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APPLICATION OF THE POTENTIAL METHOD TO MIXED BOUNDARY VALUE PROBLEMS FOR VISCOELASTIC SOLIDS WITH VOIDS

Dedicated to the 80th anniversary of George Jaiani and Roland Duduchava

Abstract. We investigate mixed boundary value problems (BVP) of the linear theory of viscoelasticity for isotropic and homogeneous Kelvin–Voigt materials with voids when on one part of the boundary of the body under consideration the Dirichlet type condition is given and on the remaining part of the boundary the Neumann type condition is prescribed. Using the potential method and the theory of pseudodifferential equations we prove the existence and uniqueness of solutions in the appropriate Sobolev–Slobodetskii, Bessel potential, and Besov spaces. Using the embedding theorems, we establish almost optimal regularity results for solutions to the mixed BVPs near the collision curves where different types of boundary conditions collide. In particular, we prove that the solutions belong to the space of Hölder continuous functions in the closed region occupied by the viscoelastic body. An efficient algebraic algorithm is described for finding the Hölder smoothness exponents which, in turn, efficiently determined the corresponding stress singularity exponents near the collision curves. It is shown that these exponents depend essentially on the material parameters.

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Key words and phrases. Viscoelasticity, potential theory, pseudodifferential equations, mixed boundary value problem, regularity of solutions.

რეზიუმე. ჩვენ ვიკვლევთ წრფივი თეორიის შერეულ სასაზღვრო ამოცანებს ბლანტი ფოროვანი იზოტროპული და ერთგვაროვანი კელვინ-ვოიგტის ტიპის მასალებისთვის, როდესაც განსახილველი სხეულის საზღვრის ერთ ნაწილზე მოცემულია დირიხლეს ტიპის პირობა, ხოლო დანარჩენ ნაწილზე – ნეიმანის ტიპის პირობა. პოტენციალთა მეთოდისა და ფსევდოდიფერენციალური განტოლებების თეორიის გამოყენებით ვამტკიცებთ ამონახსნების არსებობისა და ერთადერთობის თეორემებს სობოლევ-სლობოდეცკის, ბესელის პოტენციალისა და ბესოვის სივრცეებში. ჩართვის თეორემების გამოყენებით, შერეული სასაზღვრო ამოცანების ამონახსნებისთვის ვამტკიცებთ თითქმის ოპტიმალურ რეგულარობას სხვადასხვა ტიპის სასაზღვრო პირობების გამმიჯნავი წირების მიდამოში. კერძოდ, ვამტკიცებთ, რომ ამონახსნები ეკუთვნის ჰელდერის აზრით უწყვეტი ფუნქციების სივრცეს ბლანტი დრეკადი ფოროვანი სხეულის მიერ დაკავებულ ჩაკეტილ არეში. აღწერილია ეფექტური ალგებრული ალგორითმი ჰელდერის სიგლუვის მაჩვენებლების მოსაძებნად, რაც, თავის მხრივ, ეფექტურად განსაზღვრავს ძაბვის ვექტორის სინგულარობის მაჩვენებლებს სასაზღვრო პირობების გამმიჯნავ წირებთან. დადგენილია, რომ ეს სინგულარობის მაჩვენებლები არსებითადაა დამოკიდებული მასალის პარამეტრებზე.

1 Introduction

The development of viscoelasticity theories was initiated by Maxwell, Meyer, and Boltzmann, and further advanced by Voigt, Kelvin, Zaremba, Volterra, among others. Classical models such as the Maxwell model, the Kelvin–Voigt model, and the standard linear solid model were formulated to describe the behaviour of materials under various loading conditions (see Eringen [41], Truesdell and Noll [99], Christensen [20], Amendola et al. [4], Fabrizio and Morro [44]). Viscoelastic materials hold significant relevance across diverse fields, including civil and geotechnical engineering, technological applications, and more recently, biomechanics. Materials like amorphous polymers, semicrystalline polymers, and biopolymers can be effectively modelled to capture their stress or strain responses and time-dependent characteristics. The viscoelastic properties of bone are crucially analysed with respect to strain levels and frequency domains encountered during daily activities and diagnostic procedures (see Lakes [62]).

The study of wave propagation in viscoelastic media, as well as the attenuation of seismic waves, remains vital for geophysical exploration. Moreover, viscoelastic porous materials are extensively analysed at the nanoscale, providing detailed predictions of their behaviour. These materials have a wide range of applications, notably including potential contributions to NASA's missions involving soil behaviour prediction on the Moon and Mars (for more details, see Voyiadjis and Song [101], Polarz and Smarsly [80], Chen et al. [13], and the references therein).

Great attention has been paid to the theories considering the viscoelastic effects (see Amendola et al. [4], Fabrizio and Morro [44], Di Paola et al. [35, 36], Shaw and MacKnigh [88], Seema and Abhina [90]). Investigations concerning the existence, stability and asymptotic behaviour of solutions within the framework of linear viscoelasticity have been carried out by Fabrizio and Morro [44], Fabrizio and Lazzari [43] and Appleby et al. [6]. Important results regarding free energy in linear viscoelasticity are obtained in the papers [28–33, 45, 49].

Materials containing small-scale voids, commonly referred to as porous materials or materials with voids, play a critical role in a variety of scientific domains. The classical elasticity theories often fall short when applied to geological materials like rocks and soils, as well as biological and engineered porous structures. Besides seismology and elasticity, the viscoelasticity theories for materials with voids are widely applied in medicine, biological sciences, the oil industry, and nanotechnology. Several integral-type models for viscoelastic materials with voids have been proposed by Cowin [25], Ciarletta and Scalia [21], De Cicco and Nappa [27], and Martinez and Quintanilla [64]. More recently, the differential-type theories have garnered attention. In this context, Ieșan [54] developed a nonlinear model treating a viscoelastic composite as a mixture of a porous elastic solid and a Kelvin–Voigt material. Quintanilla [84] formulated a linearized version of this model and established the existence and exponential stability of solutions. Furthermore, Ieșan and Nappa [59] introduced nonlinear theories for heat-conducting mixtures composed of Kelvin–Voigt materials, while Chiriță et al. [14] derived exponential decay estimates for steady vibration equations.

The thermoviscoelastic behaviour of composites modelled as interacting Cosserat continua has been explored by Ieșan [56]. In [57], Ieșan expanded upon the classical theory of elastic materials with voids by establishing the fundamental equations for nonlinear thermoviscoelastic materials with voids, particularly for "virgin" materials without pre-existing stresses (see also Nunziato and Cowin [24, 77], Fabrizio and Morro [44], and Deseri et al. [31]). In the framework of the linearized theory, the uniqueness of solutions and continuous dependence on initial and external data were also proven. Additionally, Passarella et al. [79] recently developed a thermoviscoelastic theory for Kelvin–Voigt microstretch composites. Comprehensive reviews on elastic materials with voids can be found in the references [12, 22, 26, 82], Ieșan [15, 16, 55, 58, 78, 81, 84–87, 92].

Regarding the application of the potential method in the linear theory of viscoelasticity for isotropic and homogeneous Kelvin–Voigt materials with voids, we would like to mention that the basic boundary value problems, when on the whole boundary of a viscoelastic body either the Dirichlet type condition or the Neumann type condition is given, have been studied by M. M. Svanadze in the spaces of regular vector-functions, having the first order continuous derivatives in closed domains with smooth boundaries (see [93–96]).

In the present paper, using the potential method and the theory of pseudodifferential equations, we investigate the Dirichlet–Neumann type mixed boundary value problems for the system of differential equations of the linear theory of Kelvin–Voigt viscoelastic materials, when on one part of the boundary of the body under consideration the Dirichlet type condition is given and on the remaining part of the boundary the Neumann type condition is prescribed. It is well-known that, in general, this type of mixed boundary value problems have no solutions in the space of regular vector-functions even for infinitely differentiable boundary data and infinitely smooth boundary surface. The main goal of the present investigation is to establish the of existence and uniqueness of solutions for the mixed BVPs in appropriate generalized vector-function spaces and to conduct detailed analysis of their almost optimal smoothness properties in the closed domain occupied by the body with a smooth or Lipschitz boundary. To the best of our knowledge, these types of mixed boundary value problems for Kelvin–Voigt viscoelastic materials have not been studied by the potential method which is a very powerful and efficient tool to study qualitative and quantitative properties of solutions.

The paper is organized as follows. In Section 2, we present the system of basic differential equations of the theory of steady state viscoelastic vibrations, introduce the related boundary differential operators and write down the corresponding Green formulas. In Section 3, we formulate the mixed boundary value problem and prove the uniqueness theorem. In Section 4, we define the corresponding layer potentials and the generated by them boundary integral (pseudodifferential) operators, analyse their mapping and coercivity properties in the appropriate Bessel potential and Besov spaces. Section 5 is devoted to detailed investigation of the mixed BVP. Using a special representation of the sought for solution by the single layer potential, the mixed BVP under consideration is reduced equivalently to the pseudodifferential equation which lives on the Neumann part of the boundary and contains the generalized Steklov–Poincaré type operator. We prove that the boundary pseudodifferential equation is uniquely solvable, that leads to the corresponding existence results for the mixed BVP in the appropriate Sobolev–Slobodetskii, Bessel potential, and Besov spaces. Further, we establish almost optimal regularity results for solutions to the mixed BVP near the collision curve, where different types of boundary conditions collide. It is shown that the solutions belong to the space of Hölder continuous vector-functions in the closed region occupied by the viscoelastic body. The efficient algebraic algorithm is described for finding the Hölder exponents which, in turn, determine the corresponding stress singularity exponents near the collision curves. It should be mentioned that these exponents essentially depend on the material parameters. In the final part of the paper, in Appendices A, B, C, and D, for the readers convenience, we have collected the necessary auxiliary material needed for our analysis in the main text.

2 Basic differential equations and Green identities

Let an isotropic homogeneous Kelvin–Voigt material occupy a bounded three-dimensional domain $\Omega = \Omega^+ \subset \mathbb{R}^3$ with a connected boundary $\partial\Omega = S$. The unbounded complement of the domain Ω we denote by $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$.

Further, let the boundary $\partial\Omega^+ = \partial\Omega^- = S$ be divided into two connected disjoint parts, S_D and S_N, $\overline{S}_D \cup \overline{S}_N = S$, $S_D \cap S_N = \varnothing$. For simplicity, throughout the paper, we assume that $S \in C^\infty$ and $\ell = \partial S_D = \partial S_N \in C^\infty$ unless otherwise stated. Some of the results obtained in the paper are valid when S, S_D, and S_N are Lipschitz surfaces, and these cases will be singled out separately. For detailed description of properties of the Lipschitz surfaces, we refer to [76, 100].

By L_p, W^r_p, H^s_p, and $B^s_{p,q}$ ($r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov spaces of complex-valued functions of real variables, respectively (see, e.g., [7, 63, 97, 98]). Note that the relations $H^r_2 = W^r_2 = B^r_{2,2}$, $H^s_2 = B^s_{2,2}$, $W^t_p = B^t_{p,p}$, and $H^k_p = W^k_p$, hold for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

By $C^{k,\alpha}$ we denote the space of k time continuously differentiable functions whose kth order derivatives satisfy the Hölder condition with an exponent $\alpha \in (0, 1]$.

In our analysis we also use the spaces

$$\widetilde{H}^s_p(S_1) := \big\{f :\ f \in H^s_p(S),\ \ \operatorname{supp} f \subset \overline{S}_1\big\},$$
$$\widetilde{B}^s_{p,q}(S_1) := \big\{f :\ f \in B^s_{p,q}(S),\ \ \operatorname{supp} f \subset \overline{S}_1\big\},$$

$$H^s_p(S_1) := \{r_{S_1} f : \ f \in H^s_p(S)\},$$
$$B^s_{p,q}(S_1) := \{r_{S_1} f : \ f \in B^s_{p,q}(S)\},$$

where $S_1 \in \{S_D, S_N\}$, r_{S_1} is the restriction operator onto S_1. The norms in these spaces are determined naturally:

$$\|u\|_{\widetilde{H}^s_p(S_1)} = \|u\|_{H^s_p(S)}, \quad \|u\|_{\widetilde{B}^s_{p,q}(S_1)} = \|u\|_{B^s_{p,q}(S)},$$
$$\|u\|_{H^s_p(S_1)} = \inf \|v\|_{H^s_p(S)}, \ \ v \in H^s_p(S), \ \ r_{S_1} v = u,$$
$$\|u\|_{B^s_{p,q}(S_1)} = \inf \|v\|_{B^s_{p,q}(S)}, \ \ v \in B^s_{p,q}(S), \ \ r_{S_1} v = u.$$

Let a function f be defined on an open proper submanifold $S_1 \in \{S_D, S_N\}$. Let $f \in B^s_{p,q}(S_1)$ and $\widetilde{f}$ be the extension of f by zero to $S \setminus S_1$. If the extension preserves the space, that is, if $\widetilde{f} \in \widetilde{B}^s_{p,q}(S_1)$, then we write $f \in \widetilde{B}^s_{p,q}(S_1)$ instead of $f \in r_{S_1} \widetilde{B}^s_{p,q}(S_1)$, when it does not lead to misunderstanding.

Note that $\widetilde{B}^s_{p,q}(S_1)$ and $B^{-s}_{p',q'}(S_1)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ are dual spaces. Similarly, $\widetilde{H}^s_p(S_1)$ and $H^{-s}_{p'}(S_1)$ are also dual spaces (for details see [65, 71, 97, 98]).

Therefore, for the functions $f \in B^{-s}_{p',q'}(S_1)$ and $g \in \widetilde{B}^s_{p,q}(S_1)$ (resp., $f \in H^{-s}_{p'}(S_1)$ and $g \in \widetilde{H}^s_p(S_1)$) the duality relation $\langle f, \overline{g}\rangle_{S_1}$ is well defined and generalizes the classical L_2-inner product:

$$\langle f, \overline{g}\rangle_{S_1} = \overline{\langle g, \overline{f}\rangle}_{S_1} = \int\limits_{S_1} f(x)\overline{g(x)}\, dS_1 \ \text{ for } \ f, g \in L_2(S_1),$$

where the overbar denotes complex conjugation.

By the symbols $\{\,\cdot\,\}^+$ and $\{\,\cdot\,\}^-$ we denote the standard one-sided traces of functions on the surface $S = \partial\Omega^\pm$ from Ω^+ and Ω^-, respectively. In the case of Lipschitz domains, we also use the symbols $\{\,\cdot\,\}^+_{nt}$ and $\{\,\cdot\,\}^-_{nt}$ for the nontangential limiting boundary values. In [1], it is shown that if a function v belongs to $H^1(\Omega)$ and has almost everywhere on S a nontangential limiting boundary value with a square integrable maximal function, then this boundary value coincides with the trace (with the Sobolev boundary value) of v on S. For general properties of the trace operator for Sobolev spaces on the Lipschitz domains we refer to the references [23, 34].

Throughout the paper, the summation over the repeated indices is meant from 1 to 3 and the superscript $(\cdot)^\top$ denotes transposition operation.

The basic system of partial differential equations of the theory of steady state viscoelastic vibrations reads as follows (see Appendix A):

$$\mu_1 \Delta u + (\lambda_1 + \mu_1)\,\mathrm{grad}\,\mathrm{div}\, u + b_1\,\mathrm{grad}\,\varphi + \varrho\omega^2 u = -\varrho\mathcal{F}, \tag{2.1}$$
$$\alpha_1 \Delta\varphi + \varrho_0\omega^2\varphi - \beta_1\varphi - \nu_1\,\mathrm{div}\, u = -\varrho\mathcal{F}_4, \tag{2.2}$$

where Δ is the Laplace operator, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, φ is the volume fraction field, $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)^\top$ is the body force per unit mass, $\mathcal{F}_4$ is the extrinsic equilibrated forces per unit mass, ϱ is the reference mass density, $\varrho_0 = \kappa\rho$ with κ being the equilibrated inertia,

$$\begin{aligned} \lambda_1 &= \lambda - i\omega\lambda^*, \quad \mu_1 = \mu - i\omega\mu^*, \quad b_1 = b - i\omega b^*, \\ \alpha_1 &= \alpha - i\omega\alpha^*, \quad \nu_1 = b - i\omega\nu^*, \quad \beta_1 = \beta - i\omega\beta^*, \end{aligned} \tag{2.3}$$

the constants λ, λ^*, μ, μ^*, b, b^*, ν, ν^*, α, α^*, β, β^*, and κ are material parameters satisfying some inequalities that will be specified later.

Rewrite the above system in a matrix form

$$A(\partial_x, \omega)U = \Phi,$$

where $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \frac{\partial}{\partial x_j}$, $U = (u, \varphi)^\top$, $\Phi = (-\varrho\mathcal{F}, -\varrho\mathcal{F}_4)^\top$, and $-A(\partial, \omega)$ is a non-self-adjoint strongly elliptic differential operator (see Appendix B):

$$A(\partial_x, \omega) = \big[A_{kj}(\partial_x, \omega)\big]_{4\times 4},$$

$$
\begin{gathered}
A_{kj}(\partial_x,\omega)=\mu_1\delta_{kj}\Delta+(\lambda_1+\mu_1)\partial_k\partial_j+\varrho\omega^2\delta_{kj},\ \ k,j=1,2,3,\\
A_{4j}(\partial_x,\omega)=-\nu_1\partial_j,\quad A_{k4}(\partial_x,\omega)=b_1\partial_k,\ \ k,j=1,2,3,\\
A_{44}(\partial_x,\omega)=\alpha_1\Delta+(\varrho_0\omega^2-\beta_1).
\end{gathered}
$$

The formally adjoint operator of $A(\partial_x,\omega)$ without complex conjugation we denote by the symbol $A^*(\partial_x,\omega)$, that is, $A^*(\partial_x,\omega)=A^\top(-\partial_x,\omega)$.

The principal homogeneous parts of the operators $A(\partial_x,\omega)$ and $A^*(\partial_x,\omega)$ coincide, they are formally self-adjoint operators without complex conjugation and read as follows:

$$
A^{(0)}(\partial_x,\omega)=A^{*(0)}(\partial_x,\omega)=\big[A^{(0)}_{kj}(\partial_x,\omega)\big]_{4\times4}, \tag{2.4}
$$

$$
A^{(0)}_{kj}(\partial_x,\omega)=\mu_1\delta_{kj}\Delta+(\lambda_1+\mu_1)\partial_k\partial_j,\quad A_{44}(\partial_x,\omega)=\alpha_1\Delta, \tag{2.5}
$$

$$
A^{(0)}_{4j}(\partial_x,\omega)=A^{(0)}_{k4}(\partial_x,\omega)=0,\ \ k,j=1,2,3. \tag{2.6}
$$

Further, let us introduce the generalized stress-type boundary operators $P(\partial_x,n)$ and $P^*(\partial_x,n)$ associated with the differential operators $A(\partial_x,\omega)$ and $A^*(\partial_x,\omega)$:

$$
\begin{gathered}
P(\partial_x,n)=\big[P_{kj}(\partial_x,n)\big]_{4\times4},\\
P_{kj}(\partial_x,n)=\mu_1\delta_{kj}\partial_n+\mu_1n_j\partial_k+\lambda_1n_k\partial_j,\ \ k,j=1,2,3,\\
P_{4j}(\partial_x,n)=0,\quad P_{k4}(\partial_x,n)=b_1n_k,\ \ k,j=1,2,3,\\
P_{44}(\partial_x,n)=\alpha_1\partial_n,\\
P^*(\partial_x,n)=\big[P^*_{kj}(\partial_x,n)\big]_{4\times4},\\
P^*_{kj}(\partial_x,n)=\mu_1\delta_{kj}\partial_n+\mu_1n_j\partial_k+\lambda_1n_k\partial_j,\ \ k,j=1,2,3,\\
P^*_{4j}(\partial_x,n)=0,\quad P^*_{k4}(\partial_x,n)=\nu_1n_k,\ \ k,j=1,2,3,\\
P^*_{44}(\partial_x,n)=\alpha_1\partial_n,
\end{gathered} \tag{2.7}
$$

where $n=(n_1,n_2,n_3)$ is a unit outward normal vector to the boundary of the body under consideration and ∂_n denotes the normal derivative, $\partial_n=n_1\partial_1+n_2\partial_2+n_3\partial_3$.

