Memoirs on Differential Equations and Mathematical Physics

Volume 91, 2024, 151–160

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ASYMPTOTIC BEHAVIOUR OF SAME CLASSES OF SOLUTIONS OF THE NONAUTONOMOUS ORDINARY DIFFERENTIAL THIRD ORDER EQUATIONS

Abstract. Asymptotic representations of some classes of solutions of nonautonomous ordinary differential third order equations, which are somewhat close to linear equations, are established.

2020 Mathematics Subject Classification. 34D05.

Key words and phrases. Differential third order equations, non-oscillatory solutions, asymptotic representations.

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1 Introduction

We consider the differential equation

$$y''' = \alpha_0 p(t) y |\ln |y||^{\sigma}, \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$.[1]

The solution y of equation (1.1), given and different from zero on the interval $[t_y, \omega[\subset [a, \omega[$, is called $P_\omega(\lambda_0)$-solution, if it meets the following conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{either } 0, \\ \text{or } \pm\infty \end{cases} \quad (k = 0, 1, 2), \quad \lim_{t\uparrow\omega} \frac{(y''(t))^2}{y'''(t) y'(t)} = \lambda_0. \tag{1.2}$$

In [4–6], for equations (1.1), the conditions for the existence of $P_\omega(\lambda_0)$-solutions were established in case $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}\}$, and the asymptotic representations were obtained for such solutions and their derivatives up to the second order inclusive. At the same time, the question on a number of solutions with the found asymptotic representations is also clarified.

The purpose of this paper is to establish the necessary and sufficient conditions for the existence of $P_\omega(\frac{1}{2})$-solutions of the differential equation (1.1), as well as asymptotic for $t \uparrow \omega$ representations for all such solutions and their derivatives up to the second order inclusive.

Such solutions, by virtue of the results due to V. Evtukhov [1, Chapter 3, § 10, pp. 142–144], have the following a priori asymptotic properties.

Lemma 1.1. *For each $P_\omega(\frac{1}{2})$-solution of the differential equation (1.1), when $t \uparrow \omega$, there may be the following asymptotic representations:*

$$y'(t) = o\Big(\frac{y(t)}{\pi_\omega(t)}\Big), \quad y''(t) \sim -\frac{1}{\pi_\omega(t)}\, y'(t), \quad y'''(t) \sim \frac{2}{[\pi_\omega(t)]^2}\, y'(t), \tag{1.3}$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t - \omega & \text{if } \omega < +\infty. \end{cases}$$

2 Auxiliary notation

Along with the assertion of Lemma 1.1, we will also need a corollary from one of the well-known results of the existence of solutions tending to zero when $t \uparrow \omega$ for a system of quasilinear differential equations

$$\begin{cases} v_k' = h(t)\Big[f_k(t, v_1, v_2, v_3) + \sum\limits_{i=1}^{3} c_{ki} v_i + V_k(v_1, v_2, v_3)\Big] \qquad (k = 1, 2), \\ v_3' = H(t)\Big[f_3(t, v_1, v_2, v_3) + \sum\limits_{i=1}^{3} c_{3i} v_i + V_n(v_1, v_2, v_3)\Big], \end{cases} \tag{2.1}$$

where $c_{ki} \in \mathbb{R}$ $(k, i = 1, 2, 3)$, $h, H : [\tau_0, \omega[\to \mathbb{R} \setminus \{0\}$ are continuously differentiable functions, $f_k : [\tau_0, \omega[\times \mathbb{R}^3_{\frac{1}{2}}$ $(k = 1, 2, 3)$ are continuous functions satisfying the conditions

$$\lim_{t\uparrow\omega} f_k(\tau, v_1, v_2, v_3) = 0 \ \text{ uniformly in } \ (v_1, v_2, v_3) \in \mathbb{R}^3_{\frac{1}{2}},$$

where

$$\mathbb{R}^3_{\frac{1}{2}} = \Big\{(v_1, v_2, v_3) \in \mathbb{R}^3 : \ |v_i| \le \frac{1}{2} \ (i = 1, 2, 3)\Big\},$$

and $V_k : \mathbb{R}^3_{\frac{1}{2}} \to \mathbb{R}$ $(k = 1, 2, 3)$ are continuously differentiable functions such that

$$V_k(0, \dots, 0) = 0 \ (k = 1, 2, 3), \quad \frac{\partial V_k(t, 0, 0, 0)}{\partial v_i} = 0 \ (i, k = 1, 2, 3).$$

[1]We suppose that $a > 1$ for $\omega = +\infty$ and $\omega - a < 1$ for $\omega < +\infty$.

