Memoirs on Differential Equations and Mathematical Physics

Volume 91, 2024, 121–150

Mokhtar Mokhtari, Lamine Bouzettouta

GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A VISCOELASTIC BRESSE BEAM SYSTEM WITH A NONLINEAR DELAY TERMS

Abstract. In this paper, we investigate the existence and decay properties of solutions to the initial boundary value problem for the nonlinear Bresse beam system reading as in (P).

2020 Mathematics Subject Classification. 35B40, 35L70.

Key words and phrases. Bresse system, delay terms, decay rate, multiplier method.

რეზიუმე. ნაშრომში გამოკვლეულია საწყისი სასაზღვრო ამოცანის ამონახსნების არსებობა და ქრობის თვისებები არაწრფივი ბრესეს სხივის (P) სისტემისთვის.

1 Introduction

The original Bresse system is given by the following equations (see [6]):

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1\omega_{tt} = N_x - lQ + F_3, \end{cases}$$

where we use N, Q and M to denote the axial force, the shear force and the bending moment, respectively. These forces are stress-strain relations for elastic behavior and given by

$$N = Eh(\omega_x - l\varphi), \quad Q = Gh(\varphi_x + \psi + l\omega), \text{ and } M = EI\psi_x,$$

where G, E, I and h are positive constants. Finally, by the terms F_i we denote external forces.

The Bresse system without delay (i.e., $\mu_2 = \widetilde{\mu}_2 = \widetilde{\widetilde{\mu}}_2 = 0$) is more general than the well-known Timoshenko's system, where the longitudinal displacement ω is not considered, $l = 0$. There are a number of publications concerning the stabilization of Timoshenko's system with different kinds of damping (see [14, 21, 22, 28]). Raposo et al. [29] proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t = 0, \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu}_1\psi_t = 0. \end{cases}$$

Messaoudi and Mustafa [22] (see also [28]) considered the stabilization for the following Timoshenko's system with nonlinear internal feedbacks:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + g_1(\psi_t) = 0, \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + g_2(\psi_t) = 0. \end{cases}$$

Recently, Park and Kang [28] considered the stabilization of Timoshenko's system with weakly nonlinear internal feedbacks.

In [20], Liu and Rao considered a thermoelastic Bresse system that consists of three wave equations and two heat equations coupled in a certain way.

The two wave equations for the longitudinal displacement and the shear angle displacement are effectively globally damped by the dissipation from the two heat equations. The wave equation about the vertical displacement is subject to a weak thermal damping and indirectly damped through the coupling. They establish exponential energy decay rate when the vertical and the longitudinal waves have the same speed of propagation. Otherwise, a polynomial-type decay is established.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see [32]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. In recent years, the PDEs with time delay effects have become an active area of research and arise in many practical problems (see, e.g., [1, 31]). The presence of delay may be a source of instability. For example, it was proved in [9] for an arbitrarily small delay. For instance, in [26], the authors studied the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_1 and μ_2 for which the stability or, alternatively, instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and found a sequence of delays for which the solution will be instable if $\mu_2 \geq \mu_1$.

Very recently, in [4], the authors studied the Bresse system with linear internal damping terms with delays (g_1, g_2, $\widetilde{g}_1$, $\widetilde{g}_2$, $\widetilde{\widetilde{g}}_1$, $\widetilde{\widetilde{g}}_2$ are linear in problem (P)). They proved that the decay of the energy is exponential if$\mu_2 < \mu_1$, $\widetilde{\mu}_2 < \widetilde{\mu}_1$ and $\widetilde{\widetilde{\mu}}_2 < \widetilde{\widetilde{\mu}}_1$.

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to problem (P) for nonlinear damping and delay terms. To obtain global solutions to problem (P), we use an argument combining the Galerkin approximation scheme (see [18]) with the energy estimate method. The techniques based on the theory of semigroups used in [4] do not seem to be applicable in the nonlinear case with the presence of delay terms. To prove decay estimates, we use a multiplier method and some properties of convex functions (see [7, 12, 13, 21]).

2 Preliminaries and main results

In this paper, we consider the Bresse system

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1 g_1(\varphi_t(x,t)) + \mu_2 g_2(\varphi_t(x,t-\tau_1)) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \ ]0,1[\times]0,+\infty[\,, \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu}_1\widetilde{g}_1(\psi_t(x,t)) + \widetilde{\mu}_2\widetilde{g}_2(\psi_t(x,t-\tau_2)) \\ \qquad + \int\limits_0^t h(t-s)\psi_{xx}(x,s)\,ds = 0 \ \text{ in } \ ]0,1[\times]0,+\infty[\,, \\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) + \widetilde{\mu}_1\widetilde{g}_1(\omega_t(x,t)) + \widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(\omega_t(x,t-\tau_3)) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \ ]0,1[\times]0,+\infty[\,, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \ \ t \geq 0, \\ \psi(x,0) = \psi_0(x), \ \ \psi_t(x,0) = \psi_1(x), \ \ x \in]0,1[\,, \\ \varphi(x,0) = \varphi_0(x), \ \ \varphi_t(x,0) = \varphi_1(x), \ \ x \in]0,1[\,, \\ \psi_t(x,t-\tau) = f_0(x,t-\tau) \ \text{ in } \ ]0,1[\times]0,\tau[\,. \end{cases} \tag{P}$$

For the relaxation function, the damping and the delay functions, we make the following hypotheses:

(H1)

(*) $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^2 function satisfying

$$h(0) = h_0 > 0, \quad \ell = \int\limits_0^{+\infty} h(s)\,ds < b.$$

(**) There exists a nonincreasing differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$h'(s) \leq -\zeta(s)h(s) \ \ \forall\, s > 0$$

and

$$\int\limits_0^{+\infty} \zeta(s)\,ds < +\infty.$$

(H2) $g_1 : \mathbb{R} \to \mathbb{R}$ (resp. $\widetilde{g}_1 : \mathbb{R} \to \mathbb{R}$, $\widetilde{\widetilde{g}}_1(s) : \mathbb{R} \to \mathbb{R}$) is a non-decreasing function of the class $C(\mathbb{R})$ such that there exist $\epsilon_1, c_1, c_2, \widetilde{\epsilon}_1, \widetilde{c}_1, \widetilde{c}_2, \widetilde{\widetilde{\epsilon}}_1, \widetilde{\widetilde{c}}_1, \widetilde{\widetilde{c}}_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0,\infty[)$ satisfying $H(0) = 0$, and H linear on $[0,\epsilon']$ or ($H'(0) = 0$ and $H'' > 0$ on $]0,\epsilon']$) such that

$$c_1|s| \leq |g_1(s)| \leq c_2|s| \ \text{ if } \ |s| \geq \epsilon_1, \quad \widetilde{c}_1|s| \leq |\widetilde{g}_1(s)| \leq \widetilde{c}_2|s| \ \text{ if } \ |s| \geq \widetilde{\epsilon}_1,$$
$$\widetilde{\widetilde{c}}_1|s| \leq |\widetilde{\widetilde{g}}_1(s)| \leq \widetilde{\widetilde{c}}_2|s| \ \text{ if } \ |s| \geq \widetilde{\widetilde{\epsilon}}_1;$$
$$s^2 + g_1^2(s) \leq H^{-1}(sg_1(s)) \ \text{ if } \ |s| \leq \epsilon_1, \quad s^2 + \widetilde{g}\,_1^2(s) \leq H^{-1}(s\widetilde{g}_1(s)) \ \text{ if } \ |s| \leq \widetilde{\epsilon}_1,$$
$$s^2 + \widetilde{\widetilde{g}}\,_1^2(s) \leq H^{-1}(s\widetilde{\widetilde{g}}_1(s)) \ \text{ if } \ |s| \leq \widetilde{\widetilde{\epsilon}}_1.$$

$g_2 : \mathbb{R} \to \mathbb{R}$ (resp $\widetilde{g}_2 : \mathbb{R} \to \mathbb{R}$, $\widetilde{\widetilde{g}}_2 : \mathbb{R} \to \mathbb{R}$) is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2, \widetilde{c}_3, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\widetilde{c}}_3, \widetilde{\widetilde{\alpha}}_1, \widetilde{\widetilde{\alpha}}_2 > 0$,

$$|g_2'(s)| \leq c_3, \quad |\widetilde{g}_2'(s)| \leq \widetilde{c}_3, \quad |\widetilde{\widetilde{g}}_2'(s)| \leq \widetilde{\widetilde{c}}_3, \tag{2.1}$$

$$\begin{gathered} \alpha_1 \ sg_2(s) \leq G_2(s) \leq \alpha_2\, sg_1(s), \quad \widetilde{\alpha}_1 \ s\widetilde{g}_2(s) \leq \widetilde{G}_2(s) \leq \widetilde{\alpha}_2\, s\widetilde{g}_1(s), \\ \widetilde{\widetilde{\alpha}}_1\, s\widetilde{\widetilde{g}}_2(s) \leq \widetilde{\widetilde{G}}_2(s) \leq \widetilde{\widetilde{\alpha}}_2\, s\widetilde{\widetilde{g}}_1(s), \end{gathered} \tag{2.2}$$

where

$$G_2(s) = \int_0^s g_2(r)\,dr, \quad \widetilde{G}_2(s) = \int_0^s \widetilde{g}_2(r)\,dr, \quad \widetilde{\widetilde{G}}_2(s) = \int_0^s \widetilde{\widetilde{g}}_2(r)\,dr,$$

$$\alpha_2\mu_2 < \alpha_1\mu_1, \quad \widetilde{\alpha}_2\widetilde{\mu}_2 < \widetilde{\alpha}_1\widetilde{\mu}_1, \quad \widetilde{\widetilde{\alpha}}_2\widetilde{\widetilde{\mu}}_2 < \widetilde{\widetilde{\alpha}}_1\widetilde{\widetilde{\mu}}_1.$$

(H3)

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}\,.$$

We first state some Lemmas which will be needed later.

Lemma 2.1 (Sobolev–Poincaré's inequality)**.** *Let q be a number with $2 \le q < +\infty$ $(n = 1, 2)$ or $2 \le q \le 2n/(n-2)$ $(n \ge 3)$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \le c_*\|\nabla\psi\|_2 \ \text{ for } \ \psi \in H_0^1((0,1)).$$

We introduce as in [26] the new variable

$$z(x,\rho,t) = \psi_t(x, t-\tau\rho), \ \ x \in (0,1), \ \ \rho \in (0,1), \ \ t > 0.$$

Then we have

$$\tau_i z_{it}(x,\rho,t) + z_{i\rho}(x,\rho,t) = 0 \ \text{ in } (0,1)\times(0,1)\times(0,+\infty) \ \text{ for } \ i = 1,2,3.$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1 g_1(\varphi_t(x,t)) + \mu_2 g_2(z_1(x,1,t)) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \ ]0,1[\,\times\,]0,+\infty[\,, \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu}_1\widetilde{g}_1(\psi_t(x,t)) \\ \qquad + \displaystyle\int_0^t h(t-s)\psi_{xx}(x,s)\,ds + \widetilde{\mu}_2\widetilde{g}_2(z_2(x,1,t)) = 0 \ \text{ in } \ ]0,1[\,\times\,]0,+\infty[\,, \\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) + \widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega_t(x,t)) + \widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(z_2(x,1,t)) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \ ]0,1[\,\times\,]0,+\infty[\,, \\ \tau z'(x,\rho,t) + z_\rho(x,\rho,t) = 0 \ \text{ in } \ ]0,1[\,\times\,]0,1[\,\times\,]0,+\infty[\,, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \ \ t \ge 0, \\ z(x,0,t) = \psi_t(x,t) \ \text{ on } \ ]0,1[\,\times\,]0,+\infty[\,, \\ \psi(x,0) = \psi_0(x), \ \ \psi_t(x,0) = \psi_1(x), \ \ x \in\,]0,1[\,, \\ \varphi(x,0) = \varphi_0(x), \ \ \varphi_t(x,0) = \varphi_1(x), \ \ x \in\,]0,1[\,, \\ \omega(x,0) = \omega_0(x), \ \ \omega_t(x,0) = \omega_1(x), \ \ x \in\,]0,1[\,, \\ \psi_t(x,t-\tau) = f_0(x,t-\tau) \ \text{ in } \ ]0,1[\,\times\,]0,\tau[\,, \\ z(x,\rho,0) = f_0(x,-\rho\tau) \ \text{ on|; } \ ]0,1[\,\times\,]0,1[\,. \end{cases} \tag{2.3}$$

Let ξ_1 be a positive constant such that

$$\tau_1\,\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi_1 < \tau_1\,\frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}\,. \tag{2.4}$$

Let ξ_2 be a positive constant such that

$$\tau_2\,\frac{\widetilde{\mu}_2(1-\widetilde{\alpha}_1)}{\widetilde{\alpha}_1} < \xi_2 < \tau_2\,\frac{\widetilde{\mu}_1 - \widetilde{\alpha}_2\widetilde{\mu}_2}{\widetilde{\alpha}_2}\,.$$

Let ξ_3 be a positive constant such that

$$\tau_3 \frac{\tilde{\tilde{\mu}}_2(1-\tilde{\tilde{\alpha}}_1)}{\tilde{\tilde{\alpha}}_1} < \xi_3 < \tau_3 \frac{\tilde{\tilde{\mu}}_1 - \tilde{\tilde{\alpha}}_2\tilde{\tilde{\mu}}_2}{\tilde{\tilde{\alpha}}_2}.$$

We define the energy associated to the solution of problem (2.3) by the following formula:

$$\begin{aligned} E(t) = \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x+\psi+l\omega\|_2^2 + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2 \\ + \frac{1}{2}\left(b - \int_0^t h(s)\,ds\right)\int_0^t \psi_x^2\,dx + \frac{1}{2}(ho\psi)(t) + \xi\sum_{i=1}^{3}\int_0^1\int_0^1 G_i(z(x,\rho,t))\,d\rho\,dx. \quad (2.5) \end{aligned}$$

We have the following

Theorem 2.1. *Let* $(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in (H^2(0,1)\cap H_0^1(0,1)) \times H_0^1(0,1)$, $f_0 \in H_0^1((0,1);H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(\cdot,0) = \psi_1.$$

Assume that the hypotheses (H1), (H2) *and* (H3) *hold. Then problem* (P) *admits a unique weak solution*

$$\begin{gathered} \varphi,\omega \in L^\infty_{loc}(-\tau,\infty;H^2(0,1)\cap H_0^1(0,1)), \quad \varphi_t,\omega_t \in L^\infty_{loc}(-\tau,\infty;H_0^1(0,1)), \\ \varphi_{tt},\omega_{tt} \in L^\infty_{loc}(-\tau,\infty;L^2(0,1)), \\ \psi \in H^2(-\tau,0;H_0^1(0,1))\cap L^\infty_{loc}(-\tau,\infty;H^2(0,1)\cap H_0^1(0,1)), \\ \psi_t \in H^1(-\tau,0;H_0^1(0,1))\cap L^\infty_{loc}(-\tau,\infty;H^1(0,1)), \quad \psi_{tt} \in L^2(-\tau,0;H_0^1(0,1))\cap L^\infty_{loc}((0,\infty;L^2(0,1)) \end{gathered}$$

and for some constants ω_1, ω_2 *and* ω_3, ϵ_0, *we obtain the following decay property:*

$$E(t) \le \omega_1 H_1^{-1}(\omega_2 t + \omega_3) \ \ \forall t > 0, \quad (2.6)$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,ds,$$

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\epsilon'], \\ tH'(\epsilon_0 t) & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on }]0,\epsilon']. \end{cases}$$

Remark 2.1.

(1) • By the mean value Theorem, for the integrals and monotonicity of g_2, we find that

$$G_2(s) = \int_0^s g_2(r)dr \le s g_2(s).$$

Then $\alpha_1 \le \alpha_2 \le 1$.

• We need condition (2.1) only to prove the global existence, so if we study the energy decay, we can replace the linear growth order of the function $g_2(s)$ for large $|s|$ by nonlinear polynomial growth.

(2) The conditions of (H2) (with $\epsilon_1 = 1$) were introduced and employed by Lasiecka et al. [3,18] in their study of the asymptotic behavior of solutions of nonlinear wave equations where they obtained decay estimates which depend on the solution $S(t)$ of an explicit the nonlinear ordinary differential equation

$$S_t + r(S) = 0, \quad S(0) = E(0) = S_0,$$

where $r(s)$ is a continuous, monotone increasing function which behaves as H(s) at the origin. We mention here that Alabau–Boussouira [20] was the first who gave an explicit expression to the function r at the origin.

(3) We need condition (H3) only to establish a general decay estimate for the solutions of systems. Note that, in general, the condition of equality of the speeds of wave propagation does not have a physical sense, but from the mathematics point of view it is very important because the system is weakly hyperbolic of constant multiplicity of order 2. We think that the interaction of the hyperbolicity (order of multiplicity) and the number of dissipative terms affect the result.