The principal homogeneous parts of the boundary stress operators $P(\partial,n,\omega)$ and $P^*(\partial,n,\omega)$ coincide and read as follows:

$$
\begin{gathered}
P^{(0)}(\partial,n)=P^{(0)*}(\partial,n)=\big[P^{(0)}_{kj}(\partial_x,n)\big]_{4\times4},\\
P^{(0)}_{kj}(\partial_x,n)=\mu_1\delta_{kj}\partial_n+\mu_1n_j\partial_k+\lambda_1n_k\partial_j,\\
P^{(0)}_{4j}(\partial_x,n)=P^{(0)}_{k4}(\partial_x,n)=0,\quad P^{(0)}_{44}(\partial_x,n)=\alpha_1\partial_n,\ \ k,j=1,2,3.
\end{gathered}
$$

Let $U=(u,\varphi)^\top$ and $V=(v,\psi)^\top$ with $u=(u_1,u_2,u_3)^\top$ and $v=(v_1,v_2,v_3)^\top$ be regular vector-functions of the class $[C^2(\overline{\Omega})]^4$. Then the following first and the second Green formulas (cf., [61]) hold:

$$
\int\limits_{\Omega}A(\partial_x,\omega)U(x)\cdot V(x)\,dx=\int\limits_{S}\big\{P(\partial_x,n(x))U(x)\big\}^+\cdot\big\{V(x)\big\}^+\,dS-\int\limits_{\Omega}\widetilde{E}(U,V)\,dx, \tag{2.8}
$$

$$
\int\limits_{\Omega}U(x)\cdot A^*(\partial_x,\omega)V(x)\,dx=\int\limits_{S}\big\{U(x)\big\}^+\cdot\big\{P^*(\partial_x,n(x))V(x)\big\}^+\,dS-\int\limits_{\Omega}\widetilde{E}(U,V)\,dx, \tag{2.9}
$$

$$
\begin{aligned}
&\int\limits_{\Omega}\big[A(\partial_x,\omega)U(x)\cdot V(x)-U(x)\cdot A^*(\partial_x,\omega)V(x)\big]\,dx\\
&\qquad=\int\limits_{S}\Big[\big\{P(\partial_x,n(x))U(x)\big\}^+\cdot\big\{V(x)\big\}^+-\big\{U(x)\big\}^+\cdot\big\{P^*(\partial_x,n(x))V(x)\big\}^+\Big]\,dS,
\end{aligned} \tag{2.10}
$$

where for the complex valued vectors $a = (a_1, a_2, \dots, a_r)$ and $b = (b_1, b_2, \dots, b_r)$ the central dot denotes the "real" scalar product, $a \cdot b = \sum\limits_{j=1}^{r} a_j b_j$,

$$\widetilde{E}(U,V) = E_\omega(u,v) - \varrho\omega^2 u\cdot v + b_1\varphi\,\mathrm{div}\,u + \alpha_1\nabla\varphi\cdot\nabla\psi - (\varrho_0\omega^2 - \beta_1)\varphi\psi + \nu_1\psi\,\mathrm{div}\,u, \quad (2.11)$$

$$E_\omega(u,v) = E(u,v;\lambda,\mu) - i\omega E(u,v;\lambda^*,\mu^*), \quad (2.12)$$

$$\begin{aligned} E(u,v;\lambda,\mu) &= \lambda\sum_{j=1}^{3} e_{jj}(u)\sum_{j=1}^{3} e_{jj}(v) + 2\mu\sum_{l,j=1}^{3} e_{jl}(u)e_{jl}(v) \\ &= \frac{1}{3}\,(3\lambda+2\mu)\,\mathrm{div}\,u\,\mathrm{div}\,v + \mu\bigg[\frac{1}{2}\sum_{l,j=1(l\neq l)}^{3}(\partial_l u_j + \partial_j u_l)(\partial_l v_j + \partial_j v_l) \\ &\qquad + \frac{1}{3}\sum_{l,j=1}^{3}(\partial_l u_l - \partial_j u_j)(\partial_l v_l - \partial_j v_j)\bigg], \end{aligned} \quad (2.13)$$

$$\begin{aligned} E(u,v;\lambda^*,\mu^*) &= \lambda^*\sum_{j=1}^{3} e_{jj}(u)\sum_{j=1}^{3} e_{jj}(v) + 2\mu^*\sum_{l,j=1}^{3} e_{jl}(u)e_{jl}(v) \\ &= \frac{1}{3}\,(3\lambda^*+2\mu^*)\,\mathrm{div}\,u\,\mathrm{div}\,v + \mu^*\bigg[\frac{1}{2}\sum_{l,j=1(l\neq l)}^{3}(\partial_l u_j + \partial_j u_l)(\partial_l v_j + \partial_j v_l) \\ &\qquad + \frac{1}{3}\sum_{l,j=1}^{3}(\partial_l u_l - \partial_j u_j)(\partial_l v_l - \partial_j v_j)\bigg], \end{aligned} \quad (2.14)$$

where $e_{lj}(u) = \frac{1}{2}(\partial_l u_j + \partial_j u_l)$ and $e_{lj}(v) = \frac{1}{2}(\partial_l v_j + \partial_j v_l)$ are the components of the mechanical strain tensor associated with the complex-valued vector-functions u and v.

For $V = \overline{U}$, from (2.11), separating the real and imaginary parts, we get

$$\begin{aligned} \widetilde{E}(U,\overline{U}) = &\Big\{E(u,\overline{u};\lambda,\mu) - \varrho\omega^2|u|^2 + \alpha|\nabla\varphi|^2 - (\varrho_0\,\omega^2 - \beta_1)|\varphi|^2 + \mathrm{Re}\,\big(b_1\varphi\,\mathrm{div}\,\overline{u} + \nu_1\overline{\varphi}\,\mathrm{div}\,u\big)\Big\} \\ &- i\omega\Big\{E(u,\overline{u};\lambda^*,\mu^*) + \alpha^*|\nabla\varphi|^2 + \beta^*|\varphi|^2 + (b^*+\nu^*)\,\mathrm{Re}(\varphi\,\mathrm{div}\,\overline{u})\Big\} \end{aligned} \quad (2.15)$$

with

$$E(u,\overline{u};\lambda,\mu) = \lambda\Big|\sum_{j=1}^{3} e_{jj}(u)\Big|^2 + 2\mu\sum_{l,j=1}^{3}|e_{jl}|^2, \quad (2.16)$$

$$E(u,\overline{u};\lambda^*,\mu^*) = \lambda^*\Big|\sum_{j=1}^{3} e_{jj}(u)\Big|^2 + 2\mu^*\sum_{l,j=1}^{3}|e_{jl}(u)|^2. \quad (2.17)$$

Here, we used the identity

$$\mathrm{Im}\,\big[b_1\varphi\,\mathrm{div}\,\overline{u} + \nu_1\overline{\varphi}\,\mathrm{div}\,u\big] = -\omega(b^*+\nu^*)\,\mathrm{Re}(\varphi\,\mathrm{div}\,\overline{u}).$$

Lemma 2.1. *Let $U = (u,\varphi)^\top$ be a complex-valued differentiable vector-function and let the following inequalities hold:*

$$\mu^* > 0, \quad (3\lambda^*+2\mu^*)\beta^* > \frac{3}{4}\,(b^*+\nu^*)^2, \quad \alpha^* > 0, \quad \beta^* > 0, \quad \omega > 0. \quad (2.18)$$

Then there is a positive constant C_1, depending on the material parameters involved in inequalities (2.18) such that

$$-\,\mathrm{Im}\,\widetilde{E}(U,\overline{U}) \geqslant \omega C_1\Big(\sum_{l,j=1}^{3}|e_{lj}(u)|^2 + |\nabla\varphi|^2 + |\varphi|^2\Big),$$

where $e_{lj}(u) = \frac{1}{2}\,(\partial_l u_j + \partial_j u_l)$ are the components of the mechanical strain tensor.

Proof. From (2.15), with the help of relations (2.11)–(2.14), we get

$$-\operatorname{Im}\widetilde{E}(U,\overline{U})=\omega\Big\{E(u,\overline{u};\lambda^*,\mu^*)+\beta^*|\varphi|^2+(b^*+\nu^*)\operatorname{Re}\big(\varphi\operatorname{div}\overline{u}\big)+\alpha^*|\nabla\varphi|^2\Big\}. \tag{2.19}$$

Assume that $u=u'+i\,u''$ and $\varphi=\varphi'+i\,\varphi''$, where $u'=(u_1',u_2',u_3')^\top$, $u''=(u_1'',u_2'',u_3'')^\top$, φ', and φ'' are the real-valued functions. Let us introduce the notations:

$$\begin{gathered}
\zeta_1=e_{11}(u')=\partial_1u_1',\quad \zeta_2=e_{22}(u')=\partial_2u_2',\quad \zeta_3=e_{33}(u')=\partial_3u_3',\\
\zeta_4=e_{12}(u')=\frac12\,(\partial_1u_2'+\partial_2u_1'),\quad \zeta_5=e_{13}(u')=\frac12\,(\partial_1u_3'+\partial_3u_1'),\\
\zeta_6=e_{23}(u')=\frac12\,(\partial_2u_3'+\partial_3u_2'),\quad \zeta_7=\varphi',\quad \zeta_8=\partial_1\varphi',\quad \zeta_9=\partial_2\varphi',\quad \zeta_{10}=\partial_3\varphi',\\
\eta_1=e_{11}(u'')=\partial_1u_1'',\quad \eta_2=e_{22}(u'')=\partial_2u_2'',\quad \eta_3=e_{33}(u'')=\partial_3u_3'',\\
\eta_4=e_{12}(u'')=\frac12\,(\partial_1u_2''+\partial_2u_1''),\quad \eta_5=e_{13}(u'')=\frac12\,(\partial_1u_3''+\partial_3u_1''),\\
\eta_6=e_{23}(u'')=\frac12\,(\partial_2u_3''+\partial_3u_2''),\quad \eta_7=\varphi'',\quad \eta_8=\partial_1\varphi'',\quad \eta_9=\partial_2\varphi'',\quad \eta_{10}=\partial_3\varphi''.
\end{gathered}$$

Then equality (2.19) can be rewritten as a sum of two quadratic forms with respect to the real-valued 10-dimensional vectors $\zeta=(\zeta_1,\zeta_2,\dots,\zeta_{10})$ and $\eta=(\eta_1,\eta_2,\dots,\eta_{10})$:

$$-\operatorname{Im}\widetilde{E}(U,\overline{U})=\omega\big(Q(\zeta)+Q(\eta)\big), \tag{2.20}$$

where

$$\begin{aligned}
Q(z)=\lambda^*(z_1+z_2+z_3)^2+2\mu^*(z_1^2+z_2^2+z_3^2)+4\mu^*(z_4^2+z_5^2+z_6^2)&\\
+\beta^*z_7^2+(b^*+\nu^*)z_7(z_1+z_2+z_3)+\alpha^*(z_8^2+z_9^2+z_{10}^2),&\quad z\in\mathbb{R}^{10}.
\end{aligned}$$

By the Sylvester criterion it can be shown that if conditions (2.18) hold, then $Q(z)$ is a positive definite quadratic form. Therefore, from (2.20), we deduce

$$-\operatorname{Im}\widetilde{E}(U,\overline{U})\geqslant\omega C_1\big(|\zeta|^2+|\eta|^2\big),$$

which completes the proof. □

Corollary 2.1. *Let $u=(u_1,u_2,u_3)^\top$ be a complex-valued differentiable vector-function and let the following inequalities hold:*

$$\mu^*>0,\quad 3\lambda^*+2\mu^*>0. \tag{2.21}$$

Then there is a positive constant C_2^*, depending on the material parameters λ^* and μ^*, such that

$$E(u,\overline{u};\lambda^*,\mu^*)\geqslant C_2^*\sum_{l,j=1}^3|e_{lj}(u)|^2.$$

Remark 2.1. Evidently, the counterpart of Corollary 2.1 holds also true for $E(u,\overline{u};\lambda,\mu)$, i.e.,

$$E(u,\overline{u};\lambda,\mu)\geqslant C_2\sum_{l,j=1}^3|e_{lj}(u)|^2$$

if

$$\mu>0,\quad 3\lambda+2\mu>0. \tag{2.22}$$

Throughout the paper, we assume that conditions (2.18), (2.21), and (2.22) are satisfied.

Remark 2.2. If the vector-functions U and V and their first and second order derivatives decay at infinity as $\mathcal{O}(|x|^{-2})$, then the Green identities hold true also for the exterior unbounded domain Ω^-:

$$\int\limits_{\Omega^-} A(\partial_x,\omega)U(x)\cdot V(x)\,dx = -\int\limits_S \big\{P(\partial_x,n(x))U(x)\big\}^-\cdot\big\{V(x)\big\}^-\,dS - \int\limits_{\Omega^-}\widetilde{E}(U,V)\,dx, \tag{2.23}$$

$$\int\limits_{\Omega^-}\big[A(\partial_x,\omega)U(x)\cdot V(x) - U(x)\cdot A^*(\partial_x,\omega)V(x)\big]\,dx$$
$$= -\int\limits_S \Big[\big\{P(\partial_x,n(x))U(x)\big\}^-\cdot\big\{V(x)\big\}^- - \big\{U(x)\big\}^-\cdot\big\{P^*(\partial_x,n(x))V(x)\big\}^-\Big]\,dS, \tag{2.24}$$

Remark 2.3. The Green formulas (2.8), (2.10), (2.23), and (2.24) can be extended to domains with Lipschotz boundaries and to vector-functions with the properties $U\in[H^1_p(\Omega^\pm)]^4$, $V\in[H^1_{p'}(\Omega^\pm)]^4$ and $A(\partial,\omega)U\in[L_p(\Omega^\pm)]^4$, $A(\partial,\omega)V\in[L_{p'}(\Omega^\pm)]^4$ (for details, see the references [10, 65–67, 70]). In this case, the integrals over the surface S should be replaced by the appropriate duality relations.

In particular, the first Green identity reads as follows:

$$\Big\langle\big\{P(\partial_x,n(x))U(x)\big\}^+,\big\{V(x)\big\}^+\Big\rangle_S = \int\limits_\Omega A(\partial_x,\omega)U(x)\cdot V(x)\,dx + \int\limits_\Omega \widetilde{E}(U,V)\,dx,$$
$$U\in[H^1_p(\Omega)]^4,\quad A(\partial,\omega)U\in[L_p(\Omega)]^4,\quad V\in[H^1_{p'}(\Omega)]^4,\quad \frac{1}{p}+\frac{1}{p'}=1,$$

where the brackets $\langle\cdot\,,\,\cdot\rangle_S$ denote the duality between the mutually adjoint vector-function spaces $[B^{-\frac{1}{p}}_{p,p}(S)]^4$ and $[B^{\frac{1}{p}}_{p',p'}(S)]^4$. By this formula, the generalized boundary functional $\{P(\partial_x,n(x))U(x)\}^+ \in[B^{-\frac{1}{p}}_{p,p}(S)]^4$ is defined correctly, in view of the inclusion $\{V\}^+\in[B^{1-\frac{1}{p'}}_{p',p'}(S)]^4=[B^{\frac{1}{p}}_{p',p'}(S)]^4$.

In what follows, we assume that if integrals do not exist in the standard Lebesgue sense, they are then understood as functionals in the duality sense provided that the corresponding generalized functions belong to the mutually adjoint spaces. This will be clear from the context when we refer to an ordinary integral or to a functional.

Remark 2.4. From Lemma 2.1 it follows that if $\operatorname{Im}\widetilde{E}(U,\overline{U})=0$ in Ω, then $\varphi(x)=0$ and $u(x)$ is a rigid displacement, $u(x)=a\times x+b$, where a and b are arbitrary complex constant vectors and "$\times$" denotes the cross product (see, e.g., [61]).

3 Formulation of the mixed BVPs and uniqueness theorems

The mixed interior boundary value problem (BVP) in the linear theory of viscoelasticity for an isotropic homogeneous Kelvin–Voigt material occupying a bounded domain Ω is formulated as follows: Find a vector-function $U=(u,\varphi)^\top\in[H^1_p(\Omega)]^4=[W^1_p(\Omega)]^4$, $p>1$, satisfying

(i) the differential equation
$$A(\partial_x,\omega)U(x)=\Phi(x)\ \text{ in }\Omega, \tag{3.1}$$

(ii) The Dirichlet type boundary condition (on S_D)
$$\{U(x)\}^+=f(x)\ \text{ on } S_D, \tag{3.2}$$

(iii) The Neumann type boundary condition (on S_N)
$$\big\{P(\partial_x,n(x))U(x)\big\}^+=F(x)\ \text{ on } S_N, \tag{3.3}$$

where $S=\partial\Omega=\overline{S}_D\cup\overline{S}_N$, $S_D\cap S_N=\varnothing$, and the data of the problem satisfy natural inclusions

$$\Phi=(\Phi_1,\Phi_2,\Phi_3,\Phi_4)^\top\in\big[H^0_p(\Omega)\big]^4=\big[L_p(\Omega)\big]^4, \tag{3.4}$$

$$f=(f_1,f_2,f_3,f_4)^\top\in\big[B_{p,p}^{1-\frac1p}(S_D)\big]^4, \tag{3.5}$$

$$F=(F_1,F_2,F_3,F_4)^\top\in\big[B_{p,p}^{-\frac1p}(S_N)\big]^4. \tag{3.6}$$

Recall that the trace $\{h\}^+$ on $S=\partial\Omega$ of a function $h\in H_p^1(\Omega)$ belongs to the space $B_{p,p}^{1-\frac1p}(S)$.

In the formulation of the above mixed problem, equation (3.1) is understood in the weak sense, the boundary condition (3.2) is understood in the usual trace sense, while condition (3.3) is understood in the generalized functional sense defined with the help of Green's first identity (2.8) (see Remark 2.3)

$$\Big\langle\big\{P(\partial_x,n(x))U(x)\big\}^+,\big\{V(x)\big\}^+\Big\rangle_{S_N}=\int\limits_\Omega A(\partial_x,\omega)U(x)\cdot V(x)\,dx+\int\limits_\Omega \widetilde E(U,V)\,dx \tag{3.7}$$

$$\text{for all }\ V\in\big[H_{p'}^1(\Omega,S_D)\big]^4,\ \ \frac1p+\frac1{p'}=1,$$

where

$$\big[H_{p'}^1(\Omega,S_D)\big]^4=\Big\{V\in\big[H_{p'}^1(\Omega)\big]^4:\ r_{S_D}\{V\}^+=0\Big\}$$

and the brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the duality between the mutually adjoint vector-function spaces $[B_{p,p}^{-\frac1p}(S_N)]^4$ and $[\widetilde B_{p',p'}^{\frac1p}(S_N)]^4$.

Evidently, if $U\in[H_p^1(\Omega)]^4$ with $A(\partial,\omega)U\in[L_p(\Omega)]^4$, $p>1$, relation (3.7) correctly defines the generalized trace on S of the stress vector-function $\{P(\partial_x,n(x)\omega)U(x)\}^+\in[B_{p,p}^{-\frac1p}(S_N)]^4$.

For a special case $p=2$, the following uniqueness theorem holds.

Theorem 3.1. *Let $S=\partial\Omega$ be a Lipschitz surface, the material parameters meet inequalities (2.18), and let inclusions (3.4)–(3.6) hold for $p=2$. Then the mixed boundary value problem possesses at most one solution in the space $[H_2^1(\Omega)]^4=[W_2^1(\Omega)]^4$.*

Proof. Due to the linearity of the mixed boundary value problem (3.1)–(3.3), we have to show that the homogeneous problem has only the trivial solution. Let $U=(u,\varphi)^\top\in[H_2^1(\Omega)]^4$ be a solution of the homogeneous problem. Then the first Green formula (3.7) with $V=\overline U$ leads to the relation

$$\int\limits_\Omega \widetilde E(U,\overline U)\,dx=0, \tag{3.8}$$

where $\widetilde E(U,\overline U)$ is given by (2.15). Separating the imaginary part of relation (3.8), in view of Lemma 2.1, we deduce $e_{lj}(u)=0$, $l,j=1,2,3$, and $\varphi=0$ in Ω. Therefore, due to Remark 2.4, the vector-function u is a rigid displacement, $u=a\times x+b$ with a and b being arbitrary complex constant vectors. Finally, the homogeneous Dirichlet type condition (3.2) implies that $u=0$ in Ω, which completes the proof. □

The uniqueness theorem for other values of the parameter p will be shown below, in Section 5.

