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# THE $L^p$ NEUMANN PROBLEM FOR HIGHER ORDER ELLIPTIC EQUATIONS

**Abstract.** We solve the Neumann problem in the half-space $\mathbb{R}^{n+1}_+$ for higher order elliptic differential equations with variable self-adjoint $t$-independent coefficients and with boundary data in $L^p$, where $\max(1, \frac{2n}{n+2} - \varepsilon) < p < 2$.

We also establish nontangential and area integral estimates on layer potentials with inputs in $L^p$ or $\dot{W}^{\pm 1,p}$ for a similar range of $p$, based on the known bounds for $p \geq 2$; in this case, we may relax the requirement of self-adjointness.

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**Key words and phrases.** Elliptic equation, higher-order differential equation, Neumann problem, layer potentials

**რეზიუმე.** ამოხსნილია ნეიმანის ამოცანა $\mathbb{R}^{n+1}_+$ ნახევარსივრცეში მაღალი რიგის ელიფსური დიფერენციალური განტოლებებისთვის ცვლადი თვითშეუღლებული $t$-ზე დამოუკიდებელი კოეფიციენტებით და სასაზღვრო მონაცემებით $L^p$ სივცცეში, სადაც $\max(1, \frac{2n}{n+2} - \varepsilon) < p < 2$.

აგრეთვე დადგენილია არამხები და ზედაპირული ინტეგრალური შეფასებები ფენის პოტენციალებისთვის მონაცემებით $L^p$ ან $\dot{W}^{\pm 1,p}$-ში $p$-ის მსგავსი დიაპაზონისთვის, $p \geq 2$-ის ცნობილი საზღვრების საფუძველზე; ამ შემთხვევაში, შესაძლებელია თვითშეუღლებულობის მოთხოვნის შემსუბუქება.

# 1 Introduction

In this paper, we study the Neumann boundary value problem and layer potentials for higher order elliptic differential operators of the form

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u), \tag{1.1}$$

where $m$ is a positive integer, and with coefficients $\boldsymbol{A}$ that are $t$-independent in the sense that

$$\boldsymbol{A}(x,t) = \boldsymbol{A}(x,s) = \boldsymbol{A}(x) \text{ for all } x \in \mathbb{R}^n \text{ and all } s,\, t \in \mathbb{R}. \tag{1.2}$$

Our coefficients may be merely bounded measurable in the $n$ horizontal variables. Second order operators with $t$-independent coefficients have been studied extensively; see, for example, [2, 5–8, 10–12, 19, 25, 43–45, 47, 48, 51, 52, 56, 57, 64]. Higher order operators with $t$-independent coefficients have been studied by Hofmann and Mayboroda together with the author of the present paper in [15, 20–24].

Specifically, in [21, 24], we established the following result. Suppose that $L$ is an operator of the form (1.1) associated to the coefficients $\boldsymbol{A}$ that are $t$-independent, bounded, self-adjoint in the sense that $A_{\alpha\beta} = \overline{A_{\beta\alpha}}$ whenever $|\alpha| = |\beta| = m$, and satisfy the boundary Gårding inequality

$$\operatorname{Re} \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} \overline{\partial^\alpha \varphi(x,t)}\, A_{\alpha\beta}(x)\, \partial^\beta \varphi(x,t)\, dx \geq \lambda \|\nabla^m \varphi(\,\cdot\,,t)\|^2_{L^2(\mathbb{R}^n)} \tag{1.3}$$

for all $t \in \mathbb{R}$, all smooth test functions $\varphi$ that are compactly supported in $\mathbb{R}^{n+1}$, and some $\lambda > 0$ independent of $t$ and $\varphi$. Then for every $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$ there is a solution $w$, unique up to the adding polynomials of degree $m-1$, to the $L^2$ Neumann problem

$$\begin{cases} Lw = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \dot{\mathbf{M}}^+_{\boldsymbol{A}} w \ni \dot{\boldsymbol{g}}, \\ \|\mathcal{A}^+_2(t\nabla^m \partial_t w)\|_{L^2(\mathbb{R}^n)} + \|\widetilde{N}_+(\nabla^m w)\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}. \end{cases} \tag{1.4}$$

Here, $\widetilde{N}_+$ is the modified nontangential maximal operator introduced in [51] and given (in $\mathbb{R}^{n+1}_+$) by

$$\widetilde{N}_+ H(x) = \sup \left\{ \left( \fint_{B((y,s),s/2)} |H(z,t)|^2 \, dz\, dt \right)^{1/2} : \ s > 0,\ \ |x-y| < s \right\}. \tag{1.5}$$

$\mathcal{A}^+_2$ is the Lusin area integral given by

$$\mathcal{A}^+_2 H(x) = \left( \int_0^\infty \int_{|x-y|<t} |H(y,t)|^2 \, \frac{dy\, dt}{t^{n+1}} \right)^{1/2}. \tag{1.6}$$

We adopt the convention that if $t$ appears inside the argument of a tent space operator such as $\mathcal{A}^+_2$, then it denotes the $(n+1)$th coordinate function.

$\dot{\mathbf{M}}^+_{\boldsymbol{A}} w$ denotes the Neumann boundary values of $w$, and is the equivalence class of functions given by

$$\dot{\boldsymbol{g}} \in \dot{\mathbf{M}}^+_{\boldsymbol{A}} w \ \text{ if } \sum_{|\gamma|=m-1} \int_{\mathbb{R}^n} \partial^\gamma \varphi(x,0)\, g_\gamma(x)\, dx = \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^{n+1}_+} \partial^\alpha \varphi\, A_{\alpha\beta}\, \partial^\beta w \tag{1.7}$$

for all smooth test functions $\varphi$ that are compactly supported in $\mathbb{R}^{n+1}$. An integration by parts argument shows that the right-hand side depends only on the behavior of $\varphi$ near the boundary, and so $\dot{\mathbf{M}}^+_{\boldsymbol{A}} w$ is well defined as an operator on the space $\{\nabla^{m-1}\varphi|_{\partial\mathbb{R}^{n+1}_+} : \ \varphi \in C^\infty_0(\mathbb{R}^{n+1})\}$.

In the second order case $2m = 2$, $\mathrm{M}^+_{\boldsymbol{A}} w$ consists of a single distribution; however, if $m \geq 2$, then, by equality of mixed partials, $\dot{\mathbf{M}}^+_{\boldsymbol{A}} w$ contains many arrays of distributions, and so is indeed an equivalence class. This is the formulation of the Neumann boundary data used in [14, 15, 20, 21, 23, 24], and is closely related to the Neumann boundary values for the bilaplacian in [29, 60, 67, 71] and for general constant coefficient systems in [61, 72, 73]. We refer the reader to [18, 20] for further discussion of higher order Neumann boundary data.

In the present paper we extend from results for $L^2$ boundary data to $L^p$ boundary data for $p < 2$. The first of the two main results of the present paper is the following theorem. (The second main result is Theorem 1.2 below.)

**Theorem 1.1.** *Suppose that $L$ is an elliptic operator of the form (1.1) of order $2m$ associated with coefficients $\boldsymbol{A}$ that are bounded, $t$-independent in the sense of formula (1.2), satisfy the ellipticity condition (1.3), and are self-adjoint in the sense that $A_{\alpha\beta}(x) = \overline{A_{\beta\alpha}(x)}$ for all $|\alpha| = |\beta| = m$ and all $x \in \mathbb{R}^n$.*

*Then there is a positive number $\varepsilon > 0$, depending only on the dimension $n+1$, the order $2m$ of the operator $L$, the constant $\lambda$ in the bound (1.3), and $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$, with the following significance. Suppose that $p$ satisfies*

$$\max\left(1, \frac{2n}{n+2} - \varepsilon\right) < p < 2. \tag{1.8}$$

*Then for every $\dot{\boldsymbol{g}} \in L^p(\mathbb{R}^n)$, there is a solution $w$, unique up to adding polynomials of degree at most $m-1$, to the $L^p$ Neumann problem*

$$\begin{cases} Lv = 0 \ \ \text{in } \mathbb{R}^{n+1}_+, \\ \dot{\mathbf{M}}^+_{\boldsymbol{A}} v \ni \dot{\boldsymbol{g}}, \\ \|\mathcal{A}^+_2(t\nabla^m \partial_t w)\|_{L^p(\mathbb{R}^n)} + \|\widetilde{N}_+(\nabla^m w)\|_{L^p(\mathbb{R}^n)} \le C_p \|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)} \end{cases} \tag{1.9}$$

*where $C_p$ depends only on $p$, $n$, $m$, $\lambda$, and $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$.*

## 1.1 The history of the Neumann problem

We now discuss the history of the Neumann problem with boundary data in a Lebesgue space. The Neumann problem for the Laplacian with $L^p$ boundary data is traditionally the problem of finding a function $u$ such that

$$-\Delta u = 0 \ \text{ in } \Omega, \quad \nu \cdot \nabla u = g \ \text{ on } \partial\Omega, \quad \|N_\Omega(\nabla u)\|_{L^p(\partial\Omega)} \le C\|g\|_{L^p(\partial\Omega)}.$$

Here, $N_\Omega H(X) = \sup\{|H(Y)| : |X - Y| < 2\operatorname{dist}(Y, \partial\Omega)\}$ is the standard nontangential maximal operator in $\Omega$ and $\nu$ is the unit outward normal to $\partial\Omega$. We observe that if $\Delta u = 0$ in $\Omega$ and $u$ and $\partial\Omega$ are sufficiently smooth, then

$$\int_{\partial\Omega} \varphi\, \nu \cdot \nabla u \, d\sigma = \int_\Omega \nabla\varphi \cdot \nabla u$$

and so the formulation of higher order Neumann boundary values (1.7) is in the spirit of the original harmonic Neumann problem. The harmonic Neumann problem with $L^2$ boundary data was shown to be well posed in [50] for all bounded Lipschitz domains $\Omega$, and the Neumann problem with $L^p$ data for $p$ with $1 < p < 2+\varepsilon$ was shown to be well posed in [31], where $\varepsilon > 0$ depends on $\Omega$.

In [51], the $L^p$ Neumann problem for more general second order equations

$$-\operatorname{div}(\boldsymbol{A}\nabla u) = 0 \ \text{ in } \Omega, \quad \nu \cdot \boldsymbol{A}\nabla u = g \ \text{ on } \partial\Omega, \quad \|\widetilde{N}_\Omega(\nabla u)\|_{L^p(\partial\Omega)} \le C\|g\|_{L^p(\partial\Omega)}$$

was shown to be well posed for $1 < p < 2+\varepsilon$ in starlike Lipschitz domains with coefficients that are bounded, elliptic, real, symmetric, and independent of the radial coordinate. (This situation is very similar to the case of $t$-independent coefficients in the domain above a Lipschitz graph.) Here, $\widetilde{N}_\Omega$ is a suitable modification of $N_\Omega$; we remark that if $\Omega = \mathbb{R}^{n+1}_+$, then $\widetilde{N}_\Omega = \widetilde{N}_+$ is given by formula (1.5).

The case of real nonsymmetric $t$-independent coefficients was addressed in [52, 64], in which the $L^p$ Neumann problem was solved in two dimensions for all $p$ with $1 < p < 1+\varepsilon$. (As shown in the appendix to [52], there exist bounded real nonsymmetric $t$-independent coefficients for which the $L^2$ Neumann problem is ill posed.) The well posedness of the $L^2$ Neumann problem in the domain above a Lipschitz graph was shown to be stable under $t$-independent perturbation in [5] (and, under certain additional assumptions, in [2]), and some additional extrapolation type results were established in [7].

The $L^p$ Neumann problem for a second order system of equations can be written as

$$\begin{cases} (L\vec{u})_j = \sum_{\alpha=1}^{n+1}\sum_{\beta=1}^{n+1}\sum_{k=1}^{N} \partial_{x_\alpha}(A_{\alpha\beta}^{jk}\partial_{x_\beta}u_k) = 0 \text{ in } \Omega \text{ for } 1\leq j\leq N, \\ \vec{\mathrm{M}}_{\boldsymbol{A}}^{\Omega}\,\vec{u} = \vec{g}, \quad \|N_\Omega(\nabla\vec{u})\|_{L^p(\partial\Omega)} \leq C_p\|\vec{g}\|_{L^p(\partial\Omega)}, \end{cases} \tag{1.10}$$

where $\vec{\mathrm{M}}_{\boldsymbol{A}}^{\Omega}\,\vec{u}$ is given by

$$\vec{\mathrm{M}}_{\boldsymbol{A}}^{\Omega}\,\vec{u} = \vec{g} \text{ if } \sum_{j=1}^{N}\int_{\partial\Omega}\varphi_j\,g_j\,d\sigma = \sum_{\alpha=1}^{n+1}\sum_{\beta=1}^{n+1}\sum_{j=1}^{N}\sum_{k=1}^{N}\int_{\Omega}\partial_{x_\alpha}\varphi_j\,A_{\alpha\beta}^{jk}\,\partial_{x_\beta}u_k$$

for all $\vec{\varphi}\in C_0^\infty(\mathbb{R}^{n+1})$. As observed in [67], the traction boundary value problem for the Lamé system of elastostatics can be written in this form. The traction problem and the Neumann problem for the Stokes system, with boundary data in $L^p(\partial\Omega)$, $2-\varepsilon<p<2+\varepsilon$, were shown to be well posed in [34,38]; in [67], Shen observed that their arguments apply to general second order systems with real symmetric constant coefficients that satisfy an appropriate ellipticity condition. The traction boundary problem was shown to be well posed for $L^p$ boundary data, $1<p<2$, in [32]; their arguments relied on the fact that the Lamé system is defined in three dimensions, and applies to many more general three-dimensional (but not higher-dimensional) systems. In [67], Shen showed that if $\Omega\subset\mathbb{R}^{n+1}$ is a Lipschitz domain with $n+1\geq 4$, then for any second order elliptic system with real symmetric constant coefficients, the $L^p$ Neumann problem (1.10) is well posed whenever $\frac{2n}{n+2}-\varepsilon<p<2$.

Turning to higher order equations, the $L^p$ Neumann problem for the biharmonic equation is given by

$$(-\Delta)^2u=0 \text{ in } \Omega, \quad \vec{\mathrm{M}}_{\rho}^{\Omega}\,u\ni\vec{g}, \quad \|N_\Omega(\nabla^2u)\|_{L^p(\partial\Omega)}\leq C_p\|\vec{g}\|_{L^p(\partial\Omega)},$$

where

$$\vec{\mathrm{M}}_{\rho}^{\Omega}\,u\ni\vec{g} \text{ if } \int_{\Omega}\rho\Delta u\Delta\varphi+(1-\rho)\sum_{j,k=1}^{n+1}\partial_{x_jx_k}u\,\partial_{x_jx_k}\varphi = \int_{\partial\Omega}\vec{g}\cdot\nabla\varphi\,d\sigma$$

for all sufficiently smooth test functions $\varphi$. The constant $\rho$ is called the Poisson ratio; we remark that an appropriate choice of coefficients $\boldsymbol{A}_\rho$ for the biharmonic equation yields

$$\dot{\mathbf{M}}_{\boldsymbol{A}_\rho}^{+}\,u = \vec{\mathrm{M}}_{\rho}^{\mathbb{R}_+^{n+1}}\,u,$$

where $\dot{\mathbf{M}}_{\boldsymbol{A}}^{+}\,u$ is given by formula (1.7). The biharmonic Neumann problem was shown to be well posed in bounded Lipschitz domains for $p$ sufficiently close to 2 in [71] in dimension $n+1\geq 2$, and for $p$ with $\frac{2n}{n+2}-\varepsilon<p<2$ in [67] in dimension $n+1\geq 4$. (The case of $C^1$ domains in $\mathbb{R}^2$ was considered earlier in [29].)

Finally, the $L^2$ Neumann problem (1.4) was shown to be well posed in [21, 24].

We observe that Shen's paper [67] yields the well posedness of the $L^p$ Neumann problem for both the biharmonic equation and for constant coefficient second order systems, for the same range of $p$ as in our Theorem 1.1. The present paper builds heavily on our preceding paper [15], and the techniques of [15] owe much to the techniques of Shen. However, we remark that the arguments of [15] are more closely related to those of Shen's earlier paper [65] concerning the Dirichlet problem than to those of the later paper [67] concerning the Neumann problem.

Our proof of Theorem 1.1 involves the well posedness of the subregular Neumann problem as established in [15]. The subregular Neumann problem is the Neumann problem with boundary data in $\dot{W}^{-1,p}(\mathbb{R}^n)$. Here, $\dot{W}^{-1,p}(\mathbb{R}^n)$ is the dual space to $\dot{W}^{1,p'}(\mathbb{R}^n)$, the homogeneous Sobolev space in $\mathbb{R}^n$ with $\|\varphi\|_{\dot{W}^{1,p'}(\mathbb{R}^n)}=\|\nabla_\|\varphi\|_{L^{p'}(\mathbb{R}^n)}$, where $1/p+1/p'=1$ and $\nabla_\|$ denotes the gradient in $\mathbb{R}^n$

(rather than $\mathbb{R}^{n+1}$). We will discuss the main result of [15] in more detail in Section 7. Here, we only mention that subregular Neumann problems have received relatively little study; see [71] (the harmonic and biharmonic problems), [7, 10] (second order equations with $t$-independent coefficents), and [15, 21, 24] (higher order equations with $t$-independent coefficients).

The sharp range of $p$ for which a higher order $L^p$ Neumann problem is well posed is not known, even for special cases such as the biharmonic Neumann problem. However, the results for related problems are somewhat suggestive. Specifically, the range of $p$ for which the biharmonic $\dot W^{1,p}$ Dirichlet problem

$$(-\Delta)^2 u = 0 \text{ in } \Omega, \quad \nabla u = \vec f \text{ on } \partial\Omega, \quad \|N_\Omega(\nabla^2 u)\|_{L^p(\partial\Omega)} \leq C\|\vec f\|_{\dot W^{1,p}(\partial\Omega)}$$

is well posed in all Lipschitz domains $\Omega \subset \mathbb{R}^{n+1}$ is known to be $[6/5, 2]$ in dimension $n + 1 = 4$, to be $[4/3, 2]$ in dimension $n + 1 = 5$, 6, or 7, and is known to be a subset of $[4/3, 2]$ in dimension $n + 1 \geq 8$. See [65, 66, 70] for the well posedness results, [33, Section 5] and [62, Theorem 10.7] for the ill posedness results for the $L^{p'}$ Dirichlet problem, and [53] for the duality between the $L^{p'}$ and $\dot W^{1,p}$ Dirichlet problems for the bilaplacian.

This suggests that the $L^p$ Neumann problem (1.9) is probably not well posed for the full range $1 < p \leq 2$ in dimension 4 and higher.

## 1.2 Layer potentials

We will prove Theorem 1.1 by using the method of layer potentials. In the second order case $2m = 2$, the double and single layer potentials are explicitly defined integral operators given by

$$\mathcal{D}^{\boldsymbol{A}}_\Omega f(X) = \int_{\partial\Omega} \overline{\nu(Y) \cdot A^*(Y)\nabla E^{L^*}(Y, X)}\, f(Y)\, d\sigma(Y),$$

$$\mathcal{S}^L_\Omega g(X) = \int_{\partial\Omega} E^L(X, Y)\, g(Y)\, d\sigma(Y),$$

where $\nu$ is the unit outward normal vector to the domain $\Omega \subset \mathbb{R}^{n+1}$ and $E^L$ is the fundamental solution for the operator $L$ in $\mathbb{R}^{n+1}$. For reasonably well behaved domains $\Omega$ and inputs $f$ and $g$, the outputs $\mathcal{D}^{\boldsymbol{A}}_\Omega f$ and $\mathcal{S}^L_\Omega g$ are locally Sobolev functions satisfying $L(\mathcal{D}^{\boldsymbol{A}}_\Omega) = L(\mathcal{S}^L_\Omega g) = 0$ away from $\partial\Omega$. Certain other properties of layer potentials (in particular, the Green formula and jump relations) are well known. It is possible to generalize layer potentials to the case of higher order operators. This may be done by using integral kernels composed of various derivatives of higher order fundamental solutions (see [1, 28, 29, 60, 61, 67, 71]) or by using the Lax-Milgram lemma to construct operators with appropriate properties (see [14, 20] or Subsection 2.4 below).

If the operator $\dot{\boldsymbol{f}} \to \dot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, \mathcal{D}^{\boldsymbol{A}}_\Omega \dot{\boldsymbol{f}}$ is invertible $\mathfrak{D} \to \mathfrak{N}$, for some function spaces $\mathfrak{D}$ and $\mathfrak{N}$, where $\dot{\mathbf{M}}^\Omega_{\boldsymbol{A}}$ is an appropriate Neumann boundary operator, then the function $u = \mathcal{D}^{\boldsymbol{A}}_\Omega((\dot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, \mathcal{D}^{\boldsymbol{A}}_\Omega)^{-1}\dot{\boldsymbol{g}})$ is a solution to the Neumann problem

$$Lu = 0 \text{ in } \Omega, \quad \dot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u = \dot{\boldsymbol{g}}$$

with boundary data $\dot{\boldsymbol{g}}$. Furthermore, we may establish the bounds on $u$ (such as the nontangential bound $\|\widetilde N_\Omega(\nabla^m u)\|_{L^p(\partial\Omega)} \leq C_p\|\dot{\boldsymbol{g}}\|_{\mathfrak{N}}$) by establishing the corresponding bound

$$\|\widetilde N_\Omega(\nabla^m \mathcal{D}^{\boldsymbol{A}}_\Omega \dot{\boldsymbol{f}})\|_{L^p(\partial\Omega)} \leq C_p\|\dot{\boldsymbol{f}}\|_{\mathfrak{D}}$$

on the double layer potential.

Similarly, if $\dot{\boldsymbol{g}} \to \operatorname{Tr}^\Omega \nabla^{m-1}\mathcal{S}^L_\Omega \dot{\boldsymbol{g}}$ is invertible $\mathfrak{N} \to \mathfrak{D}$, then solutions to the Dirichlet problem

$$Lu = 0 \text{ in } \Omega, \quad \operatorname{Tr}^\Omega \nabla^{m-1} u = \dot{\boldsymbol{f}}$$

exist for all $\dot{\boldsymbol{f}} \in \mathfrak{D}$.

This is the classic method of layer potentials. This method of constructing solutions to the Dirichlet or Neumann problem was used in [31, 37, 39, 58, 69, 74] in the case of harmonic functions (that is, in the case $L = -\Delta$), in [34, 36, 38, 41, 67] for second order constant coefficient systems, in [2, 12, 19, 43, 47] for second order operators with variable $t$-independent coefficients, in [1,28,29,60,61,67,71] for higher order operators with constant coefficients, and in [21] for higher order operators with variable $t$-independent coefficients.

We will construct solutions to problem (1.9) by showing that $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,\mathcal{D}^{\boldsymbol{A}}$ is invertible $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,\mathcal{D}^{\boldsymbol{A}} : \dot{W}A^{1,p}_{m-1}(\mathbb{R}^n) \to (\dot{W}A^{0,p'}_{m-1}(\mathbb{R}^n))^*$, where $\dot{W}A^{j,p}_{m-1}(\mathbb{R}^n)$ is the space of all arrays of functions in $\dot{W}^{j,p}(\mathbb{R}^n)$ (or $L^p(\mathbb{R}^n)$ if $j=0$) that can arise as the gradient $\nabla^{m-1}$ of order $m-1$ of a common function. If $m\geq 2$, then by the equality of mixed partials, $\dot{W}A^{j,p}_{m-1}(\mathbb{R}^n)$ is a proper subspace of $\dot{W}^{j,p}(\mathbb{R}^n)$. Then $(\dot{W}A^{0,p'}_{m-1}(\mathbb{R}^n))^*$ is a quotient space of $L^p(\mathbb{R}^n)$ whose elements are equivalence classes of $L^p$ functions; in light of definition (1.7) of Neumann boundary values, $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,\mathcal{D}^{\boldsymbol{A}}$ is naturally such an equivalence class.

Invertibility of the operator $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,\mathcal{D}^{\boldsymbol{A}} : \dot{W}A^{1,p}_{m-1}(\mathbb{R}^n) \to (\dot{W}A^{0,p'}_{m-1}(\mathbb{R}^n))^*$ yields existence of solutions to problem (1.9) if, in addition, we have the estimates

$$\big\|\mathcal{A}^+_2(t\nabla^m\partial_t\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}})\big\|_{L^p(\mathbb{R}^n)} + \|\widetilde{N}_+(\nabla^m\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{f}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)}.$$

Thus, we have to establish these estimates for $p$ and $\boldsymbol{A}$ as in Theorem 1.1. In fact, we will establish these estimates for $\boldsymbol{A}$ satisfying weaker conditions. (In particular, we do not need $\boldsymbol{A}$ to be self-adjoint to bound the layer potentials.) Furthermore, we will establish estimates on the single layer potential and additional estimates on the double layer potential.

To discuss known the results for higher order layer potentials and to state the bounds on layer potentials to be established in this paper, we introduce some terminology. We will consider the coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition

$$\operatorname{Re}\int_{\mathbb{R}^{n+1}}\sum_{|\alpha|=|\beta|=m}\overline{\partial^\alpha\varphi(x,t)}\,A_{\alpha\beta}(x)\,\partial^\beta\varphi(x,t)\,dx\,dt \geq \lambda\|\nabla^m\varphi\|^2_{L^2(\mathbb{R}^{n+1})} \tag{1.11}$$

for all $\varphi\in C^\infty_0(\mathbb{R}^{n+1})$ and some $\lambda>0$ independent of $\varphi$. Observe that condition (1.11) is weaker than condition (1.3) of Theorem 1.1.

Meyers's reverse Hölder inequality for gradients of solutions is well known. In [9, 27], it was generalized to operators of higher order. That is, if $L$ is an operator of order $2m$, $m\geq 1$, of the form (1.1) and associated to bounded coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11), then there is a constant $\varepsilon>0$ such that if $2<p<2+\varepsilon$, then

$$\bigg(\int_{B(X_0,r)}|\nabla^m u|^p\bigg)^{1/p} \leq \frac{c(0,L,p,2)}{r^{(n+1)(1/2-1/p)}}\bigg(\int_{B(X_0,2r)}|\nabla^m u|^2\bigg)^{1/2} \tag{1.12}$$

whenever $u\in\dot{W}^{m,2}(B(X_0,2r))$ and $Lu=0$ in $B(X_0,2r)$.

In [40, Section 9, Lemma 2], it was shown that if $L=-\Delta$, then the $L^2$ norm on the right-hand side can be replaced by an $L^q$ norm for any $q<2$. The argument generalizes to arbitrary elliptic operators; see [13, Theorem 24]. Furthermore, the Gagliardo–Nirenberg–Sobolev and Caccioppoli inequalities allow us to establish bounds on the lower order derivatives; see [13, Section 4].

Thus, we define $p^+_{j,L}$ as the extended real number such that, whenever $p$ and $q$ satisfy $0<q<p<p^+_{j,L}$, there is a constant $c(j,L,p,q)<\infty$ such that

$$\bigg(\int_{B(X_0,r)}|\nabla^{m-j} u|^p\bigg)^{1/p} \leq \frac{c(j,L,p,q)}{r^{(n+1)(1/q-1/p)}}\bigg(\int_{B(X_0,2r)}|\nabla^{m-j} u|^q\bigg)^{1/q} \tag{1.13}$$

whenever $u\in\dot{W}^{m,2}(B(X_0,2r))$ and $Lu=0$ in $B(X_0,2r)$. We define $p^-_{j,L}$ by

$$\frac{1}{p^-_{j,L}}+\frac{1}{p^+_{j,L}}=1. \tag{1.14}$$

By the results mentioned above, $p^+_{j,L}$ exists whenever $0\leq j\leq m$. By [9, Theorem 49], [13, Section 4], and [15, Propositions 3.3 and 3.6], if $\boldsymbol{A}$ is bounded, $t$-independent in the sense of formula (1.2), and elliptic in the sense of formula (1.11), then there are numbers $\varepsilon > 0$ and $\widetilde{\varepsilon} > 0$, depending only on the order $2m$ of the operator $L$, the ambient dimension $n+1$, the number $\lambda$ in the ellipticity condition (1.11), and the norm $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$ of the coefficients, such that the numbers $p^+_{j,L}$ satisfy

$$\begin{cases} p^+_{0,L} = \infty, & p^+_{1,L} = \infty \qquad\qquad \text{if } n+1 = 2,\\ p^+_{0,L} \geq 2+\varepsilon, & p^+_{1,L} = \infty \qquad\qquad \text{if } n+1 = 3,\\ p^+_{0,L} \geq 2+\varepsilon, & p^+_{1,L} \geq \dfrac{2n}{n-2} + \varepsilon \quad \text{if } n+1 \geq 4.\end{cases}$$

Therefore, there is an $\widetilde{\varepsilon} > 0$ depending only on $n$ and $\varepsilon$ such that

$$\begin{cases} p^-_{0,L} = 1, & p^-_{1,L} = 1 \qquad\qquad \text{if } n+1 = 2,\\ p^-_{0,L} \leq 2-\widetilde{\varepsilon}, & p^-_{1,L} = 1 \qquad\qquad \text{if } n+1 = 3,\\ p^-_{0,L} \leq 2-\widetilde{\varepsilon}, & p^-_{1,L} \leq \dfrac{2n}{n+2} - \widetilde{\varepsilon} \quad \text{if } n+1 \geq 4.\end{cases} \tag{1.15}$$

*Remark* 1.1. If $p < 2+\varepsilon$, or if $p < \infty$ and $n+1 = 2$, then again by [13, Section 4] and [15, Section 3], the numbers $c(0, L, p, q)$ in the bound (1.13) may be bounded by constants depending only on $p$, $q$ and the standard parameters $m$, $n$, $\lambda$, and $\|\boldsymbol{A}\|_{L^\infty}$. The same is true of the numbers $c(1, L, p, q)$ if $n+1 \leq 3$ and $p < \infty$ or $n+1 \geq 4$ and $p < \frac{2n}{n-2} + \varepsilon$.

We can now discuss old and new bounds on layer potentials. In [15,20,22,24], Hofmann, Mayboroda and the author of the present paper showed that if $L$ is an operator of the form (1.1) associated to the bounded elliptic $t$-independent coefficients, then there is $\varepsilon > 0$ such that

$$\|\widetilde{N}_*(\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C(0, L, p)\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \quad 2-\varepsilon < p < p^+_{0,L}, \tag{1.16}$$

$$\|\widetilde{N}_*(\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C(0, L, p)\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)}, \quad 2-\varepsilon < p < p^+_{0,L}, \tag{1.17}$$

$$\|\mathcal{A}^*_2(t\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C(1, L, p)\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \quad 2-\varepsilon < p < p^+_{1,L}, \tag{1.18}$$

$$\|\mathcal{A}^*_2(t\nabla^m \partial_t \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C(1, L, p)\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)}, \quad 2 \leq p < p^+_{1,L}, \tag{1.19}$$

$$\|\mathcal{A}^*_2(t\nabla^m \mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C(1, L, p)\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}, \quad 2-\varepsilon < p < p^+_{1,L}, \tag{1.20}$$

$$\|\mathcal{A}^*_2(t\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C(1, L, p)\|\dot{\boldsymbol{f}}\|_{\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)}, \quad 2 \leq p < p^+_{1,L}, \tag{1.21}$$

$$\|\widetilde{N}_*(\nabla^{m-1} \mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C(1, L, p)\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}, \quad 2-\varepsilon < p < p^+_{1,L}, \tag{1.22}$$

$$\|\widetilde{N}_*(\nabla^{m-1} \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C(1, L, p)\|\dot{\boldsymbol{f}}\|_{\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)}, \quad 2-\varepsilon < p < p^+_{1,L} \tag{1.23}$$

where $p^+_{j,L}$ is as in the bound (1.13), and $C(j, L, p)$ is a constant depending only on $m$, $n$, $\lambda$, $\|\boldsymbol{A}\|_{L^\infty}$, $p$, and the number $c(j, L, p, 2)$ in the bound (1.13). These bounds played a crucial role in solving the $L^2$ Neumann problem (1.4) (and the subregular problem of [15]).