According to Theorem 2.6 from the work of V. M. Evtukhov and A. M. Samoilenko [2], for a system of differential equations of form (2.1) the following statement holds.

Lemma 2.1. *Let the functions h and H satisfy the conditions*

$$\lim_{t\uparrow\omega}\frac{H(t)}{h(t)}=0,\quad \int_{t_0}^{\omega}H(\tau)\,d\tau=\pm\infty,\quad \lim_{t\uparrow\omega}\frac{1}{H(t)}\Big(\frac{H(t)}{h(t)}\Big)'=0. \tag{2.2}$$

Let, in addition, the matrix $C_3=(c_{ki})_{k,i=1}^{3}$ satisfies the condition $\det C_3\neq 0$ and the matrix $C_2=(c_{ki})_{k,i=1}^{2}$ has no eigenvalues with zero real part. Then the system of differential equations (2.1) has at least one solution $(v_k)_{k=1}^{3}:[t_1,\omega[\,\to\mathbb{R}^3_{\frac12}$ $(t_0\le t_1\le\omega)$ tending to zero as $t\uparrow\omega$. Moreover, if among the eigenvalues of the matrix C_2 there are m eigenvalues (including multiples), and the real parts have the sign opposite to the function $h(t)$ in the interval $[t_0,\omega[$, then when on the interval $[t_0,\omega[$ the inequalities

$$H(t)(\det C_3)(\det C_2)>0$$

are fulfilled, there is an m-parameter family of solutions to system (2.1), and when the opposite inequality is fulfilled, there is an $(m+1)$-parametrer family.

3 Main results

To formulate the main result, we need the auxiliary functions

$$J_A(t)=\int_A^t\pi_\omega(\tau)p(\tau)\,d\tau,\quad I_B(t)=\int_B^t J_A(\tau)\,d\tau,$$

where

$$A=\begin{cases} a & \text{if } \displaystyle\int_a^\omega|\pi_\omega(\tau)|p(\tau)\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int_a^\omega|\pi_\omega(\tau)|p(\tau)\,d\tau<+\infty,\end{cases}\qquad B=\begin{cases} a & \text{if } \displaystyle\int_a^\omega|J_A(\tau)|\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int_a^\omega|J_A(\tau)|\,d\tau<+\infty.\end{cases}$$

Theorem 3.1. *Let $\sigma\neq1$. Then for the existence of $P_\omega(\frac12)$-solutions of the differential equation (1.1), it is necessary and sufficient to fulfill the following conditions:*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}=-1,\quad \lim_{t\uparrow\omega}|I_B(t)|^{\frac{1}{1-\sigma}}=+\infty,\quad \lim_{t\uparrow\omega}\pi_\omega(t)J_A(t)|I_B(t)|^{\frac{\sigma}{1-\sigma}}=0. \tag{3.1}$$

Moreover, for each such solution, when $t\uparrow\omega$, there may be the following asymptotic representations:

$$\ln|y(t)|=\nu_0\Big|\frac{1-\sigma}{2}\,I_B(t)\Big|^{\frac{1}{1-\sigma}}[1+o(1)], \tag{3.2}$$

$$\frac{y'(t)}{y(t)}=-\frac{\alpha_0J_A(t)}{2}\,\Big|\frac{1-\sigma}{2}\,I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}[1+o(1)], \tag{3.3}$$

$$\frac{y''(t)}{y(t)}=\frac{\alpha_0}{2}\,\frac{J_A(t)}{\pi_\omega(t)}\,\Big|\frac{1-\sigma}{2}\,I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}[1+o(1)], \tag{3.4}$$

where

$$\nu_0=-\alpha_0\,\mathrm{sign}\,\big[(1-\sigma)I_B(t)\big].$$

Additionally, if conditions (3.1) are satisfied for the differential equation (1.1), in case $\omega=+\infty$, there exists a one-parameter family of solutions with representations (3.2)–(3.4) if $\sigma<1$, and at least one solution if $\sigma>1$; and in case $\omega<\infty$, there is a 2-parameter family of such solutions if $\sigma>1$ and 3-parameter if $\sigma<1$.