We investigate the existence of solutions to the mixed boundary value problem under consideration by the potential method and the theory of pseudodifferential operators on the manifold with a boundary. To this end, in the next section, we study the mapping properties of volume and layer potentials and the corresponding boundary integral operators in different function spaces.

4 Properties of potentials and boundary integral operators

Denote by $\Gamma(x)=[\Gamma_{kj}(x)]_{4\times4}$ and $\Psi(x)=[\Psi_{kj}(x]_{4\times4}$ the fundamental matrices of the operators $A(\partial_x,\omega)$ and $A^{(0)}(\partial_x,\omega)$, respectively,

$$A(\partial_x,\omega)\Gamma(x)=\delta(x)I_4,\quad A^{(0)}(\partial_x,\omega)\Psi(x)=\delta(x)I_4,$$

where $\delta(\,\cdot\,)$ is Dirac's delta functional and I_m is $m\times m$ unite matrix. These fundamental matrices are explicitly constructed in Appendix B. By a different approach, these matrices were constructed in [93]. The matrix $\Psi(x)$ is a principal singular part of the matrix $\Gamma(x)$ in a vicinity of the origin and the following relations hold (see Appendix B):

$$\begin{gathered}\Gamma_{kj}(x)-\Psi_{kj}(x)=const+\mathcal{O}(|x|),\\ \frac{\partial^p}{\partial_1^{p_1}\partial_2^{p_2}\partial_3^{p_3}}\left[\Gamma_{kj}(x)-\Psi_{kj}(x)\right]=\mathcal{O}\big(|x|^{1-p}\big),\ \ p=p_1+p_2+p_3.\end{gathered}\tag{4.1}$$

Actually, the matrix $[\Psi_{kj}(x)]_{3\times3}$ formally coincides with the Kelvin fundamental matrix for the Lame system of the classical theory of elasticity with the complex parameters λ_1 and μ_1 defined in (2.3) (see Appendix B).

Note that at infinity the matrices $\Gamma(x)$ and $\Psi(x)$ have different behaviour. The entries of the matrix $\Gamma(x)$ decay exponentially as $|x|\to\infty$, while the entries of the matrix $\Psi(x)$ are homogeneous functions in x of order -1.

Now, let us introduce the single and double layer potentials and the Newtonian type volume potential, defined respectively by the equalities:

$$V(g)(x)=\int\limits_S\Gamma(x-y)g(y)\,dS_y,\ \ x\in\mathbb{R}^3\setminus S,\tag{4.2}$$

$$W(g)(x)=\int\limits_S\left[P^*(\partial_y,n(y))\Gamma^\top(x-y)\right]^\top g(y)\,dS_y,\ \ x\in\mathbb{R}^3\setminus S,\tag{4.3}$$

$$N_\Omega(h)(x)=\int\limits_\Omega\Gamma(x-y)h(y)\,dy,\ \ x\in\mathbb{R}^3,\tag{4.4}$$

where $P^*(\partial,n)$ is the boundary differential operator defined by (2.7), $g=(g_1,\dots,g_4)^\top$ is a density vector-function defined on S, while $h=(h_1,\dots,h_4)^\top$ is a density vector-function defined on Ω.

Note that the above-introduced potentials (4.2)–(4.4) and their derivatives of arbitrary order decay exponentially at infinity.

Let us introduce also the layer potentials associated with the operator $A^{(0)}(\partial,\omega)$ and constructed by the fundamental matrix $\Psi(x)$,

$$V^{(0)}(g)(x)=\int\limits_S\Psi(x-y)g(y)\,dS_y,\ \ x\in\mathbb{R}^3\setminus S,\tag{4.5}$$

$$W^{(0)}(g)(x)=\int\limits_S\left[P^{(0)}(\partial_y,n(y))\Psi(x-y)\right]^\top g(y)\,dS_y,\ \ x\in\mathbb{R}^3\setminus S.\tag{4.6}$$

Here, we applied the relations $\Psi(x-y)=\Psi^\top(x-y)=\Psi(y-x)$.

We have the following general integral representations in Ω of a smooth vector-function, which can be derived with the help of the second Green formula (2.10) (see [93]).

Theorem 4.1. *Let $S=\partial\Omega$ be $C^{1,\kappa}$-smooth with $0<\kappa\le1$ and let U be a regular vector-function of the class $[C^2(\overline{\Omega^+})]^4$. Then the following the integral representation formula holds:*

$$W(\{U\}^+)(x)-V(\{PU\}^+)(x)+N_\Omega(A(\partial_x,\omega)U)(x)=\begin{cases}U(x) & \textit{for }\ x\in\Omega,\\ 0 & \textit{for }\ x\in\Omega^-.\end{cases}\tag{4.7}$$

Due to the exponential decay behaviour at infinity of the fundamental matrix Ψ, using the second Green identity (2.24), one can derive similar integral representation formula in the unbounded exterior domain Ω^-.

Theorem 4.2. *Let $S=\partial\Omega^-$ be $C^{1,\kappa}$-smooth with $0<\kappa\le1$ and let U be a polynomially bounded regular vector-function of the class $[C^2(\overline{\Omega^-})]^4$ with polynomially bounded at infinity derivatives:*

$$U_l(x)=\mathcal{O}(|x|^m),\quad \partial_kU_l(x)=\mathcal{O}(|x|^m),\quad \partial_k\partial_jU_l(x)=\mathcal{O}\big(|x|^m\big),\ \ l=1,2,3,4,\ \ k,j=1,2,3,\tag{4.8}$$

where m is an integer number. Then the following the integral representation formula holds:

$$-W(\{U\}^-)(x)+V(\{PU\}^-)(x)+N_{\Omega^-}(A(\partial_x,\omega)U)(x)=\begin{cases}0 & \text{for } x\in\Omega^+,\\ U(x) & \text{for } x\in\Omega^-.\end{cases} \tag{4.9}$$

Corollary 4.1. *Let $S=\partial\Omega^\pm$ be $C^{1,\kappa}$-smooth with $0<\kappa\le 1$ and let $U\in[C^2(\overline{\Omega^\pm})]^4$ be a solution to the homogeneous equation $A(\partial,\omega)U=0$ in Ω^+ and Ω^- satisfying conditions (4.8). Then the following representation formula holds:*

$$U(x)=W([U]_S)(x)-V([PU]_S)(x),\ \ x\in\Omega^+\cup\Omega^-,$$

where $[U]_S=\{U\}^+_S-\{U\}^-_S$ and $[PU]_S=\{PU\}^+_S-\{PU\}^-_S$.

Remark 4.1. By the standard arguments applied, for example, in [65] and [8], formulas (4.7) and (4.9) can be extended to Lipschitz domains and to vector-functions satisfying the conditions $U\in[W^1_p(\Omega^\pm)]^4=[H^1_p(\Omega^\pm)]^4$ and $A(\partial,\omega)U\in[L_p(\Omega^\pm)]^4$ with $1<p<\infty$.

Now, we describe the mapping properties of layer potentials.

Theorem 4.3. *Let $S=\partial\Omega^\pm\in C^{m,\kappa}$ with integer $m\ge 1$ and $0<\kappa\le 1$. Let $k\le m-1$ be integer and $0<\kappa'<\kappa$. Then the operators*

$$\begin{aligned} &V:\big[C^{k,\kappa'}(S)\big]^4\to\big[C^{k+1,\kappa'}(\overline{\Omega^\pm})\big]^4,\\ &W:\big[C^{k,\kappa'}(S)\big]^4\to\big[C^{k,\kappa'}(\overline{\Omega^\pm})\big]^4\end{aligned} \tag{4.10}$$

are continuous.

For any $g\in[C^{0,\kappa'}(S)]^4$, $h\in[C^{1,\kappa'}(S)]^4$, and for any $x\in S$, the following jump relations hold:

$$\big\{V(g)(x)\big\}^+=\{V(g)(x)\}^-, \tag{4.11}$$

$$\big\{P(\partial_x,n(x),\omega)V(g)(x)\big\}^\pm=\Big[\mp\frac12\,I_4+\widetilde{\mathcal{K}}\Big]g(x), \tag{4.12}$$

$$\big\{W(g)(x)\big\}^\pm=\Big[\pm\frac12\,I_4+\mathcal{K}\Big]g(x), \tag{4.13}$$

$$\big\{P(\partial_x,n(x))W(h)(x)\big\}^+=\big\{P(\partial_x,n(x))W(h)(x)\big\}^-=\mathcal{L}h(x),\ \ m\ge 2, \tag{4.14}$$

where

$$\mathcal{H}g(x):=\big\{V(g)(x)\big\}^\pm=\int_S\Gamma(x-y)g(y)\,dS_y,\ \ x\in S, \tag{4.15}$$

$$\widetilde{\mathcal{K}}g(x):=\int_S\big[P(\partial_x,n(x))\Gamma(x-y)\big]g(y)\,dS_y,\ \ x\in S, \tag{4.16}$$

$$\mathcal{K}g(x):=\int_S\big[P^*(\partial_y,n(y))\Gamma^\top(x-y)\big]^\top g(y)\,dS_y,\ \ x\in S, \tag{4.17}$$

$$\begin{aligned}\mathcal{L}h(x)&:=\big\{P(\partial_x,n(x))W(h)(x)\big\}^\pm\\ &=\lim_{\Omega^\pm\ni z\to x\in S}P(\partial_z,n(x))\int_S\big[P^*(\partial_y,n(y))\Gamma^\top(z-y)\big]^\top h(y)\,dS_y,\ \ x\in S.\end{aligned} \tag{4.18}$$

Proof. The proof of relations (4.10)–(4.13) can be performed by standard arguments employed in the proof of similar theorems in [61, Chapter 5].

We demonstrate here only a simplified proof of relation (4.14), known as the *Liapunov–Tauber type theorem*. Let $h\in[C^{1,\kappa'}(S)]^6$, $S\in C^{2,\kappa}$, and consider the double layer potential $U=W(h)\in[C^{1,\kappa'}(\overline{\Omega^\pm})]^6$. Then, by Corollary 4.1 and the jump relations (4.13), we have

$$U(x)=W\big([U]_S\big)(x)-V\big([PU]_S\big)(x),\ \ x\in\Omega^\pm,$$

i.e.,

$$W(h)(x) = W(h)(x) - V\big([PW(h)]_S\big)(x), \quad x \in \Omega^{\pm},$$

since $[U]_S = \{W(h)\}^+ - \{W(h)\}^- = h$ on S, due to (4.13). Therefore, $V([PW(h)]_S) = 0$ in $\Omega^{\pm}$ and, in view of (4.12), we conclude

$$\big\{PV\big([PW(h)]_S\big)\big\}^- - \big\{PV\big([PW(h)]_S\big)\big\}^+ = \big[PW(h)\big]_S = \big\{PW(h)\big\}^+ - \big\{PW(h)\big\}^- = 0$$

on S, which completes the proof. □

Note that the operator $\mathcal{H}$ is a weakly singular integral operator, i.e., a pseudodifferential operator of order -1. The operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are the Calderón–Zygmund type singular integral operators, i.e., pseudodifferential operators of zero order. The operator $\mathcal{L}$ is an integro-differential operator, i.e., a pseudodifferential operator of order $+1$. In the case of smooth surfaces, the principal homogeneous symbols of these pseudodifferential operators will be analyzed below.

Theorem 4.4. *Let $S = \partial\Omega^{\pm} \in C^{m,\kappa}$ with the integer $m \geq 1$ and $0 < \kappa \leq 1$. Let $k \leq m-1$ be an integer and $0 < \kappa' < \kappa$. Then the operators*

$$\mathcal{H} : \big[C^{k,\kappa'}(S)\big]^4 \to \big[C^{k+1,\kappa'}(S)\big]^4, \quad m \geq 1, \tag{4.19}$$

$$\pm\frac{1}{2}\,I_4 + \mathcal{K},\ \pm\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}} : \big[C^{k,\kappa'}(S)\big]^4 \to \big[C^{k,\kappa'}(S)\big]^4, \quad m \geq 1, \tag{4.20}$$

$$\mathcal{L} : \big[C^{k,\kappa'}(S)\big]^4 \to \big[C^{k-1,\kappa'}(S)\big]^4, \quad m \geq 2, \quad k \geq 1, \tag{4.21}$$

are continuous.

Moreover, the following operator identities hold:

$$\mathcal{K}\mathcal{H} = \mathcal{H}\widetilde{\mathcal{K}}, \quad \mathcal{L}\mathcal{K} = \widetilde{\mathcal{K}}\mathcal{L}, \quad \mathcal{H}\mathcal{L} = -\frac{1}{4}\,I_4 + \mathcal{K}^2, \quad \mathcal{L}\mathcal{H} = -\frac{1}{4}\,I_4 + \widetilde{\mathcal{K}}^2. \tag{4.22}$$

Proof. The proof of mapping properties is word for word of the proofs of the counterpart theorems in [61,72,75], and [73]. Operator identities (4.24) can be obtained by taking the trace on S of formula (4.8) written for the single and double layer potentials. □

Remark 4.2. Due to relations (4.1), it is evident that the above-formulated theorems remain true for the potentials $V^{(0)}$ and $W^{(0)}$ defined by (4.5) and (4.6) and generated by them the boundary integral operators $\mathcal{H}^{(0)}$, $\mathcal{K}^{(0)}$, $\widetilde{\mathcal{K}}^{(0)}$, and $\mathcal{L}^{(0)}$, defined by relations (4.15)–(4.18) with Ψ for Γ and $P^{(0)}(\partial,n)$ for $P(\partial,n)$ and for $P^*(\partial,n)$.

The above-formulated theorems can be extended to more general function spaces. In particular, for the Lipschitz domains we have the following mapping properties.

Theorem 4.5. *Let $S = \partial\Omega$ be a Lipschitz surface. Then the operators*

$$V : \big[H_2^{-\frac{1}{2}}(S)\big]^4 \to \big[H_2^1(\Omega^{\pm})\big]^4, \tag{4.23}$$

$$W : \big[H_2^{\frac{1}{2}}(S)\big]^4 \to \big[H_2^1(\Omega^{\pm})\big]^4, \tag{4.24}$$

$$\mathcal{H} : \big[H_2^{-\frac{1}{2}}(S)\big]^4 \to \big[H_2^{\frac{1}{2}}(S)\big]^4, \tag{4.25}$$

$$\mathcal{H}^{(0)} : \big[H_2^{-\frac{1}{2}}(S)\big]^4 \to \big[H_2^{\frac{1}{2}}(S)\big]^4, \tag{4.26}$$

$$\pm\frac{1}{2}\,I_4 + \mathcal{K} : \big[H_2^{\frac{1}{2}}(S)\big]^4 \to \big[H_2^{\frac{1}{2}}(S)\big]^4, \tag{4.27}$$

$$\pm\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}} : \big[H_2^{-\frac{1}{2}}(S)\big]^4 \to \big[H_2^{-\frac{1}{2}}(S)\big]^4, \tag{4.28}$$

$$\mathcal{L} : \big[H_2^{\frac{1}{2}}(S)\big]^4 \to \big[H_2^{-\frac{1}{2}}(S)\big]^4 \tag{4.29}$$

are continuous.

Moreover, operators (4.25)–(4.29) are invertible.

Proof. The proof of the continuity of operator (4.23)–(4.29) is word for word of the proofs of the similar theorems in [1, 23, 65], and [69].

First, let us prove the invertibility of operator (4.26). Consider the single layer potential constructed by the fundamental matrix Ψ,

$$U(x) = (u,\varphi)^\top = V^{(0)}(g)(x) = \int\limits_S \Psi(x-y)g(y)\,dS_y, \quad x \in \Omega^\pm,$$

and the corresponding boundary integral operator

$$\mathcal{H}^{(0)}g(x) := \big\{V^{(0)}(g)(x)\big\}^\pm = \int\limits_S \Psi(x-y)g(y)\,dS_y, \quad x \in S.$$

Evidently, the operator

$$\mathcal{H}-\mathcal{H}^{(0)} : \big[H_2^{-\frac{1}{2}}(S)\big]^4 \to \big[H_2^{\frac{1}{2}}(S)\big]^4 \tag{4.30}$$

is compact due to (4.1). Actually, the range of the operator $\mathcal{H}-\mathcal{H}^{(0)}$ is in $[H_2^1(S)]^4$ and the compactness of (4.30) follows from the well-known Rellich–Kondrachov compact embedding theorem.

The components of the vector-function $V^{(0)}(g)(x)$ decay at infinity as $\mathcal{O}(|x|^{-1})$, while their first order derivatives decay as $\mathcal{O}(|x|^{-2})$. Therefore, we can write the first Green formula for the vector-functions $U = V^{(0)}(g) = (u,\varphi)^\top$ and $\overline{U}$ for both domains, Ω and Ω^- (cf. (2.8)),

$$\Big\langle \big\{P^{(0)}(\partial,n)V^{(0)}(g)\big\}^+, \overline{\big\{V^{(0)}(g)\big\}^+}\Big\rangle_S = \int\limits_\Omega \widetilde{E}^{(0)}_\omega\big(V^{(0)}(g), \overline{V^{(0)}(g)}\big)\,dx, \tag{4.31}$$

$$-\Big\langle \big\{P^{(0)}(\partial,n)V^{(0)}(g)\big\}^-, \overline{\big\{V^{(0)}(g)\big\}^-}\Big\rangle_S = \int\limits_{\Omega^-} \widetilde{E}^{(0)}_\omega\big(V^{(0)}(g), \overline{V^{(0)}(g)}\big)\,dx, \tag{4.32}$$

where

$$\widetilde{E}^{(0)}_\omega(U,\overline{U}) = E_\omega(u,\overline{u}) + \alpha_1|\nabla\varphi|^2 \tag{4.33}$$

with $E_\omega(u,\overline{u})$ defined by (2.12)–(2.14). Using the properties of the single layer potential, from (4.31), (4.32) we deduce

$$\Big\langle -g, \overline{\mathcal{H}^{(0)}g}\Big\rangle_S = \int\limits_{\Omega\cup\Omega^-} \widetilde{E}^{(0)}_\omega\big(V^{(0)}(g), \overline{V^{(0)}(g)}\big)\,dx,$$

i.e.,

$$\Big\langle -\mathcal{H}^{(0)}g, \overline{g}\Big\rangle_S = \int\limits_{\Omega\cup\Omega^-} \overline{\widetilde{E}^{(0)}_\omega\big(V^{(0)}(g), \overline{V^{(0)}(g)}\big)}\,dx.$$

From (4.33) we have

$$\widetilde{E}^{(0)}_\omega(U,\overline{U}) = E(u,\overline{u};\lambda,\mu) + \alpha|\nabla\varphi|^2 - i\,\omega\Big\{E(u,\overline{u};\lambda^*,\mu^*) + \alpha^*|\nabla\varphi|^2\Big\}, \tag{4.34}$$

where $E(u,\overline{u};\lambda,\mu)$ and $E(u,\overline{u};\lambda^*,\mu^*)$ are given by (2.16) and (2.17), respectively. Therefore, by Corollary 2.1 and Remark 2.1, we derive

$$\operatorname{Re}\big\langle -\mathcal{H}^{(0)}g, \overline{g}\big\rangle_S = \int\limits_{\Omega\cup\Omega^-} \big\{E(u,\overline{u};\lambda,\mu) + \alpha|\nabla\varphi|^2\big\}\,dx \geqslant C_3\Big(\sum_{l,j=1}^3 |e_{lj}(u)|^2 + |\nabla\varphi|^2\Big) \geqslant 0,$$

$$\operatorname{Im}\Big\langle -\mathcal{H}^{(0)}g, \overline{g}\Big\rangle_S = \omega\int\limits_{\Omega\cup\Omega^-} \Big\{E(u,\overline{u};\lambda^*,\mu^*) + \alpha^*|\nabla\varphi|^2\Big\}\,dx \geqslant \omega C_4\Big(\sum_{l,j=1}^3 |e_{lj}(u)|^2 + |\nabla\varphi|^2\Big) \geqslant 0,$$

where C_3 and C_4 are positive constants and $e_{lj}(u) = \frac{1}{2}\,(\partial_l u_j + \partial_j u_l)$.