Here,

$$\widetilde{N}_* H(x) = \sup\left\{\left(\fint_{B((y,s),|s|/2)} |H(z,t)|^2\,dz\,dt\right)^{1/2} : \ s \in \mathbb{R}, \ |x-y| < |s|\right\}, \tag{1.24}$$

$$\mathcal{A}^*_2 H(x) = \left(\int_{-\infty}^{\infty} \int_{|x-y|<|t|} |H(y,t)|^2 \frac{dy\,dt}{|t|^{n+1}}\right)^{1/2} \tag{1.25}$$

are two-sided analogues of the nontangential and area integral operators of formulas (1.5) and (1.6).

The second of the two main results of the present paper is the following theorem, in which we expand the range of the parameter $p$ in the bounds (1.16)–(1.23) to include more values below 2.

**Theorem 1.2.** *Suppose that $L$ is an operator of the form (1.1) of order $2m$ associated with bounded coefficients $\boldsymbol{A}$ that are $t$-independent in the sense of formula (1.2) and satisfy the ellipticity condition (1.11) for some $\lambda > 0$.*

*Then the double and single layer potentials $\mathcal{D}^{\boldsymbol{A}}$, $\mathcal{S}^L$ and $\mathcal{S}^L_\nabla$, originally defined as in Subsection 2.4 below, extend by density to operators that satisfy the following bounds for all $p$ in the given ranges and all inputs $\dot{\boldsymbol{f}}$, $\dot{\boldsymbol{g}}$, $\dot{\boldsymbol{h}}$, and $\dot{\boldsymbol{\varphi}}$ in the indicated spaces:*

$$\|\widetilde{N}_*(\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C(1,L^*,p')\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \quad p^-_{1,L^*} < p < 2, \tag{1.26}$$

$$\|\widetilde{N}_*(\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C(1,L^*,p')\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)}, \quad p^-_{1,L^*} < p < 2, \tag{1.27}$$

$$\|\mathcal{A}_2^*(t\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C(1,L^*,p')\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \quad p^-_{1,L^*} < p < 2, \tag{1.28}$$

$$\|\mathcal{A}_2^*(t\nabla^m \partial_t \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C(1,L^*,p')\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)}, \quad p^-_{1,L^*} < p < 2, \tag{1.29}$$

$$\|\mathcal{A}_2^*(t\nabla^m \mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C(0,L^*,p')\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}, \quad p^-_{0,L^*} < p < 2, \tag{1.30}$$

$$\|\mathcal{A}_2^*(t\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C(0,L^*,p')\|\dot{\boldsymbol{f}}\|_{\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)}, \quad p^-_{0,L^*} < p < 2, \tag{1.31}$$

$$\|\widetilde{N}_*(\nabla^{m-1} \mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C(0,L^*,p')\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}, \quad p^-_{0,L^*} < p < 2, \tag{1.32}$$

$$\|\widetilde{N}_*(\nabla^{m-1} \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C(0,L^*,p')\|\dot{\boldsymbol{f}}\|_{\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)}, \quad p^-_{0,L^*} < p < 2. \tag{1.33}$$

*Here, the numbers $p^-_{j,L}$ are as in formulas (1.13), (1.14) and, in particular, satisfy the bounds (1.15). The constants $C(j,L^*,p')$ depend only on the standard parameters $m$, $n$, $\lambda$, $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$, the number $p$, and the constants $c(j,L^*,p',2)$ in the bound (1.13), where $1/p+1/p'=1$.*

The use of the numbers $p^+_{j,L}$ allows us to efficiently summarize several known special cases from the case $2m=2$.

In particular, if $\boldsymbol{A}$ is constant then $p^+_{0,L}=p^+_{1,L}=\infty$. If $n+1=2$ and $\boldsymbol{A}$ is $t$-independent, then we still have that $p^+_{0,L}=p^+_{1,L}=\infty$; see [11, Théorème II.2] in the case $2m=2$ and [15, Proposition 3.3] (reproduced in the bound (1.15) above) in the general case. Thus, in either of these two special cases, Theorem 1.2, and the bounds (1.16)–(1.23), imply that all eight bounds (1.26)–(1.33) (or (1.16)–(1.23)) are valid for all $p$ with $1<p<\infty$. If $2m=2$ and if $\boldsymbol{A}$ is constant or $n+1=2$, then all eight bounds are known (see [10, Theorem 12.7]) for $1<p<\infty$.

Furthermore, if the well known De Giorgi–Nash-Moser regularity conditions are valid (which is true if $\boldsymbol{A}$ is real and $2m=2$, and which by [2, Appendix B] is true for complex $t$-independent coefficients in dimension $n+1=3$), then $p^+_{1,L}=\infty$, and so the bounds (1.28) and (1.29) are valid for $1<p<\infty$, the bounds (1.26) and (1.27) are valid for $1<p<2+\varepsilon$, and the bounds (1.30)–(1.33) are valid for $2-\varepsilon<p<\infty$. The $2+\varepsilon<p<\infty$ case of the bound (1.29) was established in [15]; the remaining bounds on layer potentials were established earlier for the second order $t$-independent operators satisfying the De Giorgi–Nash-Moser conditions in [7,43,44,46,47].

Finally, in the general case (with $n+1\geq 4$), [15, Proposition 3.6] (reproduced in the bound (1.15) above) implies that the bounds (1.28) and (1.29) are valid for $\frac{2n}{n+2}-\varepsilon<p<\frac{2n}{n-2}+\varepsilon$, the bounds (1.26) and (1.27) are valid for $\frac{2n}{n+2}-\varepsilon<p<2+\varepsilon$, and the bounds (1.30)–(1.33) are valid for $2-\varepsilon<p<\frac{2n}{n-2}+\varepsilon$. Again, the $2+\varepsilon<p<\frac{2n}{n-2}+\varepsilon$ cases of the bounds (1.28) and (1.29) are due to [15]; the remaining bounds on the layer potentials for general second order operators with $t$-independent coefficients are due to [10, Theorem 12.7].

*Remark* 1.2. In [4], Auscher identifies two numbers, which he calls $p_+(L)$ and $q_+(L)$, that govern the $L^p$ behavior of a number of operators related to the operator $L$, such as the Riesz transform $L^{1/2}$ and various Littlewood–Paley–Stein type functionals. We now remark on the connections between these numbers and the numbers $p^\pm_{k,L}$ mentioned above and governing (or at least guaranteeing) the $L^p$ behavior of layer potentials.

In [4, Corollary 5.24], Auscher identifies the number $q_+(L)$ as the supremum of the exponents $p$ for which $L$ extends to an isomorphism from $\dot{W}^{m,p}(\mathbb{R}^{n+1})$ to $\dot{W}^{-m,p}(\mathbb{R}^{n+1})$.

It is known that invertibility of $L:\dot{W}^{m,p}(\mathbb{R}^{n+1})\to\dot{W}^{-m,p}(\mathbb{R}^{n+1})$ is equivalent to the following statement: there is $\widetilde{c}(0,L,p,2)>0$ such that if $Lu=\operatorname{div}_m \dot{\boldsymbol{H}}$ in $B(X_0,2r)$, then

$$\biggl(\int_{B(X_0,r)} |\nabla^m u|^p\biggr)^{1/p} \leq \widetilde{c}(0,L,p,2)\biggl(\int_{B(X_0,2r)} |\dot{\boldsymbol{H}}|^p\biggr)^{1/p} + \frac{\widetilde{c}(0,L,p,q)}{r^{(n+1)(1/2-1/p)}}\biggl(\int_{B(X_0,2r)} |\nabla^m u|^2\biggr)^{1/2}. \tag{1.34}$$

Observe that this is a generalization of the bound (1.12). Validity of the bound 1.34 for at least some $p>2$ was proven in [59] in the second order case, and in [13, 27] in the higher order case.

The argument that invertibility of $L$ yields the bound (1.34) is clearly stated in the second order case in the proof of [25, Proposition 3.9], and is given explicitly in the higher order case in [16, Theorems 64 and 66]. The converse (that the bound (1.34) yields invertibility of $L$) may be easily established by using the invertibility of $L:\dot{W}^{m,2}(\mathbb{R}^{n+1})\to \dot{W}^{-m,2}(\mathbb{R}^{n+1})$ (which follows from the Lax-Milgram lemma), letting $r\to\infty$ and applying density (which yields the boundedness of $L^{-1}\operatorname{div}_m : L^p\to \dot{W}^{m,p}$), and using the Hahn–Banach and Riesz representation theorems to show that $\operatorname{div}_m$ is a surjection from $L^p(\mathbb{R}^{n+1})$ to $\dot{W}^{m,p}(\mathbb{R}^{n+1})$ with a bounded right inverse.

Thus, the exponent $q_+(L)$ is the supremum of the exponents $p$ for which the bound (1.34) is valid. But the bound (1.34) clearly implies the bound (1.12), and thus is valid for the same or smaller range of $p$; so,

$$q_+(L)\leq p^+_{0,L}$$

and the two numbers are closely connected.

The number $p_+(L)$ of [4] is noted in [4, Section 8.2] to satisfy

$$\frac{1}{p_+(L)}\leq \max\Bigl(0,\frac{1}{q_+(L)}-\frac{m}{n+1}\Bigr).$$

The Gagliardo–Nirenberg–Sobolev inequality and the Caccioppoli inequality readily show that the number $p^+_{m,L}$ also satisfies

$$\frac{1}{p^+_{m,L}}\leq \max\Bigl(0,\frac{1}{p^+_{0,L}}-\frac{m}{n+1}\Bigr)$$

and so the number $p^+_{m,L}$ of the present paper and the number $p_+(L)$ of [4] do satisfy similar inequalities and it is natural to conjecture that they are also related.

## 1.3 Outline

The outline of this paper is as follows. In Section 2, we will define our terminology. In Section 3, we will state some known results of the theory that we will use several times throughout the paper, and (in Subsection 3.3) will establish a number of results concerning the tent space operators, that is, the operators $\widetilde{N}_+$, $\mathcal{A}_2^+$, $\widetilde{N}_*$, $\mathcal{A}_2^*$ given by formulas (1.5), (1.6), (1.24), and (1.25), as well as the related Carleson operators $\widetilde{\mathfrak{C}}_1^\pm$, $\widetilde{\mathfrak{C}}_1^*$ given by formulas (2.2) and (2.3).

We will prove Theorem 1.2 in Section 5. We will prove it by duality with the Newton potential, and so in Section 4 we will study the Newton potential. Specifically, we will establish duality formulas relating the Newton potential to the double and single layer potentials, then bound the Newton potential using the known bounds (1.16)–(1.23) on the double and single layer potential, a decomposition argument in the spirit of [46, Lemma 4.1], and the good-$\lambda$ results of [15] modeled on those of [65].

In Section 7, we will conclude the paper by proving Theorem 1.1 using the method of layer potentials. A crucial ingredient in the proof of uniqueness of solutions is the Green formula; this formula is the subject of Section 6.

# 2 Definitions

In this section, we will provide precise definitions of the notation and concepts used throughout this paper.

We will always work with an operator $L$ of order $2m$ in the divergence form (1.1) (interpreted in the weak sense of formula (2.7) below) acting on functions defined in open sets in $\mathbb{R}^{n+1}$, $n+1 \geq 2$.

As usual, we let $B(X, r)$ denote the ball in $\mathbb{R}^{n+1}$ of radius $r$ and center $X$. We let $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$ denote the upper and lower half-spaces $\mathbb{R}^n \times (0, \infty)$ and $\mathbb{R}^n \times (-\infty, 0)$; we will identify $\mathbb{R}^n$ with $\partial\mathbb{R}^{n+1}_\pm$. If $Q$ is a cube, we will let $\ell(Q)$ be its side length, and let $cQ$ be the concentric cube of side length $c\ell(Q)$. If $E$ is a set of finite measure, let

$$\fint_E f(x)\,dx = \frac{1}{|E|}\int_E f(x)\,dx.$$

If $E$ is a measurable set in the Euclidean space and $H$ is a globally defined function, we will let $\mathbf{1}_E H = \chi_E H$, where $\chi_E$ is the characteristic function of $E$. If $H$ is defined in all of $E$, but is not globally defined, we will let $\mathbf{1}_E H$ be the extension of $H$ by zero, that is,

$$\mathbf{1}_E H(X) = \begin{cases} H(X), & X \in E, \\ 0, & \text{otherwise.} \end{cases}$$

We will use $\mathbf{1}_\pm$ as a shorthand for $\mathbf{1}_{\mathbb{R}^{n+1}_\pm}$.

## 2.1 Multiindices and arrays of functions

We will routinely work with multiindices in $(\mathbb{N}_0)^{n+1}$. (We will occasionally work with multiindices in $(\mathbb{N}_0)^n$.) Here, $\mathbb{N}_0$ denotes the nonnegative integers. If $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{n+1})$ is a multiindex, then we define $|\zeta|$ and $\partial^\zeta$ as $|\zeta| = \zeta_1 + \zeta_2 + \cdots + \zeta_{n+1}$ and $\partial^\zeta = \partial_{x_1}^{\zeta_1}\partial_{x_2}^{\zeta_2}\cdots\partial_{x_{n+1}}^{\zeta_{n+1}}$.

Recall that a vector $\vec{H}$ is a list of numbers (or functions) indexed by integers $j$ with $1 \leq j \leq N$ for some $N \geq 1$. We similarly let an array $\dot{\boldsymbol{H}}$ be a list of numbers or functions indexed by multiindices $\zeta$ with $|\zeta| = k$ for some $k \geq 1$. In particular, if $\varphi$ is a function with weak derivatives of order up to $k$, then we view $\nabla^k\varphi$ as such an array.

The inner product of two such arrays of functions $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ defined in a measurable set $\Omega$ in the Euclidean space is given by

$$\big\langle \dot{\boldsymbol{F}}, \dot{\boldsymbol{G}} \big\rangle_\Omega = \sum_{|\zeta|=k} \int_\Omega \overline{F_\zeta(X)}\, G_\zeta(X)\,dX.$$

## 2.2 Function spaces and Dirichlet boundary values

Let $\Omega$ be a measurable set in the Euclidean space. Let $C_0^\infty(\Omega)$ be the space of all smooth functions supported in a compact subset of $\Omega$. Let $L^p(\Omega)$ denote the usual Lebesgue space with respect to the Lebesgue measure with the standard norm given by

$$\|f\|_{L^p(\Omega)} = \bigg(\int_\Omega |f(x)|^p\,dx\bigg)^{1/p}.$$

If $\Omega$ is a connected open set, then we let the homogeneous Sobolev space $\dot{W}^{k,p}(\Omega)$ be the space of equivalence classes of functions $u$ that are locally integrable in $\Omega$ and have weak derivatives in $\Omega$ of order up to $k$ in the distributional sense, and whose $k$th gradient $\nabla^k u$ lies in $L^p(\Omega)$. Two functions are equivalent if their difference is a polynomial of order at most $k-1$. We impose the norm

$$\|u\|_{\dot{W}^{k,p}(\Omega)} = \|\nabla^k u\|_{L^p(\Omega)}.$$

Then $u$ is equal to a polynomial of order at most $k-1$ (and thus equivalent to zero) if and only if its $\dot{W}^{k,p}(\Omega)$-norm is zero.

If $1<p<\infty$, then $\dot W^{-1,p'}(\mathbb{R}^n)$ denotes the dual space to $\dot W^{1,p}(\mathbb{R}^n)$, where $1/p+1/p'=1$; this is a space of distributions on $\mathbb{R}^n$.

The use of a dot to denote homogeneous Sobolev spaces (as opposed to the inhomogeneous spaces $W^{k,p}(\Omega)$ with $\|u\|_{W^{k,p}(\Omega)}^p=\sum_{j=0}^k\|\nabla^j u\|_{L^p(\Omega)}^p$) is by now standard. The use of a dot to denote arrays of functions, as in Subsection 2.1, is also standard (see, for example, [1, 28, 29, 60, 61, 63, 66]). We apologize for any confusion arising from this overloading of notation, but the conventions of these fields seem to require it.

We say that $u\in L^p_{loc}(\Omega)$ or $u\in \dot W^{k,p}_{loc}(\Omega)$ if $u\in L^p(U)$ or $u\in \dot W^{k,p}(U)$ for any bounded open set $U$ with $\overline{U}\subset\Omega$.

We will need a number of more specialized norms on functions. In the introduction, we defined the nontangential maximal function $\widetilde N_+$, $\widetilde N_*$ and the Lusin area integral $\mathcal{A}_2^+$, $\mathcal{A}_2^*$. See formulas (1.5), (1.24) and (1.6), (1.25). We will also need the corresponding operators in the lower half-space; thus, we define

$$\widetilde N_\pm H(x)=\sup\biggl\{\biggl(\fint_{B((y,\pm s),s/2)}|H(z,t)|^2\,dz\,dt\biggr)^{1/2} : \ s>0,\ \ |x-y|<s\biggr\},\tag{2.1}$$

$$\mathcal{A}_2^\pm H(x)=\biggl(\int_0^\infty\int_{|x-y|<t}|H(y,\pm t)|^2\,\frac{dy\,dt}{t^{n+1}}\biggr)^{1/2}$$

for all $x\in\mathbb{R}^n$.

We will need one other tent space operator. Following [30, 46], the averaged Carleson operator is given by

$$\widetilde{\mathfrak{C}}_1^\pm H(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q\int_0^{\ell(Q)}\biggl(\fint_{B((y,\pm s),s/2)}|H(z,t)|^2\,dz\,dt\biggr)^{1/2}\frac{ds\,dy}{s},\tag{2.2}$$

where the supremum is taken over the cubes $Q$ in $\mathbb{R}^n$ containing $x$. We will let the two-sided averaged Carleson operator be given by

$$\widetilde{\mathfrak{C}}_1^* H(x)=\max(\widetilde{\mathfrak{C}}_1^+H(x),\widetilde{\mathfrak{C}}_1^-H(x)).\tag{2.3}$$

We adopt the convention that if $t$ appears inside the argument of one of the above operators, then it denotes the $(n+1)$th coordinate function.

Following [23], we define the boundary values $\operatorname{Tr}^\pm u$ of a function $u$ defined in $\mathbb{R}^{n+1}_\pm$ by

$$\operatorname{Tr}^\pm u=f\quad\text{if}\quad\lim_{t\to0^\pm}\|u(\,\cdot\,,t)-f\|_{L^1(K)}=0\tag{2.4}$$

for all compact sets $K\subset\mathbb{R}^n$. We define

$$\dot{\mathbf{Tr}}_j^\pm u=\operatorname{Tr}^\pm\nabla^j u.\tag{2.5}$$

We remark that if $\nabla u$ is locally integrable up to the boundary, then $\operatorname{Tr}^\pm u$ exists and, furthermore, $\operatorname{Tr}^\pm u$ coincides with the traditional trace in the sense of Sobolev spaces. Furthermore, if $\nabla u$ is locally integrable in a neighborhood of the boundary, then $\operatorname{Tr}^+u=\operatorname{Tr}^-u$; in this case, we will refer to the boundary values (from either side) as $\operatorname{Tr}u$.

We are interested in the functions with boundary data in the Lebesgue or Sobolev spaces. However, observe that if $j\geq1$, then the components of $\dot{\mathbf{Tr}}_j^\pm u$ are the derivatives of a common function and so must satisfy certain compatibility conditions. We thus define the following Whitney–Lebesgue, Whitney–Sobolev and Whitney-Besov spaces of arrays that satisfy these conditions.

**Definition 2.1.** Let

$$\mathfrak{D}=\bigl\{\,\dot{\mathbf{Tr}}_{m-1}\varphi:\ \ \varphi\text{ is smooth and compactly supported in }\mathbb{R}^{n+1}\bigr\}.$$

If $1 \leq p < \infty$, then we let $\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)$ be the closure of the set $\mathfrak{D}$ in $L^p(\mathbb{R}^n)$. We let $\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)$ be the closure of $\mathfrak{D}$ in $\dot{W}^{1,p}(\mathbb{R}^n)$. Finally, we let $\dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$ be the closure of $\mathfrak{D}$ in the Besov space $\dot{B}^{1/2,2}_2(\mathbb{R}^n)$; the norm in this space can be written as

$$\|f\|_{\dot{B}^{1/2,2}_2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\xi| \, d\xi\right)^{1/2}, \tag{2.6}$$

where $\widehat{f}$ denotes the Fourier transform of $f$ in $\mathbb{R}^n$.

*Remark* 2.1. It is widely known that $\dot{\boldsymbol{f}} \in \dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$ if and only if $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}^+_{m-1} F$ for some $F$ with $\nabla^m F \in L^2(\mathbb{R}^{n+1}_+)$. This was essentially proven in [49, 55]; see [22, Lemma 2.6] for further discussion.

*Remark* 2.2. There is an extensive theory of Besov spaces (see, for example, [68]). We will make use only of the Besov space $\dot{B}^{1/2,2}_2(\mathbb{R}^n)$ given by formula (2.6) and the space $\dot{B}^{-1/2,2}_2(\mathbb{R}^n)$. This space has the norm

$$\|g\|_{\dot{B}^{-1/2,2}_2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 \frac{1}{|\xi|} \, d\xi\right)^{1/2}.$$

The two important properties of this space we will use are, first, that $\dot{B}^{-1/2,2}_2(\mathbb{R}^n)$ is the dual space to $\dot{B}^{1/2,2}_2(\mathbb{R}^n)$, and, second, that $f \in \dot{B}^{1/2,2}_2(\mathbb{R}^n)$ if and only if the gradient $\nabla f$ exists in the distributional sense and satisfies $\|\nabla f\|_{\dot{B}^{-1/2,2}_2(\mathbb{R}^n)} \approx \|f\|_{\dot{B}^{1/2,2}_2(\mathbb{R}^n)}$.

## 2.3 Elliptic differential operators and Neumann boundary values

Let $\boldsymbol{A} = \big(A_{\alpha\beta}\big)$ be a matrix of measurable coefficients defined on $\mathbb{R}^{n+1}$, indexed by multtiindices $\alpha$, $\beta$ with $|\alpha| = |\beta| = m$. If $\dot{\boldsymbol{F}}$ is an array indexed by multiindices of length $m$, then $\boldsymbol{A}\dot{\boldsymbol{F}}$ is the array given by

$$(\boldsymbol{A}\dot{\boldsymbol{F}})_\alpha = \sum_{|\beta|=m} A_{\alpha\beta} F_\beta.$$

Let $L$ be the $2m$th-order divergence form operator associated with $\boldsymbol{A}$. The weak formulation of such an operator is given by

$$Lu = 0 \text{ in } \Omega \text{ in the weak sense if } \langle \nabla^m \varphi, \boldsymbol{A}\nabla^m u\rangle_\Omega = 0 \text{ for all } \varphi \in C^\infty_0(\Omega). \tag{2.7}$$

Throughout we require our coefficients to be pointwise bounded and to satisfy the Gårding inequality (1.11), which by density we may restate as

$$\operatorname{Re} \langle \nabla^m \varphi, \boldsymbol{A}\nabla^m \varphi\rangle_{\mathbb{R}^{n+1}} \geq \lambda \|\nabla^m \varphi\|^2_{L^2(\mathbb{R}^{n+1})} \text{ for all } \varphi \in \dot{W}^{m,2}(\mathbb{R}^{n+1})$$

for some $\lambda > 0$. The stronger Gårding inequality (1.3) will play a minimal role in this paper; it is needed only because the proof of the primary results of [21] required this stronger inequality, the paper [15] used the results of [21], and our proof of Theorem 1.1 uses the results of [15].

We let $L^*$ be the elliptic operator associated with the adjoint matrix $\boldsymbol{A}^*$, where $(A^*)_{\alpha\beta} = \overline{A_{\beta\alpha}}$.

Recall from the introduction that the Neumann boundary values of a solution $w$ to $Lw = 0$ in $\mathbb{R}^{n+1}_+$ that satisfies estimates as in the problem (1.4) or (1.9) are given by formula (1.7).

We will also be concerned with the solutions $u$ or $v$ to $Lu = 0$ that satisfy $u \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_+)$ or $\mathcal{A}^+_2(t\nabla^m u) \in L^{p'}(\mathbb{R}^n)$ for $p'$ with $1 < p' < \infty$.

If $u \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_+)$, then we can still use formula (1.7) to define $\dot{\mathbf{M}}^+_{\boldsymbol{A}} u$. Furthermore, by density, if $u \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_+)$ and $\dot{\mathbf{M}}^+_{\boldsymbol{A}} u$ is given by formula (1.7), then

$$\big\langle \dot{\mathbf{M}}^+_{\boldsymbol{A}} u, \dot{\mathbf{Tr}}^+_{m-1} \varphi\big\rangle_{\mathbb{R}^n} = \langle \boldsymbol{A}\nabla^m u, \nabla^m \varphi\rangle_{\mathbb{R}^{n+1}_+} \text{ for all } \varphi \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_+). \tag{2.8}$$

Thus, if $u \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_+)$, then $\dot{\mathbf{M}}^+_{\boldsymbol{A}} u$ is a bounded linear operator on $\dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$.

If $v$ satisfies $\mathcal{A}_2^+(t\nabla^m u) \in L^{p'}(\mathbb{R}^n)$, then $\nabla^m v$ may not be locally integrable up to the boundary and thus the integral on the right-hand side of formula (1.7) may not converge. Thus, the definition of $\dot{\mathbf{M}}_{\boldsymbol{A}}^+ v$ in this case is more delicate. We refer the reader to [23, Section 2.3.2] for the precise formulation of the Neumann boundary values $\dot{\mathbf{M}}_{\boldsymbol{A}}^+ v$ of a solution $v$ to $Lv = 0$ with $\mathcal{A}_2^+(t\nabla^m v) \in L^{p'}(\mathbb{R}^n)$.

The numbers $C$ and $\varepsilon$ denote the constants whose value may change from line to line, but which are always positive and depend only on the dimension $n+1$, the order $2m$ of any relevant operators, the bound $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$ on the coefficients, and the number $\lambda$ in the bound (1.11). We say that $A \approx B$ if there are some positive constants $\varepsilon$ and $C$ depending only on the above quantities such that $\varepsilon B \leq A \leq CB$.

The numbers $p_{j,L}^+$ are always as in the bound (1.13). The notation $C(j, L, p)$ denotes a constant that depends only on the standard parameters $n$, $m$, $\lambda$, $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$, the number $p$, and the constant $c(j, L, p, 2)$ in the bound (1.13). (If $p$ is small enough, then $c(j, L, p, 2)$ may be taken as depending only on $p$ and the standard parameters, and so in this case we may simply write $C_p$ rather than $C(j, L, p)$. See Remark 1.1.)

## 2.4 Potential operators

In this section, we will define the double and single layer potentials of Theorem 1.2.

We will also define the Newton potential and use the Newton potential to define the double layer potential. Furthermore, we will prove Theorem 1.2 by establishing various bounds on the Newton potential and using duality to pass to estimates on the double and single layer potentials.

For any $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$, by the Lax-Milgram lemma, there is a unique function $\Pi^L \dot{\boldsymbol{H}}$ in $\dot{W}^{m,2}(\mathbb{R}^{n+1})$ that satisfies

$$\big\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m \Pi^L \dot{\boldsymbol{H}}\big\rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m \varphi, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}} \text{ for all } \varphi \in \dot{W}^{m,2}(\mathbb{R}^{n+1}). \tag{2.9}$$

We will use the operator $\Pi^L$ operator frequently, and refer it as the Newton potential. This represents a break from tradition, as the traditional Newton potential $\mathcal{N}^L$ is usually taken to satisfy $\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m \mathcal{N}^L H\rangle_{\mathbb{R}^{n+1}} = \langle \varphi, H\rangle_{\mathbb{R}^{n+1}}$.

We record here that, by [13, Lemma 43], there is some $\varepsilon > 0$ such that if $2 - \varepsilon < r < 2 + \varepsilon$, then

$$\|\nabla^m \Pi^L \dot{\boldsymbol{H}}\|_{L^r(\mathbb{R}^{n+1})} \leq C_r \|\dot{\boldsymbol{H}}\|_{L^r(\mathbb{R}^{n+1})} \tag{2.10}$$

for all $\dot{\boldsymbol{H}} \in L^r(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1})$.

We are interested in the gradient $\nabla^{m-1}\Pi^L \dot{\boldsymbol{H}}$ of order $m-1$. However, $\Pi^L \dot{\boldsymbol{H}}$, as defined by formula (2.9), is an element of $\dot{W}^{m,2}(\mathbb{R}^{n+1})$, and as such, it is the gradient $\nabla^m \Pi^L \dot{\boldsymbol{H}}$ of order $m$ that is well defined; $\nabla^{m-1}\Pi^L \dot{\boldsymbol{H}}$ is defined only up to adding constants.

We may fix an additive normalization as follows. If $n+1 \geq 3$, then by the Gagliardo–Nirenberg–Sobolev inequality (see, for example, [35, Section 5.6]), there is a unique additive normalization of $\nabla^{m-1}\Pi^L \dot{\boldsymbol{H}}$ such that

$$\|\nabla^{m-1}\Pi^L \dot{\boldsymbol{H}}\|_{L^q(\mathbb{R}^{n+1})} \leq C \|\nabla^m \Pi^L \dot{\boldsymbol{H}}\|_{L^2(\mathbb{R}^{n+1})}, \tag{2.11}$$

where $(n+1)/q = (n+1)/2 - 1$ (and, in particular, where $q < \infty$).

If $n+1 = 2$, let $r < 2$ be as in the bound (2.10). If $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$ is compactly supported or, more generally, if $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1}) \cap L^r(\mathbb{R}^{n+1})$, then, again by the Gagliardo–Nirenberg–Sobolev inequality, there is a unique additive normalization of $\nabla^{m-1}\Pi^L \dot{\boldsymbol{H}}$ such that

$$\|\nabla^{m-1}\Pi^L \dot{\boldsymbol{H}}\|_{L^q(\mathbb{R}^2)} \leq C_r \|\nabla^m \Pi^L \dot{\boldsymbol{H}}\|_{L^r(\mathbb{R}^2)}, \tag{2.12}$$

where $2/q = 2/r - 1$ (and so again $q < \infty$).

We will use this additive normalization throughout.

We now define the double layer potential. Suppose that $\dot{\boldsymbol{f}} \in \dot{W}\!A_{m-1}^{1/2,2}(\mathbb{R}^n)$. As mentioned in Remark 2.1, $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}^- F$ for some $F \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_+)$. We define

$$\mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}} = -\Pi^L(\mathbf{1}_- \boldsymbol{A}\nabla^m F) + \mathbf{1}_- F. \tag{2.13}$$

This operator is well defined, that is, does not depend on the choice of $F$. See [20, Section 2.4] or [14, Section 4]. Using the bounds (1.17) and (1.23), we may extend $\mathcal{D}^{\boldsymbol{A}}$ by density to an operator on all of $\dot{W}A^{k,p}_{m-1}(\mathbb{R}^n)$, for $k \in \{0,1\}$ and for an appropriate range of $p$.

