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THE EXISTENCE AND HYERS–ULAM STABILITY OF SOLUTION FOR AN IMPULSIVE TYPES AMBARTSUMIAN EQUATION VIA Ξ-HILFER GENERALIZED PROPORTIONAL FRACTIONAL DERIVATIVE

Abstract. In this paper, we investigate some existence and Ulam's type stability concepts of fixed point for a class of Ambartsumian equations with impulses via Ξ-Hilfer generalized proportional fractional derivative (PFD). Our results are obtained by using standard fixed point theorems.

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Key words and phrases. Ambartsumian equation, impulsive, proportional fractional derivative, existence, uniqueness, Ulam–Hyers–Rassias stability.

რეზიუმე. ნაშრომში Ξ-ჰილფერის განზოგადებული პროპორციული წილადი წარმოებულის (PFD)-ს მეშვეობით გამოკვლეულია უძრავი წერტილის არსებობისა და ულამის ტიპის მდგრადობის კონცეფციები ამბარცუმიანის განტოლებების ერთი კლასისთვის იმპულსებით. შედეგები მიღებულია სტანდარტული უძრავი წერტილის თეორემების გამოყენებით.

1 Introduction

A fractional order differential equation (FODE) is a generalized form of an integer order differential equation. The FODE is useful in many areas, e.g., for the depiction of a physical model of various phenomena in pure and applied science (see [1, 4, 18, 23] and the references therein). The resulting equations offer inconceivable thought for researchers and analysts. The first definition of the fractional derivative was introduced at the end of nineteenth century by Liouville and Riemann, but the concept of fractional derivative and integral was mentioned already in 1695 by Leibniz and L'Hospital. Actually, FODEs are considered as an alternative model to integer differential equation. The definition of Riemann–Liouville derivative was established by Riemann in 1876. Since then, many applications of the fractional derivatives and integrals of this Riemann–Liouville type have been demonstrated in numerous fields of science and technology. The Hilfer fractional operators were suggested in 2000. The fundamental theorems of fractional calculus for the Hilfer fractional derivatives are described in [1,6]. Then, these operators were generalized by Sousa and Oliveira [5, 20]. The well-known Riemann–Liouville and Caputo fractional derivatives are the special cases of the Hilfer fractional derivative. In [3, 7–9, 15], the authors presented a new fractional derivative called Ψ-Hilfer generalized proportional fractional derivative (PFD), which generalizes most of the previous fractional derivative of a function. For more details see [10–12].

The states of many evolutionary processes are often subject to instantaneous perturbations and experience abrupt changes at certain moments of time. Usually, the duration of the changes is very short and negligible in comparison with the duration of the process under consideration, so, as a result, it is natural to study differential equations with instantaneous impulses. The mathematical investigation of impulsive ordinary differential equations began with Milman and Myshkis [16], where some general concepts of systems with an impulse effect were given and also the results for the stability of solutions were presented. Inspired by them, a number of results on the qualitative analysis for impulsive differential equations have appeared in the literature (see [13, 22] and the references therein).

In many practical problems, it is a great challenge to formulate an exact solution for certain differential equation of a physical model. Powerful numerical or analytical principles with algorithms and methods that can produce stable outcomes are necessary. In this case, stability analysis is used, which forms an incredibly obvious part of differential equation. It is important to discuss the approximate solution and determine whether it lies near the exact solution. In general, we confirm the stability of a differential equation if, for each solution of a troubled equation, an approximate solution nearby the exact solution exists. In the literature, there exist different types of stability, but recently the concept of Hyers–Ulam stability is a central topic for researchers because it is very important in approximation theory. The historical background of the Hyers–Ulam stability dates back to the nineteenth century. Ulam detailed a class of stability with respect to a functional equation, which was solved by Hyers for an additive function defined on the Banach space. There are many advantages of Hyers–Ulam type stability in solving the problems related to optimization techniques, numerical analysis, control theory and many more. Further advances in the Hyers–Ulam stability of differential equations can be found in [5,21]. Therefore, our work may broaden some of the results on other topics related to the existence theory and stability results.