Example. Let g be given by $g_1(s) = s^p(-\ln s)^q$, where $p \geq 1$ and $q \in \mathbb{R}$ on $(0, \epsilon_1]$. Then $g_1'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$ which is an increasing function in a right neighborhood of 0 (if $q = 0$, we can take $\epsilon_1 = 1$). The function H is defined in the neighborhood of 0 by

$$H(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{q}{p}} & \text{if } 0 < p < 1,\ \ q \in \mathbb{R}, \\ cs(-\ln s)^{-q} & \text{if } p = 1,\ \ q > 0, \\ c\sqrt{s}e^{-s^{\frac{1}{2q}}} & \text{if } p = 0,\ \ q < 0, \end{cases}$$

and we have

$$H'(s) = \begin{cases} cs^{\frac{1-p}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\dfrac{p+1}{2p}(-\ln s) + \dfrac{q}{p}\right) & \text{if } 0 < p \leq 1,\ \ q \in \mathbb{R},\ \ \text{when } s \text{ is near } 0, \\ c\,\dfrac{1}{\sqrt{s}}\left(1 - \dfrac{1}{q}\,s^{\frac{1}{2q}}\right)e^{-s^{\frac{1}{2q}}} & \text{if } p = 0,\ \ q < 0. \end{cases}$$

Thus

$$\varphi(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\dfrac{p+1}{2p}(-\ln s) + \dfrac{q}{p}\right) & \text{if } 0 < p \leq 1,\ \ q \in \mathbb{R},\ \ \text{when } s \text{ is near } 0, \\ c\,\dfrac{1}{\sqrt{s}}\left(1 - \dfrac{1}{q}\,s^{\frac{1}{2q}}\right)e^{-s^{\frac{1}{2q}}} & \text{if } p = 0,\ \ q < 0, \end{cases}$$

$$\psi(t) = c\int_t^1 \frac{1}{s^{\frac{p+1}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right)}\,ds$$

$$= c\int_1^{\frac{1}{t}} \frac{z^{\frac{1-3p}{2p}}}{(\ln z)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(\ln z) + \frac{q}{p}\right)}\,dz \ \ \text{when } t \text{ is near } 0.$$

and

$$\psi'(t) = c\int_t^1 \frac{1}{\frac{1}{\sqrt{s}}(1 - \frac{1}{q}\,s^{\frac{1}{2q}})e^{-s^{\frac{1}{2q}}}}\,ds = c\int_1^{\frac{1}{t}} \frac{e^{(\frac{1}{z})^{\frac{1}{2q}}}}{z^{\frac{3}{2}}(1 - \frac{1}{q}\,(\frac{1}{z})^{\frac{1}{2q}})}\,dz,\ \ p = 0,\ \ q < 0,\ \ \text{when } t \text{ is near } 0.$$

In a neighborhood of 0, we obtain

$$\psi(t) = \begin{cases} ct^{\frac{p-1}{2q}}(-\ln t)^{\frac{q}{p}} & \text{if } 0 < p < 1,\ \ q \in \mathbb{R}, \\ c(-\ln t)^{1+q} & \text{if } p = 1,\ \ q > 0, \\ ct^{\frac{q-2}{2q}}e^{t^{\frac{1}{2q}}} & \text{if } p = 0,\ \ q < 0, \end{cases}$$

and then in a neighborhood of $+\infty$,

$$\psi'(t) = \begin{cases} ct^{\frac{2p}{p-1}}(\ln t)^{-\frac{2p}{p-1}} & \text{if } 0 < p \leq 1,\ \ q \in \mathbb{R}, \\ ce^{-t^{\frac{1}{1+q}}} & \text{if } p = 1,\ \ q > 0, \\ c(\ln t)^{2q} & \text{if } p = 0,\ \ q < 0. \end{cases}$$

Using the fact that $h(t) = t$ as t tends to infinity, we can estimate

$$E(t) \leq \begin{cases} c\widetilde{\xi}(t)^{-\frac{2}{p-1}}(\ln \widetilde{\xi}(t))^{-\frac{2q}{p-1}} & \text{if } \ 0 < p \leq 1, \ \ q \in \mathbb{R}, \\ ce^{-\widetilde{\xi}(t)^{\frac{1}{1+q}}} & \text{if } \ p = 1, \ \ q < 1, \\ c(\ln \widetilde{\xi}(t))^{2q} & \text{if } \ p = 0, \ \ q < 0, \\ ce^{-e^{\widetilde{\xi}(t)}} & \text{if } \ p > 1 \ \text{ or } \ p = 1 \ \text{ and } \ q \leq 0, \end{cases}$$

where

$$\widetilde{\xi}(t) = \int_0^t \zeta(s)\, ds.$$

Proof of Theorem 2.1. We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 2.2. *Let (φ, ψ, z) be a solution of problem (2.3). Then the energy functional defined by (2.5) satisfies*

$$\begin{aligned} E'(t) \leq & -\Big(\mu_1 - \frac{\xi_1\alpha_2}{\tau_1} - \mu_2\alpha_2\Big)\int_0^1 \varphi_t g_1(\psi_t) - \Big(\frac{\xi_1}{\tau_1}\alpha_1 - \mu_2(1-\alpha_1)\Big)\int_0^1 z_1(x,1,t)g_2(z(x,1,t))\,dx \\ & - \Big(\widetilde{\mu}_1 - \frac{\xi_2\widetilde{\alpha}_2}{\tau_2} - \widetilde{\mu}_2\widetilde{\alpha}_2\Big)\int_0^1 \psi_t \widetilde{g}_1(\psi_t) - \Big(\frac{\xi_2}{\tau_2}\widetilde{\alpha}_1 - \widetilde{\mu}_2(1-\widetilde{\alpha}_1)\Big)\int_0^1 z_2(x,1,t)\widetilde{g}_2(z(x,1,t))\,dx \\ & - \Big(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3\widetilde{\widetilde{\alpha}}_2}{\tau_3} - \widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{\alpha}}_2\Big)\int_0^1 \omega_t \widetilde{\widetilde{g}}_1(\omega_t)\,dx - \Big(\frac{\xi_3}{\tau_3}\widetilde{\widetilde{\alpha}}_1 - \widetilde{\widetilde{\mu}}_2(1-\widetilde{\widetilde{\alpha}}_1)\Big)\int_0^1 z_3(x,1,t)\widetilde{\widetilde{g}}_2(z(x,1,t))\,dx \\ & - \frac{1}{2}\,h(t)\|\psi_x(x,t)\|_2^2 + \frac{1}{2}\,(h' o\psi_x)(t) \\ \leq\ & 0. \end{aligned}$$

Proof. Multiplying the first equation in (2.3) by φ_t, the second equation by ψ_t, the third equation by ω and integrating over $(0, 1)$ and using integration by parts, we get

$$\begin{aligned} & \frac{1}{2}\Big(\frac{\rho_1}{2}\,\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\,\|\psi_t\|_2^2 + \frac{\rho_1}{2}\,\|\omega_t\|_2^2 + \frac{EI}{2}\,\|\psi_x\|_2^2 + \frac{Gh}{2}\,\|\varphi_x + \psi + l\omega\|_2^2 + \frac{Eh}{2}\,\|\omega_x - l\varphi\|_2^2\Big) \\ & = -\mu_1\int_0^1 \varphi_t g_1(\varphi_t)\,dx - \mu_2\int_0^1 \varphi_t(x,t)g_2(z(x,1,t))\,dx - \widetilde{\mu}_1\int_0^1 \psi_t\widetilde{g}_1(\psi_t)\,dx - \widetilde{\mu}_2\int_0^1 \psi_t(x,t)\widetilde{g}_2(z(x,1,t))\,dx \\ & \quad - \widetilde{\widetilde{\mu}}_1\int_0^1 \omega_t\widetilde{\widetilde{g}}_1(\omega_t)\,dx - \widetilde{\widetilde{\mu}}_2\int_0^1 \omega_t(x,t)\widetilde{\widetilde{g}}_2(z(x,1,t))\,dx + \int_\Omega\int_0^t h(h-s)\psi_x(x,s)\psi_{xt}(x,t)\,ds\,dx. \end{aligned} \tag{2.7}$$

The term in the right-hand side of (2.7) can be rewritten as follows:

$$\begin{aligned} & \int_\Omega\int_0^t h(h-s)\psi_x(x,s)\psi_{xt}(x,t)\,ds\,dx + \frac{1}{2}\,h(t)\|\psi_x(x,t)\|_2^2 \\ & \qquad = \frac{1}{2}\,\frac{d}{dt}\Big[\int_0^t h(s)\,ds\|\psi_x(x,t)\|_2^2 - (ho\psi_x)(t)\Big] + \frac{1}{2}\,(h'o\psi_x)(t). \end{aligned}$$

Consequently, identity (2.7) becomes

$$\frac{1}{2}\Big(\frac{\rho_1}{2}\|\varphi_t\|_2^2+\frac{\rho_2}{2}\|\psi_t\|_2^2+\frac{\rho_1}{2}\|\omega_t\|_2^2+\frac{EI}{2}\|\psi_x\|_2^2+\frac{Gh}{2}\|\varphi_x+\psi+l\omega\|_2^2+\frac{Eh}{2}\|\omega_x-l\varphi\|_2^2\Big)$$
$$=-\mu_1\int_0^1\varphi_t g_1(\varphi_t)\,dx-\mu_2\int_0^1\varphi_t(x,t)g_2(z(x,1,t))\,dx-\widetilde{\mu}_1\int_0^1\psi_t\widetilde{g}_1(\psi_t)\,dx$$
$$-\widetilde{\mu}_2\int_0^1\psi_t(x,t)\widetilde{g}_2(z(x,1,t))\,dx-\widetilde{\widetilde{\mu}}_1\int_0^1\omega_t\widetilde{\widetilde{g}}_1(\omega_t)\,dx-\widetilde{\widetilde{\mu}}_2\int_0^1\omega_t(x,t)\widetilde{\widetilde{g}}_2(z(x,1,t))\,dx$$
$$-\frac{1}{2}h(t)\|\psi_x(x,t)\|_2^2+\frac{1}{2}(h'o\psi_x)(t)-\frac{1}{2}\Big(b-\int_0^t h(s)\,ds\Big)\int_0^1\psi_x^2\,dx.$$

Multiplying the third equation in (2.3) by $\xi_1 g_2(z(x,\rho,t))$ and integrating the result over $(0,1)\times(0,1)$, we obtain

$$\xi_1\int_0^1\int_0^1 z_1' g_2(z(x,\rho,t))\,d\rho\,dx$$
$$=-\frac{\xi_1}{\tau_1}\int_0^1\int_0^1\frac{\partial}{\partial\rho}G_2(z(x,\rho,t))\,d\rho\,dx=-\frac{\xi_1}{\tau_1}\int_0^1\big(G_2(z(x,1,t))-G_2(z(x,0,t))\big)\,dx.$$

Then

$$\xi_1\frac{d}{dt}\int_0^1\int_0^1 G_2(z_1(x,\rho,t))\,d\rho\,dx=-\frac{\xi_1}{\tau_1}\int_0^1 G_2(z_1(x,1,t))\,dx+\frac{\xi_1}{\tau_1}\int_0^1 G_2(\varphi_t)\,dx,\tag{2.8}$$

$$\xi_2\frac{d}{dt}\int_0^1\int_0^1 \widetilde{G}_2(z_2(x,\rho,t))\,d\rho\,dx=-\frac{\xi_2}{\tau_2}\int_0^1 \widetilde{G}_2(z_2(x,1,t))\,dx+\frac{\xi_2}{\tau_2}\int_0^1 \widetilde{G}_2(\psi_t)\,dx,\tag{2.9}$$

$$\xi_3\frac{d}{dt}\int_0^1\int_0^1 \widetilde{\widetilde{G}}_2(z_3(x,\rho,t))\,d\rho\,dx=-\frac{\xi_3}{\tau_3}\int_0^1 \widetilde{\widetilde{G}}_2(z_3(x,1,t))\,dx+\frac{\xi_3}{\tau_3}\int_0^1 \widetilde{\widetilde{G}}_2(\omega_t)\,dx.\tag{2.10}$$

From (2.7), (2.8), (2.9), (2.10), and using Young's inequality, we get

$$E'(t)=-\Big(\mu_1-\frac{\xi_1\alpha_2}{\tau_1}\Big)\int_0^1\varphi_t g_1(\varphi_t)\,dx-\frac{\xi_1}{\tau_1}\int_0^1 G_2(z_1(x,1,t))\,dx-\mu_2\int_0^1\varphi_t(t)g_2(z_1(x,1,t))\,dx$$
$$-\Big(\widetilde{\mu}_1-\frac{\xi_2\widetilde{\alpha}_2}{\tau_2}\Big)\int_0^1\psi_t\widetilde{g}_1(\psi_t)\,dx-\frac{\xi_2}{\tau_2}\int_0^1\widetilde{G}_2(z_2(x,1,t))\,dx-\widetilde{\mu}_2\int_0^1\psi_t(t)\widetilde{g}_2(z_2(x,1,t))\,dx$$
$$-\Big(\widetilde{\widetilde{\mu}}_1-\frac{\xi_3\widetilde{\widetilde{\alpha}}_2}{\tau_3}\Big)\int_0^1\omega_t\widetilde{\widetilde{g}}_1(\omega_t)\,dx-\frac{\xi_3}{\tau_3}\int_0^1\widetilde{\widetilde{G}}_2(z_3(x,1,t))\,dx-\widetilde{\widetilde{\mu}}_2\int_0^1\omega_t(t)\widetilde{\widetilde{g}}_2(z_3(x,1,t))\,dx$$
$$-\frac{1}{2}h(t)\|\psi_x(x,t)\|_2^2+\frac{1}{2}(h'o\psi_x)(t).\tag{2.11}$$

Let us denote G_2^* to be the conjugate function of the convex function G_2, i.e., $G_2^*(s)=\sup_{t\in\mathbb{R}^+}(st-G_2(t))$. Then G_2^* is the Legendre transform of G_2, which is given (see Arnold [3, pp. 61–62], and Lasiecka [8]) by

$$G_2^*(s)=s(G_2')^{-1}(s)-G_2[(G_2')^{-1}(s)]\ \ \forall\, s\geq 0$$

and satisfies the following inequality:

$$st \leq G_2^*(s) + G_2(t) \ \ \forall\, s,t \geq 0. \tag{2.12}$$

Next, from the definition of G_2, we get

$$G_2^*(s) = s g_2^{-1}(s) - G_2(g_2^{-1}(s)).$$

Hence

$$G_2^*(g_2(z(x,1,t))) = z(x,1,t) g_2(z(x,1,t)) - G_2(z(x,1,t)) \leq (1-\alpha_1) z(x,1,t) g_2(z(x,1,t)). \tag{2.13}$$

Making use of (2.11), (2.12) and (2.13), we have

$$\begin{aligned}
E'(t) \leq &-\Big(\mu_1 - \frac{\xi_1 \alpha_2}{\tau_1}\Big) \int_0^1 \varphi_t g_1(\varphi_t)\,dx - \frac{\xi_1}{\tau_1} \int_0^1 G_2(z_1(x,1,t))\,dx \\
&+ \mu_2 \int_0^1 \big(G_2(\varphi_t) + G_2^*(g_2(z_1(x,1,t)))\big)\,dx - \Big(\widetilde{\mu}_1 - \frac{\xi_2 \widetilde{\alpha}_2}{\tau_2}\Big) \int_0^1 \psi_t \widetilde{g}_1(\psi_t)\,dx \\
&- \frac{\xi_2}{\tau_2} \int_0^1 \widetilde{G}_2(z_2(x,1,t))\,dx + \widetilde{\mu}_2 \int_0^1 \big(\widetilde{G}_2(\psi_t) + \widetilde{G}_2^*(\widetilde{g}_2(z_2(x,1,t)))\big)\,dx \\
&- \Big(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3 \widetilde{\widetilde{\alpha}}_2}{\tau_3}\Big) \int_0^1 \omega_t \widetilde{\widetilde{g}}_1(\omega_t)\,dx - \frac{\xi_3}{\tau_3} \int_0^1 \widetilde{\widetilde{G}}_2(z_3(x,1,t))\,dx \\
&+ \widetilde{\widetilde{\mu}}_2 \int_0^1 \big(\widetilde{\widetilde{G}}_2(\omega_t) + \widetilde{\widetilde{G}}_2^*(\widetilde{\widetilde{g}}_2(z_3(x,1,t)))\big)\,dx \\
\leq &-\Big(\mu_1 - \frac{\xi_1 \alpha_2}{\tau_1} - \mu_2 \alpha_2\Big) \int_0^1 \varphi_t g_1(\varphi_t)\,dx - \frac{\xi_1}{\tau_1} \int_0^1 G_2(z_1(x,1,t))\,dx \\
&+ \mu_2 \int_0^1 G_2^*(g_2(z_1(x,1,t)))\,dx - \Big(\widetilde{\mu}_1 - \frac{\xi_2 \widetilde{\alpha}_2}{\tau_2} - \widetilde{\mu}_2 \widetilde{\alpha}_2\Big) \int_0^1 \psi_t \widetilde{g}_1(\psi_t)\,dx \\
&- \frac{\xi_2}{\tau_2} \int_0^1 \widetilde{G}_2(z_2(x,1,t))\,dx + \widetilde{\mu}_2 \int_0^1 \widetilde{G}_2^*(\widetilde{g}_2(z_2(x,1,t)))\,dx \\
&- \Big(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3 \widetilde{\widetilde{\alpha}}_2}{\tau_3} - \widetilde{\widetilde{\mu}}_2 \widetilde{\widetilde{\alpha}}_2\Big) \int_0^1 \omega_t \widetilde{\widetilde{g}}_1(\omega_t)\,dx - \frac{\xi_3}{\tau_3} \int_0^1 \widetilde{\widetilde{G}}_2(z_3(x,1,t))\,dx \\
&+ \widetilde{\widetilde{\mu}}_2 \int_0^1 \widetilde{\widetilde{G}}_2^*(\widetilde{\widetilde{g}}_2(z_3(x,1,t)))\,dx - \frac{1}{2}\, h(t) \|\psi_x(x,t)\|_2^2 + \frac{1}{2}\,(h' o \psi_x)(t).
\end{aligned}$$