Proof.

Necessity. Let $y : [t_y, \omega[\to \mathbb{R}$ be an arbitrary $P_\omega(\frac{1}{2})$-solution of the differential equation (1.1). Then, by definition of $P_\omega(\lambda_0)$-solution, there exists $t_0 \in [t_y, \omega[$ such that $\ln|y(t)| \neq 0$ on the interval $[t_0, \omega[$, and according to the first assertion of Lemma 1.1, there are asymptotic relations

$$y'''(t) \sim \frac{2}{\pi_\omega^2(t)} y'(t), \quad y''(t) \sim -\frac{y'(t)}{\pi_\omega(t)} \ \text{ as } \ t \uparrow \omega. \tag{3.5}$$

From (1.1), taking into account these relations, we obtain

$$\frac{y'(t)}{y(t)|\ln|y(t)||^\sigma} = \frac{\alpha_0}{2}\, \pi_\omega^2(t) p(t)[1 + o(1)] \ \text{ as } \ t \uparrow \omega \tag{3.6}$$

and

$$\frac{y''(t)}{y(t)|\ln|y(t)||^\sigma} = -\frac{\alpha_0}{2} \pi_\omega(t) p(t)[1 + o(1)] \ \text{ as } \ t \uparrow \omega. \tag{3.7}$$

Because

$$\Big(\frac{y'(t)}{y(t)|\ln|y(t)||^\sigma}\Big)' = \frac{y''(t)}{y(t)|\ln|y(t)||^\sigma} \Big[1 - \frac{y'^2(t)}{y''(t)y(t)} - \sigma \frac{y'^2(t)}{y''(t)y(t)\ln|y(t)|}\Big],$$

and by virtue of the definition of $P_\omega(\frac{1}{2})$-solution, the first of the asymptotic relations (1.3) and the second of (3.5), we have

$$\lim_{t\uparrow\omega} \ln|y(t)| = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{y'^2(t)}{y''(t)y(t)} = \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} \frac{y'(t)}{\pi_\omega(t)y''(t)} = 0,$$

then

$$\Big(\frac{y'(t)}{y(t)|\ln|y(t)||^\sigma}\Big)' = \frac{y''(t)}{y(t)|\ln|y(t)||^\sigma}\, [1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

Therefore, from (3.7) it follows that

$$\Big(\frac{y'(t)}{y(t)|\ln|y(t)||^\sigma}\Big)' = -\frac{\alpha_0}{2}\, \pi_\omega(t) p(t)[1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

Integrating this relation over the interval from t_0 to t, we get

$$\frac{y'(t)}{y(t)|\ln|y(t)||^\sigma} = c_0 + \frac{\alpha_0(-1)}{2} \int_{t_0}^{t} \pi_\omega(\tau) p(\tau)[1 + o(1)]\, d\tau,$$

where c_0 is some real constant, or taking into account the choice of the integration limit A in $J_A(t)$, we obtain

$$\frac{y'(t)}{y(t)|\ln|y(t)||^\sigma} = c - \frac{\alpha_0}{2}\, J_A(t)[1 + o(1)] \ \text{ as } \ t \uparrow \omega, \tag{3.8}$$

where the constant c is defined as follows:

$$c = c_0 - \frac{\alpha_0}{2} \int_{t_0}^{a} \pi_\omega(\tau) p(\tau)[1 + o(1)]\, d\tau \ \text{ if } \int_{a}^{\omega} |\pi_\omega(t)| p(t)\, dt = +\infty,$$

$$c = c_0 - \frac{\alpha_0}{2} \int_{t_0}^{\omega} \pi_\omega(\tau) p(\tau)[1 + o(1)]\, d\tau \ \text{ if } \int_{a}^{\omega} |\pi_\omega(t)| p(t)\, dt < +\infty.$$

In case $\int_a^\omega |\pi_\omega(t)| p(t)\, dt = +\infty$, relation (3.8) can be written as

$$\frac{y'(t)}{y(t)|\ln|y(t)||^\sigma} = -\frac{\alpha_0}{2}\, J_A(t)[1 + o(1)] \ \text{ as } \ t \uparrow \omega. \tag{3.9}$$

