\to\infty,$$

then

$$\|\sigma_n f - f\|_p = \begin{cases} O\Big(\dfrac{1}{2^N}\Big) & \text{as } \alpha > 1, \\ O\Big(\dfrac{N}{2^N}\Big) & \text{as } \alpha = 1, \\ O\Big(\dfrac{1}{2^{N\alpha}}\Big) & \text{as } \alpha < 1. \end{cases}$$

On the other hand, if $1 \le p < \infty$, $f \in L_p$ *and* $\|\sigma_{2^n} f - f\|_p = o(1/2^n)$ *as* $n \to \infty$, *then* f *is a constant function.*

Now, we state the following result concerning the pointwise convergence.

Theorem 9.4. *If* $x \in G$ *and* $f \in L_1(G)$ *is continuous at* x, *then*

$$\lim_{m\to\infty} \sigma_m f(x) = f(x). \tag{9.2}$$

Proof. It is easy to see that

$$\sigma_n f(x) - f(x) = \int\limits_{I_N(x)} (f(t) - f(x)) K_n(x+t)\, dt + \int\limits_{G\setminus I_N(x)} (f(t) - f(x)) K_n(x+t)\, dt.$$

Using the localization principle, we get

$$\int\limits_{G\setminus I_N(x)} (f(t) - f(x)) D_n(x+t)\, dt \longrightarrow 0, \quad n \to \infty.$$

Hence, if we use the regularity of the Fejér means, we find that

$$\int\limits_{G\setminus I_N(x)} (f(t) - f(x)) K_n(x+t)\, dt \longrightarrow 0, \quad n \to \infty.$$

On the other hand,

$$\left| \int\limits_{I_N(x)} (f(t) - f(x)) K_n(x+t)\, dt \right| \le \sup_{u\in I_N(x)} |f(u) - f(x)| \int\limits_G |K_n(x+t)|\, dt \le 2 \sup_{u\in I_N(x)} |f(u) - f(x)|.$$

Since f is continuous at x, the last term is small enough if N is large enough and we can conclude that (9.2) holds. Thus the proof is complete. □

Now, we prove that the maximal operator of Fejér means with respect to the Walsh–Fourier series is of weak-(1,1) type. In particular, the following statement is true.

Theorem 9.5. *Let* $f \in L_1(G)$. *Then* $\sup\limits_{y>0} y\mu\{\sigma^* f > y\} \le \|f\|_1$.

Proof. By Theorem 1.3, we find that the proof will be complete if we show that

$$\int\limits_{\overline{I}} |\sigma^* f|\, d\mu \le \|f\|_1 \tag{9.3}$$

for every function f, that satisfies the conditions in (1.4), where I denotes the support of the function f. Without lost of generality, we may assume that f is a function with support I and $\mu(I) = 2^N$. We may assume that $I = I_N$. It is easy to see that $\sigma_n f = \int\limits_{I_N} K_n(x+t) f(t)\, d\mu(t) = 0$ for $n \le 2^N$. Therefore, we may suppose that $n > 2^N$. Hence,

$$|\sigma^* f(x)| = \sup_{n>M_N} \left| \int\limits_{I_N} K_n(x+t) f(t)\, d\mu(t) \right|.$$

Let $t \in I_N$ and $x \in \overline{I}_N$. Then $x+t \in \overline{I}_N$, and applying Lemma 8.3, we get

$$\begin{aligned}
&\int_{\overline{I}_N} |\sigma^* f(x)| \, d\mu(x) \\
&\qquad \le \int_{\overline{I}_N} \sup_{n>2^N} \int_{I_N} |K_n(x+t) f(t)| \, d\mu(t) \, d\mu(x) \le \int_{I_N} \int_{\overline{I}_N} \sup_{n>2^N} |K_n(x+t) f(t)| \, d\mu(x) \, d\mu(t) \\
&\qquad \le \int_{I_N} |f(t)| \, d\mu(t) \int_{\overline{I}_N} \sup_{n>2^N} |K_n(x+t)| \, d\mu(x) \le \int_{I_N} |f(t)| \, d\mu(t) \int_{\overline{I}_N} \sup_{n>2^N} |K_n(x)| \, d\mu(x) \le C \|f\|_1.
\end{aligned}$$

Thus (9.3) holds and the proof is complete. □

Theorem 9.6. *Let $f \in L_1$. Then $\sigma_n f \to f$ a.e. as $n \to \infty$.*

Proof. Since $S_n P = P$ for every $P \in \mathcal{P}$, according to the regularity of Fejér means, we obtain $\sigma_n P \to P$ a.e. as $n \to \infty$, where $\mathcal{P}$ is dense in the space L_1 (see remark 6.1).

On the other hand, by Theorems 1.2 and 9.5, we obtain the almost everywhere convergence of Fejér means: $\sigma_n f \to f$ a.e. as $n \to \infty$. □

Now, we introduce the notion of the Lebesgue points and Walsh–Lebesgue points.

Definition 9.1. A point $x \in \mathbb{R}$ is called a Lebesgue point of an integrable function f if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, d\mu(t) = 0.$$

It is known that almost every point x is a Lebesgue point of $f \in L_1$ and the Fejér means $\sigma_n^T f$ of the trigonometric Fourier series of $f \in L_1$ converge to f at each Lebesgue point.

We now introduce the notion of Lebesgue points and Walsh–Lebesgue points for any $x \in G$.

Definition 9.2. A point $x \in G$ is called a Lebesgue point of an integrable function f if

$$\lim_{n \to \infty} \frac{1}{|I_n(x)|} \int_{I_n(x)} |f(t) - f(x)| \, d\mu(t) = 0.$$

Below we prove an analogical theorem for the Fejér means $\sigma_n f$ with respect to the Walsh system. First, we present the following definition.

Definition 9.3. We introduce the operator W_A defined by

$$W_A f(x) =: \sum_{s=0}^{A-1} 2^s \int_{I_A(x+e_s)} |f(t) - f(x)| \, d\mu(t).$$

A point $x \in G$ is called a Walsh–Lebesgue point of the function $f \in L_1(G)$ if $\lim\limits_{A \to \infty} W_A f(x) = 0$.

We define V_A by

$$\begin{aligned}
V_A f(x) &=: \sum_{s=0}^{A} 2^s \int_{I_A(x+e_s)} f(t) \, d\mu(t) = \sum_{s=0}^{A} \frac{2^s}{2^A} \int_G D_{2^A}(x+e_s+t) f(t) \, d\mu(t) \\
&= \int_G \Big(\sum_{s=0}^{A} \frac{2^s}{2^A} D_{2^A}(x+e_s+t) \Big) f(t) \, d\mu(t) = \int_G Y_A(x+t) f(t) \, d\mu(t),
\end{aligned}$$

where

$$Y_A(x) = \sum_{s=0}^{A} \frac{2^s}{2^A} D_{2^A}(x + e_s).$$

It is obvious that $\lim_{A\to\infty} W_A f(x) = 0$ if and only if $\lim_{A\to\infty} V_A|f - f(x)|(x) = 0$. Moreover, we define V^* by $V^* f(x) = \sup_{A\in\mathbb{N}} |V_A f(x)|$.

Theorem 9.7. *Let $N \in \mathbb{N}$. Then $\int_{G\setminus I_N} \sup_{A>N} |Y_A(x)|\, d\mu \le C < \infty$, where C is an absolute constant.*

Proof. Let $A > N$ and $x \in I_N^{k,l}$, $k = 0, \dots, N-2$, $l = k+1, \dots, N-1$. Then it is easy to prove that $x + e_s \in G \setminus I_N$. Using Lemma 5.1 (Paley's Lemma), we get $D_{2^A}(x + e_s) = 0$ and

$$|Y_A(x)| = \Big| \sum_{s=0}^{A} \frac{2^s}{2^A} D_{2^A}(x + e_s) \Big| = 0 \ \text{ for } \ A > N. \tag{9.4}$$

Using again Lemma 5.1 (Paley's Lemma), we can conclude that

$$D_{2^A}(x + e_k) = \begin{cases} 2^A, & x \in I_A(e_k), \\ 0, & x \in G \setminus I_A(e_k). \end{cases}$$

Hence,

$$|Y_A(x)| = \frac{2^k}{2^A} |D_{2^A}(x + e_k)| = \begin{cases} 2^k, & x \in I_A(e_k), \\ 0, & x \in G \setminus I_A(e_k). \end{cases} \tag{9.5}$$

Combining (1.2), (9.4) and (9.5), we find that

$$\begin{aligned} \int_{G\setminus I_N} \sup_{A>N} |Y_A(x)|\, d\mu(x) &= \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \int_{I_N^{k,l}} \sup_{A>N} |Y_A(x)|\, d\mu(x) + \sum_{k=0}^{N-1} \int_{I_N^{k,N}} \sup_{A>N} |Y_A(x)|\, d\mu(x) \\ &= \sum_{k=0}^{N-1} \int_{I_A(e_k)} \sup_{A>N} |Y_A(x)|\, d\mu(x) \le C \sum_{k=0}^{N-1} \frac{2^k}{2^N} < C < \infty. \end{aligned}$$

□

We apply this lemma to prove the following important theorem for the maximal operator V^*.

Theorem 9.8. *Let $f \in L_1(G)$. Then the operator V^* is of weak-type $(1,1)$, i.e.,*

$$\sup_{y>0} y\mu\{V^* f > y\} \le \|f\|_1.$$

Proof. Since

$$\|V^* f\|_\infty \le C\, \|f\|_\infty \sup_{A\in\mathbb{N}} \frac{1}{2^A} \sum_{s=0}^{A-1} 2^s \le C\, \|f\|_\infty,$$

we find that $V^* f$ is bounded from $L_\infty(G)$ to $L_\infty(G)$.

According to Theorem 1.3, the proof will be complete if we show that

$$\int_{\overline{I}} |V^* f|\, d\mu \le c\, \|f\|_1 \tag{9.6}$$

for every function f satisfying the conditions in (1.4), where I denotes the support of the function f. Without lost of generality, we may assume that f is a function with support I and $\mu(I) = 2^N$.

We also may assume that $I = I_N$. It is easy to see that $V_n f = 0$ when $n \leq 2^N$. Therefore, we may suppose that $n > M_N$. Hence,

$$|V^* f(x)| = \sup_{n>M_N} \left| \int_{I_N} Y_n(x+t) f(t)\, d\mu(t) \right|.$$

Let $t \in I_N$ and $x \in \overline{I}_N$. Then $x + t \in \overline{I}_N$, and applying Lemma 9.7, we get

$$\begin{aligned}\int_{\overline{I}_N} |V^* f(x)|\, d\mu(x) &\leq \int_{\overline{I}_N} \sup_{n>2^N} \int_{I_N} |Y_n(x+t) f(t)|\, d\mu(t)\, d\mu(x) \\ &\leq \int_{I_N} \int_{\overline{I}_N} \sup_{n>2^N} |Y_n(x+t) f(t)|\, d\mu(x)\, d\mu(t) = \int_{I_N} |f(t)|\, d\mu(t) \int_{\overline{I}_N} \sup_{n>2^N} |Y_n(x)|\, d\mu(x) \leq C\, \|f\|_1,\end{aligned}$$

so, (9.6) holds and the proof is complete. □

Theorem 9.9. *Let $f \in L_1(G)$. Then $\lim_{A\to\infty} W_A f(x) = 0$ a.e. $x \in G$.*

Proof. It is easy to see that $\lim_{A\to\infty} W_A f(x) = 0$ for every Walsh polynomial. Hence, since the Walsh polynomials are dense in $L_1(G)$, Theorems 1.2 and 9.8 imply the proof. □

Theorem 9.10. *Let $f \in L_1(G)$. Then $\lim_{n\to\infty} \sigma_n f(x) = f(x)$ for all Walsh–Lebesgue points of f.*

Proof. Combining (8.6) in Lemma 8.2 and (8.7) in Corollary 8.2, we get

$$\begin{aligned}|\sigma_n f(x) - f(x)| &\leq \frac{C}{n} \sum_{A=0}^{|n|} 2^A \int_G |f(t) - f(x)|\, |K_{2^A}(x+t)|\, d\mu(t) \\ &\leq \frac{C}{n} \sum_{A=0}^{|n|} 2^A \sum_{s=0} 2^s \int_{I_A(x+e_s)} |f(t) - f(x)|\, d\mu(t) \\ &\leq \frac{C}{n} \sum_{A=0}^{|n|} 2^A W_A f(x) \longrightarrow 0 \ \text{ as } \ n \to \infty.\end{aligned}$$ □

As a corollary, we obtain that the following is true.

Corollary 9.1. *Let $f \in L_1(G)$. Then $\lim_{n\to\infty} \sigma_n f(x) = f(x)$ a.e. on G.*

Theorem 9.11. *Let $f \in L_1(G)$. Then $\sup_{y>0} y\mu\{\widetilde{S}^*_{\#} f > y\} \leq \|f\|_1$, where $\widetilde{S}^*_{\#}$ is defined by (3.1).*

Proof. Combining Lemma 5.1 and (8.1) in Lemma 8.1 we get $D_{2^n} \leq 2K_{2^n}$ for any $n \in \mathbb{N}$. Hence, $|S_{2^n} f(x)| \leq 2\sigma_{2^n} |f(x)|$ and $\widetilde{S}^*_{\#} f \leq 2\widetilde{\sigma}^*_{\#} |f| \leq 2\sigma^* |f|$.

Using Theorem 9.5, we find that

$$\sup_{y>0} y\mu\{\widetilde{S}^*_{\#} f > y\} \leq \sup_{y>0} y\mu\Big\{\sigma^* |f| > \frac{y}{2}\Big\} \leq \|f\|_1.$$ □

Corollary 9.2. *Let $f \in L_1(G)$. Then $\lim_{n\to\infty} S_{2^n} f(x) = f(x)$ a.e. on G.*

Corollary 9.3. *Let $f \in L_1(G)$. Then $\lim_{n\to\infty} S_{2^n} f(x) = f(x)$ if and only if $x \in G$ is the Lebesgue point of f.*

Proof. This fact is a direct consequence of the following equalities:

$$S_{2^n}|f-f(x)| = \int_G |f(x+t)-f(x)|D_{2^n}(t)\,d\mu(t) = 2^n \int_{I_n} |f(x+t)-f(x)|\,d\mu(t)$$
$$= 2^n \int_{I_n(x)} |f(t)-f(x)|\,d\mu(t) = \frac{1}{|I_n(x)|}\int_{I_n(x)} |f(t)-f(x)|\,d\mu(t)$$

and

$$\lim_{n\to\infty} S_{2^n}|f-f(x)| = \lim_{n\to\infty} \frac{1}{|I_n(x)|}\int_{I_n(x)} |f(t)-f(x)|\,d\mu(t) = 0. \qquad \square$$

Corollary 9.4. *Let $f\in L_1(G)$. Then $\lim_{n\to\infty} S_{2^n}f(x) = f(x)$ for all Lebesgue points of f.*

Next, we prove the sharpness of Corollary 9.4 in a special sense.

Theorem 9.12. *Let $1\le p<\infty$, G be a dyadic group and $\{S_{n_k}\}$ be any subsequence of partial sums with respect to the Walsh system. If $\mathbb{E}$ is a subset of G of measure zero, then there exists a concrete $f\in L_p(G)$ such that $S_{n_k}f$ diverges on $\mathbb{E}$ as $k\to\infty$.*

Proposition 9.1. *Suppose that $\mathbb{E}$ is a subset of G such that $\mu(\mathbb{E})=0$. Cover $\mathbb{E}$ with intervals $(\mathbb{I}_i, i\in\mathbb{N})$ in a special way:*

Each point $x\in E$ belongs to an infinite number of $\mathbb{I}_k$'s and $\sum_{i=0}^{\infty}\mu(I_i)<1$. Set $n_0:=0$ and choose a sequence $\{n_i\}$ of an increasing sequence of integers so that $\sum_{k=n_i}^{\infty}\mu(\mathbb{I}_i)<M_i^{-1}$, $i\in\mathbb{N}$. Then, for the sets $\mathbb{A}_i:=\bigcup_{k=n_i}^{n_{i+1}-1}\mathbb{I}_k$, $i\in\mathbb{N}$, there exist a sequence of increasing integers $\{\alpha_i\}$ and a sequence of Walsh polynomials $\{P_i, i\in\mathbb{N}\}$ such that $sp(P_i):=\{i\in\mathbb{N}:\ \widehat{P}_i\neq 0\}\subset[2^{\alpha_i},2^{\alpha_{i+1}})$ that is,

$$P_i = \sum_{k=2^{\alpha_i}}^{2^{\alpha_{i+1}}-1} c_k^i w_k,$$
$$\|P_i\|_p^p = \mu(\mathbb{A}_i) \le 2^{-i} \tag{9.7}$$

and

$$|P_i(x)| = 1 \quad x\in\mathbb{A}_i \ \text{ for } \ i\in\mathbb{N}. \tag{9.8}$$

Proof. Let S_{n_k} be any subsequence of partial sums with respect to the Walsh system. Suppose that $\{P_i, i\in\mathbb{N}\}$ are defined as in Proposition 9.1. Then we can choose increasing integers $\{\beta_i\}$ such that $\alpha_i<\beta_i$, $i\in\mathbb{N}$ and $\{\xi_k, k\in\mathbb{N}\}\subset\{n_k, k\in\mathbb{N}\}$ such that

$$2^{\beta_i+1}<\xi_i<2^{3\beta_i}<(2^{\alpha_i}+2^{\beta_i})^3<2^{\alpha_i}+2^{\beta_{i+1}}<2^{\alpha_{i+1}}+2^{\beta_{i+1}}<2^{\beta_{i+1}+1}<\xi_{i+1}. \tag{9.9}$$

Setting

$$f:=\sum_{i=1}^{\infty} w_{2^{\beta_{i+1}}}P_i, \tag{9.10}$$

by applying (9.7), we can conclude that $f\in L_p(G)$ and (9.10) is the Walsh–Fourier series of f. Moreover, by the equality $w_{i+2^k}=w_{2^k}w_i$ for any $i<2^k$, we find that the spectra of the polynomials P_i are disjoint pairwise and

$$\widehat{f}(k)=\begin{cases} c_k^l, & \text{if } \ k=2^{\beta_{l+1}}+2^{\alpha_l},\ldots,2^{\beta_{l+1}}+2^{\alpha_{l+1}}-1, \ \ l\in\mathbb{N}_+,\\ 0, & \text{otherwise.}\end{cases}$$

Since $2^{\beta_i}+2^{\alpha_i}<2^{\beta_i+1}<\xi_i<2^{\alpha_i}+2^{\beta_{i+1}}<2^{\alpha_{i+1}}+2^{\beta_{i+1}}<2^{\beta_{i+1}+1}<\xi_{i+1}<2^{3\beta_{i+1}}<2^{\alpha_{i+1}}+2^{\beta_{i+2}}$ for any $i\in\mathbb{N}_+$, we have

$$S_{\xi_{i+1}}f(x)-S_{\xi_i}f(x)=S_{2^{\alpha_{i+1}}+2^{\beta_{i+1}}}f(x)-S_{2^{\alpha_i}+2^{\beta_{i+1}}}f(x)=w_{2^{\beta_{i+1}}}P_i(x).$$

Next, we use Proposition 9.1 to conclude that every $x\in\mathbb{E}$ belongs to an infinite number of sets $\mathbb{A}_i$. Moreover, Applying (9.8) we find that, for infinitely many i, $|S_{\xi_{i+1}}f(x)-S_{\xi_i}f(x)|=1$ for $x\in\mathbb{A}_i$ and $\limsup_{i\in\mathbb{N}}|S_{n_{i+1}}f(x)-S_{n_i}f(x)|=1$ for any $x\in\mathbb{E}$. □

Corollary 9.5. *If $f\in L_1(G)$, then $S_{2^k}f\to f$ a.e. on G as $k\to\infty$. Moreover, for any $1\le p<\infty$ and a set $\mathbb{E}\subset G$ of measure zero, there exists $f\in L_p(G)$ such that the subsequence $S_{2^k}f$ of partial sums with respect to the Walsh system diverges on $\mathbb{E}$.*

Corollary 9.6. *If $E\subseteq G$ is a set of measure zero, then there exists a function $f\in L_1(G)$ such that each $x\in E$ is not a Lebesgue point.*

We will also prove a similar result concerning the Walsh–Fejér means.

Theorem 9.13. *Let $1\le p<\infty$ and G be a dyadic group. If the subset $\mathbb{E}$ of G is of measure zero, then there exists a concrete $f\in L_p(G)$ such that its Walsh–Fejér means diverge on $\mathbb{E}$.*

Proof. Let f be defined by (9.10). Then, for $i\in\mathbb{N}_+$, we have

$$\begin{aligned}\sigma_{(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^3}f(x)-\sigma_{2^{\alpha_i}+2^{\beta_{i+1}}}f(x)&=w_{2^{\beta_{i+1}}}P_i(x)-\sum_{k=2^{\alpha_i}+2^{\beta_{i+1}}}^{2^{\alpha_{i+1}}+2^{\beta_{i+1}}-1}\frac{k}{(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^3}c_k^iw_k(x)\\&+\sum_{l=0}^{i-1}\sum_{k=2^{\alpha_l}+2^{\beta_{l+1}}}^{2^{\alpha_{l+1}}+2^{\beta_{l+1}}-1}\Big(\frac{k}{2^{\alpha_i}+2^{\beta_{i+1}}}-\frac{k}{(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^3}\Big)c_k^lw_k(x):=I-II+III.\end{aligned}$$

Since $|c_k^i|<C$ for any $i\in\mathbb{N}$, according to (9.9), for II we find that

$$\begin{aligned}|II|&\le C\sum_{k=0}^{2^{\alpha_{i+1}}+2^{\beta_{i+1}}-1}\frac{k}{(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^3}\\&\le\frac{C(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^2}{(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^3}\le\frac{C}{2^{\alpha_{i+1}}+2^{\beta_{i+1}}}\longrightarrow0\ \text{ as }\ i\to\infty.\end{aligned}\tag{9.11}$$

Analogously, applying again (9.9), we get

$$\begin{aligned}|III|&\le\sum_{l=0}^{i-1}\sum_{k=2^{\alpha_l}+2^{\beta_{l+1}}}^{2^{\alpha_{l+1}}+2^{\beta_{l+1}}}\Big(\frac{k}{2^{\alpha_i}+2^{\beta_{i+1}}}-\frac{k}{(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^3}\Big)|c_k^l|\le\sum_{k=0}^{2^{\alpha_i}+2^{\beta_i}}\frac{|c_k^l|k}{2^{\alpha_i}+2^{\beta_{i+1}}}\\&\le\frac{C}{2^{\alpha_i}+2^{\beta_{i+1}}}\sum_{k=0}^{2^{\alpha_i}+2^{\beta_i}}k\le\frac{C(2^{\alpha_i}+2^{\beta_i})^2}{2^{\alpha_i}+2^{\beta_{i+1}}}\le\frac{C}{2^{\alpha_i}+2^{\beta_i}}\longrightarrow0\ \text{ as }\ i\to\infty.\end{aligned}\tag{9.12}$$

According to the fact that each $x\in\mathbb{E}$ belongs to an infinite number of A_i, applying (9.8), we find that

$$|I|=1\ \text{ for }\ x\in\mathbb{I}_i,\ \text{ for infinitely many } i.\tag{9.13}$$

Combining (9.11), (9.12) and (9.13), for sufficiently large and infinitely many i, we have

$$\left|\sigma_{(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^3}f(x)-\sigma_{2^{\alpha_i}+2^{\beta_{i+1}}}f(x)\right|\ge|I|-|II|-|III|\ge\frac{1}{2}\,,$$

and for any $x\in\mathbb{E}$, we get

$$\limsup_{i\in\mathbb{N}}\left|\sigma_{(2^{\alpha_{i+1}}+2^{\beta_{i+1}})^3}f(x)-\sigma_{2^{\alpha_i}+2^{\beta_{i+1}}}f(x)\right|\ge\frac{1}{2}\,.$$

Hence, the Walsh–Fejér means of the function f diverge for each $x\in\mathbb{E}$. □

Corollary 9.7. *If $E \subseteq G$ is a set of measure zero, then there exists a function $f \in L_1(G)$ such that each $x \in E$ is not a Walsh–Lebesgue point.*

10 Introduction to martingale Hardy spaces

The σ-algebra generated by the intervals $I_n(x)$ of measure 2^{-n} is denoted by $\digamma_n (n \in \mathbb{N})$. The conditional exponential operator with respect to $\digamma_n (n \in \mathbb{N})$ is denoted by E_n and given by

$$E_n f(x) = S_{2^n} f(x) = \sum_{k=0}^{2^n-1} \widehat{f}(k) w_k(x) = \frac{1}{|I_n(x)|} \int\limits_{I_n(x)} f(x)\, d\mu(x),$$

where $|I_n(x)| = 2^{-n}$ denotes the length of the set $I_n(x)$.

The sequence $f = (f_n, n \in \mathbb{N})$ of functions $f_n \in L_1(G)$ is called a dyadic martingale (see [79,83]) if

(i) f_n is measurable with respect to the $\sigma-$ algebras $\digamma_n$ for any $n \in \mathbb{N}$;

(ii) $E_n f_m = f_n$ for any $n \leq m$.

The maximal function of a martingale f is defined by $f^* = \sup\limits_{n \in \mathbb{N}} |f_n|$.

In the case $f \in L_1(G)$, the maximal functions are also given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \Big| \int\limits_{I_n(x)} f(u)\, d\mu(u) \Big|.$$

For $0 < p < \infty$, the Hardy martingale space $H_p(G)$ consists of all martingales, for which $\|f\|_{H_p(G)} := \|f^*\|_p < \infty$.

It is easy to show that for the martingale $f = (f_n, n \in \mathbb{N})$ and for any $k \in \mathbb{N}$ there exists a limit

$$\widehat{f}(k) := \lim_{n \to \infty} \int\limits_G f_n(x) w_k(x)\, d\mu(x),$$

which is called k-th Walsh–Fourier coefficients of f.

If $f_0 \in L_1(G)$ and $f := (E_n f_0 : n \in \mathbb{N})$ is a regular martingale, then

$$\widehat{f}(k) = \int\limits_G f_0(x) w_k(x)\, d\mu(x) = \widehat{f}_0(k), \quad k \in \mathbb{N}.$$

A bounded measurable function a is said to be a p-atom if there exists a dyadic interval I such that $\int\limits_I a\, d\mu = 0$, $\|a\|_\infty \leq \mu(I)^{-1/p}$, $\operatorname{supp}(a) \subset I$.

The Hardy martingale space $H_p(G)$ for any $0 < p \leq 1$ can be characterized by simple functions which are called p-atoms. The following is true (for details see [128,130]).

Lemma 10.1. *A martingale $f = (f_n, n \in \mathbb{N})$ belongs to $H_p(G)$, $0 < p \leq 1$ if and only if there exists a sequence of p-atoms of $(a_k, k \in \mathbb{N})$ and a sequence of real numbers $(\mu_k, k \in \mathbb{N})$ such that for all $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f_n \tag{10.1}$$

and

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{10.2}$$

Moreover,

$$\|f\|_{H_p(G)} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (10.1).

By applying Lemma 10.1 we easily obtain that the following is true (see [130]).

Lemma 10.2. *Let $0<p\leq 1$ and T be σ-sub-linear operator such that $\int\limits_{\overline{I}}|Ta(x)|^p\,d\mu(x)\leq c_p<\infty$ for every p-atom a, where I denotes support of the atom a. In addition, if T is bounded from $L_\infty(G)$ to $L_\infty(G)$, then $\|Tf\|_p\leq c_p\|f\|_{H_p(G)}$.*

Lemma 10.3. *Suppose that an operator T is σ-sublinear and $\sup\limits_{t>0}t^p\mu\{x\in\overline{I}:\ |Ta(x)|>t\}\leq C_p<\infty$ for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then $\|Tf\|_{weak\text{-}L_p}\leq c_p\|f\|_{H_p}$.*

In the concrete cases, the norm of Hardy martingale spaces can be calculated by simpler formulas (see [128,131]).

Lemma 10.4. *If $g\in L_1(G)$ and $f:=(S_{2^n}g:n\in\mathbb{N})$ is a regular martingale, then for $0<p\leq 1$, the $H_p(G)$ norm can be calculated by $\|f\|_{H_p(G)}=\|\sup\limits_{n\in\mathbb{N}}|S_{2^n}g|\|_p$.*

The modulus of continuity in martingale Hardy spaces quantifies how "smooth" or "regular" a martingale is across scales (or filtrations). It is crucial for understanding approximation, interpolation, and embedding properties of these spaces. The definition adapts to the probabilistic, dyadic or filtration structure, and is fundamental in modern harmonic analysis and stochastic processes.

The modulus of continuity in the $H_p(G)$ spaces is defined by

$$\omega_{H_p(G)}\Big(\frac{1}{2^n},f\Big):=\|f-S_{2^n}f\|_{H_p(G)}.$$

It is important to describe how the difference $f-S_{2^n}f$ can be understood, where f is a martingale and $S_{2^n}f$ is a function.

Remark 10.1. Let $0<p\leq 1$. Since $S_{2^n}f=f_n\in L_1(G)$, where $f=(f_n:\ n\in\mathbb{N})\in H_p(G)$, and

$$\begin{aligned}(S_{2^k}f_n:\ k\in\mathbb{N})&=(S_{2^k}S_{2^n}f,k\in\mathbb{N})\\&=(S_{2^0}f,\dots,S_{2^{n-1}}f,S_{2^n}f,S_{2^n}f,\dots)=(f_0,\dots,f_{n-1},f_n,f_n,\dots),\end{aligned}$$

by the difference $f-S_{2^n}f$ we mean the following martingale: $f:=((f-S_{2^n}f)_k,k\in\mathbb{N})$, where $(f-S_{2^n}f)_k=\begin{cases}0, & k=0,\dots,n,\\ f_k-f_n, & k\geq n+1.\end{cases}$

Consequently, the norm $\|f-S_{2^n}f\|_{H_p(G)}$ is understood as an H_p-norm of

$$f-S_{2^n}f=((F-S_{2^n}f)_k,k\in\mathbb{N}).$$

It is known (see [83]) that there are strong connections between $\omega_p(1/2^n,f)$, $E_{2^n}(L_p,f)$ and $\|f-S_{2^n}f\|_p$, $p\geq 1$, $n\in\mathbb{N}$.

In particular,

$$\frac{1}{2}\,\omega_p\Big(\frac{1}{2^n},f\Big)\leq\|f-S_{2^n}f\|_p\leq\omega_p\Big(\frac{1}{2^n},f\Big)$$

and

$$\frac{1}{2}\,\|f-S_{2^n}f\|_p\leq E_{2^n}(L_p,f)\leq\|f-S_{2^n}f\|_p\,.$$

11 Important examples of p-atoms and martingales of H_p spaces

Example 11.1. Let $0<p\leq 1$ and $(\alpha_k,k\in\mathbb{N})$ be an increasing sequence of integers. Then the function $a_k=:2^{\alpha_k(1/p-1)}(D_{2^{\alpha_k+1}}-D_{2^{\alpha_k}})$ is a p-atom and

$$\|a_k\|_{H_p}\leq 1. \tag{11.1}$$

Moreover,

$$\widehat{a}_k(i) = \begin{cases} 2^{\alpha_k(1/p-1)}, & i = 2^{\alpha_k}, \dots, 2^{\alpha_k+1}-1, \\ 0, & \text{otherwise}, \end{cases} \tag{11.2}$$

$$S_i a_k = \begin{cases} 2^{\alpha_k(1/p-1)}(D_i - D_{2^{\alpha_k}}), & i = 2^{\alpha_k}+1, \dots, 2^{\alpha_k+1}-1, \\ a_k, & i \geq 2^{\alpha_k+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{11.3}$$

and

$$|\sigma_n a_k| = \frac{(n-2^{\alpha_k}) \cdot 2^{\alpha_k(1/p-1)}}{n} |K_{n-2^{\alpha_k}}| \text{ for any } 2^{\alpha_k} < n < 2^{\alpha_k+1}. \tag{11.4}$$

Proof. Since $\operatorname{supp}(a_k) = I_{\alpha_k}$, $\int_{I_{\alpha_k}} a_k \, d\mu = 0$ and

$$\|a_k\|_\infty \leq 2^{\alpha_k(1/p-1)} \cdot 2^{\alpha_k} \leq 2^{\alpha_k/p} = (\mu(\operatorname{supp} a_k))^{-1/p},$$

we conclude that a_k is a p-atom and (11.1) follows from Theorem 10.1.

On the other hand, if we apply Lemma 5.3 and Lemma 10.4, we find that

$$\begin{aligned} \|a_k\|_{H_p} &= \Big\| \sup_{n \in \mathbb{N}} S_{2^n} a_k \Big\|_{L_p} = 2^{\alpha_k(1/p-1)} \|D_{2^{\alpha_k+1}} - D_{2^{\alpha_k}}\|_p = 2^{\alpha_k(1/p-1)} \|D_{2^{\alpha_k}}\|_p \\ &= 2^{\alpha_k(1/p-1)} \Big(\int_{I_{\alpha_k}} D^p_{2^{\alpha_k}} \, d\mu \Big)^{1/p} = 2^{\alpha_k(1/p-1)} \Big(\int_{I_{\alpha_k}} 2^{p\alpha_k} \, d\mu \Big)^{1/p} \leq 2^{\alpha_k(1-1/p)}, \end{aligned} \tag{11.5}$$

and also (11.1) is proved.

Let $2^{\alpha_k} \leq i \leq 2^{\alpha_k+1}-1$. Then from the orthogonality of the Walsh functions, we immediately obtain

$$\int_G a_k w_i \, d\mu = 2^{\alpha_k(1/p-1)} \int_G \sum_{j=2^{\alpha_k}}^{2^{\alpha_k+1}-1} w_j w_i \, d\mu = 2^{\alpha_k(1/p-1)} \int_G w_i w_i \, d\mu = 2^{\alpha_k(1/p-1)}.$$

Analogously, again using the orthogonality, we get $\int_G a_k w_i \, d\mu = 0$ and hence (11.2) is proved.

Equality (11.3) follows also from the definition of partial sums of Walsh–Fourier series.

Let $2^{\alpha_k} \leq i \leq 2^{\alpha_k+1}-1$. Combining Lemma 5.3 and (11.3), we find that

$$\begin{aligned} |\sigma_n a_k| &= \frac{1}{n} \Big| \sum_{j=1}^{n} S_j a_k \Big| = \frac{1}{n} \Big| \sum_{j=2^{\alpha_k}+1}^{n} S_j a_k \Big| \\ &= \frac{1}{n} \Big| \sum_{j=2^{\alpha_k}+1}^{n} 2^{\alpha_k(1/p-1)} (D_j - D_{2^{\alpha_k}}) \Big| = \frac{1}{n} \Big| \sum_{j=1}^{n-2^{\alpha_k}} 2^{\alpha_k(1/p-1)} (D_{j+2^{\alpha_k}} - D_{2^{\alpha_k}}) \Big| \\ &= \frac{2^{\alpha_k(1/p-1)}}{n} \Big| \sum_{j=1}^{n-2^{\alpha_k}} D_j \Big| = \frac{2^{\alpha_k(1/p-1)}}{n} (n-2^{\alpha_k}) |K_{n-2^{\alpha_k}}|. \end{aligned}$$

□

Example 11.2. Let $0 < p \leq 1$, $(n_k, k \in \mathbb{N})$ be an increasing sequence of integers and $a_k = D_{2^{2n_k+1}} - D_{2^{2n_k}}$ and

$$\|a_k\|_{H_p} \leq 2^{2n_k(1-1/p)}. \tag{11.6}$$

Moreover,

$$\widehat{a}_k(i) = \begin{cases} 1, & i = 2^{2n_k}, \dots, 2^{2n_k+1}-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$S_i a_k = \begin{cases} D_i - D_{2^{2n_k}}, & i = 2^{2n_k}+1, \dots, 2^{2n_k+1}-1, \\ a_k, & i \geq 2^{2n_k+1}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$|\sigma_n a_k| = \frac{n - 2^{2n_k}}{n} |K_{n-2^{2n_k}}| \text{ for any } 2^{2n_k} \leq n \leq 2^{2n_k+1} - 1. \tag{11.7}$$

Proof. The proof follows from Example 11.1 and the fact that the norm, Fourier coefficients, partial sums and Fejér means are linear operators. So, we leave out the details. □

The next two estimates will be used frequently.

Proposition 11.1. *Let $2^k \leq n < 2^{k+1}$, $f \in H_p$ for some fixed $0 < p < \infty$ and $n \in \mathbb{N}$. If $S_n f$ denotes the n-th partial sum with respect to the Walsh system, then the following estimate holds:*

$$\|S_n f\|_{H_p} \leq C_p \|\widetilde{S}^*_{\#} f\|_p + C_p \|S_n f\|_p.$$

where $\widetilde{S}^*_{\#} f$ is defined in (3.1).

Proof. The martingale $(S_n f)$ can be written as $(S_{2^l} S_n f, l \geq 1) = (S_{2^0} f, \ldots, S_{2^k} f, S_n f, \ldots, S_n f, \ldots)$. It immediately follows that

$$\|S_n f\|_{H_p} \leq C_p \Big\| \sup_{0 \leq l \leq k} |S_{2^l} f| \Big\|_p + C_p \|S_n f\|_p \leq C_p \|\widetilde{S}^*_{\#} f\|_p + C_p \|S_n f\|_p. \qquad \square$$

Proposition 11.2. *Let $0 < p \leq 1$, $2^k \leq n < 2^{k+1}$ and $\sigma_n f$ be the Fejér means with respect to the one-dimensional Walsh–Fourier series, where $f \in H_p(G)$. Then, for any fixed $n \in \mathbb{N}$,*

$$\|\sigma_n f\|_{H_p} \leq \Big\| \sup_{0 \leq l \leq k} |\sigma_{2^l} f| \Big\|_p + \Big\| \sup_{0 \leq l \leq k} |S_{2^l} f| \Big\|_p + \|\sigma_n f\|_p \leq \|\widetilde{\sigma}^*_{\#} f\|_p + \|\widetilde{S}^*_{\#} f\|_p + \|\sigma_n f\|_p.$$

Proof. Let us consider the following martingale:

$$\begin{aligned} f_{\#} &= (S_{2^k} \sigma_n f, k \in \mathbb{N}) \\ &= \Big(\frac{2^0 \sigma_{2^0}}{n} + \frac{(n - 2^0) S_{2^0} f}{n}, \ldots, \frac{2^k \sigma_{2^k} f}{n} + \frac{(n - 2^k) S_{2^k} f}{n}, \sigma_n f, \ldots, \sigma_n f, \ldots \Big). \end{aligned}$$

Using Lemma 10.4, we immediately get

$$\|\sigma_n f\|^p_{H_p(G)} \leq \Big\| \sup_{0 \leq l \leq k} |\sigma_{2^l} f| \Big\|^p_p + \Big\| \sup_{0 \leq l \leq k} |S_{2^l} f| \Big\|^p_p + \|S_n f\|^p_p \leq \|\widetilde{\sigma}^*_{\#} f\|^p_p + \|\widetilde{S}^*_{\#} f\|^p_p + \|\sigma_n f\|^p_p. \qquad \square$$

The next five examples of martingales will be used many times to prove the sharpness of our main results. Such counterexamples first appeared in the papers of Goginava [44] (see also [45, 46]). Analogous constructions of martingales are also used in the papers [17, 18, 20, 22, 52, 53, 59, 61, 62, 66, 103, 105, 110, 115].

Example 11.3. Let $0 < p \leq 1$, $\{\lambda_k : k \in \mathbb{N}\}$ be a sequence of real numbers

$$\sum_{k=0}^{\infty} |\lambda_k|^p \leq c_p < \infty \tag{11.8}$$

and let $\{a_k : k \in \mathbb{N}\}$ be a sequence of p-atoms given by

$$a_k(x) := 2^{|\alpha_k|(1/p-1)} \big(D_{2^{|\alpha_k|+1}}(x) - D_{2^{|\alpha_k|}}(x) \big), \tag{11.9}$$

where $|\alpha_k| := \max\{j \in \mathbb{N} : (\alpha_k)_j \neq 0\}$ and $(\alpha_k)_j$ denotes the j-th binary coefficients of real number of $\alpha_k \in \mathbb{N}_+$. Then $f = (f_n : n \in \mathbb{N})$, where $f_n(x) := \sum_{\{k: \, |\alpha_k| < n\}} \lambda_k a_k(x)$ is a martingale, that belongs to $H_p(G)$ for any $0 < p \leq 1$.

It is easy to show that

$$\widehat{f}(j) = \begin{cases} \lambda_k \cdot 2^{(1/p-1)|\alpha_k|}, & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \quad k \in \mathbb{N}_+, \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}. \end{cases} \tag{11.10}$$

Let $2^{|\alpha_{l-1}|+1} \le j \le 2^{|\alpha_l|}$, $l \in \mathbb{N}_+$. Then

$$S_j f = S_{2^{|\alpha_{l-1}|+1}} = \sum_{\eta=0}^{l-1} \lambda_\eta \cdot 2^{|\alpha_\eta|(1/p-1)} (D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}). \tag{11.11}$$

Let $2^{|\alpha_l|} \le j < 2^{|\alpha_l|+1}$, $l \in \mathbb{N}_+$. Then

$$\begin{aligned} S_j f &= S_{2^{|\alpha_l|}} + \lambda_l \cdot 2^{(1/p-1)|\alpha_l|} w_{2^{|\alpha_l|}} D_{j-2^{|\alpha_l|}} \\ &= \sum_{\eta=0}^{l-1} \lambda_\eta \cdot 2^{(1/p-1)|\alpha_\eta|} (D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}) + \lambda_l \cdot 2^{(1/p-1)|\alpha_l|} w_{2^{|\alpha_l|}} D_{j-2^{|\alpha_l|}}. \end{aligned} \tag{11.12}$$

Moreover, for the modulus of continuity for $0 < p \le 1$, we have the following estimation:

$$\omega_{H_p}\Big(\frac{1}{2^n}, f\Big) = O\Big(\sum_{\{k:\ |\alpha_k| \ge n\}}^{\infty} |\lambda_k|^p\Big)^{1/p} \quad \text{as} \quad n \to \infty. \tag{11.13}$$

Proof. Since $S_{2^n} a_k = \begin{cases} a_k, & \alpha_k < n, \\ 0, & \alpha_k \ge n \end{cases}$ applying Theorem 10.1 and (11.8), we can conclude that $f \in H_p$.