The standard Ambartsumian equation (SAE) was derived by Ambartsumian [2, 14, 17] more than two decades ago. This equation describes the absorption of light by the interstellar matter. Inspired by the above results, in this paper, we aim to generalize the concepts above to the SAE with impulses via a Hilfer type generalized fractional derivative of the form

$$\begin{aligned} \mathcal{D}^{p,q,\varrho;\Xi}_{a^+}\mathcal{A}(t) &= \mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t\in J-\{t_1,t_2,t_3,\dots,t_m\},\\ \triangle\mathcal{I}^{1-\vartheta,\varrho;\Xi}_{a^+}\mathcal{A}(t_k) &= \mu_k\in\mathbb{R}, \quad k=1,2,3,4,\dots,m,\\ \mathcal{I}^{1-\vartheta,\varrho;\Xi}_{a^+}\mathcal{A}(a) &= \delta\in\mathbb{R}, \end{aligned} \tag{1.1}$$

where $J=[a,b]$, $0<p<1$ and $0\le q\le 1$, $\varrho\in(0,1]$, $\eta>1$, $\vartheta=p+q-pq$, $\mathbb{Q}:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function and Ξ is a positive increasing function, $a=t_0<t_1<t_2<\dots<t_m=b$,

$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}(\cdot)$ is the Ξ-Hilfer generalized PFD of order p and type q, $\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}$ is the Ξ-Hilfer generalized proportional integral operator,

$$\triangle \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k) = \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k^+) - \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k^-),$$
$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k^+) = \lim_{\epsilon\to 0^+}\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k+\epsilon) \text{ and } \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k^-) = \lim_{\epsilon\to 0^-}\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k+\epsilon).$$

2 Basic concepts

Let $0 \le a < b < \infty$, $J = [a,b]$ be a finite interval and ϑ be a parameter such that $n-1 \le \vartheta < n$.

Definition 2.1 ([15]). If $\varrho \in (0,1]$ and $p > 0$, then the left-sided generalized proportional fractional integral of order p of the function $\mathbb{Q}$ with respect to another function Ξ is defined by

$$(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q})(t) = \frac{1}{\varrho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}(s)\,ds, \quad t > a. \tag{2.1}$$

Definition 2.2 ([15]). If $\varrho \in (0,1]$, $p > 0$ and $\Xi \in C[a,b]$ where $\Xi'(s) > 0$, then the left-sided generalized PFD of order p of the function $\mathbb{Q}$ with respect to another function Ξ is defined by

$$(\mathcal{D}_{a^+}^{p,\varrho,\Xi}\mathbb{Q})(t) = \frac{\mathcal{D}_t^{n,\varrho,\Xi}}{\varrho^{n-p}\Gamma(n-p)}\int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{n-p-1}\Xi'(s)\mathbb{Q}(s)\,ds,$$

where $\Gamma(\cdot)$ is the gamma function.

Theorem 2.1 ([15]). *Suppose $\varrho \in (0,1]$, $p > 0$ and $q > 0$. Then, if $\mathbb{Q}$ is continuous and defined for $t \ge a$, we have*

$$\mathcal{I}_{a^+}^{p,\varrho,\Xi}(\mathcal{I}_{a^+}^{q,\varrho,\Xi}\mathbb{Q})(t) = \mathcal{I}_{a^+}^{q,\varrho,\Xi}(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q})(t) = (\mathcal{I}_{a^+}^{p+q,\varrho,\Xi}\mathbb{Q})(t).$$

Definition 2.3 ([15]). Let $J = [a,b]$, where $-\infty \le a < b \le \infty$, be an interval and let $\mathbb{Q}, \Xi \in C^n[a,b]$ be two functions such that Ξ is positive strictly increasing and $\Xi'(t) \ne 0$, $\forall\, t \in [a,b]$. The Ξ-Hilfer generalized PFDs of order p and type q of $\mathbb{Q}$ with respect to another function Ξ are defined by

$$(\mathcal{D}_{a^\pm}^{p,q,\varrho,\Xi}\mathbb{Q})(t) = \left(\mathcal{I}_{a^\pm}^{q(n-p),\varrho,\Xi}(\mathcal{D}^{n,\varrho,\Xi})\mathcal{D}_{a^\pm}^{(1-q)(n-p),\varrho,\Xi}\mathbb{Q}\right)(t), \tag{2.2}$$

where $n-1 < p < n, 0 \le q \le 1$ with $n \in \mathbb{N}$ and $\varrho \in (0,1]$. Also, $\mathcal{D}^{\varrho,\Xi}\mathbb{Q}(t) = (1-\varrho)\mathbb{Q}(t) + \varrho\,\frac{\mathbb{Q}'(t)}{\Xi'(t)}$ and $\mathcal{I}$ is the generalized proportional fractional integral defined by equation (2.1).

