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ON THE L^p HEISENBERG–PAULI–WEYL INEQUALITY FOR FLENSTED–JENSEN PARTIAL DIFFERENTIAL OPERATORS

Abstract. We establish an L^p Heisenberg–Pauli–Weyl inequality related to the Flensted-Jensen differential operators.

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რეზიუმე. დადგენილია ჰაიზენბერგ-პაული-ვეილის L^p უტოლობა, რომელიც დაკავშირებულია ფლენსტედ-იენსენის დიფერენციალურ ოპერატორებთან.

1 Introduction

The qualitative uncertainty principle states that a function and its Fourier transform cannot both be sharply localized unless the function is identically zero. This fundamental principle has multiple formulations, each highlighting different aspects of this inherent limitation.

The most quantitative formulation is the standard Heisenberg–Pauli–Weyl inequality, which states that for $u \in L^2(\mathbb{R})$, we have

$$\left(\int_{\mathbb{R}} x^2 |u(x)|^2\, dx\right)\left(\int_{\mathbb{R}} \xi^2 |\widehat{u}(\xi)|^2\, d\xi\right) \geq \frac{1}{4}\, \|u\|^4, \tag{1.1}$$

where

$$\widehat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{-i\xi x}\, dx.$$

This classical result has been extended in various settings. A notable generalization of inequality (1.1) states that for $1 \leq p \leq 2$, the following inequality holds (see [3]):

$$\|xu\|_p \|\xi \widehat{u}\|_p \geq \frac{\|u\|_2^2}{4\pi} \quad (1 \leq p \leq 2).$$

In this paper, we give an L^p-Heisenberg–Pauli–Weyl inequality associated to the Flensted-Jensen partial differential operators.

2 Preliminary

We start this section by setting some notations and collecting some basic results about the Flensted-Jensen partial differential operators.

Let $\alpha \geq 0$, we consider the system of Flensted-Jensen partial differential operators defined by

$$\begin{cases} D_\theta = \dfrac{\partial}{\partial \theta}\,, \\ D_\alpha = \dfrac{\partial^2}{\partial y^2} + \big[(2\alpha+1)\coth(y) + \tanh(y)\big] \dfrac{\partial}{\partial y} - \dfrac{1}{\cosh^2(y)} \dfrac{\partial^2}{\partial \theta^2} + (\alpha+1)^2, \end{cases}$$

where $(y, \theta) \in \mathbb{K} = [0, +\infty[\times \mathbb{R}$.

We denote $\widehat{\mathbb{K}} = L \cup \Omega$, where $L = \mathbb{R} \times [0, +\infty[$ and $\Omega = \bigcup\limits_{m \in \mathbb{N}} (D_m^+ \cup D_m^-)$ with D_m^+, D_m^- given by

$$D_m^+ = \big\{(\alpha + 2m + 1 + \eta; i\eta);\ \eta > 0\big\}$$

and

$$D_m^- = \big\{(-\alpha - 2m - 1 - \eta; i\eta);\ \eta > 0\big\}.$$

For $1 \leq p < \infty$, consider $L^p(\mathbb{K})$, the space of measurable functions f on $\mathbb{K}$ such that

$$\begin{cases} \|f\|_{p,m_\alpha} = \left(\displaystyle\int_{\mathbb{K}} |f(y,\theta)|^p\, dm_\alpha(y,\theta)\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_\infty = \operatorname*{ess\,sup}\limits_{(y,\theta) \in \mathbb{K}} |f(y,\theta)| < \infty, \end{cases}$$

where

$$dm_\alpha(y,\theta) = 2^{2(\alpha+1)} (\sin(y))^{2\alpha+1} \cosh(y)\, dy\, d\theta.$$

Now, we define the Fourier transform related to the Flensted-Jensen operators on $L^1(\mathbb{K})$ by

$$\forall\, (\lambda, \mu) \in \widehat{\mathbb{K}}: \quad \mathcal{F}_\alpha(f)(\lambda, \mu) = \int_{\mathbb{K}} f(y,\theta) \varphi_{-\lambda,\mu}(y,\theta)\, dm_\alpha(y,\theta),$$

where

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\cosh(y))^{\lambda}\varphi_{\mu}^{\alpha,\lambda}(y)$$

and

$$\varphi_{\mu}^{\alpha,\lambda}((y) = {}_2F_1\Big(\frac{\alpha+\lambda+1+i\mu}{2}, \frac{\alpha+\lambda+1-i\mu}{2}; \alpha+1; -\sinh^2(y)\Big).$$

Here, ${}_2F_1$ is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}\, z^n,$$

where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad n \geq 1,$$

and $(a)_0 = 1$.