Using (2.2) and (2.4), we obtain

$$\begin{aligned}
E'(t) \leq &-\Big(\mu_1 - \frac{\xi_1 \alpha_2}{\tau_1} - \mu_2 \alpha_2\Big) \int_0^1 \varphi_t g_1(\varphi_t)\,dx - \Big(\frac{\xi_1}{\tau_1}\,\alpha_1 - \mu_2(1-\alpha_1)\Big) \int_0^1 z_1(x,1,t) g_2(z_1(x,1,t))\,dx \\
&- \Big(\widetilde{\mu}_1 - \frac{\xi_2 \widetilde{\alpha}_2}{\tau_2} - \widetilde{\mu}_2 \widetilde{\alpha}_2\Big) \int_0^1 \psi_t \widetilde{g}_1(\psi_t)\,dx - \Big(\frac{\xi_2}{\tau_2}\,\widetilde{\alpha}_1 - \widetilde{\mu}_2(1-\widetilde{\alpha}_1)\Big) \int_0^1 z_2(x,1,t) \widetilde{g}_2(z_2(x,1,t))\,dx
\end{aligned}$$

$$
\begin{aligned}
&- \Big(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3 \widetilde{\widetilde{\alpha}}_2}{\tau_3} - \widetilde{\widetilde{\mu}}_2 \widetilde{\widetilde{\alpha}}_2\Big) \int_0^1 \omega_t \widetilde{\widetilde{g}}_1(\omega_t)\,dx - \Big(\frac{\xi_3}{\tau_3}\,\widetilde{\alpha}_1 - \widetilde{\mu}_2(1-\widetilde{\widetilde{\alpha}}_1)\Big) \int_0^1 z_3(x,1,t)\widetilde{\widetilde{g}}_2(z_3(x,1,t))\,dx \\
&- \frac{1}{2}\,h(t)\|\psi_x(x,t)\|_2^2 + \frac{1}{2}\,(h' o \psi_x)(t) \\
\le{}& 0. \qquad \square
\end{aligned}
$$

3 Global existence

We are now ready to prove Theorem 2.1 in the next two sections. Throughout this section, we assume that $\varphi_0, \psi_0 \in H^2 \cap H_0^1(0,1)$, $\varphi_1, \psi_1 \in H_0^1(0,1)$ and $f_0 \in H_0^1((0,1); H^1(0,1))$. We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$, where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2 \cap H_0^1$. Now, we define for $1 \le j \le k$ the sequence $\phi_j(x,\rho)$ as follows: $\phi_j(x,0) = w_j$. Then we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2((0,1)\times(0,1))$ and denote by Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$. We construct approximate solutions $(\varphi_k, \psi_k, \omega_k, z_{1k}, z_{2k}, z_{3k})$, $k = 1,2,3,\ldots$, in the form

$$
\begin{aligned}
\varphi_k(t) &= \sum_{j=1}^k g_{jk} w_j, \quad & z_1k(t) &= \sum_{j=1}^k h_{jk}\phi_j, \\
\psi_k(t) &= \sum_{j=1}^k \widetilde{g}_{jk} w_j, \quad & z_2k(t) &= \sum_{j=1}^k \widetilde{h}_{jk}\phi_j, \\
\omega_k(t) &= \sum_{j=1}^k \widetilde{\widetilde{g}}_{jk} w_j, \quad & z_3k(t) &= \sum_{j=1}^k \widetilde{\widetilde{h}}_{jk}\phi_j,
\end{aligned}
$$

where g_{jk}, $\widetilde{g}_{jk}$, $\widetilde{\widetilde{g}}_{jk}$, h_{jk}, $\widetilde{h}_{jk}$ and $\widetilde{\widetilde{h}}_{jk}$, $j = 1,2,\ldots,k$, are determined by the following ordinary differential equations:

$$
\begin{cases}
\rho_1(\varphi_k''(t), w_j) + Gh(\varphi_{kx}(t), w_{jx}) + Gh(\psi_k + l\omega_k)(t), w_{jx}) - lEh(\omega_{kx} - l\varphi_k)(t), w_j) \\
\qquad + \mu_1(g_1(\varphi_k'), w_j) + \mu_2(g_2(z_{1k}(\cdot,1)), w_j) = 0, \quad 1 \le j \le k, \\
z_{1k}(x,0,t) = \varphi_k'(x,t),
\end{cases} \tag{3.1}
$$

$$
\varphi_k(0) = \varphi_{0k} = \sum_{j=1}^k (\varphi_0, w_j) w_j \to \varphi_0 \ \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.2}
$$

$$
\varphi_k'(0) = \varphi_{1k} = \sum_{j=1}^k (\varphi_1, w_j) w_j \to \varphi_1 \ \text{ in } H_0^1 \text{ as } k \to +\infty; \tag{3.3}
$$

$$
\begin{cases}
\rho_2(\psi_k''(t), w_j) + El(\psi_{kx}(t), w_{jx}) + Gh((\varphi_{kx} + \psi + l\omega)(t), w_j) + \widetilde{\mu}_1(\widetilde{g}_1(\psi_k'), w_j) \\
\qquad + \widetilde{\mu}_2(\widetilde{g}_2(z_{2k}(\,\cdot\,,1)), w_j) + \displaystyle\int_0^t h(t-s)(\psi_{xx}(x,s), w_j)\,ds = 0, \quad 1 \le j \le k, \\
z_{2k}(x,0,t) = \psi_k'(x,t),
\end{cases} \tag{3.4}
$$

$$
\psi_k(0) = \psi_{0k} = \sum_{j=1}^k (\psi_0, w_j) w_j \to \psi_0 \ \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.5}
$$

$$
\psi_k'(0) = \psi_{1k} = \sum_{j=1}^k (\psi_1, w_j) w_j \to \psi_1 \ \text{ in } H_0^1 \text{ as } k \to +\infty; \tag{3.6}
$$

$$\begin{cases} \rho_1(\omega_k''(t), w_j) + Eh(\omega_{kx}(t), w_{jx}) + lEh(\varphi_{kx})(t), w_j) + lGh(\varphi_{kx} + \psi_k + l\omega_k)(t)w_j \\ \qquad + \widetilde{\widetilde{\mu}}_1(\widetilde{\widetilde{g}}_1(\omega_k'), w_j) + \widetilde{\widetilde{\mu_2}}(\widetilde{\widetilde{g}}_2(z_{3k}(\cdot,1)), w_j) = 0, 1, \quad \leq j \leq k, \\ z_{3k}(x,0,t) = \omega_k'(x,t), \end{cases} \tag{3.7}$$

$$\omega_k(0) = \omega_{0k} = \sum_{j=1}^{k} (\omega_0, w_j)w_j \to \omega_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{3.8}$$

$$\omega_k'(0) = \omega_{1k} = \sum_{j=1}^{k} (\omega_1, w_j)w_j \to \omega_1 \text{ in } H_0^1 \text{ as } k \to +\infty; \tag{3.9}$$

$$(\tau z_1 kt + z_1 k\rho, \phi_j) = 0, \quad 1 \leq j \leq k, \tag{3.10}$$

$$z_1 k(\rho, 0) = z_{01k} = \sum_{j=1}^{k} (f_1, \phi_j)\phi_j \to f_1 \text{in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty; \tag{3.11}$$

$$(\tau z_2 kt + z_2 k\rho, \phi_j) = 0, \quad 1 \leq j \leq k, \tag{3.12}$$

$$z_{2k}(\rho, 0) = z_{02k} = \sum_{j=1}^{k} (f_2, \phi_j)\phi_j \to f_2 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty; \tag{3.13}$$

$$(\tau z_{3kt} + z_3 k\rho, \phi_j) = 0, \quad 1 \leq j \leq k, \tag{3.14}$$

$$z_{3k}(\rho, 0) = z_{03k} = \sum_{j=1}^{k} (f_3, \phi_j)\phi_j \to f_3 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{3.15}$$

By virtue of the theory of ordinary differential equations, system (3.1)–(3.15) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma, since the nonlinear terms in (3.1), (3.4), (3.7) are locally Lipschitz continuous. Note that $(\varphi_k(t), \psi_k(t), \omega_k(t))$ is from the class C^2.

As the next step, we obtain a priori estimates for the solution such that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

We can use a standard compactness argument for the limiting procedure, and it suffices to derive some a priori estimates for $(\varphi_k, \psi_k, \omega_k, z_{1k}, z_{2k}, z_{3k})$.

The first estimate

Since the sequences φ_{0k}, φ_{1k}, ψ_{0k}, ψ_{1k}, ω_{0k}, ω_{1k} and z_{0k} converge, the standard calculations, using (3.1)–(3.15), similar to those used to derive (3.16), yield C independent of k such that

$$\begin{aligned} &E_k(t) + a_1 \int_0^t \int_0^1 \varphi_k' g_1(\varphi_k)\, dx\, ds + a_2 \int_0^t \int_0^1 z_{1k}(x,1,t) g_2(z_{1k}(x,1,t))\, ds\, dx \\ &\quad + b_1 \int_0^t \int_0^1 \psi_k' \widetilde{g}_1(\psi_k')\, dx\, ds + b_2 \int_0^t \int_0^1 z_{2k}(x,1,t)\widetilde{g}_2(z_{2k}(x,1,t))\, dx\, ds + c_1 \int_0^t \int_0^1 \omega_k' \widetilde{\widetilde{g}}_1(\omega_k')\, dx\, ds \\ &\quad + c_2 \int_0^t \int_0^1 z_{3k}(x,1,t)\widetilde{\widetilde{g}}_2(z_{3k}(x,1,t))\, dx\, ds + \frac{1}{2} h(t)\|\psi_x(x,t)\|_2^2 - \frac{1}{2}(h' o \psi_x)(t) \leq E_k(0) \leq C, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} E_k(t) = \frac{1}{2}\Big(&\frac{\rho_1}{2}\|\varphi_k'\|_2^2 + \frac{\rho_2}{2}\|\psi_k'\|_2^2 + \frac{\rho_1}{2}\|\omega_k'\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x + \psi + l\omega\|_2^2 + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2\Big) \\ &+ \xi_1 \int_0^1 \int_0^1 G_2(z_{1k}(x,\rho,t))\, d\rho\, dx + \xi_2 \int_0^1 \int_0^1 \widetilde{G}_2(z_{2k}(x,\rho,t))\, d\rho\, dx \end{aligned}$$

$$+\xi_3 \int_0^1 \int_0^1 \widetilde{\widetilde{G}}_2(z_{3k}(x,\rho,t))\,d\rho\,dx + \frac{1}{2}\left(b - \int_0^t h(s)\,ds\right)\int_0^1 \psi_x^2\,dx + \frac{1}{2}(h' o \psi_x)(t),$$

$$a_1 = \left(\mu_1 - \frac{\xi_1\alpha_2}{\tau_1} - \mu_2\alpha_2\right), \quad a_2 = \left(\frac{\xi_1}{\tau_1}\alpha_1 - \mu_2(1-\alpha_1)\right),$$

$$b_1 = \left(\widetilde{\mu}_1 - \frac{\xi_2\widetilde{\alpha}_2}{\tau_2} - \widetilde{\mu}_2\widetilde{\alpha}_2\right), \quad b_2 = \left(\frac{\xi_2}{\tau_2}\widetilde{\alpha}_1 - \widetilde{\mu}_2(1-\widetilde{\alpha}_1)\right),$$

$$c_1 = \left(\widetilde{\widetilde{\mu}}_1 - \frac{\xi_3\widetilde{\widetilde{\alpha}}_2}{\tau_3} - \widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{\alpha}}_2\right), \quad c_2 = \left(\frac{\xi_3}{\tau_3}\widetilde{\widetilde{\alpha}}_1 - \widetilde{\widetilde{\mu}}_2(1-\widetilde{\widetilde{\alpha}}_1)\right),$$

for some C independent of k. These estimates imply that the solution $(\varphi_k, \psi_k, \omega_k, z_{1k}, z_{2k}, z_{3k})$ exists globally in $[0, +\infty[$. Estimate (3.16) yields:

$$\varphi_k,\ \psi_k,\ \omega_k \text{ are bounded in } L^\infty_{loc}(0,\infty; H_0^1(0,1)), \tag{3.17}$$

$$\varphi_k',\ \psi_k',\ \omega_k' \text{ are bounded in } L^\infty_{loc}(0,\infty; L^2(0,1)), \tag{3.18}$$

$$\varphi_k' g_1(\varphi_k'),\ \psi_k'\widetilde{g}_1(\psi_k'),\ \omega_k'\widetilde{\widetilde{g}}_1(\omega_k') \text{ are bounded in } L^1((0,1)\times(0,\infty)), \tag{3.19}$$

$$G_{12}(z_{1k}(x,\rho,t)),\ G_{22}(z_{2k}(x,\rho,t)),\ G_{32}(z_{3k}(x,\rho,t)) \text{ are bounded in } L^\infty_{loc}(0,\infty; L^1((0,1)\times(0,1))), \tag{3.20}$$

$$z_{1k}(x,1,t)g_2(z_k(x,1,t)),\ z_{2k}(x,1,t)\widetilde{g}_2(z_{2k}(x,1,t)),\ z_{3k}(x,1,t)\widetilde{\widetilde{g}}_2(z_{3k}(x,1,t)) \text{ are bounded in } L^1((0,1)\times(0,T)). \tag{3.21}$$

The second estimate

First, we estimate $\varphi_k''(0)$, $\psi_k''(0)$ and $\omega_k''(0)$. Testing (3.1) by $g_{jk}''(t)$, (3.4) by $\widetilde{g}_{jk}''(t)$, (3.7) by $\widetilde{\widetilde{g}}_{jk}''(t)$ and choosing $t = 0$, we obtain

$$\rho_1\|\varphi_k''(0)\|_2 \le Gh\big(\|\varphi_{0kx}\|_2 + \|\psi_{0k}\|_2 + l\|\omega_{0k}\|_2\big) + lEh\big(\|\omega_{0k}\|_2 + l\|\varphi_{0k}\|_2\big) + \mu_1\|g_1(\varphi_{1k})\|_2 + \mu_2\|g_2(z_{10k})\|_2,$$

$$\rho_2\|\psi_k''(0)\|_2 \le El\|\psi_{0kxx}\|_2 + Gh\big(\|\varphi_{0k}\|_2 + \|\psi_{0k}\|_2 + l\|\omega_{0k}\|_2\big) + \widetilde{\mu}_1\|\widetilde{g}_1(\psi_{1k})\|_2 + \widetilde{\mu}_2\|\widetilde{g}_2(z_{20k})\|_2,$$

$$\rho_1\|\omega_k''(0)\|_2 \le Eh\big(\|\omega_{0kxx}\|_2 + lEh\|\varphi_{0k}\|_2\big) + lGh\big(\|\varphi_{0k}\|_2 + \|\psi_{0k}\|_2 + l\|\omega_{0k}\|_2\big) + \widetilde{\widetilde{\mu}}_1\|\widetilde{\widetilde{g}}_1(\omega_{1k})\|_2 + \widetilde{\widetilde{\mu}}_2\|\widetilde{\widetilde{g}}_2(z_{03k})\|_2.$$

Since $g_1(\varphi_{1k})$, $g_2(z_{10k})$, $\widetilde{g}_1(\psi_{1k})$, $\widetilde{g}_2(z_{20k})$, $\widetilde{\widetilde{g}}_1(\omega_{1k})$, $\widetilde{\widetilde{g}}_2(z_{30k})$ are bounded in $L^2(0,1)$ by (H1)

(3.2), (3.3) and (3.11) yield $\|\varphi_k''(0)\|_2 \le C$,

(3.5), (3.6) and (3.13) yield $\|\psi_k''(0)\|_2 \le C$,

(3.8), (3.9) and (3.15) yield $\|\omega_k''(0)\|_2 \le C$.