Let $j \notin \bigcup_{k=1}^{\infty} \{2^{2\alpha_k}, \dots, 2^{2\alpha_k+1} - 1\}$. Using the orthonormality of the Walsh functions, we find that

$$\widehat{a_k}(j) = \int_G 2^{2\alpha_k(1/p-1)} (D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}) w_i \, d\mu = 0$$

and

$$\widehat{f}(j) = \lim_{n\to\infty} \int_G \sum_{\{k:\ 2\alpha_k < n\}} \lambda_k a_k w_i \, d\mu = \lim_{n\to\infty} \sum_{\{k:\ 2\alpha_k < n\}} \lambda_k \int_G a_k w_i \, d\mu = 0. \tag{11.14}$$

Let $j \in \{2^{2\alpha_k}, \dots, 2^{2\alpha_k+1} - 1\}$ for some $k \in \mathbb{N}$. Then, for every $l \in \mathbb{N}$, we can conclude that

$$\widehat{a}_l(j) = 2^{2\alpha_l(1/p-1)} \int_G (D_{2^{2\alpha_l+1}} - D_{2^{2\alpha_l}}) w_j \, d\mu = 2^{2\alpha_l(1/p-1)} \delta_{l,k},$$

where $\delta_{l,k}$ is the Kronecker symbol. It follows that

$$\widehat{f}(j) = \lim_{n\to\infty} \sum_{\{l:\ 2\alpha_k < n\}} \lambda_l \int_G a_l w_j \, d\mu = \lim_{n\to\infty} \sum_{\{l:\ 2\alpha_l < n\}} \lambda_l \cdot 2^{2\alpha_l(1/p-1)} \delta_{l,k} = \lambda_k \cdot 2^{2\alpha_k(1/p-1)}. \tag{11.15}$$

Combining (11.14) and (11.15) we obtain (11.10).

Let $2^{2\alpha_{l-1}+1} + 1 \le j \le 2^{2\alpha_l}$. In view of (11.10), we can conclude that

$$\begin{aligned} S_j f = S_{2^{2\alpha_{l-1}+1}} f &= \sum_{v=0}^{2^{2\alpha_{l-1}+1}-1} \widehat{f}(v) w_v \\ &= \sum_{\eta=0}^{l-1} \sum_{v=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1} \lambda_\eta \cdot 2^{2\alpha_\eta(1/p-1)} w_v = \sum_{\eta=0}^{l-1} \lambda_\eta \cdot 2^{2\alpha_\eta(1/p-1)} (D_{2^{2\alpha_\eta+1}} - D_{2^{2\alpha_\eta}}). \end{aligned}$$

If $2^{2\alpha_l} \le j < 2^{2\alpha_l+1}$, then Lemma 5.3 implies

$$\begin{aligned} S_j f &= S_{2^{2\alpha_l}} f + \sum_{v=2^{2\alpha_l}}^{j-1} \widehat{f}(v) w_v = S_{2^{2\alpha_l}} f + \lambda_l \cdot 2^{2\alpha_l(1/p-1)} \sum_{v=2^{2\alpha_l}}^{j-1} w_v \\ &= S_{2^{2\alpha_l}} f + \lambda_l \cdot 2^{2\alpha_l(1/p-1)} (D_j - D_{2^{2\alpha_l}}) = S_{2^{2\alpha_l}} f + \lambda_l \cdot 2^{2\alpha_l(1/p-1)} w_{2^{2\alpha_l}} D_{j-2^{2\alpha_l}} \end{aligned} \tag{11.16}$$

and also (11.11) is proved.

Taking into account (11.11), for $j=2^{2\alpha_l}$, we can rewrite (11.16) as

$$S_jf=\sum_{\eta=0}^{l-1}\lambda_\eta\cdot 2^{2\alpha_\eta(1/p-1)}(D_{2^{2\alpha_\eta+1}}-D_{2^{2\alpha_\eta}})+\lambda_l\cdot 2^{2\alpha_l(1/p-1)}(D_j-D_{2^{2\alpha_l}}),$$

which proves (11.12).

In view of Remark 10.1, we immediately get

$$(f-S_{2^n}f)_k=\begin{cases}0, & k=0,\dots,n\\ f_k-f_n, & k\geq n+1\end{cases}=\begin{cases}0, & k=0,\dots,n,\\ \sum\limits_{l=n+1}^{k}\mu_lS_{2^n}a_l, & k\geq n+1.\end{cases} \tag{11.17}$$

According to Theorem 10.1, we can conclude that (11.17) is an atomic decomposition of the martingale $f-S_{2^n}f\in H_p$ and

$$\omega_{H_p}\Big(\frac{1}{2^n},f\Big)=O\Big(\sum_{k=n+1}^{\infty}|\mu_k|^p\Big)^{1/p}<\infty,$$

so that (11.22) holds. □

Sometimes, we will also use the martingale defined in the next example.

Example 11.4. Let $0<p\leq 1$, $(\alpha_k,k\in\mathbb{N})$ be an increasing sequence of positive integers, let $(\lambda_k,k\in\mathbb{N})$ be a sequence of real numbers such that

$$\sum_{k=0}^{\infty}|\lambda_k|^p\leq C_p<\infty \tag{11.18}$$

and $(a_k,k\in\mathbb{N})$ be a sequence of p-atoms defined by $a_k=:2^{|\alpha_k|(1/p-1)}(D_{2^{|\alpha_k|+1}}-D_{2^{|\alpha_k|}})$, where $|\alpha_k|=:\max\{j\in\mathbb{N},(\alpha_k)_j\neq 0\}$ and $(\alpha_k)_j$ denotes the j-th binary coefficient of $\alpha_k\in\mathbb{N}_+$. Then $f=(f_n,n\in\mathbb{N})\in H_p$, where $f_n=:\sum\limits_{\{k:\,|\alpha_k|<n\}}\lambda_ka_k$ and

$$\widehat{f}(j)=\begin{cases}\lambda_k\cdot 2^{|\alpha_k|(1/p-1)}, & j\in\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\ \ k\in\mathbb{N}_+,\\ 0, & j\notin\bigcup\limits_{k=1}^{\infty}\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\}.\end{cases} \tag{11.19}$$

If $2^{|\alpha_{l-1}|+1}\leq j\leq 2^{|\alpha_l|}$, $l\in\mathbb{N}_+$, then

$$S_jf=S_{2^{|\alpha_{l-1}|+1}}=\sum_{\eta=0}^{l-1}\lambda_\eta\cdot 2^{|\alpha_\eta|(1/p-1)}(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|}}). \tag{11.20}$$

If $2^{|\alpha_l|}\leq j<2^{|\alpha_l|+1}$, $l\in\mathbb{N}_+$, then

$$\begin{aligned}S_jf&=S_{2^{|\alpha_l|}}+\lambda_l\cdot 2^{|\alpha_l|(1/p-1)}w_{2^{|\alpha_l|}}D_{j-2^{|\alpha_l|}}\\ &=\sum_{\eta=0}^{l-1}\lambda_\eta\cdot 2^{|\alpha_\eta|(1/p-1)}(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|}})+\lambda_l\cdot 2^{|\alpha_l|(1/p-1)}w_{2^{|\alpha_l|}}D_{j-2^{|\alpha_l|}}.\end{aligned} \tag{11.21}$$

Moreover, for all $0<p\leq 1$,

$$\omega_{H_p}\Big(\frac{1}{2^n},f\Big)=O\Big(\sum_{\{k:\,|\alpha_k|\geq n\}}^{\infty}|\lambda_k|^p\Big)^{1/p}\quad\text{as}\ \ n\to\infty. \tag{11.22}$$

Proof. The proof is quite analogous to that of Example 11.3, so we omit the details. □

For our further investigations it is convenient to consider a fixed sequence $(\lambda_k, k \in \mathbb{N})$. In many cases, the following sequence may be such an appropriate choice: $(1/\alpha_k^{1/2}, k \in \mathbb{N})$.

Example 11.5. Let $0 < p \leq q \leq 1$ and $(\alpha_k, k \in \mathbb{N})$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{1}{\alpha_k^{p/2}} < \infty, \tag{11.23}$$

$$\sum_{\eta=0}^{k-1} \frac{2^{2\alpha_\eta/p}}{\alpha_\eta^{1/2}} < \frac{2^{2\alpha_k/p}}{\alpha_k^{1/2}} \tag{11.24}$$

and

$$\frac{2^{2\alpha_{k-1}/p+5}}{\alpha_{k-1}^{1/2}} < \begin{cases} \dfrac{2^{\alpha_k/q}}{\alpha_k^{3/2}}, & p = q \\ \dfrac{2^{\alpha_k(1/p-1/q)}}{\alpha_k^{3/2}}, & p < q. \end{cases} \tag{11.25}$$

We define a martingale $f = (f_n, n \in \mathbb{N}) \in H_p$, where $f_n =: \sum_{\{k:\ 2\alpha_k < n\}} \lambda_k a_k$, $\lambda_k = \frac{1}{\alpha_k^{1/2}}$, and $a_k =: 2^{2\alpha_k(1/p-1)}(D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}})$. Then $f \in H_p$ and

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{2\alpha_k(1/p-1)}}{\alpha_k^{1/2}}, & j \in \{2^{2\alpha_k}, \ldots, 2^{2\alpha_k+1} - 1\}, \quad k \in \mathbb{N}_+, \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{2^{2\alpha_k}, \ldots, 2^{2\alpha_k+1} - 1\}. \end{cases} \tag{11.26}$$

For every $l \in \mathbb{N}$, we have the following identities

$$S_j f = \begin{cases} S_{2^{2\alpha_{l-1}+1}}, & 2^{2\alpha_{l-1}+1} \leq j \leq 2^{2\alpha_l}, \\ S_{2^{2\alpha_l}} + \dfrac{2^{2\alpha_l(1/p-1)} w_{2^{2\alpha_l}} D_{j-2^{2\alpha_l}}}{\alpha_l^{1/2}}, & 2^{2\alpha_l} \leq j < 2^{2\alpha_l+1}. \end{cases} \tag{11.27}$$

Moreover,

$$|S_j f| \leq \frac{2^{2\alpha_{(k-1)p}+1}}{\alpha_{k-1}^{1/2}} \quad \text{for all} \quad 0 \leq j \leq 2^{2\alpha_k} \quad \text{and} \quad 0 < p \leq q \leq 1 \tag{11.28}$$

and

$$|S_{2^{2\alpha_k+1}} f| \geq \frac{2^{2\alpha_k(1/p-1)}}{4\alpha_k^{1/2}}, \quad 0 < p < q < 1. \tag{11.29}$$

Proof. First, we note that such an increasing sequence $(\alpha_k, k \in \mathbb{N})$ satisfyng the conditions (11.23)–(11.25) can be constructed (see Remark 11.2).

Applying (11.23), we conclude that condition (10.2) is satisfied and $f \in H_p$ by Theorem 10.1. Since

$$\lambda_k \cdot 2^{2\alpha_k(1/p-1)} = \frac{2^{2\alpha_k(1/p-1)}}{\alpha_k^{1/2}},$$

according to (11.14) and (11.15), we find that the Fourier coefficients are given by equality (11.26).

Let $2^{2\alpha_{l-1}+1}+1\leq j\leq 2^{2\alpha_l}$, where $1\leq l\leq k$. In view of (11.26), we can conclude that

$$S_jf=S_{2^{2\alpha_{l-1}+1}}f=\sum_{v=0}^{2^{2\alpha_{l-1}+1}-1}\widehat{f}(v)w_v$$
$$=\sum_{\eta=0}^{l-1}\sum_{v=2^{2\alpha_\eta}}^{2^{2\alpha_\eta+1}-1}\frac{2^{2\alpha_\eta(1/p-1)}}{\alpha_\eta^{1/2}}w_v=\sum_{\eta=0}^{l-1}\frac{2^{2\alpha_\eta(1/p-1)}}{\alpha_\eta^{1/2}}(D_{2^{2\alpha_\eta+1}}-D_{2^{2\alpha_\eta}}). \quad (11.30)$$

Let $2^{2\alpha_l}\leq j<2^{2\alpha_l+1}$, where $1\leq l\leq k$. We apply (11.26) again and invoke Lemma 5.3 to obtain

$$S_jf=S_{2^{2\alpha_l}}f+\sum_{v=2^{2\alpha_l}}^{j-1}\widehat{f}(v)w_v=S_{2^{2\alpha_l}}f+\frac{2^{2\alpha_l}}{\alpha_l^{1/2}}\sum_{v=2^{2\alpha_l}}^{j-1}w_v$$
$$=S_{2^{2\alpha_l}}f+\frac{2^{2\alpha_l}}{\alpha_l^{1/2}}(D_j-D_{2^{2\alpha_l}})=S_{2^{2\alpha_l}}f+\frac{2^{2\alpha_l}w_{2\alpha_l}D_{j-2^{2\alpha_l}}}{\alpha_l^{1/2}}. \quad (11.31)$$

Combining (11.30) and (11.31), we get the identities in (11.27). Moreover, if we take (11.30) for $j=2^{2\alpha_l}$, we can rewrite (11.31) as

$$S_jf=\sum_{\eta=0}^{l-1}\frac{2^{2\alpha_\eta(1/p-1)}}{\alpha_\eta^{1/2}}(D_{2^{2\alpha_\eta+1}}-D_{2^{2\alpha_\eta}})$$
$$=\sum_{\eta=0}^{l-1}\frac{2^{2\alpha_\eta(1/p-1)}}{\alpha_\eta^{1/2}}(D_{2^{2\alpha_\eta+1}}-D_{2^{2\alpha_\eta}})+\frac{2^{2\alpha_l(1/p-1)}}{\alpha_l^{1/2}}(D_j-D_{2^{2\alpha_l}}). \quad (11.32)$$

Let $2^{2\alpha_{l-1}+1}-1\leq j\leq 2^{2\alpha_l}$, where $1\leq l\leq k$. Then, according to (11.24) and (11.32), we get

$$|S_jf|\leq\sum_{\eta=0}^{l-1}\frac{2^{2\alpha_{\eta-1}/p}}{\alpha_{\eta-1}^{1/2}}=\sum_{\eta=0}^{l-2}\frac{2^{2\alpha_{\eta-1}/p}}{\alpha_{\eta-1}^{1/2}}+\frac{2^{2\alpha_{l-1}/p}}{\alpha_{l-1}^{1/2}}\leq\frac{2^{2\alpha_{l-1}/p+1}}{\alpha_{l-1}^{1/2}}\leq\frac{2^{2\alpha_{k-1}/p+1}}{\alpha_{k-1}^{1/2}}. \quad (11.33)$$

Let $2^{2\alpha_l}\leq j<2^{2\alpha_l+1}$, where $1\leq l\leq k-1$. In view of (11.24) and (11.32), we can conclude that

$$|S_jf|\ \leq\ \sum_{\eta=0}^{l-1}\frac{2^{2\alpha_{\eta-1}/p}}{\alpha_{\eta-1}^{1/2}}+\frac{2^{2\alpha_l(1/p-1)}j}{\alpha_l^{1/2}}\ \leq\ \frac{2^{2\alpha_l/p}}{\alpha_l^{1/2}}+\frac{2^{2\alpha_l/p}}{\alpha_l^{1/2}}\ =\ 2\,\frac{2^{2\alpha_l/p}}{\alpha_l^{1/2}}\ \leq\ \frac{2^{2\alpha_{k-1}/p+1}}{\alpha_{k-1}^{1/2}}. \quad (11.34)$$

Combining (11.33) and (11.34) we obtain (11.28).

Let $0<p<q<1$. Combining (11.25) and (11.26) with the expression (11.33) obtained by taking $j=2^{2\alpha_k}$, we can see that

$$|S_{2^{2\alpha_k+1}}f|=\left|\frac{2^{2\alpha_k(1/p-1)}\alpha_k^{1/2}}{w_{2^{2\alpha_k}}}+S_{2^{2\alpha_k}}f\right|=\left|\frac{2^{2\alpha_k(1/p-1)}}{\alpha_k^{1/2}}w_{2^{2\alpha_k}}+S_{2^{2\alpha_{k-1}+1}}f\right|$$
$$\geq\left|\frac{2^{2\alpha_k(1/p-1)}}{\alpha_k^{1/2}}w_{2^{2\alpha_k}}\right|-|S_{2^{2\alpha_{k-1}+1}}f|\geq\frac{2^{2\alpha_k(1/p-1)}}{\alpha_k^{1/2}}-\frac{2^{2\alpha_{k-1}/p+1}}{\alpha_{k-1}^{1/2}}$$
$$\geq\frac{2^{2\alpha_k(1/p-1)}}{\alpha_k^{1/2}}-\frac{2^{\alpha_k(1/p-1/q)}}{16\alpha_k^{3/2}}\geq\frac{2^{2\alpha_k(1/p-1)}}{\alpha_k^{1/2}}-\frac{2^{\alpha_k(1/p-1)}}{16\alpha_k^{3/2}}\geq\frac{2^{2\alpha_k(1/p-1)}}{4\alpha_k^{1/2}},$$

so (11.29) is proved and the proof is complete. □

Remark 11.1. It is obvious that if we cancel the assumption (11.25) in Example 11.5, we still get $f=(f_n,n\in\mathbb{N})\in H_p$ and equality (11.26) and estimate (11.28) are valid. Moreover, we also obtain that (11.32) and (11.33) still hold.

Remark 11.2. We now show that there exists an increasing sequence $(\alpha_k, k \in \mathbb{N})$ of positive integers which satisfies (11.23)–(11.25). First, we note that the sequences $(2^{2n\beta}/n^{1/2}, n > n_0)$ and $(2^{n\beta}/n^{3/2}, n > n_0)$ are increasing,

$$\lim_{n\to\infty} \frac{2^{2n/p}}{n^{1/2}} = \infty \text{ and } \lim_{n\to\infty} \frac{2^{n/q}}{n^{3/2}} = \infty,$$

where $\beta > 0$, in particular, $\beta = 1/p$ or $\beta = 1/p - 1/q$, $p < q$. Then (α_k) can be chosen so lacunary that (11.23) holds and

$$\frac{2^{2\alpha_0/p}}{\alpha_0^{1/2}} + \frac{2^{2\alpha_1/p}}{\alpha_1^{1/2}} + \cdots + \frac{2^{2\alpha_{k-1}/p}}{\alpha_{k-1}^{1/2}} \leq \frac{k \cdot 2^{2\alpha_{k-1}/p}}{\alpha_{k-1}^{1/2}} < \frac{2^{2\alpha_k/p}}{\alpha_k^{1/2}}.$$

Moreover,

$$\frac{2^{2\alpha_{k-1}/p+5}}{\alpha_{k-1}^{1/2}} < \frac{2^{\alpha_k/p}}{\alpha_k^{3/2}} \text{ and } \frac{2^{2\alpha_{k-1}/p+5}}{\alpha_{k-1}^{1/2}} < \frac{2^{\alpha_k(1/p-1/q)}}{\alpha_k^{3/2}}, \quad p < q,$$

which guarantee that (11.24) and (11.25) are also fulfilled.

In the next sections, we will repeatedly use a martingale with the fixed sequence $(\lambda_k, k \in \mathbb{N})$.

Example 11.6. Let $0 < p \leq 1$ and $(\alpha_k, k \in \mathbb{N})$ be an increasing sequence of positive integers and $(\Phi_k, k \in \mathbb{N})$ be a sequence of non-negative and increasing numbers such that $\Phi_k \to \infty$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} \frac{1}{\Phi_{2^{2p\alpha_k/4}}} < \infty. \tag{11.35}$$

We define a martingale $f = (f_n, n \in \mathbb{N})$ by $f_n =: \sum_k a_k$, where

$$\lambda_k = \frac{1}{\Phi_{2^{2\alpha_k}}^{1/4}} \text{ and } a_k = 2^{2\alpha_k(1/p-1)}(D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}).$$

Then $f \in H_p$ and

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{2\alpha_k(1/p-1)}}{\Phi_{2^{2\alpha_k}}^{1/4}}, & j \in \{2^{2\alpha_k}, \ldots, 2^{2\alpha_k+1} - 1\}, \quad k \in \mathbb{N}_+, \\ 0, & j \notin \bigcup\limits_{k=1}^{\infty} \{2^{2\alpha_k}, \ldots, 2^{2\alpha_k+1} - 1\}. \end{cases} \tag{11.36}$$

Proof. The proof is quite analogous to that of Example 11.3, so we omit the details. □

Sometimes, in the sequel, an appropriate choice of sequence will be the following: $(\lambda_k = 1/2^{k/p}$, $k \in \mathbb{N})$.

Example 11.7. Let $0 < p \leq 1$ and $f_n = \sum\limits_{\{k:\, 2^{k+1} < n\}} \lambda_k a_k$, where

$$\lambda_k =: \frac{1}{2^{k/p}} \text{ and } a_k =: 2^{2^{k+1}}(D_{2^{2^{k+1}}+1} - D_{2^{2^{k+1}}}).$$

Then $f = (f_n, n \in \mathbb{N}) \in H_p$,

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{2^{i+1}(1/p-1)}}{2^{i/p}}, & j \in \{2^{2^{i+1}}, \ldots, 2^{2^{i+1}+1} - 1\}, \quad i \in \mathbb{N}_+, \\ 0, & j \notin \bigcup\limits_{i=0}^{\infty} \{2^{2^{i+1}}, \ldots, 2^{2^{i+1}+1} - 1\} \end{cases} \tag{11.37}$$

and

$$\omega_{H_p}\Big(\frac{1}{2^n},f\Big)=O\Big(\frac{1}{n^{1/p}}\Big)\ \ \text{as}\ \ n\to\infty. \tag{11.38}$$

If $2^{2^{i+1}}\leq j<2^{2^{i+1}+1}$, $l\in\mathbb{N}_+$, then

$$S_jf=S_{2^{2^{i+1}}}+\frac{2^{2^{i+1}(1/p-1)}w_{2^{2^{i+1}}}D_{j-2^{2^{i+1}}}}{2^{i/p}}. \tag{11.39}$$

Proof. This example is a simple consequence of Example 11.3 if we take $\alpha_k=2^k$ and $\lambda_k=1/2^{k/p}$. Since

$$\sum_{k=0}^{\infty}|\lambda_k|^p\leq c\sum_{k=0}^{\infty}\frac{1}{2^k}<C<\infty,$$

we obtain $f=(f_n,n\in\mathbb{N})\in H_p$ for all $0<p\leq 1$, and (11.37) holds. Moreover, in view of (11.22), we can conclude that

$$\begin{aligned}\omega_{H_p}\Big(\frac{1}{2^n},f\Big)&\leq\Big(\sum_{\{k:\ 2^{k+1}\geq n\}}\frac{1}{2^k}\Big)^{1/p}\\&\leq\Big(\sum_{\{k\geq\log_2^n-1\}}\frac{1}{2^k}\Big)^{1/p}\leq\Big(\frac{1}{2^{\log_2^n-1}}\Big)^{1/p}=O\Big(\frac{1}{n^{1/p}}\Big)\longrightarrow\infty\ \ \text{as}\ \ n\to\infty,\end{aligned}$$

so (11.38) is proved as well. Finally, if we apply (11.12) with $\alpha_k=2^k$, then we immediately get equality (11.39), thus the proof is complete. □

12 Background and state of the art about the partial sums and Fejér means with respect to Walsh system in martingale Hardy spaces

The state of the art in this area up to year 2022 can be found in the book [79] by L. E. Persson, G. Tephnadze and F. Weisz. In this section, we briefly discuss some information from this book.

It is well-known (see [33, 79]) that, for any $f\in H_1(G)$, we have $\|S_nf\|_{H_1}\leq cV(n)\|f\|_{H_1}$.

In [113], it was also proved that if $f\in H_1(G)$ and $\{m_k:\ k\geq 0\}$ is an increasing sequence of natural numbers satisfying the condition

$$\omega_{H_1(G)}\Big(\frac{1}{2^{|m_k|}},f\Big)=o\Big(\frac{1}{V(m_k)}\Big)\ \ \text{as}\ \ k\to\infty,$$

then $\|S_{m_k}f-f\|_{H_1(G)}\to 0$ as $k\to\infty$. Moreover, if $\{m_k:\ k\geq 0\}$ is an increasing sequence of natural numbers satisfying the condition

$$\sup_{k\in\mathbb{N}}V(m_k)=\infty, \tag{12.1}$$

then there exist a martingale $f\in H_1(G)$ and a subsequence $\{\alpha_k:\ k\geq 0\}\subset\{m_k:\ k\geq 0\}$ for which

$$\omega_{H_1(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big)=O\Big(\frac{1}{V(\alpha_k)}\Big)\ \ \text{as}\ \ k\to\infty$$

and

$$\limsup_{k\to\infty}\|S_{\alpha_k}f-f\|_1>c>0\ \ \text{as}\ \ k\to\infty,$$

where c is an absolute constant. As a consequence, we obtain a result proved in [116]. In particular, if $f\in H_1(G)$ and

$$\omega_{H_1(G)}\Big(\frac{1}{2^N},f\Big)=o\Big(\frac{1}{N}\Big)\ \ \text{as}\ \ N\to\infty,$$

then $\|S_nf-f\|_1\to 0$ as $n\to\infty$. Moreover, there exists a martingale $f\in H_1(G)$ such that

$$\omega_{H_1(G)}\Big(\frac{1}{2^N},f\Big)=O\Big(\frac{1}{N}\Big)\ \text{ as }\ N\to\infty$$

and $\|S_nf-f\|_1\nrightarrow 0$ as $n\to\infty$.

It was proved in [113] and [111] that if $F\in H_p$, then there exists an absolute constant c_p depending only on p such that $\|S_nf\|_{H_p}\le c_p\cdot 2^{d(n)(1/p-1)}\|f\|_{H_p}$.

On the other hand, if $0<p<1$, $\{m_k:\ k\ge 0\}$ is an increasing subsequence of natural numbers such that

$$\sup_{k\in\mathbb{N}} d(m_k)=\infty \tag{12.2}$$

and $\Phi:\mathbb{N}_+\to[1,\infty)$ is a non-decreasing function satisfying the condition

$$\overline{\lim_{k\to\infty}}\frac{2^{d(m_k)(1/p-1)}}{\Phi(m_k)}=\infty,$$

then there exists a martingale $f\in H_p(G)$ such that

$$\sup_{k\in\mathbb{N}}\Big\|\frac{S_{m_k}f}{\Phi(m_k)}\Big\|_{weak\text{-}L_p(G)}=\infty.$$

In [111, 113], it was also proved that if $0<p<1$, $F\in H_p(G)$ and $\{m_k:\ k\ge 0\}$ is an increasing sequence of natural numbers satisfying the condition

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|m_k|}},f\Big)=o\Big(\frac{1}{2^{d(m_k)(1/p-1)}}\Big)\ \text{ as }\ k\to\infty,$$

then

$$\|S_{m_k}f-f\|_{H_p(G)}\longrightarrow 0\ \text{ as }\ k\to\infty. \tag{12.3}$$

On the other hand, if $\{m_k:\ k\ge 0\}$ is an increasing sequence of natural numbers satisfying condition (12.2), then there exist a martingale $f\in H_p(G)$ and a subsequence $\{\alpha_k:\ k\ge 0\}\subset\{m_k:\ k\ge 0\}$, for which

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big)=O\Big(\frac{1}{2^{d(\alpha_k)(1/p-1)}}\Big)\ \text{ as }\ k\to\infty$$

and

$$\limsup_{k\to\infty}\|S_{\alpha_k}f-f\|_{weak\text{-}L_p(G)}>c_p>0\ \text{ as }\ k\to\infty. \tag{12.4}$$

As a consequence, we obtain a result proved in [116]. In particular, if $0<p<1$, $f\in H_p(G)$ and

$$\omega_{H_p(G)}\Big(\frac{1}{2^N},f\Big)=o\Big(\frac{1}{2^{N(1/p-1)}}\Big)\ \text{ as }\ N\to\infty,$$

then $\|S_nf-f\|_p\to 0$ as $n\to\infty$. Moreover, there exists a martingale $f\in H_p(G)$, where $0<p<1$ such that

$$\omega_{H_p(G)}\Big(\frac{1}{2^N},f\Big)=O\Big(\frac{1}{2^{N(1/p-1)}}\Big)\ \text{ as }\ N\to\infty$$

and $\|S_nf-f\|_{weak\text{-}L_p(G)}\nrightarrow 0$ as $n\to\infty$.

In [98], the weighted maximal operator $\widetilde{S}^*$, defined by

$$\widetilde{S}^*f:=\sup_{n\in\mathbb{N}}\frac{|S_nf|}{\log(n+1)} \tag{12.5}$$

was investigated, and it was proved that the following inequality holds: $\|\widetilde{S}^*f\|_1\le c_p\|f\|_{H_1}$. Moreover, it was also proved that the rate of the sequence $\{(n+1)^{1/p-1}\}$, given in the denominator of (12.5), cannot be improved. In particular, for any non-decreasing function $\varphi:\mathbb{N}_+\to[1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{\log(n+1)}{\varphi(n)}=\infty,$$

there exists a martingale $f \in H_1(G)$ such that

$$\sup_{n\in\mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_1 = \infty.$$

For $0<p<1$, the weighted maximal operator $\widetilde{S}^{*,p}$, defined by

$$\widetilde{S}^{*,p} f := \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}}\,, \tag{12.6}$$

was investigated in [98], and it was proved that the following inequality holds: $\|\widetilde{S}^{*,p} f\|_p \leq c_p \|f\|_{H_p}$. Moreover, it was also proved that the rate of the sequence $\{(n+1)^{1/p-1}\}$, given in the denominator of (12.6), cannot be improved. In particular, for any non-decreasing function $\varphi : \mathbb{N}_+ \to [1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-1}}{\varphi(n)} = \infty,$$

there exists a martingale $f \in H_p(G)$ $(0<p\leq 1)$ such that

$$\sup_{n\in\mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_p = \infty.$$

In [22], it was proved that the maximal operator, defined by

$$\sup_{n\in\mathbb{N}} \frac{|S_n f|}{2^{d(n)(1/p-1)}} \tag{12.7}$$

is bounded from $H_p(G)$ to *weak*-$L_p(G)$. Moreover, if $0<p<1$, $\{m_k : \ k\geq 0\}$ is any increasing sequence of positive integers such that condition (12.2) holds and $\Phi : \mathbb{N}_+ \to \mathbb{R}$ is any non-decreasing function satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{2^{d(m_k)(1/p-1)}}{\Phi(m_k)} = \infty, \tag{12.8}$$

then there exists a martingale $f \in H_p(G)$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{S_{m_k} f}{\Phi(m_k)} \right\|_{weak\text{-}L_p} = \infty.$$

In [22], it was also proved that the weighted maximal operator defined in (12.7) is not bounded from $H_p(G)$ to the Lebesgue space $L_p(G)$ for $0<p<1$.

In [14], it was proved that if $0<p<1$, $f\in H_p$, $\{n_k : \ k\geq 0\}$ is a sequence of positive integers and $\{n_{s_i} : \ 1\leq i\leq r\} \subset \{n_k : \ k\geq 0\}$ are the integers such that $2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}$, $s\in\mathbb{N}$, then the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by

$$\widetilde{S}^{*,\nabla} f := \sup_{s\in\mathbb{N}} \sup_{2^s\leq n_{s_i}<2^{s+1}} \frac{|S_n f|}{2^{d_s(n_{s_i})(1/p-1)}}\,,$$

where $d_s(n_{s_i})$ are defined by (2.3), is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. Moreover, if $0<p<1$, $\{n_k : \ k\geq 0\}$ is a sequence of positive numbers and $\{n_{s_i} : \ 1\leq i\leq r\} \subset \{n_k : \ k\geq 0\}$ is a subsequence satisfying the condition $2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}$, $s\in\mathbb{N}$, then for any non-negative, non-decreasing function $\varphi : \mathbb{N}_+ \to \mathbb{R}$ satisfying the condition

$$\sup_{s\in\mathbb{N}} \sup_{2^s\leq n_{s_i}<2^{s+1}} \frac{2^{d_s(n_{s_i})(1/p-1)}}{\varphi(n_{s_i})} = \infty,$$

the maximal operator, defined by $\sup\limits_{s\in\mathbb{N}} \sup\limits_{2^s\leq n_{s_i}<2^{s+1}} \frac{|S_n f|}{\varphi(n_{s_i})}$ is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

In [24], it was proved that if $0<p<1$, $f\in H_p(G)$ and $\varphi:\mathbb{N}_+\to\mathbb{R}_+$ is any non-negative and non-decreasing function satisfying the condition

$$\sum_{n=1}^{\infty}\frac{1}{\varphi^p(n)}<c<\infty,$$

then for any sequence $\{n_k:\ k\geq 0\}$ of positive integers, the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by

$$\widetilde{S}^{*,\nabla}f=\sup_{k\in\mathbb{N}}\frac{|S_{n_k}f|}{2^{d(n_k)(1/p-1)}\varphi(d(n_k))} \tag{12.9}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. Moreover, for any $0<p<1$, any sequence $\{n_k:\ k\geq 0\}$ of positive numbers and any non-negative and non-decreasing function $\varphi:\mathbb{N}_+\to\mathbb{R}_+$ satisfying

$$\sum_{n=1}^{\infty}\frac{1}{\varphi^p(n)}=\infty, \tag{12.10}$$

the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by (12.9), is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. Hence, we get that if $0<p<1$ and $f\in H_p(G)$, then the weighted maximal operator $\widetilde{S}^{*,\nabla,\varepsilon}$, defined by

$$\widetilde{S}^{*,\nabla,\varepsilon}f:=\sup_{n\in\mathbb{N}}\frac{|S_nf|}{2^{d(n)(1/p-1)}(d(n)\log^{1+\varepsilon}d(n))^{1/p}},\quad \varepsilon>0,$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ for $\varepsilon>0$ and is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ for $\varepsilon=0$.

The weak $(1,1)$-type inequality for the maximal operator σ^* of Fejér means σ_n with respect to the Walsh system

$$\mu(\sigma^*f>\lambda)\leq\frac{c}{\lambda}\left\|f\right\|_1,\quad \lambda>0$$

was derived in Schipp [82] and Pál, Simon [71] (see also [20, 64] and [78]). Fujii [39] and Simon [86] verified that σ^* is bounded from $H_1(G)$ to $L_1(G)$. Weisz [129] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p>1/2$. Simon [88] gave a counterexample, which showed that boundedness does not hold for $0<p<1/2$. A counterexample for $p=1/2$ was given by Goginava [47, 48] (see also [28, 29]). Moreover, in [49] (see also [97, 108, 116]), he proved that there exists a martingale $f\in H_p(G)$, $0<p\leq 1/2$ such that $\sup_{n\in\mathbb{N}}\|\sigma_nf\|_p=\infty$. Weisz [132] proved that the maximal operator σ^* of the Fejér means is bounded from the Hardy space $H_{1/2}$ to the space *weak*-$L_{1/2}$.

In [48], Goginava (see also [100]) proved that the weighted maximal operator $\widetilde{\sigma}^*$, defined by

$$\widetilde{\sigma}^*f:=\sup_{n\in\mathbb{N}}\frac{|\sigma_nf|}{\log^2(n+1)} \tag{12.11}$$

is bounded from the Hardy space $H_{1/2}(G)$ to the Lebesgue space $L_{1/2}(G)$. Moreover, it was also proved that the rate of increase of the denominator $\{\log^2(n+1)\}$ cannot be improved. In particular, for any non-decreasing function $\varphi:\mathbb{N}_+\to[1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{\log^2(n+1)}{\varphi(n)}=\infty,$$

there exists a martingale $f\in H_{1/2}(G)$ such that

$$\sup_{n\in\mathbb{N}}\left\|\frac{\sigma_nf}{\varphi(n)}\right\|_{1/2}=\infty.$$

Similar problems for $0<p<1/2$ were investigated in [101, 102]. In particular, it was proved that the weighted maximal operator $\widetilde{\sigma}^{*,p}$, defined by

$$\widetilde{\sigma}^{*,p}f:=\sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{(n+1)^{1/p-2}}\,,\quad 0<p<\frac{1}{2} \tag{12.12}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. Moreover, for any non-decreasing function $\varphi:\mathbb{N}_+\to[1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{(n+1)^{1/p-2}}{\varphi(n)}=\infty,$$

there exists a martingale $f\in H_p(G)$, $0<p<1/2$ such that

$$\sup_{n\in\mathbb{N}}\left\|\frac{\sigma_n f}{\varphi(n)}\right\|_p=\infty.$$

Weisz [131] (see also [128]) also proved that for any $f\in H_p(G)$, $p>0$, the maximal operator, defined by $\sup\limits_{n\in\mathbb{N}}|\sigma_{2^n}f|$ is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$, but it is not bounded from the Hardy space $H_p(G)$ to the space $H_p(G)$ (see also [77, 79]).

Persson and Tephnadze [76] generalized this result and proved that if $0<p\le 1/2$ and $\{n_k:\ k\ge 0\}$ is a sequence of positive numbers such that

$$\sup_{k\in\mathbb{N}}d(n_k)<c<\infty, \tag{12.13}$$

then the maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla}f:=\sup_{k\in\mathbb{N}}|\sigma_{n_k}f| \tag{12.14}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. Moreover, if $0<p<1/2$ and $\{m_k:\ k\ge 0\}$ is a sequence of positive numbers such that condition (12.2) holds, then there exists a martingale $f\in H_p(G)$ such that $\sup\limits_{k\in\mathbb{N}}\|\sigma_{m_k}f\|_p=\infty$.

From the boundedness results of weighted maximal operators, we immediately get that for any $f\in H_p(G)$, there exists an absolute constant c_p such that for any $n\in\mathbb{N}$ the following inequalities hold:

$$\|\sigma_n f\|_{1/2}\le c\log^2(n+1)\|f\|_{H_{1/2}(G)} \tag{12.15}$$

and

$$\|\sigma_n f\|_p\le c_p n^{1/p-2}\log^{2[1/2+p]}(n+1)\|f\|_{H_p(G)}\ \text{ as }\ 0<p<\frac{1}{2}\,. \tag{12.16}$$

Applying inequalities (12.15) and (12.16), in [112] (see also [116]), the necessary and sufficient conditions for the modulus of continuity of the martingale $F\in H_p(G)$ were obtained, for which the Fejér means of the one-dimensional Walsh–Fourier series converge in the $H_p(G)$ norm $(0<p\le 1/2)$. In particular, for any $F\in H_{1/2}(G)$ if

$$\omega_{H_{1/2}(G)}\le\Big(\frac{1}{2^N},f\Big)=o\Big(\frac{1}{N^2}\Big)\ \text{ as }\ N\to\infty,$$

we have $\|\sigma_n f-f\|_{1/2}\to 0$ as $n\to\infty$. Moreover, there exists a martingale $f\in H_{1/2}(G)$, for which

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^N},f\Big)=O\Big(\frac{1}{N^2}\Big)\ \text{ as }\ N\to\infty$$

and $\|\sigma_n f-f\|_{1/2}\nrightarrow 0$ as $n\to\infty$.

For $0<p<1/2$, it was proved in [112] (see also [116]) that if $F\in H_p(G)$ and

$$\omega_{H_p(G)}\Big(\frac{1}{2^N},f\Big)=o\Big(\frac{1}{2^{N(1/p-2)}}\Big)\ \text{ as }\ N\to\infty,$$

then $\|\sigma_n f - f\|_p \to 0$ as $n \to \infty$.

Moreover, there exists a martingale $F \in H_p(G)$, for which

$$\omega_{H_p(G)}\Big(\frac{1}{2^N}, f\Big) = O\Big(\frac{1}{2^{N(1/p-2)}}\Big) \text{ as } N \to \infty$$

and $\|\sigma_n f - f\|_p \nrightarrow 0$ as $n \to \infty$.

In [112], it was also proved that if $f \in H_{1/2}(G)$, then $\|\sigma_n f\|_{H_{1/2}} \leq c V^2(n) \|f\|_{H_{1/2}}$. Moreover, if $\{m_k \in \mathbb{N} : k \geq 0\}$ is a sequence satisfying condition (12.1), $\{\varphi_n\}$ is any non-decreasing, non-negative function, satisfying the conditions $\Phi_n \uparrow \infty$, and

$$\overline{\lim_{k\to\infty}} \Big(\frac{V^2(n_k)}{\Phi_{n_k}}\Big) = \infty,$$

then there exists a martingale $f \in H_{1/2}(G)$ such that

$$\sup_{k\in\mathbb{N}} \Big\|\frac{\sigma_{n_k} f}{\Phi_{n_k}}\Big\|_{1/2} = \infty.$$

It follows that if $f \in H_{1/2}(G)$ and $\{n_k : k \geq 0\}$ is any sequence of positive numbers, then $\sigma_{n_k} f$ are bounded from the Hardy space $H_{1/2}(G)$ to the space $H_{1/2}(G)$ if and only if, for some c, the condition $\sup_{k\in\mathbb{N}} V(n_k) < c < \infty$ holds.

It was also proved in [112] that if $f \in H_{1/2}(G)$ and $\{m_k : k \geq 0\}$ is an increasing sequence of natural numbers such that

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^{|m_k|}}, f\Big) = o\Big(\frac{1}{V^2(m_k)}\Big) \text{ as } k \to \infty,$$

then $\|\sigma_{m_k} f - f\|_{H_{1/2}(G)} \to 0$ as $k \to \infty$. Moreover, if $\{m_k : k \geq 0\}$ is an increasing sequence of natural numbers such that (12.1) holds true, then there exist a martingale $f \in H_{1/2}(G)$ and a subsequence $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$ such that

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^{|\alpha_k|}}, f\Big) = O\Big(\frac{1}{V^2(\alpha_k)}\Big) \text{ as } k \to \infty$$

and $\limsup_{k\to\infty} \|\sigma_{\alpha_k} f - f\|_{1/2} > c > 0$ as $k \to \infty$, where c is an absolute constant.

When $0 < p < 1/2$, $f \in H_p(G)$ in [112] (see also [77]), it was proved that if $\{m_k : k \geq 0\}$ is an increasing sequence of natural numbers such that

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|m_k|}}, f\Big) = o\Big(\frac{1}{2^{d(m_k)(1/p-2)}}\Big) \text{ as } k \to \infty,$$

then $\|\sigma_{m_k} f - f\|_{H_p(G)} \to 0$ as $k \to \infty$.