We now define the single layer potential. Let $\dot{\boldsymbol{g}}$ be a bounded linear operator on $\dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$. Then by Remark 2.1, $F \to \langle \dot{\mathbf{Tr}}_{m-1} F, \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^n}$ is a bounded linear operator on $\dot{W}^{m,2}(\mathbb{R}^{n+1})$. By the Lax–Milgram lemma, there is a unique function $\mathcal{S}^L\dot{\boldsymbol{g}} \in \dot{W}^{m,2}(\mathbb{R}^{n+1})$ that satisfies

$$\left\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}}\right\rangle_{\mathbb{R}^{n+1}} = \langle \dot{\mathbf{Tr}}_{m-1}\,\varphi, \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^n} \text{ for all } \varphi \in \dot{W}^{m,2}(\mathbb{R}^{n+1}). \tag{2.14}$$

See [14]. We mote that formula (2.14) is also meaningful and $\mathcal{S}^L\dot{\boldsymbol{g}}$ is defined for $\dot{\boldsymbol{g}} \in \dot{B}^{-1/2,2}_2(\mathbb{R}^n)$. This definition coincides with that of $\mathcal{S}^L\dot{\boldsymbol{g}}$ involving the Newton potential given in [20, 22]. Using the bound (1.16), we may extend $\mathcal{S}^L$ by density to an operator on all of $L^p(\mathbb{R}^n)$ for all $2-\varepsilon < p < p^+_{0,L}$.

*Remark* 2.3. If $L$ is an operator of the form (2.7), then $L$ may generally be associated to many choices of coefficients $\boldsymbol{A}$; for example, if $A_{\alpha\beta} = \widetilde{A}_{\alpha\beta} + M_{\alpha\beta}$, where $M$ is a constant and $M_{\alpha\beta} = -M_{\beta\alpha}$, then the operators associated to $\boldsymbol{A}$ and $\widetilde{\boldsymbol{A}}$ are equal. The single layer potential $\mathcal{S}^L$ depends only on the operator $L$, while the double layer potential $\mathcal{D}^{\boldsymbol{A}}$ depends on the particular choice of coefficients $\boldsymbol{A}$.

In [24], the operator $\mathcal{S}^L_\nabla$ was defined in terms of integrals involving the fundamental solution. In the present paper, we simply define $\mathcal{S}^L_\nabla$ as the operator satisfying [24, formulas (4.5–4.6)]. These formulas are as follows. If $\zeta$ is a multiindex, then $\dot{\boldsymbol{e}}_\zeta$ is the unit array associated to the multiindex $\zeta$; that is,

$$(\dot{\boldsymbol{e}}_\zeta)_\zeta = 1, \quad (\dot{\boldsymbol{e}}_\zeta)_\theta = 0 \text{ whenever } |\theta| = |\zeta| \text{ and } \theta \neq \zeta. \tag{2.15}$$

Let $h \in \dot{B}^{1/2,2}_2(\mathbb{R}^n) \cap \dot{B}^{-1/2,2}_2(\mathbb{R}^n)$. Suppose that $\alpha$ and $\gamma$ are multiindices with $|\alpha| = m$ and $|\gamma| = m-1$; in particular, we require that all entries of $\gamma$ be nonnegative. Then

$$\nabla^m \mathcal{S}^L_\nabla (h\dot{\boldsymbol{e}}_\alpha)(x,t) = -\nabla^m \mathcal{S}^L((\partial_{x_j} h)\dot{\boldsymbol{e}}_\gamma)(x,t) \text{ if } 1 \leq j \leq n \text{ and } \alpha = \gamma + \vec{e}_j \tag{2.16}$$

and

$$\nabla^{m-1} \mathcal{S}^L_\nabla (h\dot{\boldsymbol{e}}_\alpha)(x,t) = -\nabla^{m-1}\partial_t \mathcal{S}^L(h\dot{\boldsymbol{e}}_\gamma)(x,t) \text{ if } \alpha = \gamma + \vec{e}_{n+1}. \tag{2.17}$$

We define $\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}$ for general $\dot{\boldsymbol{h}}$ by linearity. As shown in [24, Lemma 4.4], $\mathcal{S}^L_\nabla$ is well defined in the sense that if $1 \leq \alpha_{n+1} \leq m-1$, then we may use either formula (2.16) or (2.17) to define $\nabla^m \mathcal{S}^L_\nabla(h\dot{\boldsymbol{e}}_\alpha)$, and furthermore, if $\alpha_\ell \geq 1$ and $\alpha_k \geq 1$, then the value of the right-hand side of formula (2.16) is the same whether we choose $j = k$ or $j = \ell$.

Furthermore, by [24, Lemma 4.8], if $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n) \subset \dot{B}^{1/2,2}_2(\mathbb{R}^n) \cap \dot{B}^{-1/2,2}_2(\mathbb{R}^n)$, then there is a (necessarily unique) additive normalization of $\nabla^{m-1}\mathcal{S}^L\dot{\boldsymbol{h}}$ that satisfies

$$\lim_{t\to\pm\infty} \left\|\nabla^{m-1}\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}(\,\cdot\,,t)\right\|_{L^2(\mathbb{R}^n)} = 0.$$

Using the bound (1.22), we may extend $\mathcal{S}^L_\nabla$ by density to an operator on all of $L^p(\mathbb{R}^n)$ for $2-\varepsilon < p < p^+_{1,L}$.

# 3 Preliminaries

In this section, we will discuss a few known results and establish some general results that will be of use throughout the paper.

Specifically, in Subsection 3.1, we will discuss the change of variables $(x,t) \to (x,-t)$, and how it allows us to easily generalize from the upper half-space to the lower half-space. In Subsection 3.2, we will list some known results from the theory of solutions to elliptic equations $Lu = 0$. Finally, in Subsection 3.3, we will establish some general results involving tent spaces, that is, spaces of functions $H$ for which the tent space norms $\widetilde{N}_+ H$, $\mathcal{A}^+_2 H$ or $\widetilde{\mathfrak{C}}^+_1 H$ lie in $L^p(\mathbb{R}^n)$.

## 3.1 The lower half-space

It is often notationally convenient to establish the bounds only in the upper half-space and to use a change of variables arguments to generalize to the lower half-space.

The change of variables $(x,t)\to(x,-t)$, for $x\in\mathbb{R}^n$ and $t\in\mathbb{R}$, interchanges the upper and lower half-spaces. In [24, Section 3.3], it was shown that if $Lu=0$ in $\Omega$, then $L^-u^-=0$ in $\Omega^-$, where $u^-(x,t)=u(x,-t)$, $\Omega^-=\{(x,t):(x,-t)\in\Omega\}$, and $L^-$ is the operator of the form (2.7) associated to the coefficients $\boldsymbol{A}^-$ given by $A^-_{\alpha\beta}=(-1)^{\alpha_{n+1}+\beta_{n+1}}A_{\alpha\beta}$. Notice that if $\boldsymbol{A}$ is bounded, $t$-independent and satisfies the condition (1.11) (or (1.3)), then $\boldsymbol{A}^-$ satisfies the same conditions with $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}=\|\boldsymbol{A}^-\|_{L^\infty(\mathbb{R}^n)}$ and with the same value of $\lambda$.

We observe that by the same change of variables argument, if $j$ is an integer with $0\le j\le m$, and if $p^+_{j,L}$ and $c(j,L,p,q)$ are as in the bound (1.13), then

$$p^+_{j,L}=p^+_{j,L^-}\quad\text{and}\quad c(j,L,p,q)=c(j,L^-,p,q)\quad\text{for all}\quad 0<q<p<p^+_{j,L}.$$

Furthermore, by [24, Section 3.3],

$$\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}(x,-t)=-\mathcal{D}^{\boldsymbol{A}^-}\dot{\boldsymbol{f}}^-(x,t),\quad \mathcal{S}^L\dot{\boldsymbol{g}}(x,-t)=\mathcal{S}^{L^-}\dot{\boldsymbol{g}}^-(x,t),$$
$$\Pi^L\dot{\boldsymbol{H}}(x,-t)=\Pi^{L^-}\dot{\boldsymbol{H}}^-(x,t),\quad \mathcal{S}^L_\nabla\dot{\boldsymbol{h}}(x,-t)=\mathcal{S}^{L^-}_\nabla\dot{\boldsymbol{h}}^-(x,t),$$

where

$$f^-_\gamma(x)=(-1)^{\gamma_{n+1}}f_\gamma(x),\quad g^-_\gamma(x)=(-1)^{\gamma_{n+1}}g_\gamma(x),$$
$$H^-_\alpha(x,t)=(-1)^{\alpha_{n+1}}H_\alpha(x,-t),\quad h^-_\beta(x)=(-1)^{\beta_{n+1}}h_\beta(x).$$

It is straightforward to calculate that if $\dot{\boldsymbol{f}}=\dot{\mathbf{Tr}}^+_{m-1}\varphi$ in the sense of formula (2.5), then $\dot{\boldsymbol{f}}^-=\dot{\mathbf{Tr}}^-_{m-1}\varphi^-$. Thus, $\dot{\boldsymbol{f}}^-$ is in the distinguished subspace $\mathfrak{D}$ of Definition 2.1 if and only if $\dot{\boldsymbol{f}}$ is, and so the mapping $\dot{\boldsymbol{f}}\to\dot{\boldsymbol{f}}^-$ is an automorphism of $\dot{W}A^{s,p}_{m-1}(\mathbb{R}^n)$ for all spaces $\dot{W}A^{s,p}_{m-1}(\mathbb{R}^n)$ defined by Definition 2.1.

We observe further that if $\dot{\mathbf{M}}^+_{\boldsymbol{A}}w\ni\dot{\boldsymbol{g}}$, then by the definition (1.7) of the Neumann boundary values, if $\varphi\in C^\infty_0(\mathbb{R}^{n+1})$, then

$$\langle\dot{\mathbf{Tr}}_{m-1}\varphi,\dot{\boldsymbol{g}}^-\rangle_{\mathbb{R}^n}=\big\langle\dot{\mathbf{Tr}}_{m-1}(\varphi^-),\dot{\boldsymbol{g}}\big\rangle_{\mathbb{R}^n}=\big\langle\nabla^m(\varphi^-),\boldsymbol{A}\nabla^m w\big\rangle_{\mathbb{R}^{n+1}_+}=\big\langle\nabla^m\varphi,\boldsymbol{A}^-\nabla^m w^-\big\rangle_{\mathbb{R}^{n+1}_-},$$

and so,

$$\text{if}\quad\dot{\mathbf{M}}^+_{\boldsymbol{A}}w\ni\dot{\boldsymbol{g}}\quad\text{then}\quad\dot{\mathbf{M}}^-_{\boldsymbol{A}^-}w^-\ni\dot{\boldsymbol{g}}^-.\tag{3.1}$$

An examination of the definition of Neumann boundary values in [23, Section 2.3.2] reveals that formula (3.1) is valid if that definition of Neumann boundary values is used instead.

Thus, we may easily pass from the bounds in the upper half-space to bounds in the lower half-space.

## 3.2 Solutions to elliptic equations

It is well known that solutions to the elliptic equation $Lu=0$ display many useful properties. In this section, we will state two regularity results that will be used throughout the paper.

We begin with the higher order analogue of the Caccioppoli inequality. This lemma was proven in full generality in [13] and some important preliminary versions were established in [9, 27].

**Lemma 3.1** (The Caccioppoli inequality)**.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $u\in\dot{W}^{m,2}(B(X,2r))$ with $Lu=0$ in $B(X,2r)$. Then we have the bound*

$$\fint_{B(X,r)}|\nabla^j u(x,s)|^2\,dx\,ds\le\frac{C}{r^2}\fint_{B(X,2r)}|\nabla^{j-1}u(x,s)|^2\,dx\,ds$$

*for any $j$ with $1\le j\le m$.*

If $\boldsymbol{A}$ is $t$-independent, then solutions to $Lu = 0$ have additional regularity. In particular, the following lemma was proven in the case $m = 1$ in [2, Proposition 2.1] and generalized to the case $m \geq 2$ in [22, Lemma 3.20].

**Lemma 3.2.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $Q \subset \mathbb{R}^n$ be a cube of side length $\ell(Q)$ and let $I \subset \mathbb{R}$ be an interval with $|I| = \ell(Q)$. If $u \in \dot{W}^{m,2}_{loc}(2Q \times 2I)$ and $Lu = 0$ in $2Q \times 2I$, then*

$$\int_Q |\nabla^{m-j} \partial_t^k u(x,t)|^p \, dx \leq \frac{C(j,L,p)}{\ell(Q)} \int_{2Q} \int_{2I} |\nabla^{m-j} \partial_s^k u(x,s)|^p \, ds \, dx$$

*for any $t \in I$, any integer $j$ with $0 \leq j \leq m$, any $p$ with $0 < p < p^+_{j,L}$, and any integer $k \geq 0$.*

### 3.3 Tent spaces

Recall that Theorem 1.2 concerns nontangential maximal and area integral norms of layer potentials. Thus, in order to prove Theorem 1.2, we will need a number of results concerning the area integral, the nontangential maximal operator, and the Carleson operator of formula (2.2).

We begin with the following lemma concerning the Lebesgue norm and the area integral.

**Lemma 3.3.** *Let $\sigma > 0$, $\kappa \in \mathbb{R}$, and $0 < \theta \leq r \leq 2$. Let $\dot{\boldsymbol{F}} \in L^2_{loc}(\mathbb{R}^{n+1}_+)$ be such that $\mathcal{A}_2^+(t^\kappa \dot{\boldsymbol{F}}) \in L^\theta(\mathbb{R}^n)$.*

*If $\theta(n+1) < r(n+\theta\kappa)$, then*

$$\|\dot{\boldsymbol{F}}\|_{L^r(\mathbb{R}^n \times (\sigma,\infty))} \leq \frac{C_{n,\theta,\kappa,r}}{\sigma^{\kappa+n/\theta-1/r-n/r}} \|\mathcal{A}_2^+(t^\kappa \dot{\boldsymbol{F}})\|_{L^\theta(\mathbb{R}^n)}.$$

*If $\theta(n+1) > r(n+\theta\kappa)$, then*

$$\|\dot{\boldsymbol{F}}\|_{L^r(\mathbb{R}^n \times (0,\sigma))} \leq \frac{C_{n,\theta,\kappa,r}}{\sigma^{\kappa+n/\theta-1/r-n/r}} \|\mathcal{A}_2^+(t^\kappa \dot{\boldsymbol{F}})\|_{L^\theta(\mathbb{R}^n)}.$$

*Proof.* Our argument is largely taken from [23, Remark 5.3], where the case $r = 2$, $\kappa = 1$ was considered. Let $j$ be an integer. Then

$$\int_{\mathbb{R}^n} \int_{2^j\sqrt{n}}^{2^{j+1}\sqrt{n}} |\dot{\boldsymbol{F}}(x,t)|^r \, dt \, dx = \sum_{Q \in \mathcal{G}_j} \int_Q \int_{\sqrt{n}\ell(Q)}^{2\sqrt{n}\ell(Q)} |\dot{\boldsymbol{F}}(x,t)|^r \, dt \, dx,$$

where $\mathcal{G}_j$ is a grid of pairwise-disjoint open cubes in $\mathbb{R}^n$ of side length $2^j$ whose union is almost all of $\mathbb{R}^n$. If $r \leq 2$, then by Hölder's inequality,

$$\int_Q \int_{\ell(Q)\sqrt{n}}^{2\ell(Q)\sqrt{n}} |\dot{\boldsymbol{F}}(x,t)|^r \, dt \, dx \leq \big(|Q|\ell(Q)\sqrt{n}\big)^{1-r/2} \bigg(\int_Q \int_{\ell(Q)\sqrt{n}}^{2\ell(Q)\sqrt{n}} |\dot{\boldsymbol{F}}(x,t)|^2 \, dt \, dx\bigg)^{r/2}.$$

For every $x, y \in Q$ and every $t > \ell(Q)\sqrt{n}$, we have

$$|x-y| < \operatorname{diam} Q = \ell(Q)\sqrt{n} < t,$$

and so, for any $y \in Q$, we have

$$\int_Q \int_{\ell(Q)\sqrt{n}}^{2\ell(Q)\sqrt{n}} |\dot{\boldsymbol{F}}(x,t)|^2 \, dt \, dx \leq C_{n,\kappa} \int_0^\infty \int_{|x-y|<t} |\dot{\boldsymbol{F}}(x,t)|^2 \, \frac{\ell(Q)^{n+1-2\kappa}}{t^{n+1-2\kappa}} \, dx \, dt.$$

Thus,
$$\int_{\mathbb{R}^n}\int_{2^j\sqrt{n}}^{2^{j+1}\sqrt{n}} |\dot{\boldsymbol{F}}(x,t)|^r\,dt\,dx \leq C_{n,\kappa}\sum_{Q\in\mathcal{G}_j}\ell(Q)^{n+1-r\kappa}\biggl(\fint_Q \mathcal{A}_2^+(t^\kappa\dot{\boldsymbol{F}})(y)^\theta\,dy\biggr)^{r/\theta}.$$
If $\theta\leq r$, then
$$\sum_{Q\in\mathcal{G}_j}\biggl(\int_Q \mathcal{A}_2^+(t\dot{\boldsymbol{F}})(y)^\theta\,dy\biggr)^{r/\theta} \leq \biggl(\int_{\mathbb{R}^n} \mathcal{A}_2^+(t\dot{\boldsymbol{F}})(y)^\theta\,dy\biggr)^{r/\theta}$$
and so
$$\int_{\mathbb{R}^n}\int_{2^j\sqrt{n}}^{2^{j+1}\sqrt{n}} |\dot{\boldsymbol{F}}(x,t)|^r\,dt\,dx \leq C_{n,\kappa}\cdot 2^{j(n+1-r\kappa-nr/\theta)}\biggl(\int_{\mathbb{R}^n} \mathcal{A}_2^+(t^\kappa\dot{\boldsymbol{F}})(y)^\theta\,dy\biggr)^{r/\theta}.$$
By summing over $j$ with $2^{j+1}\sqrt{n}>\sigma$ or with $2^j\sqrt{n}<\sigma$, we complete the proof. □

We now establish the following localization lemma involving the Carleson operator (2.2).

**Lemma 3.4.** *Let $1<r\leq\infty$, let $Q\subset\mathbb{R}^n$ be a cube, and let $\dot{\boldsymbol{H}}\in L^2_{loc}(\mathbb{R}^{n+1}_+)$ be such that $\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\in L^r(16Q)$. Then*
$$\|\widetilde{\mathfrak{C}}_1^+(\mathbf{1}_{10Q\times(0,\ell(Q))}t\dot{\boldsymbol{H}})\|_{L^r(\mathbb{R}^n)} \leq C_{n,r}\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^r(16Q)}. \tag{3.2}$$
*In particular, if $\dot{\boldsymbol{H}}\in L^2(\mathbb{R}^{n+1}_+)$ is supported in a compact subset of $\mathbb{R}^{n+1}_+$, then $\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\in L^r(\mathbb{R}^n)$ for all $1<r\leq\infty$.*

*Proof.* We begin with the bound (3.2). If $x\in 16Q$, then $\widetilde{\mathfrak{C}}_1^+(\mathbf{1}_{10Q\times(0,\ell(Q))}t\dot{\boldsymbol{H}})(x)\leq\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})(x)$. Thus, we need only consider $x\notin 16Q$.

Let $\dot{\boldsymbol{\Phi}}(x,t)=t\dot{\boldsymbol{H}}(x,t)$. By formula (2.2),
$$\widetilde{\mathfrak{C}}_1^+(\mathbf{1}_{10Q\times(0,\ell(Q))}t\dot{\boldsymbol{H}})(x) = \sup_{R\ni x}\frac{1}{|R|}\int_R\int_0^{\ell(R)}\biggl(\fint_{B((y,s),s/2)}\mathbf{1}_{10Q\times(0,\ell(Q))}|\dot{\boldsymbol{\Phi}}|^2\biggr)^{1/2}\frac{ds\,dy}{s},$$
where the supremum is taken over the cubes $R\subset\mathbb{R}^n$ with $x\in R$. Observe that if
$$B\Bigl((y,s),\frac{s}{2}\Bigr)\cap\bigl(10Q\times(0,\ell(Q))\bigr)\neq\varnothing,$$
then $s<2\ell(Q)$ and $\operatorname{dist}(y,10Q)<s/2<\ell(Q)$, so $y\in 12Q$. Thus,
$$\begin{aligned}\int_{\mathbb{R}^n}\int_0^\infty\biggl(\fint_{B((y,s),s/2)}\mathbf{1}_{10Q\times(0,\ell(Q))}|\dot{\boldsymbol{\Phi}}|^2\biggr)^{1/2}\frac{ds\,dy}{s}\\ \leq\int_{12Q}\int_0^{2\ell(Q)}\biggl(\fint_{B((y,s),s/2)}|\dot{\boldsymbol{\Phi}}|^2\biggr)^{1/2}\frac{ds\,dy}{s}\leq|12Q|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})(z)\end{aligned}$$
for all $z\in 12Q$. Furthermore, if
$$\int_R\int_0^{\ell(R)}\biggl(\fint_{B((y,s),s/2)}\mathbf{1}_{10Q\times(0,\ell(Q))}|\dot{\boldsymbol{\Phi}}|^2\biggr)^{1/2}\frac{ds\,dy}{s}\neq 0,$$
then $R\cap 12Q\neq\varnothing$. But observe that if $x\notin 16Q$ and $R\ni x$ with $R\cap 12Q\neq\varnothing$, then $\sqrt{n}\ell(R)=\operatorname{diam}(R)>\operatorname{dist}(x,12Q)$. Thus, if $x\notin 16Q$, then
$$\widetilde{\mathfrak{C}}_1^+(\mathbf{1}_{10Q\times(0,\ell(Q))}t\dot{\boldsymbol{H}})(x)\leq\frac{|12Q|}{n^{n/2}\operatorname{dist}(x,12Q)^n}\biggl(\fint_{12Q}\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})(z)^r\,dz\biggr)^{1/r}.$$

A straightforward computation yields the bound (3.2) for all $r > 1$.

We now turn to the case of compactly supported $\dot{\boldsymbol{H}}$. If $\dot{\boldsymbol{H}}$ is supported in a compact subset of $\mathbb{R}^{n+1}_+$, then there is some $\varepsilon > 0$ and some $N < \infty$ such that $\dot{\boldsymbol{H}}(x,t) = 0$ whenever $t < \varepsilon$ or $t > N$. We compute that

$$\left(\fint_{B((y,s),s/2)} |t\dot{\boldsymbol{H}}(z,t)|^2\,dz\,dt\right)^{1/2} \leq C_n s^{1/2-n/2}\|\dot{\boldsymbol{H}}\|_{L^2(\mathbb{R}^{n+1}_+)}$$

and is zero if $s < 2\varepsilon/3$ or $s > 2N$. Thus, by formula (2.2),

$$\begin{aligned}\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})(x) &= \sup_{R\ni x}\fint_R \int_0^{\ell(R)} \left(\fint_{B((y,s),s/2)} |\dot{\boldsymbol{H}}(x,t)|^2\,t^2\,dx\,dt\right)^{1/2}\frac{ds\,dy}{s} \\ &\leq \int_{2\varepsilon/3}^{2N} C_n s^{-1/2-n/2}\|\dot{\boldsymbol{H}}\|_{L^2(\mathbb{R}^{n+1}_+)}\,ds.\end{aligned}$$

The right-hand side is finite and independent of $x$. Thus, if $\dot{\boldsymbol{H}}$ is supported in a compact subset of $\mathbb{R}^{n+1}_+$, then $\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})$ is bounded and so the right-hand side of formula (3.2) is finite for any fixed cube $Q$. But if $\dot{\boldsymbol{H}}$ is compactly supported, then it is supported in $10Q \times (0, \ell(Q))$ for some cube $Q$; thus, $\mathbf{1}_{10Q\times(0,\ell(Q))}\dot{\boldsymbol{H}} = \dot{\boldsymbol{H}}$, and so, by the bound (3.2), we have that $\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}}) \in L^r(\mathbb{R}^n)$, as desired. ☐

We now come to a method for bounding nontangential maximal functions by duality. This is the reason that the Carleson operator $\widetilde{\mathfrak{C}}_1^+$ is of interest in the present paper. It is well known (see [3, Theorem 5.1]) that if $1 < p < \infty$ and $1/p + 1/p' = 1$, then the dual to the space of nontangentially bounded functions

$$\big\{U :\ N_+U \in L^p(\mathbb{R}^n)\big\}, \text{ where } N_+U(x) = \sup\big\{|U(y,t)| :\ |x-y| < t\big\},$$

is the space of Borel measures

$$\big\{\mu :\ \mathfrak{C}_1^+(t|\mu|) \in L^{p'}(\mathbb{R}^n)\big\}, \text{ where } \mathfrak{C}_1^+(\mu)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q\int_0^{\ell(Q)}\frac{1}{t}\,d|\mu|(y,t).$$

We claim that a similar result is true for the spaces defined by the averaged norms $\widetilde{N}_+$ and $\widetilde{\mathfrak{C}}_1^+$ given by formulas (2.1) and (2.2). More precisely, we will use the following two lemmas.

**Lemma 3.5.** *Let $p$ satisfy $1 < p < \infty$ and let $p'$ satisfy $1/p + 1/p' = 1$. Suppose that $\dot{\boldsymbol{u}}$ and $\dot{\boldsymbol{H}}$ are such that $\widetilde{N}_+\dot{\boldsymbol{u}} \in L^p(\mathbb{R}^n)$ and $\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}}) \in L^{p'}(\mathbb{R}^n)$. Then*

$$|\langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}_+}| \leq C\|\widetilde{N}_+\dot{\boldsymbol{u}}\|_{L^p(\mathbb{R}^n)}\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}.$$

**Lemma 3.6.** *Suppose that $\dot{\boldsymbol{u}} \in L^2_{loc}(\mathbb{R}^{n+1}_+)$. Let $1 < p < \infty$ and let $1/p + 1/p' = 1$. Then*

$$\|\widetilde{N}_+\dot{\boldsymbol{u}}\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{H}}\in L^2_c(\mathbb{R}^{n+1}_+)\setminus\{\dot{\boldsymbol{0}}\}} \frac{|\langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}}$$

*provided the right-hand side is finite. Here,*

$$L^2_c(\mathbb{R}^{n+1}_+) = \Big\{\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1}_+) :\ \operatorname{supp}\dot{\boldsymbol{H}} \subset K \textit{ for some compact set } K \subset \mathbb{R}^{n+1}_+\Big\}.$$

*Proof of Lemma 3.5.* Let $F$ be an integrable function. Then

$$\int_{\mathbb{R}^{n+1}_+} F(x,t)\,dx\,dt = \int_{\mathbb{R}^{n+1}_+} F(y+sz, s+sr)\,(1+r)\,dy\,ds$$

for any $z\in\mathbb{R}^n$ and any $r>-1$. Averaging over $(z,r)\in B(0,1/2)$, we have

$$\int_{\mathbb{R}^{n+1}_+} F(x,t)\,dx\,dt = \int_{\mathbb{R}^{n+1}_+} \fint_{B(0,1/2)} F(y+sz,s+sr)\,\frac{s+sr}{s}\,dz\,dr\,dy\,ds,$$

and a change of variables yields

$$\int_{\mathbb{R}^{n+1}_+} F(x,t)\,dx\,dt = \int_{\mathbb{R}^{n+1}_+} \fint_{B((y,s),s/2)} F(x,t)\,\frac{t}{s}\,dx\,dt\,dy\,ds.$$

Let $K$ be a compact set in $\mathbb{R}^{n+1}_+$. Observe that $\dot{\boldsymbol{u}}$ and $\dot{\boldsymbol{H}}$ are both in $L^2_{loc}(\mathbb{R}^{n+1}_+)$; thus $F=\mathbf{1}_K|\dot{\boldsymbol{u}}||\dot{\boldsymbol{H}}|$ is integrable. Therefore,

$$\int_K |\dot{\boldsymbol{u}}||\dot{\boldsymbol{H}}| = \int_{\mathbb{R}^{n+1}_+} \fint_{B((y,s),s/2)} |\mathbf{1}_K\dot{\boldsymbol{u}}(x,t)|\,|t\dot{\boldsymbol{H}}(x,t)|\,dx\,dt\,\frac{dy\,ds}{s}.$$

We define

$$H(y,s)=\frac{1}{s}\biggl(\fint_{B((y,s),s/2)} |t\dot{\boldsymbol{H}}(x,t)|^2\,dx\,dt\biggr)^{1/2},\quad U(y,s)=\biggl(\fint_{B((y,s),s/2)} |\dot{\boldsymbol{u}}|^2\biggr)^{1/2}$$

so that by the definitions (2.1) and (2.2) of $\widetilde{N}_+$ and $\mathfrak{C}_1^+$,

$$N_+U=\widetilde{N}_+\dot{\boldsymbol{u}},\quad \mathfrak{C}_1^+(tH)=\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}}).$$

By Hölder's inequality,

$$\int_K |\dot{\boldsymbol{u}}||\dot{\boldsymbol{H}}| \leq \int_{\mathbb{R}^{n+1}_+} UH.$$

By the duality results discussed above (see [30, formula (2.6)]), we have that

$$\int_{\mathbb{R}^{n+1}_+} UH \leq C\|N_+U\|_{L^p(\mathbb{R}^n)}\|\mathfrak{C}_1^+(tH)\|_{L^{p'}(\mathbb{R}^n)}.$$

Thus,

$$\int_K |\dot{\boldsymbol{u}}||\dot{\boldsymbol{H}}| \leq C\|\widetilde{N}_+\dot{\boldsymbol{u}}\|_{L^p(\mathbb{R}^n)}\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}$$

provided the right-hand side is finite. Because $K$ was arbitrary, this inequality is still true if we integrate over $\mathbb{R}^{n+1}_+$ instead of $K$. Thus, $\langle\dot{\boldsymbol{u}},\dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}_+}$ represents an absolutely convergent integral that satisfies

$$|\langle\dot{\boldsymbol{u}},\dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}_+}| \leq C\|\widetilde{N}_+\dot{\boldsymbol{u}}\|_{L^p(\mathbb{R}^n)}\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)},$$

as desired. □

The following lemma will be used in the proof of Lemma 3.6.