Differentiating (3.1), (3.4) and (3.7) with respect to t, we get

$$\rho_1\varphi_k'''(t) + Gh\varphi_{kxx}'(t) + Gh\psi_{kx}'(t) + lGh\omega_k'(t) + lEh\omega_k'(t) + l^2Eh\varphi_k'(t) + \mu_1\varphi_k''(g_1'(\varphi_k') + \mu_2 z_{1k}'(x,1,t)(g_2'(z_{1k}(x,1,t)) = 0, \tag{3.22}$$

$$\rho_2\psi_k'''(t) + El\psi_{kxx}'(t) + Gh\varphi_{kx}'(t) + Gh\psi_k'(t) + lGh\omega_k'(t) + \widetilde{\mu}_1\psi_k''\widetilde{g}_1'(\psi_k') + \widetilde{\mu}_2 z_{2k}'(x,1,t)\widetilde{g}_2'(z_{2k}(x,1,t)) + \frac{d}{dt}\left(\int_0^t h(t-s)\psi_{kxx}\,ds\right) = 0, \tag{3.23}$$

$$\rho_1\omega_k'''(t) + Eh\omega_{kxx}'(t) + lEh\varphi_{kx}'(t) + lGh\varphi_{kx}' + lGh\psi_k'(t) + l^2Gh\omega_k'(t) \tag{3.24}$$

$$+ \widetilde{\widetilde{\mu}}_1\omega_k''(t)\widetilde{\widetilde{g}}_1'(\omega_k') + \widetilde{\widetilde{\mu}}_2 z_{3k}'(x,1,t)\widetilde{\widetilde{g}}_2'(z_{3k}(x,1,t)) = 0. \tag{3.25}$$

Multiplying (3.22) by $g'_{jk}(t)$, (3.23) by $\widetilde{g}'_{jk}(t)$, and (3.25) by $\widetilde{\widetilde{g}}'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2 + Gh\|\varphi'_{kx}(t)\|_2^2\right)$$
$$+ Gh\int_0^1 (\psi'_{kx}(t) + l\omega'_k(t))\varphi''_{jk}(t)\,dx + lEh\int_0^1 (\omega'_k(t) + l\varphi'_k(t)\varphi''_{jk}(t)\,dx$$
$$+ \mu_1\int_0^1 \varphi''^2_k(t)g'_1(\varphi'_k)\,dx + \mu_2\int_0^1 \varphi''_k(t)z'_{1k}(x,1,t)(g'_2(z_{1k}(x,1,t))\,dx = 0, \tag{3.26}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi''_k(t)\|_2^2 + El\|\psi'_{kx}(t)\|_2^2\right) + Gh\int_0^1 \left(\varphi'_{kx}(t) + \psi'_k(t) + l\omega'_k(t)\right)\psi''_{jk}(t)\,dx$$
$$+ \widetilde{\mu}_1\int_0^1 \psi''^2_k(t)\widetilde{g}'_1(\psi'_k)\,dx + \widetilde{\mu}_2\int_0^1 \psi''_k(t)z'_{2k}(x,1,t)\widetilde{g}'_2(z_{2k}(x,1,t))\,dx = 0, \tag{3.27}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\omega''_k(t)\|_2^2 + Eh\|\omega'_{kx}(t)\|_2^2\right) + lEh\int_0^1 \omega''_k(t)\varphi'_{kx}(t)\,dx$$
$$+ Gh\int_0^1 \omega''_k(t)\big(l\varphi'_{kx} + l\psi'_k(t) + l\omega'_k(t)\big) + \widetilde{\widetilde{\mu}}_1\int_0^1 \omega''^2_k(t)\widetilde{\widetilde{g}}'_1(\omega'_k)\,dx$$
$$+ \widetilde{\widetilde{\mu}}_2\int_0^1 \omega''_k(t)z'_{3k}(x,1,t)\widetilde{\widetilde{g}}'_2(z_{3k}(x,1,t))\,dx = 0. \tag{3.28}$$

Differentiating (3.10), (3.12) and (3.14) with respect to t, we get

$$\left(\tau z''_{1k}(t) + \frac{\partial}{\partial\rho}z'_{1k}, \phi_j\right) = 0, \tag{3.29}$$
$$\left(\tau z''_{2k}(t) + \frac{\partial}{\partial\rho}z'_{2k}, \phi_j\right) = 0, \tag{3.30}$$
$$\left(\tau z''_{3k}(t) + \frac{\partial}{\partial\rho}z'_{3k}, \phi_j\right) = 0. \tag{3.31}$$

Multiplying (3.29), (3.30) and (3.31) by $h'_{jk}(t)$, summing over j from 1 to k, we find that

$$\frac{1}{2}\tau\frac{d}{dt}\|z'_{1k}(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z'_{1k}(t)\|_2^2 = 0, \tag{3.32}$$
$$\frac{1}{2}\tau\frac{d}{dt}\|z'_{2k}(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z'_{2k}(t)\|_2^2 = 0, \tag{3.33}$$
$$\frac{1}{2}\tau\frac{d}{dt}\|z'_{3k}(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z'_{3k}(t)\|_2^2 = 0. \tag{3.34}$$

Taking the sum of (3.26), (3.27), (3.28), (3.32), (3.33) and (3.34), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2 + \rho_2\|\psi''_k(t)\|_2^2 + \rho_1\|\omega''_k(t)\|_2^2 + Gh\|\varphi'_{kx}(t)\|_2^2 + El\|\psi'_{kx}(t)\|_2^2 + Eh\|\omega'_{kx}(t)\|_2^2\right)$$
$$+ \frac{1}{2}\frac{d}{dt}lEh\Big(\|\omega'_k(t) + l\varphi'_k(t)\|_2^2 + Gh\|\varphi'_{kx}(t) + \psi'_k(t) + l\omega'_k(t)\|_2^2\Big)$$
$$+ \frac{1}{2}\tau\frac{d}{dt}\left(\|z'_{1k}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))} + \|z'_{2k}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))} + \|z'_{3k}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\right)$$

$$+ \mu_1 \int_0^1 \varphi_k''^2(t) g_1'(\varphi_k')\,dx + \widetilde{\mu}_1 \int_0^1 \psi_k''^2(t)\widetilde{g}_1'(\psi_k')\,dx + \widetilde{\widetilde{\mu}}_1 \int_0^1 \omega_k''^2(t)\widetilde{\widetilde{g}}_1'(\omega_k')\,dx$$

$$+ \frac{1}{2}\int_0^1 \Big(|z_{1k}'(x,1,t)|^2 + |z_{2k}'(x,1,t)|^2 + |z_{3k}'(x,1,t)|^2\Big)\,dx + h(0)\|u_{kx}'(t)\|_2^2$$

$$= -\mu_2 \int_0^1 \varphi_k''(t) z_{1k}'(x,1,t) g_2'(z_{1k}(x,1,t))\,dx - \widetilde{\mu}_2 \int_0^1 \psi_k''(t) z_{2k}'(x,1,t)\widetilde{g}_2'(z_{2k}(x,1,t))\,dx$$

$$- \widetilde{\widetilde{\mu}}_2 \int_0^1 \omega_k''(t) z_{3k}'(x,1,t)\widetilde{\widetilde{g}}_2'(z_{3k}(x,1,t))\,dx + \frac{1}{2}\Big(\|\varphi_k''(t)\|_2^2 + \|\psi_k''(t)\|_2^2 + \|\omega_k''(t)\|_2^2\Big)$$

$$+ h(0)\,\frac{d}{dt}\,(\psi_{kx}(t), \psi_{kx}'(t)) + \frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(t), \psi_{kx}'(t))\,ds$$

$$- h'(0)(\psi_{kx}(t), \psi_{kx}'(t)) - \frac{1}{2}\int_0^t h''(t-s)(\psi_{kx}(t), \psi_{kx}'(t))\,ds.$$

Using (2.1), Cauchy–Schwarz and Young's inequalities, we obtain

$$\big|h'(0)(\psi_{kx}(t), \psi_{kx}'(t))\big| \le \epsilon\|\psi_{kx}(t)\|_2^2 + \frac{h'^2}{4\epsilon}\,\|\psi_{kx}'(t)\|_2^2,$$

$$\Big|\int_0^t h''(t-s)(\psi_{kx}(t), \psi_{kx}'(t))\,ds\Big| \le \|\psi_{kx}'(t)\|_2 \int_0^t |h''(t-s)|\,\|\psi_{kx}(s)\|_2\,ds$$

$$\le \frac{1}{4\epsilon}\,\|\psi_{kx}'(t)\|_2^2 + \epsilon\|h''\|_{L^1}\int_0^t |h''(t-s)|\,\|\psi_{kx}(s)\|_2^2\,ds,$$

$$\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi_k''(t)\|_2^2 + \rho_2\|\psi_k''(t)\|_2^2 + \rho_1\|\omega_k''(t)\|_2^2 + Gh\|\varphi_{kx}'(t)\|_2^2 + El\|\psi_{kx}'(t)\|_2^2 + Eh\|\omega_{kx}'(t)\|_2^2\Big)$$
$$+\frac{1}{2}\,lEh\frac{d}{dt}\Big(\|\omega_k'(t) + l\varphi_k'(t)\|_2^2 + Gh\|\varphi_{kx}'(t) + \psi_k'(t) + l\omega_k'(t)\|_2^2\Big)$$
$$+\frac{1}{2}\,\tau\,\frac{d}{dt}\Big(\|z_{1k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2 + \|z_{2k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2 + \|z_{3k}'(x,\rho,t)\|_{L^2((0,1)\times(0,1))}^2\Big)$$
$$+\mu_1 \int_0^1 \varphi_k''^2(t) g_1'(\varphi_k')\,dx + \widetilde{\mu}_1 \int_0^1 \psi_k''^2(t)\widetilde{g}_1'(\psi_k')\,dx + \widetilde{\widetilde{\mu}}_1 \int_0^1 \omega_k''^2(t)\widetilde{\widetilde{g}}_1'(\omega_k')\,dx$$
$$+c\int_0^1 \Big(|z_{1k}'(x,1,t)|^2 + |z_{2k}'(x,1,t)|^2 + |z_{3k}'(x,1,t)|^2\Big) + h(0)\|u_{kx}'(t)\|_2^2\,dx$$
$$\le c'\Big(\|\varphi_k''(t)\|_2^2 + \|\psi_k''(t)\|_2^2 + \|\omega_k''(t)\|_2^2\Big) + \epsilon\|\psi_{kx}(t)\|_2^2 + \frac{h'^2}{4\epsilon}\,\|\psi_{kx}'(t)\|_2^2 + \frac{1}{4\epsilon}\|\psi_{kx}'(t)\|_2^2$$
$$+\epsilon\|h''\|_{L^1}\int_0^t |h''(t-s)|\,\|\psi_{kx}(s)\|_2^2\,ds + h(0)\,\frac{d}{dt}\,(\psi_{kx}(t), \psi_{kx}'(t)) + \frac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(t), \psi_{kx}'(t))\,ds.$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we obtain

$$\rho_1\|\varphi_k''(t)\|_2^2 + \rho_2\|\psi_k''(t)\|_2^2 + \rho_1\|\omega_k''(t)\|_2^2 + Gh\|\varphi_{kx}'(t)\|_2^2 + El\|\psi_{kx}'(t)\|_2^2 + Eh\|\omega_{kx}'(t)\|_2^2$$
$$+lEh\Big(\|\omega_k'(t) + l\varphi_k'(t)\|_2^2 + Gh\|\varphi_{kx}'(t) + \psi_k'(t) + l\omega_k'(t)\|_2^2\Big)$$

$$+\tau\Big(\|z'_{1k}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z'_{2k}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z'_{3k}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\Big)$$
$$\le e^{cT}\Big(\rho_1\|\varphi''_k(0)\|_2^2+\rho_2\|\psi''_k(0)\|_2^2+\rho_1\|\omega''_k(0)\|_2^2+Gh\|\varphi'_{kx}(0)\|_2^2+El\|\psi'_{kx}(0)\|_2^2+Eh\|\omega'_{kx}(0)\|_2^2\Big)$$
$$+e^{cT}lEh\Big(\|\omega'_k(0)+l\varphi'_k(0)\|_2^2+Gh\|\varphi'_{kx}(0)+\psi'_k(0)+l\omega'_k(0)\|_2^2\Big)$$
$$+e^{cT}\tau\Big(\|z'_{1k}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}+\|z'_{2k}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}+\|z'_{3k}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}\Big)$$
$$+c'\int_0^t\|\psi''(t)\|_2^2\,ds+h(0)(\psi_{kx}(t),\psi'_{kx}(t))-h(0)(\psi_{kx}(0),\psi'_{kx}(0))+\int_0^t h'(t-s)(\psi_{kx}(t),\psi'_{kx}(t))\,ds$$
$$+\Big(\frac{1}{4\epsilon}+\frac{h'(0)}{4\epsilon}-h(0)\Big)\int_0^t\|\psi'_{kx}(t)\|_2^2\,ds+(\epsilon+\epsilon\|h''\|_{L^1})\int_0^t\|\psi_{kx}(t)\|_2^2\,ds,$$
$$h(0)(\psi_{kx}(0),\psi'_{kx}(0))\le\epsilon\|\psi'_{kx}(t)\|_2^2+\frac{h(0)^2}{4\epsilon}\|\psi_{kx}(t)\|_2^2,$$
$$\int_0^t h'(t-s)(\psi_{kx}(t),\psi'_{kx}(t))\,ds\le\epsilon\|\psi'_{kx}(t)\|_2^2+\frac{\xi(0)\|h\|_{L^1}\|h\|_{L^\infty}}{4\epsilon}\int_0^t\|\psi_{kx}(s)\|_2^2\,ds$$

for all $t\in\mathbb{R}_+$, therefore, we conclude that

$$\varphi''_k,\ \psi''_k,\ \omega''_k\ \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2),\tag{3.35}$$
$$\varphi'_k,\ \psi'_k,\ \omega'_k\ \text{ is bounded in } L^\infty_{loc}(0,\infty;H^1_0),\tag{3.36}$$
$$z'_{1k},\ z'_{2k},\ z'_{3k}\ \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2((0,1)\times(0,1)))\tag{3.37}$$

The third estimate

Replacing w_j by $-w_{jxx}$ in (3.1), (3.4) and (3.7), multiplying the result by $g'_{jk}(t)$, $\widetilde{g}'_{jk}(t)$ and $\widetilde{\widetilde{g}}'_{jk}(t)$, summing over j from 1 to k, we obtain

$$\frac12\frac{d}{dt}\Big(\rho_1\|\varphi'_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)\|_2^2\Big)$$
$$+Gh\int_0^1(\psi_x(t)+l\omega(t))\varphi'_{kxx}(t)\,dx+lEh\int_0^1(\omega(t)+l\varphi(t)\varphi'_{kx}(t)\,dx$$
$$+\mu_1\int_0^1\varphi'^2_{kx}(t)g'_1(\varphi'_k)\,dx+\mu_2\int_0^1\varphi'_{kx}(t)z_{1kx}(x,1,t)(g'_2(z_{1k}(x,1,t))\,dx=0,\tag{3.38}$$
$$\frac12\frac{d}{dt}\Big(\rho_2\|\psi'_{kx}(t)\|_2^2+El\|\psi_{kxx}(t)\|_2^2\Big)$$
$$+Gh\int_0^1(\varphi_x+\psi+l\omega)\psi'_{kxx}(t)\,dx+\widetilde{\mu}_1\int_0^1\psi'^2_{kx}(t)\widetilde{g}'_1(\psi'_k)\,dx$$
$$+\widetilde{\mu}_2\int_0^1\psi'_{kx}(t)z_{2kx}(x,1,t)\widetilde{g}'_2(z_{2k}(x,1,t))\,dx-\int_0^t h(t-s)(\psi_{kxx}(t),\psi'_{kxx}(t))\,ds=0,\tag{3.39}$$
$$\int_0^t h(t-s)(\psi_{xx}(t),\psi'_{xx}(t))\,ds+\frac12 h(t)\|\psi_{xx}(t)\|_2^2$$
$$=\frac12\frac{d}{dt}\Big[\int_0^t h(s)\,ds\|\psi_{xx}(t)\|_2^2-(ho\psi_{xx})(t)\Big]+\frac12(h'o\psi_{xx})(t)$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\omega'_{kx}(t)\|_2^2+Eh\|\omega_{kxx}(t)\|_2^2\right)+lEh\int_0^1\omega'_{kx}(t)\varphi_{kxx}(t)\,dx+Gh\int_0^1(\varphi_x+\psi+l\omega)\psi'_{kxx}(t)\,dx$$
$$+\widetilde{\widetilde{\mu}}_1\int_0^1\omega'^2_{kx}(t)\widetilde{\widetilde{g}}'_1(\omega'_k)\,dx+\widetilde{\widetilde{\mu}}_2\int_0^1\omega'_{kx}(t)z_{3kx}(x,1,t)\widetilde{\widetilde{g}}'_2(z_{3k}(x,1,t))\,dx=0. \quad (3.40)$$

Replacing ϕ_j by $-\phi_{jxx}$ in (3.10), (3.12) and (3.14), multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k, we find that

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{1kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{1kx}(t)\|_2^2=0, \quad (3.41)$$
$$\frac{1}{2}\tau\frac{d}{dt}\|z_{2kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{2kx}(t)\|_2^2=0, \quad (3.42)$$
$$\frac{1}{2}\tau\frac{d}{dt}\|z_{3kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{3kx}(t)\|_2^2=0. \quad (3.43)$$