On the other hand, if $\{m_k : k \geq 0\}$ is an increasing sequence of natural numbers satisfying condition (12.2), then there exist a martingale $f \in H_p(G)$ and a subsequence $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$, for which

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}}, f\Big) = O\Big(\frac{1}{2^{d(\alpha_k)(1/p-2)}}\Big) \text{ as } k \to \infty$$

and

$$\limsup_{k\to\infty} \|\sigma_{\alpha_k} f - f\|_{weak\text{-}L_p(G)} > c_p > 0 \text{ as } k \to \infty,$$

where c_p is a constant depending only on p.

In [15], it was proved that if $f \in H_{1/2}(G)$ and $\{n_k : k \geq 0\}$ is a sequence of positive numbers and $\{n_{s_i} : 1 \leq i \leq r\} \subset \{n_k : k \geq 0\}$ are the numbers such that $2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}$, $s \in \mathbb{N}$, and the sets A_s, defined by (2.4), are uniformly finite for all $s \in \mathbb{N}$, that is, the cardinality of the sets A_s are uniformly finite, i.e., $\sup_{s\in\mathbb{N}} |A_s| < c < \infty$, then the restricted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} f = \sup_{k\in\mathbb{N}} |\sigma_{n_k} f|, \tag{12.17}$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$. Moreover, if $\sup_{s\in\mathbb{N}}|A_s|=\infty$, then there exists a martingale $f\in H_{1/2}(G)$ such that the maximal operator, defined by (12.17), is not bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

This result implies that if $F\in H_{1/2}(G)$ and $\{n_k:\ k\geq 0\}$ is a sequence of positive numbers, then the restricted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by (12.17), is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$ if and only if any sequence of positive numbers $\{n_k:\ k\geq 0\}$, which satisfies $n_k\in[2^s,2^{s+1})$, is uniformly finite for each $s\in\mathbb{N}_+$, and each $\{n_k:\ k\geq 0\}$ has bounded variation, i.e., $\sup_{k\in\mathbb{N}}V(n_k)<c<\infty$.

In [8], it was proved that if $f\in H_{1/2}$ and $\{n_k:\ k\geq 0\}$ is a sequence of positive numbers, then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla}f:=\sup_{k\in\mathbb{N}}\frac{|\sigma_{n_k}f|}{|A_{|n_k|}|^2}$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$. Moreover, if $\sup_{k\in\mathbb{N}}|A_{|n_k|}|=\infty$ and $\{\varphi_n\}$ is a non-decreasing sequence satisfying the condition

$$\overline{\lim_{k\to\infty}}\frac{|A_{|n_k|}|^2}{\varphi_{|n_k|}}=\infty,$$

then there exists a martingale $f\in H_{1/2}$ such that the maximal operator, defined by $\sup_{k\in\mathbb{N}}\frac{|\sigma_{n_k}f|}{\varphi_{|n_k|}}$ is not bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

From this theorem we get (see also [48, 100]) that the maximal operator $\widetilde{\sigma}^*$, defined by (12.11) is bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$. Moreover, the rate of increase of the denominator $\{\log^2(n+1)\}$ cannot be improved.

In [25] (see also [37]), the $(H_p,\textit{weak-}L_p)$ type inequalities were described for $0<p<1/2$. In particular, for any $0<p<1/2$ and $f\in H_p(G)$, the weighted maximal operator $\widetilde{\sigma}^{*,*,p}$, defined by

$$\widetilde{\sigma}^{*,*,p}f:=\sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{2^{d(n)(1/p-2)}}$$

is bounded from the Hardy space $H_p(G)$ to the space *weak*-$L_p(G)$. Moreover, if $0<p<1/2$ and $\varphi:\mathbb{N}\to[1,\infty)$ is a non-decreasing function satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{2^{d(n)(1/p-2)}}{\varphi(n)}=\infty, \tag{12.18}$$

then there exist a sequence $\{a_k,k\in\mathbb{N}_+\}$ of p-atoms and a sequence $\{q_{n_k},k\in\mathbb{N}_+\}$ of real numbers satisfying the condition $|q_{n_k}|=n_k$ such that

$$\sup_{k\in\mathbb{N}}\frac{\left\|\frac{\sigma_{q_{n_k}}a_k}{\varphi(q_{n_k})}\right\|_{weak\text{-}L_p}}{\|a_k\|_{H_p}}=\infty.$$

In [25], it was also proved that for any $0<p<1/2$, there exists a sequence $\{a_k,k\in\mathbb{N}_+\}$ of p-atoms such that

$$\sup_{k\in\mathbb{N}}\frac{\|\widetilde{\sigma}^{*,*,p}a_k\|_p}{\|a_k\|_{H_p}}=\infty.$$

In [16], the (H_p,L_p) type inequalities were investigated for $0<p<1/2$ and it was proved that for any $0<p<1/2$, $f\in H_p(G)$ and any non-negative and non-decreasing function $\varphi:\mathbb{N}_+\to\mathbb{R}$ satisfying the condition

$$\sum_{n=1}^{\infty}\frac{1}{\varphi^p(n)}<c<\infty, \tag{12.19}$$

the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{2^{d(n)(1/p-2)}\varphi(d(n))}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

Moreover, if $0<p<1/2$, $\{n_k:\ k\geq 0\}$ is a sequence of positive numbers and $\varphi:\mathbb{N}_+\to\mathbb{R}$ is any non-negative and non-decreasing function satisfying the condition

$$\sum_{n=1}^{\infty}\frac{1}{\varphi^p(n)}=\infty,$$

then there exist p-atoms a_k such that

$$\sup_{k\in\mathbb{N}}\frac{\left\|\sup_{n\in\mathbb{N}}\frac{|\sigma_n a_k|}{2^{d(n)(1/p-2)}\varphi(d(n))}\right\|_p}{\|a_k\|_{H_p}}=\infty.$$

In [19], it was also proved that if $0<p<1/2$, $f\in H_p(G)$, $\{n_k:\ k\geq 0\}$ is a sequence of positive numbers and $\{n_{s_i}:\ 1\leq i\leq r\}\subset\{n_k:\ k\geq 0\}$ are the numbers such that $2^s\leq n_{s_1}\leq n_{s_2}\leq\cdots\leq n_{s_r}\leq 2^{s+1}$, $s\in\mathbb{N}$, then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} f := \sup_{s\in\mathbb{N}}\ \sup_{2^s\leq n_{s_i}<2^{s+1}} \frac{|\sigma_{n_{s_i}} f|}{2^{d_s(n_{s_i})(1/p-2)}}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$. Moreover, if $0<p<1/2$, $f\in H_p(G)$, $\{n_k:\ k\geq 0\}$ is a sequence of positive numbers and $\{n_{s_i}:\ 1\leq i\leq r\}\subset\{n_k:\ k\geq 0\}$ are the numbers such that $2^s\leq n_{s_1}\leq n_{s_2}\leq\cdots\leq n_{s_r}\leq 2^{s+1}$, $s\in\mathbb{N}$, then, for any non-negative and non-decreasing function $\varphi:\mathbb{R}_+\to\mathbb{R}$ satisfying the condition

$$\sup_{s\in\mathbb{N}}\ \sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{2^{d_s(n_{s_i})(1/p-2)}}{\varphi(n_{s_i})}=\infty, \tag{12.20}$$

there exist p-atoms a_s such that

$$\frac{\left\|\sup_{s\in\mathbb{N}}\ \sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{|\sigma_{n_{s_i}} a_s|}{\varphi(n_{s_i})}\right\|_p}{\|a_s\|_{H_p}}\longrightarrow\infty\quad\text{as}\quad s\to\infty.$$

In [13], the martingales were constructed that ensure the sharpness of the results described above. In particular, if $0<p<1/2$ and $\varphi:\mathbb{N}\to[1,\infty)$ is any non-decreasing function that satisfies condition (12.18), then there exists a martingale $f\in H_p(G)$ such that

$$\sup_{k\in\mathbb{N}}\left\|\frac{\sigma_{n_k} f}{\varphi(n_k)}\right\|_{weak\text{-}L_p}=\infty.$$

In the same paper, it was proved that if $0<p<1/2$, $\{n_k:\ k\geq 0\}$ is a sequence of positive numbers, $\{n_{s_i}:\ 1\leq i\leq r\}\subset\{n_k:\ k\geq 0\}$ are the numbers such that $2^s\leq n_{s_1}\leq n_{s_2}\leq\cdots\leq n_{s_r}\leq 2^{s+1}$, $s\in\mathbb{N}$, and $\varphi:\mathbb{N}\to[1,\infty)$ is any non-negative and non-decreasing function satisfying condition (12.20), then there exists a martingale $f\in H_p(G)$ such that the maximal operator, defined by $\sup_{s\in\mathbb{N}}\ \sup_{2^s\leq n_{s_i}<2^{s+1}}\frac{|\sigma_{n_{s_i}} f|}{\varphi(n_{s_i})}$ is not bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

Moreover, if $0<p<1/2$, $\{n_k:\ k\geq 0\}$ is a sequence of positive numbers and $\varphi:\mathbb{N}\to[1,\infty)$ is any non-negative and non-decreasing function satisfying condition (18.34), then there exists a martingale $f\in H_p(G)$ such that the maximal operator, defined by $\sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{2^{d(n)(1/p-2)}\varphi(d(n))}$ is not bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

Similar problems for partial sums and some general summability methods can be found in Areshidze, Persson and Tephnadze [9, 10], Baramidze [12] (see also [17, 18]), Blahota [27], Blahota and Nagy [30, 32], Blahota, Nagy, Persson and Tephnadze [31], Blahota, Persson and Tephnadze [24] (see also [21–23]), Blahota and Tephnadze [34–36, 78], Gát [40–42], Goginava [44–46], Mémic, Simon and Tephnadze [62], Persson and Tephnadze [75], Nagy and Tephnadze [66–69], Simon [87, 89, 90], Simon and Weisz [91], Smith [92], Tephnadze [94–96, 99, 104, 106, 107, 109, 114, 117] (see also [51–53, 59, 61, 63, 119]), Tutberidze [118, 122–125], Weisz [130], Yano [133], Young [134].

13 Boundedness of maximal operators of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

First, we state the following sharp result.

Theorem 13.1.

(a) *Let $p > 1/2$ and $f \in H_p$. Then the maximal operator σ^* of the Walsh–Fejér means is bounded from the Hardy space H_p to the Lebesgue space L_p.*

(b) *There exist a sequence $\{a_k\}$ of 1/2-atoms and a sequence $\{q_{n_k}\}$ of real numbers, where $q_{n_k} := 2^{2n_k} + 2^{2n_k-2} + \cdots + 2^2 + 2^0$ such that*

$$\sup_{k\in\mathbb{N}} \frac{\left(\int_G |\sigma_{q_{n_k}} a_k|^{1/2}\, d\mu\right)^2}{\|a_k\|_{H_{1/2}}} = \infty. \tag{13.1}$$

Proof.

(a) First, we note that σ_n is bounded from L_∞ to L_∞ (see (8.10) in Corollary 8.3). Hence, by Theorem 10.1, the proof will be complete if we prove that

$$\int_{\overline{I}} |\sigma^* a|^p\, d\mu \leq c < \infty \tag{13.2}$$

for every p-atom a, where I denotes the support of the atom a.

Let a be an arbitrary p-atom with the support I and $\mu(I) = 2^{-M}$. We may assume that $I = I_M$. It is easy to see that $\sigma_n(a) = 0$ for $n \leq 2^M$. Therefore, we can suppose that $n > 2^M$. Since $\|a\|_\infty \leq 2^{M/p}$, it follows that

$$|\sigma_n a| \leq \int_{I_M} |a(t)|\, |K_n(x+t)|\, d\mu(t) \leq \|a\|_\infty \int_{I_M} |K_n(x+t)|\, d\mu(t) \leq 2^{N/p} \int_{I_M} |K_n(x+t)|\, d\mu(t).$$

Let $x \in I_M^{k,l}$, $0 \leq k < l \leq M$. From Corollary 8.4, we can deduce that

$$|\sigma_n a| \leq C_p \cdot 2^{M/p} \cdot \frac{2^{l+k}}{2^{2M}} = C_p \cdot 2^{M(1/p-2)} \cdot 2^{l+k}. \tag{13.3}$$

The expression on the right-hand side of (13.3) does not depend on n. Thus,

$$|\sigma^* a(x)| \leq C_p \cdot 2^{M(1/p-2)} \cdot 2^{l+k} \quad \text{for} \quad x \in I_M^{k,l}, \quad 0 \leq k < l \leq M. \tag{13.4}$$

Using (13.4) combined with identity (1.2), we obtain

$$\begin{aligned}\int\limits_{\overline{I}_M} |\sigma^* a(x)|^p \, d\mu(x) &= \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \int\limits_{I_M^{k,l}} |\sigma^* a(x)|^p \, d\mu(x) + \sum_{k=0}^{M-1} \int\limits_{I_M^{k,M}} |\sigma^* a(x)|^p \, d\mu(x) \\ &\leq c_p \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^l} \cdot 2^{M(1-2p)} \cdot 2^{(l+k)p} + c_p \sum_{k=0}^{M-1} \frac{1}{2^M} \cdot 2^{M(1-p)} \cdot 2^{pk} \\ &\leq c_p \cdot 2^{M(1-2p)} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{2^{p(k+l)}}{2^l} + c_p \sum_{k=0}^{M-1} \frac{2^{pk}}{2^{pM}} =: I + II. \quad (13.5)\end{aligned}$$

We estimate each term separately. For I, we have

$$\begin{aligned}I &= c_p \cdot 2^{M(1-2p)} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{2^{pk}}{2^{(1-p)l}} \\ &\leq c_p \cdot 2^{M(1-2p)} \sum_{k=0}^{M-2} 2^{k(2p-1)} \sum_{l=k+1}^{M-1} \frac{2^{(1-p)k}}{2^{(1-p)l}} \leq c_p \sum_{k=0}^{M-2} \frac{2^{(2p-1)k}}{2^{(2p-1)M}} \sum_{l=k+1}^{M-1} \frac{2^{(1-p)k}}{2^{(1-p)l}} \\ &\leq c_p \sum_{k=0}^{M-2} \frac{1}{2^{(M-k)(2p-1)}} \sum_{l=k+1}^{M-1} \frac{1}{2^{(p-1)(l-k)}} \leq c_p \sum_{k=0}^{M-2} \frac{[M-k]}{2^{(M-k)(2p-1)}} \leq C_p < \infty. \quad (13.6)\end{aligned}$$

It is obvious that

$$II \leq c_p \sum_{k=0}^{M-1} \frac{2^{kp}}{2^{Mp}} \leq C_p < \infty. \quad (13.7)$$

We combine (13.5)–(13.7) to see that (13.2) holds, which completes the proof of part (a).

(b) Let a_k be the 1/2-atom defined in Example 11.2. Using (11.7), we find that

$$|\sigma_{q_{n_k}} a_k| = \frac{q_{n_k - 1}}{q_{n_k}} \, |K_{q_{n_k - 1}}|.$$

Let $x \in I_{2s+1}(e_{2s-1} + e_{2s})$. Then, in view of Corollary 8.6, we can conclude that

$$|\sigma_{q_{n_k}} a_k(x)| \geq \frac{c \cdot 2^{4s}}{2^{2n_k}} \, .$$

Hence,

$$\begin{aligned}\int\limits_G |\sigma_{q_{n_k}} a_k|^{1/2} \, d\mu &\geq \sum_{s=0}^{n_k - 3} \int\limits_{I_{2s+1}(e_{2s-1}+e_{2s})} |\sigma_{q_{n_k}} a_k|^{1/2} \, d\mu \\ &\geq c \sum_{s=0}^{n_k-3} \frac{2^{2s} \, \mu(I_{2s+1}(e_{2s-1}+e_{2s}))}{2^{n_k}} \geq c \sum_{s=0}^{n_k-3} \frac{1}{2^{n_k}} \geq \frac{c \, n_k}{2^{n_k}} \, .\end{aligned}$$

From (11.6) in Example 11.2, we have

$$\frac{\left(\int\limits_G |\sigma_{q_{n_k}} a_k|^{1/2} \, d\mu \right)^2}{\|a_k\|_{H_{1/2}}} \geq \frac{c \, n_k^2}{2^{2n_k}} \cdot 2^{2n_k} \geq c \, n_k^2 \longrightarrow \infty \ \text{ as } \ k \to \infty.$$

Thus, (13.1) holds, so part (b) is proved. □

Our next sharp result reads as follows.

Theorem 13.2.

(a) *Let $f \in H_{1/2}$. Then the maximal operator σ^* of the Walsh–Fejér means is bounded from $H_{1/2}$ to the space weak-$L_{1/2}$.*

(b) *Let $0 < p < 1/2$. Then there exists a martingale $f \in H_p$ such that $\sup_{n\in\mathbb{N}} \|\sigma_n f\|_{weak\text{-}L_p} = +\infty$.*

Proof. (a) By Lemma 10.3, we have to prove that

$$t\mu\big\{x \in \overline{I}_M :\ \sigma^* a \geq t^2\big\} \leq c < \infty,\ \ t \geq 0, \tag{13.8}$$

for every 1/2-atom a. We may assume that a is an arbitrary 1/2-atom with the support I, $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $\sigma_n a(x) = 0$, when $n \leq 2^M$. Therefore, we may suppose that $n > 2^M$.

Let $t \in I_M$. Since σ_n is bounded from L_∞ to L_∞ (see (8.10) in Corollary 8.3) and $\|a\|_\infty \leq 2^{2M}$, combining (13.3) and (13.4) for $p = 1/2$, we get $|\sigma_n a(x)| \leq C \cdot 2^{k+l}$ for $x \in I_M^{k,l}$, $0 \leq k < l \leq M$, and

$$|\sigma^* a(x)| \leq C \cdot 2^{k+l}\ \text{ for }\ x \in I_M^{k,l},\ \ 0 \leq k < l \leq M. \tag{13.9}$$

It follows that for such $k < l \leq M$, we have the estimate $|\sigma_n a(x)| \leq C \cdot 2^{2M}$ for $x \in I_M^{k,l}$, and also,

$$\mu\big\{x \in I_M^{k,l} :\ \sigma^* f > C \cdot 2^{2s}\big\} = 0,\ \ s = M+1, M+2, \dots . \tag{13.10}$$

Suppose that

$$2^{l+k} > 2^{2s}\ \text{ for some }\ s \leq k < l \leq M. \tag{13.11}$$

It is evident that inequality (13.11) holds for all $l > k \geq s$, that is,

$$2^{l+k} > 2^{2s},\ \ \text{where}\ \ l > k \geq s. \tag{13.12}$$

If $l > s > k$, from (13.11) we can conclude that $l > 2s - k$. Combining (13.9) and (13.12), we get

$$\big\{x \in \overline{I}_M :\ \sigma^* f \geq C\cdot 2^{2s}\big\} \subset \bigcup_{k=s}^{M-1} \bigcup_{l=k+1}^{M} \big\{x \in I_M^{k,l} :\ \sigma^* f \geq C\cdot 2^{2s}\big\} \cup \bigcup_{k=0}^{s} \bigcup_{l=2s-k}^{M} \big\{x \in I_M^{k,l} :\ \sigma^* f \geq C\cdot 2^{2s}\big\}$$

and

$$\begin{aligned}\mu\big\{x \in \overline{I}_M :\ \sigma^* f \geq C \cdot 2^{2s}\big\} &\leq \sum_{k=s}^{M-1} \sum_{l=k+1}^{M} |I_M^{k,l}| + \sum_{k=0}^{s} \sum_{l=2s-k}^{M} |I_M^{k,l}| \\ &\leq C \sum_{k=s}^{M-1} \sum_{l=k+1}^{M} \frac{1}{2^l} + C \sum_{k=0}^{s} \sum_{l=2s-k}^{M} \frac{1}{2^l} \leq \sum_{k=s}^{M-1} \frac{1}{2^s} + \sum_{k=0}^{s} \frac{1}{2^{2s-k}} \leq \frac{C}{2^s}.\end{aligned} \tag{13.13}$$

In view of (13.10) and (13.13), we can conclude that $2^{2s}\mu\{x \in \overline{I}_M :\ \sigma^* f \geq 2^{2s}\} < C < \infty$, which shows (13.8), as does part (a).

(b) Let a_k be the atom defined in Example 11.2. Using (11.7), we find that

$$|\sigma_{2^{2n_k}+1} a_k| = \frac{1}{2^{2n_k}+1} K_1 = \frac{1}{2^{2n_k}+1} D_1 = \frac{1}{2^{2n_k}+1} w_0 = \frac{1}{2^{2n_k}+1} \geq \frac{1}{2^{2n_k+1}}$$

and

$$\mu\Big\{x \in G :\ |\sigma_{2^{2n_k}+1} a_k(x)| \geq \frac{1}{2^{2n_k+1}}\Big\} \geq \mu(G) = 1.$$

Therefore, using (11.6), we get

$$\begin{aligned}&\frac{\frac{1}{2^{2n_k+1}} \left(\mu\{x \in G :\ |\sigma_{2^{2n_k}+1} a_k(x)| \geq \frac{c}{2^{2n_k+1}}\}\right)^{1/p}}{\|a_k\|_{H_p}} \\ &\qquad \geq \frac{c}{2^{2n_k}} \cdot 2^{2n_k(1/p-1)} = c \cdot 2^{2n_k(1/p-2)} \longrightarrow \infty \ \text{ as }\ k \to \infty,\end{aligned}$$

and also part (b) is proved. □

Note the following consequence of Theorems 13.1 and 13.2.

Corollary 13.1.

(a) *There exists a martingale* $f \in H_{1/2}$ *such that* $\|\sigma^* f\|_{1/2} = \infty$.

(b) *Let* $0 < p < 1/2$. *Then there exists a martingale* $f \in H_p$ *such that* $\|\sigma^* f\|_{weak\text{-}L_p} = \infty$.

14 Boundedness of weighted maximal operators of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

Theorem 14.1.

(a) *The maximal operator* $\widetilde{\sigma}^*$, *defined by*

$$\widetilde{\sigma}^* f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ *to the Lebesgue space* $L_{1/2}$.

(b) *Let* $(\Phi_n, n \in \mathbb{N})$ *be any non-decreasing sequence satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{\log^2(n+1)}{\Phi_n} = \infty. \tag{14.1}$$

Then there exist the sequences $\{a_k\}$ *of* p*-atoms and* $\{q_{n_k}\}$ *of real numbers, where* $q_{n_k} := 2^{2n_k} + 2^{2n_k-2} + \cdots + 2^2 + 2^0$ *such that*

$$\sup_{k\in\mathbb{N}} \frac{\|\frac{\sigma_{q_{n_k}} a_k}{\Phi_{q_{n_k}}}\|_{1/2}}{\|a_k\|_{H_{1/2}}} = \infty. \tag{14.2}$$

Proof. (a) It suffices to prove that

$$\int_{\overline{I}} |\widetilde{\sigma}^* a(x)|^{1/2}\, d\mu(x) \le c < \infty \tag{14.3}$$

for every $1/2$-atom a, where I denotes the support of the atom. Let a be again an arbitrary $1/2$-atom with support $I = I_M$ and $\mu(I) = 2^{-M}$. Obviously, $\sigma_n(a) = 0$ for $n \le 2^M$. So, we can suppose that $n > 2^M$.

Since $\|a\|_\infty \le 2^{2M}$, we obtain

$$\begin{aligned}\frac{|\sigma_n a(x)|}{\log^2(n+1)} &\le \frac{1}{\log^2(n+1)} \int_{I_N} |a(t)|\, |K_n(x+t)|\, d\mu(t) \\ &\le \frac{\|a\|_\infty}{\log^2(n+1)} \int_{I_N} |K_n(x+t)|\, d\mu(t) \le \frac{2^{2N}}{\log^2(n+1)} \int_{I_N} |K_n(x+t)|\, d\mu(t).\end{aligned}$$

Let $x \in I_{l+1}(e_k + e_l)$, $0 \le k < l \le M$. Then from Corollary 8.4 it follows that

$$\frac{|\sigma_n a(x)|}{\log^2(n+1)} \le \frac{c \cdot 2^{2M}}{M^2} \frac{2^{l+k}}{2^{2M}} = \frac{c \cdot 2^{l+k}}{M^2}\,. \tag{14.4}$$

Since the expression on the right-hand side of (14.4) does not depend on n, we have

$$|\widetilde{\sigma}^* a(x)| \le \frac{c \cdot 2^{l+k}}{M^2} \quad \text{for} \quad x \in I_M^{k,l}, \quad 0 \le k < l \le M. \tag{14.5}$$

Applying (14.5) in combination with identity (1.2) we obtain

$$\int\limits_{I_M} |\widetilde{\sigma}^* a(x)|^{1/2}\, d\mu(x) = \sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\int\limits_{I_M^{k,l}} |\widetilde{\sigma}^* a(x)|^{1/2}\, d\mu(x) + \sum_{k=0}^{M-1}\int\limits_{I_M^{k,M}} |\widetilde{\sigma}^* a(x)|^{1/2}\, d\mu(x)$$
$$\leq c\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}\,\frac{2^{l+k/2}}{M} + c\sum_{k=0}^{M-1}\frac{1}{2^M}\,\frac{2^{M/2}\cdot 2^{k/2}}{M} =: I + II.$$

For the first term, we get

$$I \leq \frac{c}{M}\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\frac{2^{k/2}}{2^{l/2}} \leq \frac{c}{M}\sum_{k=0}^{M-2} 1 \leq c < \infty.$$

Moreover,

$$II \leq \frac{1}{2^{M/2}M}\sum_{k=0}^{M-1} 2^{k/2} \leq \frac{c}{M} < c < \infty.$$

We conclude that (14.3) follows just by combining the above estimates, and thus the proof of part (a) is complete.

(b) Under condition (14.1), there exists an increasing sequence $(\lambda_k, k \in \mathbb{N})$ of positive integers such that

$$\lim_{k\to\infty}\frac{\log^2(\lambda_k+1)}{\Phi_{\lambda_k}} = \infty.$$

It is evident that, for every λ_k, there exists a positive integer m_k' such that $q_{m_k'} \leq \lambda_k < q_{m_k'+1} < cq_{m_k'}$. Since Φ_n is a non-decreasing function, we have

$$\overline{\lim_{k\to\infty}}\frac{(m_k')^2}{\Phi_{q_{m_k'}}} \geq c\lim_{k\to\infty}\frac{\log^2(\lambda_k+1)}{\Phi_{\lambda_k}} = \infty.$$

Let $(n_k, k \in \mathbb{N}) \subset (m_k', k \in \mathbb{N})$ be a subsequence of positive numbers $\mathbb{N}_+$ such that $\lim_{k\to\infty}\frac{n_k^2}{\Phi_{q_{n_k}}} = \infty$ and let a_k be the 1/2-atom defined in Example 11.2. Using (11.7), we find that

$$|\sigma_{q_{n_k}} a_k| = \frac{q_{n_k-1}}{q_{n_k}}\,|K_{q_{n_k-1}}| \quad\text{and}\quad \frac{|\sigma_{q_{n_k}} a_k|}{\Phi_{q_{n_k}}} = \frac{1}{\Phi_{q_{n_k}}}\,\frac{q_{n_k-1}}{q_{n_k}}\,|K_{q_{n_k-1}}|.$$

Let $x \in I_{2s+1}(e_{2s-1}+e_{2s})$. Then, in view of Corollary 8.6, we can conclude that

$$\frac{|\sigma_{q_{n_k}} a_k(x)|}{\Phi_{q_{n_k}}} \geq \frac{c\cdot 2^{4s}}{2^{2n_k}\Phi_{q_{n_k}}}\,.$$

Hence,

$$\int\limits_G \left|\frac{\sigma_{q_{n_k}} a_k}{\Phi_{q_{n_k}}}\right|^{1/2} d\mu \geq \sum_{s=0}^{n_k-3}\int\limits_{I_{2s+1}(e_{2s-1}+e_{2s})}\left|\frac{\sigma_{q_{n_k}} a_k}{\Phi_{q_{n_k}}}\right|^{1/2} d\mu$$
$$\geq c\sum_{s=0}^{n_k-3}\frac{2^{2s}\mu(I_{2s+1}(e_{2s-1}+e_{2s}))}{\Phi_{q_{n_k}}^{1/2}\cdot 2^{n_k}} \geq c\sum_{s=0}^{n_k-3}\frac{1}{2^{n_k}\Phi_{q_{n_k}}^{1/2}} \geq \frac{c\,n_k}{2^{n_k}\Phi_{q_{n_k}}^{1/2}}\,.$$

From (11.6) in Example 11.2, we have

$$\frac{\left(\int\limits_G \left|\frac{\sigma_{q_{n_k}} a_k}{\Phi_{q_{n_k}}}\right|^{1/2} d\mu\right)^2}{\|a_k\|_{H_{1/2}}} \geq \frac{c\,n_k^2}{2^{2n_k}\Phi_{q_{n_k}}}\cdot 2^{2n_k} \geq \frac{c\,n_k^2}{\Phi_{q_{n_k}}} \longrightarrow \infty \text{ as } k\to\infty.$$

Thus, (14.2) holds, and so part (b) is proved. □

Corollary 14.1. *Let $(\Phi_n, n\in\mathbb{N})$ be any non-decreasing sequence satisfying condition (12.13). Then the maximal operator $\sup\limits_{n\in\mathbb{N}}\frac{|\sigma_n f|}{\Phi_n}$ is not bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.*

Theorem 14.2.

(a) *Let $0<p<1/2$. Then the maximal operator $\widetilde{\sigma}_p^*$, defined by*

$$\widetilde{\sigma}_p^* f := \sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{(n+1)^{1/p-2}}$$

is bounded from the Hardy martingale space H_p to the Lebesgue space L_p.

(b) *Let $(\Phi_n, n\in\mathbb{N})$ be any non-decreasing sequence satisfying the condition*

$$\overline{\lim_{n\to\infty}}\frac{(n+1)^{1/p-2}}{\Phi_n}=\infty. \tag{14.6}$$

Then there exists a sequence $\{a_k\}$ of p-atoms such that

$$\sup_{k\in\mathbb{N}}\frac{\left\|\frac{\sigma_{2^{2n_k+1}}a_k}{\Phi_{2^{2n_k+1}}}\right\|_{weak\text{-}L_p}}{\|a_k\|_{H_p}}=\infty. \tag{14.7}$$

Proof. (a) First, we note that σ_n is bounded from L_∞ to L_∞ (see (8.10) in Corollary 8.3). Hence, by Lemma 10.2, the proof will be complete if we show that

$$\int\limits_{\overline{I}}|\widetilde{\sigma}_p^*(x)|^p\,d\mu(x)\le C_p<\infty \tag{14.8}$$

for every p-atom a, where I denotes the support of the atom.

Let a be an arbitrary p-atom with support I and $\mu(I)=2^{-M}$. We may assume that $I=I_M$. It is easy to see that $\sigma_n a=0$ for $n\le 2^M$. Therefore, we can suppose that $n>2^M$. Since $\|a\|_\infty\le 2^{M/p}$, it follows that

$$\begin{aligned}\frac{|\sigma_n a|}{(n+1)^{1/p-2}}&\le\frac{1}{(n+1)^{1/p-2}}\int\limits_{I_M}|a(t)|\,|K_n(x+t)|\,d\mu(t)\\&\le\frac{\|a\|_\infty}{(n+1)^{1/p-2}}\int\limits_{I_M}|K_n(x+t)|\,d\mu(t)\le\frac{c\cdot 2^{M/p}}{(n+1)^{1/p-2}}\int\limits_{I_M}|K_n(x+t)|\,d\mu(t).\end{aligned}$$

Let $x\in I_M^{k,l}$, $0\le k<l\le M$. From Corollary 8.4, we can deduce that

$$\frac{|\sigma_n a|}{(n+1)^{1/p-2}}\le\frac{C_p\cdot 2^{M/p}}{2^M(1/p-2)}\,\frac{2^{l+k}\cdot 2^l}{2^{2M}}=C_p\cdot 2^{l+k}. \tag{14.9}$$

The expression on the right-hand side of (14.9) does not depend on n. Thus,

$$|\widetilde{\sigma}_p^* a(x)|\le C_p\cdot 2^{l+k}\ \text{ for }\ x\in I_M^{k,l},\ \ 0\le k<l\le M. \tag{14.10}$$

Using (14.10) together with identity (1.2), we obtain

$$\begin{aligned}\int\limits_{\overline{I}_M}|\widetilde{\sigma}_p^* a(x)|^p\,d\mu(x)&=\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\int\limits_{I_M^{k,l}}|\widetilde{\sigma}_p^* a(x)|^p\,d\mu(x)+\sum_{k=0}^{M-1}\int\limits_{I_M^{k,M}}|\widetilde{\sigma}_p^* a(x)|^p\,d\mu(x)\\&\le C_p\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}C_p\cdot 2^{(l+k)/p}+C_p\sum_{k=0}^{M-1}\frac{1}{2^M}\cdot 2^{pM}\cdot 2^{pk}\\&\le C_p\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\frac{2^{p(l+k)}}{2^l}+C_p\sum_{k=0}^{M-1}\frac{2^{pk}}{2^{M(1-p)}}=:I+II.\end{aligned} \tag{14.11}$$

We estimate each term separately:

$$\begin{aligned} I &= C_p \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^{l(1-2p)}} \frac{2^{p(l+k)}}{2^{2pl}} \\ &\leq C_p \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^{l(1-2p)}} \leq C_p \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^{l(1-2p)}} \leq C_p \sum_{k=0}^{M-2} \frac{1}{2^{k(1-2p)}} < C_p < \infty. \end{aligned} \tag{14.12}$$

It is obvious that

$$II \leq \frac{C_p}{2^{M(1-2p)}} \sum_{k=0}^{M-1} \frac{2^{pk}}{2^{pM}} \leq \frac{C_p}{2^{M(1-2p)}} < c < \infty. \tag{14.13}$$

Combining (14.11) with (14.12), (14.13), we obtain (14.8) and part (a).

(b) Under condition (14.6), there exists an increasing sequence of positive integers $(\lambda_k, k \in \mathbb{N})$ such that $\lim_{k\to\infty} \frac{\lambda_k^{1/p-2}}{\Phi_{\lambda_k}} = \infty$. It is evident that for every λ_k, there exists a positive integer m_k' such that $q_{m_k'} < \lambda_k < 2q_{m_k'}$. Since $(\Phi_n, n \in \mathbb{N})$ is a non-decreasing function, we have

$$\overline{\lim_{k\to\infty}} \frac{2^{2m_k'(1/p-2)}}{\Phi_{2^{2m_k'}+1}} \geq \frac{1}{2} \overline{\lim_{k\to\infty}} \frac{(2^{2m_k'}+1)^{(1/p-2)}}{\Phi_{2^{2m_k'}+1}} \geq c \lim_{k\to\infty} \frac{\lambda_k^{1/p-2}}{\Phi_{\lambda_k}} = \infty.$$

Let $(n_k, k \in \mathbb{N}) \subset (m_k', k \in \mathbb{N})$ be a sequence of positive numbers such that

$$\lim_{k\to\infty} \frac{2^{2n_k(1/p-2)}}{\Phi_{2^{2n_k}+1}} = \infty,$$

and a_k bethe atom defined in Example 11.2. Using (11.7), we find that

$$|\sigma_{2^{2n_k}+1} a_k| = \frac{1}{2^{2n_k}+1} K_1 = D_1 = w_0 = 1$$

and

$$\frac{|\sigma_{2^{2n_k}+1} a_k|}{\Phi_{2^{2n_k}+1}} = \frac{1}{\Phi_{2^{2n_k}+1}(2^{2n_k}+1)} \geq \frac{c}{2^{2n_k}\Phi_{2^{2n_k}+1}}.$$

Hence,

$$\mu\Big\{x \in G : \frac{|\sigma_{2^{2n_k}+1} a_k(x)|}{\Phi_{2^{2n_k}+1}} \geq \frac{c}{2^{2n_k}} \Phi_{2^{2n_k}+1}\Big\} \geq \mu(G) = 1.$$

Therefore, using (11.6), we get

$$\begin{aligned} &\frac{\frac{c}{2^{2n_k}\Phi_{2^{2n_k}+1}} \left(\mu\{x \in G : \frac{|\sigma_{2^{2n_k}+1} a_k(x)|}{\Phi_{2^{2n_k}+1}} \geq \frac{c}{2^{2n_k}\Phi_{2^{2n_k}+1}}\}\right)^{1/p}}{\|a_k\|_{H_p}} \\ &\qquad\geq \frac{c}{2^{2n_k}\Phi_{2^{2n_k}+1}} \cdot 2^{2n_k(1/p-1)} = \frac{c \cdot 2^{2n_k(1/p-2)}}{\Phi_{2^{2n_k}+1}} \longrightarrow \infty \text{ as } k \to \infty, \end{aligned}$$

and so part (b) is proved. $\square$

We also mention the following corollaries.

Corollary 14.2. *Let $(\Phi_n, n \in \mathbb{N})$ be any non-decreasing sequence satisfying condition (14.6). Then the maximal operator $\sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{\Phi_n}$ is not bounded from the Hardy space H_p to the space weak-L_p.*

15 Boundedness of restricted maximal operators of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

We first investigate the following restricted maximal operator.

Theorem 15.1.

(a) *Let* $0 < p \leq 1$ *and* $f \in H_p(G)$. *Then the restricted maximal operator* $\widetilde{\sigma}^{*,\sharp}$, *defined by*

$$\widetilde{\sigma}^{*}_{\#} f = \sup_{k \in \mathbb{N}} |\sigma_{2^k} f|, \tag{15.1}$$

is bounded from the Hardy space $H_{1/2}$ *to the Lebesgue space* $L_{1/2}$.

(b) *Let* $0 < p \leq 1$. *Then the operator* $|\sigma_{2^n} f|$ *is not bounded from the martingale Hardy space* H_p *to* H_p.

Proof. Since σ_n is bounded from L_∞ to L_∞, by Lemma 10.2, the proof of the theorem will be complete if we prove that

$$\int\limits_{\overline{I}_M} |\widetilde{\sigma}^{*}_{\#} f|^p \, d\mu(x) \leq c < \infty \tag{15.2}$$

for every p-atom a. We may assume that a is an arbitrary p-atom with support I, $\mu(I) = 2^{-M}$ and$I = I_M$. It is easy to see that $\sigma_{2^n}(a) = 0$ for $n < M$. Therefore, we can suppose that $n \geq M$.

Let $x \in I_M$. Since $\|a\|_\infty \leq 2^{M/p}$, we obtain

$$\begin{aligned} |\sigma_{2^n} a(x)| &\leq \int\limits_{I_M} |a(t)| \, |K_{2^n}(x+t)| \, d\mu(t) \\ &\leq \|a\|_\infty \int\limits_{I_M} |K_{2^n}(x+t)| \, d\mu(t) \leq 2^{M/p} \int\limits_{I_M} |K_{2^n}(x+t)| \, d\mu(t). \end{aligned} \tag{15.3}$$

Let $x \in I_M^{i,j}$ where $i < j < M$. Then $x + t \in I_M^{i,j}$ for $t \in I_M$ and, according to (8.1) in Lemma 8.1, since $n \geq M$, we find that

$$|K_{2^n}(x+t)| = 0. \tag{15.4}$$

Combining (15.3) and (15.4), we get

$$|\sigma_{2^n} a(x)| = 0. \tag{15.5}$$

Let $x \in I_M^{i,M}$, where $i < M$. Then $x + t \in I_M^{i,M}$ for $t \in I_M$ and, according to (8.1) in Lemma 8.1, we obtain

$$|K_{2^n}(x+t)| = 2^i. \tag{15.6}$$

Combining (15.3) and (15.6), we get

$$|\sigma_{2^n} a(x)| \leq 2^{M(1/p-1)+i}. \tag{15.7}$$

Let $0 < p \leq 1$. Combining (1.2) and (15.7), we have

$$\int\limits_{\overline{I}_M} |\widetilde{\sigma}^{*}_{\#} f|^p \, d\mu = \sum_{i=0}^{M-1} \int\limits_{I_M^{i,M}} |\widetilde{\sigma}^{*}_{\#} f|^p \, d\mu \leq \sum_{i=0}^{M-1} \int\limits_{I_M^{i,M}} 2^{M(1-p)+pi} \, d\mu \leq \sum_{i=0}^{M-1} \frac{2^{pi}}{2^{pM}} < c < \infty.$$

We conclude that (15.36) holds for $0 < p \leq 1$, so the proof of part (a) is complete.