**Lemma 3.7.** *Let $\mu$ be a nonnegative measure on $\mathbb{R}^n$. For each $(x,r)\in\mathbb{R}^{n+1}_+$, let $\dot{\boldsymbol{H}}_{(x,r)}$ be defined in $\mathbb{R}^{n+1}_+$, supported in $B((x,r),r/2)$, and satisfy*

$$\biggl(\fint_{B((x,r),r/2)} |\dot{\boldsymbol{H}}_{(x,r)}|^2\biggr)^{1/2} \leq 1.$$

*Define*

$$\dot{\boldsymbol{H}}(z,t) = \int_{\mathbb{R}^{n+1}_+} \frac{1}{r^{n+1}}\,\dot{\boldsymbol{H}}_{(x,r)}(z,t)\,d\mu(x,r).$$

*Then*

$$\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})(\widetilde{x}) \leq C\mathfrak{C}_1^+(t\mu)(\widetilde{x})$$

*for all* $\widetilde{x}\in\mathbb{R}^n$ *such that the right-hand side is finite.*

*Proof.* Let $W(y,s) = B((y,s),s/2)$ and let $V(y,s) = \{(x,r) :\ W(y,s)\cap W(x,r)\neq\varnothing\}$. Then

$$\fint_{W(y,s)} |\dot{\boldsymbol{H}}(z,t)|^2\,dz\,dt = \fint_{W(y,s)} \biggl|\int_{V(y,s)} \frac{1}{r^{n+1}}\,\dot{\boldsymbol{H}}_{(x,r)}(z,t)\,d\mu(x,r)\biggr|^2\,dz\,dt.$$

By Hölder's inequality,

$$\fint_{W(y,s)} |\dot{\boldsymbol{H}}(z,t)|^2\,dz\,dt \leq \fint_{W(y,s)}\int_{V(y,s)} \frac{\mu(V(y,s))}{r^{2n+2}}\,|\dot{\boldsymbol{H}}_{(x,r)}(z,t)|^2\,d\mu(x,r)\,dz\,dt.$$

Changing the order of integration, we see that

$$\fint_{W(y,s)} |\dot{\boldsymbol{H}}(z,t)|^2\,dz\,dt \leq \int_{V(y,s)} \frac{\mu(V(y,s))}{r^{2n+2}}\fint_{W(y,s)} |\dot{\boldsymbol{H}}_{(x,r)}(z,t)|^2\,dz\,dt\,d\mu(x,r).$$

A straightforward computation yields that $V(y,s)$ is the ellipsoid

$$V(y,s) = \Bigl\{(x,r) :\ \frac{4}{3}\,|x-y|^2 + \Bigl(r-\frac{5}{3}\,s\Bigr)^2 < \Bigl(\frac{4}{3}\,s\Bigr)^2\Bigr\}. \tag{3.3}$$

In particular, if $(x,r)\in V(y,s)$, then $\frac{1}{3}r<s<3r$. Thus, $|W(x,r)|\approx|W(y,s)|$ and so

$$\fint_{W(y,s)} |\dot{\boldsymbol{H}}(z,t)|^2\,dz\,dt \leq C\,\frac{\mu(V(y,s))}{s^{2n+2}}\int_{V(y,s)}\fint_{W(x,r)} |\dot{\boldsymbol{H}}_{(x,r)}(z,t)|^2\,dz\,dt\,d\mu(x,r).$$

Recalling the $L^2$ norm of $\dot{\boldsymbol{H}}_{(x,r)}$, we see that

$$\biggl(\fint_{W(y,s)} |\dot{\boldsymbol{H}}(z,t)|^2\,dz\,dt\biggr)^{1/2} \leq \frac{C}{s^{n+1}}\,\mu(V(y,s)) = \frac{C}{s^{n+1}}\int_{V(y,s)} d\mu(x,r).$$

Then

$$\begin{aligned}\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})(\widetilde{x}) &= \sup_{Q\ni\widetilde{x}}\frac{1}{|Q|}\int_Q\int_0^{\ell(Q)}\biggl(\fint_{W(y,s)} |t\dot{\boldsymbol{H}}(z,t)|^2\,dz\,dt\biggr)^{1/2}\frac{ds\,dy}{s}\\ &\leq \sup_{Q\ni\widetilde{x}}\frac{1}{|Q|}\int_Q\int_0^{\ell(Q)}\frac{C}{s^n}\int_{V(y,s)} d\mu(x,r)\,\frac{ds\,dy}{s}\,.\end{aligned}$$

By formula (3.3), if $(y,s)\in Q\times(0,\ell(Q))$, then $V(y,s)\subset 4Q\times(0,3\ell(Q))$. Recall that $V(y,s) = \{(x,r) : W(x,r)\cap W(y,s)\neq\varnothing\}$ and so $(x,r)\in V(y,s)$ if and only if $(y,s)\in V(x,r)$. Thus,

$$\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})(\widetilde{x}) \leq C\sup_{Q\ni\widetilde{x}}\frac{1}{|Q|}\int_{4Q}\int_0^{4\ell(Q)}\int_{V(x,r)}\frac{ds\,dy}{s^{n+1}}\,d\mu(x,r).$$

But $\int_{V(x,r)} \frac{ds\,dy}{s^{n+1}}$ is a constant. Renaming the variables $(x,r)$ to $(x,t)$, we see that

$$\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})(\widetilde{x}) \leq C \sup_{Q\ni\widetilde{x}} \frac{1}{|Q|} \int_{4Q} \int_0^{4\ell(Q)} \frac{t}{t}\, d\mu(x,t) = 4^n C \mathfrak{C}_1^+(t\mu)(\widetilde{x}),$$

as desired. □

*Proof of Lemma 3.6.* Let $\dot{\boldsymbol{u}} \in L^2_{loc}(\mathbb{R}^{n+1}_+)$ be such that

$$\sup_{\dot{\boldsymbol{H}}\in L^2_c(\mathbb{R}^{n+1}_+)\setminus\{0\}} \frac{|\langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}} < \infty.$$

Let $K_\varepsilon = \{(x,t) :\ \varepsilon \leq t \leq 1/\varepsilon,\ |x| \leq 1/\varepsilon\}$. Define $\dot{\boldsymbol{u}}_\varepsilon = \mathbf{1}_{K_\varepsilon} \dot{\boldsymbol{u}}$. By the monotone convergence theorem,

$$\|\widetilde{N}_+\dot{\boldsymbol{u}}\|_{L^p(\mathbb{R}^n)} = \lim_{\varepsilon\to 0^+} \|\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon\|_{L^p(\mathbb{R}^n)} = \sup_{\varepsilon>0} \|\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon\|_{L^p(\mathbb{R}^n)}.$$

Thus we need only bound the quantity $\|\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon\|_{L^p(\mathbb{R}^n)}$, uniformly in $\varepsilon > 0$. We observe that if $\varepsilon > 0$, then $\dot{\boldsymbol{u}}_\varepsilon \in L^2(\mathbb{R}^{n+1}_+)$, and $\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon$ is bounded and compactly supported.

We now construct a $\dot{\boldsymbol{H}}_\varepsilon$ that will allow us to bound $\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon$. Let $W(x,r) = B((x,r), r/2)$, and let $U_\varepsilon(x,r)^2 = \fint_{W(x,r)} |\dot{\boldsymbol{u}}_\varepsilon|^2$, so that $\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon = N_+ U_\varepsilon$. By [3, Theorem 5.1 and formula (2.12)], there is a (nonnegative) measure $\mu$ with $\|\mathfrak{C}_1^+(t\mu)\|_{L^{p'}(\mathbb{R}^n)} \leq C_p$ and with

$$\|N_+ U_\varepsilon\|_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^{n+1}_+} U_\varepsilon(y,s)\, d\mu(y,s).$$

Let

$$\dot{\boldsymbol{H}}^\varepsilon_{(x,r)} = \begin{cases} \dfrac{1}{U_\varepsilon(x,r)} \mathbf{1}_{W(x,r)} \dot{\boldsymbol{u}}_\varepsilon, & U_\varepsilon(x,r) > 0, \\ \dot{\mathbf{0}}, & U_\varepsilon(x,r) = 0, \end{cases}$$

so that

$$\biggl(\fint_{W(x,r)} |\dot{\boldsymbol{H}}^\varepsilon_{(x,r)}|^2\biggr)^{1/2} \leq 1, \quad U_\varepsilon(x,r) = \frac{1}{|W(x,r)|} \langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}^\varepsilon_{(x,r)}\rangle_{\mathbb{R}^{n+1}_+}.$$

Observe that there is a constant $c_n$ such that $|W(x,r)| = r^{n+1}/c_n$ for all $x \in \mathbb{R}^n$ and all $r > 0$. Then

$$\|\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon\|_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^{n+1}_+} U_\varepsilon(x,r)\, d\mu(x,r) = \int_{\mathbb{R}^{n+1}_+} \frac{c_n}{r^{n+1}} \langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}^\varepsilon_{(x,r)}\rangle_{\mathbb{R}^{n+1}_+}\, d\mu(x,r).$$

Changing the order of integration, we see that

$$\|\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon\|_{L^p(\mathbb{R}^n)} = \langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}_\varepsilon\rangle_{\mathbb{R}^{n+1}_+}, \quad \text{where } \dot{\boldsymbol{H}}_\varepsilon(z,t) = \int_{\mathbb{R}^{n+1}_+} \frac{c_n}{r^{n+1}} \dot{\boldsymbol{H}}^\varepsilon_{(x,r)}(z,t)\, d\mu(x,r).$$

We observe that $\dot{\boldsymbol{H}}_\varepsilon$ is compactly supported. By Lemma 3.7 and the assumption on $\mu$,

$$\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)} \leq C\|\mathfrak{C}_1^+(t\mu)\|_{L^{p'}(\mathbb{R}^n)} \leq CC_p$$

and so

$$\begin{aligned} \|\widetilde{N}_+\dot{\boldsymbol{u}}_\varepsilon\|_{L^p(\mathbb{R}^n)} &= \langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}_\varepsilon\rangle_{\mathbb{R}^{n+1}_+} \\ &\leq \frac{C_p}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}}_\varepsilon)\|_{L^{p'}(\mathbb{R}^n)}} \langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}_\varepsilon\rangle_{\mathbb{R}^{n+1}_+} \leq C_p \sup_{\dot{\boldsymbol{H}}\in L^2_c(\mathbb{R}^{n+1}_+)\setminus\{0\}} \frac{|\langle \dot{\boldsymbol{u}}, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}}, \end{aligned}$$

as desired. □

We will use Lemma 3.6 to prove the nontangential bounds (1.26) and (1.27). In proving the bounds (1.32) and (1.33), it will be convenient to introduce an additional derivative in the inner product on the right-hand side. Thus, we now prove the following lemma.

**Lemma 3.8.** *Let $u \in \dot W^{m,2}_{loc}(\mathbb{R}^{n+1}_+)$ satisfy $\widetilde N_+(\nabla^{m-1}u) \in L^2(\mathbb{R}^n)$. Let $p$ satisfy $1 < p < 2$ and let $1/p + 1/p' = 1$. Then*

$$\|\widetilde N_+(\nabla^{m-1}u)\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol\Psi}} \frac{|\langle \dot{\boldsymbol\Psi}, \nabla^m u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t \dot{\boldsymbol\Psi})\|_{L^{p'}(\mathbb{R}^n)}},$$

*where the supremum is taken over all compactly supported $\dot{\boldsymbol\Psi} \in L^2(\mathbb{R}^{n+1}_+)$ that are not identically zero and have a weak vertical derivative also in $L^2(\mathbb{R}^{n+1}_+)$.*

*Proof.* By Lemma 3.6,

$$\|\widetilde N_+(\nabla^{m-1}u)\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol H}\in L^2_c(\mathbb{R}^{n+1}_+)\setminus\{\dot{\boldsymbol 0}\}} \frac{|\langle \dot{\boldsymbol H}, \nabla^{m-1} u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol H})\|_{L^{p'}(\mathbb{R}^n)}}.$$

Choose some such $\dot{\boldsymbol H}$. Let $\dot{\boldsymbol\theta}(x) = \int_0^\infty \dot{\boldsymbol H}(x,t)\,dt$; since $\dot{\boldsymbol H}$ is compactly supported, we have $\dot{\boldsymbol\theta}(x) \in L^2(\mathbb{R}^n)$. Let $\dot{\boldsymbol G}_T(x,t) = \dot{\boldsymbol\theta}(x)\frac{2}{T}\chi_{(3T/4,5T/4)}(t)$, where $T>0$ is a real number and $\chi_{(3T/4,5T/4)}$ denotes the characteristic function of the interval $(3T/4, 5T/4)$. Let $\dot{\boldsymbol H}_T$ be such that $\dot{\boldsymbol H} = \dot{\boldsymbol G}_T + \dot{\boldsymbol H}_T$.

Then $\int_0^\infty \dot{\boldsymbol H}_T(x,t)\,dt = 0$ for almost every $x \in \mathbb{R}^n$. Let

$$(\Psi_T)_\alpha(x,t) = \int_0^t (H_T)_\gamma(x,s)\,ds, \quad \text{where } \alpha = \gamma + \vec e_{n+1},$$

with $\vec e_{n+1}$ as the unit vector in the $(n+1)$th direction, and let $(\Psi_T)_\alpha = 0$ if $\alpha_{n+1} = 0$. Then $\dot{\boldsymbol\Psi}_T \in L^2(\mathbb{R}^{n+1}_+)$ is compactly supported. Furthermore, $\partial_t(\dot{\boldsymbol\Psi}_T)_\alpha(x,t) = (H_T)_\gamma(x,t)$, and so

$$\langle \dot{\boldsymbol H}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+} = -\langle \dot{\boldsymbol\Psi}_T, \nabla^m u\rangle_{\mathbb{R}^{n+1}_+}.$$

Thus,

$$\frac{|\langle \dot{\boldsymbol H}, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol H})\|_{L^{p'}(\mathbb{R}^n)}} = \frac{|\langle \dot{\boldsymbol\Psi}_T, \nabla^m u\rangle_{\mathbb{R}^{n+1}_+} - \langle \dot{\boldsymbol G}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\boldsymbol\Psi}_T + t\dot{\boldsymbol G}_T)\|_{L^{p'}(\mathbb{R}^n)}}$$

for any $T > 0$.

By definition (2.2) of $\widetilde{\mathfrak{C}}_1^+$, if $x \in \mathbb{R}^n$, then

$$\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol G}_T)(x) \leq CT^{-n/2}\|\dot{\boldsymbol\theta}\|_{L^2(\mathbb{R}^n)}.$$

Suppose that $T$ is large enough that there is a cube $Q$ with $\ell(Q) = 5T/4$ and with $\operatorname{supp}\dot{\boldsymbol\theta} \subset 10Q$. By Lemma 3.4 with $r = p'$,

$$\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol G}_T)\|_{L^{p'}(\mathbb{R}^n)} \leq C_{p'}\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol G}_T)\|_{L^{p'}(16Q)} \leq C_{p'}T^{n/p'-n/2}\|\dot{\boldsymbol\theta}\|_{L^2(\mathbb{R}^n)}.$$

If $p < 2$ and $1/p + 1/p' = 1$, then $p' > 2$ and so $\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol G}_T)\|_{L^{p'}(\mathbb{R}^n)} \to 0$ as $T \to \infty$. Since $(G_T)_\gamma + \partial_{n+1}(\Psi_T)_\alpha = H_\gamma$, this implies that $\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\boldsymbol\Psi}_T)\|_{L^{p'}(\mathbb{R}^n)} \to \|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol H})\|_{L^{p'}(\mathbb{R}^n)}$ as $T \to \infty$; by assumption, $\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol H})\|_{L^{p'}(\mathbb{R}^n)} > 0$, hence

$$\frac{|\langle \dot{\boldsymbol H}, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol H})\|_{L^{p'}(\mathbb{R}^n)}} = \lim_{T\to\infty} \frac{|\langle \dot{\boldsymbol\Psi}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+} - \langle \dot{\boldsymbol G}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\boldsymbol\Psi}_T)\|_{L^{p'}(\mathbb{R}^n)}}.$$

We claim that $\langle \dot{\boldsymbol{G}}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+} \to 0$ as $T\to\infty$, as well. We compute that

$$|\langle \dot{\boldsymbol{G}}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}| \leq \fint_{3T/4}^{5T}\int_{\mathbb{R}^n} |\dot{\boldsymbol{\theta}}(x)||\nabla^{m-1}u(x,t)|\,dx\,dt \leq \|\dot{\boldsymbol{\theta}}\|_{L^2(\mathbb{R}^n)} \fint_{3T/4}^{5T} \|\nabla^{m-1}u(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)}\,dt.$$

By Hölder's inequality,

$$\fint_{3T/4}^{5T} \|\nabla^{m-1}u(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)}\,dt \leq \biggl(\fint_{3T/4}^{5T}\int_{\mathbb{R}^n} |\nabla^{m-1}u(x,t)|^2\,dx\,dt\biggr)^{1/2}.$$

Introducing a term $\fint_{|x-y|<T/4} dy$ and changing the order of integration, we see that

$$\fint_{3T/4}^{5T}\int_{\mathbb{R}^n} |\nabla^{m-1}u(x,t)|^2\,dx\,dt \leq \int_{\mathbb{R}^n}\fint_{3T/4}^{5T}\fint_{|x-y|<T/4} |\nabla^{m-1}u(x,t)|^2\,dx\,dt\,dy.$$

Observe that $\{(x,t) : \ |x-y|<T/4,\ |t-T|<T/4\}\subset B((z,T),T/2)$, and that the two regions have comparable volume. Recalling definition (1.5) of $\widetilde{N}_+$, we see that

$$\fint_{3T/4}^{5T}\int_{\mathbb{R}^n} |\nabla^{m-1}u(x,t)|^2\,dx\,dt \leq C_n \int_{\mathbb{R}^n}\biggl(\fint_{|y-z|<T} \widetilde{N}_+(\nabla^{m-1}u)(z)\,dz\biggr)^2\,dy.$$

Let

$$F_T(y) = \fint_{|y-z|<T} \widetilde{N}_+(\nabla^{m-1}u)(z)\,dz,$$

so that

$$\bigl|\langle \dot{\boldsymbol{G}}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}\bigr| \leq C_n \|\dot{\boldsymbol{\theta}}\|_{L^2(\mathbb{R}^n)}\|F_T\|_{L^2(\mathbb{R}^n)}.$$

By Hölder's inequality,

$$F_T(y) \leq C_n T^{-n/2}\|\widetilde{N}_+(\nabla^{m-1}u)\|_{L^2(\mathbb{R}^n)},$$

and so $F_T(y)\to 0$ as $T\to\infty$ pointwise for each $y\in\mathbb{R}^n$. We also have

$$F_T(y) \leq \mathcal{M}(\widetilde{N}_+(\nabla^{m-1}u))(y),$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal function. By the boundedness of $\mathcal{M}$ on $L^2(\mathbb{R}^n)$,

$$\mathcal{M}(\widetilde{N}_+(\nabla^{m-1}u)) \in L^2(\mathbb{R}^n),$$

and so, by the dominated convergence theorem, $F_T\to 0$ in $L^2(\mathbb{R}^n)$ as $T\to\infty$. Thus,

$$\lim_{T\to\infty}\bigl|\langle \dot{\boldsymbol{G}}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}\bigr| = 0.$$

Therefore,

$$\begin{aligned}\frac{|\langle \dot{\boldsymbol{H}}, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}} &= \lim_{T\to\infty}\frac{|\langle \dot{\boldsymbol{\Psi}}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+} - \langle \dot{\boldsymbol{G}}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\boldsymbol{\Psi}}_T)\|_{L^{p'}(\mathbb{R}^n)}}\\ &= \lim_{T\to\infty}\frac{|\langle \dot{\boldsymbol{\Psi}}_T, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\boldsymbol{\Psi}}_T)\|_{L^{p'}(\mathbb{R}^n)}} \leq \sup_{\dot{\boldsymbol{\Psi}}}\frac{|\langle \dot{\boldsymbol{\Psi}}, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\boldsymbol{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}}.\end{aligned}$$

Recalling that Lemma 3.6 implies

$$\|\widetilde{N}_+\dot{\boldsymbol{u}}\|_{L^p(\mathbb{R}^n)} \leq C_p\sup_{\dot{\boldsymbol{H}}}\frac{|\langle \dot{\boldsymbol{H}}, \nabla^{m-1}u\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}}$$

completes the proof. ∎

# 4 The Newton potential

We will establish the bounds on the layer potentials of Theorem 1.2 by duality with the Newton potential, as in [46] and [22, Section 9]. Thus, the present section is devoted to the duality results for the Newton potential and its bounds.

Specifically, we will establish duality between the Newton potential and the double and single layer potentials in Subsection 4.1. We will bound the Newton potential in Subsections 4.2–4.5. For ease of reference, the main bounds on the Newton potential established in the present paper are all listed in Corollary 4.1. In Section 5, we will apply the duality results of Subsection 4.1 and the bounds of Subsections 4.3–4.5 to establish bounds on the double and single layer potentials; the bounds of Subsection 4.2 will be used in Subsections 4.3–4.5.

## 4.1 Duality

In this section, we will prove the following lemma, that is, we will establish appropriate duality relations between the Newton potential and the double and single layer potentials. In Subsection 4.2, we will use these relations to establish the bounds on the Newton potential. In Section 5, we will reverse the argument and use these duality relations to establish the bounds on the double and single layer potentials.

**Lemma 4.1.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*If $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^{n+1})$ and $\dot{\boldsymbol{g}} \in (\dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n))^*$, then we have the duality relation*

$$\big\langle \dot{\mathbf{Tr}}_{m-1}\, \Pi^{L^*} \dot{\boldsymbol{\Psi}}, \dot{\boldsymbol{g}} \big\rangle_{\mathbb{R}^n} = \langle \dot{\boldsymbol{\Psi}}, \nabla^m \mathcal{S}^L \dot{\boldsymbol{g}} \rangle_{\mathbb{R}^{n+1}}. \tag{4.1}$$

*If $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^{n+1}_+)$ and $\dot{\boldsymbol{f}} \in \dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$, then we have the duality relation*

$$\big\langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\, \Pi^{L^*} (\mathbf{1}_+ \dot{\boldsymbol{\Psi}}), \dot{\boldsymbol{f}} \big\rangle_{\mathbb{R}^n} = -\langle \dot{\boldsymbol{\Psi}}, \nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}} \rangle_{\mathbb{R}^{n+1}_+}. \tag{4.2}$$

*If $\boldsymbol{A}$ is $t$-independent in the sense of formula (1.2), $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^{n+1})$ is zero in $\mathbb{R}^n \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, and if $\dot{\boldsymbol{f}} \in \dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$, $\dot{\boldsymbol{g}} \in (\dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n))^*$, and $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$, then*

$$\langle \dot{\mathbf{Tr}}_m\, \Pi^{L^*} \dot{\boldsymbol{\Psi}}, \dot{\boldsymbol{h}} \rangle_{\mathbb{R}^n} = \langle \dot{\boldsymbol{\Psi}}, \nabla^m \mathcal{S}^L_\nabla \dot{\boldsymbol{h}} \rangle_{\mathbb{R}^{n+1}}, \tag{4.3}$$

$$\big\langle \dot{\mathbf{Tr}}_{m-1}\, \partial_{n+1} \Pi^{L^*} \dot{\boldsymbol{\Psi}}, \dot{\boldsymbol{g}} \big\rangle_{\mathbb{R}^n} = -\big\langle \dot{\boldsymbol{\Psi}}, \nabla^m \partial_{n+1} \mathcal{S}^L \dot{\boldsymbol{g}} \big\rangle_{\mathbb{R}^{n+1}}, \tag{4.4}$$

$$\big\langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\, \partial_{n+1} \Pi^{L^*} (\mathbf{1}_+ \dot{\boldsymbol{\Psi}}), \dot{\boldsymbol{f}} \big\rangle_{\mathbb{R}^n} = \big\langle \dot{\boldsymbol{\Psi}}, \nabla^m \partial_{n+1} \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}} \big\rangle_{\mathbb{R}^{n+1}_+}. \tag{4.5}$$

*Proof.* By definition (2.9) of the Newton potential, if $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^{n+1})$, then $\Pi^{L^*} \dot{\boldsymbol{\Psi}} \in \dot{W}^{m,2}(\mathbb{R}^{n+1})$. By definition (2.14) of the single layer potential,

$$\big\langle \dot{\mathbf{Tr}}_{m-1}\, \Pi^{L^*} \dot{\boldsymbol{\Psi}}, \dot{\boldsymbol{g}} \big\rangle_{\partial\mathbb{R}^{n+1}_+} = \big\langle \nabla^m \Pi^{L^*} \dot{\boldsymbol{\Psi}}, \boldsymbol{A} \nabla^m \mathcal{S}^L \dot{\boldsymbol{g}} \big\rangle_{\mathbb{R}^{n+1}},$$

and by definition (2.9) of the Newton potential, we have that the relation (4.1) is valid.

If $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^{n+1}_+)$ and $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}^-_{m-1} F$ for some $F \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_-)$, then by definition of Neumann boundary values,

$$\big\langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\, \Pi^{L^*} (\mathbf{1}_+ \dot{\boldsymbol{\Psi}}), \dot{\boldsymbol{f}} \big\rangle_{\mathbb{R}^n} = \big\langle \boldsymbol{A}^* \nabla^m \Pi^{L^*} (\mathbf{1}_+ \dot{\boldsymbol{\Psi}}), \nabla^m F \big\rangle_{\mathbb{R}^{n+1}_-} = \big\langle \nabla^m \Pi^{L^*} (\mathbf{1}_+ \dot{\boldsymbol{\Psi}}), \mathbf{1}_- \boldsymbol{A} \nabla^m F \big\rangle_{\mathbb{R}^{n+1}}.$$

By [13, Lemma 42], we have

$$\langle \nabla^m \Pi^{L^*} \dot{\boldsymbol{G}}, \dot{\boldsymbol{H}} \rangle_{\mathbb{R}^{n+1}} = \langle \dot{\boldsymbol{G}}, \nabla^m \Pi^L \dot{\boldsymbol{H}} \rangle_{\mathbb{R}^{n+1}} \tag{4.6}$$

for all $\dot{\boldsymbol{G}}$, $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$. Thus,

$$\big\langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\, \Pi^{L^*} (\mathbf{1}_+ \dot{\boldsymbol{\Psi}}), \dot{\boldsymbol{f}} \big\rangle_{\mathbb{R}^n} = \big\langle \dot{\boldsymbol{\Psi}}, \nabla^m \Pi^L (\mathbf{1}_- \boldsymbol{A} \nabla^m F) \big\rangle_{\mathbb{R}^{n+1}_+}.$$

By formula (2.13), the relation (4.2) is valid.

To prove the relations (4.4) and (4.5), we review some Sobolev space theory. If $F \in L^2(\mathbb{R}^{n+1})$ and $h \neq 0$, let $F_h(x,t) = \frac{1}{h}(F(x,t+h) - F(x,t))$. Suppose that $\lim_{h\to 0} F_h$ exists in the sense of $L^2$ functions, that is,

$$\lim_{h\to 0} \|F_h - G\|_{L^2(\mathbb{R}^{n+1})} = 0$$

for some function $G \in L^2(\mathbb{R}^{n+1})$. Then, by the weak definition of a derivative, $\partial_{n+1}F$ exists and equals $G$. Conversely, if $F \in L^2(\mathbb{R}^{n+1}) \cap \dot W^{1,2}(\mathbb{R}^{n+1})$, then an argument similar to the proof of the Lebesgue differentiation theorem shows that $\lim_{h\to 0} \|F_h - \partial_{n+1}F\|_{L^2(\mathbb{R}^{n+1})} = 0$.

By linearity and $t$-independence of $\boldsymbol{A}$, $\Pi^{L^*}(\dot{\boldsymbol{\Psi}}_h) = (\Pi^{L^*}\dot{\boldsymbol{\Psi}})_h$. If $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^{n+1}) \cap \dot W^{1,2}(\mathbb{R}^{n+1})$, then taking limits as $h \to 0$ in $L^2(\mathbb{R}^{n+1})$ shows that

$$\nabla^m \Pi^L(\partial_{n+1}\dot{\boldsymbol{\Psi}}) = \partial_{n+1}(\nabla^m \Pi^L \dot{\boldsymbol{\Psi}}). \tag{4.7}$$

If, in addition, $\dot{\boldsymbol{\Psi}}$ is zero in $\mathbb{R}^n \times (-\varepsilon,\varepsilon)$ for some $\varepsilon > 0$, then formulas (4.4) and (4.5) follow from formulas (4.1) and (4.2) by integrating by parts.

To establish formulas (4.4) and (4.5) for arbitrary $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^n \times (\varepsilon,\infty))$, fix $\varepsilon > 0$, $\dot{\boldsymbol{f}} \in \dot{W}\!A^{1/2,2}_{m-1}(\mathbb{R}^n)$, and $\dot{\boldsymbol{g}} \in (\dot{W}\!A^{1/2,2}_{m-1}(\mathbb{R}^n))^*$. By formulas (2.13) and (2.14), we have that $\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}} \in \dot W^{m,2}(\mathbb{R}^{n+1}_+)$ and $\mathcal{S}^L\dot{\boldsymbol{g}} \in \dot W^{m,2}(\mathbb{R}^{n+1})$, and so, by the Caccioppoli inequality, $\nabla^m \partial_{n+1}\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}} \in L^2(\mathbb{R}^n \times (\varepsilon,\infty))$ and $\nabla^m\partial_{n+1}\mathcal{S}^L\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n \times (\varepsilon,\infty))$. Thus, the right-hand sides of formulas (4.4) and (4.5) (regarded as functions of $\dot{\boldsymbol{\Psi}}$) represent bounded linear operators on $L^2(\mathbb{R}^n \times (\varepsilon,\infty))$. Similarly, by the Caccioppoli inequality, $\partial_{n+1}\Pi^{L^*}$ is a bounded linear operator from $L^2(\mathbb{R}^n \times (\varepsilon,\infty))$ to $\dot W^{m,2}(\mathbb{R}^{n+1}_-)$, and so, if $\dot{\boldsymbol{g}} \in (\dot{W}\!A^{1/2,2}_{m-1}(\mathbb{R}^n))^*$ and $\dot{\boldsymbol{f}} \in \dot{W}\!A^{1/2,2}_{m-1}(\mathbb{R}^n)$, then the left-hand sides of formulas (4.4) and (4.5) represent bounded linear operators on $L^2(\mathbb{R}^n \times (\varepsilon,\infty))$. Thus, by density, formulas (4.4) and (4.5) are valid for all $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^n \times (\varepsilon,\infty))$. A similar argument (or the relations of Subsection 3.1) establishes formula (4.4) for $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^n \times (-\infty,\varepsilon))$.

Formula (4.3) was established in [22, Section 9] under the additional assumption that $\dot{\boldsymbol{\Psi}}$ is supported in $\mathbb{R}^{n+1}_+$. In the general case, by assumption on $\operatorname{supp}\dot{\boldsymbol{\Psi}}$, Lemma 3.2, and the bound (1.20) (with $p=2$), we have that the norms of both sides of formula (4.3) is at most

$$\frac{C}{\sqrt{\varepsilon}}\, \|\dot{\boldsymbol{\Psi}}\|_{L^2(\mathbb{R}^{n+1})} \|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}$$

and, in particular, both sides are meaningful if this quantity is finite. Thus, we need only establish formula (4.3) for $\dot{\boldsymbol{h}}$ in a dense subset of $L^2(\mathbb{R}^n)$. In particular, we only need to consider $\dot{\boldsymbol{h}}$ such that formulas (2.17), (2.16), (4.1), and (4.4) (with appropriate $\dot{\boldsymbol{g}}$) are valid, and formula (4.3) is a straightforward consequence of the given formulas. ☐

## 4.2 The boundary values of the Newton potential

In this section, we begin to establish the bounds on the Newton potential by using Lemma 4.1 and the known bounds (1.16)–(1.23). The argument is precisely dual to that of Section 5. Observe that it is the boundary values $\dot{\mathbf{Tr}}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{\Psi}}$, $\dot{\mathbf{Tr}}_m^-\,\Pi^{L^*}\dot{\boldsymbol{\Psi}}$ and $\dot{\mathbf{M}}_{\boldsymbol{A}}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}})$ that appear in the bounds (4.1)–(4.4); thus, it is the boundary values of the Newton potential that will be bounded in the present section. We remark that we will not establish all of the bounds on the Newton potential that follow from formulas (4.1)–(4.4) and the bounds (1.16)–(1.23), but only those that we will need in Subsections 4.3–4.5.