From (3.38), (3.39), (3.40), (3.41), (3.42) and (3.43), we have

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+\rho_1\|\omega'_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)\|_2^2+El\|\psi_{kxx}(t)\|_2^2+Eh\|\omega_{kxx}(t)\|_2^2\right)$$
$$+\frac{1}{2}lEh\frac{d}{dt}\left(\|\omega_{kx}(t)+l\varphi_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)+\psi_{kx}(t)+l\omega_{kx}(t)\|_2^2\right)$$
$$+\frac{1}{2}\tau\frac{d}{dt}\left(\|z_{1kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{2kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{3kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\right)$$
$$+\mu_1\int_0^1|\varphi'_{kx}(t)|^2g'_1(\varphi'_k)\,dx+\widetilde{\mu}_1\int_0^1|\psi'_{kx}(t)|^2\widetilde{g}'_1(\psi'_k)\,dx+\widetilde{\widetilde{\mu}}_1\int_0^1|\omega'_{kx}(t)|^2\widetilde{\widetilde{g}}'_1(\omega'_k)\,dx$$
$$+\frac{1}{2}\int_0^1\left(|z_{1kx}(x,1,t)|^2+|z_{2kx}(x,1,t)|^2+|z_{3kx}(x,1,t)|^2\right)dx$$
$$+\frac{d}{dt}\frac{1}{2}\left(b-\int_0^t h(s)\,ds\right)\|\psi_{kxx}(t)\|_2^2+h(t)\|\psi_{kxx}(t)\|_2^2-\frac{1}{2}(h'o\psi_{kxx})(t)$$
$$=-\mu_2\int_0^1\varphi'_{kx}(t)z_{1kx}(x,1,t)g'_2(z_{1k}(x,1,t))\,dx-\widetilde{\mu}_2\int_0^1\psi'_{kx}(t)z_{2kx}(x,1,t)\widetilde{g}'_2(z_{2k}(x,1,t))\,dx$$
$$-\widetilde{\widetilde{\mu}}_2\int_0^1\omega'_{kx}(t)z_{3kx}(x,1,t)\widetilde{\widetilde{g}}'_2(z_{3k}(x,1,t))\,dx+\frac{1}{2}\left(\|\varphi'_{kx}(t)\|_2^2+\|\psi'_{kx}(t)\|_2^2+\|\omega'_{kx}(t)\|_2^2\right)+\frac{1}{2}\|\psi'_{kx}(t)\|_2^2.$$

Using (2.1), Cauchy–Schwartz and Young's inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+\rho_1\|\omega'_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)\|_2^2+El\|\psi_{kxx}(t)\|_2^2+Eh\|\omega_{kxx}(t)\|_2^2\right)$$
$$+\frac{1}{2}lEh\frac{d}{dt}\left(\|\omega_{kx}(t)+l\varphi_{kx}(t)\|_2^2+Gh\|\varphi_{kxx}(t)+\psi_{kx}(t)+l\omega_{kx}(t)\|_2^2\right)$$
$$+\frac{1}{2}\tau\frac{d}{dt}\left(\|z_{1kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{2kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}+\|z_{3kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\right)$$
$$+\mu_1\int_0^1|\varphi'_{kx}(t)|^2g'_1(\varphi'_k)\,dx+\widetilde{\mu}_1\int_0^1|\psi'_{kx}(t)|^2\widetilde{g}'_1(\psi'_k)\,dx+\widetilde{\widetilde{\mu}}_1\int_0^1|\omega'_{kx}(t)|^2\widetilde{\widetilde{g}}'_1(\omega'_k)\,dx$$

$$+ c\int_0^1 \Big(|z_{1kx}(x,1,t)|^2+|z_{2kx}(x,1,t)|^2+|z_{3kx}(x,1,t)|^2\Big)\, dx \le c'\Big(\|\varphi'_{kx}(t)\|_2^2+\|\psi'_{kx}(t)\|_2^2+\|\omega'_{kx}(t)\|_2^2\Big).$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we have

$$\begin{aligned}
&\rho_1\|\varphi'_{kx}(t)\|_2^2 + \rho_2\|\psi'_{kx}(t)\|_2^2 + \rho_1\|\omega'_{kx}(t)\|_2^2 + Gh\|\varphi_{kxx}(t)\|_2^2 + El\|\psi_{kxx}(t)\|_2^2 + Eh\|\omega_{kxx}(t)\|_2^2\\
&\quad + lEh\Big(\|\omega_{kx}(t) + l\varphi_{kx}(t)\|_2^2 + Gh\|\varphi_{kxx}(t) + \psi_{kx}(t) + l\omega_{kx}(t)\|_2^2\Big)\\
&\quad + \tau\Big(\|z_{1kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))} + \|z_{2kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))} + \|z_{3kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\Big)\\
&\le \rho_1\|\varphi'_{kx}(0)\|_2^2 + \rho_2\|\psi'_{kx}(0)\|_2^2 + \rho_1\|\omega'_{kx}(0)\|_2^2 + Gh\|\varphi_{kxx}(0)\|_2^2 + El\|\psi_{kxx}(0)\|_2^2 + Eh\|\omega_{kxx}(0)\|_2^2\\
&\quad + lEh\Big(\|\omega_{kx}(0) + l\varphi_{kx}(0)\|_2^2 + Gh\|\varphi_{kxx}(0) + \psi_{kx}(0) + l\omega_{kx}(0)\|_2^2\Big)\\
&\quad + \tau\Big(\|z_{1kx}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))} + \|z_{2kx}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))} + \|z_{3kx}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}\Big).
\end{aligned}$$

for all $t \in \mathbb{R}_+$. Therefore, we conclude that

$$\varphi_k,\ \psi_k,\ \omega_k \ \text{ are bounded in } L^\infty_{loc}(0,\infty; H^2 \cap H_0^1(0,1)), \tag{3.44}$$

$$z_{1k},\ z_{2k},\ z_{3k} \ \text{ are bounded in } L^\infty_{loc}(0,\infty; H_0^1(0,1; L^2(0,1))). \tag{3.45}$$

Applying Dunford–Petti's theorem, from (3.17)–(3.21), (3.35)–(3.37), (3.44) and (3.45), after replacing the sequences φ_k, ψ_k and z_k with a subsequence if needed, we conclude that

$$\begin{cases}\varphi_k \to \varphi \ \text{ weak-star in } L^\infty_{loc}(0,\infty; H^2 \cap H_0^1(0,1)),\\ \psi_k \to \psi \ \text{ weak-star in } L^\infty_{loc}(0,\infty; H^2 \cap H_0^1(0,1)),\\ \omega_k \to \omega \ \text{ weak-star in } L^\infty_{loc}(0,\infty; H^2 \cap H_0^1(0,1)),\end{cases} \tag{3.46}$$

$$\begin{aligned}
\varphi'_k \to \varphi' &\ \text{ weak-star in } L^\infty_{loc}(0,\infty; H_0^1(0,1)),\\
yy\psi'_k \to \psi' &\ \text{ weak-star in } L^\infty_{loc}(0,\infty; H_0^1(0,1)),\\
\omega'_k \to \omega' &\ \text{ weak-star in } L^\infty_{loc}(0,\infty; H_0^1(0,1)),
\end{aligned}$$

$$\begin{cases}\varphi''_k \to \varphi'' \ \text{ weak-star in } L^\infty_{loc}(0,\infty; L^2(0,1)),\\ \psi''_k \to \psi'' \ \text{ weak-star in } L^\infty_{loc}(0,\infty; L^2(0,1)),\\ \omega''_k \to \omega'' \ \text{ weak-star in } L^\infty_{loc}(0,\infty; L^2(0,1)),\end{cases} \tag{3.47}$$

$$\begin{aligned}
g_1(\varphi'_k) \to \chi &\ \text{ weak-star in } L^2((0,1)\times(0,T)),\\
\widetilde{g}_1(\psi'_k) \to \widetilde{\chi} &\ \text{ weak-star in } L^2((0,1)\times(0,T)),\\
\widetilde{\widetilde{g}}_1(\omega'_k) \to \widetilde{\widetilde{\chi}} &\ \text{ weak-star in } L^2((0,1)\times(0,T)),\\
z_{1k} \to z_1 &\ \text{ weak-star in } L^\infty_{loc}(0,\infty; H_0^1((0,1); L^2(0,1))),\\
z_{2k} \to z_2 &\ \text{ weak-star in } L^\infty_{loc}(0,\infty; H_0^1((0,1); L^2(0,1))),\\
z_{3k} \to z_3 &\ \text{ weak-star in } L^\infty_{loc}(0,\infty; H_0^1((0,1); L^2(0,1))),
\end{aligned}$$

$$\begin{cases}z'_{1k} \to z'_1 \ \text{ weak-star in } L^\infty_{loc}(0,\infty; L^2((0,1)\times(0,1))),\\ z'_{2k} \to z'_2 \ \text{ weak-star in } L^\infty_{loc}(0,\infty; L^2((0,1)\times(0,1))),\\ z'_{3k} \to z'_3 \ \text{ weak-star in } L^\infty_{loc}(0,\infty; L^2((0,1)\times(0,1))),\end{cases} \tag{3.48}$$

$$\begin{aligned}
g_2(z_k(x,1,t)) \to \varphi &\ \text{ weak-star in } L^2((0,1)\times(0,T)),\\
\widetilde{g}_2(z_k(x,1,t)) \to \psi &\ \text{ weak-star in } L^2((0,1)\times(0,T)),\\
\widetilde{\widetilde{g}}_2(z_k(x,1,t)) \to \omega &\ \text{ weak-star in } L^2((0,1)\times(0,T))
\end{aligned}$$

for suitable functions $\varphi,\psi,\omega \in L^\infty(0,T; H^2 \cap H_0^1(0,1)), z_1,z_2,z_3 \in L^\infty(0,T; L^2((0,1)\times(0,1)))$, $\chi,\widetilde{\chi},\widetilde{\widetilde{\chi}} \in L^2((0,1)\times(0,T))$, $\varphi,\psi,\omega \in L^2((0,1)\times(0,T))$ for all $T \ge 0$. We have to show that $(\varphi,\psi,\omega,z_1,z_2,z_3)$ is a solution of (8).

From (3.17) and (3.18), we have that φ_k', ψ_k' and ω_k' are bounded in $L^\infty(0,T;H_0^1(0,1))$. Then φ_k',ψ_k' and ω_k' are bounded in $L^2(0,T;H_0^1)$. Since φ_k'', ψ_k'' and ω_k'' are bounded in $L^\infty(0,T;L^2(0,1))$, we have that φ_k'', ψ_k'' and ω_k'' are bounded in $L^2(0,T;L^2(0,1))$. Consequently, φ_k', ψ_k' and ω_k' are bounded in $H^1(Q)$, where $Q=(0,1)\times(0,T)$.

Since the embedding $H^1(Q)\hookrightarrow L^2(Q)$ is compact, using Aubin–Lions theorem [13], we can extract a subsequence (φ_ν) of (φ_k), (ψ_ν) of (ψ_k) and (ω_ν) of (ω_k) such that

$$\begin{aligned}\varphi_\nu'&\to\varphi' \text{ strongly in } L^2(Q),\\ \psi_\nu'&\to\psi' \text{ strongly in } L^2(Q),\\ \omega_\nu'&\to\omega' \text{ strongly in } L^2(Q).\end{aligned}$$

Therefore,

$$\begin{cases}\varphi_\nu'\to\psi' \text{ strongly and a.e. on } Q,\\ \psi_\nu'\to\psi' \text{ strongly and a.e. on } Q,\\ \omega_\nu'\to\psi' \text{ strongly and a.e. on } Q.\end{cases}\tag{3.49}$$

Similarly, we obtain

$$\begin{cases}z_{1\nu}'\to z_1' \text{ strongly and a.e. on } Q,\\ z_{2\nu}'\to z_2' \text{ strongly and a.e. on } Q,\\ z_{3\nu}'\to z_3' \text{ strongly and a.e. on } Q.\end{cases}$$

Lemma 3.1. *For each $T>0$,*

$$\begin{gathered}g_1(\varphi'),g_2(z_1(x,1,t))\in L^1(Q),\quad \widetilde{g}_1(\psi'),\widetilde{g}_2(z_2(x,1,t))\in L^1(Q),\\ \widetilde{\widetilde{g}}_1(\omega'),\widetilde{\widetilde{g}}_2(z_3(x,1,t))\in L^1(Q)\end{gathered}$$

and

$$\begin{gathered}\|g_1(\varphi')\|_{L^1(Q)},\ \|g_2(z_1(x,1,t))\|_{L^1(Q)}\le K_1,\\ \|\widetilde{g}_1(\psi')\|_{L^1(Q)},\ \|\widetilde{g}_2(z_2(x,1,t))\|_{L^1(Q)}\le K_2,\quad \|\widetilde{\widetilde{g}}_1(\omega')\|_{L^1(Q)},\ \|\widetilde{\widetilde{g}}_2(z_3(x,1,t))\|_{L^1(Q)}\le K_3,\end{gathered}$$

where K_1, K_2 and K_3 are the constants independent of t.

Proof. By (H1) and (3.49), we have

$$\begin{gathered}g_1(\varphi_k'(x,t))\to g_1(\varphi'(x,t)) \text{ a.e. in } Q,\\ \widetilde{g}_1(\psi_k'(x,t))\to\widetilde{g}_1(\psi'(x,t)) \text{ a.e. in } Q,\quad \widetilde{\widetilde{g}}_1(\omega_k'(x,t))\to\widetilde{\widetilde{g}}_1(\omega'(x,t)) \text{ a.e. in } Q,\\ 0\le g_1(\varphi_k'(x,t))\varphi_k'(x,t)\to g_1(\varphi'(x,t))\varphi'(x,t) \text{ a.e. in } Q,\\ 0\le \widetilde{g}_1(\psi_k'(x,t))\psi_k'(x,t)\to\widetilde{g}_1(\psi'(x,t))\psi'(x,t) \text{ a.e. in } Q,\\ 0\le \widetilde{\widetilde{g}}_1(\omega_k'(x,t))\omega_k'(x,t)\to\widetilde{\widetilde{g}}_1(\omega'(x,t))\omega'(x,t) \text{ a.e. in } Q.\end{gathered}$$

Hence, by (3.19) and Fatou's lemma, we have

$$\int_0^T\int_0^1\varphi'(x,t)g_1(\varphi'(x,t))\,dx\,dt\le K \text{ for } T>0,\tag{3.50}$$

$$\int_0^T\int_0^1\psi'(x,t)g_1(\psi'(x,t))\,dx\,dt\le K' \text{ for } T>0,$$

$$\int_0^T\int_0^1\omega'(x,t)g_1(\omega'(x,t))\,dx\,dt\le K'' \text{ for } T>0.$$

By the Cauchy–Schwarz inequality and using (3.50), we get

$$\int_0^T\int_0^1 |g_1(\varphi'(x,t))|\,dx\,dt \le c|Q|^{\frac12}\Big(\int_0^T\int_0^1 \varphi' g_1(\varphi')\,dx\,dt\Big)^{\frac12} \le c|Q|^{\frac12}K^{\frac12} \equiv K_1,$$

$$\int_0^T\int_0^1 |\widetilde{g}_1(\psi'(x,t))|\,dx\,dt \le c|Q|^{\frac12}\Big(\int_0^T\int_0^1 \psi'\widetilde{g}_1(\psi')\,dx\,dt\Big)^{\frac12} \le c|Q|^{\frac12}K'^{\frac12} \equiv K_2,$$

$$\int_0^T\int_0^1 |\widetilde{\widetilde{g}}_1(\omega'(x,t))|\,dx\,dt \le c|Q|^{\frac12}\Big(\int_0^T\int_0^1 \omega'\widetilde{\widetilde{g}}_1(\omega')\,dx\,dt\Big)^{\frac12} \le c|Q|^{\frac12}K''^{\frac12} \equiv K_3. \qquad \square$$

Lemma 3.2.

$$g_1(\varphi'_k)\to g_1(\varphi'),\quad \widetilde{g}_1(\psi'_k)\to\widetilde{g}_1(\psi')\ \ \textit{and}\ \ \widetilde{\widetilde{g}}_1(\omega'_k)\to\widetilde{\widetilde{g}}_1(\omega')\ \ \textit{in}\ L^1((0,1)\times(0,T)),$$
$$g_2(z_{1k})\to g_2(z_1),\quad \widetilde{g}_2(z_{2k})\to\widetilde{g}_2(z_2)\ \ \textit{and}\ \ \widetilde{\widetilde{g}}_2(z_{3k})\to\widetilde{\widetilde{g}}_2(z_3)\ \ \textit{in}\ L^1((0,1)\times(0,T)).$$

Proof. Let $E\subset(0,1)\times[0,T]$, $E'\subset(0,1)\times[0,T]$ and $E''\subset(0,1)\times[0,T]$ and set

$$E_1=\Big\{(x,t)\in E:\ \ g_1(\varphi'_k(x,t))\le\frac{1}{\sqrt{|E|}}\Big\},\quad E_2=E\setminus E_1,$$
$$E'_1=\Big\{(x,t)\in E':\ \ \widetilde{g}_1(\psi'_k(x,t))\le\frac{1}{\sqrt{|E'|}}\Big\},\quad E'_2=E'\setminus E'_1,$$
$$E''_1=\Big\{(x,t)\in E'':\ \ \widetilde{\widetilde{g}}_1(\omega'_k(x,t))\le\frac{1}{\sqrt{|E''|}}\Big\},\quad E''_2=E''\setminus E''_1,$$

where

$|E|$ is the measure of E. If $M(r):=\inf\{|s|:\ \ s\in\mathbb{R}\ \text{ and }\ |g_1(s)|\ge r\}$,