Let a_k be the p-atom from Example 11.2. Using (11.7), we find that $|\sigma_{2^{2n_k+1}}a_k| = |K_{2^{2n_k}}|$. It is evident that

$$\begin{aligned}
&S_{2^N}\left(|\sigma_{2^{2n_k+1}}a_k(x)|\right)\\
&\qquad= \int_G |\sigma_{2^{2n_k}} f_k(t)| D_{2^N}(x+t)\,d\mu(t) \geq \int_{I_{2n_k}} |\sigma_{2^{2n_k}} f_k(t)| D_{2^N}(x+t)\,d\mu(t)\\
&\qquad\qquad\geq c\int_{I_{2n_k}} |K_{2^{2n_k+1}}(t)| D_{2^N}(x+t)\,d\mu(t) \geq c\cdot 2^{2n_k}\int_{I_{2n_k}} D_{2^N}(x+t)\,d\mu(t).
\end{aligned}$$

Since $w_j(t)=1$ for $t\in I_{2n_k}$, $j=0,1,\ldots,2^{2n_k}-1$, we obtain

$$S_{2^N}\left(|\sigma_{2^{2n_k+1}}a_k(x)|\right) \geq c\cdot 2^{2n_k}\cdot\frac{1}{2^{2n_k}}D_{2^N}(x), \quad N=0,1,\ldots,2^N-1,$$

so that

$$\sup_{1\leq N<2^{2n_k}} S_{2^N}\left(|\sigma_{2^{2n_k+1}}a_k(x)|\right) \geq c\sup_{1\leq N<2^{2n_k}} S_{2^N}\left(|\sigma_{2^{2n_k+1}}a_k(x)|\right) \geq c\sup_{1\leq N<2^{2n_k}} D_{2^N}(x).$$

Let $x\in I_s\setminus I_{s+1}$ for some $s=0,1,\ldots,2n_k-1$. Then, from Lemma 5.1, it follows that

$$\sup_{1\leq N<2^{2n_k}} S_{2^N}\left(|\sigma_{2^{2n_k+1}}a_k(x)|\right) \geq c\cdot 2^s.$$

According to (11.6), for every $0<p\leq 1$, we have

$$\begin{aligned}
\left\|\,|\sigma_{2^{2n_k+1}}a_k|\,\right\|_{H_p}^p &= \left\|\sup_{1\leq N<2n_k} S_{2^N}\left(|\sigma_{2^{2n_k+1}}a_k(x)|\right)\right\|_p^p\\
&\geq \int_G \left(\sup_{1\leq N<2^{2n_k}} S_{2^N}\left(|\sigma_{2^{2n_k+1}}a_k(x)|\right)\right)^p d\mu(x)\\
&\geq \sum_{s=1}^{2n_k-1}\int_{I_s\setminus I_{s+1}} \left(\sup_{1\leq N<2^{2n_k}} S_{2^N}\left(|\sigma_{2^{2n_k+1}}a_k(x)|\right)\right)^p d\mu(x).
\end{aligned}\tag{15.8}$$

Let $0<p<1$. Then

$$\left\|\,|\sigma_{2^{2n_k+1}}a_k|\,\right\|_{H_p}^p \geq c\sum_{s=1}^{2n_k-1}\frac{2^{ps}}{2^s} = C_p > 0.$$

Let $p=1$. We find that

$$\left\|\,|\sigma_{2^{2n_k+1}}a_k|\,\right\|_{H_1} \geq c\sum_{s=1}^{2n_k-1}\frac{2^s}{2^s} \geq c\,n_k. \tag{15.9}$$

Combining (15.8) and (15.9) with (11.6) in Example 11.2, we can conclude that

$$\frac{\|\,|\sigma_{2^{2n_k+1}}a_k|\,\|_{H_p}}{\|a_k\|_{H_p}} \geq \frac{C_p}{2^{2n_k(1-1/p)}} \longrightarrow \infty \ \text{ as } \ k\to\infty, \quad 0<p<1,$$

and for $p=1$, we have $\|\,|\sigma_{2^{2n_k+1}}a_k|\,\|_{H_1}\|a_k\|_{H_1} \geq c\,n_k \to \infty$ as $k\to\infty$. □

Corollary 15.1. *Let $0<p\leq 1$. Then the maximal operator $\sigma^{\#}$, defined by $\sigma^{\#}f := \sup_{n\in\mathbb{N}}|\sigma_{2^n}f|$ is bounded from the Hardy space H_p to the Lebesgue space L_p, but it is not bounded from the Hardy space H_p to the Hardy space H_p.*

Proof. Since

$$|\sigma_{2^n}f| \leq \sigma^{\#}f, \tag{15.10}$$

this result follows from Theorem 15.1. □

Corollary 15.2. *Let $0 < p \le 1$. Then the operator $|\sigma_{2^n} f|$ is bounded from the Hardy space H_p to L_p, but is not bounded from H_p to H_p.*

Proof. Using (15.10), this result immediately follows from Theorem 15.1. □

Since $D_{2^n} \le 2K_{2^n}$ for any $n \in \mathbb{N}$, we obtain that the following statement is true.

Corollary 15.3. *Let $0 < p \le 1$ and $f \in H_p(G)$. Then, for any $n \in \mathbb{N}$, we have*

$$\|S_{2^n} f\|_p \le 2 \|\sigma_{2^n} f\|_p \le 2 \|\sigma^{\#} f\|_p \le c \|f\|_{H_p},$$

where c is an absolute constant.

Next, we investigate the case $p = 1/2$. The following result was proved in [15].

Theorem 15.2.

(a) *Let $f \in H_{1/2}(G)$ and $\{n_k : k \ge 0\}$ be a sequence of positive numbers, and let $\{n_{s_i} : 1 \le i \le r\} \subset \{n_k : k \ge 0\}$ be the numbers such that $2^s \le n_{s_1} \le n_{s_2} \le \cdots \le n_{s_r} \le 2^{s+1}$, $s \in \mathbb{N}$. If the sets A_s, defined by (2.4), are uniformly finite for all $s \in \mathbb{N}$, that is, the cardinality of the sets A_s is uniformly finite, i.e., $\sup_{s\in\mathbb{N}} |A_s| < c < \infty$, then the restricted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by*

$$\widetilde{\sigma}^{*,\nabla} f = \sup_{k\in\mathbb{N}} |\sigma_{n_k} f| \tag{15.11}$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

(b) *(Sharpness) Let $\sup_{s\in\mathbb{N}} |A_s| = \infty$. Then there exists a martingale $f \in H_{1/2}(G)$ such that the maximal operator, defined by (15.11), is not bounded from the martingale Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.*

Proof. Since σ_n is bounded from L_∞ to L_∞, the proof of theorem, according to Lemma 10.2, will be complete if we prove that

$$\int_{\overline{I}_M} \left(\sup_{s_k\in\mathbb{N}} |\sigma_{n_{s_k}} a(x)| \right)^{1/2} d\mu(x) \le c < \infty \tag{15.12}$$

for every 1/2-atom a. We may assume that a is an arbitrary 1/2-atom with support I, $\mu(I) = 2^{-M}$ and $I = I_M$ and $\sigma_n(a) = 0$ for $n < 2^M$. Therefore, we can suppose that $n_{s_k} \ge 2^M$.

Let $x \in I_M$ and $2^s \le n_{s_k} < 2^{s+1}$ for some $n_{s_k} \ge 2^M$. Since $\|a\|_\infty \le 2^{2M}$, we obtain

$$\begin{aligned} |\sigma_{n_{s_k}} a(x)| &\le \int_{I_M} |a(t)| \, |K_{n_{s_k}}(x+t)| \, d\mu(t) \\ &\le \|a\|_\infty \int_{I_M} |K_{n_{s_k}}(x+t)| \, d\mu(t) \le 2^{2M} \int_{I_M} |K_{n_{s_k}}(x+t)| \, d\mu(t). \end{aligned} \tag{15.13}$$

Using Lemma 8.5 and (2.4), we get

$$\begin{aligned} |K_{n_{s_k}}| &\le \frac{c}{n_{s_k}} \sum_{A=1}^{s} \left(2^{l_A^{s_k}} K_{2^{l_A^{s_k}}} + 2^{m_A^{s_k}} K_{2^{m_A^{s_k}}} + 2^{l_A^{s_k}} \sum_{k=l_A^{s_k}}^{\infty} D_{2^k} \right) \\ &\le \frac{c}{2^s} \sum_{A=1}^{s} \left(2^{l_A} K_{2^{l_A}} + 2^{m_A} K_{2^{m_A}} + 2^{l_A} \sum_{k=l_A}^{\infty} D_{2^k} \right) \end{aligned}$$

and

$$|\sigma_{n_{s_k}}a(x)| \leq \frac{2^M}{2^s}\left(2^M\sum_{A=1}^{r_s^1}\int_{I_M}2^{l_A^s}K_{2^{l_A^s}}(x+t)\,d\mu(t)\right)$$
$$+\frac{2^M}{2^s}\left(2^M\sum_{A=1}^{r_s^2}\int_{I_M}2^{m_A^s}K_{2^{m_A^s}}(x+t)\,d\mu(t)\right)+\frac{2^M}{2^s}\left(2^M\sum_{A=1}^{r_s^1}\int_{I_M}2^{l_A^s}\sum_{k=l_A^s}^{\infty}D_{2^k}(x+t)\,d\mu(t)\right).$$

If we denote

$$II^1_{\alpha_A^s}(x) := 2^M\int_{I_M}2^{\alpha_A^s}K_{2^{\alpha_A^s}}(x+t)\,d\mu(t),\quad \alpha=l\ \text{ or }\ \alpha=t,$$
$$II^2_{l_A^s}(x) := 2^M\int_{I_M}2^{l_A^s}\sum_{k=l_A^s}^{\infty}D_{2^k}(x+t)\,d\mu(t),$$

from (15.13) we can conclude that

$$|\sigma_{n_{s_k}}a(x)| \leq \frac{2^M}{2^s}\left(\sum_{A=1}^{r_s^1}II^1_{l_A^s}(x)+\sum_{A=1}^{r_s^2}II^1_{m_A^s}(x)+\sum_{A=1}^{r_s^1}II^2_{l_A^s}(x)\right)$$

and

$$\sup_{2^s\leq n_{s_k}<2^{s+1}}|\sigma_{n_{s_k}}a(x)| \leq \frac{2^M}{2^s}\left(\sum_{A=1}^{r_s^1}II^1_{l_A^s}(x)+\sum_{A=1}^{r_s^2}II^1_{m_A^s}(x)+\sum_{A=1}^{r_s^1}II^2_{l_A^s}(x)\right).$$

Hence,

$$\int_{\overline{I}_M}\left(\sup_{2^s\leq n_{s_k}<2^{s+1}}|\sigma_{n_{s_k}}a(x)|\right)^{1/2}d\mu$$
$$\leq \frac{2^{M/2}}{2^{s/2}}\left(\sum_{A=1}^{r_s^1}\int_{\overline{I}_M}|II^1_{l_A^s}(x)|^{1/2}\,d\mu+\sum_{A=1}^{r_s^2}\int_{\overline{I}_M}|II^1_{m_A^s}|^{1/2}\,d\mu+\sum_{A=1}^{r_s^1}\int_{\overline{I}_M}|II^2_{l_A^s}(x)|^{1/2}\,d\mu\right).\tag{15.14}$$

Since $\sup_{s\in\mathbb{N}}r_s^1<r<\infty$ and $\sup_{s\in\mathbb{N}}r_s^2<r<\infty$, we find that (15.12) holds, so part (a) will be proved if we prove that

$$\int_{\overline{I}_M}|II^2_{l_A^s}(x)|^{1/2}\,d\mu\leq c<\infty,\quad A=1,\dots,r_s^1,\tag{15.15}$$

and

$$\int_{\overline{I}_M}|II^1_{\alpha_A^s}(x)|^{1/2}\,d\mu\leq c<\infty\tag{15.16}$$

for all $\alpha_A^s=l_A^s$, $A=1,\dots,r_s^1$, and $\alpha_A^s=m_A^s$, $A=1,\dots,r_s^2$. Indeed, if (15.15) and (15.16) hold, from (15.14) we get

$$\int_{\overline{I}_M}\left(\sup_{n_{s_k}\geq 2^M}|\sigma_{n_{s_k}}a(x)|\right)^{1/2}d\mu\leq\sum_{s=M}^{\infty}\int_{\overline{I}_M}\left(\sup_{2^s\leq n_{s_k}<2^{s+1}}|\sigma_{n_{s_k}}a(x)|\right)^{1/2}d\mu\leq\sum_{s=M}^{\infty}\frac{c\cdot 2^{M/2}}{2^{s/2}}<C<\infty.$$

It remains to prove (15.15) and (15.16). Let $t\in I_M$ and $x\in I_M^{k,l}$. If $0\leq k<l<\alpha_A^s\leq M$ or $0\leq k<l<M<\alpha_A^s$, then $x+t\in I_M^{k,l}$, and if we apply (8.1) in Lemma 8.1, we obtain $K_{2^{\alpha_A^s}}(x+t)=0$ and

$$II^1_{\alpha_A^s}(x)=0.\tag{15.17}$$

Let $0 \leq k < \alpha_A^s \leq l < M$. Then $x+t \in I_M^{k,l}$, and if we use (8.1) in Lemma 8.1, we get $2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t) \leq 2^{\alpha_A^s+k}$, so that

$$II^1_{\alpha_A^s}(x) \leq 2^{\alpha_A^s+k}. \tag{15.18}$$

Analogously to (15.18), we can prove that if $0 \leq \alpha_A^s \leq k < l < M$, then $2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t) \leq 2^{2\alpha_A^s}$, $t \in I_M$, $x \in I_M^{k,l}$ so that

$$II^1_{\alpha_A^s}(x) \leq 2^{2\alpha_A^s}, \quad t \in I_M, \quad x \in I_M^{k,l}, \tag{15.19}$$

Let $t \in I_M$ and $x \in I_M^{k,M}$ and let $0 \leq k < \alpha_A^s \leq M$ or $0 \leq k < M \leq \alpha_A^s$. Since $x+t \in x \in I_M^{k,M}$, and if we apply (8.1) in Lemma 8.1, we obtain $2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t) \leq 2^{\alpha_A^s+k}$ and

$$II^1_{\alpha_A^s}(x) \leq 2^{\alpha_A^s+k}. \tag{15.20}$$

Let $0 \leq \alpha_A^s \leq k < M$. Since $x+t \in x \in I_M^{k,M}$, and if we apply (8.1) in Lemma 8.1, we find that $2^{\alpha_A^s} K_{2^{\alpha_A^s}}(x+t) \leq 2^{2\alpha_A^s}$ and

$$II^1_{\alpha_A^s}(x) \leq 2^{2\alpha_A^s}. \tag{15.21}$$

Let $0 \leq \alpha_A^s < M$. Combining (1.2) with estimates (15.17)–(15.21) for any $A = 1, \dots, s$, we have

$$\begin{aligned}
\int_{\overline{I}_M} |II^1_{\alpha_A^s}(x)|^{1/2}\, d\mu &= \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \int_{I_M^{k,l}} |II^1_{\alpha_A^s}(x)|^{1/2}\, d\mu + \sum_{k=0}^{M-1} \int_{I_M^{k,M}} |II^1_{\alpha_A^s}(x)|^{1/2}\, d\mu \\
&\leq c \sum_{k=0}^{\alpha_A^s-1} \sum_{l=\alpha_A^s} M-1 \int_{I_M^{k,l}} 2^{(\alpha_A^s+k)/2}\, d\mu + c \sum_{k=\alpha_A^s}^{M-2} \sum_{l=k+1}^{M-1} \int_{I_M^{k,l}} 2^{\alpha_A^s}\, d\mu \\
&\quad + c \sum_{k=0}^{\alpha_A^s-1} \int_{I_M^{k,M}} 2^{(\alpha_A^s+k)/2}\, d\mu + c \sum_{k=\alpha_A^s}^{M-1} \int_{I_M^{k,M}} 2^{\alpha_A^s}\, d\mu \\
&\leq c \sum_{k=0}^{\alpha_A^s-1} \sum_{l=\alpha_A^s+1}^{M-1} \frac{2^{(\alpha_A^s+k)/2}}{2^l} + c \sum_{k=\alpha_A^s}^{M-2} \sum_{l=k+1}^{M-1} \frac{2^{\alpha_A^s}}{2^l} + c \sum_{k=0}^{\alpha_A^s-1} \frac{2^{(\alpha_A^s+k)/2}}{2^M} + c \sum_{k=\alpha_A^s}^{M-1} \frac{2^{\alpha_A^s}}{2^M} \leq C < \infty.
\end{aligned}$$

Analogously, we can prove that (15.16) holds for the case $\alpha_A^s \geq M$ as well. Hence, (15.16) holds, and it remains to prove (15.15).

Let $t \in I_M$ and$x \in I_i \setminus I_{i+1}$. If $i \leq l_A^s - 1$, since $x+t \in I_i \setminus I_{i+1}$, using Lemma 5.1, we have

$$II^2_{l_A^s}(x) = 0. \tag{15.22}$$

If $l_A^s \leq i < M$, using Lemma 5.1, we obtain

$$II^2_{l_A^s}(x) \leq 2^M \int_{I_M} 2^{l_A^s} \sum_{k=l_A^s}^{i} D_{2^k}(x+t)\, d\mu(t) \leq c \cdot 2^{l_A^s+i}. \tag{15.23}$$

Let $0 \leq l_A^s < M$. Combining (1.2), (15.22) and (15.23), we get

$$\begin{aligned}
\int_{\overline{I}_M} |II^2_{l_A^s}(x)|^{1/2}\, d\mu &= \left(\sum_{i=0}^{l_A^s-1} + \sum_{i=l_A^s+1}^{M-1} \right) \int_{I_i \setminus I_{i+1}} |II^2_{l_A^s}(x)|^{1/2}\, d\mu \\
&\leq c \sum_{i=l_A^s}^{M-1} \int_{I_i \setminus I_{i+1}} 2^{(l_A^s+i)/2}\, d\mu(x) \leq c \sum_{i=l_A^s}^{M-1} 2^{(l_A^s+i)/2} \cdot \frac{1}{2^i} \leq C < \infty.
\end{aligned} \tag{15.24}$$

If $M \leq l_A^s$, then $i < M \leq l_A^s$, and applying (15.22), we get

$$\int_{\overline{I}_M} |II^2_{l_A^s}(x)|^{1/2}\, d\mu = 0. \tag{15.25}$$

Thus, (15.15) will be proved just by combining (15.24) and (15.25), and so part (a) is complete.

We now turn to the proof of part (b).

Under condition (14.6), there exists an increasing sequence $\{\alpha_k : \ k \geq 0\} \subset \{n_k : \ k \geq 0\}$ of positive integers such that

$$\sum_{k=1}^{\infty} \frac{1}{|A^2_{|\alpha_k|}} \leq c < \infty. \tag{15.26}$$

Let $f = (f_1, f_2, \dots)$ be a martingale form Example 11.3, where

$$\lambda_k := \frac{1}{|A_{|\alpha_k|}|} \quad \text{and} \quad a_k := 2^{|\alpha_k|}(D_{2^{|\alpha_k|+1}} - D_{2^{|\alpha_k|}}).$$

Applying Lemma 5.3 and (15.26), we can conclude that $f \in H_{1/2}$.

It is easy to prove that

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{|\alpha_k|}}{|A_{|\alpha_k|}|}, & j \in \{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1} - 1\}, \quad k = 0, 1, \dots, \\ 0\,, & j \notin \bigcup\limits_{k=0}^{\infty} \{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1} - 1\}. \end{cases} \tag{15.27}$$

Let $2^{|\alpha_k|} < j < \alpha_k$. Using (15.27), we get

$$S_j f = S_{2^{|\alpha_k|}} f + \sum_{v=2^{|\alpha_k|}}^{j-1} \widehat{f}(v) w_v = S_{2^{|\alpha_k|}} f + \frac{(D_j - D_{2^{|\alpha_k|}}) \cdot 2^{|\alpha_k|}}{|A_{|\alpha_k|}|}. \tag{15.28}$$

Let $2^{|\alpha_k|} \leq \alpha_{s_n} \leq 2^{|\alpha_k|+1}$. Then, using (15.28), we find that

$$\begin{aligned} \sigma_{\alpha_{s_n}} f &= \frac{1}{\alpha_{s_n}} \sum_{j=1}^{2^{|\alpha_k|}} S_j f + \frac{1}{\alpha_{s_n}} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_{s_n}} S_j f \\ &= \frac{\sigma_{2^{|\alpha_k|}} f}{\alpha_{s_n}} + \frac{(\alpha_k - 2^{|\alpha_k|}) S_{2^{|\alpha_k|}} f}{\alpha_{s_n}} + \frac{2^{|\alpha_k|}}{|A_{|\alpha_k|}| \alpha_{s_n}} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_{s_n}} (D_j - D_{2^{|\alpha_k|}}) \\ &:= III_1 + III_2 + III_3. \end{aligned} \tag{15.29}$$

Due to Lemma 5.3, we get

$$\begin{aligned} |III_3| &= \frac{2^{|\alpha_k|}}{|A_{|\alpha_k|}| \alpha_{s_n}} \Big| \sum_{j=1}^{\alpha_{s_n} - 2^{|\alpha_k|}} (D_{j+2^{|\alpha_k|}} - D_{2^{|\alpha_k|}}) \Big| \\ &= \frac{2^{|\alpha_k|}}{|A_{|\alpha_k|}| \alpha_{s_n}} \Big| \sum_{j=1}^{\alpha_{s_n} - 2^{|\alpha_k|}} D_j \Big| = \frac{2^{|\alpha_k|}}{|A_{|\alpha_k|}| \alpha_{s_n}} (\alpha_{s_n} - 2^{|\alpha_k|}) |K_{\alpha_{s_n} - 2^{|\alpha_k|}}| \\ &\geq \frac{1}{2|A_{|\alpha_k|}|} (\alpha_{s_n} - 2^{|\alpha_k|})\, |K_{\alpha_{s_n} - 2^{|\alpha_k|}}|. \end{aligned} \tag{15.30}$$

Using Corollary 15.3, we find that $\|III_1\|_{1/2} \leq C$ and $\|III_2\|_{1/2} \leq C$.

Let $2^{|\alpha_k|} \leq \alpha_{s_1} \leq \alpha_{s_2} \leq \cdots \leq \alpha_{s_r} \leq 2^{|\alpha_k|+1}$ be natural numbers generating the set

$$A_{|\alpha_k|} = \{l_1^{|\alpha_k|}, l_2^{|\alpha_k|}, \dots, l_{r^1_{|\alpha_k|}}^{|\alpha_k|}\} \cup \{m_1^{|\alpha_k|}, m_2^{|\alpha_k|}, \dots, m_{r^2_{|\alpha_k|}}^{|\alpha_k|}\},$$

and choose a number

$$\alpha_{s_n} = \sum_{i=1}^{r_n} \sum_{k=l_i^n}^{m_i^n} 2^k,$$

where

$$m_1^{|\alpha_k|} \geq l_1^{|\alpha_k|} > l_1^{|\alpha_k|} - 2 \geq m_2^{|\alpha_k|} \geq l_2^{|\alpha_k|} > l_2^{|\alpha_k|} - 2 \geq \cdots \geq m_{|\alpha_k|}^{|\alpha_k|} \geq l_{|\alpha_k|}^{|\alpha_k|} \geq 0,$$

for some $1 \leq n \leq r$ such that $l_u^{|\alpha_k|} = l_i$ for some $1 \leq u \leq r^1_{|\alpha_k|}$, $1 \leq i \leq r^1_{|\alpha_k|}$.

Since $\mu\{E_{l_i}\} \geq 1/2^{l_i+1}$, by Corollary 8.5, we get

$$\int\limits_{E_{l_i}} |\widetilde{\sigma}^{*,\nabla} f|^{1/2}\, d\mu \geq \int\limits_{E_{l_i}} |\sigma_{\alpha_{s_n}} f(x)|^{1/2}\, d\mu \geq \frac{2^{(2l_i-6)/2}}{\sqrt{2}\,(|A_{|\alpha_k|}|)^{1/2}}\, \frac{1}{2^{l_i+1}} \geq \frac{1}{2^5(|A_{|\alpha_k|}|)^{1/2}}\,. \tag{15.31}$$

On the other hand, we can also choose a number α_{s_n}, where $1 \leq n \leq r$ such that $m_u^{|\alpha_k|} = m_i$ for some $1 \leq u \leq r^2_{|\alpha_k|}$, $1 \leq i \leq r^2_{|\alpha_k|}$. According to the fact that $\mu\{E_{m_i}\} \geq 1/2^{m_i+3}$, using again Corollary 8.5 for some α_k and $1 \leq i \leq r_s^2$, we also get

$$\begin{aligned}\int\limits_{E_{m_i}} |\widetilde{\sigma}^{*,\nabla} f|^{1/2}\, d\mu &\geq \int\limits_{E_{t_i}} |\sigma_{\alpha_{s_n}} f(x)|^{1/2}\, d\mu \\ &\geq \frac{1}{\sqrt{2}\,(|A_{|\alpha_k|}|)^{1/2}} \cdots 2^{(2m_i-6)/2} \cdot \frac{1}{2^{m_i+3}} \geq \frac{1}{2^7(|A_{|\alpha_k|}|)^{1/2}}\,. \end{aligned}\tag{15.32}$$

Combining (15.31)–(15.32) with Proposition 8.7, for sufficiently large α_k, we get

$$\begin{aligned}\int\limits_{G} |\widetilde{\sigma}^{*,\nabla} f|^{1/2}\, d\mu &\geq \|III_3\|_{1/2}^{1/2} - \|III_2\|_{1/2}^{1/2} - \|III_1\|_{1/2}^{1/2} \\ &\geq \sum_{i=1}^{r^1_{|\alpha_k|}-1} \int\limits_{E_{l_i}} |\widetilde{\sigma}^{*,\nabla} f|^{1/2}\, d\mu + \sum_{i=1}^{r^2_{|\alpha_k|}-1} \int\limits_{E_{m_i}} |\widetilde{\sigma}^{*,\nabla} f|^{1/2}\, d\mu - 2\,C \\ &\geq \frac{1}{2^7(|A_{|\alpha_k|}|)^{1/2}}\,(r^1_{|\alpha_k|} + r^2_{|\alpha_k|}) - 2\,C \geq \frac{(|A_{|\alpha_k|}|)^{1/2}}{2^8} \longrightarrow \infty \text{ as } k \to \infty,\end{aligned}$$

so part (b) is also proved and the proof is complete. □

In particular, Theorem 15.2 implies the following optimal characterization.

Corollary 15.4. *Let $f \in H_{1/2}(G)$ and $\{n_k : k \geq 0\}$ be a sequence of positive numbers. Then the restricted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by (15.11), is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$ if and only if any sequence of positive numbers $\{n_k : k \geq 0\}$ which satisfies $n_k \in [2^s, 2^{s+1})$, is uniformly finite for each $s \in \mathbb{N}_+$ and each $\{n_k : k \geq 0\}$ has bounded variation, i.e., $\sup_{k\in\mathbb{N}} V(n_k) < c < \infty$.*

Theorem 15.3.

(a) *Let $0 < p \leq 1/2$ and $(\alpha_k, k \in \mathbb{N})$ be a subsequence of positive numbers such that*

$$\sup_{k\in\mathbb{N}} d(\alpha_k) = \varkappa < c < \infty, \tag{15.33}$$

where $d(n)$ is defined in (2.1). Then the maximal operator $\widetilde{\sigma}^{*,\vartriangle}$, defined by $\widetilde{\sigma}^{*,\vartriangle} f := \sup_{k\in\mathbb{N}} |\sigma_{\alpha_k} f|$ is bounded from the Hardy space H_p to the Lebesgue space L_p.

(b) *Let $0<p<1/2$ and $(\alpha_k, k\in\mathbb{N})$ be a subsequence of positive numbers satisfying the condition*

$$\sup_{k\in\mathbb{N}} d(\alpha_k)=\infty. \tag{15.34}$$

Then there exists a martingale $f\in H_p$ such that

$$\sup_{k\in\mathbb{N}} \|\sigma_{\alpha_k} f\|_{weak\text{-}L_p}=\infty. \tag{15.35}$$

Proof.

(a) Since $\widetilde{\sigma}^{*,\vartriangle}$ is bounded from L_∞ to L_∞, the proof of part (a) is complete if

$$\int_{\overline{I}_M} |\widetilde{\sigma}^{*,\vartriangle} a(x)| < c < \infty \tag{15.36}$$

holds for every p-atom a with support I_M and $\mu(I_M)=2^{-M}$. Analogously to the proof of Theorem 14.1, we may assume that $n_k>2^M$.

Since $\|a\|_\infty\le 2^{M/p}$, we find that

$$|\sigma_{n_k} a| \le \int_{I_M} |a(t)|\,|K_{n_k}(x+t)|\,d\mu(t) \le \|a\|_\infty \int_{I_M} |K_{n_k}(x+t)|\,d\mu(t) \le 2^{N/p}\int_{I_M} |K_{n_k}(x+t)|\,d\mu(t). \tag{15.37}$$

Let $x\in I_M^{i,j}$, where $i<j<\langle n_k\rangle$. Then $x+t\in I_M^{i,j}$ for $t\in I_M$ and, according to (8.1) in Lemma 8.1, we obtain $|K_{2^l}(x+t)|=0$ for all $\langle n_k\rangle\le l\le |n_k|$. Let $x\in I_M^{i,j}$, $0\le i<j<\langle n_k\rangle\le l\le |n_k|$. Combining (8.1) in Lemma 8.1, Lemma 8.2 and (15.37), we get

$$|\sigma_{n_k} a(x)| \le 2^{M/p}\sum_{l=\langle n_k\rangle}^{|n_k|}\int_{I_M} |K_{2^l}(x+t)|\,d\mu(t)=0.$$

Let $x\in I_N^{i,j}$, where $\langle n_k\rangle\le j\le M$. Then, in view of Corollary 8.4, we can conclude that

$$\int_{I_M} |K_{n_k}(x+t)|\,d\mu(t) \le \frac{c\cdot 2^{i+j}}{2^{2M}}\,.$$

Using again (15.37), we obtain $|\sigma_{n_k} a(x)|\le C_p\cdot 2^{M(1/p-2)}\cdot 2^{i+j}$. If we define $\varrho:=\min_{k\in\mathbb{N}}\langle \alpha_k\rangle$, then we have

$$|\widetilde{\sigma}^{*,\vartriangle} a(x)|=0 \ \text{ for } \ x\in I_M^{i,j}, \ \ 0\le i<j\le\varrho \tag{15.38}$$

and

$$|\widetilde{\sigma}^{*,\vartriangle} a(x)| \le C_p\cdot 2^{M(1/p-2)}\cdot 2^{i+j} \ \text{ for } \ x\in I_M^{i,j}, \ \ i<\varrho\le j\le M-1. \tag{15.39}$$

Let $p=1/2$. Combining estimates (15.38) and (15.39) with (1.2), we conclude that

$$\begin{aligned}
\int_{\overline{I}_M} |\widetilde{\sigma}^{*,\vartriangle} a|^{1/2}\,d\mu &= \sum_{i=0}^{M-2}\sum_{j=i+1}^{M-1}\int_{I_M^{i,j}} |\widetilde{\sigma}^{*,\vartriangle} a|^{1/2}\,d\mu + \sum_{i=0}^{M-1}\int_{I_M^{i,M}} |\widetilde{\sigma}^{*,\vartriangle} a|^{1/2}\,d\mu \\
&\le \sum_{i=0}^{\varrho-1}\sum_{j=\varrho}^{M-1}\int_{I_M^{i,j}} |\widetilde{\sigma}^{*,\vartriangle} a|^{1/2}\,d\mu + \sum_{i=\varrho}^{M-2}\sum_{j=i+1}^{M-1}\int_{I_M^{i,j}} |\widetilde{\sigma}^{*,\vartriangle} a|^{1/2}\,d\mu + \sum_{i=0}^{M-1}\int_{I_M^{i,M}} |\widetilde{\sigma}^{*,\vartriangle} a|^{1/2}\,d\mu \\
&\le c\sum_{i=0}^{\varrho} 2^{i/2}\sum_{j=\varrho+1}^{M-1}\frac{1}{2^{j/2}} + c\sum_{i=\varrho}^{M-2} 2^{i/2}\sum_{j=i+1}^{M-1}\frac{1}{2^{j/2}} + c\sum_{i=0}^{M-1}\frac{2^{i/2}}{2^{M/2}} \\
&\le c\cdot 2^{\varrho/2}\cdot\frac{1}{2^{\varrho/2}} + c\sum_{i=\varrho}^{M-2} 2^{i/2}\cdot\frac{1}{2^{i/2}} + c \le M-\varrho+c\le C<\infty.
\end{aligned}$$

Let $0<p<1/2$. Combining (15.38) and (15.39) with (1.2), we have

$$\begin{aligned}\int_{\overline{I}_M}|\widetilde{\sigma}^{*,\Delta}a|^p\,d\mu&=\sum_{i=0}^{M-2}\sum_{j=i+1}^{M-1}\int_{I_M^{i,j}}|\widetilde{\sigma}^{*,\Delta}a|^p\,d\mu+\sum_{i=0}^{N-1}\int_{I_M^{k,M}}|\widetilde{\sigma}^{*,\Delta}a|^p\,d\mu\\
&\leq\sum_{i=0}^{\varrho-1}\sum_{j=\varrho}^{M-1}\int_{I_M^{i,j}}|\widetilde{\sigma}^{*,\Delta}a|^p\,d\mu+\sum_{i=\varrho}^{M-2}\sum_{j=i+1}^{M-1}\int_{I_M^{i,j}}|\widetilde{\sigma}^{*,\Delta}a|^p\,d\mu+\sum_{i=0}^{M-1}\int_{I_M^{i,M}}|\widetilde{\sigma}^{*,\Delta}a|^p\,d\mu\\
&\leq C_p\cdot 2^{(1-2p)M}\sum_{i=0}^{\varrho}2^{pi}\sum_{j=\varrho+1}^{M-1}\frac{1}{2^{(1-p)i}}+2^{(1-2p)M}\sum_{i=\varrho}^{M-2}2^{pi}\sum_{j=i+1}^{M-1}\frac{1}{2^{(1-p)j}}+C_p\sum_{i=0}^{M-1}\frac{2^{pi}}{2^{pM}}\\
&\leq\frac{C_p\cdot 2^{(1-2p)M}}{2^{(1-2p)\varrho}}+C_p\leq C_p\cdot 2^{(|n_k|-\langle n_k\rangle)(1-2p)}+C_p=C_p\cdot 2^{d(n_k)(1-2p)}+C_p<\infty.\end{aligned}$$

We conclude that (15.36) holds for $0<p\leq 1/2$, so the proof of part (a) is complete.

(b) Let $(n_k, k\in\mathbb{N})$ be a sequence of positive numbers satisfying condition (15.34). Then

$$\sup_{k\in\mathbb{N}}2^{d(n_k)}=\infty. \tag{15.40}$$

Under condition (15.40), there exists a sequence $(\alpha_k,k\in\mathbb{N})\subset(n_k,k\in\mathbb{N})$ such that

$$\sum_{k=0}^{\infty}\frac{1}{2^{d(\alpha_k)(1-2p)/2}}\leq C_p<\infty.$$

Let $f=(f_n,n\in\mathbb{N})$ be the martingale defined in Example 11.4, where

$$\lambda_k=\frac{1}{2^{d(\alpha_k)(1/p-2)/2}}. \tag{15.41}$$

Applying Theorem 10.1, we can conclude that $f\in H_p$. Using (11.19) with λ_k defined by (15.41), we obtain

$$\widehat{f}(j)=\begin{cases}2^{|\alpha_k|/2p}\cdot 2^{\langle\alpha_k\rangle(1/p-2)/2}, & j\in\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\quad k\in\mathbb{N}_+,\\ 0, & j\notin\bigcup_{k=0}^{\infty}\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\}.\end{cases} \tag{15.42}$$

Therefore,

$$\sigma_{\alpha_k}f=\frac{1}{\alpha_k}\sum_{j=1}^{2^{|\alpha_k|}}S_jf+\frac{1}{\alpha_k}\sum_{j=2^{|\alpha_k|}+1}^{\alpha_k}S_jf=:I+II.$$

Let $2^{|\alpha_k|}<j\leq\alpha_k$. Then, according to (15.42), we have

$$S_jf=S_{2^{|\alpha_k|}}f+2^{|\alpha_k|/2p}\cdot 2^{\langle\alpha_k\rangle(1/p-2)/2}D_{j-2^{|\alpha_k|}}. \tag{15.43}$$

Applying now (15.43), we can rewrite II as

$$II=\frac{\alpha_k-2^{|\alpha_k|}}{\alpha_k}S_{2^{|\alpha_k|}}f+\frac{2^{|\alpha_k|/2p}\cdot 2^{\langle\alpha_k\rangle(1/p-2)/2}w_{2^{|\alpha_k|}}}{\alpha_k}\sum_{j=2^{|\alpha_k|}+1}^{\alpha_k}D_{j-2^{|\alpha_k|}}=:II_1+II_2.$$

Using Corollary 15.3, we get

$$\|I\|_{weak\text{-}L_p}^p=\left(\frac{2^{|\alpha_k|}}{\alpha_k}\right)^p\|\sigma_{2^{|\alpha_k|}}f\|_{weak\text{-}L_p}^p\leq\|\sigma_{2^{|\alpha_k|}}f\|_{weak\text{-}L_p}^p\leq C_p\|f\|_{H_p}^p<\infty \tag{15.44}$$

and

$$\|II_1\|^p_{weak\text{-}L_p} \leq \left(\frac{\alpha_k - 2^{|\alpha_k|}}{\alpha_k}\right)^p \|S_{2^{|\alpha_k|}} f\|^p_{weak\text{-}L_p} \leq C_p \|\sigma_{2^{|\alpha_k|}} f\|^p_{weak\text{-}L_p} \leq C_p \|f\|^p_{H_p} < \infty. \tag{15.45}$$

Under condition (15.34), we can conclude $\langle \alpha_k \rangle \neq |\alpha_k|$ and $\langle \alpha_k - 2^{|\alpha_k|} \rangle = \langle \alpha_k \rangle$. Let $x \in I^{\langle \alpha_k \rangle - 1, \langle \alpha_k \rangle}_{\langle \alpha_k \rangle + 1}$. According to Corollary 8.7, we find that

$$\begin{aligned} |II_2| &= \frac{2^{|\alpha_k|/2p} \cdot 2^{\langle \alpha_k \rangle (1/p-2)/2}}{\alpha_k} \Big| \sum_{j=1}^{\alpha_k - 2^{|\alpha_k|}} D_j \Big| = \frac{2^{|\alpha_k|/2p} \cdot 2^{\langle \alpha_k \rangle (1/p-2)/2}}{\alpha_k} (\alpha_k - 2^{|\alpha_k|}) |K_{\alpha_k - 2^{|\alpha_k|}}| \\ &\geq c \cdot 2^{|\alpha_k|(1/2p-1)} \cdot 2^{\langle \alpha_k \rangle (1/p-2)/2} (\alpha_k - 2^{|\alpha_k|}) |K_{\alpha_k - 2^{|\alpha_k|}}| \geq c \cdot 2^{|\alpha_k|(1/2p-1)} \cdot 2^{\langle \alpha_k \rangle (1/p+2)/2}. \end{aligned}$$

It follows that

$$\begin{aligned} \|II_2\|^p_{weak\text{-}L_p} &\geq C_p \big(2^{|\alpha_k|(1/p-2)/2} \cdot 2^{\langle \alpha_k \rangle (1/p+2)/2}\big)^p \\ &\quad \times \mu\big\{x \in G: \ |II_2| \geq C_p \cdot 2^{|\alpha_k|(1/p-2)/2} \cdot 2^{\langle \alpha_k \rangle (1/p+2)/2}\big\} - C_p \\ &\geq C_p \cdot 2^{|\alpha_k|(1/2-p)} \cdot 2^{\langle \alpha_k \rangle (1/2+p)} \mu\big\{I^{\langle \alpha_k \rangle - 1, \langle \alpha_k \rangle}_{\langle \alpha_k \rangle + 1}\big\} \geq C_p \cdot 2^{d(\alpha_k)(1/2-p)} \longrightarrow \infty \ \text{ as } \ k \to \infty. \end{aligned} \tag{15.46}$$

Combining now (15.45) and (15.46), we can conclude that

$$\begin{aligned} \|\sigma_{\alpha_k} f\|^p_{weak\text{-}L_p} &\geq \|II_2\|^p_{weak\text{-}L_p} - \|II_1\|^p_{weak\text{-}L_p} - \|I\|^p_{weak\text{-}L_p} \\ &\geq \frac{1}{2} \|II_2\|^p_{weak\text{-}L_p} \geq C_p \cdot 2^{d(\alpha_k)(1/2-p)} \longrightarrow \infty \ \text{ as } \ k \to \infty, \end{aligned}$$

which proofs part (b). □

16 Convergence in norm of Fejér means with respect to the one-dimensional Walsh–Fourier series in the martingale Hardy spaces

Theorem 16.1. *Let $0 < p < 1$ and $f \in H_p$. Then there exists an absolute constant C_p depending only on p such that $\|\sigma_{2^n} f\|_{H_p} \leq C_p \|f\|_{H_p}$.*

Proof. In view of Corollary 15.1 and Proposition 11.2, we conclude that

$$\|\sigma_{2^n} f\|_{H_p} \leq \Big\| \sup_{0 \leq l \leq n} |\sigma_{2^l} f| \Big\|_p \leq C_p \|f\|_{H_p}.$$ □

Theorem 16.2. *There exists a martingale $f \in H_{1/2}$ such that*

$$\sup_{n \in \mathbb{N}} \|\sigma_n f\|_{1/2} = \infty. \tag{16.1}$$

Proof. Let $f = (f_n, n \in \mathbb{N})$ be the martingale defined in Example 11.5 in the case $p = q = 1/2$. We have

$$\sigma_{q_{\alpha_k}} f = \frac{1}{q_{\alpha_k}} \sum_{j=1}^{2^{2\alpha_k}} S_j f + \frac{1}{q_{\alpha_k}} \sum_{j=2^{2\alpha_k}+1}^{q_{\alpha_k}} S_j f =: III + IV. \tag{16.2}$$

According to (11.25) and (11.28) for $p = q = 1/2$, we conclude that

$$|III| \leq \frac{1}{q_{\alpha_k}} \sum_{j=1}^{2^{2\alpha_k}} |S_i f| \leq \frac{2^{4\alpha_{k-1}+1}}{\alpha_{k-1}^{1/2}} \frac{2^{2\alpha_k}}{q_{\alpha_k}} \leq \frac{2^{4\alpha_{k-1}+1}}{\alpha_{k-1}^{1/2}} \leq \frac{2^{2\alpha_k}}{16\alpha_k^{3/2}}. \tag{16.3}$$

Applying (11.27) obtained by letting $l = k$, we can rewrite IV as

$$IV = \frac{(q_{\alpha_k} - 2^{2\alpha_k})S_{2^{2\alpha_k}} f}{q_{\alpha_k}} + \frac{2^{2\alpha_k} w_{2^{2\alpha_k}}}{\alpha_k^{1/2} q_{\alpha_k}} \sum_{j=2^{2\alpha_k}+1}^{q_{\alpha_k}} D_{j-2^{2\alpha_k}} =: IV_1 + IV_2.$$

According to (11.28) for $j = 2^{2\alpha_k}$ and invoke estimate (11.25) for $p = q = 1/2$, we find that

$$|IV_1| \leq |S_{2^{2\alpha_k}} f| \leq \frac{2^{4\alpha_{k-1}+1}}{\alpha_{k-1}^{1/2}} \leq \frac{2^{2\alpha_k}}{16\alpha_k^{3/2}}. \tag{16.4}$$

Let $x \in I_{2\eta+1}(e_{2\eta-1} + e_{2\eta})$, $\eta = [\frac{2\alpha_k}{3}] + 1, \ldots, \alpha_k - 3$. Applying Corollary 8.6, we have

$$144 q_{\alpha_k - 1} |K_{q_{\alpha_k - 1}}(x)| \geq 2^{4\eta}.$$

Hence, for IV_2, we readily get

$$|IV_2| = \frac{2^{2\alpha_k}}{\alpha_k^{1/2} q_{\alpha_k}} \left| w_{2^{2\alpha_k}} \sum_{j=1}^{q_{\alpha_k - 1}} D_j \right| = \frac{2^{2\alpha_k}}{q_{\alpha_k}} \frac{q_{\alpha_k - 1}}{\alpha_k^{1/2}} |K_{q_{\alpha_k} - 1}| \geq \frac{q_{\alpha_k - 1} |K_{q_{\alpha_k} - 1}|}{2\alpha_k^{1/2}} \geq \frac{2^{4\eta}}{288\alpha_k^{1/2}}. \tag{16.5}$$

Therefore, using (16.2), (16.3), (16.4) and (16.5), we find that

$$|\sigma_{q_{\alpha_k}} f(x)| \geq |IV_2| - \big(|III| + |IV_1|\big) \geq \frac{c}{\alpha_k^{1/2}} \Big(2^{4\eta} - \frac{1442^{2\alpha_k}}{\alpha_k}\Big). \tag{16.6}$$

Thus, for a sufficiently large k, we have $2^{2\eta} \geq 1442^{\alpha_k}$, where $\alpha_k > 1$, and it follows that for some $c > 0$,

$$|\sigma_{q_{\alpha_k}} f(x)| \geq \frac{c \cdot 2^{4\eta}}{\alpha_k^{1/2}}, \quad x \in I_{2\eta+1}(e_{2\eta-1} + e_{2\eta}), \quad \eta = \left[\frac{2\alpha_k}{3}\right] + 1, \ldots, \alpha_k - 3.$$

Hence,

$$\begin{aligned}
\int_G |\sigma_{q_{\alpha_k}} f(x)|^{1/2} \, d\mu(x) &\geq \sum_{\eta=[2\alpha_k/3]+1}^{\alpha_k - 3} \int_{I_{2\eta+1}(e_{2\eta-1}+e_{l_{2\eta}})} |\sigma_{q_{\alpha_k}} f(x)|^{1/2} \, d\mu(x) \\
&\geq \frac{c}{\alpha_k^{1/4}} \sum_{\eta=[2\alpha_k/3]+1}^{\alpha_k - 3} \big| I_{2\eta+1}(e_{2\eta-1} + e_{l_{2\eta}}) \big| \cdot 2^{2\eta} \geq \frac{c}{\alpha_k^{1/4}} \sum_{\eta=[2\alpha_k/3]+1}^{\alpha_k - 3} 1 \geq \alpha_k^{3/4} \longrightarrow \infty \;\text{ as }\; k \to \infty.
\end{aligned}$$

Thus (16.1) holds. □

We also point out the following consequence of Theorem 14.1 which will be used later.