It is known that the inverse of $\mathcal{F}_\alpha$ is given by

$$\mathcal{F}_\alpha^{-1}(f)(y,\theta) = \int_{\widehat{\mathbb{K}}} f(\lambda,\mu)\varphi_{\lambda,\mu}(y,\theta)\, d\gamma_\alpha(\lambda,\mu),$$

where $\gamma_\alpha(\lambda,\mu)$ is the Plancherel measure defined on $\widehat{\mathbb{K}}$ by

$$\begin{aligned}\int_{\widehat{\mathbb{K}}} g\, d\gamma_\alpha(\lambda,\mu) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}\times[0,+\infty[} g(\lambda,\mu)\, \frac{d\lambda\, d\mu}{|C_1(\lambda,\mu)|^2} \\ &\quad + \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \int_0^{\infty} \Big\{ g(\alpha+2m+1+\eta, i\eta) C_2(\alpha+2m+1+\eta, i\eta)\, d\eta \\ &\qquad + \int_0^{\infty} g(-\alpha-2m-1-\eta, i\eta) C_2(-\alpha-2m-1-\eta, i\eta)\, d\eta \Big\},\end{aligned}$$

where

$$C_1(\lambda,\mu) = \frac{2^{\alpha+1-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{\alpha+\lambda+1+i\mu}{2})\Gamma(\frac{\alpha-\lambda+1+i\mu}{2})}, \quad (\lambda,\mu) \in L,$$

and

$$C_2(\lambda,\mu) = -2i\pi\, \mathrm{Res}_{z=\mu}\, \big[C_1(\lambda,z)C_1(\lambda,-z)\big]^{-1}, \quad (\lambda,\mu) \in \Omega.$$

We have (see [2])

$$C_2(\lambda,\mu) = 2^{-2(n-1)}\pi\eta\, \frac{(m+1)_{n-1}(\eta+m+1)_{n-1}}{((n-1)!)^2},$$

where $n = \alpha+1 \in \mathbb{N}^*$.

Now, let $L^p(\widehat{\mathbb{K}})$ be the space of functions f on $\widehat{\mathbb{K}}$ such that

$$\begin{cases} \|f\|_{p,\gamma_\alpha} = \Big(\int_{\widehat{\mathbb{K}}} |f(\lambda,\mu)|^p\, d\gamma_\alpha(\lambda,\mu)\Big)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{\infty,\gamma_\alpha} = \operatorname*{ess\,sup}_{(\lambda,\mu)\in\widehat{\mathbb{K}}} |f(\lambda,\mu)| < \infty. \end{cases}$$

For $f \in L^1(\mathbb{K})$, we have

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\gamma_\alpha} \leq \|f\|_{1,m_\alpha}. \tag{2.1}$$

For $f \in L^2(\mathbb{K})$, we also have the Plancherel theorem

$$\|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}. \tag{2.2}$$

Hence, using the Marcinkiewicz interpolation with (2.1) and (2.2), for every $1 \leq p \leq 2$ and $f \in L^p(\mathbb{K})$, we obtain

$$\|\mathcal{F}_\alpha(f)\|_{q,\gamma_\alpha} \leq \|f\|_{p,m_\alpha}, \tag{2.3}$$

where $q = \frac{p}{p-1}$.

For all $(y, \theta) \in \mathbb{K}$, we denote $|y, \theta| = |y| + |\theta| = y + |\theta|$ and $(\lambda, \mu) \in \widehat{\mathbb{K}}: \ |\lambda, \mu| = |\lambda| + |\mu|$.