$|E'|$ is the measure of E'. If $M'(r):=\inf\{|s|:\ \ s\in\mathbb{R}\ \text{ and }\ |\widetilde{g}_1(s)|\ge r\}$,

$|E''|$ is the measure of E''. If $M''(r):=\inf\{|s|:\ \ s\in\mathbb{R}\ \text{ and }\ |\widetilde{\widetilde{g}}_1(s)|\ge r\}$,

$$\int_E |g_1(\varphi'_k)|\,dx\,dt\le\sqrt{|E|}+\Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1}\int_{E_2}|\varphi'_k g_1(\varphi'_k)|\,dx\,dt,$$
$$\int_{E'} |\widetilde{g}_1(\psi'_k)|\,dx\,dt\le\sqrt{|E'|}+\Big(M'\Big(\frac{1}{\sqrt{|E'|}}\Big)\Big)^{-1}\int_{E'_2}|\psi'_k\widetilde{g}_1(\psi'_k)|\,dx\,dt,$$
$$\int_{E''} |\widetilde{\widetilde{g}}_1(\omega'_k)|\,dx\,dt\le\sqrt{|E''|}+\Big(M''\Big(\frac{1}{\sqrt{|E''|}}\Big)\Big)^{-1}\int_{E''_2}|\omega'_k\widetilde{\widetilde{g}}_1(\omega'_k)|\,dx\,dt.$$

Applying (3.19), we deduce that

$$\sup_k\int_E|g_1(\varphi'_k)|\,dx\,dt\to0\ \text{ as }\ |E|\to0,$$
$$\sup_k\int_{E'}|\widetilde{g}_1(\psi'_k)|\,dx\,dt\to0\ \text{ as }\ |E'|\to0,$$
$$\sup_k\int_{E''}|\widetilde{\widetilde{g}}_1(\omega'_k)|\,dx\,dt\to0\ \text{ as }\ |E''|\to0.$$

From Vitali's convergence theorem, we deduce that $g_1(\varphi'_k) \to g_1(\varphi')$, $\widetilde{g}_1(\psi'_k) \to \widetilde{g}_1(\psi')$ and $\widetilde{\widetilde{g}}_1(\omega'_k) \to \widetilde{\widetilde{g}}_1(\omega')$ in $L^1((0,1)\times(0,T))$, hence

$$\begin{aligned} g_1(\varphi'_k) &\to g_1(\varphi') \ \text{ weak star in } L^2(Q),\\ \widetilde{g}_1(\psi'_k) &\to \widetilde{g}_1(\psi') \ \text{ weak star in } L^2(Q),\\ \widetilde{\widetilde{g}}_1(\omega'_k) &\to \widetilde{\widetilde{g}}_1(\omega') \ \text{ weak star in } L^2(Q). \end{aligned}$$

Similarly, we have

$$\begin{aligned} g_2(z'_{1k}) &\to g_2(z'_1) \ \text{ weak star in } L^2(Q),\\ \widetilde{g}_2(z'_{2k}) &\to \widetilde{g}_2(z'_2) \ \text{ weak star in } L^2(Q),\\ \widetilde{\widetilde{g}}_2(z'_{3k}) &\to \widetilde{\widetilde{g}}_2(z'_3) \ \text{ weak star in } L^2(Q), \end{aligned}$$

and this implies that

$$\int_0^T\int_0^1 g_1(\varphi'_k)v\,dx\,dt \to \int_0^T\int_0^1 g_1(\varphi')v\,dx\,dt \ \text{ for all } \ v\in L^2(0,T;H_0^1), \tag{3.51}$$

$$\int_0^T\int_0^1 \widetilde{g}_1(\psi'_k)v\,dx\,dt \to \int_0^T\int_0^1 \widetilde{g}_1(\psi')v\,dx\,dt \ \text{ for all } \ v\in L^2(0,T;H_0^1), \tag{3.52}$$

$$\int_0^T\int_0^1 \widetilde{\widetilde{g}}_1(\omega'_k)v\,dx\,dt \to \int_0^T\int_0^1 \widetilde{\widetilde{g}}_1(\omega')v\,dx\,dt \ \text{ for all } \ v\in L^2(0,T;H_0^1), \tag{3.53}$$

$$\int_0^T\int_0^1 g_2(z_{1k})v\,dx\,dt \to \int_0^T\int_0^1 g_2(z_1)v\,dx\,dt \ \text{ for all } \ v\in L^2(0,T;H_0^1), \tag{3.54}$$

$$\int_0^T\int_0^1 \widetilde{g}_2(z_{2k})v\,dx\,dt \to \int_0^T\int_0^1 \widetilde{g}_2(z_2)v\,dx\,dt \ \text{ for all } \ v\in L^2(0,T;H_0^1), \tag{3.55}$$

$$\int_0^T\int_0^1 \widetilde{\widetilde{g}}_2(z_{3k})v\,dx\,dt \to \int_0^T\int_0^1 \widetilde{\widetilde{g}}_2(z_3)v\,dx\,dt \ \text{ for all } \ v\in L^2(0,T;H_0^1) \tag{3.56}$$

as $k\to+\infty$. It follows at once from (3.46), (3.47), (3.51)–(3.56) and (3.48) that for each fixed $u,v,w\in L^2(0,T;H_0^1(0,1))$ and $w_1,w_2,w_3\in L^2(0,T;H_0^1((0,1)\times(0,1)))$,

$$\int_0^T\int_0^1 \Big(\rho_1\varphi''_k + Gh\Big(\varphi_{kxx} + Gh(\psi_k+l\omega_k) + lEh(\omega_k+l\varphi_k) + \mu_1 g_1(\varphi'_k) + \mu_2 g_2(z_1k)\Big)\Big)u\,dx\,dt$$

$$\to \int_0^T\int_0^1 \Big(\rho_1\varphi'' + Gh\Big(\varphi_{xx} + Gh(\psi+l\omega) + lEh(\omega+l\varphi) + \mu_1 g_1(\varphi') + \mu_2 g_2(z_1)\Big)\Big)u\,dx\,dt,$$

$$\int_0^T\int_0^1 \Big(\rho_2\psi''_k + El\psi_{kxx} + Gh\Big(\varphi_{kx} + \psi_k + l\omega_k + \widetilde{\mu}_1\widetilde{g}_1(\psi'_k) + \widetilde{\mu}_2\widetilde{g}_2(z_{2k})\Big)\Big)v\,dx\,dt$$

$$\to \int_0^T\int_0^1 \Big(\rho_2\psi'' + El\psi_{xx} + Gh\Big(\varphi_x + \psi + l\omega + \widetilde{\mu}_1\widetilde{g}_1(\psi') + \widetilde{\mu}_2\widetilde{g}_2(z_2)\Big)\Big)v\,dx\,dt,$$

$$\int_0^T\int_0^1\Big(\rho_1\omega_k''+Eh\omega_{kxx}+lEh\varphi_{kx}+lGh(\varphi_{kx}+\psi_k+l\omega_k)+\widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega_k')+\widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(z_3k)\Big)w\,dx\,dt$$

$$\to\int_0^T\int_0^1\Big(\rho_1\omega''+Eh\omega_{xx}+lEh\varphi_x+lGh(\varphi_x+\psi+l\omega)+\widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega')+\widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(z_3))w\,dx\,dt,$$

$$\int_0^T\int_0^1\int_0^1\Big(\tau z_{1k}'+\frac{\partial}{\partial\rho}z_{1k}\Big)w_1\,dx\,d\rho\,dt\to\int_0^T\int_0^1\int_0^1\Big(\tau z_1'+\frac{\partial}{\partial\rho}z_1\Big)w_1\,dx\,d\rho\,dt,$$

$$\int_0^T\int_0^1\int_0^1\Big(\tau z_{2k}'+\frac{\partial}{\partial\rho}z_{2k}\Big)w_2\,dx\,d\rho\,dt\to\int_0^T\int_0^1\int_0^1\Big(\tau z_2'+\frac{\partial}{\partial\rho}z_2\Big)w_2\,dx\,d\rho\,dt,$$

$$\int_0^T\int_0^1\int_0^1\Big(\tau z_{3k}'+\frac{\partial}{\partial\rho}z_{3k}\Big)w_3\,dx\,d\rho\,dt\to\int_0^T\int_0^1\int_0^1\Big(\tau z_3'+\frac{\partial}{\partial\rho}z_3\Big)w_3\,dx\,d\rho\,dt$$

as $k\to+\infty$. Hence

$$\int_0^T\int_0^1\Big(\rho_1\varphi''+Gh\Big(\varphi_{xx}+Gh(\psi+l\omega)+lEh(\omega+l\varphi)+\mu_1g_1(\varphi')+\mu_2g_2(z_1)\Big)\Big)u\,dx\,dt=0,$$

$$\int_0^T\int_0^1\Big(\rho_2\psi''+El\psi_{xx}+Gh\Big(\varphi_x+\psi+l\omega+\widetilde{\mu}_1\widetilde{g}_1(\psi')+\widetilde{\mu}_2\widetilde{g}_2(z_2)\Big)\Big)v\,dx\,dt=0,$$

$$\int_0^T\int_0^1\Big(\rho_1\omega''+Eh\omega_{xx}+lEh\varphi_x+lGh(\varphi_x+\psi+l\omega)+\widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega)+\widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(z_3)\Big)w\,dx\,dt=0,$$

$$\int_0^T\int_0^1\int_0^1\Big(\tau u'+\frac{\partial}{\partial\rho}z\Big)w\,dx\,d\rho\,dt=0,\quad w_1,w_2,w_3\in L^2(0,T;H_0^1((0,1)\times(0,1))).$$

Thus problem (P) admits a global weak solution (φ,ψ,ω).

Uniqueness. Let $(\varphi_1,\psi_1,\omega_1,z_1,z_2,z_3)$ and $(\widetilde{\varphi}_1,\widetilde{\psi}_1,\widetilde{\omega}_1,\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3)$ be two solutions of problem (2.3). Then

$$(\widetilde{\widetilde{\varphi}},\widetilde{\widetilde{\psi}},\widetilde{\widetilde{\omega}},\widetilde{\widetilde{z}}_1,\widetilde{\widetilde{z}}_2,\widetilde{\widetilde{z}}_3)=(\varphi_1,\psi_1,\omega_1,z_1,z_2,z_3)-(\widetilde{\varphi}_1,\widetilde{\psi}_1,\widetilde{\omega}_1,\widetilde{z}_1,\widetilde{z}_2,\widetilde{z}_3)$$

verifies

$$\begin{cases}
\rho_1\widetilde{\widetilde{\varphi}}_{tt} - Gh(\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}})_x - lEh(\widetilde{\widetilde{\omega}}_x - l\widetilde{\widetilde{\varphi}}) + \mu_1 g_1(\widetilde{\varphi}'(x,t)) \\
\qquad -\mu_1 g_1(\varphi'(x,t)) + \mu_2 g_2(\widetilde{z}_1(x,1,t)) - \mu_2 g_2(z_1(x,1,t)) = 0 \ \text{ in }]0,1[\times]0,+\infty[\,, \\
\tau_1 \widetilde{\widetilde{z}}_{1t}(x,\rho,t) + \widetilde{\widetilde{z}}_{1\rho}(x,\rho,t) = 0 \ \text{ in }]0,1[\times]0,1[\times]0,+\infty[\,, \\
\rho_2\widetilde{\widetilde{\psi}}_{tt} - EI\widetilde{\widetilde{\psi}}_{xx} + Gh(\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}}) + \widetilde{\mu}_1\widetilde{g}_1(\widetilde{\psi}'(x,t)) - \widetilde{\mu}_1\widetilde{g}_1(\psi'(x,t)) + \widetilde{\mu}_2\widetilde{g}_2(\widetilde{z}_2(x,1,t)) \\
\qquad -\widetilde{\mu}_2\widetilde{g}_2(z_2(x,1,t)) + \int\limits_0^t h(t-s)\widetilde{\widetilde{\psi}}_{xx}(x,s)\,ds = 0 \ \text{ in }]0,1[\times]0,+\infty[\,, \\
\tau_2 \widetilde{\widetilde{z}}_{2t}(x,\rho,t) + \widetilde{\widetilde{z}}_{2\rho}(x,\rho,t) = 0 \ \text{ in }]0,1[\times]0,1[\times]0,+\infty[\,, \\
\rho_1\widetilde{\widetilde{\omega}}_{tt} - Eh(\widetilde{\widetilde{\omega}}_x - l\widetilde{\widetilde{\varphi}})_x + lGh(\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}}) + \widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\widetilde{\omega}'(x,t)) - \widetilde{\widetilde{\mu}}_1\widetilde{\widetilde{g}}_1(\omega'(x,t)) \\
\qquad +\widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(\widetilde{z}_3(x,1,t)) - \widetilde{\widetilde{\mu}}_2\widetilde{\widetilde{g}}_2(z_3(x,1,t)) = 0 \ \text{ in }]0,1[\times]0,+\infty[\,, \\
\tau_3 \widetilde{\widetilde{z}}_{3t}(x,\rho,t) + \widetilde{\widetilde{z}}_{3\rho}(x,\rho,t) = 0 \ \text{ in }]0,1[\times]0,1[\times]0,+\infty[\,, \\
\widetilde{\widetilde{\varphi}}(0,t) = \widetilde{\widetilde{\varphi}}(1,t) = \widetilde{\widetilde{\psi}}(0,t) = \widetilde{\widetilde{\psi}}(1,t) = 0, \ \ t \geq 0, \\
\widetilde{\widetilde{\psi}}(x,0,t) = \widetilde{\psi}'(x,t) - \psi'(x,t), \ \ x \in]0,1[\times]0,+\infty[\,, \\
\widetilde{\widetilde{\varphi}}(x,0,t) = \widetilde{\varphi}'(x,t) - \varphi'(x,t), \ \ x \in]0,1[\times]0,+\infty[\,, \\
\widetilde{\widetilde{\omega}}(x,0,t) = \widetilde{\omega}'(x,t) - \omega'(x,t), \ \ x \in]0,1[\times]0,+\infty[\,, \\
\widetilde{\widetilde{\varphi}}(x,0) = \widetilde{\widetilde{\varphi}}'(x,0) = \widetilde{\widetilde{\psi}}(x,0) = \widetilde{\widetilde{\psi}}'(x,0) = \widetilde{\widetilde{\omega}}(x,0) = \widetilde{\widetilde{\omega}}'(x,0) = 0, \ \ x \in]0,1[\,, \\
\widetilde{\widetilde{z}}_1(x,\rho,0) = \widetilde{\widetilde{z}}_2(x,\rho,0) = \widetilde{\widetilde{z}}_3(x,\rho,0) = 0, \ \ x \in]0,1[\times]0,1[\,.
\end{cases} \tag{3.57}$$

Multiplying the first equation (3.57) by $\widetilde{\widetilde{\varphi}}'$ the third equation by $\widetilde{\widetilde{\psi}}'$ and the fifth equation by $\widetilde{\widetilde{\omega}}'$, integrating over $(0,1)\times(0,1)$, we get

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\widetilde{\widetilde{\varphi}}_t\|_2^2\right) + Gh\int\limits_0^1 (\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}})_x)\widetilde{\widetilde{\varphi}}'\,dx + lEh\int\limits_0^1 (\widetilde{\widetilde{\omega}}_x - l\widetilde{\widetilde{\varphi}}')\widetilde{\widetilde{\varphi}}'\,dx$$
$$+ \mu_1\big(g_1(\widetilde{\varphi}'(x,t)) - g_1(\varphi'(x,t)), \widetilde{\widetilde{\varphi}}'\big) + \mu_2\big(g_2(\widetilde{z}_1(x,1,t) - g_2(z_1(x,1,t))), \widetilde{\widetilde{\varphi}}'\big) = 0, \tag{3.58}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\widetilde{\widetilde{\psi}}_t\|_2^2 + EI\|\widetilde{\widetilde{\psi}}_x\|_2^2\right) + Gh\int\limits_0^1 (\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}})\widetilde{\widetilde{\psi}}'\,dx$$
$$+ \widetilde{\mu}_1\big(\widetilde{g}_1(\widetilde{\psi}'(x,t)) - \widetilde{g}_1(\psi'(x,t)), \widetilde{\widetilde{\psi}}'\big) + \widetilde{\mu}_2\big(\widetilde{g}_2(\widetilde{z}_2(x,1,t)) - \widetilde{g}_2(z_2(x,1,t)), \widetilde{\widetilde{\psi}}'\big) = 0, \tag{3.59}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\widetilde{\widetilde{\omega}}_t\|_2^2\right) + Eh\int\limits_0^1 (\widetilde{\widetilde{\omega}}_x - l\widetilde{\widetilde{\varphi}})_x)\widetilde{\widetilde{\omega}}'\,dx + lGh\int\limits_0^1 (\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}})\widetilde{\widetilde{\omega}}'\,dx$$
$$+ \widetilde{\widetilde{\mu}}_1\big(\widetilde{\widetilde{g}}_1(\widetilde{\omega}'(x,t)) - \widetilde{\widetilde{g}}_1(\omega'(x,t)), \widetilde{\widetilde{\omega}}'\big) + \widetilde{\widetilde{\mu}}_2\big(\widetilde{\widetilde{g}}_2(\widetilde{z}_3(x,1,t)) - \widetilde{\widetilde{g}}_2(z_3(x,1,t)), \widetilde{\widetilde{\omega}}'\big) = 0. \tag{3.60}$$