Corollary 16.1. *Let $f \in H_{1/2}$. Then there exists an absolute constant c such that*

$$\|\sigma_n f\|_{1/2} \leq c \log^2(n+1) \|f\|_{H_{1/2}}, \quad n \in \mathbb{N}_+. \tag{16.7}$$

Proof. According to part (a) of Theorem 14.1, we readily conclude that

$$\left\| \frac{\sigma_n f}{\log^2(n+1)} \right\|_{1/2} \leq \left\| \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{\log^2(n+1)} \right\|_{1/2} \leq c \, \|f\|_{H_{1/2}}, \quad n \in \mathbb{N}_+,$$

which implies that (16.7) holds, and the proof is complete. □

The following statement, which is a consequence of theorem 14.2, will be needed later.

Corollary 16.2. *Let $0 < p < 1/2$ and $f \in H_p$. Then there exists an absolute constant C_p depending only on p such that*

$$\|\sigma_n f\|_p \leq C_p (n+1)^{1/p - 2} \|f\|_{H_p}, \quad n \in \mathbb{N}_+. \tag{16.8}$$

Proof. According to part (a) of Theorem 14.2, we conclude that

$$\left\|\frac{\sigma_n f}{(n+1)^{1/p-2}}\right\|_p \leq \left\|\sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{(n+1)^{1/p-2}}\right\|_p \leq C_p\|f\|_{H_p}, \quad n\in\mathbb{N}_+,$$

which implies that (16.8) holds. □

Corollary 16.3. *Let $p>0$ and $f\in H_p(G)$. Then $\|\sigma_{2^k}f-f\|_{H_p(G)}\to 0$ as $k\to\infty$ and $\|\sigma_{2^k}f\|_{H_p(G)}\leq C_p\|f\|_{H_p(G)}$, where C_p is an absolute constant.*

Since $D_{2^n}\leq 2K_{2^n}$ for all $k\in\mathbb{N}$, we find that the following statement is true.

Corollary 16.4. *Let $p>0$ and $f\in H_p(G)$. Then $\|S_{2^k}f-f\|_{H_p(G)}\to 0$ as $k\to\infty$ and $\|S_{2^k}f-f\|_{H_p(G)}\leq C_p\|f\|_{H_p(G)}$, where C_p is an absolute constant.*

17 Modulus of continuity and convergence in norm of Fejér means with respect to the one-dimensional Walsh–Fourier series in the martingale Hardy spaces

In this section, we find the necessary and sufficient conditions for the modulus of continuity in the martingale Hardy spaces H_p that ensure the convergence of partial sums. We also describe and motivate the sharpness of these results.

Theorem 17.1. *Let $0<p<1/2$ and $f\in H_p$. Then there exists an absolute constant C_p depending only on p such that $\|\sigma_n f\|_{H_p}\leq C_p n^{1/p-2}\|f\|_{H_p}$.*

Proof. According to Proposition 11.2 and Corollary 16.2, we arrive at

$$\|\sigma_n f\|_{H_p}\leq \|S^{\#}f\|_p+\|\sigma^{\#}f\|_p+\|\sigma_n f\|_p\leq \|f\|_{H_p}+C_p n^{1/p-2}\|f\|_{H_p}\leq C_p n^{1/p-2}\|f\|_{H_p}. \tag{17.1}$$

The proof is complete. □

Remark 17.1. Note that the asymptotic behaviour of the sequence $(n^{1/p-2}, n\in\mathbb{N})$ in Theorem 17.1 cannot be improved (cf. part (b) of Theorem 14.2).

Theorem 17.2. *Let $f\in H_{1/2}$. Then there exists an absolute constant C such that $\|\sigma_n f\|_{H_{1/2}}\leq C\log^2 n\|f\|_{H_{1/2}}$.*

Proof. According to Theorem 17.1 and Corollaries 15.1 and 16.1, if we invoke Proposition 11.2, we find that

$$\|\sigma_n f\|_{H_{1/2}}\leq \|S^{\#}f\|_{1/2}+\|\sigma^{\#}f\|_{1/2}+\|\sigma_n f\|_{1/2}\leq \|f\|_{H_{1/2}}+c\log^2 n\|f\|_{H_{1/2}}\leq C\log^2 n\|f\|_{H_{1/2}}.$$ □

Remark 17.2. Note that the asymptotic behaviour of the sequence $(\log^2 n, n\in\mathbb{N})$ in Theorem 17.1 cannot be improved (cf. part (b) of Theorem 14.1).

Theorem 17.3.

(a) *Let $0<p<1/2$, $f\in H_p$ and*

$$\omega_{H_p}\left(\frac{1}{2^n},f\right)=o\left(\frac{1}{2^{n(1/p-2)}}\right) \quad \text{as } n\to\infty. \tag{17.2}$$

Then $\|\sigma_n f-f\|_{H_p}\to 0$ as $n\to\infty$.

(b) *Let $0<p<1/2$. Then there exists a martingale $f\in H_p$ for which*

$$\omega\left(\frac{1}{2^n},f\right)_{H_p}=O\left(\frac{1}{2^{n(1/p-2)}}\right) \quad \text{as } n\to\infty$$

and $\|\sigma_n f-f\|_{weak\text{-}L_p}\nrightarrow 0$ as $n\to\infty$.

Proof.

(a) Let $f \in H_p$, $0 < p < 1/2$ and $2^N < n \le 2^{N+1}$. Making some routine calculations, we get

$$\begin{aligned}\sigma_n S_{2^N} f - S_{2^N} f &= \frac{1}{n}\sum_{k=0}^{2^N} S_k S_{2^N} f + \frac{1}{n}\sum_{k=2^N+1}^{n} S_k S_{2^N} f - S_{2^N} f \\ &= \frac{1}{n}\sum_{k=0}^{2^N} S_k f + \frac{1}{n}\sum_{k=2^N+1}^{n} S_{2^N} f - S_{2^N} f = \frac{1}{n}\sum_{k=0}^{2^N} S_k f + \frac{n-2^N}{n} S_{2^N} f - S_{2^N} f \\ &= \frac{2^N}{n}\sigma_{2^N} f - \frac{2^N}{n} S_{2^N} f = \frac{2^N}{n}(S_{2^N}\sigma_{2^N} f - S_{2^N} f) = \frac{2^N}{n} S_{2^N}(\sigma_{2^N} f - f).\end{aligned}$$

Hence,

$$\begin{aligned}\|\sigma_n f - f\|_{H_p}^p &\le \|\sigma_n f - \sigma_n S_{2^N} f\|_{H_p}^p + \|\sigma_n S_{2^N} f - S_{M_N} f\|_{H_p}^p + \|S_{2^N} f - f\|_{H_p}^p \\ &= \|\sigma_n (S_{2^N} f - f)\|_{H_p}^p + \|S_{2^N} f - f\|_{H_p}^p + \|\sigma_n S_{2^N} f - S_{2^N} f\|_{H_p}^p \\ &\le C_p (n^{1-2p}+1)\omega_{H_p}^p\Big(\frac{1}{2^N}, f\Big) + \|S_{2^N}(\sigma_{2^N} f - f)\|_p^p.\end{aligned}$$

Then, according to Corollary 15.3 and Theorem 16.1, we obtain

$$\|S_{2^k} f - f\|_{H_p} \to 0 \ \text{ as } \ k \to \infty, \tag{17.3}$$
$$\|\sigma_{2^k} f - f\|_{H_p} \to 0 \ \text{ as } \ k \to \infty. \tag{17.4}$$

Thus,

$$\begin{aligned}\|\sigma_n S_{2^N} f - S_{2^N} f\|_{H_p}^p &= \frac{2^{pN}}{n^p}\|S_{2^N}(\sigma_{2^N} f - f)\|_{H_p}^p \\ &\le \|S_{2^N}(\sigma_{2^N} f - f)\|_{H_p}^p \le \|\sigma_{2^N} f - f\|_{H_p}^p \longrightarrow 0 \ \text{ as } \ k \to \infty.\end{aligned}$$

It follows that, under condition (17.2), $\|\sigma_n f - f\|_{H_p} \to 0$ as $n \to \infty$, so, the proof of part (a) is complete.

(b) Let $f = (f_n, n \in \mathbb{N})$ be the martingale defined in Lemma 11.3, where $\lambda_k = \frac{1}{2^{2\alpha_k(1/p-2)}}$. Combining (11.22) in Lemma 11.3, we conclude that $f \in H_p$ and

$$\omega_{H_p}\Big(\frac{1}{2^n}, f\Big) \le \Big(\sum_{\{k, 2\alpha_k \ge n\}} \frac{1}{2^{2\alpha_k(1-2p)}}\Big)^{1/p} = O\Big(\frac{1}{2^{n(1/p-2)}}\Big) \ \text{ as } \ n \to \infty.$$

A simple calculation gives

$$\widehat{f}(j) = \begin{cases} 2^{2\alpha_i}, & j \in \{2^{2\alpha_i}, \dots, 2^{2\alpha_i+1}-1\}, \ \ i \in \mathbb{N}_+, \\ 0, & j \notin \bigcup_{i=0}^{\infty}\{2^{2\alpha_i}, \dots, 2^{2\alpha_i+1}-1\}. \end{cases} \tag{17.5}$$

By (17.5) we obtain

$$\begin{aligned}&\|\sigma_{2^{2\alpha_k}+1} f - f\|_{weak\text{-}L_p} \\ &= \Big\|\frac{2^{2\alpha_k}\sigma_{2^{2\alpha_k}} f}{2^{2\alpha_k}+1} + \frac{S_{2^{2\alpha_k}} f}{2^{2\alpha_k}+1} + \frac{2^{2\alpha_k} w_{2^{2\alpha_k}+1}}{2^{2\alpha_k}+1} - \frac{2^{2\alpha_k} f}{2^{2\alpha_k}+1} - \frac{f}{2^{2\alpha_k}+1}\Big\|_{weak\text{-}L_p} \\ &\ge \frac{2^{2\alpha_k}}{2^{2\alpha_k}+1}\|w_{2^{2\alpha_k}}\|_{weak\text{-}L_p} - \frac{2^{2\alpha_k}}{2^{2\alpha_k}+1}\|\sigma_{2^{2\alpha_k}} f - f\|_{weak\text{-}L_p} - \frac{1}{2^{2\alpha_k}+1}\|S_{2^{2\alpha_k}} f - f\|_{weak\text{-}L_p} \\ &\ge \frac{2^{2\alpha_k}}{2^{2\alpha_k}+1} - o(1) \ \text{ as } \ k \to \infty.\end{aligned}$$

Therefore,

$$\limsup_{k\to\infty}\|\sigma_{2^{2\alpha_k}+1}f-f\|_{weak\text{-}L_p}\geq\limsup_{k\to\infty}\frac{2^{2\alpha_k}}{2^{2\alpha_k}+1}\geq c>0.$$

Hence part (b) is proved, and so the proof is complete. □

Theorem 17.4.

(a) *Let $f\in H_{1/2}$ and*

$$\omega_{H_{1/2}}\Big(\frac{1}{2^n},f\Big)=o\Big(\frac{1}{n^2}\Big)\quad as\ \ n\to\infty. \tag{17.6}$$

Then $\|\sigma_n f-f\|_{H_{1/2}}\to 0$ as $n\to\infty$.

(b) *There exists a martingale $f\in H_{1/2}$ for which*

$$\omega_{H_{1/2}}\Big(\frac{1}{2^n},f\Big)=O\Big(\frac{1}{n^2}\Big)\quad as\ \ n\to\infty$$

and $\|\sigma_n f-f\|_{1/2}\nrightarrow 0$ as $n\to\infty$.

Proof. Let $f\in H_{1/2}$ and $2^N<n\leq 2^{N+1}$. Using identity (9.1), we find that

$$\begin{aligned}\|\sigma_n f-f\|_{H_{1/2}}^{1/2}&\leq\|\sigma_n f-\sigma_n S_{2^N}f\|_{H_{1/2}}^{1/2}+\|\sigma_n S_{2^N}f-S_{2^N}f\|_{H_{1/2}}^{1/2}+\|S_{2^N}f-f\|_{H_{1/2}}^{1/2}\\&=\|\sigma_n(S_{2^N}f-f)\|_{H_{1/2}}^{1/2}+\|S_{2^N}f-f\|_{H_{1/2}}^{1/2}+\|\sigma_n S_{2^N}f-S_{2^N}f\|_{H_{1/2}}^{1/2}\\&\leq c(\log^2 n+1)\omega_{H_{1/2}}^{1/2}\Big(\frac{1}{2^N},f\Big)+\|\sigma_n S_{2^N}f-S_{2^N}f\|_{H_{1/2}}^{1/2}.\end{aligned}$$

Combining (17.3), (17.4) with equality (9.1), we have

$$\|\sigma_n S_{2^N}f-S_{2^N}f\|_{H_{1/2}}^{1/2}=\frac{2^N}{n}\|S_{2^N}(\sigma_{2^N}f-f)\|_{H_{1/2}}^{1/2}\longrightarrow 0\ \text{ as }\ k\to\infty.$$

It follows that under condition (17.6), $\|\sigma_n f-f\|_{H_{1/2}}\to 0$ as $n\to\infty$, so part (a) is proved.

(b) Let $f=(f_n,n\in\mathbb{N})$ be a martingale from Lemma 11.7, where $p=1/2$. Then $f\in H_{1/2}$ and

$$\omega_{H_{1/2}}\Big(\frac{1}{2^n},f\Big)=O\Big(\frac{1}{n^2}\Big)\ \text{ as }\ n\to\infty.$$

Hence,

$$\sigma_{q_{2^k}}f-f=\frac{2^{2^{k+1}}\sigma_{2^{2^{k+1}}}f}{q_{2^k}}+\frac{1}{q_{2^k}}\sum_{j=2^{2^{k+1}}+1}^{q_{2^k}}S_jf-\frac{2^{2^{k+1}}f}{q_{2^k}}-\frac{q_{2^k-1}f}{q_{2^k}}. \tag{17.7}$$

Let $2^{2^{k+1}}<j\leq q_{2^k}$. Combining (11.37) and (11.39), we have

$$\begin{aligned}\frac{1}{q_{2^k}}\sum_{j=2^{2^{k+1}}+1}^{q_{2^k}}S_jf&=\frac{q_{2^k-1}S_{2^{2^{k+1}}}f}{q_{2^k}}+\frac{2^{2^{k+1}}w_{2^{2^{k+1}}}}{q_{2^k}\cdot 2^{2k}}\sum_{j=1}^{q_{2^k-1}}D_j\\&=\frac{q_{2^k-1}S_{2^{2^{k+1}}}f}{q_{2^k}}+\frac{2^{2^{k+1}}w_{2^{2^{k+1}}}q_{2^k-1}K_{q_{2^k-1}}}{q_{2^k}\cdot 2^{2k}}.\end{aligned}$$

Due to (17.7), we obtain

$$\begin{aligned}&\|\sigma_{q_{2^k}}f-f\|_{1/2}^{1/2}\\&\geq\frac{c}{2^k}\|q_{2^k-1}K_{q_{2^k-1}}\|_{1/2}^{1/2}-\Big(\frac{2^{k+1}}{q_{2^k}}\Big)^{1/2}\|\sigma_{2^{2^{k+1}}}f-f\|_{1/2}^{1/2}-\Big(\frac{q_{2^k-1}}{q_{2^k}}\Big)^{1/2}\|S_{2^{2^{k+1}}}f-f\|_{1/2}^{1/2}.\end{aligned} \tag{17.8}$$

Let $x \in I_{2s+1}(e_{2s-1}+e_{2s})$, $s = 1, \ldots, 2^k-2$. Moreover, applying Corollary 8.6, we find that for some $c > 0$, $q_{2k-1}|K_{q_{2^k-1}}(x)| \geq c2^{4s}$ so that

$$\int\limits_G |q_{2^k-1}K_{q_{2^k-1}}|^{1/2}\,d\mu \geq c\sum_{s=0}^{2^k-2}\int\limits_{I_{2s+1}^{2s-1,2s}} |q_{2^k-1}K_{q_{2^k-1}}|^{1/2}\,d\mu \geq c\sum_{s=1}^{2^k-2}\frac{1}{2^{2s+1}}\cdot 2^{2s} \geq c\cdot 2^k. \tag{17.9}$$

Combining (17.8) and (17.9) we obtain $\limsup_{k\to\infty}\|\sigma_{q_{2k}}f-f\|_{1/2} \geq c > 0$, which implies part (b). □

18 Boundedness of restricted and weighted maximal operators of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

Our first theorem of this section was proved in [8].

Theorem 18.1.

(a) *Let $f \in H_{1/2}$ and $\{n_k : k \geq 0\}$ be a sequence of positive numbers. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by*

$$\widetilde{\sigma}^{*,\nabla}f := \sup_{k\in\mathbb{N}}\frac{|\sigma_{n_k}f|}{|A_{|n_k|}|^2}$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

(b) *(Sharpness) Let $\sup_{k\in\mathbb{N}}|A_{|n_k|}| = \infty$ and $\{\varphi_n\}$ be a non-decreasing sequence satisfying the condition*

$$\overline{\lim_{k\to\infty}}\frac{|A_{|n_k|}|^2}{\varphi_{|n_k|}} = \infty.$$

Then there exist the p-atoms a_{n_k} such that

$$\frac{\left\|\sup_{k\in\mathbb{N}}\frac{|\sigma_{n_k}a_k|}{\varphi_{|n_k|}}\right\|_p}{\|a_k\|_{H_p}} = \infty.$$

Proof. In view of Lemma 10.3, the proof of part (a) will be completed if we prove that

$$\int\limits_{\overline{I}_M}\left(\sup_{k\in\mathbb{N}}\frac{|\sigma_{n_{s_k}}a(x)|}{|A_{|n_{s_k}|}|^2}\right)^{1/2}d\mu(x) \leq c < \infty \quad \text{for every } 1/2\text{-atom } a. \tag{18.1}$$

We may assume that a is an arbitrary 1/2-atom, with support I, where $\mu(I) = 2^{-M}$ and $I = I_M$. Since $\sigma_n a = 0$, when $n < 2^M$, we can suppose that $n_{s_k} \geq 2^M$.

Let $x \in I_M$ and $2^s \leq n_{s_k} < 2^{s+1}$ for some $n_{s_k} \geq 2^M$. According to $\|a\|_\infty \leq 2^{2M}$ and $|n_{s_k}| = s$, using Lemma 8.4, we get

$$\begin{aligned}
\frac{|\sigma_{n_{s_k}}a(x)|}{|A_{|n_{s_k}|}|^2} &\leq \frac{2^{2M}}{|A_s|^2}\int\limits_{I_M}|K_{n_{s_k}}(x+t)|\,d\mu(t)\\
&\leq \frac{2^M}{|A_s|^2 2^s}\left(2^M\sum_{A=1}^{r_s^1}\int\limits_{I_M}2^{l_A^s}|K_{2^{l_A^s}}(x+t)|\,d\mu(t)\right) + \frac{2^M}{|A_s|^2 2^s}\left(2^M\sum_{A=1}^{r_s^2}\int\limits_{I_M}2^{t_A^s}|K_{2^{t_A^s}}(x+t)|\,d\mu(t)\right)\\
&\quad + \frac{2^M}{|A_s|^2 2^s}\left(2^M\sum_{A=1}^{r_s^1}\int\limits_{I_M}2^{l_j^s}\sum_{k=l_j^s}^{\infty}D_{2^k}(x+t)\,d\mu(t)\right).
\end{aligned} \tag{18.2}$$

Defining

$$II^1_{\alpha^s_A}(x) := 2^M \int_{I_M} 2^{\alpha^s_A} |K_{2^{\alpha^s_A}}(x+t)|\, d\mu(t), \quad \alpha = l \text{ or } \alpha = t,$$

$$II^2_{l^s_A}(x) := 2^M \int_{I_M} 2^{l^s_A} \sum_{k=l^s_A}^{\infty} D_{2^k}(x+t)\, d\mu(t),$$

from (18.2) we can see that

$$\sup_{2^s \le n_{s_k} < 2^{s+1}} \frac{|\sigma_{n_{s_k}} a|}{|A_{|n_{s_k}|}|^2} \le \frac{2^M}{|A_s|^2 2^s} \Big(\sum_{A=1}^{r^1_s} II^1_{l^s_A} + \sum_{A=1}^{r^2_s} II^1_{t^s_A} + \sum_{A=1}^{r^1_s} II^2_{l^s_A} \Big).$$

Hence,

$$\int_{\overline{I}_M} \Big(\sup_{2^s \le n_{s_k} < 2^{s+1}} \frac{|\sigma_{n_{s_k}} a|}{|A_{|n_{s_k}|}|^2} \Big)^{1/2} d\mu$$
$$\le \frac{2^{M/2}}{2^{s/2}|A_s|} \Big(\sum_{A=1}^{r^1_s} \int_{\overline{I}_M} |II^1_{l^s_A}|^{1/2}\, d\mu + \sum_{A=1}^{r^2_s} \int_{\overline{I}_M} |II^1_{t^s_A}|^{1/2}\, d\mu + \sum_{A=1}^{r^1_s} \int_{\overline{I}_M} |II^2_{l^s_A}|^{1/2}\, d\mu \Big). \quad (18.3)$$

Since $\sup_{s\in\mathbb{N}} r_s \le |A_s|$ and $\sup_{s\in\mathbb{N}} r^2_s \le |A_s|$, we find that (18.1) holds, so the theorem will be proved if we prove that

$$\int_{\overline{I}_M} |II^2_{l^s_A}(x)|^{1/2}\, d\mu(x) \le c < \infty, \quad A = 1, \dots, r^1_s, \quad (18.4)$$

and

$$\int_{\overline{I}_M} |II^1_{\alpha^s_A}(x)|^{1/2}\, d\mu(x) \le c < \infty \quad (18.5)$$

for all $\alpha^s_A = l^s_A$, $A = 1, \dots, r^1_s$ and $\alpha^s_A = t^s_A$, $A = 1, \dots, r^2_s$.

Indeed, if (18.4) and (18.5) hold, from (18.3) we get

$$\int_{\overline{I}_M} \Big(\sup_{n_{s_k} \ge 2^M} \frac{|\sigma_{n_{s_k}} a|}{|A_{|n_{s_k}|}|^2} \Big)^{1/2} d\mu \le \sum_{s=M}^{\infty} \int_{\overline{I}_M} \Big(\sup_{2^s \le n_{s_k} < 2^{s+1}} \frac{|\sigma_{n_{s_k}} a|}{|A_{|n_{s_k}|}|^2} \Big)^{1/2}$$
$$\le \sum_{s=M}^{\infty} \frac{c \cdot 2^{M/2}}{2^{s/2}} \frac{1}{A_s} \cdot (2r^1_s + r^2_s) \le \sum_{s=M}^{\infty} \frac{c \cdot 2^{M/2}}{2^{s/2}} < C < \infty.$$

It remains to prove (18.4) and (18.5).

Let $t \in I_M$ and $x \in I^{k,l}_M$. If $0 \le k < l < \alpha^s_A \le M$ or $0 \le k < l < M < \alpha^s_A$, then $x + t \in I^{k,l}_M$, and if we apply (8.1) in Lemma 8.1, we obtain

$$K_{2^{\alpha^s_A}}(x+t) = 0 \text{ and } II^1_{\alpha^s_A}(x) = 0. \quad (18.6)$$

Let $0 \le k < \alpha^s_A \le l < M$. Then $x + t \in I^{k,l}_M$, and if we use (8.1) in Lemma 8.1, we get $2^{\alpha^s_A}|K_{2^{\alpha^s_A}}(x+t)| \le 2^{\alpha^s_A + k}$ so,

$$II^1_{\alpha^s_A}(x) \le 2^{\alpha^s_A + k}. \quad (18.7)$$

Analogously to (18.7), we can prove that if $0 \le \alpha^s_A \le k < l < M$, then $2^{\alpha^s_A}|K_{2^{\alpha^s_A}}(x+t)| \le 2^{2\alpha^s_A}$, $t \in I_M$, $x \in I^{k,l}_M$ so that

$$II^1_{\alpha^s_A}(x) \le 2^{2\alpha^s_A}, \quad t \in I_M, \quad x \in I^{k,l}_M. \quad (18.8)$$

Let $t \in I_M$ and $x \in I_M^{k,M}$ for $0 \leq k < \alpha_A^s \leq M$ or $0 \leq k < M \leq \alpha_A^s$. Since $x + t \in x \in I_M^{k,M}$, if using again (8.1) in Lemma 8.1, we find that $2^{\alpha_A^s}|K_{2^{\alpha_A^s}}(x+t)| \leq 2^{\alpha_A^s + k}$, so,

$$II^1_{\alpha_A^s}(x) \leq 2^{\alpha_A^s + k}. \tag{18.9}$$

Let $0 \leq \alpha_A^s \leq k < M$. Since $x + t \in x \in I_M^{k,M}$, in view of (8.1) in Lemma 8.1, we have $2^{\alpha_A^s}|K_{2^{\alpha_A^s}}(x+t)| \leq 2^{2\alpha_A^s}$, so,

$$II^1_{\alpha_A^s}(x) \leq 2^{2\alpha_A^s}. \tag{18.10}$$

Let $0 \leq \alpha_A^s < M$, $A = 1, \ldots, s$. Combining (1.2) with (18.6)–(18.10), we get

$$\begin{aligned}
\int\limits_{\overline{I}_M} |II^1_{\alpha_A^s}|^{1/2}\, d\mu &= \sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1} \int\limits_{I_M^{k,l}} |II^1_{\alpha_A^s}|^{1/2}\, d\mu + \sum_{k=0}^{M-1} \int\limits_{I_M^{k,M}} |II^1_{\alpha_A^s}|^{1/2}\, d\mu \\
&\leq c \sum_{k=0}^{\alpha_A^s - 1}\sum_{l=\alpha_A^s}^{M-1} \int\limits_{I_M^{k,l}} 2^{(\alpha_A^s + k/2}\, d\mu + c\sum_{k=\alpha_A^s}^{M-2}\sum_{l=k+1}^{M-1} \int\limits_{I_M^{k,l}} 2^{\alpha_A^s}\, d\mu \\
&\quad + c\sum_{k=0}^{\alpha_A^s - 1} \int\limits_{I_M^{k,M}} 2^{(\alpha_A^s + k)/2}\, d\mu + c\sum_{k=\alpha_A^s}^{M-1} \int\limits_{I_M^{k,M}} 2^{\alpha_A^s}\, d\mu \\
&\leq c\sum_{k=0}^{\alpha_A^s - 1}\sum_{l=\alpha_A^s+1}^{M-1} \frac{2^{(\alpha_A^s + k)/2}}{2^l} + c\sum_{k=\alpha_A^s}^{M-2}\sum_{l=k+1}^{M-1} \frac{2^{\alpha_A^s}}{2^l} + c\sum_{k=0}^{\alpha_A^s - 1} \frac{2^{(\alpha_A^s + k)/2}}{2^M} + c\sum_{k=\alpha_A^s}^{M-1} \frac{2^{\alpha_A^s}}{2^M} \leq C < \infty,
\end{aligned}$$

which means that (15.16) holds for $\alpha_A^s < M$.

Analogously, we can prove that (15.16) holds for $\alpha_A^s \geq M$ as well. Hence, (15.16) holds and it remains to prove (15.15).

Next, we prove the boundedness of $II^2_{l_A^s}$. Let $t \in I_M$ and $x \in I_i \setminus I_{i+1}$. If $i \leq l_A^s - 1$, since $x + t \in I_i \setminus I_{i+1}$, by Lemma 5.1, we can conclude that

$$II^2_{l_A^s}(x) = 0. \tag{18.11}$$

If $l_A^s \leq i < M$, then using again Lemma 5.1, we obtain

$$II^2_{l_A^s}(x) \leq 2^M \int\limits_{I_M} 2^{l_A^s} \sum_{k=l_A^s}^{i} D_{2^k}(x+t)\, d\mu(t) \leq c \cdot 2^{l_A^s + i}. \tag{18.12}$$

Hence, for $0 \leq l_A^s < M$, Combining (1.2), (18.11) and (18.12), we get

$$\begin{aligned}
\int\limits_{\overline{I}_M} |II^2_{l_A^s}|^{1/2}\, d\mu &= \Big(\sum_{i=0}^{l_A^s - 1} + \sum_{i=l_A^s+1}^{M-1}\Big) \int\limits_{I_i \setminus I_{i+1}} |II^2_{l_A^s}|^{1/2}\, d\mu \\
&\leq c\sum_{i=l_A^s}^{M-1} \int\limits_{I_i \setminus I_{i+1}} 2^{(l_A^s + i)/2}\, d\mu \leq c\sum_{i=l_A^s}^{M-1} 2^{(l_A^s+i)/2} \cdot \frac{1}{2^i} \leq C < \infty.
\end{aligned} \tag{18.13}$$

If $M \leq l_A^s$, then $i < M \leq l_A^s$, and applying (18.11), we arrive at

$$\int\limits_{\overline{I}_M} |II^2_{l_A^s}(x)|^{1/2}\, d\mu(x) = 0, \tag{18.14}$$

and (15.15) holds for this case too. So, (15.15) is proved by combining (18.13) and (18.14). Thus, the proof of part (a) is complete and we turn to the proof of part (b).

Under condition (14.6), there exists an increasing sequence $\{\alpha_k :\ k \geq 0\}$ of positive integers such that

$$\lim_{k\to\infty} \frac{|A_{|\alpha_k|}|^{1/2}}{\varphi_{|\alpha_k|}^{1/4}} = \infty$$

and

$$\sum_{k=1}^{\infty} \frac{\varphi_{|\alpha_k|}^{1/4}}{|A_{|\alpha_k|}|^{1/2}} \leq c < \infty. \tag{18.15}$$

Let a_k be a p-atom from Example 11.1 and let $2^{|\alpha_k|} \leq \alpha_{s_n} \leq 2^{|\alpha_k|+1}$. Using (11.4), we find that

$$|\sigma_{\alpha_{s_n}} a_k| = \frac{2^{|\alpha_k|}}{\alpha_{s_n}} (\alpha_{s_n} - 2^{|\alpha_k|})|K_{\alpha_{s_n} - 2^{|\alpha_k|}}| \geq (\alpha_{s_n} - 2^{|\alpha_k|})|K_{\alpha_{s_n} - 2^{|\alpha_k|}}|.$$

Let $2^{|\alpha_k|} \leq \alpha_{s_1} \leq \alpha_{s_2} \leq \cdots \leq \alpha_{s_r} \leq 2^{|\alpha_k|+1}$ be natural numbers generating the set

$$A_{|\alpha_k|} = \{l_1^{|\alpha_k|}, l_2^{|\alpha_k|}, \dots, l_{r^1_{|\alpha_k|}}^{|\alpha_k|}\} \cup \{m_1^{|\alpha_k|}, m_2^{|\alpha_k|}, \dots, m_{r^2_{|\alpha_k|}}^{|\alpha_k|}\},$$

where $m_1^{|\alpha_k|} \geq l_1^{|\alpha_k|} > l_1^{|\alpha_k|} - 2 \geq m_2^{|\alpha_k|} \geq l_2^{|\alpha_k|} > l_2^{|\alpha_k|} - 2 \geq \cdots \geq m_{|\alpha_k|}^{|\alpha_k|} \geq l_{|\alpha_k|}^{|\alpha_k|} \geq 0$, and for any $1 \leq i \leq r^1_{|\alpha_k|}$, there exists $\alpha_{s_n} \in A_{|\alpha_k|}$, for some $1 \leq n \leq r$ such that $\alpha_{s_n} = \sum_{i=1}^{r_n} \sum_{k=l_i^n}^{m_i^n} 2^k$, where $l_u^{|\alpha_k|} = l_i$ for some $1 \leq u \leq r^1_{|\alpha_k|}$.

By using Corollary 8.5, we get

$$\sup_{k\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_n}} a_k|}{\varphi_{|\alpha_{s_k}|}} \geq \frac{c \cdot 2^{l_i}}{\varphi_{|\alpha_{s_k}|}}.$$

Since $\mu\{E_{l_i}\} \geq 1/2^{l_i+1}$, we have

$$\int_{E_{l_i}} \left(\sup_{k\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_k}} f|}{\varphi_{|\alpha_{s_k}|}}\right)^{1/2} d\mu \geq \int_{E_{l_i}} \left(\frac{\sigma_{\alpha_{s_n}} f}{\varphi_{|\alpha_{s_n}|}}\right)^{1/2} d\mu \geq \frac{c}{\varphi_{|\alpha_k|+1}^{1/2}}. \tag{18.16}$$

On the other hand, for any $1 \leq i \leq r^2_{|\alpha_k|}$, there exists a number α_{s_n}, for some $1 \leq n \leq r$, such that $m_u^{|\alpha_k|} = m_i$ for some $1 \leq u \leq r^2_{|\alpha_k|}$. According to the fact that $\mu\{E_{t_i}\} \geq 1/2^{t_i+3}$, using again Corollary 8.5 for some α_{s_n} and $1 \leq i \leq r_s^2$, we also get

$$\int_{E_{m_i}} \left(\sup_{k\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_k}} f|}{\varphi_{|\alpha_{s_k}|}}\right)^{1/2} d\mu \geq \int_{E_{m_i}} \left(\frac{\sigma_{\alpha_{s_n}} f}{\varphi_{|\alpha_{s_n}|}}\right)^{1/2} d\mu \geq \frac{c}{\varphi_{|\alpha_k|+1}^{1/2}}. \tag{18.17}$$

Combining equation (18.59)–(18.17)) for sufficiently large α_k, we obtain

$$\begin{aligned}
&\int_G \left(\sup_{k\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_k}} f|}{\varphi_{|\alpha_{s_k}|}}\right)^{1/2} d\mu \\
&\quad \geq c \sum_{i=1}^{r^1_{|\alpha_k|}-1} \int_{E_{l_i}} \left(\sup_{k\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_k}} f|}{\varphi_{|\alpha_{s_k}|}}\right)^{1/2} d\mu + c \sum_{i=1}^{r^2_{|\alpha_k|}-1} \int_{E_{m_i}} \left(\sup_{k\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_k}} f|}{\varphi_{|\alpha_{s_k}|}}\right)^{1/2} d\mu \\
&\quad \geq \frac{c(r^1_{|\alpha_k|} + r^2_{|\alpha_k|})}{\varphi_{|\alpha_k|}^{1/2}} \geq \frac{c|A_{|\alpha_k|}|}{\varphi_{|\alpha_k|}^{1/2}} \longrightarrow \infty \ \text{ as } \ k \to \infty,
\end{aligned}$$

so part (b) is proved and the proof is complete. □

From Theorem 18.1, we get the following result of Goginava [48] (see also [100]).

Corollary 18.1.

(a) *The maximal operator* $\widetilde{\sigma}^*$, *defined by* (12.11), *is bounded from the Hardy space* $H_{1/2}(G)$ *to the space* $L_{1/2}(G)$.

(b) *The rate of increase of the denominator* $\{\log^2(n+1)\}$ *cannot be improved.*

To enable comparison with other results presented in the literature, we also state the following statement.

Corollary 18.2. *Let* $f \in H_{1/2}(G)$. *Then the restricted maximal operators* $\widetilde{\sigma}_i^{*,\nabla}$, $i = 1,2,3,4,5$, *defined by*

$$\widetilde{\sigma}_1^{*,\nabla} f := \sup_{k\in\mathbb{N}} |\sigma_{2^k} f|, \quad \widetilde{\sigma}_2^{*,\nabla} f := \sup_{k\in\mathbb{N}} |\sigma_{2^k+2^{k-1}} f|, \tag{18.18}$$

$$\widetilde{\sigma}_3^{*,\nabla} f := \sup_{k\in\mathbb{N}} |\sigma_{2^k+1} f|, \quad \widetilde{\sigma}_4^{*,\nabla} f := \sup_{k\in\mathbb{N}} |\sigma_{2^{2k}+2^k} f|, \quad \widetilde{\sigma}_5^{*,\nabla} f := \sup_{k\in\mathbb{N}} |\sigma_{2^{3k}+2^k} f|, \tag{18.19}$$

are bounded from the space $H_{1/2}(G)$ *to the Lebesgue space* $L_{1/2}(G)$.

Next, we describe the $(H_p, \textit{weak-}L_p)$ type inequalities for $0 < p < 1/2$ which are proved in [25].

Theorem 18.2.

(a) *Let* $0 < p < 1/2$ *and* $f \in H_p(G)$. *Then the weighted maximal operator* $\widetilde{\sigma}^{*,*,p}$, *defined by*

$$\widetilde{\sigma}^{*,*,p} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{2^{d(n)(1/p-2)}}$$

is bounded from the Hardy space $H_p(G)$ *to the space weak-*$L_p(G)$.