Since $U = V^{(0)}(g) \in [H^1(\mathbb{R}^3)]^4$, by exactly the same arguments as in the proof of Theorem 4.6 in [10, Section 4.2], we deduce the following coercivity properties:

$$\operatorname{Re}\big\langle -\mathcal{H}^{(0)}g,\overline{g}\big\rangle_S \geqslant C_5\|g\|^2_{[H^{-\frac{1}{2}}(S)]^4},$$
$$\operatorname{Im}\big\langle -\mathcal{H}^{(0)}g,\overline{g}\big\rangle_S \geqslant \omega C_6\|g\|^2_{[H^{-\frac{1}{2}}(S)]^4},$$

with some positive constants C_5 and C_6. Therefore, by the Lax–Milgram theorem, the operator

$$\mathcal{H}^{(0)} : \big[H_2^{-\frac{1}{2}}(S)\big]^4 \to \big[H_2^{\frac{1}{2}}(S)\big]^4$$

is invertible and, consequently, operator (4.25) is Fredholm operator with zero index. It remains to show that the null space of operator (4.25) is trivial.

Let $g \in [H^{-\frac{1}{2}}(S)]^4$ be a solution of the homogeneous equation $\mathcal{H}g = 0$ on S. Then the vector-function $U(x) = V(g)(x)$ belongs to the spaces $[H_2^1(\Omega]^4$ and $[H_2^1(\Omega^-)]^4$ and is a solution of the homogeneous interior and exterior Dirichlet type BVPs:

$$A(\partial_x,\omega)U(x) = 0 \ \text{ in } \Omega,$$

and

$$A(\partial_x,\omega)U(x) = 0 \ \text{ in } \Omega^-,$$
$$\big\{U(x)\big\}^- = 0 \ \text{ on } S.$$

As in the proof of Theorem 3.1, with the help of the corresponding Green identities, we can show that these BVPs possess only the trivial solution, i.e., $U(x) = V(g)(x) = 0$ for $x \in \Omega\cup\Omega^-$, implying $g = \{P(\partial,n,\omega)V(g)\}^- - \{P(\partial,n,\omega)V(g)\}^- = 0$ on S. Consequently, operator (4.25) is invertible.

Now, we prove the invertibility of operator (4.29).

To this end, let us recall that the operator $\mathcal{L}^{(0)}$ is defined by the relation

$$\mathcal{L}^{(0)}h(x) := \big\{P^{(0)}(\partial_x,n(x))W^{(0)}(h)(x)\big\}^{\pm}$$
$$= \lim_{\Omega^\pm\ni z\to x} P^{(0)}(\partial_z,n(x))\int\limits_S \big[P^{(0)}(\partial_y,n(y))\Psi(z-y)\big]^\top h(y)\,dS_y, \ \ x\in S.$$

Evidently, $\mathcal{L}^{(0)}$ is a principal singular part of the operator $\mathcal{L}$, implying that the operator

$$\mathcal{L}-\mathcal{L}^{(0)} : \big[H_2^{\frac{1}{2}}(S)\big]^4 \to \big[H_2^{-\frac{1}{2}}(S)\big]^4 \tag{4.35}$$

is compact due to relations (4.1). Indeed, since the boundary S is a Lipschitz manifold, the unite normal vector n is an essentially bounded vector-function with respect to $x\in S$ and the range of the operator $\mathcal{L}-\mathcal{L}^{(0)}$ is in $[L_2(S)]^4$. Therefore, as in the previous case, the compactness of (4.35) follows from the Rellich–Kondrachov compact embedding theorem.

Further, we write two Green formulas of type (4.31), (4.32) with the double layer potential $U = (u,\varphi)^\top = W^{(0)}(g)$ in the place of a single layer potential $V^{(0)}(g)$. By adding these two relations and using the properties of double layer potential, we obtain

$$\big\langle\mathcal{L}^{(0)}g,\overline{g}\big\rangle_S = \int\limits_{\Omega\cup\Omega^-} \widetilde{E}^{(0)}_\omega\big(W^{(0)}(g),\overline{W^{(0)}(g)}\big)\,dx, \tag{4.36}$$

where $\widetilde{E}^{(0)}_\omega(W^{(0)}(U),\overline{U})$ is given by (4.34) with $U = (u,\varphi)^\top = W^{(0)}(g)$.

From (4.36), by Corollary 2.1 and Remark 2.1, we get

$$\operatorname{Re}\big\langle\mathcal{L}^{(0)}g,\overline{g}\big\rangle_S = \int\limits_{\Omega\cup\Omega^-} \big\{E(u,\overline{u};\lambda,\mu)+\alpha|\nabla\varphi|^2\big\}\,dx \geqslant C_7\Big(\sum_{l,j=1}^3 |e_{lj}(u)|^2 + |\nabla\varphi|^2\Big) \geqslant 0, \tag{4.37}$$

$$\operatorname{Im}\big\langle -\mathcal{L}^{(0)}g,\overline{g}\big\rangle_S=\omega\int\limits_{\Omega\cup\Omega^-}\Big\{E(u,\overline{u};\lambda^*,\mu^*)+\alpha^*|\nabla\varphi|^2\Big\}\,dx\geqslant\omega C_8\Big(\sum_{l,j=1}^{3}|e_{lj}(u)|^2+|\nabla\varphi|^2\Big)\geqslant 0, \quad (4.38)$$

where C_7 and C_8 are positive constants and $e_{lj}(u)=\frac{1}{2}\,(\partial_l u_j+\partial_j u_l)$.

Since $U=(u,\varphi)^\top=W^{(0)}(g)\in[H^1(\Omega^\pm)]^4$, by the same arguments as in [10, Section 5.5], one can derive the following Gårding type inequalities:

$$\operatorname{Re}\big\langle\mathcal{L}^{(0)}g,\overline{g}\big\rangle_S\geqslant C_9\|g\|^2_{[H^{\frac{1}{2}}(S)]^4}-C_{10}\|g\|^2_{[H^{-\frac{1}{2}}(S)]^4},$$
$$\operatorname{Im}\big\langle -\mathcal{L}^{(0)}g,\overline{g}\big\rangle_S\geqslant\omega C_{11}\|g\|^2_{[H^{\frac{1}{2}}(S)]^4}-C_{12}\|g\|^2_{[H^{-\frac{1}{2}}(S)]^4},$$

with some positive constants C_9, C_{10}, C_{11}, and C_{12}. Therefore, the operator

$$\mathcal{L}^{(0)}:\big[H_2^{\frac{1}{2}}(S)\big]^4\to\big[H_2^{-\frac{1}{2}}(S)\big]^4$$

is a Fredholm operator with zero index (see, e.g., [65, Theorem 2.34]. Then operator (4.29) is also a Fredholm operator with zero index due to the compactness of operator (4.35).

Further, we show that the corresponding null space is trivial. To this end, let $g\in[H^{\frac{1}{2}}(S)]^4$ be a solution of the homogeneous equation $\mathcal{L}g=0$ on S. Then the vector-function $U(x)=W(g)(x)$ belongs to the spaces $[H_2^1(\Omega)]^4$ and $[H_2^1(\Omega^-)]^4$ and is a solution of the homogeneous interior and exterior Neumann type BVPs:

$$A(\partial_x,\omega)U(x)=0\ \text{ in } \Omega,$$
$$\big\{P(\partial,n)U(x)\big\}^+=0\ \text{ on } S,$$

and

$$A(\partial_x,\omega)U(x)=0\ \text{ in } \Omega^-,$$
$$\big\{P(\partial,n)U(x)\big\}^-=0\ \text{ on } S.$$

As in the proof of Theorem 3.1, we can show that these BVPs possess only the trivial solution, i.e., $U(x)=W(g)(x)=0$ for $x\in\Omega\cup\Omega^-$, implying $g=\{W(g)\}^+-\{W(g)\}^-=0$ on S. Consequently, operator (4.29) is invertible.

In the proof of the invertibility of operators (4.27) and (4.28), the crucial point is that they are the Calderón–Zygmund type singular integral operators (see, e.g., [61, 65]). The null spaces of these operators and their adjoint operators have the trivial null spaces, which follows from the fact that the interior and exterior Dirichlet and Neumann type homogeneous boundary value problems for the differential operators $A(\partial,\omega)$ and $A^*(\partial,\omega)$ possess only the trivial solutions due to the Green formulas (2.8) and (2.9) and Lemma 2.1. Consequently, operators (4.27) and (4.28) are invertible. □

The next assertion is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without a boundary (see, e.g., [1–3, 8, 10, 38–40, 42, 52, 53, 74, 89] and the references therein).

Theorem 4.6. *Let $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$, and let $S\in C^\infty$. Then the layer potential operators (4.10) and the boundary operators (4.19)–(4.21) can be extended to the following continuous operators:*

$$V:\big[B^s_{p,p}(S)\big]^4\to\big[H_p^{s+1+\frac{1}{p}}(\Omega^\pm)\big]^4\quad\Big[\big[B^s_{p,q}(S)\big]^4\to\big[B_{p,q}^{s+1+\frac{1}{p}}(\Omega^\pm)\big]^4\Big], \quad (4.39)$$
$$W:\big[B^s_{p,p}(S)\big]^4\to\big[H_p^{s+\frac{1}{p}}(\Omega^\pm)\big]^4\quad\Big[\big[B^s_{p,q}(S)\big]^4\to\big[B_{p,q}^{s+\frac{1}{p}}(\Omega^\pm)\big]^4\Big], \quad (4.40)$$
$$\mathcal{H}:\big[H^s_p(S)\big]^4\to\big[H_p^{s+1}(S)\big]^4\quad\Big[\big[B^s_{p,q}(S)\big]^4\to\big[B_{p,q}^{s+1}(S)\big]^4\Big], \quad (4.41)$$
$$\mathcal{H}^{(0)}:\big[H^s_p(S)\big]^4\to\big[H_p^{s+1}(S)\big]^4\quad\Big[\big[B^s_{p,q}(S)\big]^4\to\big[B_{p,q}^{s+1}(S)\big]^4\Big], \quad (4.42)$$

$$\pm\frac{1}{2}\,I_4+\mathcal{K}:\big[H^s_p(S)\big]^4\to\big[H^s_p(S)\big]^4\quad\Big[\,\big[B^s_{p,q}(S)\big]^4\to\big[B^s_{p,q}(S)\big]^4\,\Big],\tag{4.43}$$

$$\pm\frac{1}{2}\,I_4+\widetilde{\mathcal{K}}:\big[H^s_p(S)\big]^4\to\big[H^s_p(S)\big]^4\quad\Big[\,\big[B^s_{p,q}(S)\big]^4\to\big[B^s_{p,q}(S)\big]^4\,\Big],\tag{4.44}$$

$$\mathcal{L}:\big[H^{s+1}_p(S)\big]^4\to\big[H^s_p(S)\big]^4\quad\Big[\,\big[B^{s+1}_{p,q}(S)\big]^4\to\big[B^s_{p,q}(S)\big]^4\,\Big].\tag{4.45}$$

Operators (4.41)–(4.45) are invertible.

The jump relations (4.11)–(4.14) and formulas (4.22) remain valid in the appropriate function spaces. In particular,

(i) *if $g\in[B^{-\frac{1}{p}}_{p,q}(S)]^4$, relation (4.11) is valid in the sense of the space $[B^{1-\frac{1}{p}}_{p,q}(S)]^4$, while relations (4.12) are understood in the sense of the space $[B^{-\frac{1}{p}}_{p,q}(S)]^4$;*

(ii) *if $g,h\in[B^{1-\frac{1}{p}}_{p,q}(S)]^4$, then relation (4.13) is valid in the sense of the space $[B^{1-\frac{1}{p}}_{p,q}(S)]^4$, while relation (4.14) is understood in the sense of the space $[B^{-\frac{1}{p}}_{p,q}(S)]^4$.*

Proof. Mapping properties (4.39) and (4.40) can be shown by word for word arguments applied in the proofs of the similar theorems in [8, 37–39], and [40].

Further, in accordance with Appendix C and Theorem 4.5, due to the properties of their principal homogeneous symbols, operators (4.41), (4.42), and (4.45) are strongly elliptic pseudodifferential operators respectively of order -1, -1 and $+1$ with trivial null-spaces, while operators (4.43) and (4.44) are the Calderón–Zygmund type singular integral operators of normal type, i.e., they are pseudodifferential operators of zero order with non-degenerate symbol matrices and with trivial null spaces. Therefore, the invertibility of operators (4.25)–(4.29) for particular values of parameters stated in Theorem 4.7 implies their invertibility for all $s\in\mathbb{R}$, $p>1$, and $q>1$ if $S\in C^\infty$. □

Corollary 4.2. *Let $S\in C^\infty$ and $p>1$. Then the Dirichlet problem*

$$A(\partial_x,\omega)U(x)=0\ \ in\ \Omega,\quad U=(u,\varphi)^\top\in\big[H^1_p(\Omega)\big]^4=\big[W^1_p(\Omega)\big]^4,$$
$$\big\{U(x)\big\}^+=\widetilde{f}(x)\ \ on\ S,\ \ \widetilde{f}\in\big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^4,$$

is uniquely solvable for arbitrary $\widetilde{f}$ and the solution can be represented by the single layer potential $U(x)=V(\mathcal{H}^{-1}\widetilde{f})(x)$, where the operator

$$\mathcal{H}^{-1}:\big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^4\to\big[B^{-\frac{1}{p}}_{p,p}(S)\big]^4$$

is the inverse of the operator

$$\mathcal{H}:\big[B^{-\frac{1}{p}}_{p,p}(S)\big]^4\to\big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^4.\tag{4.46}$$

For the Lipschitz domains, the result holds true for $p=2$.

Proof. The proof follows from the fact that the invertibility of operator (4.46) for $p=2$ implies its invertibility for arbitrary $p>1$ due to the general theory of pseudodifferential equations on smooth closed manifolds. In turn, this leads to the unique solvability of the Dirichlet problem in the space $[H^1_p(\Omega)]^4=[W^1_p(\Omega)]^4$. □

Further, we describe the mapping properties of the Newtonian volume potential.

First of all, let us note that the Newtonian volume potential defined by (4.4) with a compactly supported density decays exponentially at infinity.

Using the same arguments as in [51, 61, 68], and [5], one can show that if $S=\partial\Omega\in C^{2,\alpha}$, then the volume potential operator N_Ω possesses the following mapping properties:

$$N_\Omega:\big[L_\infty(\Omega)\big]^4\to\big[C^{1,\gamma}(\mathbb{R}^3)\big]^4\ \text{ for all }\ 0<\gamma<1,$$
$$N_\Omega:\big[C^{0,\beta}(\overline{\Omega})\big]^4\to\big[C^{2,\beta}(\overline{\Omega})\big]^4,\ \ 0<\beta<1,$$

$$N_\Omega : \big[C^{1,\beta}(\overline{\Omega})\big]^4 \to \big[C^{3,\beta}(\overline{\Omega})\big]^4, \ \ 0 < \beta < \alpha \leqslant 1.$$

Further, let $h \in [L_{p,comp}(\mathbb{R}^3)]^4$ with $p > 1$. Then (cf. [61, Chapter 5, Section 10], [68, Chapter 11])

$$N_{\mathbb{R}^3}(h) \in [W^2_{p,loc}(\mathbb{R}^3)]^4, \ \ A(\partial,\omega)N_{\mathbb{R}^3}(h) = h \ \text{ almost everywhere in } \mathbb{R}^3.$$

Moreover, the following assertion holds.

Theorem 4.7. *Let Ω be a bounded open three–dimensional region of $\mathbb{R}^3$ with a simply connected, closed, infinitely smooth boundary $S = \partial\Omega$, and $1 < p, q < \infty$. The following operators are continuous:*

$$N_\Omega : \big[\widetilde{H}^s_p(\Omega)\big]^4 \to \big[H^{s+2}_p(\Omega)\big]^4 \quad \Big[\,\big[\widetilde{B}^s_{p,q}(\Omega)\big]^4 \to \big[B^{s+2}_{p,q}(\Omega)\big]^4\,\Big], \ \ s \in \mathbb{R}, \tag{4.47}$$

$$N_\Omega : \big[H^s_p(\Omega)\big]^4 \to \big[H^{s+2}_p(\Omega)\big]^4 \quad \Big[\,\big[B^s_{p,q}(\Omega)\big]^4 \to \big[B^{s+2}_{p,q}(\Omega)\big]^4\,\Big], \ \ s > -1 + \frac{1}{p}\,. \tag{4.48}$$

Proof. Since the Newtonian operator is a pseudodifferential operator with a rational symbol, the mapping properties (4.47) and (4.48) can be shown by using exactly the same arguments applied in the proof of Theorem 3.8 in [19]). □

Remark 4.3. Note that $\mathcal{K}^{(0)}$ and $\widetilde{\mathcal{K}}^{(0)}$ are mutually adjoint operators without complex conjugation, while the operators $\mathcal{H}^{(0)}$ and $\mathcal{L}^{(0)}$ are symmetric operators.

5 Existence and regularity theorems

In this section, for the sake of simplicity, we assume that the boundary surface S is sufficiently smooth, say $S \in C^\infty$. We use the potential method and the theory of pseudodifferential equations on manifolds with a boundary and prove the existence and regularity theorems for the mixed boundary value problem (3.1)–(3.3), provided inclusions (3.4)–(3.6) are satisfied. Since the Newtonian potential $N_\Omega(\Phi)$ is a particular solution of the nonhomogeneous equation (3.1), in what follows, without loss of generality, we consider the homogeneous equation (3.1), i.e., we assume that $\Phi = 0$. Therefore, the mixed problem under consideration reads as follows: Find a complex-valued vector-function $U = (u,\varphi)^\top \in [H^1_p(\Omega)]^4 = [W^1_p(\Omega)]^4$, $p > 1$, satisfying the relations:

$$A(\partial_x,\omega)U(x) = 0 \ \text{ in } \Omega, \tag{5.1}$$

$$\big\{U(x)\big\}^+ = f(x) \ \text{ on } S_D, \tag{5.2}$$

$$\big\{P(\partial_x, n(x))U(x)\big\}^+ = F(x) \ \text{ on } S_N, \tag{5.3}$$

where

$$f = (f_1,f_2,f_3,f_4)^\top \in \big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^4, \tag{5.4}$$

$$F = (F_1,F_2,F_3,F_4)^\top \in \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^4. \tag{5.5}$$

Denote by $f^{(e)}$ a fixed extension of the vector-function f from S_D onto the whole S preserving the functional space,

$$f^{(e)} \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^4, \quad r_{_{S_D}} f^{(e)} = f \ \text{ on } S_D. \tag{5.6}$$

Recall that $r_{_\mathcal{M}}$ denotes the restriction operator onto $\mathcal{M}$. If $f = 0$ on S_D, we always choose in the role of a fixed extension the zero function $f^{(e)} = 0$ on S.

Evidently, an arbitrary extension $\widetilde{f}$ of f onto the whole S, which preserves the function space, can be then represented as

$$\widetilde{f} = f^{(e)} + \widetilde{g} \ \text{ with } \ \widetilde{g} \in \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^4. \tag{5.7}$$

In accordance with Corollary 4.2, we can seek a solution in the form

$$U = V\big(\mathcal{H}^{-1}(f^{(e)} + \widetilde{g})\big),$$

where $\widetilde{g} \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)]^4$ is an unknown vector-function and $\mathcal{H}^{-1}$ is a pseudodifferential operator, inverse to the operator $\mathcal{H} : [B_{p,p}^{-\frac{1}{p}}(S)]^4 \to [B_{p,p}^{1-\frac{1}{p}}(S)]^4$, i.e.,

$$\mathcal{H}^{-1} : \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^4 \to \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^4.$$

In view of (5.6), (5.7), and Theorem 4.6, the vector-function $U = V(\mathcal{H}^{-1}(f^{(e)} + \widetilde{g}))$ belongs to the space $[H_p^1(\Omega)]^4 = [W_p^1(\Omega)]^4$ and the differential equation (5.1) and the Dirichlet condition (5.2) on S_D are satisfied automatically. It remains to satisfy the Neumann condition (5.3) on S_N, which leads to the following pseudodifferential equation for the unknown vector-function $\widetilde{g}$:

$$r_{S_N}\Big(-\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}}\Big)\mathcal{H}^{-1}(f^{(e)} + \widetilde{g}) = F \ \text{ on } S_N. \tag{5.8}$$

Let us introduce the Steklov–Poincaré type operator

$$\mathcal{A} = \Big(-\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}}\Big)\mathcal{H}^{-1},$$

which is a pseudodifferential operator of order 1 and has the following mapping property:

$$\mathcal{A} \ : \ \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^4 \to \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^4,$$

due to Theorem 4.6. Denote

$$F^{(0)} := F - r_{S_N}\mathcal{A}f^{(e)} \in \big[B_{p,p}^{-\frac{1}{p}}(S_N)\big]^4$$

and rewrite equation (5.8) as

$$r_{S_N}\mathcal{A}\widetilde{g} = F^{(0)} \ \text{ on } S_N, \tag{5.9}$$

which is a pseudodifferential equation on the submanifold S_N with the boundary ∂S_N. We would like to investigate the solvability of equation (5.9) in the appropriate function spaces.