Let us show that in case $\int\limits_a^\omega |\pi_\omega(t)|p(t)\,dt < +\infty$, it also admits such a form, i.e., in (3.8) $c=0$. Indeed, if this integral converges and $c \neq 0$, then it follows from (3.8) that

$$\frac{y'(t)}{y(t)|\ln|y(t)||^\sigma} = c + o(1) \ \text{ as } \ t \uparrow \omega,$$

and according to (3.5)

$$\pi_\omega^2(t)p(t) = 2\alpha_0[c+o(1)] \ \text{ as } \ t \uparrow \omega.$$

Thus

$$\pi_\omega(t)p(t) = \frac{2\alpha_0[c+o(1)]}{\pi_\omega(t)} \ \text{ as } \ t \uparrow \omega$$

and then

$$\int\limits_a^t \pi_\omega(\tau)p(\tau)\,d\tau \sim 2\alpha_0[c+o(1)]\ln|\pi_\omega(t)| \to \pm\infty \ \text{ as } \ t \uparrow \omega,$$

which contradicts the convergence of the improper integral $\int\limits_a^\omega |\pi_\omega(t)|p(t)\,dt$.

Hence, in each of the two possible cases under consideration, the asymptotic relation (3.9) holds.

Due to (3.9) and (3.6), we have

$$-\frac{\pi_\omega^2 p(t)}{J_A(t)} \to 1 \ \text{ as } \ t \uparrow \omega,$$

therefore, the first of conditions (3.1) is satisfied.

Moreover, integrating relation (3.9) on the interval from t_0 to t, taking into account the condition $\lim\limits_{t\uparrow\omega} \ln|y(t)| = \pm\infty$, we get

$$\frac{|\ln|y(t)||^{1-\sigma}}{1-\sigma}\ \operatorname{sign}\ln|y(t)| = -\frac{\alpha_0}{2} I_B(t)[1+o(1)] \ \text{ as } \ t \uparrow \omega.$$

Whence we find that

$$\operatorname{sign}\ln|y(t)| = -\alpha_0 \operatorname{sign}\big[(1-\sigma)I_B(t)\big]$$

and

$$\ln|y(t)| = -\alpha_0\big[\operatorname{sign}(1-\sigma)I_B(t)\big]\,\Big|\frac{1-\sigma}{2}\,I_B(t)\Big|^{\frac{1}{1-\sigma}}[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

i.e., the asymptotic relation (3.2) holds. Taking into account the first condition in (1.2) (for $k=0$), we obtain the validity of the second condition (3.1).

Substituting the found value $\ln|y(t)|$ into formula (3.8), we get the representation

$$\frac{y'(t)}{y(t)} = -\frac{\alpha_0}{2}\,J_A(t)\Big|\frac{1-\sigma}{2}I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}[1+o(1)] \ \text{ as } \ t \uparrow \omega,$$

which, by virtue of the first of the asymptotic relations (1.3), directly follows from the validity of the third of conditions (3.1). Using this relation and the second of relations (1.3), we also find that

$$\frac{y'(t)}{y(t)} \sim -\frac{\alpha_0}{2}\,J_A(t)\Big|\frac{1-\sigma}{2}I_B(t)\Big|^{\frac{\sigma}{1-\sigma}},$$

$$\frac{y''(t)}{y(t)} \sim -\frac{1}{\pi_\omega(t)}\,\frac{y'(t)}{y(t)} \sim \frac{\alpha_0}{2}\,\frac{J_A(t)}{\pi_\omega(t)}\,\Big|\frac{1-\sigma}{2}I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}$$

as $t \uparrow \omega$, i.e., the asymptotic representations (3.3) and (3.4) hold as $t \uparrow \omega$.

Sufficiency. Let $\sigma \neq 1$ and conditions (3.1) be satisfied. Let us show that in this case the differential equation (1.1) has $P_\omega(\frac{1}{2})$-solutions admitting the asymptotic representations (3.2) and (3.3) as $t \uparrow \omega$, and find out the question of a number of solutions with such representations.