(b) *Let* $0 < p < 1/2$ *and* $\varphi : \mathbb{N} \to [1,\infty)$ *be a non-decreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{2^{d(n)(1/p-2)}}{\varphi(n)} = \infty. \tag{18.20}$$

Then there exist a sequence $\{a_k, k \in \mathbb{N}_+\}$ *of* p*-atoms and a sequence* $\{q_{\alpha_k}, k \in \mathbb{N}_+\}$ *of real numbers satisfying the condition* $|q_{\alpha_k}| = \alpha_k$ *such that*

$$\sup_{k\in\mathbb{N}} \frac{\left\| \frac{\sigma_{q_{\alpha_k}} a_k}{\varphi(q_{\alpha_k})} \right\|_{weak\text{-}L_p}}{\|a_k\|_{H_p}} = \infty.$$

(c) *Let* $0 < p < 1/2$. *There exists a sequence* $\{a_k, k \in \mathbb{N}_+\}$ *of* p*-atoms such that*

$$\sup_{k\in\mathbb{N}} \frac{\|\widetilde{\sigma}^{*,*,p} a_k\|_p}{\|a_k\|_{H_p}} = \infty.$$

Proof. Since σ_n is bounded from L_∞ to L_∞, by Lemma 10.3, the proof of the theorem will be completed if we show that

$$t\mu\left\{x \in \overline{I_M} : \ \widetilde{\sigma}^{*,\nabla} a(x) \geq t^{1/p}\right\} \leq c < \infty, \ \ t \geq 0, \tag{18.21}$$

for every p-atom a. We may assume that a is an arbitrary p-atom, with support I, $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $\sigma_n a(x) = 0$, when $n < 2^M$. Therefore, we can suppose that $n \geq 2^M$. Since $\|a\|_\infty \leq 2^{M/p}$, we obtain

$$\frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}} \leq \frac{1}{2^{d(n)(1/p-2)}} \|a\|_\infty \int_{I_M} |K_n(x+t)|\, d\mu(t) \leq \frac{1}{2^{d(n)(1/p-2)}} \cdot 2^{M/p} \int_{I_M} |K_n(x+t)|\, d\mu(t).$$

Let $x \in I_M^{k,l}$, $0 \le k,l \le [n] \le M$ or $0 \le k,l \le M < [n]$. Then it is easy to see that $x+t \in I_M^{k,l}$ for $t \in I_M$, and if we combine Lemma 5.1 and (8.1) in Lemma 8.1, we get $K_n(x+t)=0$ for $t \in I_M$ and

$$\frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}} = 0. \tag{18.22}$$

Let $x \in I_M^{k,l}$, $[n] \le k,l \le M$ or $k \le [n] \le l \le M$. Using Corollary 8.4, we can conclude that

$$\begin{aligned}\frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}} &\le c_p \cdot 2^{M/p} \cdot \frac{2^{k+l-2M}}{2^{d(n)(1/p-2)}} \\ &\le c_p \frac{2^{[n](1/p-2)+k+l+M(1/p-2)}}{2^{|n|(1/p-2)}} \le c_p \cdot 2^{[n](1/p-2)+k+l} \le c_p \cdot 2^{k+l(1/p-1)}.\end{aligned} \tag{18.23}$$

Applying (18.22) and (18.23) for any $x \in I_M^{k,l}$, $1 \le k < l \le M$, we find that

$$\widetilde{\sigma}^{*,\nabla} a(x) = \sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}} \le c_p \cdot 2^{k+l(1/p-1)}.$$

It immediately follows that for such $k < l \le M$, we have the following estimate: $\widetilde{\sigma}^{*,\nabla} a(x) \le C_p \cdot 2^{M/p}$ for $x \in I_M^{k,l}$, and also

$$\mu\big\{x \in I_N^{k,l} : \ \widetilde{\sigma}^{*,\nabla} a(x) > C_p \cdot 2^{s/p}\big\} = 0, \quad s = M+1, M+2, \dots . \tag{18.24}$$

Suppose that

$$2^{k+l(1/p-1)} > 2^{s/p} \ \text{ for some } \ s \le M. \tag{18.25}$$

It is evident that inequality (18.25) does not hold for $k < l \le s$. On the other hand, inequality (18.25) holds for all $l > k \ge s$, that is,

$$2^{k+l(1/p-1)} > 2^{s/p}, \ \text{ where } \ l > k \ge s. \tag{18.26}$$

If $l > s > k$, from (13.11) we can conclude that

$$k + l\Big(\frac{1}{p}-1\Big) > \frac{s}{p}, \quad l > \frac{\frac{s}{p}-k}{\frac{1}{p}-1}$$

and

$$2^{k+l(1/p-1)} > 2^{s/p}, \ \text{ where } \ s > k, \ \ l > \frac{\frac{s}{p}-k}{\frac{1}{p}-1}. \tag{18.27}$$

Combining (1.2), (18.26) and (18.27), we get

$$\begin{aligned}&\big\{x \in \overline{I}_M : \ \widetilde{\sigma}^{*,\nabla} a(x) \ge C_p \cdot 2^{s/p}\big\} \\ &\qquad \subset \Big(\bigcup_{k=s}^{M-1} \bigcup_{l=k+1}^{M} \big\{x \in I_M^{k,l} : \ \widetilde{\sigma}^{*,\nabla} a(x) \ge C_p \cdot 2^{s/p}\big\}\Big) \\ &\qquad\qquad \cup \Big(\bigcup_{k=0}^{s} \bigcup_{l>(s/p-k)(1/p-1)}^{M} \big\{x \in I_M^{k,l} : \ \widetilde{\sigma}^{*,\nabla} a(x) \ge C_p \cdot 2^{s/p}\big\}\Big)\end{aligned}$$

and

$$\begin{aligned}&\mu\big\{x \in \overline{I}_M : \ \widetilde{\sigma}^{*,\nabla} a(x) \ge C_p \cdot 2^{s/p}\big\} \le \sum_{k=s}^{M-1} \sum_{l=k+1}^{M} \mu(I_M^{k,l}) + \sum_{k=0}^{s} \sum_{l>(s/p-k)/(1/p-1)}^{M} \mu(I_M^{k,l}) \\ &\quad \le \sum_{k=s}^{M-1} \sum_{l=k+1}^{M} \frac{1}{2^l} + \sum_{k=0}^{s} \sum_{l>(s/p-k)/(1/p-1)}^{M} \frac{1}{2^l} \le \sum_{k=s}^{M-1} \frac{1}{2^k} + \sum_{k=0}^{s} \frac{1}{2^{(s/p-k)/(1/p-1)-1}} \le \frac{c_p}{2^s}.\end{aligned} \tag{18.28}$$

In view of (13.10) and (18.28), we have $2^s\mu\{x \in \overline{I}_M : \widetilde{\sigma}^{*,\nabla}a(x) \geq C_p \cdot 2^{s/p}\} < c_p < \infty$, which implies (18.21) and part (a) as well.

Let q_{α_k} be a sequence such that $|q_{\alpha_k}| = \alpha_k$ and $[q_{\alpha_k}] = s_k$ and

$$\lim_{k\to\infty} \frac{2^{d(q_{\alpha_k})(1/p-2)}}{\varphi(q_{\alpha_k})} = \infty. \tag{18.29}$$

Let a_k be a p-atom from Example 11.1. Due to (11.4), we find that

$$|\sigma_{q_{\alpha_k}} a_k(x)| = \frac{2^{\alpha_k(1/p-1)}(q_{\alpha_k} - 2^{\alpha_k} - 1)}{q_{\alpha_k}} |K_{q_{\alpha_k}-2^{\alpha_k}-1}(x)|. \tag{18.30}$$

Let $x \in I_{[q_{\alpha_k}]+1}(e_{[q_{\alpha_k}]-1} + e_{[q_{\alpha_k}]})$. Using now Corollary 8.7, we obtain

$$|\sigma_{q_{\alpha_k}} a_k(x)| \geq c \cdot 2^{2[q_{\alpha_k}]+\alpha_k(1/p-2)} \quad \text{and} \quad \frac{|\sigma_{q_{\alpha_k}} a_k(x)|}{\varphi(q_{\alpha_k})} \geq \frac{c \cdot 2^{2[q_{n_k}]+\alpha_k(1/p-2)}}{\varphi(q_{\alpha_k})}.$$

Hence, we can conclude that

$$\mu\Big\{x \in G : \frac{|\sigma_{q_{\alpha_k}} a_k(x)|}{\varphi(q_{\alpha_k})} \geq \frac{c \cdot 2^{2[q_{\alpha_k}]+\alpha_k(1/p-2)}}{\varphi(q_{\alpha_k})}\Big\} \geq \mu\big(I_{[q_{\alpha_k}]+1}(e_{[q_{\alpha_k}]-1} + e_{[q_{\alpha_k}]})\big) > \frac{c}{2^{[q_{n_k}]}}. \tag{18.31}$$

Combining (11.1), (18.29) and (18.31), we get

$$\frac{\frac{c\cdot 2^{2[q_{\alpha_k}]+\alpha_k(1/p-2)}}{\varphi(q_{\alpha_k})}\big(\mu\{x \in G : \frac{|\sigma_{q_{\alpha_k}} a_k(x)|}{\varphi(q_{\alpha_k})} \geq \frac{c\cdot 2^{2[q_{\alpha_k}]+\alpha_k(1/p-2)}}{\varphi(q_{\alpha_k})}\}\big)^{1/p}}{\|a_k(x)\|_{H_p}}$$
$$\geq \frac{c_p \cdot 2^{2[q_{\alpha_k}]+\alpha_k(1/p-2)}}{\varphi(q_{\alpha_k})} \frac{1}{2^{[q_{\alpha_k}]/p}} = \frac{c_p \cdot 2^{\alpha_k(1/p-2)}}{2^{[q_{\alpha_k}](1/p-2)}\varphi(q_{\alpha_k})} = \frac{c_p \cdot 2^{d(q_{\alpha_k})(1/p-2)}}{\varphi(q_{\alpha_k})} \longrightarrow \infty \quad \text{as} \quad k \to \infty.$$

The proof of part (b) is complete.

Let a_k be a p-atom from Example 11.2. If we replace q_{n_k} by $q^s_{n_k} = 2^{2n_k} + 2^{2s}$, from (11.7) we find that

$$|\sigma_{q^s_{n_k}} a_k(x)| = \frac{2^{2s} K_{2^{2s}}(x)}{q^s_{n_k}}.$$

Using Corollary 8.7, we get

$$|\sigma_{q^s_{n_k}} a_k(x)| \geq \frac{c \cdot 2^{4s}}{2^{2n_k}} \quad \text{for} \quad x \in I_{2s+1}(e_{2s-1} + e_{2s})$$

and

$$\frac{|\sigma_{q^s_{n_k}} a_k(x)|}{2^{(1/p-2)d(q^s_{n_k})}} \geq \frac{c_p \cdot 2^{2s/p}}{2^{2n_k(1/p-1)}} \quad \text{for} \quad x \in I_{s+1}(e_{2s-1} + e_{2s}).$$

Hence,

$$\int_G \Big(\sup_{k\in\mathbb{N}} \frac{|\sigma_{q^s_{n_k}} a_k(x)|}{2^{(1/p-2)d(q^s_{n_k})}}\Big)^p d\mu(x)$$
$$\geq \sum_{s=1}^{n_k-1} \int_{I_{2s+1}(e_{2s-1}+e_{2s})} \Big(\frac{|\sigma_{q_{n^s_k}} a_k(x)|}{2^{(1/p-2)d(q^s_{n_k})}}\Big)^p d\mu(x) \geq c_p \sum_{s=1}^{n_k-1} \frac{1}{2^{2s}} \frac{2^{2s}}{2^{2n_k(1-p)}} \geq \frac{C_p n_k}{2^{2n_k(1-p)}}. \tag{18.32}$$

Finally, Combining (11.6) and (18.32), we have

$$\frac{\big(\int_G \big(\sup_{k\in\mathbb{N}} \sup_{0<s<n_k} \frac{|\sigma_{q^s_{n_k}} a_k(x)|}{2^{(1/p-2)\,d(q^s_{n_k})}}\big)^p d\mu(x)\big)^{1/p}}{\|a_k\|_{H_p}} \geq \frac{\big(\frac{C_p n_k}{2^{2n_k(1-p)}}\big)^{1/p}}{2^{2n_k(1/p-1)}} \geq c_p {n_k}^{1/p} \longrightarrow \infty \quad \text{as} \quad k \to \infty. \qquad \square$$

In [16], the (H_p, L_p) type inequalities were investigated for $0 < p < 1/2$ and the following sharp result was proved.

Theorem 18.3.

(a) *Let $0 < p < 1/2$, $f \in H_p(G)$ and $\varphi : \mathbb{N}_+ \to \mathbb{R}$ be any non-negative and non-decreasing function satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi^p(n)} < c < \infty. \tag{18.33}$$

Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by

$$\widetilde{\sigma}^{*,\nabla} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{2^{d(n)(1/p-2)}\varphi(d(n))}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

(b) *Let $0 < p < 1/2$, $\{n_k : k \geq 0\}$ be a sequence of positive numbers and $\varphi : \mathbb{N}_+ \to \mathbb{R}$ be any non-negative and non-decreasing function satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi^p(n)} = \infty. \tag{18.34}$$

Then there exist the p-atoms a_k such that

$$\sup_{k\in\mathbb{N}} \frac{\left\| \sup_{n\in\mathbb{N}} \frac{|\sigma_n a_k|}{2^{d(n)(1/p-2)}\varphi(d(n))} \right\|_p}{\|a_k\|_{H_p}} = \infty.$$

Proof. Since σ_n is bounded from L_∞ to L_∞ by Lemma 10.3, the proof will be completed if we prove that

$$\int_{\overline{I}} \left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}\varphi(d(n))} \right)^p d\mu \leq c_p < \infty \tag{18.35}$$

for every p-atom a. We may assume that a is an arbitrary p-atom with support I, $\mu(I) = 2^{-M}$and $I = I_M$. It is easy to see that $\sigma_n a(x) = 0$, when $n < 2^M$. Therefore, we can suppose that $n \geq 2^M$. Since $\|a\|_\infty \leq 2^{M/p}$, we find that

$$\begin{aligned} \frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}\varphi(d(n))} &\leq \frac{1}{2^{d(n)(1/p-2)}\varphi(d(n))} \|a\|_\infty \int_{I_M} |K_n(x+t)|\, d\mu(t) \\ &\leq \frac{1}{2^{d(n)(1/p-2)}\varphi(d(n))} \cdot 2^{M/p} \int_{I_M} |K_n(x+t)|\, d\mu(t). \end{aligned} \tag{18.36}$$

Let $x \in I_M^{k,l}$, $0 \leq k < l < [n] \leq M$. Then $x + t \in I_M^{k,l}$ and, applying (8.4) in Lemma 8.2, we get $K_n(x+t) = 0$ for $t \in I_M$, and from (18.36) it follows that

$$\frac{1}{2^{d(n)(1/p-2)}\varphi(d(n))} |\sigma_n a(x)| = 0. \tag{18.37}$$

Let $x \in I_M^{k,l}$, $[n] \leq k < l < M$ or $k < [n] \leq l < M$. Since $|n| \geq M$, due to (18.36) and Corollary 8.4, we can conclude that

$$\begin{aligned} \frac{1}{2^{d(n)(1/p-2)}\varphi(d(n))} |\sigma_n a(x)| &\leq \frac{c \cdot 2^{M(1/p-2)+k+l}}{2^{d(n)(1/p-2)}\varphi(d(n))} = \frac{c \cdot 2^{[n](1/p-2)} \cdot 2^{M(1/p-2)+k+l}}{2^{|n|(1/p-2)}\varphi(d(n))} \\ &\leq \frac{c \cdot 2^{M(1/p-2)}}{2^{|n|(1/p-2)}} \frac{2^{[n](1/p-2)+k+l}}{\varphi(M-l)} \leq \frac{c \cdot 2^{M(1/p-2)}}{2^{|n|(1/p-2)}} \frac{2^{k+l(1/p-1)}}{\varphi(M-l)} \leq \frac{c \cdot 2^{k+l(1/p-1)}}{\varphi(M-l)}. \end{aligned} \tag{18.38}$$

Applying (18.37) and (18.38) for any $x \in I_M^{k,l}$, $0 \le k < l < M$, we arrive at

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}\varphi(d(n))} \le \frac{c \cdot 2^{k+l(1/p-1)}}{\varphi(M-l)}. \tag{18.39}$$

Let $x \in I_M^{k,M}$, $0 \le k < M$. Using again (18.36) and Corollary 8.4 for $k = l$, we conclude that

$$\frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}\varphi(d(n))} \le c \cdot 2^{M/p} \int_{I_M} |K_n(x+t)|\, d\mu(t) \le c \cdot 2^{M/p} \cdot \frac{2^k}{2^M} = c \cdot 2^{k+M(1/p-1)}$$

and

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}\varphi(d(n))} \le c \cdot 2^{k+M(1/p-1)}. \tag{18.40}$$

Combining (1.2), (18.39) and (18.40), we obtain

$$\begin{aligned}
&\int_{\overline{I}_M} \left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}\varphi(d(n))}\right)^p d\mu(x) \\
&= \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \int_{I_M^{k,l}} \left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}\varphi(d(n))}\right)^p d\mu(x) + \sum_{k=0}^{M-1} \int_{I_M^{k,M}} \left(\sup_{n\in\mathbb{N}} \frac{|\sigma_n a(x)|}{2^{d(n)(1/p-2)}\varphi(d(n))}\right)^p d\mu(x) \\
&\le c_p \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^l} \frac{2^{pk+l(1-p)}}{\varphi^p(M-l)} + c_p \sum_{k=0}^{M} \frac{1}{2^M} \cdot 2^{pk+M(1-p)} := I + II.
\end{aligned} \tag{18.41}$$

Hence,

$$\begin{aligned}
I &\le c_p \sum_{k=0}^{M-2} 2^{pk} \sum_{l=k+1}^{M-1} \frac{1}{2^{pl}\varphi^p(M-l)} \\
&= c_p \sum_{k=0}^{M-2} 2^{pk} \sum_{l=k+1}^{[(k+M)/2]} \frac{1}{2^{pl}\varphi^p(M-l)} + c_p \sum_{k=0}^{M-2} 2^{pk} \sum_{l=[(k+M)/2]+1}^{M-1} \frac{1}{2^{pl}\varphi^p(M-l)} := I_1 + I_2.
\end{aligned} \tag{18.42}$$

Using now (18.33) for I_1, we get

$$I_1 \le c_p \sum_{k=0}^{M-2} \frac{2^{pk}}{\varphi^p([(M-k)/2])} \sum_{l=k+1}^{[(k+M)/2]} \frac{1}{2^{pl}} \le c_p \sum_{k=0}^{M-2} \frac{1}{\varphi^p([(M-k)/2])} < c_p < \infty.$$

For I_2, we find that

$$I_2 \le c_p \sum_{k=0}^{M-2} 2^{pk} \sum_{l=[(k+M)/2]+1}^{M-1} \frac{1}{2^{pl}} \le c_p \sum_{k=0}^{M-2} 2^{pk} \frac{1}{2^{p[(k+M)/2]}} \le c_p \sum_{k=0}^{M-2} \frac{2^{pk/2}}{2^{pM/2}} < c_p < \infty.$$

For II, we can conclude that

$$II \le c_p \sum_{k=0}^{M-2} \frac{2^{pk}}{2^{pM}} < c_p < \infty. \tag{18.43}$$

Combining (18.41)–(18.43), we come to the conclusion that (18.35) holds, and so the proof is complete. $\square$

Proof of Theorem 18.1. In view of (18.34), we have

$$\Big(\sum_{s=1}^{n_k-1}\frac{1}{\varphi^p(s)}\Big)^{1/p}\longrightarrow\infty \ \text{ as } \ k\to\infty. \tag{18.44}$$

Let a_k be a p-atom from Example 11.1. Let $q^s_{\alpha_k}\in\mathbb{N}$ be such that $2^{\alpha_k}\le q^s_{\alpha_k}\le 2^{\alpha_k+1}$ and $[q^s_{\alpha_k}]=s$, where $0\le s<\alpha_k$. Using (11.4), we find that

$$|\sigma_{q^s_{\alpha_k}}a_k(x)|=\frac{2^{\alpha_k(1/p-1)}(q^s_{\alpha_k}-2^{\alpha_k})}{q^s_{\alpha_k}}\,|K_{q^s_{\alpha_k}-2^{\alpha_k}}(x)|.$$

Let $x\in I_{s+1}(e_{s-1}+e_s)$. By Lemma 8.7, we have $|\sigma_{q^s_{\alpha_k}}a_k(x)|\ge c\cdot 2^{2s+\alpha_k(1/p-2)}$ and

$$\frac{|\sigma_{q^s_{\alpha_k}}a_k(x)|}{2^{(1/p-2)d(q^s_{\alpha_k})}\varphi(d(q^s_{\alpha_k}))}\ge\frac{c_p\cdot 2^{s/p}}{\varphi(\alpha_k-s)}\,.$$

Hence,

$$\begin{aligned}\int_G\Big(\sup_{k\in\mathbb{N}}\Big|\frac{|\sigma_{q^s_{\alpha_k}}a_k(x)|}{2^{(1/p-2)d(q^s_{\alpha_k})}\varphi(d(q^s_{\alpha_k}))}\Big|\Big)^p d\mu(x)&\\ \ge\frac12\sum_{s=0}^{\alpha_k-1}\int_{I_{s+1}(e_{s-1}+e_s)}\Big(\frac{|\sigma_{q_{\alpha^s_k}}a_k(x)|}{2^{(1/p-2)d(q^s_{\alpha_k})}\varphi(d(q^s_{\alpha_k}))}\Big)^p d\mu(x)&\\ \ge c_p\sum_{s=0}^{\alpha_k-1}\frac{1}{2^s}\,\frac{2^s}{\varphi^p(\alpha_k-s)}\ge c_p\sum_{s=1}^{\alpha_k}\frac{1}{\varphi^p(s)}\,.&\end{aligned}$$

Finally, using this estimate combined with (11.1) and (18.44), we find that

$$\frac{\Big(\int_G\Big(\sup_{k\in\mathbb{N}}\sup_{0\le s<\alpha_k}\Big|\frac{|\sigma_{q^s_{\alpha_k}}a_k(x)|}{2^{(1/p-2)d(q^s_{\alpha_k})}\varphi(d(q^s_{\alpha_k}))}\Big|\Big)^p d\mu(x)\Big)^{1/p}}{\|a_k\|_{H_p}}\ge c_p\Big(\sum_{s=1}^{\alpha_k}\frac{1}{\varphi^p(s)}\Big)^{1/p}\longrightarrow\infty \ \text{ as } \ k\to\infty.\qquad\square$$

If we take $\varphi(n)=n^{(1+\varepsilon)/p}$ for any $\varepsilon>0$, we get that condition (18.33) is fulfilled. On the other hand, if we take $\varphi(n)=n^{1/p}$, then condition (18.34) holds. Hence, Theorem 18.3 implies the following sharp result.

Corollary 18.3. *Let $0<p<1/2$ and $f\in H_p(G)$. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla,\varepsilon}$, defined by*

$$\widetilde{\sigma}^{*,\nabla,\varepsilon}f:=\sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{2^{d(n)(1/p-2)}(d(n))^{(1+\varepsilon)/p}}\,,\quad\varepsilon\ge0,$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ for $\varepsilon>0$ and is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ for $\varepsilon=0$.

In [19], it was also proved that the following theorem holds.

Theorem 18.4.

(a) *Let $0<p<1/2$, $f\in H_p(G)$ and $\{n_k:\ k\ge0\}$ be a sequence of positive numbers and let $\{n_{s_i}:\ 1\le i\le r\}\subset\{n_k:\ k\ge0\}$ be the numbers such that $2^s\le n_{s_1}\le n_{s_2}\le\cdots\le n_{s_r}\le 2^{s+1}$, $s\in\mathbb{N}$. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by*

$$\widetilde{\sigma}^{*,\nabla}f:=\sup_{s\in\mathbb{N}}\sup_{2^s\le n_{s_i}<2^{s+1}}\frac{|\sigma_{n_{s_i}}f|}{2^{d_s(n_{s_i})(1/p-2)}}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

(b) *Let $0 < p < 1/2$, $f \in H_p(G)$, $\{n_k : k \geq 0\}$ be a sequence of positive numbers and let $\{n_{s_i} : 1 \leq i \leq r\} \subset \{n_k : k \geq 0\}$ be the numbers such that $2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}$, $s \in \mathbb{N}$. Then for any non-negative and non-decreasing function $\varphi : \mathbb{N}_+ \to \mathbb{R}$ satisfying the condition*

$$\sup_{s\in\mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{2^{d_s(n_{s_i})(1/p-2)}}{\varphi(n_{s_i})} = \infty, \tag{18.45}$$

there exist p-atoms f_s such that

$$\frac{\left\| \sup_{s\in\mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} f_s|}{\varphi(n_{s_i})} \right\|_p}{\|f_s\|_{H_p}} \longrightarrow \infty \quad \textit{as} \quad s \to \infty.$$

Proof. Since σ_n is bounded from L_∞ to L_∞, by Lemma 10.3, the proof of the theorem will be completed if we prove that

$$\int_{\overline{I}_M} \left(\sup_{s\in\mathbb{N}} \sup_{2^s \leq n_{s_i} < 2^{s+1}} \left| \frac{\sigma_{n_{s_k}} a}{2^{\rho_s(n_{s_k})(1/p-2)}} \right| \right)^p d\mu \leq c < \infty \tag{18.46}$$

for every p-atom a. We may assume that a is an arbitrary p-atom with support I, $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $\sigma_{n_{s_k}} a = 0$, when $n_{s_k} < 2^M$. Therefore, we can suppose that $n_{s_k} \geq 2^M$. Let $2^s \leq n_{s_i} < 2^{s+1}$ for some $s \geq M$. Since $\|a\|_\infty \leq 2^{M/p}$, we find that

$$\begin{aligned} \frac{|\sigma_{n_{s_i}} a|}{2^{\rho_s(n_{s_i})(1/p-2)}} &\leq \frac{1}{2^{\rho_s(n_{s_i})(1/p-2)}} \|a\|_\infty \int_{I_M} |K_{n_{s_i}}(x+t)| \, d\mu(t) \\ &\leq \frac{1}{2^{\rho_s(n_{s_i})(1/p-2)}} \cdot 2^{M/p} \int_{I_M} |K_{n_{s_i}}(x+t)| \, d\mu(t). \end{aligned}$$

Let $x \in I_M^{k,l}$, $0 \leq k < l \leq [n_{s_i}] \leq M$. Then $x + t \in I_M^{k,l}$, $0 \leq k < l \leq [n_{s_i}] \leq M$, and applying (8.4) in Lemma 8.2, we get $K_{n_{s_i}}(x+t) = 0$ for $t \in I_M$ and $|\sigma_{n_{s_i}} a| = 0$ for any $2^s \leq n_{s_i} < 2^{s+1}$. Since $[n_{s_i}] \geq s_-$, we have

$$\sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} a|}{2^{\rho_s(n_{s_i})(1/p-2)}} = 0 \;\; \text{for} \;\; x \in I_{l+1}(e_k + e_l), \;\; 0 \leq k < l < s_-. \tag{18.47}$$

Next, we suppose that $x \in I_M^{k,l}$, where either $[n_{s_i}] \leq k < l \leq M$ or $k \leq [n_{s_i}] < l \leq M$. Since $n_{s_i} \geq 2^M$ and $|n_{s_i}| = s$, applying Corollary 8.4, we find that

$$\frac{|\sigma_{n_{s_i}} a|}{2^{\rho_s(n_{s_i})(1/p-2)}} \leq \frac{2^{M(1/p-2)+k+l}}{2^{\rho_s(n_{s_i})(1/p-2)}} \leq \frac{2^{M(1/p-2)}}{2^{s(1/p-2)}} \cdot 2^{k+l+s_-(1/p-2)}. \tag{18.48}$$

Applying (18.48) for any $x \in I_M^{k,l}$, $s_- \leq k < l \leq M$ or $k \leq s_- < l \leq M$, we obtain

$$\sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} a|}{2^{\rho_s(n_{s_i})(1/p-2)}} \leq \frac{2^{M(1/p-2)}}{2^{s(1/p-2)}} \cdot 2^{k+l+s_-(1/p-2)}. \tag{18.49}$$

If we now define $\widetilde{\sigma}_s^*$ by

$$\widetilde{\sigma}_s^* f := \sup_{2^s \leq n_{s_i} < 2^{s+1}} \frac{|\sigma_{n_{s_i}} f|}{2^{\rho_s(n_{s_i})(1/p-2)}},$$

then we can conclude that

$$\left(\sup_{s\in\mathbb{N}} \sup_{k\in\mathbb{N}} \left| \frac{\sigma_{n_{s_k}} a}{2^{\rho_s(n_{s_k})(1/p-2)}} \right| \right)^p \leq \sum_{s=M}^{\infty} |\widetilde{\sigma}_s^* a|^p.$$

Hence, combining (1.2), (18.47) and (18.49), we get

$$\int_{\overline{I}_M}\left(\sup_{s\in\mathbb{N}}\sup_{k\in\mathbb{N}}\left|\frac{\sigma_{n_{s_k}}a}{2^{\rho_s(n_{s_k})(1/p-2)}}\right|\right)^p d\mu \le \sum_{s=M}^{\infty}\int_{\overline{I}_M}|\widetilde{\sigma}_s^* a|^p\,d\mu$$

$$=\sum_{s=M}^{\infty}\left(\sum_{k=0}^{s_--2}\sum_{l=k+1}^{s_--1}+\sum_{k=0}^{s_--1}\sum_{l=s_-}^{M-1}+\sum_{k=s_-}^{M-2}\sum_{l=k+1}^{M-1}\right)\int_{I_M^{k,l}}|\widetilde{\sigma}_s^* a|^p\,d\mu+\sum_{s=M}^{\infty}\sum_{k=0}^{M-1}\int_{I_M^{k,M}}|\widetilde{\sigma}_s^* a|^p\,d\mu$$

$$\le c_p\sum_{s=M}^{\infty}\left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}\right)^p\sum_{k=0}^{s_--1}\sum_{l=s_-+1}^{M-1}\frac{1}{2^l}\cdot 2^{s_-(1-2p)}\cdot 2^{p(k+l)}$$

$$+c_p\sum_{s=M}^{\infty}\left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}\right)^p\sum_{k=s_-}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}\cdot 2^{s_-(1-2p)}\cdot 2^{p(k+l)}$$

$$+c_p\sum_{s=M}^{\infty}\left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}\right)^p\frac{2^{s_-(1-2p)}}{2^M}\sum_{k=0}^{s_-}2^{p(k+M)}:=I+II+III. \tag{18.50}$$

Since

$$\sum_{s=M}^{\infty}\left(\frac{2^{M(1/p-2)}}{2^{s(1/p-2)}}\right)^p<c<\infty,$$

it follows that

$$I\le c_p\sum_{k=0}^{s_--1}\sum_{l=s_-+1}^{M-1}\frac{1}{2^l}\cdot 2^{s_-(1-2p)}\cdot 2^{p(k+l)}<c_p<\infty, \tag{18.51}$$

$$II\le c_p\sum_{k=s_-}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}\cdot 2^{s_-(1-2p)}\cdot 2^{p(k+l)}<c_p<\infty \tag{18.52}$$

and

$$III\le\frac{c_p\cdot 2^{s_-(1-2p)}}{2^M}\sum_{k=0}^{s_-}2^{p(k+M)}<c_p<\infty. \tag{18.53}$$

Combining (18.50)–(18.53), we conclude that (18.46) holds, so the proof of part (a) is complete.

Let $n_{s_k}:=q_{\alpha_k}^s\in\mathbb{N}$ be such that $2^{\alpha_k}\le q_{\alpha_k}^s\le 2^{\alpha_k+1}$, where $0\le s<2^{\alpha_k}$. If s_0 is such that $q_{\alpha_k}^{s_0}=s_-$, then we get $\rho_{\alpha_k}(q_{\alpha_k^s})=\alpha_k-s_-=\rho(q_{\alpha_k}^{s_0})$. In view of (18.34), we have

$$\frac{2^{\rho(q_{\alpha_k}^{s_0})(1/p-2)}}{\varphi(q_{\alpha_k}^{s_0})}\longrightarrow\infty\quad\text{as}\quad k\to\infty. \tag{18.54}$$

Let a_k be a p-atom from the Example 11.1. By (11.4), we find that

$$|\sigma_{q_{\alpha_k}^s}a_k=\frac{2^{\alpha_k(1/p-1)}}{q_{\alpha_k}^s}(q_{\alpha_k}^s-2^{\alpha_k}-1)|K_{q_{\alpha_k}^s-2^{\alpha_k}-1}|.$$

Let $q_{\alpha_k}^{s_0}$ be α_{s_j}, so that $[q_{\alpha_k}^{s_0}]=s_-$, that is, $s_0=s_-$ and $x\in I_{s_0+1}(e_{s_0-1}+e_{s_0})$. Using Lemma 8.7, we obtain

$$|\sigma_{q_{\alpha_k}^{s_0}}a_k|\ge c\cdot 2^{2s_0+\alpha_k(1/p-2)}\quad\text{and}\quad\frac{|\sigma_{q_{\alpha_k}^{s_0}}a_k|}{\varphi(q_{\alpha_k}^{s_0})}\ge\frac{c\cdot 2^{2s_0+\alpha_k(1/p-2)}}{\varphi(q_{\alpha_k}^{s_0})}.$$

Hence,

$$\int_G\left(\sup_{k\in\mathbb{N}}\frac{|\sigma_{q_{\alpha_k}^s}a_k|}{\varphi(q_{\alpha_k}^s)}\right)^p d\mu\ge\int_{I_{s_0+1}(e_{s_0-1}+e_{s_0})}\left(\frac{|\sigma_{q_{\alpha_k}^{s_0}}a_k|}{\varphi(q_{\alpha_k}^{s_0})}\right)^p d\mu$$

$$\ge c_p\frac{1}{2^{s_0}}\frac{c\cdot 2^{2ps_0+\alpha_k(1-2p)}}{\varphi^p(q_{\alpha_k}^{s_0})}\ge\frac{C_p\cdot 2^{(2p-1)s_0}}{2^{(1-2p)\alpha_k}\varphi^p(q_{\alpha_k}^{s_0})}\ge\frac{c_p\cdot 2^{\rho(q_{\alpha_k}^{s_0})(1-2p)}}{\varphi^p(q_{\alpha_k}^{s_0})}.$$

Finally, combining this inequality, (11.1) and (18.54), we find that

$$\frac{\left(\int\limits_G \left(\sup\limits_{k\in\mathbb{N}}\sup\limits_{0\le s<n_k}\frac{|\sigma_{q^s_{\alpha_k}}a_k|}{\varphi(q^s_{\alpha_k})}\right)^p d\mu\right)^{1/p}}{\|a_k\|_{H_p}} \geq \frac{C_p\cdot 2^{(2-1/p)s_0}}{2^{\alpha_k}\varphi(q^{s_0}_{\alpha_k})}\frac{1}{2^{\alpha_k(1-1/p)}} \geq \frac{c_p\cdot 2^{\rho(q^{s_0}_{\alpha_k})(1/p-2)}}{\varphi(q^{s_0}_{\alpha_k})} \longrightarrow \infty \text{ as } k\to\infty. \qquad \square$$

The following result of Tephnadze [101] follows immediately from Theorem 18.4.

Corollary 18.4.

(a) *Let $0<p<1/2$ and $f\in H_p(G)$. Then the weighted maximal operator $\widetilde{\sigma}^*$, defined by (12.12), is bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.*

(b) *The rate of increase of the denominator $\{(n+1)^{1/p-2}\}$ cannot be improved.*

From Theorem 18.4, we also obtain the results of Persson and Tephnadze [76].

Corollary 18.5. *If $0<p<1/2$, $f\in H_p(G)$ and $\{n_k:\ k\ge 0\}$ is any sequence of positive numbers, then the maximal operator, defined by (12.14), is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ if and only if condition (12.13) is fulfilled.*

We also have the following new consequences of Theorem 18.4.

Corollary 18.6. *Let $p>0$ and $f\in H_p(G)$. Then the restricted maximal operators $\widetilde{\sigma}^{*,\nabla}_i$, $i=1,2$, defined by (18.18), are both bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.*

Corollary 18.7. *Let $0<p<1/2$ and $f\in H_p(G)$. Then the restricted maximal operators $\widetilde{\sigma}^{*,\nabla}_i$, $i=3,4,5$, defined by (18.19), are not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.*

Corollary 18.8. *Let $0<p<1/2$ and $f\in H_p(G)$. Then the restricted maximal operators $\widetilde{\sigma}^{*,\nabla}_i$, $i=6,7,8$, defined by*

$$\widetilde{\sigma}^{*,\nabla}_6 f := \sup_{k\in\mathbb{N}}\frac{|\sigma_{2^k+1}f|}{2^{k(1/p-2)}}, \quad \widetilde{\sigma}^{*,\nabla}_7 f := \sup_{k\in\mathbb{N}}\frac{|\sigma_{2^{2k}+2^k}f|}{2^{k(1/p-2)}}, \quad \widetilde{\sigma}^{*,\nabla}_8 f := \sup_{k\in\mathbb{N}}\frac{|\sigma_{2^{3k}+2^k}f|}{2^{2k(1/p-2)}}, \tag{18.55}$$

are bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

Remark 18.1. Let $0<p<1/2$ and $f\in H_p(G)$. Then from Corollary 18.8 we conclude that the restricted maximal operators $\widetilde{\sigma}^{*,\nabla}_i$, $i=6,7,8$, defined by (18.55), are bounded from the space $H_p(G)$ to the Lebesgue space *weak*-$L_p(G)$.

In [13], the martingales were constructed, which provide the sharpness of our main results described above.

Theorem 18.5.

(a) *Let $\sup\limits_{k\in\mathbb{N}}|A_{|n_k|}|=\infty$ and $\{\varphi_n\}$ be a non-decreasing sequence satisfying the condition*

$$\overline{\lim_{k\to\infty}}\frac{|A_{|n_k|}|^2}{\varphi_{|n_k|}}=\infty.$$

Then there exists a martingale $f\in H_{1/2}$ such that the maximal operator, defined by $\sup\limits_{k\in\mathbb{N}}\frac{|\sigma_{n_k}f|}{\varphi_{|n_k|}}$ is not bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

(b) *Let $0<p<1/2$ and $\varphi:\mathbb{N}\to[1,\infty)$ be any non-decreasing function satisfying condition* (18.20). *Then there exists a martingale $f\in H_p(G)$ such that*

$$\sup_{k\in\mathbb{N}}\left\|\frac{\sigma_{n_k}f}{\varphi(n_k)}\right\|_{weak\text{-}L_p}=\infty.$$

(c) *Let $0<p<1/2$, $\{n_k:\ k\geq 0\}$ be a sequence of positive numbers, $\{n_{s_i}:\ 1\leq i\leq r\}\subset\{n_k:\ k\geq 0\}$ be the numbers such that $2^s\leq n_{s_1}\leq n_{s_2}\leq\cdots\leq n_{s_r}\leq 2^{s+1}$, $s\in\mathbb{N}$, and $\varphi:\mathbb{N}\to[1,\infty)$ be any non-negative and non-decreasing function satisfying condition* (18.45). *Then there exists a martingale $f\in H_p(G)$ such that the maximal operator, defined by $\sup\limits_{s\in\mathbb{N}}\sup\limits_{2^s\leq n_{s_i}<2^{s+1}}\frac{|\sigma_{n_{s_i}}f|}{\varphi(n_{s_i})}$, is not bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.*

(d) *Let $0<p<1/2$, $\{n_k:\ k\geq 0\}$ be a sequence of positive numbers and $\varphi:\mathbb{N}\to[1,\infty)$ be any non-negative and non-decreasing function satisfying condition* (18.34). *Then there exists a martingale $f\in H_p(G)$ such that the maximal operator, defined by $\sup\limits_{n\in\mathbb{N}}\frac{|\sigma_n f|}{2^{d(n)(1/p-2)}\varphi(d(n))}$ is not bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.*

Proof. Under condition (14.6), there exists an increasing sequence $\{\alpha_k:\ k\geq 0\}$ of positive integers such that

$$\lim_{k\to\infty}\frac{|A_{|\alpha_k|}|^{1/2}}{\varphi^{1/4}_{|\alpha_k|}}=\infty$$

and

$$\sum_{k=1}^{\infty}\frac{\varphi^{1/4}_{|\alpha_k|}}{|A_{|\alpha_k|}|^{1/2}}\leq c<\infty. \tag{18.56}$$

Let

$$f_A:=\sum_{\{k;\ |\alpha_k|<A\}}\lambda_k a_k,\quad \lambda_k:=\frac{\varphi^{1/2}_{|\alpha_k|}}{|A_{|\alpha_k|}|},\quad a_k:=2^{|\alpha_k|}(D_{2^{|\alpha_k|+1}}-D_{2^{|\alpha_k|}}).$$

If we apply Lemma 5.3 and (18.56), we conclude that $F\in H_{1/2}$. Moreover,

$$\widehat{f}(j)=\begin{cases}\dfrac{2^{|\alpha_k|}\varphi^{1/2}_{|\alpha_k|}}{|A_{|\alpha_k|}|}, & j\in\{2^{|\alpha_k|},\ldots,2^{|\alpha_k|+1}-1\},\quad k\in\mathbb{N},\\ 0, & j\notin\bigcup\limits_{k=0}^{\infty}\{2^{|\alpha_k|},\ldots,2^{|\alpha_k|+1}-1\}.\end{cases} \tag{18.57}$$

Let $2^{|\alpha_k|}<j<\alpha_k$. Using (18.57), we get

$$S_jf=S_{2^{|\alpha_k|}}f+\sum_{v=2^{|\alpha_k|}}^{j-1}\widehat{f}(v)w_v=S_{2^{|\alpha_k|}}f+\frac{(D_j-D_{2^{|\alpha_k|}})\cdot 2^{|\alpha_k|}}{|A_{|\alpha_k|}|}. \tag{18.58}$$

Let $2^{|\alpha_k|}\leq\alpha_{s_n}\leq 2^{|\alpha_k|+1}$. Then, according to (18.58), we have

$$\begin{aligned}\sigma_{\alpha_{s_n}}f&=\frac{1}{\alpha_{s_n}}\sum_{j=1}^{2^{|\alpha_k|}}S_jf+\frac{1}{\alpha_{s_k}}\sum_{j=2^{|\alpha_k|}+1}^{\alpha_{s_n}}S_jf=\frac{\sigma_{2^{|\alpha_k|}}f}{\alpha_{s_k}}+\frac{(\alpha_{s_n}-2^{|\alpha_k|})S_{2^{|\alpha_k|}}f}{\alpha_{s_n}}\\&\quad+\frac{2^{|\alpha_k|}\varphi^{1/2}_{|\alpha_k|}}{|A_{|\alpha_k|}|\alpha_{s_n}}\sum_{j=2^{|\alpha_k|}+1}^{\alpha_{s_n}}(D_j-D_{2^{|\alpha_k|}}):=III_1+III_2+III_3.\end{aligned} \tag{18.59}$$

Using Lemma 5.3, we get

$$|III_3| = \frac{2^{|\alpha_k|}\varphi_{|\alpha_n|}^{1/2}}{|A_{|\alpha_k|}|\alpha_{s_n}}\Big|\sum_{j=1}^{\alpha_{s_n}-2^{|\alpha_k|}} D_j\Big| \geq \frac{\varphi_{|\alpha_k|}^{1/2}}{2|A_{|\alpha_k|}|}(\alpha_{s_n}-2^{|\alpha_k|})|K_{\alpha_{s_n}-2^{|\alpha_k|}}|.$$

Combining Corollaries 16.3 and 16.4, we find that $\|III_1\|_{1/2} \leq C$ and $\|III_2\|_{1/2} \leq C$.

Let $2^{|\alpha_k|} \leq \alpha_{s_1} \leq \alpha_{s_2} \leq \cdots \leq \alpha_{s_r} \leq 2^{|\alpha_k|+1}$ be natural numbers that generate the set $A_{|\alpha_k|} = \{l_1^{|\alpha_k|}, l_2^{|\alpha_k|}, \ldots, l_{r^1_{|\alpha_k|}}^{|\alpha_k|}\} \cup \{m_1^{|\alpha_k|}, m_2^{|\alpha_k|}, \ldots, m_{r^2_{|\alpha_k|}}^{|\alpha_k|}\}$, where

$$m_1^{|\alpha_k|} \geq l_1^{|\alpha_k|} > l_1^{|\alpha_k|} - 2 \geq m_2^{|\alpha_k|} \geq l_2^{|\alpha_k|} > l_2^{|\alpha_k|} - 2 \geq \cdots \geq m_{|\alpha_k|}^{|\alpha_k|} \geq l_{|\alpha_k|}^{|\alpha_k|} \geq 0,$$

and for any $1 \leq i \leq r^1_{|\alpha_k|}$, there exists $\alpha_{s_n} \in A_{|\alpha_k|}$, for some $1 \leq n \leq r$ such that $\alpha_{s_n} = \sum_{i=1}^{r_n}\sum_{k=l_i^n}^{m_i^n} 2^k$, where $l_u^{|\alpha_k|} = l_i$ for some $1 \leq u \leq r^1_{|\alpha_k|}$.