To this end, we first prove the following assertion.

Theorem 5.1. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $S \in C^\infty$. Then the operators*

$$\mathcal{A} : \big[H_p^{s+1}(S)\big]^4 \to \big[H_p^{s}(S)\big]^4, \tag{5.10}$$

$$\mathcal{A} : \big[B_{p,q}^{s+1}(S)\big]^4 \to \big[B_{p,q}^{s}(S)\big]^4 \tag{5.11}$$

are invertible.

The principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{A};x,\xi')$ of the operator $\mathcal{A}$ is strongly elliptic and the following inequalities hold:

$$\operatorname{Re}\big[\mathfrak{S}(\mathcal{A};x,\xi')\eta\cdot\overline{\eta}\big] \ge \delta_1|\xi'|\,|\eta|^2 \ \textit{ for all } \ x \in S, \ \ \xi' \in \mathbb{R}^2 \setminus \{0\}, \ \ \eta \in \mathbb{C}^4, \tag{5.12}$$

$$-\operatorname{Im}\big[\mathfrak{S}(\mathcal{A};x,\xi')\eta\cdot\overline{\eta}\big] \ge \delta_2|\xi'|\,|\eta|^2 \ \textit{ for all } \ x \in S, \ \ \xi' \in \mathbb{R}^2 \setminus \{0\}, \ \ \eta \in \mathbb{C}^4, \tag{5.13}$$

with the positive constants δ_1 and δ_2.

Proof. The invertibility of operators (5.10) and (5.11) follows from Theorem 4.8, since they are the compositions of invertible operators.

To prove inequalities eqref5.16 and (5.13), let us note that the principal homogeneous symbol matrices of the operators $\mathcal{A} = (-\frac{1}{2}\,I_6 + \widetilde{\mathcal{K}})\mathcal{H}^{-1}$ and $\mathcal{A}^{(0)} = (-\frac{1}{2}\,I_6 + \widetilde{\mathcal{K}}^{(0)})[\mathcal{H}^{(0)}]^{-1}$ are the same, since the difference $\mathcal{A} - \mathcal{A}^{(0)}$ is a compact operator between the spaces shown in (5.10) and (5.11) in view of relations (4.1).

Due to the local principal technique, it suffices to prove inequalities (5.12) and (5.13) in the case of a half-space with the outward unit normal vector $n = (n_!, n_2, n_3)$ to the boundary plane.

Without loss of generality, let us consider the upper half-space domain $\Omega = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : \ x_3 > 0\}$ with the boundary $S = \mathbb{R}^2$. Let $n = (0,0,-1)$ be the outward-oriented unit normal vector to the boundary $S = \mathbb{R}^2$.

Let $\mathcal{F}_{x'\to\xi'}$ and $\mathcal{F}^{-1}_{\xi'\to x'}$ denote the direct and inverse generalized two-dimensional Fourier transforms in the space of tempered distributions (Schwartz space $\mathcal{S}'(\mathbb{R}^2)$), which for regular summable functions f and g read as follows:

$$\mathcal{F}_{x'\to\xi'}[f]=\widehat{f}(\xi')=\int\limits_{\mathbb{R}^2} f(x')e^{ix'\cdot\xi'}\,dx',\quad \mathcal{F}^{-1}_{\xi'\to x'}[g]=\frac{1}{(2\pi)^2}\int\limits_{\mathbb{R}^2} g(\xi')e^{-ix'\cdot\xi'}\,d\xi',\ \ x',\xi'\in\mathbb{R}^2.$$

Assume that g belongs to the Schwartz space of rapidly decreasing at infinity functions, $g\in[\mathcal{S}(\mathbb{R}^2)]^4$, and consider the single layer potential with the integration surface $S=\mathbb{R}^2$,

$$\begin{aligned}V^{(0)}\big([\mathcal{H}^{(0)}]^{-1}g\big)(x)&=\int\limits_{\mathbb{R}^2}\Psi(x-y)\big([\mathcal{H}^{(0)}]^{-1}g(y')\big)\,dy'\\&=\int\limits_{\mathbb{R}^2}\mathcal{F}^{-1}_{\xi'\to(x'-y')}\big[\Pi(\xi',x_3)\big]\mathcal{F}^{-1}_{\xi'\to y'}\big[\mathfrak{S}(\mathcal{H}^{(0)};\xi')\widehat{g}(\xi')\big]\,dy',\end{aligned}\tag{5.14}$$

where $x=(x',x_3)\in\mathbb{R}^3_+$, $x'=(x_1,x_2)\in\mathbb{R}^2$, $y=(y',0)$, $y'=(y_1,y_2)\in\mathbb{R}^2$, $[\mathcal{H}^{(0)}]^{-1}$ is a pseudodifferential operator with the homogeneous symbol matrix $[\mathfrak{S}(\mathcal{H}^{(0)};\xi')]^{-1}=[\mathfrak{S}(\mathcal{H};\xi')]^{-1}$ of order $+1$, and the matrix $\Pi(\xi',x_3)$ is introduced in Appendix C (see (C.1)),

$$\Pi(\xi',x_3)=\frac{1}{2\pi}\int\limits_{\mathbb{R}^1}\big\{A^{(0)}(-i\xi,\omega)\big\}^{-1}e^{-ix_3\xi_3}\,d\xi_3=\frac{1}{2\pi}\int\limits_{\ell^-}\big[A^{(0)}(-i\xi,\omega)\big]^{-1}e^{-ix_3\xi_3}\,d\xi_3.$$

Note that the Fourier transform of g again belongs to the Schwartz space, $\widehat{g}\in[\mathcal{S}(\mathbb{R}^2)]^4$.

From (5.14) we have

$$P^{(0)}(\partial_x,n)V^{(0)}\big([\mathcal{H}^{(0)}]^{-1}g\big)(x)=\int\limits_{\mathbb{R}^2}\big[P^{(0)}(\partial_x,n)\Psi(x-y)\big]\big([\mathcal{H}^{(0)}]^{-1}g(y')\big)\,dy',\tag{5.15}$$

implying the relation for the one-sided limit on $S=\mathbb{R}^2$ from $\Omega=\mathbb{R}^3_+$,

$$\Big\{P^{(0)}(\partial_x,n)V^{(0)}\big([\mathcal{H}^{(0)}]^{-1}g\big)(x)\Big\}^+=\Big(-\frac{1}{2}\,I_4+\widetilde{\mathcal{K}}^{(0)}\Big)\big[\mathcal{H}^{(0)}\big]^{-1}g(x')=\mathcal{A}^{(0)}g(x').\tag{5.16}$$

With the help of equalities (5.14)–(5.16), we deduce

$$\mathcal{A}^{(0)}g(x')=\int\limits_{\mathbb{R}^2}\mathcal{F}^{-1}_{\xi'\to(x'-y')}\big[\widetilde{\Pi}(\xi',0)\big]\mathcal{F}^{-1}_{\xi'\to y'}\big[\mathfrak{S}\big([\mathcal{H}^{(0)}]^{-1};\xi'\big)\widehat{g}(\xi')\big]\,dy',\tag{5.17}$$

where $\widetilde{\Pi}(\xi',0)$ is the symbol matrix of the operator $-\frac{1}{2}\,I_4+\widetilde{\mathcal{K}}^{(0)}$ (cf. (C.3)),

$$\widetilde{\Pi}(\xi',0)=\frac{1}{2\pi}\int\limits_{\ell^-}P^{(0)}(-i\xi,n)\big[A^{(0)}(-i\xi,\omega)\big]^{-1}\,d\xi_3=\frac{i}{2\pi}\int\limits_{\ell^-}P^{(0)}(\xi,n)\big[A^{(0)}(\xi,\omega)\big]^{-1}\,d\xi_3.\tag{5.18}$$

Taking into consideration that (5.17) is a convolution type operator, we find

$$\mathcal{F}^{-1}_{x'\to\xi'}\big[\mathcal{A}^{(0)}g(x')\big]=\widetilde{\Pi}(\xi',0)\mathfrak{S}\big([\mathcal{H}^{(0)}]^{-1};\xi'\big)\widehat{g}(\xi').$$

Therefore, the symbol matrix of the operator $\mathcal{A}^{(0)}$ reads as

$$\mathfrak{S}(\mathcal{A}^{(0)};\xi')=\widetilde{\Pi}(\xi',0)\mathfrak{S}\big([\mathcal{H}^{(0)}]^{-1};\xi'\big).\tag{5.19}$$

Now, we show that the matrix defined by (5.19) meets conditions (5.12) and (5.13). To this end, let us consider the Neumann type boundary value problem for the system of ordinary differential equations on the unbounded interval $(0,+\infty)$:

$$A^{(0)}(-i\xi',\partial_{x_3},\omega)\widehat{U}(x_3)=0\ \text{ for }\ x_3>0,\ \ \widehat{U}=(\widehat{u},\widehat{\varphi})^\top,\tag{5.20}$$

$$\big\{P^{(0)}(-i\xi',\partial_{x_3},n)\widehat{U}(x_3)\big\}^+_{x_3=0}=\widehat{F}(\xi')\ \text{ for }\ x_3=0,\qquad\qquad (5.21)$$
$$\lim_{x_3\to+\infty}\widehat{U}(x_3)=0,$$

where $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ is a parameter, $n=(0,0,-1)$, and $F\in\mathbb{C}^4$ is a given vector-function of the parameter ξ'.

It is evident that problem (5.20), (5.21) is obtained by the Fourier transform $\mathcal{F}_{x'\to\xi'}$ from the counterpart Neumann type boundary value problem for the partial differential equation $A^{(0)}(\partial_x,\omega)U(x)=0$ in $\mathbb{R}^3_+$ with the boundary condition $\{P^{(0)}(\partial_x,n)U(x)\}^+=F$ on $\mathbb{R}^2$.

Using the Cauchy theorem for analytic functions, it can easily be checked that the columns of the matrix $\Pi(\xi',x_3)$ are linearly independent solutions of equation (5.20), since $\Pi(\xi',0)$ is a non-degenerate matrix. The matrix $\Pi(\xi',0)$ is the principal homogenous symbol of the operator $\mathcal{H}^{(0)}$ (see (C.2)). The components of these solutions decay exponentially at infinity due to the Cauchy residue theorem. Therefore, any solution of the problem can be represented in the form

$$\widehat{U}(x_3)=\Pi(\xi',x_3)C,$$

where $C\in\mathbb{C}^4$ is an arbitrary complex constant vector with respect to x_3, it may depend on ξ'. In particular, since the matrix $\mathfrak{S}([\mathcal{H}^{(0)}]^{-1};\xi')$ is non-degenerate, we can look for a solution to problem (5.20), (5.21) in the form

$$\widehat{U}(x_3)=\Pi(\xi',x_3)\mathfrak{S}([\mathcal{H}^{(0)}]^{-1};\xi')C,$$

where the unknown vector $C\in\mathbb{C}^4$ is to be determined from the following system of linear algebraic equations due to the boundary condition (5.21):

$$\widetilde{\Pi}(\xi',0)\mathfrak{S}([\mathcal{H}^{(0)}]^{-1};\xi')C=F,$$

where $\widetilde{\Pi}(\xi',0)$ is defined in (5.18).

In view of (5.19), this equation can be rewritten as

$$\mathfrak{S}(\mathcal{A}^{(0)};\xi')C=F.\qquad\qquad (5.22)$$

Further, we show that the determinant of the matrix $\mathfrak{S}(\mathcal{A}^{(0)};\xi')$ is different from zero.

For this purpose, we write the Green type identity associated with the operator $A^{(0)}(\partial_x,\omega)$, which is a counterpart of formula (4.31) and reads as follows:

$$\int\limits_0^\infty \mathcal{A}^{(0)}(-i\xi',\partial_{x_3})\widehat{U}\cdot\overline{\widehat{W}}\,dx=\big\{P^{(0)}(-i\xi',\partial_{x_3},n)\widehat{U}\cdot\overline{\widehat{W}}\big\}^+_{x_3=0}-\int\limits_0^\infty \mathcal{E}(\widehat{U},\overline{\widehat{W}})\,dx_3,\qquad\qquad (5.23)$$

where we assume that the improper integrals involved in (5.23) are convergent, $\widehat{U}=(\widehat{u},\widehat{\varphi})^\top$, $\widehat{W}=(\widehat{w},\widehat{\psi})^\top$, and $\mathcal{E}(\widehat{U},\overline{\widehat{W}})$ has the form (cf. (2.13), (2.14))

$$\begin{aligned}\mathcal{E}(\widehat{U},\overline{\widehat{W}})=\lambda\sum_{j=1}^3\varepsilon_{jj}(\widehat{u})\sum_{j=1}^3\overline{\varepsilon_{jj}(\widehat{w})}+2\mu\sum_{l,j=1}^3\varepsilon_{lj}(\widehat{u})\overline{\varepsilon_{lj}(\widehat{w})}+\alpha\Big(|\xi'|^2\widehat{\varphi}\overline{\widehat{\psi}}+\partial_{x_3}\widehat{\varphi}\partial_{x_3}\overline{\widehat{\psi}}\Big)\\ -i\omega\Big[\lambda^*\sum_{j=1}^3\varepsilon_{jj}(\widehat{u})\sum_{j=1}^3\overline{\varepsilon_{jj}(\widehat{w})}+2\mu^*\sum_{l,j=1}^3\varepsilon_{lj}(\widehat{u})\overline{\varepsilon_{lj}(\widehat{w})}+\alpha^*\big(|\xi'|^2\widehat{\varphi}\overline{\widehat{\psi}}+\partial_{x_3}\widehat{\varphi}\partial_{x_3}\overline{\widehat{\psi}}\big)\Big]\end{aligned}$$

with

$$\varepsilon_{lj}(\widehat{v})=\varepsilon_{jl}(\widehat{v})=-\frac{i}{2}\,(\xi_l\widehat{v}_j+\xi_j\widehat{v}_l),\ \ l,j=1,2,\quad \varepsilon_{33}(\widehat{v})=\frac{\partial\widehat{v}_3}{\partial x_3}\,,$$
$$\varepsilon_{j3}(\widehat{v})=\varepsilon_{3j}(\widehat{v})=\frac{1}{2}\Big(\frac{\partial\widehat{v}_j}{\partial x_3}-i\xi_j\widehat{v}_3\Big),\ \ j=1,2.$$

From Green's identity, substituting $\widehat{U}=\widehat{W}=(\widehat{u},\widehat{\varphi})^\top=\Pi(\xi',x_3)\mathfrak{S}([\mathcal{H}^{(0)}]^{-1};\xi')C$ and keeping in mind that $\widehat{U}$ solves equation (5.20), we obtain the relation

$$\big\{P^{(0)}(-i\xi',\partial_{x_3},n)\widehat{U}\cdot\overline{\widehat{U}}\big\}^+_{x_3=0}=\int\limits_0^\infty \mathcal{E}(\widehat{U},\overline{\widehat{U}})\,dx_3, \tag{5.24}$$

where

$$\begin{aligned}\mathcal{E}(\widehat{U},\overline{\widehat{U}})=\lambda\Big|\sum_{j=1}^3\varepsilon_{jj}(\widehat{u})\Big|^2+2\mu\sum_{l,j=1}^3|\varepsilon_{lj}(\widehat{u})|^2+\alpha\big(|\xi'|^2|\widehat{\varphi}|^2+|\partial_{x_3}\widehat{\varphi}|^2\big)\\ -i\omega\Big[\lambda^*\Big|\sum_{j=1}^3\varepsilon_{jj}(\widehat{u})\Big|^2+2\mu^*\sum_{l,j=1}^3|\varepsilon_{lj}(\widehat{u})|^2+\alpha^*\big(|\xi'|^2|\widehat{\varphi}|^2+|\partial_{x_3}\widehat{\varphi}|^2\big)\Big].\end{aligned} \tag{5.25}$$

Let us show that the homogeneous problem (5.20), (5.21) with $\widehat{F}(\xi')=0$ possesses only the trivial solution. We prove it by contradiction. Assume that $\widehat{U}=(\widehat{u},\widehat{\varphi})^\top$ is a nontrivial solution to the homogeneous problem. Then, by Corollary 2.1 and Remark 2.1, from the Green formula (5.24), we deduce $\operatorname{Re}\mathcal{E}(\widehat{U},\overline{\widehat{U}})=0$ and $\operatorname{Im}\mathcal{E}(\widehat{U},\overline{\widehat{U}})=0$ in $[0,+\infty)$, implying

$$\varepsilon_{lj}(\widehat{u})=0,\ \ l,j=1,2,3,\ \ \widehat{\varphi}=0,$$

which, in turn, imply $\widehat{u}=0$ and $\widehat{\varphi}=0$ in the interval $[0,+\infty)$. Thus the homogeneous problem (5.20), (5.21) possesses only the trivial solution. Consequently, the determinant of the matrix $\mathfrak{S}(\mathcal{A}^{(0)};\xi')$ is different from zero and system (5.22) is uniquely solvable.

Now, employing the equalities (cf. (C.1), (C.2), (5.19)):

$$\Pi(\xi',0)=\mathfrak{S}(\mathcal{H}^{(0)};\xi'),\quad \widetilde{\Pi}(\xi',0)\mathfrak{S}([\mathcal{H}^{(0)}]^{-1};\xi')=\mathfrak{S}(\mathcal{A}^{(0)};\xi'),$$

for the vector-function $\widehat{U}(x_3)=\Pi(\xi',x_3)\mathfrak{S}([\mathcal{H}^{(0)}]^{-1};\xi')\eta$ with $\eta\in\mathbb{C}^4\setminus\{0\}$, we have

$$\begin{gathered}\{\widehat{U}\}^+_{x_3}=\big\{\Pi(\xi',x_3)\mathfrak{S}\big([\mathcal{H}^{(0)}]^{-1};\xi'\big)\eta\big\}^+_{x_3=0}=\eta,\\ \big\{P^{(0)}(-i\xi',\partial_{x_3},n)\widehat{U}\big\}^+_{x_3=0}=\widetilde{\Pi}(\xi',0)\mathfrak{S}\big([\mathcal{H}^{(0)}]^{-1};\xi'\big)\eta=\mathfrak{S}(\mathcal{A}^{(0)};\xi')\eta.\end{gathered}$$

Therefore, equality (5.24) can be rewritten as

$$\mathfrak{S}(\mathcal{A}^{(0)};\xi')\eta\cdot\overline{\eta}=\int\limits_0^\infty\mathcal{E}(\widehat{U},\overline{\widehat{U}})\,dx_3. \tag{5.26}$$

Since for nonzero η the real part of the right-hand side expression in (5.26) is strictly positive, while the imaginary part is strictly negative, due to (5.25), we conclude that

$$\begin{gathered}\operatorname{Re}\big[\mathfrak{S}\big(\mathcal{A}^{(0)};\xi'\big)\eta\cdot\overline{\eta}\big]>0,\quad -\operatorname{Im}\big[\mathfrak{S}\big(\mathcal{A}^{(0)};\xi'\big)\eta\cdot\overline{\eta}\big]>0\\ \text{for arbitrary }\ \xi'\in\mathbb{R}^2\setminus\{0\}\ \text{ and for arbitrary }\ \eta\in\mathbb{C}^4\setminus\{0\}.\end{gathered} \tag{5.27}$$

Thus, $\mathcal{A}^{(0)}$ is a strongly elliptic pseudodifferential operator.

Since $\mathfrak{S}(\mathcal{A}^{(0)};\xi')$ is a homogeneous function of order $+1$ in ξ', inequalities (5.12) and (5.13) follow from (5.27) with the constants δ_1 and δ_2 defined by the relations

$$\begin{aligned}\delta_1&=\inf\big\{\operatorname{Re}\big[\mathfrak{S}(\mathcal{A}^{(0)};\xi')\eta\cdot\overline{\eta}\big]\big\}>0,\ \ |\xi'|=1,\ \ |\eta|=1,\\ \delta_2&=\inf\big\{-\operatorname{Im}\big[\mathfrak{S}(\mathcal{A}^{(0)};\xi')\eta\cdot\overline{\eta}\big]\big\}>0,\ \ |\xi'|=1,\ \ |\eta|=1.\end{aligned}$$

This completes the proof. □

Remark 5.1. By the same arguments, one can show that the Steklov–Poincaré type operator associated with the exterior unbounded domain Ω^-

$$\mathcal{A}^{(-)} = -\Big(\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}}\Big)\mathcal{H}^{-1}$$

and its principal homogeneous symbol matrix have exactly the same properties stated in Theorem 5.1 for the operator $\mathcal{A}$.