Because of

$$\pi_\omega(t)J_A(t)|I_B(t)|^{\frac{\sigma}{1-\sigma}} = \frac{\pi_\omega(t)J_A(t)}{|I_B(t)|}\,|I_B(t)|^{\frac{1}{1-\sigma}},$$

it also follows from condition (3.1) that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A(t)}{I_B(t)} = 0. \tag{3.10}$$

Applying to equation (1.1) the transformation

$$\begin{aligned}
&\frac{y^{(k)}(t)}{y(t)} = \frac{\alpha_0(-1)^{2-k}}{2}\,\frac{J_A(t)}{\pi_\omega^{k-1}(t)}\,\Big|\frac{1-\sigma}{2}\,I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}[1+v_k(t)] \quad (k=1,2),\\
&\ln|y(t)| = \nu_0\Big|\frac{1-\sigma}{2}\,I_B(t)\Big|^{\frac{1}{1-\sigma}}[1+v_3(t)],
\end{aligned} \tag{3.11}$$

we get the system of differential equations

$$\begin{cases}
v_1' = -\dfrac{1}{\pi_\omega(t)}\,(1+v_2) + \dfrac{\alpha_0}{2}\,J_A(t)\Big|\dfrac{1-\sigma}{2}\,I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}(1+v_1)^2\\
\qquad\qquad -\Big[\dfrac{J_A'(t)}{J_A(t)} + \dfrac{\sigma}{1-\sigma}\,\dfrac{J_A(t)}{I_B(t)}\Big](1+v_1),\\
v_2' = \dfrac{2J_A'(t)}{J_A(t)}\,(1+v_3)^\sigma + \dfrac{\alpha_0}{2}\,J_A(t)\Big|\dfrac{1-\sigma}{2}\,I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}(1+v_1)(1+v_2)\\
\qquad\qquad -\Big[\dfrac{J_A'(t)}{J_A(t)} - \dfrac{1}{\pi_\omega(t)} + \dfrac{\sigma}{1-\sigma}\,\dfrac{J_A(t)}{I_B(t)}\Big](1+v_2),\\
v_3' = \dfrac{J_A(t)}{(1-\sigma)I_B(t)}\,(1+v_1) - \dfrac{J_A(t)}{(1-\sigma)I_B(t)}\,(1+v_3).
\end{cases}$$

We consider this system of equations on the set

$$\Omega = [t_0,\omega[\,\times\mathbb{R}^3_{\frac12}, \ \text{ where } \ \mathbb{R}^3_{\frac12} = \Big\{(v_1,v_2,v_3)\in\mathbb{R}^3: \ \ |v_i| \le \frac{1}{2}, \ \ i=1,2,3\Big\}$$

and t_0 is any number from the interval $[a,\omega[$.

On the given set, the right-hand sides of the system are defined and continuous. Considering now

$$\begin{aligned}
&h(t) = \frac{1}{\pi_\omega(t)}\,, \quad H(t) = \frac{J_A(t)}{(1-\sigma)I_B(t)}\,, \quad \delta_1(t) = \frac{\sigma}{1-\sigma}\,\frac{\pi_\omega(t)J_A(t)}{I_B(t)}\,,\\
&\delta_2(t) = \frac{\pi_\omega(t)J_A'(t)}{J_A(t)} + 1\,, \quad \delta_3(t) = -\frac{\alpha_0}{2}\,\pi_\omega(t)J_A(t)\Big|\frac{1-\sigma}{2}I_B(t)\Big|^{\frac{\sigma}{1-\sigma}},
\end{aligned}$$

rewrite the resulting system of differential equations in the form

$$\begin{cases}
v_1' = h(t)\big[f_k(t,v_1,v_2,v_3) + v_1 - v_2\big],\\
v_2' = h(t)\big[f_2(t,v_1,v_2,v_3) + 2v_2 - 2\sigma v_n + V(v_3)\big],\\
v_3' = H(t)[v_1 - v_3],
\end{cases} \tag{3.12}$$

where

$$\begin{aligned}
f_1(t,v_1,v_2,v_3) &= -(1+v_1)\big[\delta_1(t)+\delta_2(t)+\delta_3(t)(1+v_1)\big],\\
f_2(t,v_1,v_2,v_3) &= 2\delta_2(t)(1+v_3)^\sigma - (1+v_2)\big[\delta_1(t)+\delta_2(t)+)\delta_3(t)(1+v_1)\big],\\
V(v_3) &= -2\big[(1+v_3)^\sigma - 1 - \sigma v_3\big].
\end{aligned}$$

Since due to conditions (3.1) and (3.10)

$$\lim_{t\uparrow\omega} \delta_i(t) = 0 \ \ (i = 1, 2, 3),$$

the functions f_k $(k = 1, 2)$ are such that

$$\lim_{t\uparrow\omega} f_k(t, v_1, v_2, v_3) = 0 \ \text{ evenly across } \ (v_1, v_2, v_3) \in \mathbb{R}^3_{\frac{1}{2}}.$$

Taking into account that $V'(v_3) = -2[\sigma(1 + v_3)^{\sigma-1} - \sigma]$, we also have

$$V(0) = 0 \ \text{ and } \ V'(0) = 0.$$

This means that the system of differential equations (3.12) is a system of type (2.1).