From now on, we suppose $\alpha = 1$. In order to prove our main result, we begin by establishing the following Lemma.

Lemma 2.1. *For $q > 1$ and $t > 0$, we have*

$$\left\|e^{-t(|\lambda,\mu|^2+4)}\right\|_{q,\gamma} \leq h(t) = \begin{cases} \dfrac{C}{t^{\frac{4}{q}}} & \text{if } 0 < t < 1, \\ \dfrac{Ce^{-\frac{4t}{q}}}{t^{\frac{1}{q}}} & \text{if } 1 \leq t. \end{cases}$$

Proof. We have

$$\begin{aligned} &\|e^{-t(|\lambda,\mu|^2+4)}\|_{q,\gamma}^q \\ &\leq Ce^{-4t}\Bigg(\int\limits_{\mathbb{R}\times[0,+\infty[} e^{-qt|\lambda,\mu|^2}\frac{d\lambda\, d\mu}{|C_1(\lambda,\mu)|^2} + \frac{2}{(2\pi)^2}\sum_{m=0}^{\infty}\int_0^\infty e^{-qt(2+2m+2\eta)^2}C_2(2+2m+\eta, i\eta)\, d\eta\Bigg). \end{aligned}$$

According to [7, p. 50], we have $|C_1(\lambda,\mu)|^{-2} \leq (1+|\lambda|^2+|\mu|^2)^{2[\alpha+\frac{1}{2}]+1}$. Then

$$\begin{aligned} \int\limits_{\mathbb{R}\times[0,+\infty[} e^{-qt(|\lambda,\mu|^2+4)}\frac{d\lambda\, d\mu}{|C_1(\lambda,\mu)|^2} &\leq e^{-4t}\int\limits_{\mathbb{R}\times[0,+\infty[} e^{-qt|\lambda,\mu|^2}\left(1+|\lambda|^2+|\mu|^2\right)^3 d\lambda\, d\mu \\ &\leq e^{-4t}\int\limits_{\mathbb{R}\times[0,+\infty[} e^{-qt(|\lambda|^2+|\mu|^2)}(1+|\lambda|^2+|\mu|^2)^3\, d\lambda\, d\mu \leq Ce^{-4t}\int_0^\infty e^{-qts^2}(1+s^2)^3 s\, ds \\ &\leq Ce^{-4t}\Big(\frac{3}{(qt)^4}+\frac{3}{(qt)^3}+\frac{3}{2(qt)^2}+\frac{1}{2(qt)}\Big) \leq \begin{cases} \dfrac{C}{t^4} & \text{if } 0 < t < 1, \\ \dfrac{Ce^{-4t}}{t} & \text{if } 1 \leq t. \end{cases} \end{aligned}$$

It remains to show that for all $t > 0$,

$$S(qt) \leq \begin{cases} \dfrac{C}{t^4} & \text{if } 0 < t < 1, \\ \dfrac{Ce^{-4t}}{t} & \text{if } 1 \leq t, \end{cases}$$

where

$$\begin{aligned} S(t) =& \frac{\pi}{4}\sum_{m=0}^{\infty}\int_0^\infty e^{-t(2+2m+2\eta)^2}C_2(2+2m+\eta, i\eta)\, d\eta \\ =& \frac{\pi}{4}\sum_{m=0}^{\infty}\int_0^\infty (m+1)e^{-t(2+2m+2\eta)^2}\eta(\eta+m+1)\, d\eta. \end{aligned}$$

Making a change of variables $s = 2\sqrt{t}\,(1 + m + \eta)$ and integrating, we obtain

$$\int_0^\infty e^{-t(2+2m+2\eta)^2}\eta(\eta + m + 1)\, d\eta = erfc\big(2\sqrt{t}\,(m+1)\big),$$

where $erfc$ is the complementary error function defined by

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\, dt, \quad x \in \mathbb{R}.$$

Therefore, we have

$$S(t) = \frac{\pi\sqrt{\pi}}{128}\sum_{m=1}^{\infty} m\, \frac{erfc(2\sqrt{t}\, m)}{t^{\frac{3}{2}}}\,.$$