Since $\mu\{E_{l_i}\} \geq 1/2^{l_i+1}$, using Corollary 8.5, we get

$$\int_{E_{l_i}} \Big(\sup_{k\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_k}} f|}{\varphi_{|\alpha_{s_k}|}}\Big)^{1/2} d\mu \geq \int_{E_{l_i}} \Big(\frac{\sigma_{\alpha_{s_n}} f}{\varphi_{|\alpha_{s_n}|}}\Big)^{1/2} d\mu \geq \frac{1}{2^5|A_{|\alpha_k|}|^{1/2}\varphi_{|\alpha_k|}^{1/4}}. \tag{18.60}$$

On the other hand, for any $1 \leq i \leq r^2_{|\alpha_k|}$ there exists a number α_{s_n}, for some $1 \leq n \leq r$, such that $m_u^{|\alpha_k|} = m_i$ for some $1 \leq u \leq r^2_{|\alpha_k|}$. According to the fact that $\mu\{E_{t_i}\} \geq 1/2^{t_i+3}$, using again Corollary 8.5 for some α_{s_n} and $1 \leq i \leq r_s^2$, we also get

$$\int_{E_{m_i}} \Big(\sup_{k\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_k}} f|}{\varphi_{|\alpha_{s_k}|}}\Big)^{1/2} d\mu \geq \int_{E_{m_i}} \Big(\frac{\sigma_{\alpha_{s_n}} f}{\varphi_{|\alpha_{s_n}|}}\Big)^{1/2} d\mu \geq \frac{1}{2^7|A_{|\alpha_k|}|^{1/2}\varphi_{|\alpha_k|}^{1/4}}. \tag{18.61}$$

Combining (18.59)–(18.61) for sufficiently large α_k, we obtain

$$\begin{aligned}
\int_G \Big(\sup_{n\in\mathbb{N}} \frac{|\sigma_{\alpha_{s_n}} f|}{\varphi_{|\alpha_{s_n}|}}\Big)^{1/2} d\mu &\geq \|III_3\|_{1/2}^{1/2} - \|III_2\|_{1/2}^{1/2} - \|III_1\|_{1/2}^{1/2} \\
&\geq c\sum_{i=1}^{r^1_{|\alpha_k|}-1} \int_{E_{l_i}} \frac{1}{|A_{|\alpha_k|}|^{1/4}\varphi_{|\alpha_k|}^{1/8}} d\mu + c\sum_{i=1}^{r^2_{|\alpha_k|}-1} \int_{E_{m_i}} \frac{1}{|A_{|\alpha_k|}|^{1/4}\varphi_{|\alpha_k|}^{1/8}} d\mu - 2C \\
&\geq \frac{c(r^1_{|\alpha_k|} + r^2_{|\alpha_k|})}{2^7|A_{|\alpha_k|}|^{1/2}\varphi_{|\alpha_k|}^{1/4}} - 2C \geq \frac{c|A_{|\alpha_k|}|^{1/2}}{2^8\varphi_{|\alpha_k|}^{1/4}} \longrightarrow \infty \text{ as } k \to \infty.
\end{aligned}$$

So, part (b) is proved and the proof is complete. □

19 Boundedness of subsequences of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

In this section, we study the boundedness of subsequences of Fejér means with respect to the one-dimensional Walsh–Fourier series in the martingale Hardy spaces (for details see [112]).

First, we consider the case $p = 1/2$. The following statement is true.

Theorem 19.1.

(a) *Let $f \in H_{1/2}(G)$. Then there exists an absolute constant c such that*

$$\|\sigma_n f\|_{H_{1/2}(G)} \leq c\, V^2(n) \|f\|_{H_{1/2}(G)}.$$

(b) *Let $\{n_k : k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers such that $\sup_{k\in\mathbb{N}_+} V(n_k) = \infty$ and let $\Phi : \mathbb{N}_+ \to [1, \infty)$ be a non-decreasing function satisfying the conditions $\Phi(n) \uparrow \infty$ and*

$$\overline{\lim_{k\to\infty}} \frac{V^2(n_k)}{\Phi(n_k)} = \infty. \tag{19.1}$$

Then there exists a martingale $f \in H_{1/2}(G)$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{\sigma_{n_k} f}{\Phi(n_k)} \right\|_{1/2} = \infty.$$

Proof. Suppose that

$$\left\| \frac{\sigma_n f}{V^2(n)} \right\|_{1/2} \leq c\, \|f\|_{H_{1/2}(G)}. \tag{19.2}$$

Combining Propositions (11.1) and (11.2), we conclude that

$$\begin{aligned} \left\| \frac{\sigma_n f}{V^2(n)} \right\|_{H_{1/2}(G)}^{1/2} &\leq \left\| \frac{\sigma_n f}{V^2(n)} \right\|_{1/2}^{1/2} + \frac{1}{V^2(n)} \|\sigma^*_\# f\|_{1/2}^{1/2} + \frac{1}{V^2(n)} \|\widetilde{S}^*_\#\|_{1/2}^{1/2} \\ &\leq \left\| \frac{\widetilde{\sigma}_n f}{V^2(n)} \right\|_{1/2}^{1/2} + \|\widetilde{\sigma}^*_\# f\|_{1/2}^{1/2} + \|\widetilde{S}^*_\# f\|_{1/2}^{1/2} \leq c\, \|f\|_{H_{1/2}(G)}^{1/2}. \end{aligned} \tag{19.3}$$

By Lemma 10.2 and (19.3), the theorem will be proved if we show that

$$\int_{\overline{I}_M} \left(\frac{|\sigma_n a|}{V^2(n)} \right)^{1/2} d\mu \leq c < \infty$$

for any 1/2-atom a.

Without loss of generality, we may assume that a is a 1/2-atom with support I, for which $\mu(I) = 2^{-M}, I = I_M$. It is easy to check that $\sigma_n(a) = 0$, when $n \leq 2^M$. Therefore, we may assume that $n > 2^M$. Set

$$II^1_{\alpha_A}(x) := 2^M \int_{I_M} 2^{\alpha_A} |K_{2^{\alpha_A}}(x+t)|\, d\mu(t),$$

$$II^2_{l_A}(x) = 2^M \int_{I_M} 2^{l_A} \sum_{k=l_A}^{m_A} D_{2^k}(x+t)\, d\mu(t).$$

Let $x \in I_M$. Since σ_n is bounded from $L_\infty(G)$ to $L_\infty(G)$, for $n > 2^M$ and $\|a\|_\infty \leq 2^{2M}$, using Lemma 8.4, we can conclude that

$$\begin{aligned} \frac{|\sigma_n a(x)|}{V^2(n)} &\leq \frac{c}{V^2(n)} \int_{I_M} |a(x)|\, |K_n(x+t)|\, d\mu(t) \\ &\leq \frac{c\, \|a\|_\infty}{V^2(n)} \int_{I_M} |K_n(x+t)|\, d\mu(t) \leq \frac{c \cdot 2^{2M}}{V^2(n)} \int_{I_M} |K_n(x+t)|\, d\mu(t) \\ &\leq \frac{c \cdot 2^M}{V^2(n)} \left\{ \sum_{A=1}^{s} \int_{I_M} 2^{l_A} |K_{2^{l_A}}(x+t)|\, d\mu(t) + \int_{I_M} 2^{m_A} |K_{2^{m_A}}(x+t)|\, d\mu(t) \right\} \end{aligned}$$

$$+\frac{c\cdot 2^M}{V^2(n)}\sum_{A=1}^{s}\int_{I_M}2^{l_A}\sum_{k=l_A}^{m_A}D_{2^k}(x+t)\,d\mu(t)+\frac{c\cdot 2^M}{V^2(n)}\int_{I_M}V(n)\,d\mu(t)$$

$$=\frac{c}{V^2(n)}\sum_{A=1}^{s}\left(II^1_{l_A}(x)+II^1_{m_A}(x)+II^2_{l_A}(x)\right)+c.$$

Hence,

$$\int_{\overline{I}_M}\left|\frac{\sigma_n a(x)}{V^2(n)}\right|^{1/2}d\mu(x)$$

$$\leq\frac{c}{V(n)}\left(\sum_{A=1}^{s}\int_{\overline{I}_M}|II^1_{l_A}(x)|^{1/2}\,d\mu(x)+\int_{\overline{I}_M}|II^1_{m_A}(x)|^{1/2}\,d\mu(x)+\int_{\overline{I}_M}|II^2_{l_A}(x)|^{1/2}\,d\mu(x)\right)+c.$$

Since $s\leq 4V(n)$, Theorem 19.1 will be proved if we show that

$$\int_{\overline{I}_M}|II^1_{\alpha_A}(x)|^{1/2}\,d\mu(x)\leq c<\infty,\quad \int_{\overline{I}_M}|II^2_{l_A}(x)|^{1/2}\,d\mu(x)\leq c<\infty, \tag{19.4}$$

where $\alpha_A=l_A$ or $\alpha_A=m_A$, $A=1,\dots,s$.

Let $t\in I_M$ and $x\in I_M^{k,l}$, $0\leq k<l<\alpha_A\leq M$ or $0\leq k<l\leq M\leq\alpha_A$. Since $x+t\in I_M^{k,l}$, applying (8.1) in Lemma 8.1, we conclude that

$$K_{2^{\alpha_A}}(x+t)=0\ \text{ and }\ II^1_{\alpha_A}(x)=0. \tag{19.5}$$

Let $x\in I_M^{k,l}$, $0\leq k<\alpha_A\leq l\leq M$. Then $x+t\in I_M^{k,l}$, where $t\in I_M$, and if we apply again (8.1) in Lemma 8.1, we get

$$2^{\alpha_A}|K_{2^{\alpha_A}}(x+t)|\leq 2^{\alpha_A+k}\ \text{ and }\ II^1_{\alpha_A}(x)\leq 2^{\alpha_A+k}. \tag{19.6}$$

Analogously to (19.6), for $0\leq\alpha_A\leq k<l\leq M$, we can prove that

$$2^{\alpha_A}|K_{2^{\alpha_A}}(x+t)|\leq 2^{2\alpha_A},\quad II^1_{\alpha_A}(x)\leq 2^{2\alpha_A},\quad t\in I_M,\quad x\in I_M^{k,l}. \tag{19.7}$$

Let $0\leq\alpha_A\leq M-1$, where $A=1,\dots,s$. According to (1.2) and (19.5)–(19.7), we find that

$$\int_{\overline{I}_M}|II^1_{\alpha_A}(x)|^{1/2}\,d\mu(x)=\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\int_{I_M^{k,l}}|II^1_{\alpha_A}(x)|^{1/2}\,d\mu(x)+\sum_{k=0}^{M-1}\int_{I_M^{k,M}}|II^1_{\alpha_A}(x)|^{1/2}\,d\mu(x)$$

$$\leq c\sum_{k=0}^{\alpha_A-1}\sum_{l=\alpha_A+1}^{M-1}\int_{I_M^{k,l}}2^{(\alpha_A+k)/2}\,d\mu(x)+c\sum_{k=\alpha_A}^{M-2}\sum_{l=k+1}^{M-1}\int_{I_M^{k,l}}2^{\alpha_A}\,d\mu(x)$$

$$+c\sum_{k=0}^{\alpha_A-1}\int_{I_M^{k,M}}2^{(\alpha_A+k)/2}\,d\mu(x)+c\sum_{k=\alpha_A}^{M-1}\int_{I_M^{k,M}}2^{\alpha_A}\,d\mu(x)$$

$$\leq c\sum_{k=0}^{\alpha_A-1}\sum_{l=\alpha_A+1}^{M-1}\frac{2^{(\alpha_A+k)/2}}{2^l}+c\sum_{k=\alpha_A}^{M-2}\sum_{l=k+1}^{M-1}\frac{2^{\alpha_A}}{2^l}$$

$$+c\sum_{k=0}^{\alpha_A-1}\frac{2^{(\alpha_A+k)/2}}{2^M}+c\sum_{k=\alpha_A}^{M-1}\frac{2^{\alpha_A}}{2^M}\leq c<\infty.$$

Let $\alpha_A\geq M$. Analogously to $II^1_{\alpha_A}(x)$, we can prove (19.4) for $A=1,\dots,s$.

Now, we prove the boundedness of $II_{l_A}^2$. Let $t \in I_M$ and $x \in I_i \setminus I_{i+1}$, $i \leq l_A - 1$. Since $x + t \in I_i \setminus I_{i+1}$, applying the first equality of Lemma 10.3, we get

$$II_{l_A}^2(x) = 0. \tag{19.8}$$

Let $x \in I_i \setminus I_{i+1}$, $l_A \leq i \leq m_A$. Since $n \geq 2^M$ and $t \in I_M$, applying the first equality of Lemma 10.3, we get

$$II_{l_A}^2(x) \leq 2^M \int_{I_M} 2^{l_A} \sum_{k=l_A}^{i} D_{2^k}(x+t)\, d\mu(t) \leq c \cdot 2^{l_A+i}. \tag{19.9}$$

Let $x \in I_i \setminus I_{i+1}$, $m_A < i \leq M-1$. Then $x + t \in I_i \setminus I_{i+1}$ for any $t \in I_M$, and due to the first equality of Lemma 10.3, we obtain

$$II_{l_A}^2(x) \leq c \cdot 2^M \int_{I_M} 2^{l_A+m_A} \leq c \cdot 2^{l_A+m_A}. \tag{19.10}$$

Let $0 \leq l_A \leq m_A \leq M$. Then, in the view of (1.2) and (19.8)–(19.10), we can conclude that

$$\begin{aligned}
\int_{\overline{I}_M} |II_{l_A}^2(x)|^{1/2}\, d\mu(x) &= \Big(\sum_{i=0}^{l_A-1} + \sum_{i=l_A}^{m_A} + \sum_{i=m_A+1}^{M-1} \Big) \int_{I_i \setminus I_{i+1}} |II_{l_A}^2(x)|^{1/2}\, d\mu(x) \\
&\leq c \sum_{i=l_A}^{m_A} \int_{I_i \setminus I_{i+1}} 2^{(l_A+i)/2}\, d\mu(x) + c \sum_{i=m_A+1}^{M-1} \int_{I_i \setminus I_{i+1}} 2^{(l_A+m_A)/2}\, d\mu(x) \\
&\leq c \sum_{i=l_A}^{m_A} 2^{(l_A+i)/2} \cdot \frac{1}{2^i} + c \sum_{i=m_A+1}^{M-1} 2^{(l_A+m_A)/2} \cdot \frac{1}{2^i} \leq c < \infty.
\end{aligned}$$

Analogously, we can prove the same estimations in the cases $0 \leq l_A \leq M < m_A$ and $M \leq l_A \leq m_A$.

Now, we prove part (b) of Theorem 19.1. According to (19.1), there exists an increasing sequence $\{\alpha_k : k \in \mathbb{N}_+\} \subset \{n_k : k \in \mathbb{N}_+\}$ of natural numbers such that

$$\sum_{k=1}^{\infty} \frac{\Phi^{1/4}(\alpha_k)}{V^{1/2}(\alpha_k)} \leq c < \infty. \tag{19.11}$$

Let $f = (f_n, n \in \mathbb{N}_+)$ be a martingale from Example 11.3, where $\lambda_k := \Phi^{1/2}(\alpha_k)/V(\alpha_k)$. According to (19.11), we get that condition (11.8) is fulfilled and thus $f = (f_n, n \in \mathbb{N}_+)$. Applying (11.10), we get

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{V(\alpha_k)}, & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \quad k \in \mathbb{N}_+, \\ 0, & j \notin \bigcup\limits_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}. \end{cases} \tag{19.12}$$

Let $2^{|\alpha_k|} < j < \alpha_k$. If we apply (11.12), we arrive at

$$S_j f = S_{2^{|\alpha_k|}} f + \frac{w_{2^{|\alpha_k|}} D_{j-2^{|\alpha_k|}} \Phi^{1/2}(\alpha_k)}{V(\alpha_k)}. \tag{19.13}$$

Hence,

$$\frac{\sigma_{\alpha_k} f}{\Phi(\alpha_k)} = \frac{1}{\Phi(\alpha_k)\alpha_k} \sum_{j=1}^{2^{|\alpha_k|}} S_j f + \frac{1}{\Phi(\alpha_k)\alpha_k} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_k} S_j f = \frac{\sigma_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)\alpha_k} + \frac{(\alpha_k - 2^{|\alpha_k|}) S_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)\alpha_k}$$

$$+\frac{w_{2^{|\alpha_k|}}\cdot 2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V(\alpha_k)\alpha_k}\sum_{j=2^{|\alpha_k|}+1}^{\alpha_k} D_{j-2^{|\alpha_k|}} = III_1 + III_2 + III_3. \quad (19.14)$$

For III_3, we have

$$|III_3| = \frac{2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V(\alpha_k)\alpha_k}\Big|\sum_{j=1}^{\alpha_k-2^{|\alpha_k|}} D_j\Big|$$
$$= \frac{2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V(\alpha_k)\alpha_k}(\alpha_k-2^{|\alpha_k|})|K_{\alpha_k-2^{|\alpha_k|}}| \geq \frac{c(\alpha_k-2^{|\alpha_k|})|K_{\alpha_k-2^{|\alpha_k|}}|}{\Phi^{1/2}(\alpha_k)V(\alpha_k)}. \quad (19.15)$$

Let $\alpha_k = \sum_{i=1}^{r_k}\sum_{k=l_i^k}^{m_i^k} 2^k$, where $m_1^k \geq l_1^k > l_1^k-2 \geq m_2^k \geq l_2^k > l_2^k-2 \geq \cdots \geq m_s^k \geq l_s^k \geq 0.$

Since (see Corollaries 16.3 and 16.4) $\|III_1\|_{1/2} \leq c$, $\|III_2\|_{1/2} \leq c$ and $\mu\{E_{l_i^k}\} \geq \frac{1}{2^{l_i^k-1}}$, combining (19.14), (19.15) and Lemma 8.6, we get

$$\int_G \Big|\frac{\sigma_{\alpha_k}f(x)}{\Phi(\alpha_k)}\Big|^{1/2} d\mu(x) \geq \|III_3\|_{1/2}^{1/2} - \|III_2\|_{1/2}^{1/2} - \|III_1\|_{1/2}^{1/2}$$
$$\geq c\sum_{i=2}^{r_k-2}\int_{E_{l_i^k}}\Big|\frac{2^{2l_i^k}}{\Phi^{1/2}(\alpha_k)V(\alpha_k)}\Big|^{1/2} d\mu(x) - 2c \geq c\sum_{i=2}^{r_k-2}\frac{1}{V^{1/2}(\alpha_k)\Phi^{1/4}(\alpha_k)} - 2c$$
$$\geq \frac{c\,r_k}{V^{1/2}(\alpha_k)\Phi^{1/4}(\alpha_k)} \geq \frac{c\,V^{1/2}(\alpha_k)}{\Phi^{1/4}(\alpha_k)} \longrightarrow \infty \text{ as } k\to\infty. \qquad \square$$

Theorem 19.2.

(a) *Let $0<p<1/2$, $f\in H_p(G)$. Then there exists an absolute constant c_p depending only on p such that*
$$\|\sigma_n f\|_{H_p(G)} \leq c_p\cdot 2^{d(n)(1/p-2)}\|f\|_{H_p(G)}.$$

(b) *Let $0<p<1/2$ and $\Phi(n):\mathbb{N}_+\to[1,\infty)$ be a non-decreasing function such that*
$$\sup_{k\in\mathbb{N}_+} d(n_k) = \infty, \qquad \overline{\lim_{k\to\infty}}\frac{2^{d(n_k)(1/p-2)}}{\Phi(n_k)} = \infty. \quad (19.16)$$

Then there exists a martingale $f\in H_p(G)$ such that
$$\sup_{k\in\mathbb{N}_+}\Big\|\frac{\sigma_{n_k}f}{\Phi(n_k)}\Big\|_{weak\text{-}L_p(G)} = \infty.$$

Proof. Let $n\in\mathbb{N}$. Analogously to (19.3), it is sufficient to prove that $\int_{\overline{I}_M}\left(2^{d(n)(2-1/p)}|\sigma_n(a)|\right)^p d\mu \leq c_p < \infty$ for every p-atom a, where I denotes the support of the atom.

Analogously to Theorem 19.1, we may assume that a is a p-atom with support $I=I_M$, $\mu(I_M)=2^{-M}$ and $n>2^M$. Since $\|a\|_\infty \leq 2^{M/p}$, we can conclude that

$$2^{d(n)(2-1/p)}|\sigma_n a| \leq 2^{d(n)(2-1/p)}\|a\|_\infty\int_{I_M}|K_n(x+t)|\,d\mu(t) \leq 2^{d(n)(2-1/p)}\cdot 2^{M/p}\int_{I_M}|K_n(x+t)|\,d\mu(t).$$

Let $x\in I_M^{k,l}$, $0\leq k,l\leq [n]\leq M$. Then, applying (8.4) in Lemma 8.2, we get $K_n(x+t)=0$, where $t\in I_M$, and hence,
$$2^{d(n)(2-1/p)}|\sigma_n a| = 0. \quad (19.17)$$

Let $x \in I_M^{k,l}$, $[n] \leq k,l \leq M$ or $k \leq [n] \leq l \leq M$. Then by Lemma 8.5 it follows that

$$2^{d(n)(2-1/p)}|\sigma_n a| \leq 2^{d(n)(2-1/p)} \cdot 2^{M(1/p-2)+k+l} \leq c_p \cdot 2^{[n](1/p-2)+k+l}. \tag{19.18}$$

Combining (1.2), (19.17) and (19.18), we can conclude that

$$\begin{aligned}
&\int\limits_{\overline{I}_M} \left|2^{d(n)(2-1/p)}\sigma_n a(x)\right|^p d\mu(x) \\
&\leq \Bigg(\sum_{k=0}^{[n]-2} \sum_{l=k+1}^{[n]-1} + \sum_{k=0}^{[n]-1} \sum_{l=[n]}^{M-1} + \sum_{k=[n]}^{M-2} \sum_{l=k+1}^{M-1} \Bigg) \int\limits_{I_M^{k,l}} \left|2^{d(n)(2-1/p)}\sigma_n a(x)\right|^p d\mu(x) \\
&+ \sum_{k=0}^{M-1} \int\limits_{I_M^{k,M}} \left|2^{d(n)(2-1/p)}\sigma_n a(x)\right|^p d\mu(x) \leq c_p \sum_{k=[n]}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^l} \cdot 2^{[n](2p-1)} \cdot 2^{p(k+l)} \\
&+ c_p \sum_{k=0}^{[n]} \sum_{l=[n]+1}^{M-1} \frac{1}{2^l} \cdot 2^{[n](2p-1)} \cdot 2^{p(k+l)} + \frac{c_p \cdot 2^{[n](2p-1)}}{2^M} \sum_{k=0}^{[n]} 2^{p(k+M)} < c_p < \infty.
\end{aligned}$$

Now, we prove part (b) of Theorem 19.2. According to (19.16), there exists an increasing sequence of natural numbers $\{\alpha_k : \ k \in \mathbb{N}_+\} \subset \{n_k : \ k \in \mathbb{N}_+\}$ such that $\alpha_0 \geq 3$ and

$$\sum_{\eta=0}^{\infty} u^{-p}(\alpha_\eta) < c_p < \infty, \quad u(\alpha_k) = \frac{2^{d(\alpha_k)(1/p-2)/2}}{\Phi^{1/2}(\alpha_k)}. \tag{19.19}$$

Let f be a martingale from Example 11.3, where $\lambda_k = u^{-1}(\alpha_k)$.

If we apply (19.19), we get that (11.8) is fulfilled, and it follows that $f \in H_p(G)$. According to (11.10), we have

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{|\alpha_k|(1/p-1)}}{u(\alpha_k)}, & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1}-1\}, \ \ k \in \mathbb{N}_+, \\ 0, & j \notin \bigcup\limits_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1}-1\}. \end{cases} \tag{19.20}$$

Let $2^{|\alpha_k|} < j < \alpha_k$. Then, analogously to (19.13) and (19.14), applying (19.20), we get

$$\begin{aligned}
\frac{\sigma_{\alpha_k} f}{\Phi(\alpha_k)} &= \frac{\sigma_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)\alpha_k} + \frac{(\alpha_k - 2^{|\alpha_k|})S_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)\alpha_k} \\
&+ \frac{2^{|\alpha_k|(1/p-1)}}{\Phi(\alpha_k)u(\alpha_k)\alpha_k} \sum_{j=2^{|\alpha_k|}}^{\alpha_k-1} (D_j - D_{2^{|\alpha_k|}}) = IV_1 + IV_2 + IV_3.
\end{aligned}$$

Let $\alpha_k \in \mathbb{N}$ and $E_{[\alpha_k]} := I_{[\alpha_k]+1}(e_{[\alpha_k]-1} + e_{[\alpha_k]})$. Since $[\alpha_k - 2^{|\alpha_k|}] = [\alpha_k]$, analogously to (19.15), applying Lemma 8.6 for IV_3, we have the following estimation:

$$\begin{aligned}
|IV_3| &= \frac{2^{|\alpha_k|(1/p-1)}}{\Phi(\alpha_k)u(\alpha_k)\alpha_k} (\alpha_k - 2^{|\alpha_k|})|K_{\alpha_k - 2^{|\alpha_k|}}| \\
&= \frac{2^{|\alpha_k|(1/p-1)}}{\Phi(\alpha_k)u(\alpha_k)\alpha_k} \cdot \left|2^{[\alpha_k]} K_{[\alpha_k]}\right| \geq \frac{2^{|\alpha_k|(1/p-2)} \cdot 2^{2[\alpha_k]-4}}{\Phi(\alpha_k)u(\alpha_k)} \geq \frac{2^{|\alpha_k|(1/p-2)/2} \cdot 2^{2[\alpha_k]-4}}{\Phi^{1/2}(\alpha_k)}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\|IV_3\|^p_{weak\text{-}L_p(G)} &\geq \Big(\frac{2^{|\alpha_k|(1/p-2)/2}\cdot 2^{2[\alpha_k]-4}}{\Phi^{1/2}(\alpha_k)}\Big)^p \mu\Big\{x\in G:\ |IV_3|\geq \frac{2^{|\alpha_k|(1/p-2)/2}\cdot 2^{2[\alpha_k]-4}}{\Phi^{1/2}(\alpha_k)}\Big\}\\
&\geq c_p\Big(\frac{2^{2[\alpha_k]+|\alpha_k|(1/p-2)/2}}{\Phi^{1/2}(\alpha_k)}\Big)^p \mu(E_{[\alpha_k]}) \geq c_p\Big(\frac{2^{(|\alpha_k|-[\alpha_k])(1/p-2)}}{\Phi(\alpha_k)}\Big)^{p/2}\\
&= c_p\Big(\frac{2^{d(\alpha_k)(1/p-2)}}{\Phi(\alpha_k)}\Big)^{p/2} \longrightarrow \infty \ \text{ as } \ k\to\infty.
\end{aligned}
$$

Combining Corollaries 16.3 and 16.4, we find that

$$\|IV_1\|_{weak\text{-}L_p(G)} \leq c_p < \infty, \quad \|IV_2\|_{weak\text{-}L_p(G)} \leq c_p < \infty.$$

On the other hand, for sufficiently large n, we can conclude that

$$
\begin{aligned}
\|\sigma_{\alpha_k} f\|^p_{weak\text{-}L_p(G)} &\geq \|IV_3\|^p_{weak\text{-}L_p(G)} - \|IV_2\|^p_{weak\text{-}L_p(G)} - \|IV_1\|^p_{weak\text{-}L_p(G)}\\
&\geq \frac{1}{2}\,\|IV_3\|^p_{weak\text{-}L_p(G)} \longrightarrow \infty \ \text{ as } \ k\to\infty.
\end{aligned}
$$

□

Corollary 19.1. *Let $p>0$ and $f\in H_p(G)$. Then*

$$\|\sigma_{2^k+2^{k-1}} f - f\|_{H_p(G)} \to 0 \ \text{ as } \ k\to\infty.$$

Corollary 19.2. *Let $0<p<1/2$. Then there exists a martingale $f\in H_p(G)$ such that*

$$\|\sigma_{2^k+1} f - f\|_{weak\text{-}L_p(G)} \nrightarrow 0 \ \text{ as } \ k\to\infty.$$

On the other hand, for any $f\in H_{1/2}(G)$,

$$\|\sigma_{2^k+1} f - f\|_{H_{1/2}(G)} \to 0 \ \text{ as } \ k\to\infty.$$

20 Modulus of continuity and convergence in norm of subsequences of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

In this section, we apply Theorem 19.1 and Theorem 19.2 to find the necessary and sufficient conditions for the modulus of continuity of the martingale $f\in H_p$, for which subsequences of Fejér means with respect to the one-dimensional Walsh–Fourier series converge in the H_p-norm.

First, we prove the following result.

Theorem 20.1.

(a) *Let $f\in H_{1/2}(G)$, $\sup_{k\in\mathbb{N}_+} V(n_k)=\infty$ and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|n_k|}}, f\Big) = o\Big(\frac{1}{V^2(n_k)}\Big) \ \text{ as } \ k\to\infty. \tag{20.1}$$

Then $\|\sigma_{n_k} f - f\|_{H_{1/2}(G)}\to 0$ as $k\to\infty$.

(b) *Let $\sup_{k\in\mathbb{N}_+} V(n_k)=\infty$. Then there exists a martingale $f\in H_{1/2}(G)$ such that*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^{|n_k|}}, f\Big) = O\Big(\frac{1}{V^2(n_k)}\Big) \ \text{ as } \ k\to\infty \tag{20.2}$$

and

$$\|\sigma_{n_k} f - f\|_{H_{1/2}(G)} \nrightarrow 0 \ \text{ as } \ k\to\infty. \tag{20.3}$$

Proof. Let $f \in H_{1/2}(G)$ and $2^k < n \le 2^{k+1}$. Then

$$\begin{aligned}\|\sigma_n f - f\|_{H_{1/2}(G)}^{1/2} &\le \|\sigma_n f - \sigma_n S_{2^k} f\|_{H_{1/2}(G)}^{1/2} + \|\sigma_n S_{2^k} f - S_{2^k} f\|_{H_{1/2}(G)}^{1/2} + \|S_{2^k} f - f\|_{H_{1/2}(G)}^{1/2} \\ &= \|\sigma_n (S_{2^k} f - f)\|_{H_{1/2}(G)}^{1/2} + \|S_{2^k} f - f\|_{H_{1/2}(G)}^{1/2} + \|\sigma_n S_{2^k} f - S_{2^k} f\|_{H_{1/2}(G)}^{1/2} \\ &\le c\,(V(n)+1)\omega_{H_{1/2}(G)}^{1/2}\Big(\frac{1}{2^k}, f\Big) + \|\sigma_n S_{2^k} f - S_{2^k} f\|_{H_{1/2}(G)}^{1/2}.\end{aligned}$$

It is evident that

$$\sigma_n S_{2^k} f - S_{2^k} f = \frac{2^k}{n}\,(S_{2^k}\sigma_{2^k} f - S_{2^k} f) = \frac{2^k}{n}\,S_{2^k}(\sigma_{2^k} f - f).$$

Let $p > 0$. Combining Corollaries 16.3 and 16.4, we conclude that

$$\|\sigma_n S_{2^k} f - S_{2^k} f\|_{H_{1/2}(G)}^{1/2} \le \frac{2^{k/2}}{n^{1/2}}\,\|S_{2^k}(\sigma_{2^k} f - f)\|_{H_{1/2}(G)}^{1/2} \le \|\sigma_{2^k} f - f\|_{H_{1/2}(G)}^{1/2} \longrightarrow 0 \ \text{ as } \ k \to \infty.$$

Now, we prove part (b) of Theorem 20.1. Since $\sup\limits_{k\in\mathbb{N}_+} V(\alpha_k) = \infty$, there exists a martingale $\{\alpha_k :\ k \in \mathbb{N}_+\} \subset \{n_k :\ k \in \mathbb{N}_+\}$ such that $V(\alpha_k) \uparrow \infty$ as $k \to \infty$ and

$$V^2(\alpha_k) \le V(\alpha_{k+1}). \tag{20.4}$$

Let f be a martingale from Example 11.3, where $\lambda_k = V^{-2}(\alpha_k)$.

Applying (20.4), we find that condition (10.2) is fulfilled and it follows that $f \in H_p(G)$. Using (11.10) we obtain

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{|\alpha_k|}}{V^2(\alpha_k)}\,, & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \ \ k \in \mathbb{N}_+ \\ 0\,, & j \notin \bigcup\limits_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}. \end{cases} \tag{20.5}$$

Combining (11.13) and (20.4), we conclude

$$w_{H_{1/2}(G)}\Big(\frac{1}{2^n}, f\Big) = \|f - S_{2^n} f\|_{H_{1/2}(G)} \le \sum_{i=n+1}^{\infty} \frac{1}{V^2(\alpha_i)} = O\Big(\frac{1}{V^2(\alpha_n)}\Big) \ \text{ as } \ n \to \infty. \tag{20.6}$$

Let $2^{|\alpha_k|} < j \le \alpha_k$. By (11.12), we get

$$S_j f = S_{2^{|\alpha_k|}} f + \frac{2^{|\alpha_k|} w_{2^{|\alpha_k|}} D_{j-2^{|\alpha_k|}}}{V^2(\alpha_k)}\,.$$

Hence,

$$\sigma_{\alpha_k} f - f = \frac{2^{|\alpha_k|}}{\alpha_k}\,(\sigma_{2^{|\alpha_k|}} f - f) + \frac{\alpha_k - 2^{|\alpha_k|}}{\alpha_k}\,(S_{2^{|\alpha_k|}} f - f) + \frac{2^{|\alpha_k|} w_{2^{|\alpha_k|}} (\alpha_k - 2^{|\alpha_k|}) K_{\alpha_k - 2^{|\alpha_k|}}}{\alpha_k V^2(\alpha_k)}\,. \tag{20.7}$$

According to (20.7), we have

$$\begin{aligned}\|\sigma_{\alpha_k} f - f\|_{1/2}^{1/2} \ge \frac{c}{V(\alpha_k)}\,&\big\|(\alpha_k - 2^{|\alpha_k|}) K_{\alpha_k - 2^{|\alpha_k|}}\big\|_{1/2}^{1/2} \\ &- \Big(\frac{2^{|\alpha_k|}}{\alpha_k}\Big)^{1/2} \|\sigma_{2^{|\alpha_k|}} f - f\|_{1/2}^{1/2} - \Big(\frac{\alpha_k - 2^{\alpha_k|}}{\alpha_k}\Big)^{1/2} \|S_{2^{|\alpha_k|}} f - f\|_{1/2}^{1/2}.\end{aligned} \tag{20.8}$$

Let $\alpha_k = \sum\limits_{i=1}^{r_k} \sum\limits_{k=l_i^k}^{m_i^k} 2^k$, where $m_1^k \ge l_1^k > l_1^k - 2 \ge m_2^k \ge l_2^k > l_2^k - 2 > \cdots > m_s^k \ge l_s^k \ge 0$ and $E_{l_i^k} := I_{l_i^k+1}(e_{l_i^k-1} + e_{l_i^k})$.

Using Lemma 8.6, we get

$$\int\limits_G \left|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}(x)\right|^{1/2}d\mu$$
$$\geq \frac{1}{16}\sum_{i=2}^{r_k-2}\int\limits_{E_{l_i^k}}\left|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}(x)\right|^{1/2}d\mu(x)\geq \frac{1}{16}\sum_{i=2}^{r_k-2}\frac{1}{2^{l_i^k}}\cdot 2^{l_i^k}\geq c\,r_k\geq c\,V(\alpha_k). \quad (20.9)$$

Combining estimations (20.8), (20.9), Corollaries 16.3 and 16.4, we get that (20.3) holds true and Theorem 20.1 is proved. □

Theorem 20.2.

(a) *Let* $0<p<1/2$, $f\in H_p(G)$, $\sup\limits_{k\in\mathbb{N}_+} d(n_k)=\infty$ *and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|n_k|}},f\Big)=o\Big(\frac{1}{2^{d(n_k)(1/p-2)}}\Big)\ \text{ as }\ k\to\infty. \quad (20.10)$$

Then

$$\|\sigma_{n_k}f-f\|_{H_p(G)}\to 0\ \text{ as }\ k\to\infty. \quad (20.11)$$

(b) *Let* $\sup\limits_{k\in\mathbb{N}_+} d(n_k)=\infty$. *Then there exists a martingale* $f\in H_p(G)$, $0<p<1/2$ *such that*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|n_k|}},f\Big)=O\Big(\frac{1}{2^{d(n_k)(1/p-2)}}\Big)\ \text{ as }\ k\to\infty \quad (20.12)$$

and $\|\sigma_{n_k}f-f\|_{weak\text{-}L_p(G)}\nrightarrow 0$ *as* $k\to\infty$.

Proof. Let $0<p<1/2$. Then under condition (20.10) if we repeat steps of the proof of Theorem 20.1, we immediately get that (20.11) holds.

Let us prove part (b) of Theorem 20.2. Since $\sup\limits_{k\in\mathbb{N}_+} d(n_k)=\infty$, there exists $\{\alpha_k:\ k\in\mathbb{N}_+\}\subset\{n_k:\ k\in\mathbb{N}_+\}$ such that $\sup\limits_{k\in\mathbb{N}_+} d(\alpha_k)=\infty$ and

$$2^{2d(\alpha_k)(1/p-2)}\leq 2^{d(\alpha_{k+1})(1/p-2)}. \quad (20.13)$$

Let f be a martingale from Lemma 11.3, where $\lambda_k=2^{-(1/p-2)d(\alpha_i)}$.

Using (20.13), we conclude that condition (10.2) is fulfilled and thus $f\in H_p(G)$.

According to (11.10), we get

$$\widehat{f}(j)=\begin{cases}2^{(1/p-2)[\alpha_k]}, & j\in\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\ \ k\in\mathbb{N}_+,\\ 0\,, & j\notin\bigcup\limits_{n=0}^{\infty}\{2^{|\alpha_n|},\dots,2^{|\alpha_n|+1}-1\}.\end{cases} \quad (20.14)$$

Combining (11.13) and (20.13), we have

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big)\leq\sum_{i=k}^{\infty}\frac{1}{2^{d(\alpha_i)(1/p-2)}}=O\Big(\frac{1}{2^{d(\alpha_k)(1/p-2)}}\Big)\ \text{ as }\ k\to\infty. \quad (20.15)$$

Analogously to the proof of the previous theorem, for a sufficiently large k, we can conclude that

$$\|\sigma_{\alpha_k}f-f\|^p_{weak\text{-}L_p(G)}\geq 2^{(1-2p)[\alpha_k]}\left\|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}\right\|^p_{weak\text{-}L_p(G)}$$
$$-\Big(\frac{2^{|\alpha_k|}}{\alpha_k}\Big)^p\|\sigma_{2^{|\alpha_k|}}f-f\|^p_{weak\text{-}L_p(G)}-\Big(\frac{\alpha_k-2^{|\alpha_k|}}{\alpha_k}\Big)^p\|S_{2^{|\alpha_k|}}f-f\|^p_{weak\text{-}L_p(G)}$$
$$\geq 2^{(1-2p)[\alpha_k]-1}\left\|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}\right\|^p_{weak\text{-}L_p(G)}. \quad (20.16)$$

Let $x \in E_{[\alpha_k]}$. Lemma 8.6 implies that

$$\mu\Big(x \in G:\ (\alpha_k - 2^{|\alpha_k|})|K_{\alpha_k - 2^{|\alpha_k|}}| \geq 2^{2[\alpha_k]-4}\Big) \geq \mu(E_{[\alpha_k]}) \geq \frac{1}{2^{[\alpha_k]-4}}$$

and

$$2^{2p[\alpha_k]-4}\mu\Big(x \in G:\ (\alpha_k - 2^{|\alpha_k|})|K_{\alpha_k - 2^{|\alpha_k|}}| \geq 2^{2[\alpha_k]-4}\Big) \geq 2^{(2p-1)[\alpha_k]-4}. \tag{20.17}$$

Hence, combining (20.16) and (20.17) and Corollaries 16.3 and 16.4, we get $\|\sigma_{n_k} f - f\|_{weak\text{-}L_p(G)} \nrightarrow 0$ as $k \to \infty$. □

Using Theorem 20.2, we easily get an important result proved in [116].

Corollary 20.1.

(a) *Let $f \in H_{1/2}(G)$ and*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^k}, f\Big) = o\Big(\frac{1}{k^2}\Big) \quad as \ \ k \to \infty.$$

Then $\|\sigma_k f - f\|_{H_{1/2}(G)} \to 0$ as $k \to \infty$.

(b) *There exists a martingale $f \in H_{1/2}(G)$, for which*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^k}, f\Big) = O\Big(\frac{1}{k^2}\Big) \quad as \ \ k \to \infty$$

and $\|\sigma_k f - f\|_{1/2} \nrightarrow 0$ as $k \to \infty$.

Corollary 20.2.

(a) *Let $0 < p < 1/2$, $f \in H_p(G)$ and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^k}, f\Big) = o\Big(\frac{1}{2^{k(1/p-2)}}\Big) \quad as \ \ k \to \infty.$$

Then $\|\sigma_k f - f\|_{H_p(G)} \to 0$ as $k \to \infty$.

(b) *There exists a martingale $f \in H_p(G)$, $0 < p < 1/2$, for which*

$$\omega_{H_p(G)}\Big(\frac{1}{2^k}, f\Big) = O\Big(\frac{1}{2^{k(1/p-2)}}\Big) \quad as \ \ k \to \infty$$

and $\|\sigma_k f - f\|_{weak\text{-}L_p(G)} \nrightarrow 0$ as $k \to \infty$.

21 Strong convergence of Fejér means with respect to the one-dimensional Walsh–Fourier series in the martingale Hardy spaces

In this section, we consider results on the strong convergence of Fejér means with respect to the one-dimensional Walsh–Fourier series in the martingale Hardy spaces, when $0 < p \leq 1/2$ (for details see [97]).

The following statement is true.

Theorem 21.1.