Let us show that for that system all conditions of Lemma 2.1 are satisfied. Let us first establish the validity of conditions (2.2). Due to the type of functions I_B and J_A,

$$\int\limits_{t_0}^{t} H(\tau)\, d\tau \sim \frac{1}{1-\sigma} \ln |J_A(t)| \to \pm\infty \ \text{ as } \ t \uparrow \omega.$$

This implies that the second of conditions (2.2) is fulfilled. Besides,

$$\frac{H(t)}{h(t)} = \frac{\pi_\omega(t) J_A(t)}{(1-\sigma) I_B(t)}, \quad \frac{1}{H(t)} \Big(\frac{H(t)}{h(t)}\Big)' = 1 + \frac{\pi_\omega(t) J_A'(t)}{J_A(t)} - \frac{\pi_\omega(t) J_A(t)}{I_B(t)}$$

and therefore, in view of the fulfillment of the first of conditions (3.1) and condition (3.10),

$$\lim_{t\uparrow\omega} \frac{H(t)}{h(t)} = 0, \quad \lim_{t\uparrow\omega} \frac{1}{H(t)} \Big(\frac{H(t)}{h(t)}\Big)' = 0,$$

i.e., the first and the last of conditions (2.2) are satisfied.

Note that the matrices C_2 and C_3 of dimensions 2×2 and 3×3 (respectively) from Lemma 2.1 for the system of differential equations (3.12) have the form

$$C_2 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -2\sigma \\ 1 & 0 & -1 \end{pmatrix}.$$

The eigenvalues of the matrix C_2 are the roots of the algebraic equation

$$(\lambda - 1)(\lambda - 2) = 0,$$

i.e., the numbers $\lambda_1 = 1 > 0$, $\lambda_2 = 2 > 0$. Besides,

$$\det C_2 = 2, \quad \det C_3 = 2(\sigma - 1).$$

Hence the system of differential equations (3.12) satisfies all the conditions of Lemma 2.1. According to this lemma, the system of equations (3.12) has at least one solution $(v_k)_{k=1}^3 : [t_1, \omega[\to \mathbb{R}^3$ $(t_1 \in [t_0, \omega[)$ tending to zero as $t \uparrow \omega$. Moreover, since all eigenvalues of the matrix C_2 are positive and $\det C_2 \det C_3 = 4(\sigma - 1)$, by Lemma 1.1, for such solutions in the case when $h(t) < 0$ on the interval $[t_0, \omega[$ there exists the 3-parameter family if $(\sigma - 1)H(t) < 0$ when $t \in [t_0, \omega[$, and 2-parameter family if the opposite inequality holds, while in the case when $h(t) > 0$ on the interval $[t_0, \omega[$, there is a one-parameter family of such solutions if $(\sigma - 1)H(t) < 0$ when $t \in [t_0, \omega[$, and at least one solution if the opposite inequality holds.

Finally, to solve the problem of a number of solutions of system (3.12) that disappear when $t \uparrow \omega$, it is necessary to determine the signs of the functions h and H on the interval $[t_0, \omega[$.

Since $h(t) = \pi_\omega^{-1}(t)$, according to the form of the function $\pi_\omega(t)$, we have

$$\operatorname{sign} h(t) = \begin{cases} 1 & \text{if } \omega = +\infty, \\ -1 & \text{if } \omega = +\infty. \end{cases}$$

For the function H, by virtue of the definition of the function I_B, we have

$$H(t) = \frac{J_A(t)}{(1-\sigma)I_B(t)} = \frac{|J_A(t)|}{(1-\sigma)\int\limits_B^t |J_A(\tau)|\,d\tau}\,.$$