Multiplying the second equation in (3.57) by $\widetilde{\widetilde{z}}_1$, the fourth equation by $\widetilde{\widetilde{z}}_2$ and the sixth equation by $\widetilde{\widetilde{z}}_3$, integrating over $(0,1)\times(0,1)$, we get

$$\tau_1\frac{1}{2}\frac{d}{dt}\int\limits_0^1 \|\widetilde{\widetilde{z}}'_1\|_2^2\,d\rho + \frac{1}{2}\left(\|\widetilde{\widetilde{z}}_1(x,1,t)\|_2^2 - \|\widetilde{\widetilde{\varphi}}'\|_2^2\right) = 0, \tag{3.61}$$

$$\tau_2\frac{1}{2}\frac{d}{dt}\int\limits_0^1 \|\widetilde{\widetilde{z}}'_2\|_2^2\,d\rho + \frac{1}{2}\left(\|\widetilde{\widetilde{z}}_2(x,1,t)\|_2^2 - \|\widetilde{\widetilde{\psi}}'\|_2^2\right) = 0, \tag{3.62}$$

$$\tau_3\,\frac{1}{2}\,\frac{d}{dt}\int\limits_0^1 \|\widetilde{\widetilde{z}}'_3\|_2^2\,d\rho + \frac{1}{2}\left(\|\widetilde{\widetilde{z}}_3(x,1,t)\|_2^2 - \|\widetilde{\widetilde{\omega}}'\|_2^2\right) = 0. \tag{3.63}$$

From (3.58)–(3.63) and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\Big(\rho_1\|\widetilde{\widetilde{\varphi}}_t\|_2^2 + \rho_2\|\widetilde{\widetilde{\psi}}_t\|_2^2 + \rho_1\|\widetilde{\widetilde{\omega}}_t\|_2^2 + Gh\|\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}}\|_2^2 + Eh\|\widetilde{\widetilde{\omega}}_x - l\widetilde{\widetilde{\varphi}}'^2_2 + EI\|\widetilde{\widetilde{\psi}}_x\|_2^2\Big)\\
&\qquad + \frac{1}{2}\,\frac{d}{dt}\Big(\tau_1\int\limits_0^1 \|\widetilde{\widetilde{z}}'_1\|_2^2\,d\rho + \tau_2\int\limits_0^1 \|\widetilde{\widetilde{z}}'_2\|_2^2\,d\rho + \tau_3\int\limits_0^1 \|\widetilde{\widetilde{z}}'_3\|_2^2\,d\rho\Big)\\
&\quad + \frac{1}{2}\,\frac{d}{dt}\Big(\|\widetilde{\widetilde{z}}_1(x,1,t)\|_2^2 + \|\widetilde{\widetilde{z}}_2(x,1,t)\|_2^2 + \|\widetilde{\widetilde{z}}_3(x,1,t)\|_2^2\Big) + \mu_1\big(g_1(\widetilde{\varphi}'(x,t)) - g_1(\varphi'(x,t)), \widetilde{\widetilde{\varphi}}'\big)\\
&\qquad + \widetilde{\mu}_1\big(\widetilde{g}_1(\widetilde{\psi}'(x,t)) - \widetilde{g}_1(\psi'(x,t)), \widetilde{\widetilde{\psi}}'\big) + \widetilde{\widetilde{\mu}}_1\big(\widetilde{\widetilde{g}}_1(\widetilde{\omega}'(x,t)) - \widetilde{\widetilde{g}}_1(\omega'(x,t)), \widetilde{\widetilde{\omega}}'\big)\\
&\quad = -\mu_2\big(g_2(\widetilde{z}_1(x,1,t)) - g_2(z_1(x,1,t)), \widetilde{\widetilde{\varphi}}'\big) - \widetilde{\mu}_2\big(\widetilde{g}_2(\widetilde{z}_2(x,1,t)) - \widetilde{g}_2(z_2(x,1,t)), \widetilde{\widetilde{\psi}}'\big)\\
&\qquad - \widetilde{\widetilde{\mu}}_2\big(\widetilde{\widetilde{g}}_2(\widetilde{z}_3(x,1,t)) - \widetilde{\widetilde{g}}_2(z_3(x,1,t)), \widetilde{\widetilde{\omega}}'\big) + \frac{1}{2}\,\frac{d}{dt}\Big(\|\widetilde{\widetilde{\varphi}}'\|_2^2 + \|\widetilde{\widetilde{\psi}}'\|_2^2 + \|\widetilde{\widetilde{\omega}}'\|_2^2\Big)\\
&\quad \le \frac{1}{2}\Big(\|\widetilde{\widetilde{\varphi}}'\|_2^2 + \|\widetilde{\widetilde{\psi}}'\|_2^2 + \|\widetilde{\widetilde{\omega}}'\|_2^2\Big) + \big\|g_2(\widetilde{z}_1(x,1,t) - g_2(z_1(x,1,t))\big\|_2\|\widetilde{\widetilde{\varphi}}'\|_2\\
&\qquad + \big\|\widetilde{g}_2(\widetilde{z}_2(x,1,t) - \widetilde{g}_2(z_2(x,1,t))\big\|_2\|\widetilde{\widetilde{\psi}}'\|_2 + \big\|\widetilde{\widetilde{g}}_2(\widetilde{z}_3(x,1,t)) - \widetilde{\widetilde{g}}_2(z_3(x,1,t))\big\|_2\|\widetilde{\widetilde{\omega}}'\|_2.
\end{aligned}$$

Using condition (2.2) and Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\Big(\rho_1\|\widetilde{\widetilde{\varphi}}_t\|_2^2 + \rho_2\|\widetilde{\widetilde{\psi}}_t\|_2^2 + \rho_1\|\widetilde{\widetilde{\omega}}_t\|_2^2 + Gh\|\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}}\|_2^2 + Eh\|\widetilde{\widetilde{\omega}}_x - l\widetilde{\widetilde{\varphi}}'^2_2 + EI\|\widetilde{\widetilde{\psi}}_x\|_2^2\Big)\\
&\quad + \frac{1}{2}\,\frac{d}{dt}\Big(\tau_1\int\limits_0^1 \|\widetilde{\widetilde{z}}'_1\|_2^2\,d\rho + \tau_2\int\limits_0^1 \|\widetilde{\widetilde{z}}'_2\|_2^2\,d\rho + \tau_3\int\limits_0^1 \|\widetilde{\widetilde{z}}'_3\|_2^2\,d\rho\Big) \le c\Big(\|\widetilde{\widetilde{\varphi}}'\|_2^2) + \|\widetilde{\widetilde{\psi}}'\|_2^2) + \|\widetilde{\widetilde{\omega}}'\|_2^2\Big),
\end{aligned}$$

where c is a positive constant. Then integrating over $(0,t)$ and using Gronwall's lemma, we conclude that

$$\begin{aligned}
&\rho_1\|\widetilde{\widetilde{\varphi}}_t\|_2^2 + \rho_2\|\widetilde{\widetilde{\psi}}_t\|_2^2 + \rho_1\|\widetilde{\widetilde{\omega}}_t\|_2^2 + Gh\|\widetilde{\widetilde{\varphi}}_x + \widetilde{\widetilde{\psi}} + l\widetilde{\widetilde{\omega}}\|_2^2\\
&\quad + Eh\|\widetilde{\widetilde{\omega}}_x - l\widetilde{\widetilde{\varphi}}'\|_2^2 + EI\|\widetilde{\widetilde{\psi}}_x\|_2^2 + \tau_1\int\limits_0^1 \|\widetilde{\widetilde{z}}'_1\|_2^2\,d\rho + \tau_2\int\limits_0^1 \|\widetilde{\widetilde{z}}'_2\|_2^2\,d\rho + \tau_3\int\limits_0^1 \|\widetilde{\widetilde{z}}'_3\|_2^2\,d\rho = 0. \qquad \square
\end{aligned}$$

4 Asymptotic behavior

First, we state and prove a lemma that will be needed to establish the asymptotic behavior.

Lemma 4.1. *There exists a positive constant C such that for every $(\varphi, \psi, \omega) \in (H_0^1(0,1))^3$, the following inequality holds:*

$$\int\limits_0^1 \left(|\varphi_x|^2 + |\psi_x|^2 + |\omega_x|^2\right) dx \le C\int\limits_0^1 \Big(EI|\psi_x|^2 + Gh|\varphi_x + \psi + l\omega|^2 + Eh|\omega_x - l\varphi|^2\Big)\,dx. \tag{4.1}$$

Proof. We will argue by contradiction. Indeed, let us suppose that (4.1) is not true. So, we can find a sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ in $(H_0^1(0,L))^3$ satisfying

$$\int\limits_0^1 \Big(EI|\psi_{\nu x}|^2 + Gh|\varphi_{\nu x} + \psi + l\omega\nu|^2 + Eh|\omega_{\nu x} - l\varphi\nu|^2\Big)\,dx \le \frac{1}{\nu} \tag{4.2}$$

and

$$\int_0^1 \left(|\varphi_{\nu x}|^2 + |\psi_{\nu x}|^2 + |\omega_{\nu x}|^2 \right) dx = 1. \tag{4.3}$$

From (4.3), the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ is bounded in $(H_0^1(0,1))^3$. Since the embedding $H_0^1(0,1) \hookrightarrow L^2(0,1)$ is compact, the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ converges strongly in $(L^2(0,1))^3$.

From (4.2),

$$\varphi_{\nu x} \to 0 \ \text{ strongly in } L^2(0,1).$$

Using Poincaré's inequality, we can conclude that

$$\varphi_\nu \to 0 \ \text{ strongly in } L^2(0,1).$$

Now, setting $\varphi_\nu \to \varphi$ and $\omega_\nu \to \omega$ strongly in $L^2(0,1)$, from (4.2), we have

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu \to 0 \ \text{ strongly in } L^2(0,1).$$

Then

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu = \varphi_{\nu x} + \psi_\nu + l(\omega_\nu - \omega) + l\omega \to 0 \ \text{ strongly in } L^2(0,1),$$

which implies that

$$\varphi_{\nu x} \to -l\omega \ \text{ strongly in } L^2(0,1). \tag{4.4}$$

Then $\{\varphi_\nu\}_n$ is a Cauchy sequence in $H^1(0,L)$. Therefore, $\{\varphi_\nu\}_n$ converges to a function φ_1 in $H^1(0,L)$. Consequently, $\{\varphi_\nu\}_n$ converges to φ_1 in $L^2(0,L)$. Thus by the uniqueness of the limit $\varphi_1 = \varphi$. Moreover, $\varphi \in H_0^1(0,L)$.

From (4.4), we deduce that

$$\varphi_x + l\omega = 0 \ \text{ a.e. } x \in (0,1). \tag{4.5}$$

Similarly, we have

$$\omega_x - l\varphi = 0 \ \text{ a.e. } x \in (0,1) \tag{4.6}$$

and $\omega \in H_0^1(0,1)$.

(4.5) and (4.6) provide us $\varphi = \omega = 0$, contradicting (4.3). □

From now on, we denote by c various positive constants which may be different at different occurrences.

Multiplying the first equation in (2.3) by $\frac{\varphi(E)}{E}\varphi$, the third equation by $\frac{\varphi(E)}{E}\psi$ and the fifth equation by $\frac{\varphi(E)}{E}\omega$, we obtain

$$0 = \int_S^T \frac{\varphi(E)}{E} \int_0^1 \varphi\Big(\rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t + \mu_2 z_1(x,1,t)\Big)\, dx\, dt,$$

$$\begin{aligned}
0 = {} & \left[\frac{\varphi(E)}{E}\rho_1 \int_0^1 \varphi\varphi_t\, dx\right]_S^T - \int_S^T \rho_1 \Big(\frac{\varphi(E)}{E}\Big)' \int_0^1 \varphi\varphi_t\, dx\, dt - \rho_1 \int_S^T \frac{\varphi(E)}{E} \|\varphi_t\|_2^2\, dt \\
& - \int_S^T \frac{\varphi(E)}{E} \int_0^1 \varphi_x Gh(\varphi_x + \psi + l\omega)\, dx\, dt - \int_S^T \frac{\varphi(E)}{E} \int_0^1 \varphi(lEh)(\omega_x - l\varphi)\, dx\, dt \\
& + \mu_1 \int_S^T \frac{\varphi(E)}{E} \int_0^1 \varphi_t\varphi\, dx\, dt + \mu_2 \int_S^T \frac{\varphi(E)}{E} \int_0^1 \varphi z_1(x,1,t)\, dx\, dt,
\end{aligned}$$

$$0 = \int_S^T \frac{\varphi(E)}{E} \int_0^1 \psi\Big(\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu}_1\psi_t + \widetilde{\mu}_2 z_2(x,1,t)\Big)\, dx\, dt$$

$$+\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi \int_0^t h(t-s)\varphi_{xx}(x,s)\,ds\,dx\,dt,$$

$$\begin{aligned}0 &= \Big[\frac{\varphi(E)}{E}\rho_2\int_0^1 \psi\psi_t\,dx\Big]_S^T - \int_S^T \rho_2\Big(\frac{\varphi(E)}{E}\Big)'\int_0^1 \psi\psi_t\,dx\,dt - \rho_2\int_S^T \frac{\varphi(E)}{E}\,\|\psi_t\|_2^2\,dt\\ &\quad+\int_S^T \frac{\varphi(E)}{E}\,EI\|\psi_x\|_2^2\,dt + \int_S^T \frac{\varphi(E)}{E}\int_0^1 \psi\psi Gh(\varphi_x+\psi+l\omega))\,dx\,dt\\ &\quad+\widetilde{\mu}_1\int_S^T \frac{\varphi(E)}{E}\int_0^1 \psi\psi_t\,dx\,dt + \widetilde{\mu}_2\int_S^T \frac{\varphi(E)}{E}\int_0^1 \psi z_2(x,1,t)\,dx\,dt\\ &\quad+\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi\int_0^t h(t-s)\varphi_{xx}(x,s)\,ds\,dx\,dt;\end{aligned}$$

$$0 = \int_S^T \frac{\varphi(E)}{E}\int_0^1 \omega\Big(\rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x+\psi+l\omega) + \widetilde{\widetilde{\mu}}_1\omega_t + \widetilde{\widetilde{\mu}}_2 z_3(x,1,t)\Big)\,dx\,dt,$$

$$\begin{aligned}0 &= \Big[\frac{\varphi(E)}{E}\rho_1\int_0^1 \omega\omega_t\,dx\Big]_S^T - \int_S^T \rho_1\Big(\frac{\varphi(E)}{E}\Big)'\int_0^1 \omega\omega_t\,dx\,dt - \rho_1\int_S^T \frac{\varphi(E)}{E}\,\|\omega_t\|_2^2\,dt\\ &\quad+\int_S^T \frac{\varphi(E)}{E}\int_0^1 Eh\omega_x(\omega_x - l\varphi)\,dx\,dt + \int_S^T \frac{\varphi(E)}{E}\int_0^1 \omega(lGh)(\varphi_x+\psi+l\omega)\,dx\,dt\\ &\quad+\widetilde{\widetilde{\mu}}_1\int_S^T \frac{\varphi(E)}{E}\int_0^1 \omega_t\omega\,dx\,dt + \widetilde{\widetilde{\mu}}_2\int_S^T \frac{\varphi(E)}{E}\int_0^1 \omega z_3(x,1,t)\,dx\,dt.\end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned}0 &= \Big[\frac{\varphi(E)}{E}\rho_1\int_0^1 \varphi\varphi_t\,dx\Big]_S^T + \Big[\frac{\varphi(E)}{E}\rho_2\int_0^1 \psi\psi_t\,dx\Big]_S^T + \Big[\frac{\varphi(E)}{E}\rho_1\int_0^1 \omega\omega_t\,dx\Big]\\ &\quad-\int_S^T \rho_1\Big(\frac{\varphi(E)}{E}\Big)'\int_0^1 (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1\omega\omega_t)\,dx\,dt\\ &\quad-2\rho_1\int_S^T \frac{\varphi(E)}{E}\,\|\varphi_t\|_2^2\,dt - 2\rho_2\int_S^T \frac{\varphi(E)}{E}\,\|\psi_t\|_2^2\,dt - 2\rho_1\int_S^T \frac{\varphi(E)}{E}\,\|\omega_t\|_2^2\,dt\\ &\quad+\int_S^T \frac{\varphi(E)}{E}\Big(\rho_1\|\varphi_t\|_2^2 + \rho_2\|\psi_t\|_2^2 + \rho_1\|\omega_t\|_2^2 + Gh\|\varphi_x+\psi+l\omega\|_2^2 + EI\|\psi_t\|_2^2 + Eh\|\omega_x - l\psi\|_2^2\Big)\\ &\quad+\mu_1\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi_t\varphi\,dx\,dt + \mu_2\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi z_1(x,1,t)\,dx\,dt + \widetilde{\mu}_1\int_S^T \frac{\varphi(E)}{E}\int_0^1 \psi\psi_t\,dx\,dt\\ &\quad+\widetilde{\mu}_2\int_S^T \frac{\varphi(E)}{E}\int_0^1 \psi z_2(x,1,t)\,dx\,dt + \widetilde{\widetilde{\mu}}_1\int_S^T \frac{\varphi(E)}{E}\int_0^1 \omega_t\omega\,dx\,dt\end{aligned}$$

$$+\widetilde{\widetilde{\mu}}_2\int_S^T\frac{\varphi(E)}{E}\int_0^1\omega z_3(x,1,t)\,dx\,dt+\int_S^T\frac{\varphi(E)}{E}\int_0^L\varphi\int_0^t h(t-s)\varphi_{xx}(x,s)\,ds\,dx\,dt. \quad (4.7)$$