Remark 5.2. Using identities (4.22), we can show that

$$\mathcal{A} = \Big(-\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}}\Big)\mathcal{H}^{-1} = \mathcal{H}^{-1}\Big(-\frac{1}{2}\,I_4 + \mathcal{K}\Big). \tag{5.28}$$

From Theorem 5.1, Remark 5.1, and relations (5.28), (C.6), (C.7), it follows that the principal homogeneous symbols of the operators $\pm\frac{1}{2}\,I_4 + \ \mathcal{K}$ and $\pm\frac{1}{2}\,I_4 + \widetilde{\mathcal{K}}$ are non-degenerate.

In our analysis we need the following auxiliary lemmas (cf. [10]).

Lemma 5.1. *If $\lambda_j(x)$, $j = 1, 2, 3, 4$, are the eigenvalues of the matrix*

$$M = \big[\mathfrak{S}(\mathcal{A}^{(0)}; x, 0, +1)\big]^{-1}\mathfrak{S}(\mathcal{A}^{(0)}; x, 0, -1), \quad x \in S = \partial\Omega, \tag{5.29}$$

then $\operatorname{Re}\lambda_j(x) > 0$ *and*

$$-\frac{1}{4} < \delta_j(x) = \operatorname{Re}\big[(2\pi i)^{-1}\ln\lambda_j(x)\big] = \frac{\arg\lambda_j(x)}{2\pi} < \frac{1}{4} \ \ \textit{for all} \ \ x \in S, \tag{5.30}$$

where $\ln\zeta$ *denotes the branch of the logarithm analytic in the complex plane cut along* $(-\infty, 0]$.

Proof. Let λ be an eigenvalue of matrix (5.29) and let $\eta \in \mathbb{C}^4 \setminus \{0\}$ be the corresponding eigenvector,

$$M\eta = \lambda\eta, \ \text{ i.e., } \ \mathfrak{S}(\mathcal{A}^{(0)}; x, 0, -1)\eta = \lambda\mathfrak{S}(\mathcal{A}^{(0)}; x, 0, +1)\eta. \tag{5.31}$$

Denote

$$\begin{aligned} \operatorname{Re}\mathfrak{S}(\mathcal{A}^{(0)}; x, 0, -1) = M^{(1)}, &\quad -\operatorname{Im}\mathfrak{S}(\mathcal{A}^{(0)}; x, 0, -1) = M^{(2)},\\ \operatorname{Re}\mathfrak{S}(\mathcal{A}^{(0)}; x, 0, 1) = M^{(3)}, &\quad -\operatorname{Im}\mathfrak{S}(\mathcal{A}^{(0)}; x, 0, 1) = M^{(4)}. \end{aligned}$$

From (5.31) we obtain

$$M^{(1)}\eta\cdot\overline{\eta} - iM^{(2)}\eta\cdot\overline{\eta} = \lambda\big[M^{(3)}\eta\cdot\overline{\eta} - iM^{(4)}\eta\cdot\overline{\eta}\big],$$

implying the relation

$$\lambda = \frac{(M^{(1)}\eta\cdot\overline{\eta} - iM^{(2)}\eta\cdot\overline{\eta})(M^{(3)}\eta\cdot\overline{\eta} + iM^{(4)}\eta\cdot\overline{\eta})}{|M^{(3)}\eta\cdot\overline{\eta} - iM^{(4)}\eta\cdot\overline{\eta}|^2}\,.$$

Therefore, in view (5.27), we deduce

$$\operatorname{Re}\lambda = \frac{(M^{(1)}\eta\cdot\overline{\eta})(M^{(3)}\eta\cdot\overline{\eta}) + (M^{(2)}\eta\cdot\overline{\eta})(M^{(4)}\eta\cdot\overline{\eta})}{|M^{(3)}\eta\cdot\overline{\eta} - iM^{(4)}\eta\cdot\overline{\eta}|^2} > 0.$$

This completes the proof. □

Introduce the notation

$$a_1 = \inf_{x\in\partial S_N,\, 1\le j\le 4}\delta_j(x), \quad a_2 = \sup_{x\in\partial S_N,\, 1\le j\le 4}\delta_j(x), \tag{5.32}$$

where δ_j is defined in (5.30).

Lemma 5.2. *Let* $\mathbf{Q}$ *be the set of all non-singular* $k\times k$, $k\in\mathbb{N}$, *square matrices with complex-valued entries having the structure*

$$\begin{bmatrix} [Q_{lj}]_{(k-1)\times(k-1)} & \{0\}_{(k-1)\times 1} \\ \{0\}_{1\times(k-1)} & Q_{kk} \end{bmatrix}_{k\times k} \quad \text{with } Q_{kk}\neq 0. \tag{5.33}$$

If $X,Y\in\mathbf{Q}$, *then* $XY\in\mathbf{Q}$ *and* $X^{-1}\in\mathbf{Q}$. *Moreover, if, in addition,* $X=[X_{jl}]_{k\times k}$ *and* $Y=[Y_{jl}]_{k\times k}$ *are strongly elliptic, i.e.,*

$$\mathrm{Re}(X\zeta\cdot\zeta)>0,\quad \mathrm{Re}(Y\zeta\cdot\zeta)>0 \ \text{ for all } \zeta\in\mathbb{C}^k\setminus\{0\},$$

and $X_{kk}Y_{kk}>0$ *are real numbers, then at least one eigenvalue of the matrix* XY *is positive.*

In particular, if $X_{kk}Y_{kk}=1$, *then* $\lambda=1$ *is an eigenvalue of the matrix* XY.

Taking into consideration that the principal homogeneous symbol matrices defined by formulas (C.3)–(C.5) have the structure (5.33) with $k=4$, we can show that one of the eigenvalues of the matrix M given by (5.29) equals 1, say $\lambda_4=1$. Consequently, the argument of λ_4 equals zero and the corresponding $\delta_4=0$. Therefore, for the numbers a_1 and a_2 defined by (5.32) the inequalities

$$-\frac{1}{4}<a_1\leqslant 0\leqslant a_2\leqslant\frac{1}{4} \tag{5.34}$$

hold.

Now, we are ready to formulate and prove the main results of the paper.

Theorem 5.2. *The operators*

$$r_{S_N}\mathcal{A}:\big[\widetilde{H}^s_p(S_N)\big]^4\to\big[H^{s-1}_p(S_N)\big]^4, \tag{5.35}$$

$$r_{S_N}\mathcal{A}:\big[\widetilde{B}^s_{p,q}(S_N)\big]^4\to\big[B^{s-1}_{p,q}(S_N)\big]^4,\ \ q\geq 1, \tag{5.36}$$

are invertible if

$$\frac{1}{p}-\frac{1}{2}+a_2<s<\frac{1}{p}+\frac{1}{2}+a_1, \tag{5.37}$$

where a_1 *and* a_2 *are given by* (5.32).

Proof. The mapping properties (5.35) and (5.36) follow from Theorems 4.5 and 4.6.

To prove the invertibility of operators (5.35) and (5.36), we first consider the particular values of the parameters $s=1/2$ and $p=q=2$, which fall into the region defined by inequalities (5.37), and show that the null space of the operator

$$r_{S_N}\mathcal{A}:\big[\widetilde{H}^{\frac{1}{2}}_2(S_N)\big]^4\to\big[H^{-\frac{1}{2}}_2(S_N)\big]^4 \tag{5.38}$$

is trivial, i.e., the equation

$$r_{S_N}\mathcal{A}\widetilde{g}=0 \ \text{ on } S_N \tag{5.39}$$

admits only the trivial solution in the space $[\widetilde{H}^{\frac{1}{2}}_2(S_N)]^4$. Recall that $\widetilde{H}^s_2(S_N)=\widetilde{B}^s_{2,2}(S_N)$ and $H^s_2(S_N)=B^s_{2,2}(S_N)$ for $s\in\mathbb{R}$.

Let $\widetilde{g}\in[\widetilde{H}^{\frac{1}{2}}(S_N)]^4$ be a solution of the homogeneous equation (5.39). It is evident that the vector-function $U=V(\mathcal{H}^{-1}\widetilde{g})$ belongs to the space $[H^1_2(\Omega)]^4=[W^1_2(\Omega)]^4$ and solves the homogeneous mixed BVP (5.1)–(5.3) with $f=0$ and $F=0$. Therefore, $U(x)=V(\mathcal{H}^{-1}\widetilde{g})(x)=0$ for $x\in\Omega^+$, due to the uniqueness Theorem 3.1, and, consequently, $\{U\}^+=\widetilde{g}=0$ on S.

Since the principal homogeneous symbol matrix of the operator $\mathcal{A}$ is strongly elliptic, by Theorem D.1 (see Appendix D) we conclude that for all values of the parameters satisfying inequalities (5.37), operators (5.35) and (5.36) are Fredholm with zero index and with trivial null spaces. Therefore, they are invertible. □

Remark 5.3. From Theorems 3.1 and 4.5 and from the proof of the above theorem, it follows that Theorem 5.2 remains true for the Lipschitz surfaces with $s = \frac{1}{2}$ and $p = 2$, that is, operator (5.38) is invertible.

Indeed, let us write Green's first identity of type (4.31) for the vector-function $U^{(0)} = V^{(0)}([\mathcal{H}^{(0)}]^{-1}g) = (u,\varphi)^\top$ with $g \in [\widetilde{H}^{\frac{1}{2}}(S_N)]^4$:

$$\Big\langle \{P^{(0)}(\partial,n)U^{(0)}\}^+, \overline{\{U^{(0)}\}^+} \Big\rangle_S = \int\limits_\Omega \widetilde{E}_\omega^{(0)}\big(U^{(0)}, \overline{U^{(0)}}\big)\, dx,$$

implying the relation (see (4.33))

$$\begin{aligned}\big\langle r_{s_N}\, \mathcal{A}^{(0)} g, \overline{g}\big\rangle_S &= \int\limits_\Omega \big[E_\omega(u,\overline{u}) + \alpha_1|\nabla\varphi|^2\big]\, dx \\ &= \int\limits_\Omega \big[E(u,\overline{u};\lambda,\mu) + \alpha|\nabla\varphi|^2\big]\, dx - i\omega \int\limits_\Omega \big[E(u,\overline{u};\lambda^*,\mu^*) + \alpha^*|\nabla\varphi|^2\big]\, dx, \end{aligned} \tag{5.40}$$

where $E(u,\overline{u};\lambda,\mu)$ and $E(u,\overline{u};\lambda^*,\mu^*)$ are defined in (2.16) and (2.17), respectively. Using the well-known Korn and Poincaré inequalities, from (5.40) we deduce (see, e.g., [60])

$$\operatorname{Re}\big\langle r_{s_N}\mathcal{A}^{(0)} g, \overline{g}\big\rangle_S \geqslant C_1\big(\|U^{(0)}\|_{[H^1(\Omega)]^4} - \|U^{(0)}\|_{[L_2(\Omega)]^4}\big), \tag{5.41}$$

$$-\operatorname{Im}\big\langle r_{s_N}\, \mathcal{A}^{(0)} g, \overline{g}\big\rangle_S \geqslant \omega C_2\big(\|U^{(0)}\|_{[H^1(\Omega)]^4} - \|U^{(0)}\|_{[L_2(\Omega)]^4}\big), \tag{5.42}$$

where C_1 and C_2 are positive constants depending on material parameters.

The trace theorem in the Lipschitz domains and the properties of the single layer potential $U^{(0)} = V^{(0)}([\mathcal{H}^{(0)}]^{-1}g)$ lead to the following relations (see, e.g., [34, 46], [66, Section 6]):

$$\begin{gathered}\|g\|_{[H^{\frac{1}{2}}(S)]^4} = \|\{U^{(0)}\}^+\|_{[H^{\frac{1}{2}}(S)]^4} \leqslant C_3\|U^{(0)}\|_{[H^1(\Omega)]^4}, \\ \|U^{(0)}\|_{[L_2(\Omega)]^4} \leqslant \|U^{(0)}\|_{[H^{\frac{1}{2}+\varepsilon}(\Omega)]^4} \leqslant C_4\|[\mathcal{H}]^{-1}g\|_{[H^{\varepsilon-1}(S)]^4} \leqslant C_5\|g\|_{[H^{\varepsilon}(S)]^4},\end{gathered}$$

where ε is a sufficiently small positive number.

Now, from (5.41) and (5.42), we conclude that there are the positive constants C_6 and C_7 such that

$$\begin{gathered}\operatorname{Re}\big\langle r_{s_N}\mathcal{A}^{(0)} g, \overline{g}\big\rangle_S \geqslant C_6\big(\|g\|^2_{[H^{\frac{1}{2}}(S)]^4} - \|g\|^2_{[H^{\varepsilon}(S)]^4}\big), \\ -\operatorname{Im}\big\langle r_{s_N}\mathcal{A}^{(0)} g, \overline{g}\big\rangle_S \geqslant C_7\big(\|g\|^2_{[H^{\frac{1}{2}}(S)]^4} - \|g\|^2_{[H^{\varepsilon}(S)]^4}\big).\end{gathered}$$

Therefore, the operator $r_{s_N}\mathcal{A}^{(0)} : [\widetilde{H}_2^{\frac{1}{2}}(S_N)]^4 \to [H_2^{-\frac{1}{2}}(S_N)]^4$ is a Fredholm operator with zero index (see, e.g., [65, Theorem 2.34]). Taking into consideration that in the case of Lipschitz surfaces the operator $r_{s_N}\mathcal{A} - r_{s_N}\mathcal{A}^{(0)}$ is again a compact operator in view of relations (4.1), we infer that operator (5.38) is a Fredholm operator with zero index. Due to the uniqueness Theorem 3.1, it follows that the corresponding null space is trivial, implying that operator (5.38) is invertible.

Theorem 5.3. *Let conditions* (5.4), (5.5) *be fulfilled and*

$$\frac{4}{3-2a_2} < p < \frac{4}{1-2a_1}\,, \tag{5.43}$$

where a_1 *and* a_2 *are defined by* (5.32).

Then the mixed boundary value problem (5.1)–(5.3) *possesses a unique solution* $U \in [W_p^1(\Omega)]^4$, *which is representable in the form of single layer potential*

$$U = V\big(\mathcal{H}^{-1}(f^{(e)} + \widetilde{g})\big),$$

where $f^{(e)} \in [B^{1-1/p}_{p,p}(S)]^4$ *is a fixed extension of the vector-function* $f \in [B^{1-1/p}_{p,p}(S_D)]^4$ *from* S_D *onto* S *preserving the functional space and* $\widetilde{g} \in [\widetilde{B}^{1-1/p}_{p,p}(S_N)]^4$ *is defined by the uniquely solvable pseudodifferential equation*

$$r_{S_N}\mathcal{A}\widetilde{g} = F^{(0)} \ \text{ on } S_N \tag{5.44}$$

with

$$F^{(0)} := F - r_{S_N}\mathcal{A}f^{(e)} \in \big[B^{-1/p}_{p,p}(S_N)\big]^4.$$

Proof. First, we note that, in accordance with Theorem 5.2, equation (5.44) is uniquely solvable for $s = 1 - \frac{1}{p}$, where p meets inequality (5.43), since inequalities (5.37) are fulfilled. This implies that the mixed boundary value problem (5.1)–(5.3) is solvable in the space $[W^1_p(\Omega)]^4$ with p satisfying (5.43).

Next, we show the uniqueness of the solution in the space $[W^1_p(\Omega)]^4$ for arbitrary p satisfying (5.43). Note that $p = 2$ belongs to the interval defined by inequality (5.43) and the uniqueness for $p = 2$ was proved in Theorem 3.1. Now, let $U \in [W^1_p(\Omega)]^4$ be a solution of the homogeneous mixed boundary value problem (5.1)–(5.3). Evidently, then

$$\{U\}^+ \in \big[\widetilde{B}^{1-1/p}_{p,p}(S_N)\big]^6. \tag{5.45}$$

By Corollary 4.2, we have the representation

$$U(x) = V\big(\mathcal{H}^{-1}\{U\}^+\big)(x), \ \ x \in \Omega.$$

Since U satisfies the homogeneous Neumann condition (5.3) on S_N, we arrive at the equation

$$r_{S_N}\mathcal{A}^+\{U\}^+ = 0 \ \text{ on } S_N,$$

whence $\{U\}^+ = 0$ on S follows due to inclusion (5.45), Theorem 5.2, and inequality (5.43) implying conditions (5.37). Therefore, $U = 0$ in Ω. □

Note that due to (5.34) we have the inclusion

$$\Big(\frac{8}{5}\,,\frac{8}{3}\Big) \subset \Big(\frac{4}{3\text{–}2a_2}\,,\frac{4}{1\text{–}2a_1}\Big).$$

Remark 5.4. Using Remark 5.3, one can easily show that Theorem 5.3 remains true in the case of Lipschitz domains for $p = 2$.

Further, we prove almost the best regularity results for solutions to the mixed type boundary value problems.

Theorem 5.4. *Let conditions* (5.4), (5.5) *hold and let*

$$\frac{4}{3-2a_2} < p < \frac{4}{1-2a_1}\,, \ \ 1 < r < \infty, \ \ 1 \le q \le \infty, \ \ \frac{1}{r} - \frac{1}{2} + a_2 < s < \frac{1}{r} + \frac{1}{2} + a_1, \tag{5.46}$$

with a_1 *and* a_2 *defined by* (5.32).

Further, let $U \in [W^1_p(\Omega)]^4$ *be a unique solution to the mixed boundary value problem* (5.1)–(5.3). *Then the following hold:*

(i) *if*

$$f \in \big[B^s_{r,r}(S_D)\big]^4, \quad F \in \big[B^{s-1}_{r,r}(S_N)\big]^4,$$

then $U \in [H^{s+\frac{1}{r}}_r(\Omega)]^4 \cap [B^{s+\frac{1}{r}}_{r,r}(\Omega)]^4$;

(ii) *if*

$$f \in \big[B^s_{r,q}(S_D)\big]^4, \quad F \in \big[B^{s-1}_{r,q}(S_N)\big]^4,$$

then

$$U \in \big[B^{s+\frac{1}{r}}_{r,q}(\Omega^+)\big]^4; \tag{5.47}$$

(iii) *if*

$$f \in \left[C^{\beta}(\overline{S_D})\right]^4, \quad F \in \left[B^{\beta-1}_{\infty,\infty}(S_N)\right]^4, \ \ 0 < \beta < 1, \tag{5.48}$$

then

$$U \in \bigcap_{\beta' < \kappa_m} \left[C^{\beta'}(\overline{\Omega^+})\right]^4,$$

where

$$0 < \kappa_m = \min\left\{\beta, a_1 + \frac{1}{2}\right\} \leqslant \frac{1}{2}.$$

Proof. The proofs of items (i) and (ii) follow easily from Theorems 5.2, 5.3, and D.1.

To prove (iii), we use the following embedding (see, e.g., [98, Sections 2.3.5, 3.3.1], [7, Theorem 6.2.4], [97, Section 4.6], [83, Section 2.1.2]):

$$C^{\beta}(\mathcal{M}) = B^{\beta}_{\infty,\infty}(\mathcal{M}) \subset B^{\beta-\varepsilon}_{\infty,1}(\mathcal{M}) \subset B^{\beta-\varepsilon}_{\infty,q}(\mathcal{M}) \subset B^{\beta-\varepsilon}_{r,q}(\mathcal{M}) \subset C^{\beta-\varepsilon-k/r}(\mathcal{M}), \tag{5.49}$$

where ε is an arbitrary small positive number, $\mathcal{M} \subset \mathbb{R}^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with a smooth boundary, $1 \le q \le \infty$, $1 < r < \infty$, $\beta - \varepsilon - \frac{k}{r} > 0$, and β and $\beta - \varepsilon - \frac{k}{r}$ are not integers.

From (5.48) and embedding (5.49), condition (5.47) follows with any $s \le \beta - \varepsilon$.