Since $t \to S(t)$ is continuous on $\mathbb{R}_+^*$, it suffices to show that $\lim\limits_{t\to 0^+} t^4 S(t) \in \mathbb{R}^+$ and $\lim\limits_{t\to\infty} tS(t) \in \mathbb{R}$. Towards this end, we set $g(x) = xerfc(2\sqrt{t}\, x)$, $x \geq 1$. One has

$$g'(x) = e^{-4t^2x^2}\Big(e^{4t^2x^2} erfc\big(2\sqrt{t}\, x\big) - 4\sqrt{\frac{t}{\pi}}\, x\Big).$$

But (see [1, p. 303])

$$\frac{2}{\sqrt{\pi}}\,\frac{e^{-x^2}}{x + \sqrt{x^2+2}} < erfc(x) < \frac{2}{\sqrt{\pi}}\,\frac{e^{-x^2}}{x + \sqrt{x^2+\frac{\pi}{4}}}\,, \quad x \geq 0.$$

Hence

$$\lim_{x\to+\infty} e^{4t^2x^2} erfc\big(2\sqrt{t}\, x\big) - 4\sqrt{\frac{t}{\pi}}\, x = -\infty,$$

and so $\exists\, m_0 \in \mathbb{N} :\ x \geq m_0 \Longrightarrow g'(x) < 0$.

Thus g is a decreasing function on $[m_0, \infty[$, henceforth it follows that

$$\forall\, t > 0: \quad \int_{m_0+1}^\infty x\, erfc\big(2\sqrt{t}\, x\big)\, dx \leq \sum_{n=m_0+1}^\infty n\, erfc\big(2\sqrt{t}\, n\big) \leq \int_{m_0}^\infty x\, erfc\big(2\sqrt{t}\, x\big)\, dx.$$

But

$$\frac{2}{\sqrt{\pi}}\,\frac{e^{-x^2}}{x + \sqrt{x^2+2}} < erfc(x) < \frac{2}{\sqrt{\pi}}\,\frac{e^{-x^2}}{x + \sqrt{x^2+\frac{\pi}{4}}}\,, \quad x \geq 0,$$

so $\lim\limits_{x\to\infty} x^2\, erfc(x) = 0$ and by integration by parts we obtain

$$\int_m^\infty x\, erfc\big(2\sqrt{t}\, x\big)\, dx$$
$$= \frac{1}{16t\sqrt{\pi}}\Big(-8m^2 t\sqrt{\pi}\, erfc\big(2\sqrt{t}\, m\big)e^{-4tm^2} + 4m\sqrt{t}\, e^{-4tm^2} + \sqrt{\pi}\, erfc\big(2\sqrt{t}\, m\big)\Big),$$

where $m \in \mathbb{N}$.

Hence

$$\lim_{t\to 0^+} t^4 S(t) = 0.$$

On the other hand, we have

$$\forall\, t > 0: \quad 0 \leq tS(t) \leq Ct\sum_{k=0}^\infty \int_0^\infty (k+1)e^{-t(1+k+\eta)}\eta(\eta + k + 1)\, d\eta.$$

Since for $s > 0$,

$$\sum_{m=0}^{\infty} m e^{-ms} = \frac{e^{-s}}{(e^{-s}-1)^2}\,,$$
$$\sum_{m=0}^{\infty} m^2 e^{-ms} = \frac{e^{-s}(1+e^{-s})}{(1-e^{-s})^3}\,,$$
$$\sum_{m=0}^{\infty} e^{-ms} = \frac{1}{1-e^{-s}}\,,$$

it follows that

$$t\sum_{m=0}^{\infty}\int_0^{\infty}(m+1)e^{-t(1+m+\eta)}\eta(\eta+m+1)\,d\eta = \frac{e^t(te^t+2e^t-2+t)}{t^2(e^t-1)^3}\,.$$

Consequently,

$$\lim_{t\to\infty} tS(t) = 0.$$

Thus the proof of lemma is complete. □

Let $B_r = \{(y,\theta)\in\mathbb{K}:\ |y,\theta| < r\}$ and $B_r^c = \mathbb{K}\setminus B_r$ for some $r>0$. Denote by χ_{B_r} and $\chi_{B_r^c}$ the characteristic functions.