Similarly, we multiply the equation of (2.3) by $\frac{\varphi(E)}{E}\,\xi_i e^{-2\tau_i\rho}z_i(x,\rho,t)$ and get

$$\begin{aligned}
0 &= \int_S^T\frac{\varphi(E)}{E}\int_0^1\int_0^1 e^{-2\tau_i\rho}\xi_i z_i(\tau_i z_{it}+z_{i\rho}\,d\rho\,dx\,dt\\
&= \left[\frac{1}{2}\,\xi_i\tau_i\,\frac{\varphi(E)}{E}\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\right]_S^T-\frac{\tau_i\xi_i}{2}\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\,dt\\
&\quad+\int_S^T\frac{\varphi(E)}{E}\xi_i\int_0^1\int_0^1\frac{e^{-2\tau_i\rho}}{2}\frac{d}{d\rho}\left(z_i^2\right)d\rho\,dx\,dt,\\
0 &= \left[\frac{1}{2}\,\xi_i\tau_i\,\frac{\varphi(E)}{E}\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\right]_S^T-\frac{\tau_i\xi_i}{2}\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\,dt\\
&\quad+\frac{\xi_i}{2}\int_S^T\frac{\varphi(E)}{E}\int_0^1\int_0^1\left[\frac{d}{d\rho}\left(e^{-2\tau_i\rho}z_i^2\right)+2\tau_i e^{-2\tau_i\rho}z_i^2\right]d\rho\,dx\,dt,\\
0 &= \left[\frac{1}{2}\,\xi_i\tau_i\,\frac{\varphi(E)}{E}\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\right]_S^T-\frac{\tau_i\xi_i}{2}\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\,dt\\
&\quad+\frac{\xi_i}{2}\int_S^T\frac{\varphi(E)}{E}\int_0^1\left[e^{-2\tau_i}z_i^2(x,1,t)-z_i^2(x,0,t)\right]dx\,dt+\xi_i\tau_i\int_S^T\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\,dt. \quad (4.8)
\end{aligned}$$

From (4.7) and (4.8), we get

$$\begin{aligned}
A\int_S^T\varphi(E)\,dt &\le -\left[\rho_1\frac{\varphi(E)}{E}\int_0^1\varphi\varphi_t\,dx\right]_S^T-\left[\rho_2\frac{\varphi(E)}{E}\int_0^1\psi\psi_t\,dx\right]_S^T-\left[\rho_1\frac{\varphi(E)}{E}\int_0^1\omega\omega_t\,dx\right]_S^T\\
&+\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^1(\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1\omega\omega_t)\,dx\,dt+2\int_S^T\frac{\varphi(E)}{E}\left(\rho_1\|\varphi_t\|_2^2+\rho_2\|\psi_t\|_2^2+\rho_1\|\omega_t\|_2^2\right)dt\\
&-\mu_1\int_S^T\frac{\varphi(E)}{E}\int_0^1\varphi_t\varphi\,dx\,dt-\mu_2\int_S^T\frac{\varphi(E)}{E}\int_0^1\varphi z_1(x,1,t)\,dx\,dt-\widetilde{\mu}_1\int_S^T\frac{\varphi(E)}{E}\int_0^1\psi\psi_t\,dx\,dt\\
&-\widetilde{\mu}_2\int_S^T\frac{\varphi(E)}{E}\int_0^1\psi z_2(x,1,t)\,dx\,dt-\widetilde{\widetilde{\mu}}_1\int_S^T\frac{\varphi(E)}{E}\int_0^1\omega_t\omega\,dx\,dt-\widetilde{\widetilde{\mu}}_2\int_S^T\frac{\varphi(E)}{E}\int_0^1\omega z_3(x,1,t)\,dx\,dt\\
&-\sum_{i=1}^3\left[\frac{1}{2}\,\xi_i\tau_i\,\frac{\varphi(E)}{E}\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\right]_S^T+\sum_{i=1}^3\frac{\tau_i\xi_i}{2}\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_0^1\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\,dt\\
&-\sum_{i=1}^3\frac{\xi_i}{2}\int_S^T\frac{\varphi(E)}{E}\,e^{-2\tau_i}\int_0^1 z_i^2(x,1,t)\,dx\,dt+\sum_{i=1}^3\frac{\xi_i}{2}\int_S^T\frac{\varphi(E)}{E}\,\|z_i(x,0,t)\|_2^2\,dt,
\end{aligned}$$

where $A = 2\min\{1, 2\tau_1 e^{-2\tau_1}, 2\tau_2 e^{-2\tau_2}, 2\tau_3 e^{-2\tau_3}\}$. Since E is non-increasing, we find that

$$-\Big[\frac{\varphi(E)}{E}\int_0^1 \varphi\varphi_t\,dx\Big]_S^T = \frac{\varphi(E(S))}{E(S)}\int_0^1 \varphi(S)\varphi'(S)\,dx - \frac{\varphi(E(T))}{E(T)}\int_0^1 \varphi(T)\varphi'(T)\,dx \le C\varphi(E(S)),$$

$$\Big|\int_S^T \Big(\frac{\varphi(E)}{E}\Big)'\int_0^1 (\rho_1\varphi\varphi' + \rho_2\psi\psi' + \rho_1\omega\omega')\,dx\,dt\Big| \le c\int_S^T (-E')\frac{\varphi(E)}{E}\,dt \le c\varphi(E(S)),$$

$$\Big|\frac{1}{2}\,\xi_i\tau_i\,\frac{\varphi(E)}{E}\int_0^1\int_0^1 e^{-2\tau_i\rho} z_i^2\,dx\,d\rho\Big| \le c\varphi(E(S)) \ \ \forall\, t\ge S,$$

$$\int_S^T \frac{\varphi(E)}{E}\int_0^1 u'^2\,dx\,dt \le c\int_S^T \frac{\varphi(E)}{E}(-E')\,dt \le c\varphi(E(S)),$$

$$\int_S^T \frac{\varphi(E)}{E}\,\xi_i\int_0^1 e^{-2\tau_i} z_i^2(x,1,t)\,dx\,dt \le c\int_S^T \frac{\varphi(E)}{E}(-E')\,dt \le c\varphi(E(S)),$$

$$\frac{1}{2}\int_S^T \frac{\varphi(E)}{E}\,\xi_i\int_0^1 z_i^2(x,0,t))\,dx\,dt = \frac{1}{2}\int_S^T \frac{\varphi(E)}{E}\,\xi_i\int_0^1 \varphi'^2\,dx\,dt \le c\varphi(E(S)),$$

$$\Big|\frac{\tau_i\xi_i}{2}\int_S^T \Big(\frac{\varphi(E)}{E}\Big)'\int_0^1\int_0^1 e^{-2\tau_i\rho} z_i^2\,dx\,d\rho\,dt\Big| \le c\int_S^T (-E')\frac{\varphi(E)}{E}\,dt \le c\varphi(E(S)),$$

$$\begin{aligned}\Big|\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi\varphi'\,dx\,dt\Big| &\le \varepsilon\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi^2\,dx\,dt + c(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi'^2\,dx\,dt \le \varepsilon c\int_0^1 \varphi(E)\,dt + c(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi'^2\,dx\,dt\\ &\le \varepsilon c\int_0^1 \varphi(E)\,dt + c(\varepsilon)\int_S^T \frac{\varphi(E)}{E}\,(-E')\,dt \le \varepsilon c\int_0^1 \varphi(E)\,dt + c(\varepsilon)E(S)^{q+1} \qquad (4.9)\end{aligned}$$

and

$$\begin{aligned}\Big|\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi z_1(x,1,t)\,dx\,dt\Big| &\le \varepsilon_1\int_S^T \frac{\varphi(E)}{E}\int_0^1 \varphi^2\,dx\,dt + c(\varepsilon_1)\int_S^T \frac{\varphi(E)}{E}\int_0^1 z_1(x,1,t)^2\,dx\,dt\\ &\le \varepsilon_1 c\int_{\mathbb{R}^n} \varphi(E)\,dt + c(\varepsilon_1)\int_S^T \frac{\varphi(E)}{E}\int_0^1 z_1(x,1,t)^2\,dx\,dt\\ &\le \varepsilon_1 c\int_0^1 \varphi(E)\,dt + c(\varepsilon_1)\int_S^T \frac{\varphi(E)}{E}(-E')\,dt \le \varepsilon_1 c\int_0^1 \varphi(E)\,dt + c(\varepsilon_1)\varphi(E(S)), \qquad (4.10)\end{aligned}$$

$$\Big|\int_S^T \frac{\varphi(E)}{E}\int_0^1 \psi\psi'\,dx\,dt\Big| \le \varepsilon' c\int_0^1 \varphi(E)\,dt + c(\varepsilon')\varphi(E(S)), \qquad (4.11)$$

$$\Big|\int_S^T \frac{\varphi(E)}{E}\int_0^1 \psi z_2(x,1,t)\,dx\,dt\Big| \le \varepsilon_1' c\int_0^1 \varphi(E)\,dt + c(\varepsilon_1')\varphi(E(S)), \qquad (4.12)$$

$$\left|\int_S^T \frac{\varphi(E)}{E}\int_0^1 \omega\omega'\,dx\,dt\right| \le \varepsilon'' c\int_0^1 \varphi(E)\,dt + c(\varepsilon'')\varphi(E(S)), \tag{4.13}$$

$$\left|\int_S^T \frac{\varphi(E)}{E}\int_0^1 \omega z_3(x,1,t)\,dx\,dt\right| \le \varepsilon_1'' c\int_0^1 E^{q+1}\,dt + c(\varepsilon_1'')\varphi(E(S)). \tag{4.14}$$

Choosing ε, ε_1, ε', ε_1', ε'' and ε_1'' small enough, we deduce from (4.8), (4.9)–(4.14) that

$$\int_S^T \varphi(E)\,dt \le c\varphi(E(S)),$$

where c is a positive constant independent of $E(0)$. Hence we deduce from Lemma 4.1 that

$$E(t) \le cE(0)e^{-\omega t}, \quad t \ge 0.$$

This ends the proof of Theorem 2.1. □

References

[1] C. Abdallah, P. Dorato, J. Benites-Read and R. Byrne, Delayed positive feedback can stabilize oscillatory systems. *Proceedings of 1993 American Control Conference (San Francisco, CA, USA, 1993)*, pp. 3106–3107, IEEE, 1993.

[2] F. Alabau-Boussouira, Asymptotic behavior for Timoshenko beams subject to a single nonlinear feedback control. *NoDEA Nonlinear Differential Equations Appl.* **14** (2007), no. 5-6, 643–669.

[3] V. I. Arnol'd, *Mathematical methods of Classical Mechanics*. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein. Corrected reprint of the second (1989) edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1989.

[4] A. Benaissa, M. Miloudi and M. Mokhtari, Global existence and energy decay of solutions to a Bresse system with delay terms. *Comment. Math. Univ. Carolin.* **56** (2015), no. 2, 169–186.

[5] L. Bouzettouta, S. Zitouni, Kh. Zennir and A. Guesmia, Stability of Bresse system with internal distributed delay. *J. Math. Comput. Sci.* **7** (2017), no. 1, 92–118.

[6] J. A. C. Bresse, *Cours de Mécanique Appliquée: Professé à L'École Impériale des Ponts et Chaussées. Hydraulique*. Seconde partie. Vol. 2. Mallet-Bachelier, 1860.

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti and I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping—source interaction. *J. Differential Equations* **236** (2007), no. 2, 407–459.

[8] M. Daoulatli, I. Lasiecka and D. Toundykov, Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth restrictions. *Discrete Contin. Dyn. Syst. Ser. S* **2** (2009), no. 1, 67–94.

[9] R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.* **24** (1986), no. 1, 152–156.

[10] J. Dieudonné, *Calcul Infinitésimal.* (French) Hermann, Paris 1968 479 pp. (loose errata).

[11] Z.-J. Han and G.-Q. Xu, Exponential stability of Timoshenko beam system with delay terms in boundary feedbacks. *ESAIM Control Optim. Calc. Var.* **17** (2011), no. 2, 552–574.

[12] A. Haraux, Two remarks on hyperbolic dissipative problems. Nonlinear partial differential equations and their applications. *Collège de France seminar, Vol. VII (Paris, 1983–1984)*, 6, 161–179, Res. Notes in Math., 122, *Pitman, Boston, MA*, 1985.

[13] V. Komornik, *Exact Controllability and Stabilization.* The multiplier method. RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.

[14] J. U. Kim and Y. Renardy, Boundary control of the Timoshenko beam. *SIAM J. Control Optim.* **25** (1987), no. 6, 1417–1429.

[15] H. E. Khochemane, S. Zitouni and L. Bouzettouta, Stability result for a nonlinear damping porous-elastic system with delay term. *Nonlinear Stud.* **27** (2020), no. 2, 487–503.

[16] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential Integral Equations* **6** (1993), no. 3, 507–533.

[17] B. Said-Houari and Y. Laskri, A stability result of a Timoshenko system with a delay term in the internal feedback. *Appl. Math. Comput.* **217** (2010), no. 6, 2857–2869.

[18] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*. Dunod, Paris, 1969.

[19] W.-J. Liu and E. Zuazua, Decay rates for dissipative wave equations. Papers in memory of Ennio De Giorgi (Italian). *Ricerche Mat.* **48** (1999), suppl., 61–75.

[20] Z. Liu and B. Rao, Energy decay rate of the thermoelastic Bresse system. *Z. Angew. Math. Phys.* **60** (2009), no. 1, 54–69.

[21] S. A. Messaoudi and M. I. Mustafa, On the stabilization of the Timoshenko system by a weak nonlinear dissipation. *Math. Methods Appl. Sci.* **32** (2009), no. 4, 454–469.

[22] S. A. Messaoudi and M. I. Mustafa, On the internal and boundary stabilization of Timoshenko beams. *NoDEA Nonlinear Differential Equations Appl.* **15** (2008), no. 6, 655–671.

[23] J. E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems-global existence and exponential stability. *J. Math. Anal. Appl.* **276** (2002), no. 1, 248–278.

[24] J. E. Muñoz Rivera and R. Racke, Global stability for damped Timoshenko systems. *Discrete Contin. Dyn. Syst.* **9** (2003), no. 6, 1625–1639.

[25] M. Nakao, Decay of solutions of some nonlinear evolution equations. *J. Math. Anal. Appl.* **60** (1977), no. 2, 542–549.

[26] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.* **45** (2006), no. 5, 1561–1585.

[27] J. Y. Park and S. H. Park, General decay for a nonlinear beam equation with weak dissipation. *J. Math. Phys.* **51** (2010), no. 7, 073508, 8 pp.

[28] J.-H. Park and J.-R. Kang, Energy decay of solutions for Timoshenko beam with a weak nonlinear dissipation. *IMA J. Appl. Math.* **76** (2011), no. 2, 340–350.

[29] C. A. Raposo, J. Ferreira, M. L. Santos and N. N. O. Castro, Exponential stability for the Timoshenko system with two weak dampings. *Appl. Math. Lett.* **18** (2005), no. 5, 535–541.

[30] W. Rudin, *Real and Complex Analysis*. Second edition. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York–Düsseldorf–Johannesburg, 1974.

[31] H. Suh and Z. Bien, Use of time-delay actions in the controller design. *IEEE Trans. Autom. Control* **25** (1980), no. 3, 600–603.

[32] F. G. Shinskey, *Process Control Systems*. McGraw-Hill Book Company, New York, 1967.

[33] S. P. Timoshenko, On the correction for shear of the differential equation for transverse Vibrations of prismatic bars. *Phil. Mag.* **41** (1921), 744–746.

(Received 29.05.2022; revised 24.11.2022; accepted 13.12.2022)

Authors' addresses:

Mokhtar Mokhtari

Department of Economic Sciences, Ibn-Khaldoun Tiaret University, BP P 78 zaâroura 14000, Tiaret, Algérie.

E-mail: `mokhtar.mokhtari@univ-tiaret.dz`

Lamine Bouzettouta

University 20 August 1955, B.P.26 route d'El-Hadaiek Skikda 21000 Algeria, Skikda.

E-mail: `lami_750000@yahoo.fr`