**Lemma 4.2.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Then there is some $\varepsilon$ with the following significance. Suppose that $\dot{\boldsymbol{\Psi}} \in L^2(\mathbb{R}^{n+1})$ is supported in a compact subset of $\mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-$. If $1/p + 1/p' = 1$ and $p$ lies in the indicated ranges, then*

$$\|\dot{\mathbf{Tr}}_m\,\Pi^{L^*}\dot{\boldsymbol{\Psi}}\|_{L^{p'}} \leq C(1,L,p)\|\mathcal{A}_2^*\dot{\boldsymbol{\Psi}}\|_{L^{p'}},\quad 2-\varepsilon < p < p^+_{1,L}, \tag{4.8}$$

$$\big\|\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}})\big\|_{(\dot{W}\!A^{0,p}_{m-1})^*} \leq C(1,L,p)\|\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}\|_{L^{p'}},\quad 2 \leq p < p^+_{1,L}. \tag{4.9}$$

*Suppose $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$ is supported in a compact subset of $\mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-$. If we normalize $\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}$ as in formulas (2.11) and (2.12), then*

$$\|\dot{\mathbf{Tr}}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}\|_{L^{p'}} \leq C(0,L,p)\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}}, \quad 2-\varepsilon < p < p^+_{0,L}. \tag{4.10}$$

*Proof.* We use Lemmas 3.5 and 3.6 to establish the bound (4.10). We need a similar formula involving the area integral to establish the bounds (4.8) and (4.9). Let $T^p_2 = \{\psi : \mathcal{A}^+_2\psi \in L^p(\mathbb{R}^n)\}$ with the natural norm. By [30, p. 316], if $1 < p < \infty$, then under the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^{n+1}_+} f(x,t)\,g(x,t)\,\frac{dx\,dt}{t},$$

the dual space to $T^p_2$ is $T^{p'}_2$. Thus,

$$\frac{1}{C_p}\,\|\mathcal{A}^+_2(t\dot{\boldsymbol{u}})\|_{L^p(\mathbb{R}^n)} \leq \sup_{\dot{\boldsymbol{\Psi}}} \frac{|\langle\dot{\boldsymbol{\Psi}},\dot{\boldsymbol{u}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\mathcal{A}^+_2\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}} \leq C\|\mathcal{A}^+_2(t\dot{\boldsymbol{u}})\|_{L^p(\mathbb{R}^n)}, \tag{4.11}$$

where the supremum is taken over all $\dot{\boldsymbol{\Psi}} \in L^2_{loc}(\mathbb{R}^{n+1}_+)$ such that $\mathcal{A}^+_2\dot{\boldsymbol{\Psi}} \in L^{p'}(\mathbb{R}^n)$ and such that the denominator is positive. A similar formula is valid for $\mathcal{A}^-_2$ and $\mathcal{A}^*_2$.

*Remark* 4.1. We may take the supremum only over all $\dot{\boldsymbol{\Psi}} \in L^2_c(\mathbb{R}^{n+1}_+) \setminus \{\dot{\mathbf{0}}\}$, where $L^2_c$ is as in Lemma 3.6.

If $1 < p < \infty$, then by formula (4.3) and by density of $L^p \cap L^2$ in $L^p$,

$$\|\dot{\mathbf{Tr}}_m\,\Pi^{L^*}\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)} = \sup_{\dot{\boldsymbol{h}}\in L^p\cap L^2\setminus\{\dot{\mathbf{0}}\}} \frac{|\langle\dot{\boldsymbol{\Psi}}, \nabla^m \mathcal{S}^L_\nabla \dot{\boldsymbol{h}}\rangle_{\mathbb{R}^{n+1}}|}{\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}}.$$

By the bound (4.11), if $1 < p < \infty$, then

$$\|\dot{\mathbf{Tr}}_m\,\Pi^{L^*}\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)} \leq C \sup_{\dot{\boldsymbol{h}}\in L^p\cap L^2\setminus\{0\}} \frac{\|\mathcal{A}^*_2\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}\|\mathcal{A}^*_2(t\nabla^m\mathcal{S}^L_\nabla\dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)}}{\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}},$$

and by the bound (1.20), if $2-\varepsilon < p < p^+_{1,L}$, then the bound (4.8) is valid.

Similarly, by formula (4.2) and the bounds (4.11) and (1.21), if $2 \leq p < p^+_{1,L}$ and $\dot{\boldsymbol{f}} \in \dot{W}A^{0,p}_{m-1}(\mathbb{R}^n) \cap \dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$, then

$$\big|\langle\dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}), \dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}\big| \leq C(1,L,p)\|\mathcal{A}^+_2\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}\|\dot{\boldsymbol{f}}\|_{\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)}.$$

By density of $\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n) \cap \dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$ in $\dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$, the bound (4.9) is valid.

We now turn to the bound (4.10). Let $\dot{\boldsymbol{\gamma}} \in \dot{B}^{-1/2,2}_2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p$ with $2-\varepsilon < p < p^+_{0,L}$. Then by formula (4.1), Lemma 3.5, and the bound (1.16), we have

$$\big|\langle\dot{\mathbf{Tr}}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}, \dot{\boldsymbol{\gamma}}\rangle_{\mathbb{R}^n}\big| \leq C(0,L,p)\|\dot{\boldsymbol{\gamma}}\|_{L^p(\mathbb{R}^n)}\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}.$$

By density of $\dot{B}^{-1/2,2}_2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, there is an $\dot{\boldsymbol{f}} \in L^{p'}(\mathbb{R}^n)$ with

$$\|\dot{\boldsymbol{f}}\|_{L^{p'}(\mathbb{R}^n)} \leq C(0,L,p)\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}$$

such that $\langle\dot{\boldsymbol{f}}, \dot{\boldsymbol{\gamma}}\rangle_{\mathbb{R}^n} = \langle\dot{\mathbf{Tr}}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}, \dot{\boldsymbol{\gamma}}\rangle_{\mathbb{R}^n}$ for all $\dot{\boldsymbol{\gamma}} \in \dot{B}^{-1/2,2}_2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. We need only establish that $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}$.

We normalize $\Pi^{L^*}\dot{\boldsymbol{H}}$ as in formulas (2.11) and (2.12). Then there is some $q < \infty$ such that $\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}} \in L^q(\mathbb{R}^{n+1})$. By Lemma 3.2 and because $\dot{\boldsymbol{H}}$ is supported away from $\partial\mathbb{R}^{n+1}_\pm$, we have that $\dot{\mathbf{Tr}}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}} \in L^q(\mathbb{R}^n)$ (and is, in particular, locally integrable).

If $\dot{\boldsymbol{\varphi}}\in C_0^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}\dot{\boldsymbol{\varphi}}=0$, then $\dot{\boldsymbol{\varphi}}\in \dot{B}_2^{-1/2,2}(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, and so

$$\langle \dot{\boldsymbol{\varphi}}, \dot{\mathbf{T}}\mathbf{r}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}-\dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}=0.$$

Thus, $\dot{\mathbf{T}}\mathbf{r}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}-\dot{\boldsymbol{f}}$ is a constant.

We have seen that $\dot{\boldsymbol{f}}\in L^{p'}(\mathbb{R}^n)$, $\dot{\mathbf{T}}\mathbf{r}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}\in L^q(\mathbb{R}^n)$, for $p'$, $q<\infty$, and $\dot{\boldsymbol{f}}-\dot{\mathbf{T}}\mathbf{r}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}$ is constant; therefore, $\dot{\boldsymbol{f}}=\dot{\mathbf{T}}\mathbf{r}_{m-1}\,\Pi^{L^*}\dot{\boldsymbol{H}}$, as desired. □

## 4.3 Inputs satisfying area integral estimates

In this section, we will continue to establish the bounds on the Newton potential. The two main results of this section are Lemma 4.3, in which we establish the $L^2$ bound

$$\big\|\widetilde{N}_*(\nabla^m\Pi^{L^*}\dot{\mathbf{\Psi}})\big\|_{L^2(\mathbb{R}^n)}\leq C\|\mathcal{A}_2^*\dot{\mathbf{\Psi}}\|_{L^2(\mathbb{R}^n)},$$

and Lemma 4.5, in which we establish the $L^{p'}$ bound

$$\big\|\widetilde{N}_*(\nabla^{m-j}\partial_t^j\Pi^{L^*}\dot{\mathbf{\Psi}})\big\|_{L^{p'}(\mathbb{R}^n)}\leq C(j,L^*,p')\|\mathcal{A}_2^*\dot{\mathbf{\Psi}}\|_{L^{p'}(\mathbb{R}^n)},\quad p_{j,L^*}^-<p\leq 2.$$

The proof of Lemma 4.3 involves the bound (4.8) and some techniques from the proof of [46, Lemma 4.1], while the proof of Lemma 4.5 involves Lemma 4.3 and some techniques from [15].

**Lemma 4.3.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $\dot{\mathbf{\Psi}}\in L^2(\mathbb{R}^{n+1})$ be supported in a compact subset of $\mathbb{R}^{n+1}_+\cup\mathbb{R}^{n+1}_-$. Then*

$$\big\|\widetilde{N}_*(\nabla^m\Pi^{L^*}\dot{\mathbf{\Psi}})\big\|_{L^2(\mathbb{R}^n)}\leq C\|\mathcal{A}_2^*\dot{\mathbf{\Psi}}\|_{L^2(\mathbb{R}^n)}.$$

*Proof.* Let $z\in\mathbb{R}^n$ and let $(x_0,t_0)$ satisfy $|z-x_0|<|t_0|$. Let $B=B((x_0,t_0),|t_0|/2)$. We wish to bound

$$\fint_B|\nabla^m\Pi^{L^*}\dot{\mathbf{\Psi}}|^2$$

by a quantity depending only on $z$ and $\dot{\mathbf{\Psi}}$, and not on $x_0$ or $t_0$.

Let $\Delta(x,r)=\{y\in\mathbb{R}^n:|x-y|<r\}$ denote a disk in $\mathbb{R}^n$ (not $\mathbb{R}^{n+1}$). Let $E_k=\Delta(x_0,2^{k+2}|t_0|)\times(-2^{k+2}|t_0|,2^{k+2}|t_0|)$ be a cylinder in $\mathbb{R}^{n+1}$. We define

$$\dot{\mathbf{\Psi}}_0=\mathbf{1}_{E_0}\dot{\mathbf{\Psi}},\quad \dot{\mathbf{\Psi}}_k=\mathbf{1}_{E_k\setminus E_{k-1}}\mathbf{1}_+\dot{\mathbf{\Psi}},\quad k\geq 1,$$

and let

$$w_k=\Pi^{L^*}\dot{\mathbf{\Psi}}_k,\quad k\geq 0.$$

Then $\Pi^{L^*}\dot{\mathbf{\Psi}}=w_0+\sum_{k=1}^\infty w_k$.

We begin with bounding $\nabla^m w_0$. By the $L^2$ boundedness of $\nabla^m\Pi^{L^*}$,

$$\fint_B|\nabla^m w_0|^2\leq \frac{C}{|t_0|^{n+1}}\,\|\dot{\mathbf{\Psi}}_0\|^2_{L^2(\mathbb{R}^{n+1})}.$$

By Lemma 3.3 with $r=2$ and $\kappa=0$, if $\frac{2n}{n+1}<\theta\leq 2$, then

$$\|\dot{\mathbf{\Psi}}_0\|_{L^2(\mathbb{R}^{n+1})}=\|\dot{\mathbf{\Psi}}_0\|_{L^2(\mathbb{R}^n\times(-4|t_0|,4|t_0|))}\leq\frac{C_\theta}{|t_0|^{n/\theta-1/2-n/2}}\,\|\mathcal{A}_2^*\dot{\mathbf{\Psi}}_0\|_{L^\theta(\mathbb{R}^n)}.$$

But $\mathcal{A}_2^*\dot{\mathbf{\Psi}}_0=0$ outside of $\Delta(x_0,8|t_0|)\subset\Delta(z,9|t_0|)$, and $\mathcal{A}_2^*\dot{\mathbf{\Psi}}_0(x)\leq\mathcal{A}_2^*\dot{\mathbf{\Psi}}(x)$ for all $x\in\Delta(z,9|t_0|)$. Thus,

$$\fint_B|\nabla^m w_0|^2\leq\frac{C_\theta}{|t_0|^{2n/\theta}}\bigg(\int_{\Delta(z,9|t_0|)}(\mathcal{A}_2^*\dot{\mathbf{\Psi}})^\theta\bigg)^{2/\theta}.$$

Let $\mathcal{M}$ denote the Hardy-Littlewood maximal operator (in $\mathbb{R}^n$) given by

$$\mathcal{M}f(z) = \sup_{r>0} \fint_{|y-z|<r} |f(y)|\,dy.$$

We then have

$$\fint_B |\nabla^m w_0|^2 \leq C_\theta \mathcal{M}((\mathcal{A}_2^* \dot{\mathbf{\Psi}})^\theta)(z)^{2/\theta} \tag{4.12}$$

whenever $\frac{2n}{n+1} < \theta \leq 2$.

We now turn to $\dot{\mathbf{\Psi}}_k$, $k \geq 1$. Let $\widetilde{w} = \sum_{k=1}^\infty w_k$. Observe that $L^* \widetilde{w} = 0$ in $E_0$.

The following lemma may be proven by using the same argument as [24, Lemma 3.19], in which the case of cubes (rather than cylinders) was considered.

**Lemma 4.4.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $x_0 \in \mathbb{R}^n$ and let $r > 0$, $c > 0$, and $\sigma > 1$. Let $E = \Delta(x_0, r) \times (-cr, cr)$ and $\widetilde{E} = \Delta(x_0, \sigma r) \times (-c\sigma r, c\sigma r)$. Suppose that $u \in \dot{W}^{m,2}(\widetilde{E})$ and $Lu = 0$ in $\widetilde{E}$. Let $0 \leq j \leq m$. Then there is a constant $C_{c,\sigma}$ depending only on $c$, $\sigma$ and the standard parameters (in particular, independent of $x_0$ and $r$) such that*

$$\fint_E |\nabla^j u(x,t)|^2\,dt\,dx \leq C_{c,\sigma}\bigg(r \fint_{\widetilde{E}} |\partial_t^{j+1} u(x,t)|\,dt\,dx\bigg)^2 + C_{c,\sigma}\bigg(\fint_{\Delta(x_0,\sigma r)} |\dot{\mathbf{Tr}}_j\, u(x)|\,dx\bigg)^2.$$

Observe that $B = B((x_0,t_0), |t_0|/2) \subset \Delta(x_0, |t_0|/2) \times (-3|t_0|/2, 3|t_0|/2)$. Thus, by Lemma 4.4,

$$\bigg(\fint_B |\nabla^m \widetilde{w}|^2\bigg)^{1/2} \leq C \fint_{\Delta(x_0,|t_0|)} \int_{-2|t_0|}^{2|t_0|} |\partial_{n+1}^{m+1} \widetilde{w}| + C \fint_{\Delta(x_0,|t_0|)} |\dot{\mathbf{Tr}}_m\, \widetilde{w}|. \tag{4.13}$$

Recall that $\theta$ is a number with $\frac{2n}{n+1} < \theta \leq 2$. If $n \geq 1$, then $\theta > 1$. Thus, by Hölder's inequality,

$$\fint_{\Delta(x_0,|t_0|)} |\dot{\mathbf{Tr}}_m\, \widetilde{w}| \leq C\mathcal{M}(\dot{\mathbf{Tr}}_m\, w)(z) + \bigg(\fint_{\Delta(x_0,|t_0|)} |\dot{\mathbf{Tr}}_m\, w_0|^\theta\bigg)^{1/\theta}.$$

By the bound (1.15), $p_{1,L}^- \leq \max(1, \frac{2n}{n+2}) \leq \frac{2n}{n+1}$, and so, if $\frac{2n}{n+1} < \theta \leq 2$, then the bound (4.8) is valid for $p' = \theta$. Furthermore, the constant $c(1, L, \theta', 2)$ of the bound (1.13) may be bounded by a constant depending only on $\theta$ and the standard parameters. Thus,

$$\fint_{\Delta(x_0,|t_0|)} |\dot{\mathbf{Tr}}_m\, w_0|^\theta \leq \frac{1}{|\Delta(x_0,|t_0|)|} \int_{\mathbb{R}^n} |\dot{\mathbf{Tr}}_m\, w_0|^\theta \leq C_\theta \frac{1}{|\Delta(x_0,|t_0|)|} \int_{\mathbb{R}^n} (\mathcal{A}_2^* \dot{\mathbf{\Psi}}_0)^\theta.$$

As before, $\mathcal{A}_2^* \dot{\mathbf{\Psi}}_0 \leq \mathcal{A}_2^* \dot{\mathbf{\Psi}}$ and $\mathcal{A}_2^* \dot{\mathbf{\Psi}}_0 = 0$ outside of $\Delta(z, 9|t_0|)$, and so

$$\fint_{\Delta(x_0,|t_0|)} |\dot{\mathbf{Tr}}_m\, \widetilde{w}| \leq C\mathcal{M}(\dot{\mathbf{Tr}}_m\, w)(z) + C_\theta \mathcal{M}((\mathcal{A}_2^* \dot{\mathbf{\Psi}})^\theta)(z)^{1/\theta}. \tag{4.14}$$

We are left with the term involving $\partial_{n+1}^{m+1} \widetilde{w}$.

Choose some $k \geq 1$. Let $(x,t) \in \Delta(x_0, 2|t_0|) \times (-2|t_0|, 2|t_0|) = E_{-1} \subseteq E_{k-2}$. Observe that since $\boldsymbol{A}$ (and thus $\boldsymbol{A}^*$) is $t$-independent, we have that $L^*(\partial_{n+1}^{m+1} w_k) = 0$ in $E_{k-1}$ for each $k \geq 1$. By [13, formula (29)], if $2m > n+1$, then

$$|\partial_t^{m+1} w_k(x,t)| \leq C\bigg(\fint_{E_{k-3/2}} |\partial_s^{m+1} w_k(y,s)|^2\,dy\,ds\bigg)^{1/2}.$$

Recall that $w_k = \Pi^{L^*}\dot{\mathbf{\Psi}}_k$. By the Caccioppoli inequality and the boundedness of the Newton potential $L^2(\mathbb{R}^{n+1}) \to \dot{W}^{m,2}(\mathbb{R}^{n+1})$,

$$|\partial_t^{m+1} w_k(x,t)| \leq \frac{C}{(2^k|t_0|)^{1+(n+1)/2}} \|\dot{\mathbf{\Psi}}_k\|_{L^2(\mathbb{R}^{n+1})}.$$

Observe that $\mathcal{A}_2^*\dot{\mathbf{\Psi}}_k \leq \mathcal{A}_2^*\dot{\mathbf{\Psi}}$ and that $\mathcal{A}_2^*\dot{\mathbf{\Psi}}_k = 0$ outside of $\Delta(x_0, 2^{k+3}|t_0|) \subset \Delta(z, 2^{k+4}|t_0|)$. As before, by Lemma 3.3 with $r = 2$ and $\kappa = 0$,

$$\|\dot{\mathbf{\Psi}}_k\|_{L^2(\mathbb{R}^{n+1})} \leq \frac{C_\theta}{(2^k|t_0|)^{n/\theta-1/2-n/2}} \|\mathcal{A}_2^*\dot{\mathbf{\Psi}}_k\|_{L^\theta(\mathbb{R}^n)} \leq C_\theta (2^k|t_0|)^{(n+1)/2} \mathcal{M}((\mathcal{A}_2^*\dot{\mathbf{\Psi}})^\theta)(z)^{1/\theta}.$$

Thus,

$$\fint_{\Delta(x_0,|t_0|)} \int_{-2|t_0|}^{2|t_0|} |\partial_{n+1}^{m+1}\widetilde{w}| \leq \sum_{k=1}^\infty \frac{C_\theta}{2^k} \mathcal{M}((\mathcal{A}_2^*\dot{\mathbf{\Psi}})^\theta)(z)^{1/\theta}.$$

Summing and applying the bounds (4.12)–(4.14), we see that if $2m > n+1$ and $\frac{2n}{n+1} < \theta \leq 2$, then

$$\begin{aligned}\left(\fint_B |\nabla^m \Pi^{L^*}\dot{\mathbf{\Psi}}|^2\right)^{1/2} &\leq \left(\fint_B |\nabla^m w_0|^2\right)^{1/2} + \left(\fint_B |\nabla^m \widetilde{w}|^2\right)^{1/2} \\ &\leq C_\theta \mathcal{M}((\mathcal{A}_2^*\dot{\mathbf{\Psi}})^\theta)(z)^{1/\theta} + C\mathcal{M}(\dot{\mathbf{Tr}}_m \Pi^{L^*}\dot{\mathbf{\Psi}})(z).\end{aligned}$$

The right-hand side depends only on $z$, not on $x_0$ or $t_0$, and so

$$\widetilde{N}_*(\nabla^m \Pi^{L^*}\dot{\mathbf{\Psi}})(z) \leq C_\theta \mathcal{M}((\mathcal{A}_2^*\dot{\mathbf{\Psi}})^\theta)(z)^{1/\theta} + C\mathcal{M}(\dot{\mathbf{Tr}}_m \Pi^{L^*}\dot{\mathbf{\Psi}})(z).$$

By the bound (4.8), we have that $\|\dot{\mathbf{Tr}}_m \Pi^{L^*}\dot{\mathbf{\Psi}}\|_{L^2(\mathbb{R}^n)} \leq C\|\mathcal{A}_2^*\dot{\mathbf{\Psi}}\|_{L^2(\mathbb{R}^n)}$. Choose $\theta = (2n+1)/(n+1) < 2$. By boundedness of $\mathcal{M}$ on $L^2(\mathbb{R}^n)$ and on $L^{2/\theta}(\mathbb{R}^n)$, we have

$$\|\widetilde{N}_*(\nabla^m \Pi^{L^*}\Psi)\|_{L^2(\mathbb{R}^n)} \leq C\|\mathcal{A}_2^*\dot{\mathbf{\Psi}}\|_{L^2(\mathbb{R}^n)}.$$

This completes the proof in the case $2m > n+1$.

Suppose now that $2m \leq n+1$. Let $\widetilde{L} = \Delta^M L \Delta^M$ for some large integer $M$. As shown in the proof of [13, Theorem 62], there are the constants $a_\xi$ such that

$$\Pi^L \dot{\mathbf{\Psi}} = \Delta^M \Pi^{\widetilde{L}} \dot{\widetilde{\mathbf{\Psi}}}, \quad \text{where} \quad \dot{\widetilde{\mathbf{\Psi}}} = \sum_{|\xi|=2M} \sum_{|\beta|=m} a_\xi \, \Psi_\beta \, \dot{\boldsymbol{e}}_{\beta+\xi},$$

where $\dot{\boldsymbol{e}}_{\beta+\xi}$ is given by formula (2.15). Thus,

$$\widetilde{N}_*(\nabla^m \Pi^{L^*}\dot{\mathbf{\Psi}})(z) = \widetilde{N}_+(\nabla^m \Delta^M \Pi^{\widetilde{L}^*}\dot{\widetilde{\mathbf{\Psi}}})(z)$$

and if we choose $M$ such that $2m + 4M > n+1$, then

$$\big\|\widetilde{N}_*(\nabla^{m+2M} \Pi^{\widetilde{L}^*}\dot{\widetilde{\mathbf{\Psi}}})\big\|_{L^2(\mathbb{R}^n)} \leq C\|\mathcal{A}_2^*\dot{\widetilde{\mathbf{\Psi}}}\|_{L^2(\mathbb{R}^n)} \leq C^2\|\mathcal{A}_2^*\dot{\mathbf{\Psi}}\|_{L^2(\mathbb{R}^n)},$$

as desired. □

We now extend to the bounds for $\mathcal{A}_2^*\dot{\mathbf{\Psi}} \in L^{p'}(\mathbb{R}^n)$, $p < 2$.

**Lemma 4.5.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $j$ be an integer with $0 \leq j \leq m$. Let $p$ satisfy $p_{j,L^*}^- < p < 2$ and $1/p + 1/p' = 1$. Let $\dot{\mathbf{\Psi}} \in L^2(\mathbb{R}^{n+1})$ be supported in a compact subset of $\mathbb{R}_+^{n+1} \cup \mathbb{R}_-^{n+1}$. Then*

$$\big\|\widetilde{N}_*(\nabla^{m-j}\partial_t^j \Pi^{L^*}\dot{\mathbf{\Psi}})\big\|_{L^{p'}(\mathbb{R}^n)} \leq C(j, L^*, p')\|\mathcal{A}_2^*\dot{\mathbf{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}, \quad p_{j,L^*}^- < p \leq 2.$$

*Proof.* The $p=2$ case is Lemma 4.3. Let

$$u = \partial_t^j \Pi^{L^*} \dot{\mathbf{\Psi}}, \quad u_Q = \partial_t^j \Pi^{L^*} \big(\mathbf{1}_{10Q\times(-\ell(Q),\ell(Q))} \dot{\mathbf{\Psi}}\big), \quad \Phi_1 = \mathcal{A}_2^* \dot{\mathbf{\Psi}},$$

where $Q$ is any cube in $\mathbb{R}^n$. Hereafter, the proof closely parallels that of [15, Theorem 4.12] and, in fact, will use many results of [15]. Choose some $p$ with $p_{j,L^*}^- < p < 2$. By standard self-improvement properties of reverse Hölder estimates (see, for example, [42, Chapter V, Theorem 1.2]), there is a $p_2 > p'$ such that the bound (1.13) is valid for solutions $u$ to $L^* u = 0$ and for $p = p_2$. That is, there is $p_2 > p'$ such that $p_2 < p_{j,L^*}^+$ with $p_2$ and $c(j, L^*, p_2, 2)$ depending only on $p$ and $c(j, L^*, p', 2)$.

We have that $u - u_Q \in \dot W_{loc}^{m,2}(10Q \times (-\ell(Q), \ell(Q)))$ and $L^*(u - u_Q) = 0$ in $10Q \times (-\ell(Q), \ell(Q))$. By [15, Lemma 4.11] with $v = u - u_Q$, we obtain

$$\bigg(\fint_{8Q} \widetilde{N}_n^\ell (\nabla^{m-j}(u - u_Q))^{p_2}\bigg)^{1/p_2} \leq C(j, L^*, p_2) \bigg(\fint_{10Q} \widetilde{N}_n^{3\ell} (\nabla^{m-j}(u - u_Q))^2\bigg)^{1/2},$$

where $\ell = \ell(Q)/4$ and $\widetilde{N}_n^\ell$ is as given in [15, Section 4]. In particular, $\widetilde{N}_n^{3\ell}(\nabla^{m-j} u) \leq \widetilde{N}_*(\nabla^{m-j} u)$. By Lemma 4.3, we have

$$\|\widetilde{N}_*(\nabla^{m-j} u)\|_{L^2(\mathbb{R}^n)} \leq C \|\Phi_1\|_{L^2(\mathbb{R}^n)} < \infty.$$

Observe that $\mathcal{A}_2^*(\mathbf{1}_{10Q\times(-\ell(Q),\ell(Q))} \dot{\mathbf{\Psi}})(x) \leq \mathcal{A}_2^* \dot{\mathbf{\Psi}}(x) = \Phi_1(x)$ and is zero if $x \notin 12Q$; thus, again by Lemma 4.3, we have that

$$\|\widetilde{N}_*(\nabla^{m-j} u_Q)\|_{L^2(\mathbb{R}^n)} \leq C \|\Phi_1\|_{L^2(16Q)}.$$

These bounds imply that

$$\big\|\widetilde{N}_n^\ell (\nabla^{m-j}(u - u_Q))\big\|_{L^{p_2}(8Q)} \leq \frac{C(j, L^*, p_2)}{|Q|^{1/2-1/p_2}} \Big(\|\Phi_1\|_{L^2(16Q)} + \|\widetilde{N}_*(\nabla^{m-j} u)\|_{L^2(10Q)}\Big).$$

The conditions of [15, Lemma 4.3] with $\dot{\boldsymbol{u}} = \nabla^{m-j} u$ and $\dot{\boldsymbol{u}}_Q = \nabla^{m-j} u_Q$ are thus satisfied, and so,

$$\|\widetilde{N}_+(\nabla^{m-j} u)\|_{L^{p'}(\mathbb{R}^n)} \leq C(j, L^*, p') \|\Phi_1\|_{L^{p'}(\mathbb{R}^n)},$$

as desired. □

## 4.4 Inputs satisfying Carleson estimates

In this section, we will continue to establish the bounds on the Newton potential. In Lemma 4.6, we will establish the area integral bound

$$\|\mathcal{A}_2^*(t \nabla^m \Pi^{L^*} \dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)} \leq C(0, L, p) \|\widetilde{\mathfrak{C}}_1^*(t \dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}, \quad 2 \leq p < p_{0,L}^+,$$

and in Lemmas 4.7 and 4.8, we will establish the nontangential bound

$$\|\widetilde{N}_*(\nabla^{m-1} \Pi^{L^*} \dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)} \leq \widetilde{C}_p \|\widetilde{\mathfrak{C}}_1^*(t \dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}, \;\; p_{1,L^*}^- < p < p_{0,L}^+$$

for an appropriate constant $\widetilde{C}_p$.

Lemma 4.6 will be proven by a simple duality argument. The proof of Lemma 4.7 will use some techniques similar to those of Lemma 4.3. Most of the proof of Lemma 4.8 will be omitted, since once some notation has been established it can be proved in the same fashion as [15, Theorem 4.12] or Lemma 4.5.

**Lemma 4.6.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$ be supported in a compact subset of $\mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-$. Let $p_{0,L}^+$ be as in the bound (1.13). If $2 \leq p < p_{0,L}^+$ and $1/p + 1/p' = 1$, then*

$$\big\|\mathcal{A}_2^*(t \nabla^m \Pi^{L^*} \dot{\boldsymbol{H}})\big\|_{L^{p'}(\mathbb{R}^n)} \leq C(0, L, p) \|\widetilde{\mathfrak{C}}_1^*(t \dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}, \;\; 2 \leq p < p_{0,L}^+.$$

*Proof.* By the bound (4.11),

$$\big\|\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})\big\|_{L^{p'}(\mathbb{R}^n)} \approx \sup_{\dot{\boldsymbol{\Psi}}} \frac{|\langle\dot{\boldsymbol{\Psi}},\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}}|}{\|\mathcal{A}_2^*\dot{\boldsymbol{\Psi}}\|_{L^p(\mathbb{R}^n)}}.$$

We may take the supremum over $\dot{\boldsymbol{\Psi}}$ supported in a compact subset of $\mathbb{R}^{n+1}_+\cup\mathbb{R}^{n+1}_-$ such that the denominator is positive and finite. Thus, we may assume that $\dot{\boldsymbol{\Psi}}\in L^2(\mathbb{R}^{n+1})$. By [13, Lemma 42] (reproduced as formula (4.6) above),

$$\big\|\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})\big\|_{L^{p'}(\mathbb{R}^n)} \approx \sup_{\dot{\boldsymbol{\Psi}}} \frac{|\langle\nabla^m\Pi^{L}\dot{\boldsymbol{\Psi}},\dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}}|}{\|\mathcal{A}_2^*\dot{\boldsymbol{\Psi}}\|_{L^p(\mathbb{R}^n)}},$$

and by Lemma 3.5,

$$\big\|\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})\big\|_{L^{p'}(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{\Psi}}} \frac{\|\widetilde{N}_*(\nabla^m\Pi^{L}\dot{\boldsymbol{\Psi}})\|_{L^p(\mathbb{R}^n)}\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}}{\|\mathcal{A}_2^*\dot{\boldsymbol{\Psi}}\|_{L^p(\mathbb{R}^n)}}.$$

Using Lemma 4.5 with $j=0$ and permuting $p$, $p'$ and $L$, $L^*$ we complete the proof. □

We now establish nontangential estimates.