Proposition 2.1. *Let $1 < p \le 2$, $q = \frac{p}{p-1}$ and $0 < a < \frac{5}{q}$. Then for all $f \in L^p(\mathbb{K})$ and $t > 0$, we have*

$$\left\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha f\right\|_{q,\gamma} \le Ct^{-\frac{4a}{5}}\|\,|y,\theta|^a f\|_{p,m_\alpha}.$$

Proof. We have

$$\begin{aligned}\left\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha(f\chi_{B_r})\right\|_{q,\gamma} &\le \left\|e^{-t(|\lambda,\mu|^2+4)}\right\|_{q,\gamma}\|\mathcal{F}_\alpha(f\chi_{B_r})\|_{\infty,\gamma}\\ (\text{by } (2.1)) &\le \left\|e^{-t(|\lambda,\mu|^2+4)}\right\|_{q,\gamma}\|f\chi_{B_r}\|_1\\ (\text{by Hölder's inequality}) &\le h(t)\|\,|y,\theta|^{-a}\chi_{B_r}\|_{q,m_\alpha}\|\,|y,\theta|^a f\|_{p,m_\alpha}.\end{aligned}$$

But

$$4(\sinh(y))^3(\cosh(y)) \sim_{0^+} 4y^3 \ \text{ and }\ 4(\sinh(y))^3(\cosh(y)) \sim_{+\infty} 4e^{4y},$$

so if $r < 1$, we have

$$\begin{aligned}\left\|\,|y,\theta|^{-a}\chi_{B_r}\right\|_{q,m_\alpha}^q &= 16\int_{\mathbb{R}_+\times\mathbb{R}}\chi_{B_r}(|y|+|\theta|)^{-aq}\sin^3(y)\cosh(y)\,dy\,d\theta\\ &\le C\int_{\mathbb{R}_+\times\mathbb{R}}\chi_{B_r}(y,\theta)(|y|+|\theta|)^{-aq+3}\,dy\,d\theta \le C\int_0^r s^{4-aq}\,ds \le Cr^{5-aq},\end{aligned}$$

and if $r \ge 1$, we have

$$\begin{aligned}\left\|\,|y,\theta|^{-a}\chi_{B_r}\right\|_{q,m_\alpha}^q &\le C + C'\int_{\{(y,\theta)\in\mathbb{R}_+\times\mathbb{R}:\ 1\le|y|+|\theta|<r\}}(|y|+|\theta|)^{-aq}e^{4y}\,dy\,d\theta\\ &\le C + C'e^{4r}\int_{\{(y,\theta)\in\mathbb{R}_+\times\mathbb{R}:\ 1\le|y|+|\theta|<r\}}(|y|+|\theta|)^{-aq}\,dy\,d\theta\\ &\le C + C'e^{4r}\int_{\{\frac12\le s<r\}}s^{-aq}\,ds \le Cr^{-aq+1}e^{4r}.\end{aligned}$$

In the last line, we have used the fact that the function $y \to y^{1-aq}e^{4y}$ is increasing for $y \geq \frac{aq-1}{4}$ and $r \geq 1 > \frac{aq-1}{4}$. Thus

$$\big\| |y,\theta|^{-a}\chi_{B_r}\big\|_{q,m_\alpha} \leq r^{-a}V(r),$$

where

$$V(r) = \begin{cases} Cr^{\frac{5}{q}} & \text{if } 0 < r < 1, \\ Cr^{\frac{1}{q}}e^{\frac{4r}{q}} & \text{if } 1 \leq r. \end{cases}$$

Therefore,

$$\big\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha(f\chi_{B_r})\big\|_{q,\gamma} \leq CV(r)h(t)\||y,\theta|^a f\|_{p,m_\alpha},$$

and we have

$$\begin{aligned}\big\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha(f\chi_{B_r^c})\big\|_{q,\gamma} &\leq \|e^{-t|\lambda,\mu|^2}\|_{\infty,\gamma}\|\mathcal{F}_\alpha(f\chi_{B_r^c})\|_{q,\gamma} \\ (\text{by } (2.3)) &\leq \|f\chi_{B_r^c}\|_{p,m_\alpha} \\ &\leq r^{-a}\||y,\theta|^a f\|_{p,m_\alpha}.\end{aligned}$$