Bearing in mind (5.46) and taking r sufficiently large and ε sufficiently small, we can put

$$s = \beta - \varepsilon \ \text{ if } \ \frac{1}{r} - \frac{1}{2} + a_2 < \beta - \varepsilon < \frac{1}{r} + \frac{1}{2} + a_1, \tag{5.50}$$

and

$$s \in \left(\frac{1}{r} - \frac{1}{2} + a_2, \frac{1}{r} + \frac{1}{2} + a_1\right) \ \text{ if } \ \frac{1}{r} + \frac{1}{2} + a_1 < \beta - \varepsilon. \tag{5.51}$$

By (5.47), for the solution vector, we have $U \in [B^{s+\frac{1}{r}}_{r,q}(\Omega^+)]^4$ with

$$s + \frac{1}{r} = \beta - \varepsilon + \frac{1}{r}$$

if (5.50) holds, and with

$$s + \frac{1}{r} \in \left(\frac{2}{r} - \frac{1}{2} + a_2, \frac{2}{r} + \frac{1}{2} + a_1\right)$$

if (5.51) holds. In the last case, we can take

$$s + \frac{1}{r} = \frac{2}{r} + \frac{1}{2} + a_1 - \varepsilon.$$

Therefore, we have either

$$U \in \left[B^{\beta-\varepsilon+\frac{1}{r}}_{r,q}(\Omega^+)\right]^4,$$

or

$$U \in \left[B^{\frac{1}{2}+\frac{2}{r}+a_1-\varepsilon}_{r,q}(\Omega^+)\right]^4,$$

in accordance with inequalities (5.50) and (5.51). The last embedding in (5.49) (with $k = 3$) yields then that either

$$U \in \left[C^{\beta-\varepsilon-\frac{2}{r}}(\overline{\Omega^+})\right]^4,$$

or

$$U \in \left[C^{\frac{1}{2}-\varepsilon+a_1-\frac{1}{r}}(\overline{\Omega^+})\right]^4.$$

These relations lead to the inclusion

$$U \in \left[C^{\kappa_m-\varepsilon-\frac{2}{r}}(\overline{\Omega^+})\right]^4, \tag{5.52}$$

where $\kappa_m = \min\{\beta, a_1 + \frac{1}{2}\}$ and $0 < \kappa_m \leqslant \frac{1}{2}$ due to inequalities (5.34). Since r is sufficiently large and ε is sufficiently small, inclusions (5.52) accomplish the proof. □

Remark 5.5. Using the approach developed in [9,11,17,18], for investigating the asymptotic behaviour of solutions to the mixed boundary value problems near the collision curve ℓ, where the different types of boundary conditions collide, one can characterise optimal regularity results for the displacement vector in the closed domain under consideration and find a possible maximal Hölder continuous exponent explicitly with the help of the eigenvalues of matrix (5.29). Analyzing the first dominant terms of the asymptotic expansion, it can be shown that, in general, the Hölder continuous exponent does not exceed the number κ_m defined in Theorem 5.4(iii), but in the case of simple eigenvalues the Hölder exponent equals κ_m. Therefore, Theorem 5.4 gives an almost optimal regularity result for the displacement vector.

6 Appendix A: Dynamical field equations

The complete system of the dynamical model of the linear theory of viscoelasticity for a homogeneous and isotropic Kelvin–Voigt material with voids consists of the following field equations (for details see Ieşan [57]):

(i) The constitutive equations:

$$\begin{aligned} t_{lj} &= 2\mu e_{lj} + \lambda e_{rr}\delta_{lj} + b\varphi\delta_{lj} + 2\mu^* \dot{e}_{lj} + \lambda^* \dot{e}_{rr}\delta_{lj} + b^* \dot{\varphi}\delta_{lj}, \\ H_j &= \alpha\partial_j\varphi + \alpha^*\partial_j\dot{\varphi}, \\ H_0 &= -be_{rr} - \xi\varphi - \nu^*\dot{e}_{rr} - \xi^*\dot{\varphi}, \quad l,j = 1,2,3. \end{aligned}$$

(ii) The equations of motion:

$$\begin{aligned} \partial_j t_{lj} &= \rho(\ddot{u}_l - \mathcal{F}_l), \quad l = 1,2,3, \\ \partial_j H_j + H_0 &= \rho_0\ddot{\varphi} - \rho\mathcal{F}_4, \end{aligned}$$

where the superposed dot denotes differentiation with respect to the time variable t.

(iii) The geometrical equations

$$e_{lj}(u) = \frac{1}{2}\,(\partial_l u_j + \partial_j u_l), \quad l,j = 1,2,3.$$

These relations lead to the following partial differential equations of dynamics:

$$\begin{aligned} \mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u + b\,\mathrm{grad}\,\varphi + \mu^*\Delta\dot{u} + (\lambda^*+\mu^*)\,\mathrm{grad}\,\mathrm{div}\,\dot{u} + b^*\,\mathrm{grad}\,\dot{\varphi} = \rho(\ddot{u} - \mathcal{F}), \\ (\alpha\Delta - \xi)\varphi - b\,\mathrm{div}\, u + (\alpha^*\Delta - \xi^*)\dot{\varphi} - \nu^*\,\mathrm{div}\,\dot{u} = \rho_0\ddot{\varphi} - \rho\mathcal{F}_4. \end{aligned}$$

In the above relations, t_{lj} are the components of the total stress tensor, e_{lj} are the components of the strain tensor, H_j are the components of the equilibrated stress vector, H_0 is the intrinsic equilibrated body force, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, φ is the volume fraction field, $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)^\top$ is the body force per unit mass, $\mathcal{F}_4$ is the extrinsic equilibrated body force per unit mass, ϱ is the reference mass density, κ is the equilibrated inertia, $\varrho_0 = \varrho\kappa$. The real-valued constants λ, λ^*, μ, μ^*, b, b^*, ν, ν^*, α, α^*, β, β^*, and κ are material parameters satisfying inequalities (2.18).

All the functions involved in the above dynamical equations depend on the spatial variables $x = (x_1, x_2, x_3)$ and the time variable t. If all the functions are harmonic time dependent, i.e., they are products of functions of the spacial variables $x = (x_1, x_2, x_3)$ and the function of time variable $e^{i\omega t}$, where ω is a real-valued frequency parameter and $i = \sqrt{-1}$ is the imaginary unit, then the dynamical equations can be rewritten as system (2.1), (2.2).

7 Appendix B: Fundamental matrices

Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized three-dimensional Fourier transforms in the space of tempered distributions (Schwartz space $\mathcal{S}'(\mathbb{R}^3)$), which for regular summable functions

f and g read as follows:

$$\mathcal{F}_{x\to\xi}[f]=\widehat{f}(\xi)=\int\limits_{\mathbb{R}^3} f(x)e^{ix\cdot\xi}\,dx,\quad \mathcal{F}^{-1}_{\xi\to x}[g]=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} g(\xi)e^{-ix\cdot\xi}\,d\xi,\ \ x,\xi\in\mathbb{R}^3.$$

First, we calculate the inverse Fourier transform of the regular functional $(|\xi|^2-\tau^2)^{-1}$, where $\tau=\omega+i\,\varepsilon$ with $\omega\in\mathbb{R}$ and $\varepsilon>0$. This will be employed below to construct explicitly the fundamental matrix of the operator $A(\partial,\omega)$. Since the function under consideration is square integrable, we can write (see, e.g., [47])

$$\begin{aligned}H(x,\tau)&:=\mathcal{F}^{-1}_{\xi\to x}\big[\big(|\xi|^2-\tau^2\big)^{-1}\big]\\&=\mathcal{F}^{-1}_{\xi\to x}\big[\big(|\xi|^2+(\varepsilon-i\omega)^2\big)^{-1}\big]=\frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_{|\xi|<R}\frac{e^{-ix\cdot\xi}}{|\xi|^2+(\varepsilon-i\omega)^2}\,d\xi.\end{aligned}\tag{B.1}$$

Let $\Lambda(\widetilde{x})=[\Lambda_{kj}(\widetilde{x})]_{3\times3}$ with $\widetilde{x}=x/|x|$ be an orthogonal matrix with the properties

$$\det\Lambda(\widetilde{x})=1,\quad \Lambda^\top(\widetilde{x})x=(0,0,|x|)^\top.$$

Using the substitution $\xi=\Lambda(\widetilde{x})\eta$ and keeping in mind that $x\cdot\Lambda(\widetilde{x})\eta=|x|\eta_3$, $|\xi|=|\eta|$ and $d\xi=d\eta$, from (B.1) we get

$$H(x,\tau)=\frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_{|\eta|<R}\frac{e^{-i|x|\eta_3}}{|\eta|^2+(\varepsilon-i\omega)^2}\,d\eta.\tag{B.2}$$

Introduce the spherical co-ordinates

$$\eta_1=\varrho\cos\varphi\sin\vartheta,\quad \eta_2=\varrho\sin\varphi\sin\vartheta,\quad \eta_3=\varrho\cos\vartheta,\quad \varrho=|\eta|,\ \ \varphi\in[0,2\pi],\ \ \vartheta\in[0,\pi],$$

and rewrite (B.2) as follows:

$$\begin{aligned}H(x,\tau)&=\frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_0^R\int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{e^{-i|x|\varrho\cos\vartheta}}{\varrho^2+(\varepsilon-i\omega)^2}\,\varrho^2\sin\vartheta\,d\vartheta\,d\varphi\,d\varrho\\&=\frac{1}{4\pi^2}\lim_{R\to\infty}\int\limits_0^R\frac{\varrho^2}{\varrho^2+(\varepsilon-i\omega)^2}\,d\varrho\int\limits_0^{\pi}\frac{1}{i|x|\varrho}\Big(\frac{\partial}{\partial\vartheta}\,e^{-i|x|\varrho\cos\vartheta}\Big)\,d\vartheta\\&=-\frac{i}{4\pi^2|x|}\lim_{R\to\infty}\int\limits_0^R\frac{\varrho}{\varrho^2+(\varepsilon-i\omega)^2}\,\big[e^{i|x|\varrho}-e^{-i|x|\varrho}\big]\,d\varrho\\&=\frac{1}{2\pi^2|x|}\lim_{R\to\infty}\int\limits_0^R\frac{\varrho\sin(|x|\varrho)}{\varrho^2+(\varepsilon-i\omega)^2}\,d\varrho\\&=\frac{1}{2\pi^2|x|}\int\limits_0^\infty\frac{\varrho\sin(|x|\varrho)}{\varrho^2+(\varepsilon-i\omega)^2}\,d\varrho.\end{aligned}$$

With the help of the formula [48, 3.723.3]

$$\int\limits_0^\infty\frac{t\sin(at)}{t^2+\beta^2}\,dt=\frac{\pi}{2}\,e^{-a\beta}\ \ \text{for}\ \ a>0,\ \ \operatorname{Re}\beta>0,$$

we finally get

$$H(x,\tau)=\frac{e^{-|x|(\varepsilon-i\omega)}}{4\pi|x|}=\frac{e^{i\tau|x|}}{4\pi|x|}\,,$$

i.e.,

$$\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-\tau^2}\Big]=\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-(\omega+i\varepsilon)^2}\Big]=\frac{e^{i\tau|x|}}{4\pi|x|}\,, \tag{B.3}$$

where $\tau=\omega+i\varepsilon$ with $\varepsilon>0$ and $\omega\in\mathbb{R}$.

Quite analogously we can derive the similar formula

$$\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-\tau^2}\Big]=\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{1}{|\xi|^2-(\omega-i\varepsilon)^2}\Big]=\frac{e^{-i\tau|x|}}{4\pi|x|} \tag{B.4}$$

for $\tau=\omega-i\,\varepsilon$ with $\varepsilon>0$ and $\omega\in\mathbb{R}$.

The fundamental matrix $\Gamma(x)$ of the operator $A(\partial,\omega)$ can be constructed by the Fourier transform technique:

$$\Gamma(x)=\mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi)\big]=\frac{1}{8\pi^3}\lim_{R\to\infty}\int\limits_{|\xi|<R}A^{-1}(-i\xi)e^{-ix\cdot\xi}\,d\xi, \tag{B.5}$$

where $A^{-1}(-i\xi)$ is the inverse of the symbol matrix $A(-i\xi)$ of the differential operator $A(\partial,\omega)$. Evidently,

$$\begin{gathered}
A(-i\xi,\omega)=\big[A_{kj}(-i\xi,\omega)\big]_{4\times4},\ \ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3,\\
A_{kj}(-i\xi,\omega)=-\mu_1|\xi|^2\delta_{kj}-(\lambda_1+\mu_1)\xi_k\xi_j+\varrho\omega^2\delta_{kj},\ \ k,j=1,2,3,\\
A_{4j}(-i\xi,\omega)=i\nu_1\xi_j,\quad A_{k4}(-i\xi,\omega)=-ib_1\xi_k,\ \ k,j=1,2,3,\\
A_{44}(-i\xi,\omega)=-\alpha_1|\xi|^2-\beta_1+\varrho_0\omega^2,
\end{gathered}$$

where λ_1, μ_1, b_1, α_1, ν_1, and β_1 are given by (2.3). The determinant of the matrix $A(-i\xi,\omega)$ is an eighth order polynomial in $|\xi|$:

$$|A(\xi,\omega)|=\big(-\mu_1|\xi|^2+\varrho\omega^2\big)^2D(|\xi|,\omega), \tag{B.6}$$

$$\begin{aligned}
D(|\xi|,\omega)&=\alpha_1(\lambda_1+2\mu_1)|\xi|^4\\
&\quad-\big[b_1\nu_1+\alpha_1\varrho\omega^2+(\lambda_1+2\mu_1)(\varrho_0\omega^2-\beta_1)\big]|\xi|^2+(\varrho_0\omega^2-\beta_1)\varrho\omega^2.
\end{aligned} \tag{B.7}$$

Note that all cofactors of the matrix $A(-i\xi,\omega)$ contain the common factor $-\mu_1|\xi|^2+\varrho\omega^2$. Therefore, the entries of the inverse matrix $A^{-1}(-i\xi,\omega)$ read as follows:

$$\begin{gathered}
A^{-1}(-i\xi,\omega)=\big[A^{-1}_{kj}(-i\xi,\omega)\big]_{4\times4},\\
A^{-1}_{kj}=\frac{B_{kj}}{D_1}\,,\quad A^{-1}_{4j}=\frac{B_{4j}}{D}\,,\quad A^{-1}_{k4}=\frac{B_{k4}}{D}\,,\quad A^{-1}_{44}=\frac{B_{44}}{D}\,,\ \ k,j=1,2,3,\\
D_1(|\xi|,\omega)=\big(-\mu_1|\xi|^2+\varrho\omega^2\big)D(|\xi|,\omega),\\
B_{11}(\xi,\omega)=-b_1\nu_1(\xi_2^2+\xi_3^2)+\big[\mu_1\xi_1^2+(\lambda_1+2\mu_1)(\xi_2^2+\xi_3^2)-\varrho\omega^2\big](\alpha_1|\xi|^2+\beta_1-\varrho_0\omega^2),\\
B_{12}(\xi,\omega)=B_{21}(\xi,\omega)=\xi_1\xi_2\big[b_1\nu_1-(\lambda_1+\mu_1)(\alpha_1|\xi|^2+\beta_1\varrho_0\omega^2)\big],\\
B_{13}(\xi,\omega)=B_{31}(\xi,\omega)=\xi_1\xi_3\big[b_1\nu_1-(\lambda_1+\mu_1)(\alpha_1|\xi|^2+\beta_1-\varrho_0\omega^2)\big],\\
B_{22}(\xi,\omega)=-b_1\nu_1(\xi_1^2+\xi_3^2)+\big[\mu_1\xi_2^2+(\lambda_1+2\mu_1)(\xi_1^2+\xi_3^2)-\varrho\omega^2\big](\alpha_1|\xi|^2+\beta_1-\varrho_0\omega^2),\\
B_{23}(\xi,\omega)=B_{32}=\xi_2\xi_3\big[b_1\nu_1-(\lambda_1+\mu_1)(\alpha_1|\xi|^2+\beta_1-\varrho_0\omega^2)\big],\\
B_{33}(\xi,\omega)=-b_1\nu_1(\xi_1^1+\xi_2^2)+\big[\mu_1\xi_3^2+(\lambda_1+2\mu_1)(\xi_1^2+\xi_2^2)-\varrho\omega^2\big](\alpha_1|\xi|^2+\beta_1-\varrho_0\omega^2),\\
B_{14}(\xi,\omega)=-ib_1\xi_1,\quad B_{24}(\xi,\omega)=ib_1\xi_2,\quad B_{34}(\xi,\omega)=ib_1\xi_3,\\
B_{41}(\xi,\omega)=-i\nu_1\xi_1,\quad B_{42}(\xi,\omega)=-i\nu_1\xi_2,\quad B_{43}(\xi,\omega)=-i\nu_1\xi_3,\\
B_{44}(\xi,\omega)=\varrho\,\omega^2-(\lambda_1+2\mu_1)\,|\xi|^2.
\end{gathered} \tag{B.8}$$

The symbol matrix of the operator $A^{(0)}(\partial,\omega)$ defined by (2.4)–(2.6) has the following form:

$$A^{(0)}(-i\xi,\omega)=\big[A^{(0)}_{kj}(-i\xi,\omega)\big]_{4\times4},\ \ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3,$$

$$A^{(0)}_{kj}(-i\xi,\omega) = -\mu_1|\xi|^2\delta_{kj} - (\lambda_1+\mu_1)\xi_k\xi_j, \quad A^{(0)}_{44}(-i\xi,\omega) = -\alpha_1|\xi|^2,$$
$$A^{(0)}_{4j}(-i\xi,\omega) = A^{(0)}_{k4}(-i\xi,\omega) = 0, \quad k,j=1,2,3.$$

If

$$\mu>0, \;\; 3\lambda+2\mu>0, \;\; \alpha>0, \;\; \mu^*>0, \;\; 3\lambda^*+2\mu^*>0, \;\; \alpha^*>0, \;\; \omega>0,$$

then there are the positive constants C_1 and C_2, depending on the material parameters, such that

$$\begin{gathered}\mathrm{Re}\,\big(-A^{(0)}(-i\xi)\zeta\cdot\overline{\zeta}\big) = \mathrm{Re}\,\big(A^{(0)}(\xi)\zeta\cdot\overline{\zeta}\big) \geq C_1|\xi|^2|\zeta|^2,\\ -\mathrm{Im}\,\big(-A^{(0)}(-i\xi)\zeta\cdot\overline{\zeta}\big) = -\mathrm{Im}\,\big(A^{(0)}(\xi)\zeta\cdot\overline{\zeta}\big) \geq \omega C_2|\xi|^2|\zeta|^2,\\ \text{for all } \;\xi\in\mathbb{R}^3 \;\text{ and for all }\; \zeta\in\mathbb{C}^4.\end{gathered} \tag{B.9}$$

Therefore, $-A^{(0)}(\partial_x,\omega)$ and $-A(\partial_x,\omega)$ are strongly elliptic differential operators.

Note that relations (B.9) imply the following inequalities:

$$\begin{gathered}\mathrm{Re}\,\big(-[A^{(0)}(-i\xi)]^{-1}\zeta\cdot\overline{\zeta}\big) = \mathrm{Re}\,\big([A^{(0)}(\xi)]^{-1}\zeta\cdot\overline{\zeta}\big) \geq C_3|\xi|^{-2}|\zeta|^2,\\ \mathrm{Im}\,\big(-[A^{(0)}(-i\xi)]^{-1}\zeta\cdot\overline{\zeta}\big) = \mathrm{Im}\,\big([A^{(0)}(\xi)]^{-1}\zeta\cdot\overline{\zeta}\big) \geq \omega C_4|\xi|^{-2}|\zeta|^2,\\ \text{for all } \;\xi\in\mathbb{R}^3 \;\text{ and for all }\; \zeta\in\mathbb{C}^4,\end{gathered} \tag{B.10}$$

where C_3 and C_4 are positive constants depending on the material parameters.