In particular, if $n = 1$, then $0 < p < 1$ and $0 \le q \le 1$, so equation (2.2) becomes

$$(\mathcal{D}_{a^\pm}^{p,q,\varrho,\Xi}\mathbb{Q})(t) = \left(\mathcal{I}_{a^\pm}^{q(1-p),\varrho,\Xi}(\mathcal{D}^{1,\varrho,\Xi})\mathcal{D}_{a^\pm}^{(1-q)(1-p),\varrho,\Xi}\mathbb{Q}\right)(t).$$

Lemma 2.1 ([15]). *Let $n-1 < p < n$ with $n \in \mathbb{N}$, $0 \le q \le 1$, $\varrho \in (0,1]$ and $\vartheta = p + q(n-p)$. If $\mathbb{Q} \in C_{n-\vartheta}^{\vartheta}[a,b]$ and $\mathcal{I}_{a^+}^{n-q(n-p),\varrho,\Xi}\mathbb{Q} \in C_{n-\vartheta,\Xi}^{n}[a,b]$, then $\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}$ exists in $(a,b]$ and*

$$\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}(t) = \mathbb{Q}(t), \quad t \in (a,b].$$

Lemma 2.2 ([15]). *Let $n-1 < p < n$ with $n \in \mathbb{N}, 0 \le q \le 1, \varrho \in (0,1]$ with $\vartheta = p + q(n-p)$ such that $n-1 < \vartheta < n$. If $\mathbb{Q} \in C_{\vartheta}[a,b]$ and $\mathcal{I}_{a^+}^{n-\vartheta,\varrho,\Xi}\mathbb{Q} \in C_{\vartheta,\Xi}^{n}[a,b]$, then*

$$(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q})(t) = \mathbb{Q}(t) - \left[\sum_{k=0}^{n}\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-k}}{\varrho^{\vartheta-k}\Gamma(\vartheta-k+1)}(\mathcal{I}_{a^+}^{k-\vartheta,\varrho,\Xi}\mathbb{Q})(a)\right].$$

Let us consider the weighted spaces

$$C_{1-\vartheta,\Xi}(J)=\Big\{\mathbb{Q}:\ \ J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}:(\Xi(t)-\Xi(a))^{1-\vartheta}\mathcal{A}(t)\in C(J)\Big\},\ \ 0<\vartheta<1.$$

Define the weighted space of a piecewise continuous function as follows:

$$\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})=\Big\{\mathbb{Q}:\ \ J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}:\mathbb{Q}\in C_{1-\vartheta;\Xi}((t_k,t_{k+1}],\mathbb{R}),\ \ k=0,1,2,\dots,m\Big\}.$$

Clearly, $\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})$ is a Banach space with the norm

$$\|\mathcal{A}\|_{\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})}=\sup_{t\in J}(\Xi(t)-\Xi(a))^{1-\vartheta}\mathcal{A}(t).$$

Theorem 2.2 ([13]; The $\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})$ type Arzelà–Ascoli theorem)**.** *Let X be a Banach space and $\mathcal{W}_{1-\vartheta;\Xi}\subset\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})$. If the following conditions are satisfied:*

- *$\mathcal{W}_{1-\vartheta;\Xi}$ is uniformly bounded subset of $\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})$;*
- *$\mathcal{W}_{1-\vartheta;\Xi}$ is equicontinuous in $(t_k,t_{k+1}),k=0,1,2,\dots,m$;*
- *$\mathcal{W}_{1-\vartheta;\Xi}=\big\{\mathcal{A}(t):\mathcal{A}\in\mathcal{W}_{1-\vartheta;\Xi}t\in J-t_1\cdots t_m\big\}$, $\mathcal{W}_{1-\vartheta;\Xi}(t_k^+)=\mathcal{A}(t_k^+):\mathcal{A}\in\mathcal{W}_{1-\vartheta;\Xi}$ and $\mathcal{W}_{1-\vartheta;\Xi}(t_k^-)=\mathcal{A}(t_k^-):\mathcal{A}\in\mathcal{W}_{1-\vartheta;\Xi}$ are relatively compact subset of X,*

then $\mathcal{W}_{1-\vartheta;\Xi}$ is a relatively compact subset of $\mathcal{P}C_{1-\vartheta,\Xi}(J,X)$.