Hence

$$\begin{aligned}\big\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha(f)\big\|_{q,\gamma} &\leq \big\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha(f\chi_{B_r})\big\|_{q,\gamma} + \big\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha(f\chi_{B_r^c})\big\|_{q,\gamma} \\ &\leq C\Big(V(r)h(t)r^{-a}\|\,|y,\theta|^a f\|_{p,m_\alpha} + r^{-a}\|\,|y,\theta|^a f\|_{p,m_\alpha}\Big) \\ &\leq C(1+V(r)h(t))r^{-a}\|\,|y,\theta|^a f\|_{p,m_\alpha}.\end{aligned}$$

For $r = t^{\frac{4}{5}}$, we obtain

$$\big\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha(f)\big\|_{q,\gamma} \leq Ct^{\frac{-4a}{5}}\||y,\theta|^a f\|_{p,m_\alpha}.$$

□

3 L^p-Heisenberg inequality

In order to prove our results we need the following Lemma.

Lemma 3.1. *Let $1 < p \leq 2$, $q = \frac{p}{p-1}$, $0 < a < \frac{5}{q}$ and $b > 0$. Then for $f \in L^p(\mathbb{K})$ one has*

$$\|\mathcal{F}_\alpha(f)\|_{q,\gamma} \leq C(a,b)\|\,|y,\theta|^a f\|_{p,m_\alpha}^{\frac{b}{a+b}}\big\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\gamma}^{\frac{a}{a+b}},$$

where $C(a,b)$ is a positive constant.

Proof. We have

$$\|\mathcal{F}_\alpha(f)\|_{q,\gamma} \leq \big\|e^{-t(|\lambda,\mu|^2+4)}\mathcal{F}_\alpha(f)\big\|_{q,\lambda} + \big\|(1-e^{-t(|\lambda,\mu|^2+4)})\mathcal{F}_\alpha(f)\big\|_{q,\gamma},$$

since $x \to (1-e^{-x})x^{-\frac{4b}{5}}$ is bounded for $x \geq 0$ if $b \leq \frac{5}{4}$. Further,

$$\begin{aligned}\big\|(1-e^{-t(|\lambda,\mu|^2+4)})\mathcal{F}_\alpha(f)\big\|_{q,\gamma} &= t^{\frac{4b}{5}}\Big\|\big(-t(|\lambda,\mu|^2+4)\big)^{-\frac{4b}{5}}\big(1-e^{-t(|\lambda,\mu|^2+4)}\big)(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\Big\|_{q,\gamma} \\ &\leq Ct^{\frac{4b}{5}}\big\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\gamma}.\end{aligned}$$

So, by Proposition 2.1, we have

$$\|\mathcal{F}_\alpha(f)\|_{q,\gamma} \leq C\Big(t^{-\frac{4a}{5}}\||y,\theta|^a f\|_{p,m} + t^{\frac{4b}{5}}\big\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\gamma}\Big).$$

Now, choosing

$$t = \Big(\frac{a}{b}\,\frac{\|\,|y,\theta|^a f\|_{p,m}}{\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\|_{q,\gamma}}\Big)^{\frac{5}{4(a+b)}},$$

we obtain

$$\|\mathcal{F}_\alpha(f)\|_{q,\gamma} \leq C(a,b)\| \, |y,\theta|^a f\|_{p,m_\alpha}^{\frac{b}{a+b}} \big\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\lambda}^{\frac{a}{a+b}}.$$

If $b > \frac{5}{4}$, letting $b' < \frac{5}{4}$, we have

$$\forall\, \epsilon > 0: \quad (|\lambda,\mu|^2+4)^{\frac{4b'}{5}} \leq \epsilon^{b'} + \epsilon^{b'-b}(|\lambda,\mu|^2+4)^{\frac{4b}{5}}.$$