**Lemma 4.7.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $\dot{\boldsymbol{H}}\in L^2(\mathbb{R}^{n+1})$ be supported in a compact subset of $\mathbb{R}^{n+1}_+\cup\mathbb{R}^{n+1}_-$. Let $p^+_{0,L}$ be as in the bound (1.13). If $2\leq p<p^+_{0,L}$ and $1/p+1/p'=1$, then*

$$\big\|\widetilde{N}_*(\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}})\big\|_{L^{p'}(\mathbb{R}^n)} \leq C(0,L,p)\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)},\quad 2\leq p<p^+_{0,L}.$$

*Proof.* As in the proof of Lemma 4.3, let $\Delta(x,r)=\{y\in\mathbb{R}^n: |x-y|<r\}$, let $z\in\mathbb{R}^n$, and let $B=B((x_0,t_0),|t_0|/2)$ be a Whitney ball with $|x_0-z|<|t_0|$. Let

$$\dot{\boldsymbol{H}}=\dot{\boldsymbol{H}}_n+\dot{\boldsymbol{H}}_f,\quad\text{where}\quad \dot{\boldsymbol{H}}_n=\mathbf{1}_{\Delta(x_0,4|t_0|)\times(-4|t_0|,4|t_0|)}\dot{\boldsymbol{H}}.$$

Our goal is thus to show that

$$\bigg(\fint_B|\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_n|^2\bigg)^{1/2}+\bigg(\fint_B|\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_f|^2\bigg)^{1/2}$$

may be bounded by a quantity depending only on $z$ and $\dot{\boldsymbol{H}}$, not on $x_0$ and $t_0$.

Fix some $p$ with $2\leq p<p^+_{0,L}$. As in the proof of Lemma 4.5, by standard self-improvement properties of the reverse Hölder estimates (see, for example, [42, Chapter V, Theorem 1.2]), there is a $\theta>p$ such that the bound (1.13) is valid for solutions $u$ to $Lu=0$ and for $p=\theta$. That is, there is $\theta$ such that $p<\theta<p^+_{0,L}$ with $\theta$ and $c(0,L,\theta,2)$ depending only on $p$ and $c(0,L,p,2)$.

If $n+1\geq 3$, let $r=2$; if $n+1=2$, let $r$ satisfy $\theta'<r<2$ and be close enough to 2 that the bound (2.10) is valid. Let $q$ be as in the bound (2.11) or (2.12). Observe that $r>1$ and so $q>2$.

We begin with $\Pi^{L^*}\dot{\boldsymbol{H}}_n$. By Hölder's inequality,

$$\bigg(\fint_B|\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_n|^2\bigg)^{1/2}\leq C|t_0|^{-(n+1)/q}\|\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_n\|_{L^q(\mathbb{R}^n\times I(t_0))},$$

where $I(t_0)=(t_0/2,\infty)$ if $t_0>0$, and where $I(t_0)=(-\infty,t_0/2)=(-\infty,-|t_0|/2)$ if $t_0<0$.

Recall that $\dot{\boldsymbol{H}}\in L^2(\mathbb{R}^{n+1})$ and $r\leq 2$, and so $\dot{\boldsymbol{H}}_n\in L^r(\mathbb{R}^{n+1})$. By the bound (2.10), we have $\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}}_n\in L^r(\mathbb{R}^n\times I(t_0))$. By the Gagliardo–Nirenberg–Sobolev inequality and standard extension theorems of Sobolev spaces on a half-space, we find that there is a constant $\dot{\boldsymbol{c}}$ such that

$$\|\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_n-\dot{\boldsymbol{c}}\|_{L^q(\mathbb{R}^n\times I(t_0))}\leq C_r\|\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}}_n\|_{L^r(\mathbb{R}^n\times I(t_0))}.$$

By the bound (2.11) or (2.12), $\|\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_n\|_{L^q(\mathbb{R}^{n+1})}$ is finite, and so $\dot{\boldsymbol{c}}=0$.

Recall that $1<\theta'<r\leq 2$ and so $\theta'(n+1)/(n+\theta')<r$. By Lemma 3.3 with $\kappa=1$,

$$\|\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}}_n\|_{L^r(\mathbb{R}^n\times I(t_0))}\leq \frac{C_{\theta,r}}{|t_0|^{1+n/\theta'-1/r-n/r}}\big\|\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}}_n)\big\|_{L^{\theta'}(\mathbb{R}^n)}.$$

By Lemma 4.6, if $2<\theta<p_{0,L}^+$, then

$$\big\|\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}}_n)\big\|_{L^{\theta'}(\mathbb{R}^n)}\leq C(0,L,\theta)\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}_n)\|_{L^{\theta'}(\mathbb{R}^n)}.$$

Thus,

$$\bigg(\fint_B \big|\nabla^{m-1}\Pi^{L^*}(\dot{\boldsymbol{H}}_n)\big|^2\bigg)^{1/2}\leq\frac{C(0,L,\theta)}{|t_0|^{n/\theta'}}\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}_n)\|_{L^{\theta'}(\mathbb{R}^n)}.$$

By Lemma 3.4 with $r=\theta'$, we have

$$\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}_n)\|_{L^{\theta'}(\mathbb{R}^n)}\leq C_\theta|t_0|^{n/\theta'}\mathcal{M}((\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}))^{\theta'})(z)^{1/\theta'},$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal function. Thus,

$$\bigg(\fint_B \big|\nabla^{m-1}\Pi^{L^*}(\dot{\boldsymbol{H}}_n)\big|^2\bigg)^{1/2}\leq C(0,L,\theta)\mathcal{M}((\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}))^{\theta'})(z)^{1/\theta'}. \tag{4.15}$$

We now turn to $\Pi^{L^*}\dot{\boldsymbol{H}}_f$. Recall that $\dot{\boldsymbol{H}}_f=0$ in $\Delta(x_0,4|t_0|)\times(-4|t_0|,4|t_0|)$. By Lemma 4.4,

$$\bigg(\fint_B \big|\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_f\big|^2\bigg)^{1/2}\leq C\fint_{\Delta(x_0,|t_0|)}|\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_f|+C\fint_{-2|t_0|}^{2|t_0|}\fint_{\Delta(x_0,|t_0|)}|\partial_{n+1}^m\Pi^{L^*}\dot{\boldsymbol{H}}_f|.$$

We begin by bounding the trace. We have

$$\fint_{\Delta(x_0,|t_0|)}\big|\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_f\big|\leq C\mathcal{M}(\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}})(z)+\fint_{\Delta(x_0,|t_0|)}|\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_n|.$$

By the bound (4.10), we have that $\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}\in L^{\theta'}(\mathbb{R}^n)$, and

$$\big\|\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_n\big\|_{L^{\theta'}(\mathbb{R}^n)}\leq C(0,L,\theta)\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}_n)\|_{L^{\theta'}(\mathbb{R}^n)}.$$

Thus, by Hölder's inequality and Lemma 3.4,

$$\fint_{\Delta(x_0,|t_0|)}\big|\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_n\big|\leq C(0,L,\theta)t_0^{-n/\theta'}\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}_n)\|_{L^{\theta'}(\mathbb{R}^n)}\leq C(0,L,\theta)\mathcal{M}((\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}))^{\theta'})(z)^{1/\theta'}.$$

Therefore,

$$\begin{aligned}\bigg(\fint_B|\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_f|^2\bigg)^{1/2}&\leq C\mathcal{M}(\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}})(z)\\&\quad+C(0,L,\theta)\mathcal{M}((\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}))^{\theta'})(z)^{1/\theta'}+C\fint_{-2|t_0|}^{2|t_0|}\fint_{\Delta(x_0,|t_0|)}|\partial_{n+1}^m\Pi^{L^*}\dot{\boldsymbol{H}}_f|.\end{aligned}$$

We are left with the term involving $\partial_{n+1}^m\Pi^{L^*}\dot{\boldsymbol{H}}_f$. Let $w=\partial_{n+1}^m\Pi^{L^*}\dot{\boldsymbol{H}}_f$. By the bound (1.13), if $0<\mu<\infty$, then

$$\fint_{-2|t_0|}^{2|t_0|}\fint_{\Delta(x_0,|t_0|)}|w|\leq C_\mu\bigg(\fint_{-3|t_0|}^{3|t_0|}\fint_{\Delta(x_0,2|t_0|)}|w|^\mu\bigg)^{1/\mu}.$$

Choose $\mu = 1/2$. By Lemma 3.3 with $\theta = r = 1/2$ and $\kappa = 1$, it follows that

$$\Big(\fint_{-3|t_0|}^{3|t_0|}\fint_{\Delta(x_0,2|t_0|)} |w|^{1/2}\Big)^2 \leq C t_0^{-2n-1} \|\mathcal{A}_2^*(t\mathbf{1}_E w)\|_{L^{1/2}(\mathbb{R}^n)},$$

where $E$ is the region of integration on the left-hand side. Observe that $\mathcal{A}_2^*(t\mathbf{1}_E w) = 0$ outside of $\Delta(x_0, 5|t_0|) \subset \Delta(z, 6|t_0|)$. By Hölder's inequality, if $\theta' > 1/2$, then

$$\int_{-2|t_0|}^{2|t_0|}\fint_{\Delta(x_0,|t_0|)} |w| \leq C\Big(\fint_{\Delta(z,6|t_0|)} \mathcal{A}_2^*(tw)(y)^{\theta'}\,dy\Big)^{1/\theta'}.$$

Recalling the definitions of $w$ and $\dot{\boldsymbol{H}}_f$, we see that if $\theta' \geq 1$, then

$$\Big(\fint_{\Delta(z,6|t_0|)} \mathcal{A}_2^*(tw)^{\theta'}\Big)^{1/\theta'} \leq \Big(\fint_{\Delta(z,6|t_0|)} \mathcal{A}_2^*(t\nabla^m \Pi^{L^*}\dot{\boldsymbol{H}}_n)^{\theta'}\Big)^{1/\theta'} + \Big(\fint_{\Delta(z,6|t_0|)} \mathcal{A}_2^*(t\nabla^m \Pi^{L^*}\dot{\boldsymbol{H}})^{\theta'}\Big)^{1/\theta'}.$$

By Lemma 4.6, if $2 \leq \theta < p_{0,L}^+$, then

$$\Big(\fint_{\Delta(z,6|t_0|)} \mathcal{A}_2^*(tw)^{\theta'}\Big)^{1/\theta'} \leq \frac{C(0,L,\theta)}{t_0^{n/\theta'}} \|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}_n)\|_{L^{\theta'}(\mathbb{R}^n)} + \mathcal{M}(\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})^{\theta'})(z)^{1/\theta'},$$

and by Lemma 3.4 with $r = \theta'$,

$$\frac{1}{t_0^{n/\theta'}} \|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}_n)\|_{L^{\theta'}(\mathbb{R}^n)} \leq C\mathcal{M}((\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}))^{\theta'})(z)^{1/\theta'}.$$

Thus,

$$\begin{aligned}\Big(\fint_B |\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}}_f|^2\Big)^{1/2} &\leq C\mathcal{M}(\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}})(z)\\ &\quad + C\mathcal{M}(\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})^{\theta'})(z)^{1/\theta'} + C(0,L,\theta)\mathcal{M}((\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}))^{\theta'})(z)^{1/\theta'}.\end{aligned}$$

Combining this estimate with the bound (4.15) yields

$$\begin{aligned}&\widetilde{N}_*(\nabla^{m-1}\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{H}}))(z)\\ &\quad\leq C\mathcal{M}(\dot{\mathbf{Tr}}_{m-1}\Pi^{L^*}\dot{\boldsymbol{H}})(z) + C\mathcal{M}(\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})^{\theta'})(z)^{1/\theta'} + C(0,L,\theta)\mathcal{M}((\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}))^{\theta'})(z)^{1/\theta'}.\end{aligned}$$

Recall that $p < \theta < p_{0,L}^+$, so that $p'/\theta' > 1$, and that $c(0,L,\theta,2)$ depends only on $p$ and $c(0,L,p,2)$. By the bound (4.10), Lemma 4.6, and the $L^{p'}$ and $L^{p'/\theta'}$-boundedness of $\mathcal{M}$, we find that the lemma follows from the above bound. □

The techniques of [15] allow us to extend the range of $p$ in our nontangential bound.

**Lemma 4.8.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$ be supported in a compact subset of $\mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-$. Let $p_{j,L}^+$ be as in the bound (1.13). If $p_{1,L^*}^- < p < 2$ and $1/p + 1/p' = 1$, then*

$$\|\widetilde{N}_*(\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)} \leq C(1,L^*,p')\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)},\quad p_{1,L^*}^- < p < 2.$$

*Proof.* Let

$$u = \Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{H}}),\quad u_Q = \Pi^{L^*}(\mathbf{1}_{10Q\times(-\ell(Q),\ell(Q))}\dot{\boldsymbol{H}}),\quad \Phi_1 = \widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}),\quad j=1.$$

The proof is similar to that of [15, Theorem 4.12] or Lemma 4.5 and will be omitted. □

## 4.5 Area integral estimates on the Newton potential

In this section, we will establish area integral estimates on the Newton potential beyond Lemma 4.6. We recall that the Fatou-type estimates on the Neumann boundary values established in [23] involve area integral estimates but not nontangential estimates; thus, in light of formulas (4.2) and (4.5), area integral estimates are necessary to bound the double layer potential. We will also expand the range of $p$ in the nontangential bound of Lemma 4.5. For ease of reference, all our nontangential and area integral bounds on the Newton potential are listed in Corollary 4.1.

**Lemma 4.9.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $\dot{\boldsymbol\Psi} \in L^2(\mathbb{R}^{n+1}_+)$ be compactly supported. Then*

$$\big\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}))\big\|_{L^2(\mathbb{R}^n)} \leq C\|\mathcal{A}_2^+\dot{\boldsymbol\Psi}\|_{L^2(\mathbb{R}^n)}.$$

*Proof.* By the boundedness of the Newton potential (see Subsection 2.4), $\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}) \in \dot W^{m,2}(\mathbb{R}^{n+1})$. By definition (2.9) of $\Pi^{L^*}$, $L^*(\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})) = 0$ in $\mathbb{R}^{n+1}_-$. By [14, Lemma 5.2] or [20, formula (2.26)], we have the Green formula

$$\mathbf{1}_-\nabla^m\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}) = \nabla^m\mathcal{D}^{\boldsymbol{A}^*}(\dot{\mathbf{Tr}}_{m-1}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})) + \nabla^m\mathcal{S}^{L^*}(\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}))$$

away from $\partial\mathbb{R}^{n+1}_\pm$. This formula can also be derived from formula (2.13) for the double layer potential (with $F = \Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})$), and from the definitions (2.8), (2.9) and (2.14) of $\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-$, $\Pi^{L^*}$, and $\mathcal{S}^{L^*}$.

Thus,

$$\begin{aligned}&\big\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}))\big\|_{L^2(\mathbb{R}^n)}\\ &\quad\leq \big\|\mathcal{A}_2^-(t\nabla^m\partial_t\mathcal{D}^{\boldsymbol{A}^*}(\dot{\mathbf{Tr}}_{m-1}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})))\big\|_{L^2(\mathbb{R}^n)} + \big\|\mathcal{A}_2^-(t\nabla^m\partial_t\mathcal{S}^{L^*}(\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})))\big\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

By the bound (1.19) with $p=2$,

$$\big\|\mathcal{A}_2^-\big(t\nabla^m\partial_t\mathcal{D}^{\boldsymbol{A}^*}(\dot{\mathbf{Tr}}_{m-1}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}))\big)\big\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\mathbf{Tr}}_{m-1}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})\|_{\dot WA^{1,2}_{m-1}(\mathbb{R}^n)},$$

and by definition of $\dot WA^{1,2}_{m-1}(\mathbb{R}^n)$ and the bound (4.8),

$$\|\,\dot{\mathbf{Tr}}_{m-1}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})\|_{\dot WA^{1,2}_{m-1}(\mathbb{R}^n)} \leq \|\dot{\mathbf{Tr}}_m^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})\|_{L^2(\mathbb{R}^n)} \leq C\|\mathcal{A}_2^+\dot{\boldsymbol\Psi}\|_{L^2(\mathbb{R}^n)},$$

as desired.

We apply a similar argument to the second term. By the bound (4.9) with $p=2$,

$$\big|\langle\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}),\dot{\boldsymbol f}\rangle_{\mathbb{R}^n}\big| \leq C\|\mathcal{A}_2^+\dot{\boldsymbol\Psi}\|_{L^2(\mathbb{R}^n)}\|\dot{\boldsymbol f}\|_{\dot WA^{0,2}_{m-1}(\mathbb{R}^n)}.$$

By the boundedness of the Newton potential $L^2(\mathbb{R}^{n+1}) \mapsto \dot W^{m,2}(\mathbb{R}^{n+1})$, and by definition (2.8) of the Neumann boundary data, we also have that

$$\big|\langle\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}),\dot{\boldsymbol f}\rangle_{\mathbb{R}^n}\big| \leq C\|\dot{\boldsymbol\Psi}\|_{L^2(\mathbb{R}^{n+1}_+)}\|\dot{\boldsymbol f}\|_{\dot WA^{1/2,2}_{m-1}(\mathbb{R}^n)}.$$

Thus, $\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi})$ extends to a bounded linear operator on $\dot WA^{0,2}_{m-1}(\mathbb{R}^n) + \dot WA^{1/2,2}_{m-1}(\mathbb{R}^n)$, and by the Hahn–Banach theorem, extends to a bounded linear operator on $L^2(\mathbb{R}^n)+\dot B^{1/2,2}_2(\mathbb{R}^n)$. By standard duality arguments, there is a $\dot{\boldsymbol g} \in L^2(\mathbb{R}^n)\cap\dot B^{-1/2,2}_2(\mathbb{R}^n)$ such that $\langle\dot{\boldsymbol g},\dot{\boldsymbol\varphi}\rangle_{\mathbb{R}^n} = \langle\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}),\dot{\boldsymbol\varphi}\rangle_{\mathbb{R}^n}$ for all $\dot{\boldsymbol\varphi} \in \dot WA^{1/2,2}_{m-1}(\mathbb{R}^n)$. We may ensure that $\|\dot{\boldsymbol g}\|_{L^2(\mathbb{R}^n)} \leq C\|\mathcal{A}_2^+\dot{\boldsymbol\Psi}\|_{L^2(\mathbb{R}^n)}$ by carefully choosing the norm in $L^2(\mathbb{R}^n)+\dot B^{1/2,2}_2(\mathbb{R}^n)$.

By definition (2.14) of the single layer potential, we have that $\mathcal{S}^{L^*}\dot{\boldsymbol g} = \mathcal{S}^{L^*}(\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\,\Pi^{L}(\mathbf{1}_+\dot{\boldsymbol\Psi}))$. Thus,

$$\big\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol\Psi}))\big\|_{L^2(\mathbb{R}^n)} \leq C\|\mathcal{A}_2^+\dot{\boldsymbol\Psi}\|_{L^2(\mathbb{R}^n)} + \|\mathcal{A}_2^-(t\nabla^m\partial_t\mathcal{S}^{L^*}\dot{\boldsymbol g})\|_{L^2(\mathbb{R}^n)},$$

and the given bound on $\|\dot{\boldsymbol g}\|_{L^2(\mathbb{R}^n)}$ and the bound (1.18) complete the proof. ∎

We now establish area integral estimates for a wider range of $p$.

**Lemma 4.10.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $\dot{\boldsymbol{H}}$ and $\dot{\boldsymbol{\Psi}}$ be elements of $L^2(\mathbb{R}^{n+1})$ that are supported in compact subsets of $\mathbb{R}^{n+1}_+\cup\mathbb{R}^{n+1}_-$ and $\mathbb{R}^{n+1}_+$, respectively. Let $p^-_{j,L}$ be as in formula (1.14). If $1/p+1/p'=1$ and $p^-_{1,L^*}<p<2$, then we have the bounds*

$$\|\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}\leq C(1,L^*,p')\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)},\quad p^-_{1,L^*}<p<2,$$
$$\big\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}))\big\|_{L^{p'}(\mathbb{R}^n)}\leq C(1,L^*,p')\|\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)},\quad p^-_{1,L^*}<p<2.$$

*Proof.* We will use [15, Lemma 6.2].

For ease of notation, we consider only $\mathcal{A}_2^-$ in both cases; a similar argument or Subsection 3.1 establishes the bound on $\mathcal{A}_2^+(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})$. We make one of the following two choices of notation:

$$u=\Pi^{L^*}\dot{\boldsymbol{H}},\quad u_Q=\Pi^{L^*}\dot{\boldsymbol{H}}_Q,\quad \Phi_1=\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}),$$

or

$$u=\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}),\quad u_Q=\partial_t\Pi^{L^*}\dot{\boldsymbol{\Psi}}_Q,\quad \Phi_1=\mathcal{A}_2^+\dot{\boldsymbol{\Psi}},$$

where

$$\dot{\boldsymbol{H}}_Q=\mathbf{1}_{10Q\times(-\ell(Q),\ell(Q))}\dot{\boldsymbol{H}},\quad \dot{\boldsymbol{\Psi}}_Q=(\mathbf{1}_{11Q\times(0,2\ell(Q))}\dot{\boldsymbol{\Psi}}).$$

Observe that $\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}_Q(x)\leq\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}(x)$ and $\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}_Q(x)=0$ whenever $x\notin 15Q$, while by Lemma 3.4, we have that $\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}}_Q)\|_{L^r(\mathbb{R}^n)}\leq C\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^r(16Q)}$ for any $1<r<\infty$.

By definition of $\Pi^{L^*}$ and the Caccioppoli inequality, we have

$$u-u_Q\in\dot{W}^{m,2}(10Q\times(-\ell(Q),\ell(Q))),$$
$$L^*(u-u_Q)=0\ \text{ in } 10Q\times(-\ell(Q),\ell(Q)).$$

By Lemmas 4.6 and 4.9, we get

$$\mathcal{A}_2^-(t\nabla^m u)\in L^2(\mathbb{R}^n),$$
$$\|\mathcal{A}_2^-(t\nabla^m u_Q)\|_{L^2(\mathbb{R}^n)}\leq C\|\Phi_1\|_{L^2(16Q)}.$$

By Lemmas 4.5 and 4.8, if $p^-_{1,L^*}<p\leq 2$, then

$$\|\widetilde{N}_*(\nabla^{m-1}u)\|_{L^{p'}(\mathbb{R}^n)}\leq C(1,L^*,p')\|\Phi_1\|_{L^{p'}(\mathbb{R}^n)},$$
$$\|\widetilde{N}_*(\nabla^{m-1}u_Q)\|_{L^2(10Q)}\leq C\|\Phi_1\|_{L^2(16Q)}.$$

By [15, Lemma 6.2], the conclusion is valid. □

For the sake of completeness, we establish a few final bounds on the Newton potential; for ease of reference, we list all our nontangential and area integral bounds on the Newton potential in the following corollary.

**Corollary 4.1.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $\dot{\boldsymbol{\Psi}}\in L^2(\mathbb{R}^{n+1})$ and $\dot{\boldsymbol{H}}\in L^2(\mathbb{R}^{n+1})$ be supported in compact subsets of $\mathbb{R}^{n+1}_+\cup\mathbb{R}^{n+1}_-$. Let $j$ be an integer with $0\leq j\leq m$. Let $1/p+1/p'=1$. If $p$ lies in the given ranges, then*

$$\|\widetilde{N}_*(\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}\leq\widetilde{C}_p\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)},\quad p^-_{1,L^*}<p<p^+_{0,L},\tag{4.16}$$
$$\|\mathcal{A}_2^*(t\nabla^m\Pi^{L^*}\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}\leq\widetilde{C}_p\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)},\quad p^-_{1,L^*}<p<p^+_{0,L},\tag{4.17}$$
$$\|\widetilde{N}_*(\nabla^{m-j}\partial_t^j\Pi^{L^*}\dot{\boldsymbol{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}\leq\widetilde{C}_p\|\mathcal{A}_2^*\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)},\quad p^-_{j,L^*}<p<p^+_{1,L},\tag{4.18}$$
$$\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}))\|_{L^{p'}(\mathbb{R}^n)}\leq\widetilde{C}_p\|\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)},\quad p^-_{1,L^*}<p<p_{1,L},\tag{4.19}$$

*where $\widetilde{C}_p$ depends only on the standard parameters, $p$, and the constants $c(k,L,p,2)$ (if $p>2$) or $c(k,L^*,p',2)$ (if $p<2$) in the bound (1.13), for appropriate values of $k$.*

*Proof.* The bounds (4.16) and (4.17) were established in Lemmas 4.7 and 4.8 and in Lemmas 4.6 and 4.10, respectively.

The $p \leq 2$ case of the bound (4.18) was established in Lemma 4.5. To establish the $p > 2$ case, we may take $j = 0$. The bound then follows from Lemma 3.6, formula (4.6), and the bounds (4.11) and (4.17), as in the proof of Lemma 4.6.

The $p \leq 2$ case of the bound (4.19) was established in Lemmas 4.9, 4.10. As in the proof of the bound (4.18), we will establish the $p > 2$ case by duality. By formulas (4.6) and (4.7), if $\dot{\boldsymbol{G}}$ and $\dot{\boldsymbol{H}}$ are in $L^2(\mathbb{R}^{n+1}) \cap \dot{W}^{1,2}(\mathbb{R}^{n+1})$, then

$$\langle \partial_{n+1} \nabla^m \Pi^{L^*} \dot{\boldsymbol{G}}, \dot{\boldsymbol{H}} \rangle_{\mathbb{R}^{n+1}} = -\langle \dot{\boldsymbol{G}}, \partial_{n+1} \nabla^m \Pi^L \dot{\boldsymbol{H}} \rangle_{\mathbb{R}^{n+1}}. \tag{4.20}$$

If $\dot{\boldsymbol{G}}$ is supported in $J$ and $\dot{\boldsymbol{H}}$ is supported in $K$, where $J$ and $K$ are disjoint compact sets, then by the Caccioppoli inequality, both the left-hand and right-hand sides are at most $C_{J,K} \|\dot{\boldsymbol{G}}\|_{L^2(J)} \|\dot{\boldsymbol{H}}\|_{L^2(K)}$; thus, by density, formula (4.20) is valid whenever $\dot{\boldsymbol{G}} \in L^2(\mathbb{R}^{n+1})$ and $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$ have disjoint compact support.

We may now see that the $p > 2$ case of the bound (4.19) follows from the bound (4.11), formula (4.20), and the $p < 2$ case of the bound (4.19) (that is, Lemma 4.10). ☐

*Remark* 4.2. The nontangential bounds (4.16) and (4.18) and the area integral estimate (4.17) involve the two-sided operators $\widetilde{N}_*$ and $\mathcal{A}_2^*$, while the bound (4.19) involves one-sided operators $\mathcal{A}_2^+$ and $\mathcal{A}_2^-$.

This restriction cannot be removed. Let $F$ be a function that is smooth and supported in the Whitney ball $B((0,1),1/4)$. Let $\dot{\boldsymbol{\Psi}} = \boldsymbol{A} \nabla^m F$. It follows from the definition of $\Pi^L$ in Subection 2.4 that $F = \Pi^L(\boldsymbol{A}\nabla^m F) = \Pi^L \dot{\boldsymbol{\Psi}}$. Thus,

$$\|\mathcal{A}_2^+(t \nabla^m \partial_t \Pi^L \dot{\boldsymbol{\Psi}})\|_{L^p(\mathbb{R}^n)} = \|\mathcal{A}_2^+(t \nabla^m \partial_t F)\|_{L^p(\mathbb{R}^n)}.$$

By the ellipticity condition (1.11) and the definition (1.6) of $\mathcal{A}_2^+$, if $0 < p < \infty$, then

$$\|\mathcal{A}_2^+(\nabla^m F)\|_{L^p(\mathbb{R}^n)} \approx \|\nabla^m F\|_{L^2(B((0,1),1/4))} \approx \|\mathcal{A}_2^+ \dot{\boldsymbol{\Psi}}\|_{L^p(\mathbb{R}^n)},$$

where the constants of approximation depend on $p$. Thus, $\|\mathcal{A}_2^+(\nabla^m F)\|_{L^p(\mathbb{R}^n)} \leq C_p \|\mathcal{A}_2^+ \dot{\boldsymbol{\Psi}}\|_{L^p(\mathbb{R}^n)}$.

But for any fixed number $\widetilde{C}$, we may choose $F$ so that

$$\|\mathcal{A}_2^+(t \nabla^m \partial_t F)\|_{L^p(\mathbb{R}^n)} \not\leq \frac{\widetilde{C}}{C_p} \|\mathcal{A}_2^+(\nabla^m F)\|_{L^p(\mathbb{R}^n)}$$

and so

$$\|\mathcal{A}_2^+(t \nabla^m \partial_t \Pi^L \dot{\boldsymbol{\Psi}})\|_{L^p(\mathbb{R}^n)} \not\leq \widetilde{C} \|\mathcal{A}_2^+ \dot{\boldsymbol{\Psi}}\|_{L^p(\mathbb{R}^n)}.$$

Thus, no two-sided analogue to the bound (4.19) is possible.

# 5 The double and single layer potentials

In this section, we will prove Theorem 1.2.

We will establish estimates on the double and single layer potentials using the duality results of Lemma 4.1 and the bounds on the Newton potential of Corollary 4.1. Recall that Lemma 4.1 involves the Dirichlet and Neumann boundary values of the Newton potential along $\mathbb{R}^n = \partial \mathbb{R}^{n+1}_\pm$, while Corollary 4.1 yields the nontangential and area integral bounds, that is, the bounds in the interior of $\mathbb{R}^{n+1}_\pm$. Thus, we will need Fatou type theorems to pass from Corollary 4.1 to the useful estimates on boundary values.

We will list three Fatou type theorems from [15, 23] in Subsection 5.1. These theorems suffice to prove the bounds (1.26)–(1.31); the arguments will be given in Subsection 5.2. The bounds (1.28) and (1.30) allow us to eliminate a technical assumption in certain results of [23]; these simplified theorems will be stated in Subsection 5.3, after the bounds (1.28) and (1.30) have been established, and will be used in Subsection 5.4 to establish the bounds (1.32) and (1.33).

## 5.1 Fatou type theorems

In this section, we list some known results concerning the boundary values of functions that satisfy nontangential or area integral estimates.

We begin with the following theorem concerning the Dirichlet boundary values.