Theorem 2.3 ([23]) (Krasnoselskii's fixed point theorem)**.** *Let B be a non-empty bounded closed convex subset of a Banach space X. Let $N,M:B\to X$ be two continuous operators satisfying the conditions:*

- *$Nx+My\in B$, whenever $x,y\in B$;*
- *N is compact and continuous;*
- *M is a contraction mapping.*

Then there exists $u\in B$ such that $u=Nu+Mu$.

3 Main results

The following Lemma helps us to construct an equivalent fractional integral equation of our proposed problem (1.1).

Lemma 3.1. *Let $0<p<1$ and $0\le q\le 1$, $\vartheta=p+q-pq$ and $\mathbb{Q}:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous. Then for any $b\in J$, a function $\mathcal{A}\in C_{1-\vartheta,\Xi}(J,\mathbb{R})$ defined by*

$$\mathcal{A}(t)=\eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big\{\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(b)-\mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=b}\Big\}+\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\quad(3.1)$$

is the solution of the following differential equation:

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t)=\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big),\ \ t\in J,$$

where

$$h_{\Xi}^{\varrho,\vartheta}(t,a)=\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)}.$$

Proof. Applying $\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}$ to both sides of equation (3.1), we obtain

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) = \Big\{\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(b) - \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=b}\Big\}\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\eth_{\Xi}^{\varrho,\vartheta}(t,a) + \mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t\in J.$$

For $0<\vartheta<1$, we get

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}(\Xi(t)-\Xi(a))^{\vartheta-1} = 0, \quad 0<\vartheta<1, \tag{3.2}$$

and by Lemma 2.1, we get

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) = \mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t\in J.$$

This completes the proof. □

Now, we will obtain the equivalent fractional integral equation of the proposed problem (1.1) by using Lemma 3.1.

Lemma 3.2. *Let $\mathbb{Q}: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Then a function $\mathcal{A}\in\mathcal{PC}_{1-\vartheta,\Xi}(J,\mathbb{R})$ is a solution of the impulsive type Ambartsumian equation with the Ξ-Hilfer generalized PFD,*

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) = \mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t\in J-\{t_1,t_2,t_3,\dots,t_m\}, \tag{3.3}$$

$$\triangle\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k) = \mu_k\in\mathbb{R}, \quad k=1,2,3,4,\dots,m, \tag{3.4}$$

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(a) = \delta\in\mathbb{R}, \tag{3.5}$$

if and only if $\mathcal{A}$ is a solution of the following fractional integral equation:

$$\mathcal{A}(t) = \begin{cases} \eth_{\Xi}^{\varrho,\vartheta}(t,a)\delta + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), & t\in[a,t_1],\\ \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(\delta+\sum_{i=1}^{k}\mu_i\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), & t\in(t_k,t_{k+1}], \quad k=1,2,\dots,m. \end{cases} \tag{3.6}$$

Proof. Let us assume that $\mathcal{A}\in\mathcal{PC}_{1-\vartheta,\Xi}(J,\mathbb{R})$ satisfies (3.3)–(3.5).

If $t\in[a,t_1]$, then

$$\begin{aligned} \mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) &= \mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big),\\ \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(a) &= \delta. \end{aligned} \tag{3.7}$$

Thus problem (3.7) is equivalent to the fractional integral equation

$$\mathcal{A}(t) = \eth_{\Xi}^{\varrho,\vartheta}(t,a)\delta + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t\in[a,t_1]. \tag{3.8}$$

For any $t\in(t_1,t_2]$, we have

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) = \mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t\in(t_1,t_2],$$

with

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_1^+) - \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_1^-) = \mu_1.$$

By Lemma 3.1, we have

$$\begin{aligned} \mathcal{A}(t) &= \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big\{\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_1^+) - \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=t_1}\Big\} + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\\ &= \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big\{\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_1^-) + \mu_1 - \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=t_1}\Big\}\\ &\quad + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t\in(t_1,t_2]. \end{aligned} \tag{3.9}$$