(a) *Let $0 < p \leq 1/2$ and $f \in H_p(G)$. Then there exists a constant c_p, depending only on p, such that*

$$\frac{1}{\log^{[1/2+p]} n} \sum_{m=1}^{n} \frac{\|\sigma_m f\|_{H_p(G)}^p}{m^{2-2p}} \leq c_p \|f\|_{H_p(G)}^p.$$

(b) *Let* $0<p<1/2$, $\Phi:\mathbb{N}_+\to[1,\infty)$ *be a non-decreasing function such that* $\Phi(n)\uparrow\infty$ *and*

$$\overline{\lim_{k\to\infty}}\frac{k^{2-2p}}{\Phi(k)}=\infty.$$

Then there exists a martingale $f\in H_p(G)$ *such that*

$$\sum_{m=1}^{\infty}\frac{\|\sigma_m f\|^p_{weak\text{-}L_p(G)}}{\Phi(m)}=\infty.$$

Proof. Suppose that

$$\frac{1}{\log^{[1/2+p]}n}\sum_{m=1}^{n}\frac{\|\sigma_m f\|_p^p}{m^{2-2p}}\leq c_p\|f\|^p_{H_p(G)}.$$

Combining Propositions (11.1) and (11.2), we can conclude that

$$\frac{1}{\log^{[1/2+p]}n}\sum_{m=1}^{n}\frac{\|\sigma_m f\|^p_{H_p(G)}}{m^{2-2p}}$$
$$\leq\frac{1}{\log^{[1/2+p]}n}\sum_{m=1}^{n}\frac{\|\sigma_m f\|_p^p}{m^{2-2p}}+\|\widetilde{\sigma}^*_{\#}f\|_{H_p(G)}+\|\widetilde{S}^*_{\#}f\|_{H_p(G)}\leq c_p\|f\|^p_{H_p(G)}. \qquad (21.1)$$

According to Lemma 10.2 and (21.1), Theorem 21.1 will be proved if we show that

$$\frac{1}{\log^{[1/2+p]}n}\sum_{m=1}^{n}\frac{\|\sigma_m a\|_p^p}{m^{2-2p}}\leq c<\infty,\quad m=2,3,\ldots,$$

for any p-atom a. We may assume that a is p-atom with support I, $\mu(I)=2^{-M}$ and $I=I_M$. It is obvious that $\sigma_n(a)=0$, when $n\leq 2^M$. Therefore, we may assume that $n>2^M$.

Let $x\in I_M$. Since σ_n is bounded from $L_\infty(G)$ to $L_\infty(G)$ (the boundedness follows from (8.10) in Corollary 8.3) and $\|a\|_\infty\leq 2^{M/p}$, we have

$$\int_{I_M}|\sigma_m a(x)|^p\,d\mu(x)\leq\frac{\|\sigma_m a\|^p_\infty}{2^M}\leq\frac{\|a\|^p_\infty}{2^M}\leq c<\infty,\quad 0<p\leq\frac{1}{2}.$$

Let $0<p\leq 1/2$. Then

$$\frac{1}{\log^{[1/2+p]}n}\sum_{m=1}^{n}\frac{\int_{I_M}|\sigma_m a(x)|^p\,d\mu(x)}{m^{2-2p}}\leq\frac{c}{\log^{[1/2+p]}n}\sum_{m=1}^{n}\frac{1}{m^{2-2p}}\leq c<\infty.$$

It is evident that

$$|\sigma_m a(x)|\leq\int_{I_M}|a(t)|\,|K_m(x+t)|\,d\mu(t)\leq 2^{M/p}\int_{I_M}|K_m(x+t)|\,d\mu(t).$$

Equalities (8.1) in Lemma 8.1 and Lemma 8.2 imply that

$$|\sigma_m a(x)|\leq\frac{c\cdot 2^{k+l}\cdot 2^{M(1/p-1)}}{m},\quad x\in I_M^{k,l},\quad 0\leq k<l<M, \qquad (21.2)$$

and

$$|\sigma_m a(x)|\leq c\cdot 2^{M(1/p-1)}\cdot 2^k,\quad x\in I_M^{k,M},\quad 0\leq k<M. \qquad (21.3)$$

If we use identities (1.2), (21.2) and (21.3), we get

$$\int\limits_{\overline{I}_M} |\sigma_m a(x)|^p \, d\mu(x) = \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \int\limits_{I_M^{k,l}} |\sigma_m a(x)|^p \, d\mu(x) + \sum_{k=0}^{M-1} \int\limits_{I_M^{k,M}} |\sigma_m a(x)|^p \, d\mu(x)$$

$$\leq c \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^l} \frac{2^{p(k+l)} \cdot 2^{M(1-p)}}{m^p} + c \sum_{k=0}^{M-1} \frac{1}{2^M} \cdot 2^{M(1-p)} \cdot 2^{pk}$$

$$\leq \frac{c \cdot 2^{M(1-p)}}{m^p} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{2^{p(k+l)}}{2^l} + c \sum_{k=0}^{M-1} \frac{2^{pk}}{2^{pM}} \leq \frac{c \cdot 2^{M(1-p)} M^{[1/2+p]}}{m^p} + c. \tag{21.4}$$

Hence,

$$\frac{1}{\log^{[1/2+p]} n} \sum_{m=2^M+1}^{n} \frac{\int_{\overline{I}_M} |\sigma_m a(x)|^p \, d\mu(x)}{m^{2-2p}}$$

$$\leq \frac{1}{\log^{[1/2+p]} n} \left(\sum_{m=2^M+1}^{n} \frac{c \cdot 2^{M(1-p)} M^{[1/2+p]}}{m^{2-p}} + \sum_{m=2^M+1}^{n} \frac{c}{m^{2-2p}} \right) < c < \infty.$$

The proof of part (a) of Theorem 21.1 is complete.

Now, we prove part (b) of Theorem 21.1. Let $\Phi(n)$ be a non-decreasing function satisfying the condition

$$\lim_{k\to\infty} \frac{2^{(|n_k|+1)(2-2p)}}{\Phi(2^{|n_k|+1})} = \infty. \tag{21.5}$$

According to (21.5), there exists an increasing sequence $\{\alpha_k : \ k \in \mathbb{N}_+\} \subset \{n_k : \ k \in \mathbb{N}_+\}$ such that

$$|\alpha_k| \geq 2, \quad \text{where} \ \ k \in \mathbb{N}_+ \tag{21.6}$$

and

$$\sum_{\eta=0}^{\infty} \frac{\Phi^{1/2}(2^{|\alpha_\eta|+1})}{2^{|\alpha_\eta|(1-p)}} = 2^{1-p} \sum_{\eta=0}^{\infty} \frac{\Phi^{1/2}(2^{|\alpha_\eta|+1})}{2^{(|\alpha_\eta|+1)(1-p)}} < c < \infty. \tag{21.7}$$

Let $f = (f_n, n \in \mathbb{N}_+) \in H_p(G)$ be a martingale from Example 11.3, where

$$\lambda_k = \frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{2^{(|\alpha_k|)(1/p-1)}}.$$

Using (21.7), we find that $f \in H_p(G)$. According to (11.10), we have

$$\widehat{f}(j) = \begin{cases} \Phi^{1/2p}(2^{|\alpha_k|+1}), & \text{if } j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \ \ k \in \mathbb{N}_+, \\ 0, & \text{if } j \notin \bigcup\limits_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}. \end{cases} \tag{21.8}$$

Let $2^{|\alpha_k|} < n < 2^{|\alpha_k|+1}$. Then

$$\sigma_n f = \frac{1}{n} \sum_{j=1}^{2^{|\alpha_k|}} S_j f + \frac{1}{n} \sum_{j=2^{|\alpha_k|}+1}^{n} S_j f = III + IV. \tag{21.9}$$

It is evident that

$$S_j f = 0, \ \text{ if } \ 0 \leq j \leq 2^{|\alpha_1|}. \tag{21.10}$$

Let $2^{|\alpha_s|} < j \le 2^{|\alpha_s|+1}$, where $s = 1, 2, \dots, k$. If we apply (11.12), we get

$$S_j f = \sum_{\eta=0}^{s-1} \Phi^{1/2p}(2^{|\alpha_\eta|+1})(D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}) + \Phi^{1/2p}(2^{|\alpha_s|+1}) w_{2^{|\alpha_s|}} D_{j-2^{|\alpha_s|}}. \tag{21.11}$$

Let $2^{|\alpha_s|+1} \le j \le 2^{|\alpha_{s+1}|}$, $s = 0, 1, \dots, k-1$. Then, using (11.11), we can conclude that

$$S_j f = \sum_{\eta=0}^{s} \Phi^{1/2p}(2^{|\alpha_\eta|+1})(D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}). \tag{21.12}$$

Let $x \in I_2(e_0 + e_1)$. Since (see Lemma 10.3 and (8.1)) in Lemma 8.1 we obtain that

$$D_{2^n}(x) = K_{2^n}(x) = 0, \quad \text{where} \quad n \ge 2, \tag{21.13}$$

Combining (21.6) and (21.10)–(21.13), we get

$$\begin{aligned} III &= \frac{1}{n} \sum_{\eta=0}^{k-1} \Phi^{1/2p}(2^{|\alpha_\eta|+1}) \sum_{v=2^{|\alpha_\eta|}+1}^{2^{|\alpha_\eta|+1}} D_v(x) \\ &= \frac{1}{n} \sum_{\eta=0}^{k-1} \Phi^{1/2p}(2^{|\alpha_\eta|+1}) \big(2^{|\alpha_\eta|+1} K_{2^{|\alpha_\eta|+1}}(x) - 2^{|\alpha_\eta|} K_{2^{|\alpha_\eta|}}(x)\big) = 0. \end{aligned} \tag{21.14}$$

If we use (21.11), when $s = k$, for IV we can write

$$\begin{aligned} IV = \frac{n - 2^{|\alpha_k|}}{n} &\sum_{\eta=0}^{k-1} \Phi^{1/2p}(2^{|\alpha_\eta|+1})(D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}) \\ &+ \frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{n} \sum_{j=2^{|\alpha_k|}+1}^{n} w_{2^{|\alpha_k|}} D_{j-2^{|\alpha_k|}} = IV_1 + IV_2. \end{aligned} \tag{21.15}$$

Combining (21.6) and (21.13), we conclude that

$$IV_1 = 0, \quad \text{where} \quad x \in I_2(e_0 + e_1). \tag{21.16}$$

Let $\alpha_k \in \mathbb{A}_{0,2}$, $2^{|\alpha_k|} < n < 2^{|\alpha_k|+1}$ and $x \in I_2(e_0 + e_1)$. Since $n - 2^{|\alpha_k|} \in \mathbb{A}_{0,2}$, Lemma 5.3, Lemma 8.2 and (21.13) imply that

$$\begin{aligned} |IV_2| &= \frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{n} \Big| \sum_{j=1}^{n-2^{|\alpha_k|}} D_j(x) \Big| \\ &= \frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{n} \big|(n - 2^{|\alpha_k|}) K_{n-2^{|\alpha_k|}}(x)\big| \ge \frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{2^{|\alpha_k|+1}}. \end{aligned} \tag{21.17}$$

Let $0 < p < 1/2$ and $n \in \mathbb{A}_{0,2}$. Combining (21.9)–(21.17), we get

$$\begin{aligned} \|\sigma_n f\|^p_{weak\text{-}L_p(G)} &\ge \frac{c_p \Phi^{1/2}(2^{|\alpha_k|+1})}{2^{p(|\alpha_k|+1)}} \mu\Big\{ x \in I_2(e_0 + e_1) : \ |\sigma_n f| \ge \frac{c_p \Phi^{1/2p}(2^{|\alpha_k|+1})}{2^{|\alpha_k|+1}} \Big\} \\ &\ge \frac{c_p \Phi^{1/2}(2^{|\alpha_k|+1})}{2^{p(|\alpha_k|+1)}} \mu\{I_2(e_0 + e_1)\} \ge \frac{c_p \Phi^{1/2}(2^{|\alpha_k|+1})}{2^{p(|\alpha_k|+1)}}. \end{aligned} \tag{21.18}$$

Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\|\sigma_n f\|^p_{weak\text{-}L_p(G)}}{\Phi(n)} &\ge \sum_{\{n \in \mathbb{A}_{0,2} :\ 2^{|\alpha_k|} < n < 2^{|\alpha_k|+1}\}} \frac{\|\sigma_n f\|^p_{weak\text{-}L_p(G)}}{\Phi(n)} \\ &\ge \frac{1}{\Phi^{1/2}(2^{|\alpha_k|+1})} \sum_{\{n \in \mathbb{A}_{0,2} :\ 2^{|\alpha_k|} < n < 2^{|\alpha_k|+1}\}} \frac{1}{2^{p(|\alpha_k|+1)}} \ge \frac{c_p \cdot 2^{(1-p)(|\alpha_k|+1)}}{\Phi^{1/2}(2^{|\alpha_k|+1})} \longrightarrow \infty \ \text{ as } \ k \to \infty. \end{aligned}$$

The proof of Theorem 21.1 is complete. □

Theorem 21.2. *Let $f \in H_{1/2}(G)$. Then*

$$\sup_{n\in\mathbb{N}_+} \sup_{\|f\|_{H_p}\le 1} \frac{1}{n}\sum_{m=1}^{n} \|\sigma_m f\|_{1/2}^{1/2} = \infty.$$

Proof. Let $0 < p \le 1$ and a_k be the p-atom from Example 11.1. Using (11.4) for $0 < n < 2^{\alpha_k}$, we have

$$|\sigma_{n+2^{\alpha_k}} a_k| = \frac{2^{\alpha_k}}{n+2^{\alpha_k}}\, n|K_n|.$$

Let $n = \sum_{i=1}^{s}\sum_{k=l_i}^{m_i} 2^k$, where $0 \le l_1 \le m_1 \le l_2 - 2 < l_2 \le m_2 \le \cdots \le l_s - 2 < l_s \le m_s$. Applying Lemma 8.6 and (21), we find that $|\sigma_{n+2^{\alpha_k}} a_k(x)| \ge c \cdot 2^{2l_i}$, where $x \in I_{l_i+1}(e_{l_i-1}+e_{l_i})$. Hence,

$$\int_G |\sigma_{n+2^{\alpha_k}} a_k(x)|^{1/2}\, d\mu(x) \ge \sum_{i=0}^{s} \int_{I_{l_i+1}(e_{l_i-1}+e_{l_i})} |\sigma_{n+2^{\alpha_k}} a_k(x)|^{1/2}\, d\mu(x) \ge c\sum_{i=0}^{s} \frac{1}{2^{l_i}} \cdot 2^{l_i} \ge cs \ge c\,V(n).$$

According to the second estimation of (4.2), we can conclude that

$$\sup_{n\in\mathbb{N}_+} \sup_{\|f\|_{H_p}\le 1} \frac{1}{n}\sum_{j=1}^{n} \|\sigma_j f\|_{1/2}^{1/2} \ge \frac{1}{2^{\alpha_k+1}} \sum_{j=2^{\alpha_k}+1}^{2^{\alpha_k+1}-1} \|\sigma_j a_k\|_{1/2}^{1/2}$$
$$\ge \frac{c}{2^{\alpha_k+1}} \sum_{j=2^{\alpha_k}+1}^{2^{\alpha_k+1}-1} V(j-2^{\alpha_k}) \ge \frac{c}{2^{\alpha_k+1}} \sum_{j=1}^{2^{\alpha_k}-1} V(j) \ge c\log\alpha_k \longrightarrow \infty \quad \text{as} \quad k\to\infty. \qquad \square$$

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References

[1] G. Abesadze, N. Anakidze, T. Nasibov and G. Tephnadze, Classical inequalities and an approximate identity in homogeneous Banach spaces. *Acta Math. Acad. Paed. Nyíreg.* **36** (2025), no. 3, 53–63.

[2] D. Adamadze and T. Kopaliani, Vilenkin-Fourier series in variable Lebesgue spaces. *Math. Inequal. Appl.* **26** (2023), no. 4, 977–993.

[3] G. N. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli and A. I. Rubinshteĭn, *Multiplicative Systems of Functions And Harmonic Analysis on Zero-Dimensional Groups.* (Russian) 1‘Èlm”, Baku, 1981.

[4] N. Anakidze and N. Areshidze, On the approximation by Vilenkin–Fejér means in Lebesgue spaces. *Trans. A. Razmadze Math. Inst.*, 2026 (to appear).

[5] N. Anakidze, N. Areshidze, L.-E. Persson and G. Tephnadze, Approximation by T means of Walsh-Fourier series in Lebesgue spaces and Lipschitz classes. *Ann. Univ. Sci. Budapest. Sect. Comput.* **56** (2024), 53–68.

[6] N. Anakidze, N. Areshidze, L.-E. Persson and G. Tephnadze, Approximation by T means with respect to Vilenkin system in Lebesgue spaces. *Eurasian Math. J.* **16** (2025), no. 4, 8–22.

[7] N. Anakidze, L.-E. Persson, G. Tephnadze and M. F. Yimer, Norm and almost everywhere convergence of subsequences of T means of Vilenkin-Fourier series with non-decreasing sequences. *Nonlinear Stud.* (to appear).

[8] N. Areshidze, D. Baramidze, L.-E. Persson and G. Tephnadze, Maximal operators related to Fejér means of Walsh–Fourier series in the martingale Hardy space $H_{1/2}$. *Mem. Differ. Equ. Math. Phys.*, 2026 (to appear).

[9] N. Areshidze, L.-E. Persson and G. Tephnadze, On divergence of Fejér means with respect to Walsh system on sets of measure zero. *Tbilisi analysis and PDE seminar*, 21–29, Trends Math., Res. Perspect. Ghent Anal. PDE Cent., 7, *Birkhäuser/Springer, Cham*, 2024.

[10] N. Areshidze, L.-E. Persson and G. Tephnadze, Convergence almost everywhere of partial sums and Fejér means of Vilenkin–Fourier series. *Mediterr. J. Math.* **22** (2025), no. 1, Paper no. 15, 17 pp.

[11] M. Avdispahić and N. Memić, On the Lebesgue test for convergence of Fourier series on unbounded Vilenkin groups. *Acta Math. Hungar.* **129** (2010), no. 4, 381–392.

[12] D. Baramidze, Sharp (H_p, L_p) and $(H_p, \text{weak-}L_p)$ type inequalities of weighted maximal operators of T means with respect to Vilenkin systems. *Trans. A. Razmadze Math. Inst.* **176** (2022), no. 2, 173–182.

[13] D. Baramidze, Martingale Hardy spaces and divergence of maximal operators associated with Walsh–Fejér means. *Nonlinear Stud.* **32** (2025), no. 1, 201–214.

[14] D. Baramidze, Martingale Hardy spaces and some new weighted maximal operators of partial sums of Walsh-Fourier series. *Georgian Math. J.* **33** (2026), no. 1, 31–40.

[15] D. Baramidze, L. Baramidze, L.-E. Perssson and G. Tephnadze, Some new restricted maximal operators of Fejér means of Walsh-Fourier series. *Banach J. Math. Anal.* **17** (2023), no. 4, Paper no. 75, 20 pp.

[16] D. Baramidze, I. Blahota, G. Tephnadze and R. Toledo, Martingale Hardy spaces and some new weighted maximal operators of Fejér means of Walsh–Fourier series. *J. Geom. Anal.* **34** (2024), no. 1, Paper no. 3, 17 pp.

[17] D. Baramidze, Z. Dvalashvili and G. Tutberidze, Convergence of Nörlund means with respect to Vilenkin systems of integrable functions. *Mem. Differ. Equ. Math. Phys.* **86** (2022), 1–14.

[18] D. Baramidze, N. Gogolashvili and N. Nadirashvili, Convergence of T means with respect to Vilenkin systems of integrable functions. *Georgian Math. J.* **29** (2022), no. 4, 481–491.

[19] D. Baramidze, D. Lukkassen, A. Meidell and L.-E. Persson, Some new (H_p-L_p) type inequalities for weighted maximal operators of Fejér means of Walsh–Fourier series. *Trans. A. Razmadze Math. Inst.* **180** (2026), no. 1, 25–34.

[20] D. Baramidze, N. Nadirashvili, L.-E. Persson and G. Tephnadze, Some weak type inequalities and almost everywhere convergence of Vilenkin–Nörlund means. *J. Inequal. Appl.* **2023**, Paper no. 66, 17 pp.

[21] D. Baramidze, L.-E. Persson, H. Singh and G. Tephnadze, Some new results and inequalities for subsequences of Nörlund logarithmic means of Walsh–Fourier series. *J. Inequal. Appl.* **2022**, Paper no. 30, 13 pp.

[22] D. Baramidze, L.-E. Persson, H. Singh and G. Tephnadze, Some new weak $(H_p - L_p)$-type inequality for weighted maximal operators of partial sums of Walsh-Fourier series. *Mediterr. J. Math.* **20** (2023), no. 5, Paper no. 284, 13 pp.

[23] D. Baramidze, L.-E. Persson, K. Tangrand and G. Tephnadze, $(H_p - L_p)$-type inequalities for subsequences of Nörlund means of Walsh–Fourier series. *J. Inequal. Appl.* **2023**, Paper no. 52, 13 pp.

[24] D. Baramidze, L.-E. Persson and G. Tephnadze, Some new $(H_p - L_p)$ type inequalities for weighted maximal operators of partial sums of Walsh–Fourier series. *Positivity* **27** (2023), no. 3, Paper no. 38, 14 pp.

[25] D. Baramidze and G. Tephnadze, Some new weak-$(H_p - L_p)$ type inequalities for weighted maximal operators of Fejér means of Walsh–Fourier series. *Acta Math. Hungar.* **171** (2023), no. 2, 267–283.

[26] N. K. Bary, *Trigonometric Series.* (Russian) Gos. Izd. Fiz. Mat. Lit., Moscow, 1961.

[27] I. Blahota, Approximation by Vilenkin–Fourier sums in $L^p(G_m)$. *Acta Acad. Paedagog. Nyházi., Mat.-Inform. Közl.* **13/D** (1992), 35–39.

[28] I. Blahota, G. Gát and U. Goginava, Maximal operators of Fejér means of Vilenkin–Fourier series. *JIPAM. J. Inequal. Pure Appl. Math.* **7** (2006), no. 4, Article 149, 7 pp.

[29] I. Blahota, G. Gát and U. Goginava, Maximal operators of Fejér means of double Vilenkin–Fourier series. *Colloq. Math.* **107** (2007), no. 2, 287–296.

[30] I. Blahota and K. Nagy, Approximation by Θ-means of Walsh–Fourier series. *Anal. Math.* **44** (2018), no. 1, 57–71.

[31] I. Blahota, K. Nagy, L.-E. Persson and G. Tephnadze, A sharp boundedness result for restricted maximal operators of Vilenkin–Fourier series on martingale Hardy spaces. *Georgian Math. J.* **26** (2019), no. 3, 351–360.

[32] I. Blahota, K. Nagy and G. Tephnadze, Approximation by Marcinkiewicz Θ-means of double Walsh–Fourier series. *Math. Inequal. Appl.* **22** (2019), no. 3, 837–853.

[33] I. Blahota, L. E. Persson and G. Tephnadze, Two-sided estimates of the Lebesgue constants with respect to Vilenkin systems and applications. *Glasg. Math. J.* **60** (2018), no. 1, 17–34.

[34] I. Blahota and G. Tephnadze, On the (C,α)-means with respect to the Walsh system. *Anal. Math.* **40** (2014), no. 3, 161–174.

[35] I. Blahota and G. Tephnadze, Strong convergence theorem for Vilenkin–Fejér means. *Publ. Math. Debrecen* **85** (2014), no. 1-2, 181–196.

[36] I. Blahota, G. Tephnadze and R. Toledo, Strong convergence theorem of Cesàro means with respect to the Walsh system. *Tohoku Math. J. (2)* **67** (2015), no. 4, 573–584.

[37] I. Blahota, G. Tephnadze and R. Toledo, Some new weak-(H_p-L_p)-type inequalities for weighted maximal operators of Fejér means of Vilenkin–Fourier series. *Georgian Math. J.*, 2026; doi: 10.1515/gmj-2026-2007/html

[38] N. J. Fine, On the Walsh functions. *Trans. Amer. Math. Soc.* **65** (1949), 372–414.

[39] N. Fujii, A maximal inequality for H^1-functions on a generalized Walsh–Paley group. *Proc. Amer. Math. Soc.* **77** (1979), no. 1, 111–116.

[40] G. Gát, Investigations of certain operators with respect to the Vilenkin system. *Acta Math. Hungar.* **61** (1993), no. 1-2, 131–149.

[41] G. Gát. Cesàro means of integrable functions with respect to unbounded Vilenkin systems. *J. Approx. Theory* **124** (2003), no. 1, 25–43.

[42] G. Gát and U. Goginava, Uniform and L-convergence of logarithmic means of Walsh–Fourier series. *Acta Math. Sin. (Engl. Ser.)* **22** (2006), no. 2, 497–506.

[43] G. Gát, U. Goginava and G. Tkebuchava, Convergence in measure of logarithmic means of quadratical partial sums of double Walsh–Fourier series. *J. Math. Anal. Appl.* **323** (2006), no. 1, 535–549.

[44] U. Goginava, The maximal operator of Marcinkiewicz–Fejér means of the d-dimensional Walsh–Fourier series. *East J. Approx.* **12** (2006), no. 3, 295–302.

[45] U. Goginava, The maximal operator of the (C,α) means of the Walsh–Fourier series. *Ann. Univ. Sci. Budapest. Sect. Comput.* **26** (2006), 127–135.

[46] U. Goginava, The maximal operator of the Fejér means of the character system of the p-series field in the Kaczmarz rearrangement. *Publ. Math. Debrecen* **71** (2007), no. 1-2, 43–55.

[47] U. Goginava, Maximal operators of Fejér means of double Walsh-Fourier series. *Acta Math. Hungar.* **115** (2007), no. 4, 333–340.

[48] U. Goginava, Maximal operators of Fejér-Walsh means. *Acta Sci. Math. (Szeged)* **74** (2008), no. 3-4, 615–624.

[49] U. Goginava, The martingale Hardy type inequality for Marcinkiewicz–Fejér means of two-dimensional conjugate Walsh–Fourier series. *Acta Math. Sin. (Engl. Ser.)* **27** (2011), no. 10, 1949–1958.

[50] U. Goginava and G. Tkebuchava, Convergence of subsequences of partial sums and logarithmic means of Walsh–Fourier series. *Acta Sci. Math. (Szeged)* **72** (2006), no. 1-2, 159–177.

[51] N. Gogolashvili, K. Nagy and G. Tephnadze, Strong convergence theorem for Walsh–Kaczmarz–Fejér means. *Mediterr. J. Math.* **18** (2021), no. 2, Paper no. 37, 17 pp.

[52] N. Gogolashvili and G. Tephnadze, Maximal operators of T means with respect to Walsh–Kaczmarz system. *Math. Inequal. Appl.* **24** (2021), no. 3, 737–750.

[53] N. Gogolashvili and G. Tephnadze, On the maximal operators of T means with respect to Walsh-Kaczmarz system. *Studia Sci. Math. Hungar.* **58** (2021), no. 1, 119–135.

[54] B. Golubov, A. Efimov and V. Skvortsov, *Walsh Series and Transforms. Theory and Applications.* Mathematics and its Applications (Soviet Series), 64. Kluwer Academic Publishers Group, Dordrecht, 1991.

[55] J. Gosselin, Almost everywhere convergence of Vilenkin–Fourier series. *Trans. Amer. Math. Soc.* **185** (1973), 345–370 (1974).

[56] N. V. Guličev, Approximation to continuous functions by Walsh-Fourier sums. *Anal. Math.* **6** (1980), no. 4, 269–280.

[57] I. Joó, On some problems of M. Horváth (saturation theorems for Walsh–Fourier expansions). *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **31** (1988), 243–260 (1989).

[58] T. Kopaliani, G. Tephnadze and F. Weisz, On the approximation by Nörlund means with respect to Walsh system in homogeneous Banach spaces. *Trends Math.*, 2026 (to appear).

[59] D. Lukkassen, L. E. Persson, G. Tephnadze and G. Tutberidze, Some inequalities related to strong convergence of Riesz logarithmic means. *J. Inequal. Appl.* **2020**, Paper no. 79, 17 pp.

[60] S. F. Lukomskii, Lebesgue constants for characters of the compact zero-dimensional abelian group. *East J. Approx.* **15** (2009), no. 2, 219–231.

[61] N. Memić, L. E. Persson and G. Tephnadze, A note on the maximal operators of Vilenkin–Nörlund means with non-increasing coefficients. *Studia Sci. Math. Hungar.* **53** (2016), no. 4, 545–556.

[62] N. Memić, I. Simon and G. Tephnadze, Strong convergence of two-dimensional Vilenkin–Fourier series. *Math. Nachr.* **289** (2016), no. 4, 485–500.

[63] N. Nadirashvili, L.-E. Persson, G. Tephnadze and F. Weisz, Vilenkin–Lebesgue points and almost everywhere convergence for some classical summability methods. *Mediterr. J. Math.* **19** (2022), no. 5, Paper no. 239, 16 pp.

[64] N. Nadirashvili, G. Tephnadze and G. Tutberidze, The norm and almost everywhere convergence of approximate identity and Fejér means of trigonometric and Vilenkin systems. *Trans. A. Razmadze Math. Inst.* **177** (2023), no. 3, 451–461.

[65] K. Nagy, Approximation by Cesàro means of negative order of Walsh–Kaczmarz–Fourier series. *East J. Approx.* **16** (2010), no. 3, 297–311.

[66] K. Nagy and G. Tephnadze, Approximation by Marcinkiewicz means of Walsh–Kaczmarz–Fourier series in the Hardy space $H_{2/3}$. *Bull. TICMI* **18** (2014), no. 1, 110–121.

[67] K. Nagy and G. Tephnadze, Walsh–Marcinkiewicz means and Hardy spaces. *Cent. Eur. J. Math.* **12** (2014), no. 8, 1214–1228.

[68] K. Nagy and G. Tephnadze, Strong convergence theorem for Walsh–Marcinkiewicz means. *Math. Inequal. Appl.* **19** (2016), no. 1, 185–195.

[69] K. Nagy and G. Tephnadze, The Walsh–Kaczmarz–Marcinkiewicz means and Hardy spaces. *Acta Math. Hungar.* **149** (2016), no. 2, 346–374.

[70] C. W. Onneweer, On L_1-convergence of Walsh–Fourier series. *Internat. J. Math. Math. Sci.* **1** (1978), no. 1, 47–56.

[71] J. Pál and P. Simon, On a generalization of the concept of derivative. *Acta Math. Acad. Sci. Hungar.* **29** (1977), no. 1-2, 155–164.

[72] L.-E. Persson, N. Samko and G. Tephnadze, On generalized sharpness of some Hardy-type inequalities. *Tbilisi analysis and PDE seminar*, 171–181, Trends Math., Res. Perspect. Ghent Anal. PDE Cent., 7, *Birkhäuser/Springer, Cham*, 2024.

[73] L.-E. Persson, F. Schipp, G. Tephnadze and F. Weisz, An analogy of the Carleson–Hunt theorem with respect to Vilenkin systems. *J. Fourier Anal. Appl.* **28** (2022), no. 3, Paper no. 48, 29 pp.

[74] L.-E. Persson, F. Schipp, G. Tephnadze and F. Weisz, A note on Carleson–Hunt type theorems for Vilenkin–Fourier series. Extended abstracts 2021/2022 – Methusalem lectures, 157–167, Trends Math., Res. Perspect. Ghent Anal. PDE Cent., 3, *Birkhäuser/Springer, Cham*, 2024.

[75] L.-E. Persson and G. Tephnadze, A note on Vilenkin–Fejér means on the martingale Hardy spaces H_p. *Bull. TICMI* **18** (2014), no. 1, 55–64.

[76] L.-E. Persson and G. Tephnadze, A sharp boundedness result concerning some maximal operators of Vilenkin–Fejér means. *Mediterr. J. Math.* 13 (2016), no. 4, 1841–1853.

[77] L.-E. Persson, G. Tephnadze and G. Tutberidze, On the boundedness of subsequences of Vilenkin–Fejér means on the martingale Hardy spaces. *Oper. Matrices* **14** (2020), no. 1, 283–294.

[78] L.-E. Persson, G. Tephnadze and P. Wall, Maximal operators of Vilenkin–Nörlund means. *J. Fourier Anal. Appl.* **21** (2015), no. 1, 76–94.

[79] L.-E. Persson, G. Tephnadze and F. Weisz, *Martingale Hardy Spaces and Summability of One-Dimensional Vilenkin–Fourier Series.* Birkhäuser/Springer, Cham, 2022.

[80] L. S. Pontryagin, *Topological Groups.* Gordon and Breach Science Publishers, Inc., New York–London–Paris, 1966.

[81] L.-E. Persson, N. Samko and G. Tephnadze, Sharpness of some Hardy-type inequalities. *J. Inequal. Appl.* 2023, Paper no. 155, 16 pp.

[82] F. Schipp, Certain rearrangements of series in the Walsh system. (Russian) *Mat. Zametki* **18** (1975), no. 2, 193–201.

[83] F. Schipp, W. R. Wade, P. Simon, *Walsh Series. An Introduction to Dyadic Harmonic Analysis.* With the collaboration of J. Pál. Adam Hilger, Ltd., Bristol, 1990.

[84] A. Shalukhina, Self-improving boundedness of the maximal operator on quasi-Banach lattices over spaces of homogeneous type. *J. Math. Anal. Appl.* **548** (2025), no. 2, Paper no. 129419, 22 pp.

[85] A. Shalukhina, The maximal function on spaces of homogeneous type, or adjacent dyadic cubes do good. *Anal. Math. Phys.* (to appear).

[86] P. Simon, Investigations with respect to the Vilenkin system. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **27** (1984), 87–101 (1985).

[87] P. Simon, Strong convergence of certain means with respect to the Walsh–Fourier series. *Acta Math. Hungar.* **49** (1987), no. 3-4, 425–431.

[88] P. Simon, Cesaro summability with respect to two-parameter Walsh systems. *Monatsh. Math.* **131** (2000), no. 4, 321–334.

[89] P. Simon, A note on the Sunouchi operator with respect to Vilenkin system. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **43** (2000), 101–116 (2001).

[90] P. Simon, Strong convergence theorem for Vilenkin–Fourier series. *J. Math. Anal. Appl.* **245** (2000), no. 1, 52–68.

[91] P. Simon and F. Weisz, Weak inequalities for Cesàro and Riesz summability of Walsh–Fourier series. *J. Approx. Theory* **151** (2008), no. 1, 1–19.

[92] B. Smith, A strong convergence theorem for $H^1(T)$. *Banach spaces, harmonic analysis, and probability theory (Storrs, Conn., 1980/1981)*, 169–173, Lecture Notes in Math., 995, *Springer, Berlin–New York*, 1983.

[93] E. M. Stein and R. Shakarchi, *Fourier Analysis. An Introduction.* Princeton Lectures in Analysis, 1. Princeton University Press, Princeton, NJ, 2003.

[94] G. Tephnadze, Convergence and summability in classical and martingale Hardy spaces. *Tbilisi analysis and PDE seminar*, 191–199, Trends Math., Res. Perspect. Ghent Anal. PDE Cent., 7, *Birkhäuser/Springer, Cham*, 2024.

[95] G. Tephnadze, Sharp strong convergence result of the two-dimensional Walsh-Fourier series in martingale Hardy spaces. *Anal. Math. Phys.* **15** (2025), no. 3, Paper no. 78, 23 pp.

[96] G. Tephnadze, The maximal operators of logarithmic means of one-dimensional Vilenkin–Fourier series. *Acta Math. Acad. Paedagog. Nyházi* **27** (2011), no. 2, 245–256.

[97] G. Tephnadze, Fejér means of Vilenkin–Fourier series. *Studia Sci. Math. Hungar.* **49** (2012), no. 1, 79–90.

[98] G. Tephnadze, A note on the Fourier coefficients and partial sums of Vilenkin–Fourier series. *Acta Math. Acad. Paedagog. Nyházi* **28** (2012), no. 2, 167–176.

[99] G. Tephnadze, A note on the strong convergence of two-dimensional Walsh–Fourier series. *Proc. A. Razmadze Math. Inst.* **162** (2013), 93–97.

[100] G. Tephnadze, On the maximal operators of Vilenkin–Fejér means. *Turkish J. Math.* **37** (2013), no. 2, 308–318.

[101] G. Tephnadze, On the maximal operators of Vilenkin–Fejér means on Hardy spaces. *Math. Inequal. Appl.* **16** (2013), no. 1, 301–312.

[102] G. Tephnadze, On the maximal operators of Walsh–Kaczmarz–Fejér means. *Period. Math. Hungar.* **67** (2013), no. 1, 33–45.

[103] G. Tephnadze, On the Vilenkin–Fourier coefficients. *Georgian Math. J.* **20** (2013), no. 1, 169–177.

[104] G. Tephnadze, Strong convergence of two-dimensional Walsh–Fourier series. *Ukrainian Math. J.* **65** (2013), no. 6, 914–927.

[105] G. Tephnadze, Approximation by Walsh–Kaczmarz–Fejér means on the Hardy space. *Acta Math. Sci. Ser. B (Engl. Ed.)* **34** (2014), no. 5, 1593–1602.

[106] G. Tephnadze, A note on the norm convergence by Vilenkin–Fejér means. *Georgian Math. J.* **21** (2014), no. 4, 511–517.

[107] G. Tephnadze, On the maximal operators of Riesz logarithmic means of Vilenkin–Fourier series. *Studia Sci. Math. Hungar.* **51** (2014), no. 1, 105–120.

[108] G. Tephnadze, Strong convergence theorems for Walsh–Fejér means. *Acta Math. Hungar.* **142** (2014), no. 1, 244–259.

[109] G. Tephnadze, Martingale Hardy spaces and summability of the one dimensional Vilenkin–Fourier series. *PhD thesis, Luleå Tekniska Universitet, Luleå*, 2015; https://www.diva-portal.org/smash/record.jsf?dswid=9214

[110] G. Tephnadze, On the maximal operators of Walsh–Kaczmarz–Nörlund means. *Acta Math. Acad. Paedagog. Nyházi* **31** (2015), no. 2, 259–271.

[111] G. Tephnadze, On the partial sums of Walsh–Fourier series. *Colloq. Math.* **141** (2015), no. 2, 227–242.

[112] G. Tephnadze, On the convergence of Fejér means of Walsh–Fourier series in the space H_p. (Russian) *Izv. Nats. Akad. Nauk Armenii Mat.* **51** (2016), no. 2, 54–70; translation in *J. Contemp. Math. Anal.* **51** (2016), no. 2, 90–102.

[113] G. Tephnadze, On the convergence of partial sums with respect to Vilenkin system on the Martingale Hardy spaces. (Russian) *Izv. Nats. Akad. Nauk Armenii Mat.* **53** (2018), no. 5, 77–94; translation in *J. Contemp. Math. Anal.* **53** (2018), no. 5, 294–306.

[114] G. Tephnadze, Convergence and strong summability of the two-dimensional Vilenkin–Fourier series. *Nonlinear Stud.* **26** (2019), no. 4, 973–989.

[115] G. Tephnadze, A note on the strong summability of two-dimensional Walsh–Fourier series. *Georgian Math. J.* **28** (2021), no. 3, 477–482.

[116] G. Tephnadze, Martingale Hardy spaces and partial sums and Fejér means with respect to the one-dimensional Walsh–Fourier series. *Mem. Differ. Equ. Math. Phys.* **88** (2023), 109–158.

[117] G. Tephnadze, A note on the new strong convergence result of the two-dimensional Walsh–Fourier series in martingale Hardy spaces. *Trends Math.*, 2025 (to appear).

[118] G. Tephnadze and G. Tutberidze, A note on the maximal operators of the Nörlund logaritmic means of Vilenkin–Fourier series. [Corrected title: A note on the maximal operators of the Nörlund logarithmic means of Vilenkin–Fourier series] *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 1, 107–112.

[119] G. Tephnadze and F. Weisz, Modulus of continuity and convergence of Fejér means of Vilenkin-Fourier series in the variable martingale Hardy space $H_{p(\cdot)}$. In: *Tbilisi analysis and PDE seminar*, 201–210, Trends Math., Res. Perspect. Ghent Anal. PDE Cent., 7, Birkhäuser/Springer, Cham, [2024], ©2024.

[120] R. Toledo, On the boundedness of the L^1-norm of Walsh–Fejér kernels. *J. Math. Anal. Appl.* **457** (2018), no. 1, 153–178.

[121] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis.* Pure and Applied Mathematics, 123. Academic Press, Inc., Orlando, FL, 1986.

[122] G. Tutberidze, A note on the strong convergence of partial sums with respect to Vilenkin system. (Russian) *Izv. Nats. Akad. Nauk Armenii Mat.* **54** (2019), no. 6, 81–87; translation in *J. Contemp. Math. Anal.* **54** (2019), no. 6, 365–370.

[123] G. Tutberidze, Maximal operators of T means with respect to the Vilenkin system. *Nonlinear Stud.* **27** (2020), no. 4, 1157–1167.

[124] G. Tutberidze, Modulus of continuity and convergence of subsequences of Vilenkin–Fejér means in martingale Hardy spaces. *Georgian Math. J.* **29** (2022), no. 1, 153–162.

[125] G. Tutberidze, Sharp (H_p, L_p) type inequalities of maximal operators of T means with respect to Vilenkin systems. *Mediterr. J. Math.* **19** (2022), no. 2, Paper no. 81, 23 pp.

[126] Ch. Watari, On generalized Walsh Fourier series. *Tohoku Math. J. (2)* **10** (1958), 211–241.

[127] F. Weisz, Convergence of singular integrals. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **32** (1989), 243–256 (1990).

[128] F. Weisz, *Martingale Hardy Spaces and their Applications in Fourier Analysis.* Lecture Notes in Mathematics, 1568. Springer-Verlag, Berlin, 1994.

[129] F. Weisz, Cesàro summability of one- and two-dimensional Walsh–Fourier series. *Anal. Math.* **22** (1996), no. 3, 229–242.

[130] F. Weisz, Hardy spaces and Cesàro means of two-dimensional Fourier series. *Approximation theory and function series (Budapest, 1995)*, 353–367, Bolyai Soc. Math. Stud., 5, *János Bolyai Math. Soc., Budapest*, 1996.

[131] F. Weisz, S*ummability of Multi-Dimensional Fourier Series and Hardy Spaces.* Mathematics and its Applications, 541. Kluwer Academic Publishers, Dordrecht, 2002.

[132] F. Weisz, Weak type inequalities for the Walsh and bounded Ciesielski systems. *Anal. Math.* **30** (2004), no. 2, 147–160.

[133] Sh. Yano, Cesàro summability of Walsh–Fourier series. *Tohoku Math. J. (2)* **9** (1957), 267–272.

[134] W. S. Young, Mean convergence of generalized Walsh–Fourier series. *Trans. Amer. Math. Soc.* **218** (1976), 311–320.

[135] A. Zygmund, *Trigonometric Series.* 2nd ed. Vols. I, II. Cambridge University Press, New York, 1959.

Contents

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