**Lemma 5.1** ([15, Lemma 5.1]). *Let $\dot{\boldsymbol{u}}$ be defined and locally square integrable in $\mathbb{R}^{n+1}_+$. Suppose that $\widetilde{N}_+\dot{\boldsymbol{u}} \in L^p(\mathbb{R}^n)$ for some $p$ with $1 < p \leq \infty$. Suppose also that $\mathrm{Tr}^+ \dot{\boldsymbol{u}}$ exists in the sense of formula (2.4); that is, there is an array of functions $\mathrm{Tr}^+ \dot{\boldsymbol{u}}$ such that*

$$\lim_{t\to 0^+} \int_K \left|\dot{\boldsymbol{u}}(x,t) - \mathrm{Tr}^+ \dot{\boldsymbol{u}}(x)\right| dx = 0$$

*for any compact set $K \subset \mathbb{R}^n$. Then $\mathrm{Tr}^+ \dot{\boldsymbol{u}}$ satisfies*

$$\|\mathrm{Tr}^+ \dot{\boldsymbol{u}}\|_{L^p(\mathbb{R}^n)} \leq \|\widetilde{N}_+\dot{\boldsymbol{u}}\|_{L^p(\mathbb{R}^n)}.$$

We will also need the following Fatou type theorems for the Neumann boundary values. Note that in [23], these theorems are stated for the solutions $v$, $w$ to $Lv = Lw = 0$ in $\mathbb{R}^{n+1}_+$ with $\mathcal{A}_2^+(t\nabla^m v)$, $\mathcal{A}_2^+(t\nabla^m \partial_t w) \in L^p(\mathbb{R}^n)$. We will usually apply these theorems to the solutions $v$, $w$ to $L^*v = L^*w = 0$ in $\mathbb{R}^{n+1}_-$ with $\mathcal{A}_2^-(t\nabla^m v)$, $\mathcal{A}_2^-(t\nabla^m \partial_t w) \in L^{p'}(\mathbb{R}^n)$; we have modified the theorem statements accordingly.

**Theorem 5.1** ([23, Theorem 6.1]). *Let $L$ be an operator of order $2m$ of the form (2.7) associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Let $v$ satisfy $\mathcal{A}_2^-(t\nabla^m v) \in L^{p'}(\mathbb{R}^n)$ and $L^*v = 0$ in $\mathbb{R}^{n+1}_-$. If $p < 2$, suppose further that $v \in \dot{W}^{m,2}(\mathbb{R}^n \times (-\infty, -\sigma))$ for all $\sigma > 0$, albeit possibly with norms that approach $\infty$ as $\sigma \to 0^+$.*

*Then for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$, we have*

$$\left|\langle \dot{\mathbf{Tr}}_{m-1}\, \varphi, \dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\, v\rangle_{\mathbb{R}^n}\right| \leq C_p \|\dot{\mathbf{Tr}}_{m-1}\,\varphi\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)} \|\mathcal{A}_2^-(t\nabla^m v)\|_{L^{p'}(\mathbb{R}^n)},$$

*where $\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\, v$ is as in [23, Section 2.3.2]. In particular, if $v \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_-)$, then, by [23, Lemma 2.4], $\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\, v$ is as in formula (2.8).*

The theorem as stated in [23] requires that $v \in \dot{W}^{m,2}(\mathbb{R}^n \times (-\infty, -\sigma))$ for all $p$; however, if $p \geq 2$, then this condition follows from Lemma 3.3 or its predecessor [23, Remark 5.3].

**Theorem 5.2** ([23, Theorem 6.2]). *Let $L$ be an operator of order $2m$ of the form (2.7) associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Let $w$ satisfy $\mathcal{A}_2^-(t\nabla^m \partial_t w) \in L^{p'}(\mathbb{R}^n)$, $\widetilde{N}_-(\nabla^m w) \in L^{p'}(\mathbb{R}^n)$, and $L^*w = 0$ in $\mathbb{R}^{n+1}_-$. If $p < 2$, we impose the additional condition $\partial_{n+1} w \in \dot{W}^{m,2}(\mathbb{R}^n \times (-\infty, -\sigma))$ for all $\sigma > 0$.*

*Then for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$, we have*

$$\left|\langle \dot{\mathbf{Tr}}_{m-1}\, \varphi, \dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\, w\rangle_{\mathbb{R}^n}\right| \leq C_p \|\dot{\mathbf{Tr}}_{m-1}\,\varphi\|_{\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)} \big(\|\mathcal{A}_2^-(t\nabla^m \partial_t w)\|_{L^{p'}(\mathbb{R}^n)} + \|\widetilde{N}_-(\nabla^m w)\|_{L^{p'}(\mathbb{R}^n)}\big)$$

*where $\dot{\mathbf{M}}_{\boldsymbol{A}^*}^-\, w$ is as in formula (1.7).*

## 5.2 The bounds (1.26)–(1.31)

In this section, we will prove most of Theorem 1.2; specifically, we will establish the estimates (1.26)–(1.31). Throughout this section, we will let $L$ and $\boldsymbol{A}$ be as in Theorem 1.2; that is, $L$ is an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).

**The estimate (1.26)**

By Lemma 3.6, if $1 < p < \infty$, then

$$\|\widetilde{N}_*(\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{H}}} \frac{|\langle \dot{\boldsymbol{H}}, \nabla^m \mathcal{S}^L \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^{n+1}}|}{\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}},$$

where the supremum is taken over all $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$ supported in a compact subset of $\mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-$ such that the denominator is positive. By formula (4.1), if $\dot{\boldsymbol{g}} \in \dot{B}^{-1/2,2}_2(\mathbb{R}^n)$, then

$$\|\widetilde{N}_*(\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{H}}} \frac{|\langle \dot{\mathbf{Tr}}_{m-1} \Pi^{L^*} \dot{\boldsymbol{H}}, \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^n}|}{\|\widetilde{\mathfrak{C}}_1^*(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}}.$$

Since $\Pi^{L^*}\dot{\boldsymbol{H}} \in \dot{W}^{m,2}(\mathbb{R}^{n+1})$, we find that $\dot{\mathbf{Tr}}_{m-1} \Pi^{L^*}\dot{\boldsymbol{H}}$ exists in the sense of Sobolev spaces, and thus in the sense of formulas (2.4) and (2.5). By Lemma 5.1,

$$\|\dot{\mathbf{Tr}}_{m-1} \Pi^{L^*}\dot{\boldsymbol{H}}\|_{L^{p'}(\mathbb{R}^n)} \leq \|\widetilde{N}_*(\nabla^{m-1}\Pi^{L^*}\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)},$$

and so, by Lemmas 4.7 and 4.8, if $p^-_{1,L^*} < p < p^+_{0,L}$ and $\dot{\boldsymbol{g}} \in \dot{B}^{-1/2,2}_2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then

$$\|\widetilde{N}_*(\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq \widetilde{C}_p \|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)},$$

where $\widetilde{C}_p$ is as in Corollary 4.1. By density, the bound (1.26) is valid.

**The estimate (1.27)**

By Lemma 3.6 and formula (4.2), if $\dot{\boldsymbol{\varphi}} \in \dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$, then

$$\|\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{H}}} \frac{|\langle \dot{\boldsymbol{H}}, \nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}} = C_p \sup_{\dot{\boldsymbol{H}}} \frac{|\langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*} \Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{H}}), \dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n}|}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}}.$$

By Theorem 5.1, if $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m-1} \Phi$ for some $\Phi \in C^\infty_0(\mathbb{R}^{n+1})$, then

$$\|\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{H}}} \frac{\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)} \|\mathcal{A}_2^-(t\nabla^m \Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{H}}))\|_{L^{p'}(\mathbb{R}^n)}}{\|\widetilde{\mathfrak{C}}_1^+(t\dot{\boldsymbol{H}})\|_{L^{p'}(\mathbb{R}^n)}}.$$

By Lemmas 4.6 and 4.10, if $p^-_{1,L^*} < p < p^+_{0,L}$, then

$$\|\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq \widetilde{C}_p \|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)}.$$

We establish a bound on $\widetilde{N}_-(\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})$ using Subsection 3.1 and extend to all $\dot{\boldsymbol{\varphi}} \in \dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)$ by density. This completes the proof of the bound (1.27).

**The estimate (1.28)**

By the bound (4.11) and formula (4.4), if $1 < p < \infty$ and $\dot{\boldsymbol{g}} \in \dot{B}^{-1/2,2}_2(\mathbb{R}^n)$, then

$$\|\mathcal{A}_2^*(t\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{\Psi}}} \frac{|\langle \dot{\boldsymbol{\Psi}}, \nabla^m \partial_{n+1} \mathcal{S}^L \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^{n+1}}|}{\|\mathcal{A}_2^* \dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}} = C_p \sup_{\dot{\boldsymbol{\Psi}}} \frac{|\langle \dot{\mathbf{Tr}}_{m-1} \partial_{n+1} \Pi^{L^*}\dot{\boldsymbol{\Psi}}, \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^n}|}{\|\mathcal{A}_2^* \dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}.$$

We may take $\dot{\boldsymbol{\Psi}}$ to be supported away from $\partial\mathbb{R}^{n+1}_\pm$. By Lemma 3.2, $\dot{\mathbf{Tr}}_m \Pi^{L^*}\dot{\boldsymbol{\Psi}}$ exists in the sense of formula (2.5), and so, by Lemma 5.1,

$$\|\mathcal{A}_2^*(t\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{\Psi}}} \frac{\|\widetilde{N}_-(\nabla^{m-1}\partial_{n+1}\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}))\|_{L^{p'}(\mathbb{R}^n)} \|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}}{\|\mathcal{A}_2^+ \dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}.$$

By the bound (4.18) with $j=1$, if $p^-_{1,L^*}<p<p^+_{1,L}$ and $\dot{\boldsymbol{g}}\in\dot B^{-1/2,2}_2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, we have

$$\|\mathcal{A}_2^*(t\nabla^m\partial_t\mathcal{S}^L\dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)}\leq\widetilde{C}_p\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)},\quad p^-_{1,L^*}<p<p^+_{1,L}. \tag{5.1}$$

By density, the bound (1.28) is valid.

**The estimate (1.29)**

By the bound (4.11) and formula (4.5), if $\dot{\boldsymbol{\varphi}}\in\dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$, then

$$\big\|\mathcal{A}_2^+(t\nabla^m\partial_t\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}})\big\|_{L^p(\mathbb{R}^n)}\leq C_p\sup_{\dot{\boldsymbol{\Psi}}}\frac{|\langle\dot{\boldsymbol{\Psi}},\nabla^m\partial_t\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}=C_p\sup_{\dot{\boldsymbol{\Psi}}}\frac{|\langle\dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\partial_{n+1}\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}),\dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n}|}{\|\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}.$$

By Theorem 5.1 and the bound (4.19), if $p^-_{1,L^*}<p<p^+_{1,L}$ and $\dot{\boldsymbol{\varphi}}=\dot{\mathbf{Tr}}^+_{m-1}\,\Phi$ for some $\Phi\in C^\infty_0(\mathbb{R}^{n+1})$, it follows that

$$\begin{aligned}\big\|\mathcal{A}_2^+(t\nabla^m\partial_t\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}})\big\|_{L^p(\mathbb{R}^n)}&\leq C_p\sup_{\dot{\boldsymbol{\Psi}}}\frac{\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}))\|_{L^{p'}(\mathbb{R}^n)}\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)}}{\|\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}\\&\leq C_p\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)}.\end{aligned}$$

As before, we may use density arguments and Subsection 3.1 to complete the proof of the bound (1.29).

**The estimate (1.30)**

By the bound (4.11), formula (4.3), and Lemma 5.1, if $1<p<\infty$ and $\dot{\boldsymbol{h}}\in L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, then

$$\begin{aligned}\|\mathcal{A}_2^*(t\nabla^m\mathcal{S}^L_\nabla\dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)}&\leq C_p\sup_{\dot{\boldsymbol{\Psi}}}\frac{|\langle\dot{\boldsymbol{\Psi}},\nabla^m\mathcal{S}^L_\nabla\dot{\boldsymbol{h}}\rangle_{\mathbb{R}^{n+1}}|}{\|\mathcal{A}_2^*\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}=C_p\sup_{\dot{\boldsymbol{\Psi}}}\frac{\langle\dot{\mathbf{Tr}}_m\,\Pi^{L^*}\dot{\boldsymbol{\Psi}},\dot{\boldsymbol{h}}\rangle_{\mathbb{R}^n}}{\|\mathcal{A}_2^*\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}\\&\leq C_p\sup_{\dot{\boldsymbol{\Psi}}}\frac{\|\widetilde{N}_*(\nabla^m\Pi^{L^*}\dot{\boldsymbol{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}}{\|\mathcal{A}_2^*\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}.\end{aligned}$$

By density and the bound (4.18) with $j=0$, if $p^-_{0,L^*}<p<p^+_{1,L}$, then

$$\|\mathcal{A}_2^*(t\nabla^m\mathcal{S}^L_\nabla\dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)}\leq\widetilde{C}_p\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)},\quad p^-_{0,L^*}<p<p^+_{1,L}. \tag{5.2}$$

Thus, the bound (1.30) is valid.

**The estimate (1.31)**

By the bound (4.11) and formula (4.2), if $1<p<\infty$ and $\dot{\boldsymbol{f}}\in\dot{W}A^{1/2,2}_{m-1}(\mathbb{R}^n)$, then

$$\|\mathcal{A}_2^+(t\nabla^m\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)}\approx\sup_{\dot{\boldsymbol{\Psi}}}\frac{|\langle\dot{\boldsymbol{\Psi}},\nabla^m\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}=\sup_{\dot{\boldsymbol{\Psi}}}\frac{|\langle\dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}),\dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}|}{\|\mathcal{A}_2^+\dot{\boldsymbol{\Psi}}\|_{L^{p'}(\mathbb{R}^n)}}.$$

By Theorem 5.2, if $\dot{\boldsymbol{f}}=\dot{\mathbf{Tr}}_{m-1}\,F$ for some $F\in C^\infty_0(\mathbb{R}^{n+1})$, we get

$$\big|\langle\dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}),\dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}\big|\leq C_p\big\|\widetilde{N}_-(\nabla^m\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}))+\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\boldsymbol{\Psi}}))\big\|_{L^{p'}(\mathbb{R}^n)}\|\dot{\boldsymbol{f}}\|_{\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)}$$

provided the right-hand side is finite. Thus, by the bounds (4.18) and (4.19), if $p^-_{0,L^*}<p<p^+_{1,L}$, then

$$\|\mathcal{A}_2^+(t\nabla^m\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)}\leq C\|\dot{\boldsymbol{f}}\|_{\dot{W}A^{0,p}_{m-1}(\mathbb{R}^n)}.$$

By density and due to Subsection 3.1, we conclude that the bound (1.31) is valid.

### 5.3 Further Fatou type theorems

In order to establish the bounds (1.32) and (1.33), we will need further Fatou type theorems.

The Fatou theorems [23, Theorems 5.1 and 5.2] contain a technical assumption involving the single layer potential. As observed in [23, Remark 5.3], this technical assumption is true if $p \geq 2$; given that the bounds (1.28) and (1.30) are established (see the bounds (5.1) and (5.2) above), we find that this technical assumption is true for a wider range of $p$. Thus, we will now restate the parts of [23, Theorems 5.1, 5.2 and 6.2] necessary for the proofs of the bounds (1.32) and (1.33). As in Subsection 5.1, we have interchanged the roles of $L$ and $L^*$, $p$ and $p'$, and $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$ relative to their roles in [23].

In [23], $p_j^+$ is defined as $p_j^+ = \min(p_{j,L}^+, p_{j,L^*}^+)$; however, a careful examination of the proofs in [23] yields that the results are valid for $p_{j,L}^\pm$ and $p_{j,L^*}^\pm$, as indicated below.

**Theorem 5.3** ([23, Theorem 5.1])**.** *Let $L$ be an operator of order $2m$ of the form (2.7) associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $p_{1,L^*}^- < p < \infty$ and $1/p + 1/p' = 1$. Let $v$ satisfy $\mathcal{A}_2^-(t\nabla^m v) \in L^{p'}(\mathbb{R}^n)$ and $L^* v = 0$ in $\mathbb{R}^{n+1}_-$. If $p < 2$, suppose further that $v \in \dot W^{m,2}(\mathbb{R}^n \times (-\infty, -\sigma))$ for all $\sigma > 0$, albeit possibly with norms that approach $\infty$ as $\sigma \to 0^+$.*

*Then $\dot{\mathbf{Tr}}_{m-1}^- v$ exists in the sense of formula (2.4), and there is some constant array $\dot{\boldsymbol{c}}$ such that*

$$\|\dot{\mathbf{Tr}}_{m-1}^- v - \dot{\boldsymbol{c}}\|_{L^{p'}(\mathbb{R}^n)} \leq C(1, L^*, p') \|\mathcal{A}_2^-(t\nabla^m v)\|_{L^{p'}(\mathbb{R}^n)}.$$

**Theorem 5.4** ([23, Theorems 5.2 and 6.2])**.** *Let $L$ be an operator of order $2m$ of the form (2.7) associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $p_{0,L^*}^- < p < \infty$ and $1/p + 1/p' = 1$. Let $w \in \dot W^{m,2}_{loc}(\mathbb{R}^{n+1}_-)$ satisfy $L^* w = 0$ in $\mathbb{R}^{n+1}_-$ and $\mathcal{A}_2^-(t\nabla^m \partial_t w) \in L^{p'}(\mathbb{R}^n)$. If $p < 2$, we impose the additional condition $\partial_{n+1} w \in \dot W^{m,2}(\mathbb{R}^n \times (-\infty, \sigma))$ for all $\sigma > 0$.*

*If there is some $t < 0$ such that $\nabla^m w(\,\cdot\,, t) \in L^{p'}(\mathbb{R}^n)$, then $\dot{\mathbf{Tr}}_m^- w$ exists in the sense of formula (2.4) and satisfies*

$$\|\dot{\mathbf{Tr}}_m^- w\|_{L^{p'}(\mathbb{R}^n)} \leq C(0, L^*, p') \|\mathcal{A}_2^-(t\nabla^m \partial_t w)\|_{L^{p'}(\mathbb{R}^n)}.$$

*We also have the uniform bound*

$$\sup_{t>0} \|\nabla^m w(\,\cdot\,, t)\|_{L^{p'}(\mathbb{R}^n)} \leq C(0, L^*, p') \|\mathcal{A}_2^-(t\nabla^m \partial_t w)\|_{L^{p'}(\mathbb{R}^n)}$$

*and the limits*

$$\lim_{t\to\infty} \|\nabla^m w(\,\cdot\,, t)\|_{L^{p'}(\mathbb{R}^n)} = \lim_{t\to 0^+} \|\nabla^m w(\,\cdot\,, t) - \dot{\mathbf{Tr}}_m^- w\|_{L^{p'}(\mathbb{R}^n)} = 0.$$

*Finally, we have that $\dot{\mathbf{M}}_{\boldsymbol{A}^*}^- w$ exists in the sense of formula (1.7) and*

$$\big|\langle \dot{\mathbf{Tr}}_{m-1} \varphi, \dot{\mathbf{M}}_{\boldsymbol{A}^*}^- w\rangle_{\mathbb{R}^n}\big| \leq C(0, L^*, p') \|\dot{\mathbf{Tr}}_{m-1} \varphi\|_{\dot W A^{0,p}_{m-1}(\mathbb{R}^n)} \|\mathcal{A}_2^-(t\nabla^m \partial_t w)\|_{L^{p'}(\mathbb{R}^n)}$$

*for every $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$.*

### 5.4 The bounds (1.32) and (1.33)

In this section, we will complete the proof of Theorem 1.2 by establishing the bounds (1.32) and (1.33). As in Subsection 5.2, throughout this section, we will let $L$ and $\boldsymbol{A}$ be as in Theorem 1.2.

We begin with the bound (1.32). Let $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p$ with $p_{0,L^*}^- < p < 2$. By the bound (1.22) with $p = 2$, we may apply Lemma 3.8 with $u = \mathcal{S}^L_\nabla \dot{\boldsymbol{h}}$; by Lemma 3.8 and formula (4.3),

$$\|\widetilde N_+(\nabla^{m-1} \mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\boldsymbol{\Psi}}} \frac{|\langle \dot{\boldsymbol{\Psi}}, \nabla^m \mathcal{S}^L_\nabla \dot{\boldsymbol{h}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t \dot{\boldsymbol{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}} = C_p \sup_{\dot{\boldsymbol{\Psi}}} \frac{|\langle \dot{\mathbf{Tr}}_m^- \Pi^{L^*}(\mathbf{1}_+ \dot{\boldsymbol{\Psi}}), \dot{\boldsymbol{h}}\rangle_{\mathbb{R}^n}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t \dot{\boldsymbol{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}},$$

where the supremum is taken over all $\dot{\mathbf{\Psi}} \in L^2(\mathbb{R}^{n+1}_+)$ that are supported in a compact subset of $\mathbb{R}^{n+1}_+$ and have a weak vertical derivative in $L^2(\mathbb{R}^{n+1}_+)$.

By the definition of the Newton potential and the Caccioppoli inequality, we have

$$\partial_{n+1}\Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}}) \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_-).$$

By Lemma 3.2 and the bound (4.18), we have that $\nabla^m\Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}})(\,\cdot\,,t) \in L^{p'}(\mathbb{R}^n)$ for any (hence some) $t<0$. Thus, we can apply Theorem 5.4 with $w = \Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}})$ and see that

$$\big|\langle \dot{\mathbf{Tr}}_m^- \,\Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}}), \dot{\boldsymbol{h}}\rangle_{\mathbb{R}^n}\big| \leq C(0,L^*,p')\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}}))\|_{L^{p'}(\mathbb{R}^n)}\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}\,.$$

By formula (4.7) and the bound (4.17),

$$\big\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}}))\big\|_{L^{p'}(\mathbb{R}^n)} = \big\|\mathcal{A}_2^-(t\nabla^m\Pi^{L^*}(\mathbf{1}_+\partial_t\dot{\mathbf{\Psi}}))\big\|_{L^{p'}(\mathbb{R}^n)} \leq C(1,L^*,p')\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\mathbf{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}.$$

Thus, if $p^-_{0,L^*} < p < 2$, then

$$\|\widetilde{N}_+(\nabla^{m-1}\mathcal{S}^L_\nabla\dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C(0,L^*,p')\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}.$$

By density and Subsection 3.1, the bound (1.32) is valid.

Similarly, let $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}\,F$ for some $F \in C_0^\infty(\mathbb{R}^{n+1})$. By the bound (1.23) with $p=2$, Lemma 3.8, and formula (4.2), if $1<p<2$, then

$$\|\widetilde{N}_+(\nabla^{m-1}\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C_p \sup_{\dot{\mathbf{\Psi}}} \frac{|\langle\dot{\mathbf{\Psi}}, \nabla^m\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}\rangle_{\mathbb{R}^{n+1}_+}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\mathbf{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}} = C_p \sup_{\dot{\mathbf{\Psi}}} \frac{|\langle\dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}}), \dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}|}{\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\mathbf{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}}\,.$$

By Theorem 5.4, formula (4.7) and the bound (4.17), if $p^-_{0,L^*} < p < 2$, then

$$\begin{aligned}\big|\langle\dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}}), \dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}\big| &\leq C(0,L^*,p')\big\|\mathcal{A}_2^-(t\nabla^m\partial_t\Pi^{L^*}(\mathbf{1}_+\dot{\mathbf{\Psi}}))\big\|_{L^{p'}(\mathbb{R}^n)}\|\dot{\boldsymbol{f}}\|_{L^p(\mathbb{R}^n)}\\ &\leq C(0,L^*,p')\|\widetilde{\mathfrak{C}}_1^+(t\,\partial_t\dot{\mathbf{\Psi}})\|_{L^{p'}(\mathbb{R}^n)}\|\dot{\boldsymbol{f}}\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

By density and due to Subsection 3.1, the bound (1.33) is valid. This completes the proof of Theorem 1.2.

# 6 The Green formula

A useful tool in the theory of higher order equations, and one of the reasons layer potentials are of interest, is the Green formula

$$\mathbf{1}_+\nabla^m u = -\nabla^m\mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}^+_{m-1}\,u) + \nabla^m\mathcal{S}^L(\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,u). \tag{6.1}$$

This formula is valid for all $u \in \dot{W}^{m,2}(\mathbb{R}^{n+1}_+)$ that satisfy $Lu=0$ in $\mathbb{R}^{n+1}_+$. See [14, Lemma 5.2] or [20, formula (2.26)]. It is also valid if $Lu=0$ in $\mathbb{R}^{n+1}_+$, $\mathcal{A}_2^+(t\nabla^m\partial_t u) \in L^2(\mathbb{R}^n)$ and $\nabla^m u(\,\cdot\,,t) \in L^2(\mathbb{R}^n)$ for some $t>0$; see [21, Theorem 4.3]. This Green's formula was used in [21] to establish the uniqueness of solutions to the $L^2$ Neumann problem (1.4); the corresponding formula in the lower half-space was used to prove Lemma 4.9 above.

In this section, we will show that the Green formula is still valid if $Lu=0$ in $\mathbb{R}^{n+1}_+$, $\mathcal{A}_2^+(t\nabla^m\partial_t u) \in L^p(\mathbb{R}^n)$ and $\sup_{t>0}\|\nabla^m u(\,\cdot\,,t)\|_{L^p(\mathbb{R}^n)} < \infty$ for some $p$ with $p^-_{1,L^*} < p \leq 2$. The Green formula for such solutions will be used in Section 7 to establish the uniqueness of solutions to the Neumann problem (1.9).

We begin with some useful auxiliary lemmas. Specifically, recall from Theorem 5.4 that $\nabla^m w(\,\cdot\,,t) \to \dot{\mathbf{Tr}}_m\,w$ as $t\to 0^+$ and $\nabla^m w(\,\cdot\,,t)\to 0$ as $t\to\infty$. We wish to prove a similar result for the Neumann boundary values. Our argument will follow the proof of [21, Lemma 4.2].

**Lemma 6.1.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $p$ satisfy $0 < p \leq 2$ and let $j$ be an integer with $0 \leq j \leq m$. Let $u \in \dot{W}^{m,2}_{loc}(\mathbb{R}^{n+1}_+)$ be such that $Lu = 0$ in $\mathbb{R}^{n+1}_+$ and $\mathcal{A}_2^+(t\nabla^j u) \in L^p(\mathbb{R}^n)$. Define $u_\varepsilon(x,t) = u(x,t+\varepsilon)$. If $\varepsilon > 0$, then*

$$\|\mathcal{A}_2^+(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)} \leq C\|\mathcal{A}_2^+(t\nabla^j u)\|_{L^p(\mathbb{R}^n)}$$

*and*

$$\lim_{\varepsilon\to 0^+} \big\|\mathcal{A}_2^+(t\nabla^j(u-u_\varepsilon))\big\|_{L^p(\mathbb{R}^n)} = \lim_{T\to\infty} \|\mathcal{A}_2^+(t\nabla^j u_T)\|_{L^p(\mathbb{R}^n)} = 0.$$

*Proof.* We define

$$\mathcal{A}_f^\ell H(x) = \bigg(\int_\ell^\infty \int_{|x-y|<t} |H(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\bigg)^{1/2},$$

$$\mathcal{A}_n^\ell H(x) = \bigg(\int_0^\ell \int_{|x-y|<t} |H(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\bigg)^{1/2},$$

so that

$$\mathcal{A}_2^+ H(x)^2 = \mathcal{A}_f^\ell H(x)^2 + \mathcal{A}_n^\ell H(x)^2.$$

Let $c > 1$ be a constant to be chosen later. We start with analyzing $\mathcal{A}_n^{\varepsilon/c}(t\nabla^j u_\varepsilon)$. Let $\mathcal{G}$ be a grid of pairwise-disjoint open cubes in $\mathbb{R}^n$ of side length $\varepsilon/c$ whose union is almost all of $\mathbb{R}^n$. Then

$$\|\mathcal{A}_n^{\varepsilon/c}(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)}^p = \sum_{Q\in\mathcal{G}} \int_Q \mathcal{A}_n^{\varepsilon/c}(t\nabla^j u_\varepsilon)(x)^p\,dx.$$

By Hölder's inequality,

$$\|\mathcal{A}_n^{\varepsilon/c}(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)}^p \leq \sum_{Q\in\mathcal{G}} |Q|^{1-p/2} \bigg(\int_Q \mathcal{A}_n^{\varepsilon/c}(t\nabla^j u_\varepsilon)(x)^2\,dx\bigg)^{p/2}.$$

By definition of $u_\varepsilon$ and $\mathcal{A}_n^\ell$,

$$\|\mathcal{A}_n^{\varepsilon/c}(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)}^p \leq \sum_{Q\in\mathcal{G}} |Q|^{1-p/2} \bigg(\int_Q \int_0^{\varepsilon/c} \int_{|x-y|<t} |\nabla^j u(y,t+\varepsilon)|^2 \frac{dy\,dt}{t^{n-1}}\,dx\bigg)^{p/2}.$$

Changing the order of integration and evaluating the integral $dx$, we have

$$\|\mathcal{A}_n^{\varepsilon/c}(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)}^p \leq \alpha_n^{p/2} \sum_{Q\in\mathcal{G}} |Q|^{1-p/2} \bigg(\int_0^{\varepsilon/c} \int_{3Q} t|\nabla^j u(y,t+\varepsilon)|^2\,dy\,dt\bigg)^{p/2},$$

where $\alpha_n$ is the area of the unit disk in $\mathbb{R}^n$.