From (3.8), we get

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t) = \delta + \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big).$$

Hence

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t^-) - \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=t_1} = \delta. \tag{3.10}$$

From (3.9) and (3.10), we get

$$\mathcal{A}(t) = \eth_{\Xi}^{\varrho,\vartheta}(t,a)(\delta+\mu_1) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \ \ t \in (t_1, t_2]. \tag{3.11}$$

If $t \in (t_2, t_3]$, then

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) = \mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \ \ t \in (t_2, t_3]$$

with

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_2^+) - \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_2^-) = \mu_2.$$

Again, by Lemma 3.1, we have

$$\begin{aligned}\mathcal{A}(t) &= \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big\{\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_2^+) - \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=t_2}\Big\} + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big) \\ &= \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big\{\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_2^-) + \mu_2 - \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big\}, \ \ t \in (t_2, t_3].\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we get

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t) = (\delta+\mu_1) + \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big).$$

Hence

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_2^-) - \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=t_2} = \delta + \mu_1. \tag{3.13}$$

From (3.12) and (3.13), we get

$$\mathcal{A}(t) = \eth_{\Xi}^{\varrho,\vartheta}(t,a)(\delta+\mu_1+\mu_2) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \ \ t \in (t_2, t_3].$$

Continuing the above process, we obtain

$$\mathcal{A}(t) = \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(\delta + \sum_{i=1}^{k}\mu_i\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \ \ t \in (t_k, t_{k+1}], \ \ k = 1, 2, 3, \dots, m.$$

Conversely, let $\mathcal{A} \in \mathcal{P}C_{1-\vartheta,\Xi}(J, \mathbb{R})$ satisfy the fractional integral equation (3.6). Then for $t \in [a, t_1]$, we have

$$\mathcal{A}(t) = \eth_{\Xi}^{\varrho,\vartheta}(t,a)\delta + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big).$$

Now, applying the $\Xi-$ Hilfer generalized PFD to both sides of the above equation, we get

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) = \delta\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\eth_{\Xi}^{\varrho,\vartheta}(t,a) + \mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big).$$

By using equation (3.2),

$$\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) = \mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \ \ t \in [a, t_1].$$

For any $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, we have

$$\mathcal{A}(t) = \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(\delta + \sum_{i=1}^{k}\mu_i\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \ \ t \in (t_k, t_{k+1}], \ \ k = 1, 2, 3, \dots, m.$$

Now, applying the $\Xi-$ Hilfer generalized PFD to both sides of the above equation, we get

$$\begin{aligned}\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t) &= \Big(\delta + \sum_{i=1}^{k}\mu_i\Big)\mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\eth_{\Xi}^{\varrho,\vartheta}(t,a) + \mathcal{D}_{a^+}^{p,q,\varrho;\Xi}\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big),\\ &= \mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big).\end{aligned}$$

We have proved that $\mathcal{A}$ satisfies equation (3.3). Now, we have to prove that $\mathcal{A}$ satisfies (3.4) and (3.5).

Applying the integral operator $\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}$ to both sides of equation (3.8), we get

$$\begin{aligned}\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t) &= \delta\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\eth_{\Xi}^{\varrho,\vartheta}(t,a) + \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\\ &= \delta + \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big),\end{aligned}$$

from which we obtain

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t) = \delta.$$

Hence we have proved equation (3.5).

Now, from equation (3.6), for $t \in (t_k, t_{k+1}]$, we get

$$\begin{aligned}\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t) &= \Big(\delta + \sum_{i=1}^{k}\mu_i\Big)\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\eth_{\Xi}^{\varrho,\vartheta}(t,a) + \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big),\\ &= \delta + \sum_{i=1}^{k}\mu_i + \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \qquad (3.14)\end{aligned}$$

and for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned}\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t) &= \Big(\delta + \sum_{i=1}^{k-1}\mu_i\Big)\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\eth_{\Xi}^{\varrho,\vartheta}(t,a) + \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big),\\ &= \delta + \sum_{i=1}^{k-1}\mu_i + \mathcal{I}_{a^+}^{1-\vartheta+p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big). \qquad (3.15)\end{aligned}$$

Hence from (3.14) and (3.15), we obtain

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k^+) - \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k^-) = \sum_{i=1}^{k}\mu_i - \sum_{i=1}^{k-1}\mu_i = \mu_k.$$

which is condition (3.4). Hence $\mathcal{A}$ satisfies problem (3.3)–(3.5). □

4 Existence theory

Let us consider the following hypotheses:

(H_1) Let $\mathbb{Q} : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $\mathbb{Q} \in \mathcal{PC}_{1-\vartheta,\Xi}^{q(1-p)}[a,b]$ for any $\mathcal{A} \in \mathcal{PC}_{1-\vartheta,\Xi}^{\vartheta}[a,b]$.