Here, the numerator is positive on the interval $[t_0, \omega[$, and the integral in the denominator is positive if $B = a$ and negative if $B = \omega$. By virtue of the second of conditions (3.1), this integral satisfies the inequality

$$(1-\sigma)\int\limits_B^t |J_A(\tau)|\,d\tau > 0 \ \text{ as } \ t \in [t_0, \omega[\,.$$

From this and the previous equality it follows that

$$\operatorname{sign} H(t) = 1 \ \text{ as } \ t \in [t_0, \omega[\,.$$

Taking into account the obtained sign conditions for the functions h and H, arrive at the following final conclusion about the number of solutions of the system of differential equations (3.12) that vanish when $t \uparrow \omega$:

(1) if $\omega = +\infty$, then for $\sigma < 1$, the system of differential equations (3.12) has a one-parameter family of solutions vanishing as $t \uparrow \omega$, and for $\sigma > 1$, it has at least one such solution;

(2) if $\omega < +\infty$, then for $\sigma < 1$, the system of differential equations has 3–parameter family of solutions vanishing as $t \uparrow \omega$, and for $\sigma > 1$, it has two-parameter family.

To each of the solutions $(v_k)_{k=1}^n : [t_1, \omega[\to \mathbb{R}^n$ of the system of differential equations (3.12) tending to zero, due to (3.11), there corresponds a solution $y : [t_1, \omega[\to \mathbb{R}$ of differential equation (1.1) admitting asymptotic representations (3.2), (3.3) as $t \uparrow \omega$. Using these representations and conditions (3.1), it is also easy to verify that each such solution is a $P_\omega(\frac{1}{2})$-solution of the differential equation (1.1). ☐

Remark 3.1. Since the function V from the proof of sufficiency satisfies the local Lipschitz condition, it can be proved under conditions (3.1) that for $\omega = +\infty$ and $\sigma > 1$, the differential equation (1.1) has the only solution admitting the asymptotic representations (3.2), (3.3), (3.4) when $t \uparrow \omega$.

Remark 3.2. When checking the fulfillment of conditions (3.1), we can take into account that by virtue of the first of them, the second and third are equivalent to the conditions

$$\lim_{t\uparrow\omega}\bigg|\int\limits_B^t \pi_\omega^2(\tau)p(\tau)\,d\tau\bigg|^{\frac{1}{1-\sigma}} = +\infty, \quad \lim_{t\uparrow\omega}\pi_\omega^3(t)p(t)\bigg|\int\limits_B^t \pi_\omega^2(\tau)p(\tau)\,d\tau\bigg|^{\frac{\sigma}{1-\sigma}} = 0.$$

In conclusion, we note that Theorem 3.1 covers the case $\sigma = 0$, i.e., when equation (1.1) is a linear differential equation of the form

$$y''' = \alpha_0 p(t) y. \tag{3.13}$$

For this equation, by virtue of Theorem 3.1, the following assertion holds.

Consequence. *For the differential equation (3.13) to have $P_\omega(\frac{1}{2})$-solutions, it is necessary and sufficient that the conditions*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_A'(t)}{J_A(t)} = -1, \quad \lim_{t\uparrow\omega}|I_B(t)| = +\infty, \quad \lim_{t\uparrow\omega}\pi_\omega(t)J_A(t) = 0, \tag{3.14}$$

are satisfied. Moreover, for each such solution, when $t \uparrow \omega$, the following asymptotic representations hold:

$$\ln|y(t)| = \nu_0 \Big| \frac{1-\sigma}{2}\, I_B(t) \Big| [1 + o(1)], \tag{3.15}$$

$$\frac{y'(t)}{y(t)} = -\frac{\alpha_0 J_A(t)}{2}\, [1 + o(1)], \tag{3.16}$$

$$\frac{y''(t)}{y(t)} = \frac{\alpha_0}{2}\, \frac{J_A(t)}{\pi_\omega(t)}\, [1 + o(1)], \tag{3.17}$$

where

$$\nu_0 = -\alpha_0 \operatorname{sign}\big[(1-\sigma) I_B(t)\big].$$

Moreover, if conditions (3.14) are satisfied for the differential equation (3.13), in case $\omega = +\infty$, there exists a one-parameter family of solutions, and in case $\omega < \infty$, there is a two-parameter family of solutions with representations (3.15)–(3.17).

The obtained asymptotic representations are consistent with the already known results for linear differential equations (see [3, Chapter 1]).

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(Received 05.05.2023; accepted 19.10.2023)

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