Making a change of variables, we see that

$$\|\mathcal{A}_n^{\varepsilon/c}(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)}^p \leq \alpha_n^{p/2} \sum_{Q\in\mathcal{G}} |Q|^{1-p/2} \bigg(\int_\varepsilon^{\varepsilon+\varepsilon/c} \int_{3Q} (t-\varepsilon)|\nabla^j u(y,t)|^2\,dy\,dt\bigg)^{p/2}.$$

Let $c = 2\sqrt{n} = \sqrt{4n}$. If $x \in Q$, $y \in 3Q$, and $t \in (\varepsilon, \varepsilon+\varepsilon/c)$, then $|x-y| < 2\sqrt{n}\,\ell(Q) = \varepsilon < t$. Thus, if $x \in Q$, then

$$\bigg(\int_\varepsilon^{\varepsilon+\varepsilon/\sqrt{4n}} \int_{3Q} (t-\varepsilon)|\nabla^j u(y,t)|^2\,dy\,dt\bigg)^{p/2} \leq \bigg(\Big(\varepsilon + \frac{\varepsilon}{\sqrt{4n}}\Big)^n \int_\varepsilon^{\varepsilon+\varepsilon/\sqrt{4n}} \int_{|x-y|<t} |t\nabla^j u(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\bigg)^{p/2}.$$

The right-hand side is at most

$$(C_n|Q|)^{p/2}\min\big(\mathcal{A}_n^{\varepsilon+\varepsilon/\sqrt{4n}}(t\nabla^j u)(x),\mathcal{A}_f^{\varepsilon}(t\nabla^j u)(x)\big)^p.$$

For ease of notation, we replace $\varepsilon+\varepsilon/\sqrt{4n}$ by $2\varepsilon$. Thus,

$$\|\mathcal{A}_n^{\varepsilon/\sqrt{4n}}(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)}^p \leq C_n^{p/2}\sum_{Q\in\mathcal{G}}\int_Q \min\big(\mathcal{A}_n^{2\varepsilon}(t\nabla^j u)(x),\mathcal{A}_f^{\varepsilon}(t\nabla^j u)(x)\big)^p\,dx.$$

Summing over $Q$, we have

$$\|\mathcal{A}_n^{\varepsilon/\sqrt{4n}}(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)} \leq C_n\min\big(\|\mathcal{A}_n^{2\varepsilon}(t\nabla^j u)\|_{L^p(\mathbb{R}^n)},\|\mathcal{A}_f^{\varepsilon}(t\nabla^j u)\|_{L^p(\mathbb{R}^n)}\big). \tag{6.2}$$

We now turn to $\mathcal{A}_f^{\varepsilon/\sqrt{4n}}$. By definition of $u_\varepsilon$,

$$\begin{aligned}\mathcal{A}_f^{\varepsilon/\sqrt{4n}}(t\nabla^j(u-u_\varepsilon))(x) &= \bigg(\int_{\varepsilon/\sqrt{4n}}^\infty\int_{|x-y|<t}|\nabla^j(u(y,t)-u(y,t+\varepsilon))|^2\frac{dy\,dt}{t^{n-1}}\bigg)^{1/2}\\ &= \bigg(\int_{\varepsilon/\sqrt{4n}}^\infty\int_{|x-y|<t}\bigg|\int_t^{t+\varepsilon}\nabla^j\partial_s u(y,s)\,ds\bigg|^2\frac{dy\,dt}{t^{n-1}}\bigg)^{1/2}.\end{aligned}$$

Applying Hölder's inequality and changing the order of integration, we obtain

$$\begin{aligned}\mathcal{A}_f^{\varepsilon/\sqrt{4n}}(t\nabla^j(u-u_\varepsilon))(x) &\leq \bigg(\int_{\varepsilon/\sqrt{4n}}^\infty\int_{|x-y|<t}\varepsilon\int_t^{t+\varepsilon}|\nabla^j\partial_s u(y,s)|^2\,ds\,\frac{dy\,dt}{t^{n-1}}\bigg)^{1/2}\\ &\leq C_n\bigg(\varepsilon^2\int_{\varepsilon/\sqrt{4n}}^\infty\int_{|x-y|<s}|\nabla^j\partial_s u(y,s)|^2\frac{dy\,ds}{s^{n-1}}\bigg)^{1/2}.\end{aligned}$$

By the Caccioppoli inequality,

$$\mathcal{A}_f^{\varepsilon/\sqrt{4n}}(t\nabla^j(u-u_\varepsilon))(x) \leq C\bigg(\int_{\varepsilon/\sqrt{16n}}^\infty\frac{\varepsilon^2}{s^2}\int_{|x-y|<2s}|\nabla^j u(y,s)|^2\frac{dy\,ds}{s^{n-1}}\bigg)^{1/2}.$$

Now, define

$$\mathcal{A}_2^r H(x) = \bigg(\int_0^\infty\int_{|x-y|<rt}|H(y,t)|^2\frac{dy\,dt}{t^{n+1}}\bigg)^{1/2}$$

for any $r>0$, so that $\mathcal{A}_2^+H=\mathcal{A}_2^1 H$. It is well known (see [30, Proposition 4] or [26, Theorem 3.4]) that if $0<p<\infty$, then $\|\mathcal{A}_2^2 H\|_{L^p(\mathbb{R}^n)}\leq C_p\|\mathcal{A}_2^1 H\|_{L^p(\mathbb{R}^n)}$. Thus,

$$\|\mathcal{A}_f^{\varepsilon/\sqrt{4n}}(t\nabla^j(u-u_\varepsilon))\|_{L^p(\mathbb{R}^n)} \leq C_p\|\mathcal{A}_f^{\varepsilon/\sqrt{16n}}(\varepsilon\nabla^j u)\|_{L^p(\mathbb{R}^n)}. \tag{6.3}$$

The bound $\|\mathcal{A}_2^+(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)}\leq C_p\|\mathcal{A}_2^+(t\nabla^j u)\|_{L^p(\mathbb{R}^n)}$ follows from the bounds (6.2) and (6.3). We now use these bounds to bound $\mathcal{A}_2^+(t\nabla^j u_T)$ as $T\to\infty$ and $\mathcal{A}_2^+(t\nabla^j(u-u_\varepsilon))$ as $\varepsilon\to 0^+$.

First, by definition of $\mathcal{A}_n^\ell$ and $\mathcal{A}_f^\ell$,

$$\|\mathcal{A}_2^+(t\nabla^j u_T)\|_{L^p(\mathbb{R}^n)} \leq \|\mathcal{A}_n^{T/\sqrt{4n}}(t\nabla^j u_T)\|_{L^p(\mathbb{R}^n)} + \|\mathcal{A}_f^{T/\sqrt{4n}}(t\nabla^j u_T)\|_{L^p(\mathbb{R}^n)}.$$

Next, by the bounds (6.2) and (6.3),

$$\|\mathcal{A}_2^+(t\nabla^j u_T)\|_{L^p(\mathbb{R}^n)} \leq C_p \|\mathcal{A}_f^{T/\sqrt{16n}}(t\nabla^j u)\|_{L^p(\mathbb{R}^n)}.$$

If $\mathcal{A}_2^+(t\nabla^j u)(x) < \infty$, then $\mathcal{A}_f^{T/\sqrt{16n}}(t\nabla^j u)(x) \to 0$ as $T \to \infty$, and so, by the dominated convergence theorem, if $\mathcal{A}_2^+(t\nabla^j u) \in L^p(\mathbb{R}^n)$, then $\mathcal{A}_f^{T/\sqrt{16n}}(t\nabla^j u) \to 0$ in $L^p(\mathbb{R}^n)$ as $T \to \infty$. Thus, $\|\mathcal{A}_2^+(t\nabla^j u_T)\|_{L^p(\mathbb{R}^n)} \to 0$ as $T \to \infty$, as desired.

We now turn to $u - u_\varepsilon$. By definition of $\mathcal{A}_n^\ell$ and $\mathcal{A}_f^\ell$,

$$\begin{aligned}\|\mathcal{A}_2^+(t\nabla^j(u-u_\varepsilon))\|_{L^p(\mathbb{R}^n)}& \\ \leq \|\mathcal{A}_n^{\varepsilon/\sqrt{4n}}(t\nabla^j u)\|_{L^p(\mathbb{R}^n)} &+ \|\mathcal{A}_n^{\varepsilon/\sqrt{4n}}(t\nabla^j u_\varepsilon)\|_{L^p(\mathbb{R}^n)} + \|\mathcal{A}_f^{\varepsilon/\sqrt{4n}}(t\nabla^j(u-u_\varepsilon))\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

By the bounds (6.2) and (6.3),

$$\|\mathcal{A}_2^+(t\nabla^j(u-u_\varepsilon))\|_{L^p(\mathbb{R}^n)} \leq C\|\mathcal{A}_n^{2\varepsilon}(t\nabla^j u)\|_{L^p(\mathbb{R}^n)} + C_p\|\mathcal{A}_f^{\varepsilon/\sqrt{16n}}(\varepsilon\nabla^j u)\|_{L^p(\mathbb{R}^n)}.$$

Both terms converge to zero by the dominated convergence theorem and hence the proof is complete. □

Combining Lemma 6.1 with Theorem 5.4 (or, for more notational convenience, [23, Theorem 6.2]) yields the following corollary.

**Corollary 6.1.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Suppose that $w \in \dot W^{m,2}_{loc}(\mathbb{R}^{n+1}_+)$ satisfies $Lw = 0$ in $\mathbb{R}^{n+1}_+$, $\mathcal{A}_2^+(t\nabla^m\partial_t w) \in L^p(\mathbb{R}^n)$ for some $p$ with $1 < p \leq 2$, and $\nabla^m w(\,\cdot\,,t) \in L^p(\mathbb{R}^n)$ for some $t > 0$.*

*Let $w_\varepsilon(x,t) = w(x,t+\varepsilon)$. Then*

$$\lim_{T\to\infty} \|\dot{\mathbf{M}}_{\boldsymbol{A}}^+ w_T\|_{(\dot W A^{0,p'}_{m-1}(\mathbb{R}^n))^*} = 0, \qquad \lim_{\varepsilon\to 0^+} \|\dot{\mathbf{M}}_{\boldsymbol{A}}^+ (w-w_\varepsilon)\|_{(\dot W A^{0,p'}_{m-1}(\mathbb{R}^n))^*} = 0.$$

We are now in a position to prove the Green formula.

**Theorem 6.1.** *Let $L$ be an operator of the form (2.7) of order $2m$ associated to bounded $t$-independent coefficients $\boldsymbol{A}$ that satisfy the ellipticity condition (1.11).*

*Let $p$ satisfy $p^-_{1,L^*} < p \leq 2$, where $p^-_{1,L^*}$ is as in formula (1.14). Suppose that $w \in \dot W^{m,2}_{loc}(\mathbb{R}^{n+1}_+)$ satisfies $Lw = 0$ in $\mathbb{R}^{n+1}$, $\mathcal{A}_2^+(t\nabla^m\partial_t w) \in L^p(\mathbb{R}^n)$, and $\nabla^m w(\,\cdot\,,t) \in L^p(\mathbb{R}^n)$ for some $t > 0$.*

*Then we have the Green formula*

$$\mathbf{1}_+\nabla^m w = -\nabla^m \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}^+_{m-1} w) + \nabla^m \mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+ w).$$

*Proof.* Let $w_\varepsilon(x,t) = w(x,t+\varepsilon)$ and let $w_{\varepsilon,T} = w_\varepsilon - w_T$. If $\boldsymbol{A}$ is $t$-independent, then $Lw_{\varepsilon,T} = 0$ in $\mathbb{R}^{n+1}_+$ for any $T > \varepsilon > 0$. By Lemma 3.3 or [23, Remark 5.3], if $T > \varepsilon > 0$, then $w_{\varepsilon,T} \in \dot W^{m,2}(\mathbb{R}^{n+1}_+)$.

Recall that formula (6.1) is valid for all solutions in $\dot W^{m,2}(\mathbb{R}^{n+1}_+)$. Thus, we have

$$\mathbf{1}_+\nabla^m w_{\varepsilon,T} = -\nabla^m \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}^+_{m-1} w_{\varepsilon,T}) + \nabla^m \mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+ w_{\varepsilon,T}).$$

Let $B = B((x_0,t_0),|t_0|/2)$ be a Whitney ball in $\mathbb{R}^{n+1}_\pm$. By Theorem 5.4, we find that $\dot{\mathbf{Tr}}^+_{m-1} w_{\varepsilon,T} \to \dot{\mathbf{Tr}}^+_{m-1} w$ in $\dot W A^{1,p}_{m-1}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$ and $T \to \infty$, and by Corollary 6.1, $\dot{\mathbf{M}}_{\boldsymbol{A}}^+ w_{\varepsilon,T} \to \dot{\mathbf{M}}_{\boldsymbol{A}}^+ w$ in $(\dot W A^{0,p'}_{m-1}(\mathbb{R}^n))^*$ as $\varepsilon \to 0^+$ and $T \to \infty$. By the bounds (1.27) and (1.26) established in Subsection 5.2, we have

$$-\nabla^m \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}^+_{m-1} w_{\varepsilon,T}) + \nabla^m \mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+ w_{\varepsilon,T}) \longrightarrow -\nabla^m \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}^+_{m-1} w) + \nabla^m \mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+ w)$$

in $L^2(B)$ as $T \to \infty$ and $\varepsilon \to 0^+$.

Since $\mathcal{A}_2^+(t\nabla^m\partial_t w) \in L^p(\mathbb{R}^n)$, we have that $w_\varepsilon \to w$ as $\varepsilon \to 0^+$ in $\dot W^{m,2}(B)$. By Theorem 5.4, $\nabla^m w_T(\,\cdot\,,t) \to 0$ in $L^p(\mathbb{R}^n)$ for any fixed $t > 0$, uniformly for $t$ in $(-3|t_0|/2, 3|t_0|/2)$. Therefore, $w_T \to 0$ in $\dot W^{m,p}(B)$; by the bound (1.13), $w_T \to 0$ in $\dot W^{m,2}(B)$ as $T \to \infty$.

Thus, taking appropriate limits we obtain the Green formula, as desired. □

# 7 The Neumann problem

In this section, we prove Theorem 1.1, that is, establish the well posedness of the Neumann problem with boundary data in $L^p(\mathbb{R}^n)$ for the operators with bounded elliptic $t$-independent self-adjoint coefficients.

Our proof of Theorem 1.1 is based on a duality argument. That is, we show that the well posedness of the Neumann problem with the boundary data in $\dot W^{-1,p'}(\mathbb{R}^n)$ implies the well posedness with the boundary data in $L^p(\mathbb{R}^n)$ for adjoint coefficients; as well posedness of the subregular Neumann problem was established in [15], this implies the well posedness of the $L^p$ Neumann problem.

We begin with precisely stating the well posedness result of [15].

**Theorem 7.1** ( [15])**.** *Let $L$ and $\boldsymbol{A}$ satisfy the conditions given in Theorem 1.1.*

*Then there is some $\varepsilon_1 > 0$, depending only on the standard parameters $n$, $m$, $\lambda$, and $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$, with the following significance. If*

$$\max\left(0, \frac{1}{2} - \frac{1}{n} - \varepsilon_1\right) < \frac{1}{p'} \leq \frac{1}{2}, \tag{7.1}$$

*then for every $\dot{\boldsymbol{h}}$ in $\dot W^{-1,2}(\mathbb{R}^n) \cap \dot W^{-1,p'}(\mathbb{R}^n)$, there is a solution $v$, unique up to adding polynomials of degree $m-2$, to the subregular Neumann problem*

$$\begin{cases} L^* v = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \dot{\mathbf{M}}^+_{\boldsymbol{A}^*} v \ni \dot{\boldsymbol{h}}, \\ \|\mathcal{A}_2^+(t\nabla^m v)\|_{L^2(\mathbb{R}^n)} + \|\widetilde N_+(\nabla^{m-1} v)\|_{L^2(\mathbb{R}^n)} \leq C_2 \|\dot{\boldsymbol{h}}\|_{\dot W^{-1,2}(\mathbb{R}^n)}, \\ \|\mathcal{A}_2^+(t\nabla^m v)\|_{L^{p'}(\mathbb{R}^n)} + \|\widetilde N_+(\nabla^{m-1} v)\|_{L^{p'}(\mathbb{R}^n)} \leq C_{p'} \|\dot{\boldsymbol{h}}\|_{\dot W^{-1,p'}(\mathbb{R}^n)}. \end{cases} \tag{7.2}$$

*The numbers $C_2$ and $C_{p'}$ depend only on $n$, $m$, $\lambda$, $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$, and $p'$.*

We note that the $p' = 2$ case, like the $L^2$ Neumann problem (1.4), is from [21,24]. Here, $\dot{\mathbf{M}}^+_{\boldsymbol{A}^*} v$ is as given in [23, Section 2.3.2].

If $\boldsymbol{A}$ is self-adjoint, then $\boldsymbol{A} = \boldsymbol{A}^*$ and $L = L^*$; however, we have phrased the problem (7.2) in terms of $\boldsymbol{A}^*$ and $L^*$ for ease of notation for duality arguments. We now state our duality theorem; Theorem 1.1 will follow easily from Theorem 7.2.

**Theorem 7.2.** *Suppose that $L$ is an elliptic operator of the form (2.7) of order $2m$ associated to the coefficients $\boldsymbol{A}$ that are bounded, $t$-independent in the sense of formula (1.2), and satisfy the ellipticity condition (1.11).*

*Let $p$ and $p'$ satisfy $p^-_{1,L^*} < p < 2$ and $1/p + 1/p' = 1$, where $p^-_{1,L^*}$ is as in formulas (1.14) and (1.13). Suppose that for every $\dot{\boldsymbol{h}} \in \dot W^{-1,2}(\mathbb{R}^n) \cap \dot W^{-1,p'}(\mathbb{R}^n)$, there is a unique solution $v$ to the Neumann problem (7.2) for $L^*$.*

*Then for every $\dot{\boldsymbol{g}} \in L^p(\mathbb{R}^n)$, there is a solution $w$, unique up to adding polynomials of degree at most $m-1$, to the $L^p$-Neumann problem*

$$\begin{cases} Lv = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \dot{\mathbf{M}}^+_{\boldsymbol{A}} v \ni \dot{\boldsymbol{g}}, \\ \|\mathcal{A}_2^+(t\nabla^m \partial_t w)\|_{L^p(\mathbb{R}^n)} + \|\widetilde N_+(\nabla^m w)\|_{L^p(\mathbb{R}^n)} \leq C_p \|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \end{cases} \tag{7.3}$$

*where $C_p$ depends only on $p$, $n$, $m$, $\lambda$, $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)}$, the number $c(1, L^*, p', 2)$ in formula (1.13), and the constants $C_2$ and $C_{p'}$ in the problem (7.2).*

*Proof.* Fix some such $p$ and $p'$. We use the method of layer potentials of [17,19,69], specifically as formulated in [14].

Let $\mathfrak{X}^\pm_p$ and $\widetilde{\mathfrak{X}}^\pm_{p'}$ be the spaces of all equivalence classes of functions such that the appropriate norms

$$\|w\|_{\mathfrak{X}^\pm_p} = \|\widetilde N_\pm(\nabla^m w)\|_{L^p(\mathbb{R}^n)} + \|\mathcal{A}_2^\pm(t\nabla^m \partial_t w)\|_{L^p(\mathbb{R}^n)},$$

$$\|v\|_{\widetilde{\mathfrak{X}}^\pm_{p'}} = \|\widetilde N_\pm(\nabla^{m-1} v)\|_{L^{p'}(\mathbb{R}^n)} + \|\mathcal{A}_2^\pm(t\nabla^m v)\|_{L^{p'}(\mathbb{R}^n)}$$

are finite.

We define the following function spaces:

$$\begin{aligned}
\mathfrak{Y}^\pm &= \{w_p + w_2 : \ w_p \in \mathfrak{X}_p^\pm,\ w_2 \in \mathfrak{X}_2^\pm,\ Lw_p = Lw_2 = 0 \text{ in } \mathbb{R}^{n+1}_\pm\},\\
\widetilde{\mathfrak{Y}}^\pm &= \{v \in \widetilde{\mathfrak{X}}_{p'}^\pm \cap \widetilde{\mathfrak{X}}_2^\pm : \ L^* v = 0 \text{ in } \mathbb{R}^{n+1}_\pm\},\\
\mathfrak{D} &= \dot{W}\!A^{1,p}_{m-1}(\mathbb{R}^n) + \dot{W}\!A^{1,2}_{m-1}(\mathbb{R}^n),\\
\widetilde{\mathfrak{D}} &= \dot{W}\!A^{0,p'}_{m-1}(\mathbb{R}^n) \cap \dot{W}\!A^{0,2}_{m-1}(\mathbb{R}^n),\\
\mathfrak{N} &= (\dot{W}\!A^{0,p'}_{m-1}(\mathbb{R}^n))^* + (\dot{W}\!A^{0,2}_{m-1}(\mathbb{R}^n))^*,\\
\widetilde{\mathfrak{N}} &= (\dot{W}\!A^{1,p}_{m-1}(\mathbb{R}^n))^* \cap (\dot{W}\!A^{1,2}_{m-1}(\mathbb{R}^n))^*.
\end{aligned}$$

We are interested in a family of norms on these function spaces. For each number $\delta > 0$, let

$$\begin{aligned}
\|w\|_{\mathfrak{Y}^\pm_\delta} &= \inf\Big\{\|w_p\|_{\mathfrak{X}_p^\pm} + \frac{1}{\delta}\|w_2\|_{\mathfrak{X}_2^\pm} : \ w = w_p + w_2,\ \ Lw_p = Lw_2 = 0\Big\},\\
\|\dot{\boldsymbol\varphi}\|_{\mathfrak{D}_\delta} &= \inf\Big\{\|\dot{\boldsymbol\varphi}_p\|_{\dot{W}\!A^{1,p}_{m-1}(\mathbb{R}^n)} + \frac{1}{\delta}\|\dot{\boldsymbol\varphi}_2\|_{\dot{W}\!A^{1,2}_{m-1}(\mathbb{R}^n)} : \ \dot{\boldsymbol\varphi} = \dot{\boldsymbol\varphi}_p + \dot{\boldsymbol\varphi}_2\Big\},\\
\|\dot{\boldsymbol G}\|_{\mathfrak{N}_\delta} &= \inf\Big\{\|\dot{\boldsymbol G}_p\|_{(\dot{W}\!A^{0,p'}_{m-1}(\mathbb{R}^n))^*} + \frac{1}{\delta}\|\dot{\boldsymbol G}_2\|_{(\dot{W}\!A^{0,2}_{m-1}(\mathbb{R}^n))^*} : \ \dot{\boldsymbol G} = \dot{\boldsymbol G}_p + \dot{\boldsymbol G}_2\Big\},\\
\|v\|_{\widetilde{\mathfrak{Y}}^\pm_\delta} &= \|v\|_{\widetilde{\mathfrak{X}}^\pm_{p'}} + \delta\|v\|_{\widetilde{\mathfrak{X}}^\pm_2},\\
\|\dot{\boldsymbol f}\|_{\widetilde{\mathfrak{D}}_\delta} &= \|\dot{\boldsymbol f}\|_{\dot{W}\!A^{0,p'}_{m-1}(\mathbb{R}^n)} + \delta\|\dot{\boldsymbol f}\|_{\dot{W}\!A^{0,2}_{m-1}(\mathbb{R}^n)},\\
\|\dot{\boldsymbol H}\|_{\widetilde{\mathfrak{N}}_\delta} &= \|\dot{\boldsymbol H}\|_{(\dot{W}\!A^{1,p}_{m-1}(\mathbb{R}^n))^*} + \delta\|\dot{\boldsymbol H}\|_{(\dot{W}\!A^{1,2}_{m-1}(\mathbb{R}^n))^*}.
\end{aligned}$$

Then $\mathfrak{N}_\delta = (\widetilde{\mathfrak{D}}_\delta)^*$ and $\widetilde{\mathfrak{N}}_\delta = (\mathfrak{D}_\delta)^*$. See [54, formula (1.3) and Theorem 1.7].

By Theorems 5.1, 5.2, 5.3 and 5.4, the operators

$$\dot{\mathbf{Tr}}^\pm_{m-1} : \mathfrak{Y}^\pm_\delta \to \mathfrak{D}_\delta, \quad \dot{\mathbf{Tr}}^\pm_{m-1} : \widetilde{\mathfrak{Y}}^\pm_\delta \to \widetilde{\mathfrak{D}}_\delta, \quad \dot{\mathbf{M}}^\pm_{\boldsymbol A} : \mathfrak{Y}^\pm_\delta \to \mathfrak{N}_\delta, \quad \dot{\mathbf{M}}^\pm_{\boldsymbol A^*} : \widetilde{\mathfrak{Y}}^\pm_\delta \to \widetilde{\mathfrak{N}}_\delta$$

are bounded with bounds depending only on $p$ and the standard parameters, and in particular, not on $\delta$ provided $\delta > 0$. By Theorem 1.2 and the bounds (1.20)–(1.23) we have that the double and single layer potentials are bounded

$$\mathcal{D}^{\boldsymbol A} : \mathfrak{D}_\delta \to \mathfrak{Y}^\pm_\delta, \quad \mathcal{D}^{\boldsymbol A^*} : \widetilde{\mathfrak{D}}_\delta \to \widetilde{\mathfrak{Y}}^\pm_\delta, \quad \mathcal{S}^L : \mathfrak{N}_\delta \to \mathfrak{Y}^\pm_\delta, \quad \mathcal{S}^{L^*} : \widetilde{\mathfrak{N}}_\delta \to \widetilde{\mathfrak{Y}}^\pm_\delta$$

with bounds independent of $\delta$.

By [21, Theorem 4.3], and by Theorem 6.1 and Subsection 3.1, we find that if $v \in \widetilde{\mathfrak{Y}}^\pm_\delta$ and $w \in \mathfrak{Y}^\pm_\delta$, then the Green formulas

$$\begin{aligned}
\mathbf{1}_\pm \nabla^m v &= \mp\nabla^m \mathcal{D}^{\boldsymbol A^*}(\dot{\mathbf{Tr}}^\pm_{m-1} v) + \nabla^m \mathcal{S}^{L^*}(\dot{\mathbf{M}}^\pm_{\boldsymbol A^*} v),\\
\mathbf{1}_\pm \nabla^m w &= \mp\nabla^m \mathcal{D}^{\boldsymbol A^*}(\dot{\mathbf{Tr}}^\pm_{m-1} w) + \nabla^m \mathcal{S}^{L^*}(\dot{\mathbf{M}}^\pm_{\boldsymbol A^*} w)
\end{aligned}$$

are valid.

Finally, the jump relations

$$\dot{\mathbf{Tr}}^+_{m-1} \mathcal{D}^{\boldsymbol A} \dot{\boldsymbol f} - \dot{\mathbf{Tr}}^-_{m-1} \mathcal{D}^{\boldsymbol A} \dot{\boldsymbol f} = -\dot{\boldsymbol f}, \quad \dot{\mathbf{Tr}}^+_{m-1} \mathcal{S}^L \dot{\boldsymbol g} - \dot{\mathbf{Tr}}^-_{m-1} \mathcal{S}^L \dot{\boldsymbol g} = \dot{\mathbf{0}}, \tag{7.4}$$

$$\dot{\mathbf{M}}^+_{\boldsymbol A} \mathcal{D}^{\boldsymbol A} \dot{\boldsymbol f} + \dot{\mathbf{M}}^-_{\boldsymbol A} \mathcal{D}^{\boldsymbol A} \dot{\boldsymbol f} \ni \dot{\mathbf{0}}, \quad \dot{\mathbf{M}}^+_{\boldsymbol A} \mathcal{S}^L \dot{\boldsymbol g} + \dot{\mathbf{M}}^-_{\boldsymbol A} \mathcal{S}^L \dot{\boldsymbol g} = \dot{\boldsymbol g} \tag{7.5}$$

of [14, Conditions 6.19–6.22] are valid for all $\dot{\boldsymbol f} \in \dot{W}\!A^{1/2,2}_{m-1}(\mathbb{R}^n)$ and all $\dot{\boldsymbol g} \in (\dot{W}\!A^{1/2,2}_{m-1}(\mathbb{R}^n))^*$; see [14, Lemma 5.4]. By density, the relations (7.4) and (7.5) are true for all $\dot{\boldsymbol f}$ in $\mathfrak{D}_\delta$ or $\widetilde{\mathfrak{D}}_\delta$ and all $\dot{\boldsymbol g}$ in $\mathfrak{N}_\delta$ or $\widetilde{\mathfrak{N}}_\delta$.

Thus, [14, Conditions 6.14–6.22] are valid for the spaces $\widetilde{\mathfrak{Y}}^\pm_\delta$, $\widetilde{\mathfrak{D}}_\delta$ and $\widetilde{\mathfrak{N}}_\delta$, and so, by [14, Theorems 6.23 and 6.24] and the well posedness of the Neumann problem (7.2), we get that $\dot{\mathbf{M}}^+_{\boldsymbol A^*} \mathcal{D}^{\boldsymbol A^*}$ is invertible $\widetilde{\mathfrak{D}}_\delta \to \widetilde{\mathfrak{N}}_\delta$ and $\|(\dot{\mathbf{M}}^+_{\boldsymbol A^*} \mathcal{D}^{\boldsymbol A^*})^{-1}\|$ is independent of $\delta$. By duality (see [14, Lemma 5.3]), $\dot{\mathbf{M}}^+_{\boldsymbol A} \mathcal{D}^{\boldsymbol A}$

is invertible $\mathfrak{D}_\delta \to \mathfrak{N}_\delta$. Furthermore, the norm is independent of $\delta$ and the value of $(\dot{\mathbf{M}}_{\boldsymbol{A}}^+ \mathcal{D}^{\boldsymbol{A}})^{-1}\dot{\boldsymbol{g}}$ is independent of $\delta$.

Let $w = \mathcal{D}^{\boldsymbol{A}}((\dot{\mathbf{M}}_{\boldsymbol{A}}^+ \mathcal{D}^{\boldsymbol{A}})^{-1}\dot{\boldsymbol{G}})$, $\dot{\boldsymbol{G}} \in L^p(\mathbb{R}^n) \subset \mathfrak{N}_\delta$.

Then $w \in \mathfrak{Y}_\delta^+$ and so $w = w_p^\delta + w_2^\delta$ for some $w_p^\delta \in \mathfrak{X}_p^+$, $w_2^\delta \in \mathfrak{X}_2^+$ with $Lw_p^\delta = Lw_2^\delta = 0$ in $\mathbb{R}^{n+1}_+$ and with

$$\|w_p^\delta\|_{\mathfrak{X}_p^+} + \frac{1}{\delta}\|w_2^\delta\|_{\mathfrak{X}_2^+} \leq C\|\dot{\boldsymbol{G}}\|_{\widetilde{\mathfrak{N}}_\delta} \leq C\|\dot{\boldsymbol{G}}\|_{L^p(\mathbb{R}^n)}.$$

Taking the limit as $\delta \to 0^+$, we see that $w_2^\delta \to 0$ in $\dot{W}^{m,2}_{loc}(\mathbb{R}^{n+1}_+)$. Thus $w = \lim_{\delta\to 0^+} w_p^\delta$ and so

$$\|\widetilde{N}_\pm(\nabla^m w) + \mathcal{A}_2^\pm(t\nabla^m \partial_t w)\|_{L^{p'}(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{G}}\|_{L^p(\mathbb{R}^n)},$$

as desired.

Thus, the solutions to the Neumann problem (7.3) exist. We have seen that $\dot{\mathbf{M}}_{\boldsymbol{A}}^+ \mathcal{D}^{\boldsymbol{A}}$ is one-to-one on $\mathfrak{D} = \dot{W}A^{1,p}_{m-1}(\mathbb{R}^n) + \dot{W}A^{1,2}_{m-1}(\mathbb{R}^n)$, and so it is also one-to-one on the subspace $\dot{W}A^{1,p}_{m-1}(\mathbb{R}^n)$. The Green formula of Theorem 6.1 allows us to apply [14, Theorem 6.13] to see that the solutions to the problem (7.3) are unique, as desired. □

We conclude the paper by proving Theorem 1.1.

*Proof of Theorem 1.1.* The ellipticity condition (1.3) in Theorem 1.1 implies that the condition (1.11) in Theorem 7.2 is valid. Thus, if $L$ and $\boldsymbol{A}$ satisfy the conditions of Theorem 1.1, then they satisfy the conditions of Theorems 7.1 and 7.2, as well.

There is an $\varepsilon > 0$, depending only on $n$ and the number $\varepsilon_1$ in formula (7.1), such that if $p$ satisfies the bound (1.8), then $p'$ satisfies the bound (7.1). Thus, if $\varepsilon > 0$ is small enough and the conditions of Theorem 1.1 are satisfied, then the subregular Neumann problem (7.2) is well posed.

Recall from formula (1.15) that there is some $\widetilde{\varepsilon} > 0$ depending on the standard parameters such that

$$p^-_{1,L^*} \leq \max\Big(1, \frac{2n}{n+2} - \widetilde{\varepsilon}\Big).$$

By Remark 1.1, if $\max\big(1, \frac{2n}{n+2} - \widetilde{\varepsilon}\big) < p < 2$, then $c(1, L^*, p, 2)$ depends only on $p$ and the standard parameters.

Thus, if $\varepsilon$ is small enough and $p$ satisfies the condition (1.8) of Theorem 1.1, then $p$ and $L$ also satisfy the conditions of Theorem 7.2. Thus, the Neumann problem (7.3) (or (1.9)) is well posed. □

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