To find the explicit form of the fundamental matrix $\Gamma(x)$, we have to characterize the roots of the equation $D_1(\tau,\omega)=0$ and then employ formulas (B.3), (B.4). It can be shown that the equation has no positive roots with respect to τ^2 (see [93]). Denote these roots by τ_1^2, τ_2^2, and τ_3^2, where

$$\tau_1^2 = \frac{\varrho\,\omega^2}{\mu_1} = \frac{\varrho\,\omega^2\,(\mu+i\omega\mu^*)}{|\mu_1|^2}\,,$$

while τ_2^2 and τ_3^2 are solutions of the biquadratic equation $D(\tau,\omega)=0$ with respect to τ^2. By τ_1, τ_2, and τ_3 we denote those complex roots of the equation $D_1(\tau,\omega)=0$ with respect to τ, which have positive imaginary parts, that is,

$$\tau_j = \tau_j' + i\tau_j'', \;\; \tau_j'\neq 0, \;\; \tau_j''>0, \;\; j=1,2,3. \tag{B.11}$$

Evidently, another triplet of roots is $\{-\tau_1,-\tau_2,-\tau_3\}$. Moreover, we assume that $\tau_j\neq\tau_k$ for $j\neq k$, $k,j=1,2,3$.

Using the relations

$$\begin{aligned}D_1(|\xi|,\omega) &= \big(-\mu_1|\xi|^2+\varrho\omega^2\big)D(|\xi|,\omega) = -\mu_1\alpha_1(\lambda_1+2\mu_1)\big(|\xi|^2-\tau_1^2\big)\big(|\xi|^2-\tau_2^2\big)\big(|\xi|^2-\tau_3^2\big),\\ D(|\xi|,\omega) &= \alpha_1(\lambda_1+2\mu_1)\big(|\xi|^2-\tau_2^2\big)\big(|\xi|^2-\tau_3^2\big),\end{aligned}$$

we derive the following identities:

$$\begin{aligned}\frac{1}{D_1(|\xi|,\omega)} &= -\frac{1}{\mu_1\alpha_1(\lambda_1+2\mu_1)}\Big(\frac{A_1}{|\xi|^2-\tau_1^2}+\frac{A_2}{|\xi|^2-\tau_2^2}+\frac{A_3}{|\xi|^2-\tau_3^2}\Big),\\ \frac{1}{D(|\xi|,\omega)} &= \frac{1}{\alpha_1\,(\lambda_1+2\mu_1)}\Big(\frac{A_4}{|\xi|^2-\tau_2^2}+\frac{A_5}{|\xi|^2-\tau_3^2}\Big),\end{aligned}$$

where

$$A_1 = \frac{1}{(\tau_1^2-\tau_2^2)(\tau_1^2-\tau_3^2)}\,, \quad A_2 = \frac{1}{(\tau_2^2-\tau_1^2)(\tau_2^2-\tau_3^2)}\,,$$
$$A_3 = \frac{1}{(\tau_3^2-\tau_1^2)(\tau_3^2-\tau_2^2)}\,, \quad A_4 = \frac{1}{\tau_2^2-\tau_3^2}, \quad A_5 = \frac{1}{\tau_3^2-\tau_2^2}\,.$$

Now, in view of (B.3), we obtain

$$\Lambda^{(1)}(x) := \mathcal{F}_{\xi\to x}\Big[\frac{1}{D_1(|\xi|,\omega)}\Big] = -\frac{1}{4\pi\mu_1\alpha_1(\lambda_1+2\mu_1)}\sum_{j=1}^{3} A_j\,\frac{e^{i\tau_j|x|}}{|x|}\,, \tag{B.12}$$

$$\Lambda^{(2)}(x) := \mathcal{F}_{\xi\to x}\Big[\frac{1}{D(|\xi|,\omega)}\Big] = \frac{1}{4\pi\alpha_1(\lambda_1+2\mu_1)}\Big(A_4\,\frac{e^{i\tau_2|x|}}{|x|} + A_5\,\frac{e^{i\tau_3|x|}}{|x|}\Big). \tag{B.13}$$

Therefore, for the fundamental matrix $\Gamma(x)$, due to (B.5) and (B.8), we obtain the following formula:

$$\Gamma(x) = \big[\Gamma_{kj}(x)\big]_{4\times 4} = \mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi)\big], \tag{B.14}$$

where

$$\begin{gathered}\Gamma_{kj}(x) = B_{kj}(i\partial_x,\omega)\Lambda^{(1)}(x), \quad \Gamma_{4j}(x) = B_{4j}(i\partial_x,\omega)\Lambda^{(2)}(x),\\ \Gamma_{k4} = B_{k4}(i\partial_x,\omega)\Lambda^{(2)}(x), \quad \Gamma_{44} = B_{44}(i\partial_x,\omega)\Lambda^{(2)}(x),\ \ k,j=1,2,3,\end{gathered} \tag{B.15}$$

the differential operators $B_{kj}(i\partial_x,\omega)$ are defined by relations (B.8) with $i\partial_x$ for ξ.

Note that $\Gamma^*(x) = \Gamma^\top(-x)$ is a fundamental solution of the adjoint differential operator $A^*(\partial,\omega) = A^\top(-\partial,\omega)$.

It is evident that the entries of the fundamental matrix $\Gamma(x)$ decay exponentially at infinity and in a vicinity of the origin have the singularities of type $\mathcal{O}(|x|^{-1})$.

The fundamental matrix $\Psi(x)$ of the operator $A^{(0)}(\partial,\omega)$ defined by (2.4)–(2.6) can be constructed similarly:

$$\Psi(x) = \mathcal{F}^{-1}_{\xi\to x}\big[\big(A^{(0)}(-i\xi,\omega)\big)^{-1}\big] = \frac{1}{8\pi^3}\lim_{R\to\infty}\int\limits_{|\xi|<R}\big[A^{(0)}(-i\xi,\omega)\big]^{-1}e^{-ix\cdot\xi}\,d\xi,$$

where the entries of the inverse matrix $\big[A^{(0)}(-i\xi,\omega)\big]^{-1}$ are obtained from relations (B.6)–(B.8) if we keep only the highest order terms in the corresponding polynomials.

Using the formula

$$\mathcal{F}^{-1}_{\xi\to x}\big[|\xi|^{-2}\big] = \frac{1}{4\pi|x|}\,,$$

we find the explicit expression for $\Psi(x)$ (cf. [61, Chapter II]):

$$\begin{gathered}\Psi(x) = [\Psi_{kj}(x)]_{4\times 4}, \quad \Psi_{kj}(x) = \frac{a\delta_{kj}}{|x|} + \frac{b x_k x_j}{|x|^3}\,,\ \ k,j=1,2,3,\\ \Psi_{44}(x) = -\frac{1}{4\pi\alpha_1}\,\frac{1}{|x|}\,, \quad \Psi_{4j}(x) = \Psi_{k4}(x) = 0,\ \ k,j=1,2,3,\\ a = -\frac{\lambda_1+3\mu_1}{8\pi\mu_1(\lambda_1+2\mu_1)}\,, \quad b = -\frac{\lambda_1+\mu_1}{8\pi\mu_1(\lambda_1+2\mu_1)}\,.\end{gathered}$$

Evidently, $\Psi(x) = \Psi^\top(x) = \Psi(-x)$, which implies that $\mathcal{H}^{(0)}$ is a symmetric operator.

Using the equalities

$$\sum_{k=1}^{3} A_k = 0, \quad \sum_{k=1}^{3} A_k\tau_k^2 = 0, \quad \sum_{k=1}^{3} A_k\tau_k^4 = 1, \quad A_4 + A_5 = 0, \quad A_4\tau_2^2 + A_5\tau_3^2 = 1,$$

one can show that in a vicinity of the origin the following relations hold:

$$\begin{gathered}\Gamma_{kj}(x) - \Psi_{kj}(x) = const + \mathcal{O}(|x|),\\ \frac{\partial^p}{\partial_1^{p_1}\partial_2^{p_2}\partial_3^{p_3}}\big[\Gamma_{kj}(x) - \Psi_{kj}(x)\big] = \mathcal{O}\big(|x|^{1-p}\big),\ \ p = p_1+p_2+p_3.\end{gathered} \tag{B.16}$$

In view of relations (B.11) and (B.12)–(B.15), it is evident that the entries of the matrix $\Gamma(x)$ decay exponentially at infinity.

8 Appendix C: Explicit expressions for the principal homogeneous symbol matrices

Here, we present the explicit expressions for the principal homogeneous symbol matrices of the boundary pseudodifferential operators introduced in Section 4. The principal homogeneous symbol matrix of a boundary pseudodifferential operator $\mathcal{A}$ defined on S we denote by $\mathfrak{S}(\mathcal{A};x,\xi')$, where $x\in S$ and $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2$.

Note that the principal homogeneous symbols of the boundary pseudodifferential operators constructed by the fundamental matrices $\Gamma(x-y)$ and $\Psi(x-y)$ coincide in view of relations (B.16).

Using the Cauchy integral theorem for analytic functions, we can represent the fundamental matrix $\Psi(x)$ in the form

$$\Psi(x)=\mathcal{F}^{-1}_{\xi\to x}\big[\{A^{(0)}(-i\xi,\omega)\}^{-1}\big]=\mathcal{F}^{-1}_{\xi'\to x'}\big[\mathcal{F}^{-1}_{\xi_3\to x_3}\{A^{(0)}(-i\xi,\omega)\}^{-1}\big]=\mathcal{F}^{-1}_{\xi'\to x'}\big[\Pi(\xi',x_3)\big],$$
$$\xi'=(\xi_1,\xi_2),\quad x'=(x_1',x_2'),$$

with

$$\Pi(\xi',x_3)=\frac{1}{2\pi}\int\limits_{\mathbb{R}^1}\{A^{(0)}(-i\xi,\omega)\}^{-1}e^{-ix_3\xi_3}\,d\xi_3$$
$$=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int\limits_{\ell^+}\{A^{(0)}(-i\xi,\omega)\}^{-1}e^{-ix_3\xi_3}\,d\xi_3 & \text{for } x_3\le 0,\\ \dfrac{1}{2\pi}\displaystyle\int\limits_{\ell^-}\{A^{(0)}(-i\xi,\omega)\}^{-1}e^{-ix_3\xi_3}\,d\xi_3 & \text{for } x_3\ge 0,\end{cases}\tag{C.1}$$

where ℓ_- (ℓ_+) is a closed contours in the lower (upper) complex $\xi_3=\xi_3'+i\xi_3''$ half-plane, orientated clockwise (counterclockwise) and enclosing all roots with negative (positive) imaginary parts of the equation $\det A^{(0)}(\xi)=0$ with respect to ξ_3, while $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ is to be considered as a parameter.

Using the approach described in the monograph [10], one can write the explicit formulas for the principal homogeneous symbols of the boundary integral operators generated by the single and double layer potentials. These formulas read as follows:

$$\mathfrak{S}(\mathcal{H};x,\xi')=H(x,\xi')=\big[H_{pq}(x,\xi')\big]_{4\times4}=\begin{bmatrix}[H_{kj}(x,\xi')]_{3\times3} & [0]_{3\times1}\\ [0]_{1\times3} & H_{44}(x,\xi')\end{bmatrix}_{4\times4}$$
$$:=-\frac{1}{2\pi}\int\limits_{\ell_\pm}\big[A^{(0)}(B\xi,\omega)\big]^{-1}d\xi_3=-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\big[A^{(0)}(B\xi,\omega)\big]^{-1}d\xi_3,\tag{C.2}$$

$$\mathfrak{S}(\pm2^{-1}I_4+\widetilde{\mathcal{K}};x,\xi')=K^{(\pm)}(x,\xi')=\big[K^{(\pm)}_{pq}(x,\xi')\big]_{4\times4}=\begin{bmatrix}[K^{(\pm)}_{kj}(x,\xi')]_{3\times3} & [0]_{3\times1}\\ [0]_{1\times3} & \pm2^{-1}\end{bmatrix}_{4\times4}$$
$$:=\frac{i}{2\pi}\int\limits_{\ell_\pm}P^{(0)}(B\xi,n)\big[A^{(0)}(B\xi,\omega)\big]^{-1}d\xi_3,\tag{C.3}$$

$$\mathfrak{S}(\pm2^{-1}I_4+\mathcal{K};x,\xi')=\widetilde{K}^{(\pm)}x,\xi')=\big[\widetilde{K}^{(\pm)}_{pq}(x,\xi')\big]_{4\times4}=\begin{bmatrix}[\widetilde{K}^{(\pm)}_{kj}(x,\xi')]_{3\times3} & [0]_{3\times1}\\ [0]_{1\times3} & \pm2^{-1}\end{bmatrix}_{4\times4}$$
$$:=-\frac{i}{2\pi}\int\limits_{\ell_\mp}\big[A^{(0)}(B\xi,\omega)\big]^{-1}\big[P^{(0)}(B\xi,n)\big]^\top d\xi_3,\tag{C.4}$$

$$\mathfrak{S}(\mathcal{L};x,\xi')=L(x,\xi')=\big[L_{pq}(x,\xi')\big]_{4\times4}=\begin{bmatrix}[L_{kj}(x,\xi')]_{3\times3} & [0]_{3\times1}\\ [0]_{1\times3} & L_{44}(x,\xi')\end{bmatrix}_{4\times4}$$

$$:= -\frac{1}{2\pi}\int\limits_{\ell_\pm} P^{(0)}(B\xi,n)\big[A^{(0)}(B\xi,\omega)\big]^{-1}\big[P^{(0)}(B\xi,n)\big]^\top\,d\xi_3, \tag{C.5}$$

where $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$, $\xi=(\xi_1,\xi_2,\xi_3)^\top$, the matrices $A^{(0)}(\,\cdot\,)$ and $P^{(0)}(\,\cdot\,,\,\cdot\,)$ are

$$B(x)=\begin{bmatrix} l_1(x) & m_1(x) & n_1(x)\\ l_2(x) & m_2(x) & n_2(x)\\ l_3(x) & m_3(x) & n_3(x)\end{bmatrix}$$

is an orthogonal matrix with $\det B(x)=1$ for $x\in\partial\Omega^\pm=S$; here, $n(x)$ is the exterior unit normal vector to S, while $l(x)$ and $m(x)$ are orthogonal unit vectors in the tangential plane associated with some local chart; ℓ_- (ℓ_+) is a closed contour in the lower (upper) complex $\xi_3=\xi_3'+i\xi_3''$ half-plane, orientated clockwise (counterclockwise) and enclosing all roots with negative (positive) imaginary parts of the equation $\det A^{(0)}(B\xi)=0$ with respect to ξ_3, while $\xi'=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$ is to be considered as a parameter. Let $R>0$ be a sufficiently large positive number such that the circle C_R, centered at the origin and of radius R, encloses all the roots. Then, without loss of generality, we can take

$$\ell_-=[-R,+R]\cup C_R^{(-)},\quad \ell_+=[-R,+R]\cup C_R^{(+)},$$

where $C_R^{(-)}\subset C_R$ is a semi-circle in the lower half-plane orientated clockwise and $C_R^{(+)}\subset C_R$ is a semi-circle in the upper half-plane orientated counterclockwise.

In (C.3) and (C.4), we employed the fact that $\widetilde{\mathcal{K}}_{44}$ and $\mathcal{K}_{44}$ are weakly singular integral operators, since their kernel functions, the normal derivatives $\partial_{n(x)}\Psi_{44}(x-y)$ and $\partial_{n(y)}\Gamma_{44}(x-y)$, are weakly singular functions of type $\mathcal{O}(|x-y|^{-2+\kappa})$ on a $C^{1,\kappa}$ smooth surface S with $0<\kappa\leqslant 1$.

The entries of the matrices $H(x,\xi')$ and $L(x,\xi')$ are homogeneous functions in ξ' of order -1 and $+1$, respectively, while the entries of the matrices $K^{(\pm)}(x,\xi')$ and $\widetilde{K}^{(\pm)}(x,\xi')$ are homogeneous functions in $\xi'=(\xi_1,\xi_2)$ of order 0.

With the help of relations (B.10), it can easily be shown that there are positive constants δ_k, $k=1,2$, depending on the material parameters such that the following inequalities hold:

$$\operatorname{Re}\big[-\mathfrak{S}(\mathcal{H};x,\xi_1,\xi_2)\eta\cdot\eta\big]\ge\delta_1|\xi|^{-1}|\eta|^2\ \text{ for all }\ x\in S,\ \ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\},\ \ \eta\in\mathbb{C}^4, \tag{C.6}$$

$$\operatorname{Im}\big[-\mathfrak{S}(\mathcal{H};x,\xi_1,\xi_2)\eta\cdot\eta\big]\ge\delta_2|\xi|^{-1}|\eta|^2\ \text{ for all }\ x\in S,\ \ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\},\ \ \eta\in\mathbb{C}^4, \tag{C.7}$$

By the approach described in the proof of Theorem 5.1 and using relations (4.36)–(4.38) (cf. (5.24), (5.26), (5.27)), we show that there are the positive constants δ_3 and δ_4, depending on the material parameters, such that

$$\operatorname{Re}\big[\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)\eta\cdot\eta\big]\ge\delta_3|\xi|\,|\eta|^2\ \text{ for all }\ x\in S,\ \ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\},\ \ \eta\in\mathbb{C}^4, \tag{C.8}$$

$$\operatorname{Im}\big[\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)\eta\cdot\eta\big]\ge\delta_4|\xi|\,|\eta|^2\ \text{ for all }\ x\in S,\ \ (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\},\ \ \eta\in\mathbb{C}^4. \tag{C.9}$$

These inequalities imply that the principal homogeneous symbol matrices $-\mathfrak{S}(\mathcal{H};x,\xi_1,\xi_2)$ and $\mathfrak{S}(\mathcal{L};x,\xi_1,\xi_2)$ are strongly elliptic. Thus, the operators $-\mathcal{H}$ and $\mathcal{L}$ are strongly elliptic pseudodifferential operators.

From (C.6)–(C.9) and the last two equalities in (4.22), it follows that the principal homogeneous symbol matrices (C.3) and (C.4) are elliptic, which means that the operators $\pm 2^{-1}I_4+\mathcal{K}$ and $\pm 2^{-1}I_4+\widetilde{\mathcal{K}}$ are singular integral operators of normal type, i.e., the determinants of the principal homogeneous symbol matrices are different form zero (see also Remark 5.2).

9 Appendix D: Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary

Here, we present some results from the theory of strongly elliptic pseudodifferential equations on manifolds with a boundary (see, e.g., [42, 50, 91]), which play a crucial role for proving existence theorems by the potential method for mixed boundary value problems, mixed boundary-transmission and crack type problems.

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, non-self-intersecting manifold with the boundary $\partial\mathcal{M} \in C^\infty$, and let $\mathbf{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathbf{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathbf{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \dots, \lambda_N(x)$ be the eigenvalues of the matrix

$$\big[\mathfrak{S}(\mathbf{A}; x, 0, \dots, 0, +1)\big]^{-1}\big[\mathfrak{S}(\mathbf{A}; x, 0, \dots, 0, -1)\big], \quad x \in \partial\mathcal{M}.$$

Introduce the notation

$$\delta_j(x) = \operatorname{Re}\big[(2\pi i)^{-1} \ln \lambda_j(x)\big], \quad j = 1, \dots, N,$$

where $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$. Due to the strong ellipticity of $\mathbf{A}$, we have the strict inequality

$$-frac12 < \delta_j(x) < frac12 \;\text{ for }\; x \in \overline{\mathcal{M}}, \;\; j = 1, \dots, N.$$

The numbers $\delta_j(x)$ do not depend on a particular choice of the local coordinate system at a fixed point $x \in \partial\mathcal{M}$. Note that in particular cases, when $\mathfrak{S}(\mathbf{A}; x, \xi)$ is an even matrix function in ξ or $\mathfrak{S}(\mathbf{A}; x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_j(x) = 0$ for $j = 1, \dots, N$, since all the eigenvalues $\lambda_j(x)$ ($j = 1, \dots, N$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with a boundary are given in the following theorem.

Theorem D.1. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and let $\mathbf{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\operatorname{Re}\mathfrak{S}(\mathbf{A}; x, \xi)\eta \cdot \eta \ge c_0|\eta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.

Then the operators

$$\mathbf{A} : \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\nu}_p(\mathcal{M})\big]^N, \tag{D.1}$$

$$\mathbf{A} : \big[\widetilde{B}^s_{p,t}(\mathcal{M})\big]^N \to \big[B^{s-\nu}_{p,t}(\mathcal{M})\big]^N, \tag{D.2}$$

are Fredholm with zero index if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\, 1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\, 1\le j\le N} \delta_j(x). \tag{D.3}$$

Moreover, the null–spaces and indices of operators (D.1) *and* (D.2) *are the same* (*for all values of the parameter $t \in [1, +\infty]$*) *provided p and s satisfy inequality* (D.3).

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