(H_2) There exists a constant $0 < L \leq \frac{\Gamma(p+\vartheta)}{2\Gamma(\vartheta)(\Xi(T)-\Xi(a))^p}$ satisfying

$$\big|\mathbb{Q}(t,m,u) - \mathbb{Q}(t,\overline{m},\overline{u})\big| \leq k\big(|m-\overline{m}| + |u-\overline{u}|\big), \quad u,\overline{u},m,\overline{m} \in \mathbb{R}, \quad t \in J.$$

Theorem 4.1. *Let $0 < p < 1$, $0 \leq q \leq 1$ and $\vartheta = p + q(1-p)$. Suppose that the assumptions (H_1) and (H_2) hold. Then the impulsive problem (1.1) has at least one solution in the space $\mathcal{PC}_{1-\vartheta}[a,b]$.*

Proof. In view of Lemma 3.2, the fractional integral equation corresponding to the impulsive Ξ-Hilfer fractional differential equation (Ξ-HFDE) is given by

$$\mathcal{A}(t) = \eth_{\Xi}^{\vartheta}(t,a)\Big(\delta + \sum_{a<t_k<t} \mu_k\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t \in J. \tag{4.1}$$

Also, consider the set

$$B_r = \Big\{\mathcal{A} \in \mathcal{PC}_{1-\vartheta;\Xi}(J,\mathbb{R}) : \mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(a) = \delta, \quad \|\mathcal{A}\|_{\mathcal{PC}_{1-\vartheta}(J,\mathbb{R})} \le r\Big\}$$

with

$$r \ge 2\bigg(\frac{1}{\vartheta}\Big\{|\delta| + \sum_{i=1}^{m}|\mu_i|\Big\} + \frac{\mathcal{M}(\Xi(T)-\Xi(a))^{1-\vartheta+p}}{\Gamma(p+1)}\bigg),$$

where

$$\mathcal{M} = \sup_{\sigma\in J}|\mathbb{Q}(\sigma,0,0)|.$$

For all $t \in J$, consider the operators $\mathcal{G}$ and $\mathcal{H}$ defined on B_r by

$$(\mathcal{H}\mathcal{A})(t) = \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t \in J,$$

$$(\mathcal{G}\mathcal{A})(t) = \eth_{\Xi}^{\vartheta}(t,a)\Big(\delta + \sum_{a<t_k<t}\mu_k\Big), \quad t \in J.$$

By virtue of the operators $\mathcal{G}\mathcal{A}(t)$ and $\mathcal{H}\mathcal{A}(t)$, the fractional integral equation (4.1) can be rewritten as

$$\mathcal{A}(t) = \mathcal{G}\mathcal{A}(t) + \mathcal{H}\mathcal{A}(t), \quad \mathcal{A} \in \mathcal{PC}_{1-\vartheta}^{\vartheta}(J,\mathbb{R}).$$

Step 1. We prove that $\mathcal{G}\mathcal{A} + \mathcal{H}\overline{\mathcal{A}} \in B_r$ for any $\mathcal{A}, \overline{\mathcal{A}} \in B_r$:

$$\begin{aligned}
&\big|\big((\Xi(t)-\Xi(a))^{1-\vartheta}\mathcal{G}\mathcal{A}(t) + \mathcal{H}\overline{\mathcal{A}}(t)\big)\big| \\
&\quad = \Big|(\Xi(t)-\Xi(a))^{1-\vartheta}\Big\{\eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(\delta + \sum_{a<t_k<t}\mu_k\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \overline{\mathcal{A}}(t), \overline{\mathcal{A}}\Big(\frac{t}{\eta}\Big)\Big)\Big\}\Big| \\
&\quad \le \frac{1}{\Gamma(\vartheta)}\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\Big|\mathbb{Q}\Big(\sigma, \overline{\mathcal{A}}(\sigma), \overline{\mathcal{A}}\Big(\frac{\sigma}{\eta}\Big)\Big)\Big|\,d\sigma \\
&\le \frac{1}{\Gamma(\vartheta)}\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\Big|\mathbb{Q}\Big(\sigma, \overline{\mathcal{A}}(\sigma), \overline{\mathcal{A}}\Big(\frac{\sigma}{\eta}\Big)\Big) - \mathbb{Q}(\sigma,0,0)\Big|\,d\sigma \\
&\qquad + \frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)|\mathbb{Q}(\sigma,0,0)| \\
&\quad \le \frac{1}{\Gamma(\vartheta)}\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{L(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)|\overline{\mathcal{A}}(\sigma)|\,d\sigma \\
&\qquad + \frac{\mathcal{M}(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\,d\sigma \le \frac{1}{\Gamma(\vartheta)}\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) \\
&\quad + \frac{L(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)L(\Xi(\sigma)-\Xi(a))^{\vartheta-1}\big|L(\Xi(\sigma)-\Xi(a))^{1-\vartheta}\overline{\mathcal{A}}(\sigma)\big|\,d\sigma \\
&\qquad + \frac{\mathcal{M}(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\,\frac{(\Xi(t)-\Xi(a))^p}{p} \le \frac{1}{\Gamma(\vartheta)}\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ L(\Xi(t)-\Xi(a))^{1-\vartheta}\|\overline{\mathcal{A}}\|_{\mathcal{P}C_{1-\vartheta}(J,\mathbb{R})}\mathcal{I}_{a^+}^{p,\varrho;\Xi}(\Xi(t)-\Xi(a))^{\vartheta-1} + \frac{\mathcal{M}(\Xi(t)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)}\\
&\leq \frac{1}{\Gamma(\vartheta)}\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{L\Gamma(\vartheta)}{\Gamma(p+\vartheta)}(\Xi(t)-\Xi(a))^p\|\overline{\mathcal{A}}\|_{\mathcal{P}C_{1-\vartheta}(J,\mathbb{R})} + \frac{\mathcal{M}(\Xi(t)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)}\\
&\leq \frac{1}{\Gamma(\vartheta)}\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{L\Gamma(\vartheta)}{\Gamma(p+\vartheta)}(\Xi(T)-\Xi(a))^p r + \frac{\mathcal{M}(\Xi(T)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)}.
\end{aligned}
$$

Since

$$L \leq \frac{\Gamma(p+\vartheta)}{2\Gamma(\vartheta)(\Xi(T)-\Xi(a))^p}$$

and

$$r \geq 2\bigg(\frac{1}{\vartheta}\Big\{|\delta| + \sum_{i=1}^{m}|\mu_i|\Big\} + \frac{\mathcal{M}(\Xi(T)-\Xi(a))^{1-\vartheta+p}}{\Gamma(p+1)}\bigg),$$

we have

$$\Big|\Big((\Xi(t)-\Xi(a))^{1-\vartheta}\mathcal{G}\mathcal{A}(t) + \mathcal{H}\overline{\mathcal{A}}(t)\Big)\Big| \leq r, \quad t \in J.$$

Therefore,

$$\|\mathcal{G}\mathcal{A}(t) + \mathcal{H}\overline{\mathcal{A}}(t)\|_{\mathcal{P}C_{1-\vartheta;\Xi}(J,\mathbb{R})} \leq r.$$

We have proved that $\mathcal{G}\mathcal{A} + \mathcal{H}\overline{\mathcal{A}} \in B_k$.

Step 2. Clearly, $\mathcal{G}$ is a contraction with the contraction constant zero.

Step 3. $\mathcal{H}$ is compact and continuous.

The continuity of $\mathcal{H}$ follows from the continuity of $\mathbb{Q}$. Next, we prove that $\mathcal{H}$ is uniformly bounded on B_r.