So,

$$\big\|(|\lambda,\mu|^2+4)^{\frac{4b'}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\gamma} \leq \epsilon^{b'}\|\mathcal{F}_\alpha(f)\|_{q,\gamma} + \epsilon^{b'-b}\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\gamma}.$$

Optimizing in ϵ, we get

$$\big\|(|\lambda,\mu|^2+4)^{\frac{4b'}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\gamma} \leq C\|\mathcal{F}_\alpha(f)\|_{q,\gamma}^{1-\frac{b'}{b}} \big\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\gamma}^{\frac{b'}{b}}.$$

Therefore,

$$\begin{aligned}\|\mathcal{F}_\alpha(f)\|_{q,\gamma} &\leq C(a,b')\| \, |y,\theta|^a f\|_{p,m}^{\frac{b'}{a+b'}} \big\|(|\lambda,\mu|^2+4)^{\frac{4b'}{5}}\mathcal{F}_\alpha(f)\big\|_{q,\lambda}^{\frac{a}{a+b'}} \\ &\leq C(a,b)\| \, |y,\theta|^a f\|_{p,m}^{\frac{b'}{a+b'}} \|\mathcal{F}_\alpha(f)\|_{q,\gamma}^{(1-\frac{b'}{b})\,\frac{a}{a+b'}} \big\|(|\lambda,\mu|^2+4)^{b}\mathcal{F}_\alpha(f)\big\|_{q,\gamma}^{\frac{b'}{b}\,\frac{a}{a+b'}},\end{aligned}$$

which gives the result for $b > \frac{5}{4}$. □

Now, we can give an L^2 Heisenberg inequality for the Flensted-Jensen partial differential operators.

Theorem 3.1. *Let $a, b > 0$. Then for $f \in L^2(\mathbb{K})$, one has*

$$\|f\|_{2,m_\alpha} \leq C(a,b)\| \, |y,\theta|^a f\|_{2,m_\alpha}^{\frac{b}{a+b}} \big\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{2,\gamma}^{\frac{a}{a+b}}, \tag{3.1}$$

where $C(a,b)$ is a positive constant.

Proof. By Lemma 3.1 and the Plancherel formula, inequality (3.1) holds for $p = 2$, when $0 < a < \frac{5}{q}$. If $a \geq \frac{5}{q}$, let $0 < a' < \frac{5}{q}$, for $x \geq 0$, we have

$$\forall\, \epsilon > 0: \quad \Big(\frac{x}{\epsilon}\Big)^{a'} \leq 1 + \Big(\frac{x}{\epsilon}\Big)^{a}.$$

So, for $f \in L^2(\mathbb{K})$, one has

$$\| \, |y,\theta|^{a'} f\|_{2,m_\alpha} \leq \epsilon^{a'}\|f\|_{2,m_\alpha} + \epsilon^{a'-a}\||y,\theta|^a f\|_{2,m_\alpha}.$$

Optimizing in ϵ, we get

$$\| \, |y,\theta|^{a'} f\|_{2,m_\alpha} \leq C\|f\|_{2,m_\alpha}^{\frac{a-a'}{a}} \| \, |y,\theta|^a f\|_{2,m_\alpha}^{\frac{a'}{a}}. \tag{3.2}$$

Therefore, using Lemma 3.1 for a' and b and inequality (3.2), we obtain

$$\begin{aligned}\|f\|_{2,m_\alpha} &\leq C(a',b)\| \, |y,\theta|^{a'} f\|_{2,m_\alpha}^{\frac{b}{a'+b}} \big\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{2,\gamma}^{\frac{a'}{a'+b}} \\ &\leq C(a',b)\|f\|_{2,m_\alpha}^{\frac{b(a'-a)}{a(a'+b)}} \| \, |y,\theta|^a f\|_{2,m_\alpha}^{\frac{a'b}{a(a'+b)}} \big\|(|\lambda,\mu|^2+4)^{\frac{4b}{5}}\mathcal{F}_\alpha(f)\big\|_{2,\gamma}^{\frac{a'}{a'+b}},\end{aligned}$$

which leads to (3.1). □

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