Let $\mathcal{A} \in B_r$. Then by (H_2), for any $t \in J$, we have

$$
\begin{aligned}
\big|(\Xi(t)-\Xi(a))^{1-\vartheta}\mathcal{H}\mathcal{A}(t)\big| &= \Big|(\Xi(t)-\Xi(a))^{1-\vartheta}\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|\\
&\leq \frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\Big|\mathbb{Q}\Big(\sigma,\mathcal{A}(\sigma),\mathcal{A}\big(\frac{\sigma}{\eta}\big)\Big)\Big|\,d\sigma\\
&\leq \frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\Big|\mathbb{Q}\Big(\sigma,\mathcal{A}(\sigma),\mathcal{A}\Big(\frac{\sigma}{\eta}\Big)\Big) - \mathbb{Q}(\sigma,0,0)\Big|\,d\sigma\\
&\quad + \frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)|\mathbb{Q}(\sigma,0,0)|\,d\sigma\\
&\leq \frac{L(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)|\mathcal{A}(\sigma)|\,d\sigma + \frac{\mathcal{M}(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t \mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\,d\sigma\\
&\leq \frac{L\Gamma(\vartheta)}{\vartheta^p\Gamma(p+\vartheta)}(\Xi(t)-\Xi(a))^p\|\overline{\mathcal{A}}\|_{\mathcal{P}C_{1-\vartheta}(J,\mathbb{R})} + \frac{\mathcal{M}(\Xi(t)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)}\\
&\leq \frac{L\Gamma(\vartheta)}{\vartheta^p\Gamma(p+\vartheta)}(\Xi(T)-\Xi(a))^p r + \frac{\mathcal{M}(\Xi(T)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)}.
\end{aligned}
$$

Therefore,

$$\|\mathcal{H}\mathcal{A}\|_{\mathcal{P}C_{1-\vartheta}(J,\mathbb{R})} \leq \frac{L\Gamma(\vartheta)}{\vartheta^p\Gamma(p+\vartheta)}(\Xi(T)-\Xi(a))^p r + \frac{\mathcal{M}(\Xi(T)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)}.$$

This proves that $\mathcal{H}$ is uniformly bounded on B_r. Next, we show that $\mathcal{H}B_r$ is equicontinuous.

Let $\mathcal{A} \in B_r$ and $t_1, t_2 \in (t_k, t_{k+1}]$ for some k, $k = 0, 1, 2, \ldots, m$ with $t_1 < t_2$. Then,

$$\begin{aligned}
&\big|(\mathcal{H}\mathcal{A}(t_2)) - (\mathcal{H}\mathcal{A}(t_1))\big| = \left|\Big(\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=t_2}\Big) - \Big(\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|_{t=t_1}\Big)\right| \\
&\leq \frac{1}{\vartheta^p\Gamma(p)}\int_a^{t_2}\mathcal{L}_{\Xi}^{p,\varrho}(t_2,\sigma)\Big|\mathbb{Q}\Big(\sigma, \mathcal{A}(\sigma), \mathcal{A}\Big(\frac{\sigma}{\eta}\Big)\Big)\Big|\,d\sigma - \frac{1}{\vartheta^p\Gamma(p)}\int_a^{t_1}\mathcal{L}_{\Xi}^{p,\varrho}(t_1,\sigma)\Big|\mathbb{Q}\Big(\sigma, \mathcal{A}(\sigma), \mathcal{A}\Big(\frac{\sigma}{\eta}\Big)\Big)\Big|\,d\sigma \\
&= \frac{1}{\vartheta^p\Gamma(p)}\int_a^{t_2}\mathcal{L}_{\Xi}^{p,\varrho}(t_2,\sigma)(\Xi(t)-\Xi(a))^{\vartheta-1}\Big|(\Xi(t)-\Xi(a))^{1-\vartheta}\mathbb{Q}\Big(\sigma, \mathcal{A}(\sigma), \mathcal{A}\Big(\frac{\sigma}{\eta}\Big)\Big)\Big|\,d\sigma \\
&\quad - \frac{1}{\vartheta^p\Gamma(p)}\int_a^{t_1}\mathcal{L}_{\Xi}^{p,\varrho}(t_1,\sigma)(\Xi(t)-\Xi(a))^{\vartheta-1}\Big|(\Xi(t)-\Xi(a))^{1-\vartheta}\mathbb{Q}\Big(\sigma, \mathcal{A}(\sigma), \mathcal{A}\Big(\frac{\sigma}{\eta}\Big)\Big)\Big|\,d\sigma \\
&\leq \Big\{\Big(\mathcal{I}_{a^+}^{p,\varrho;\Xi}(\Xi(t)-\Xi(a))^{\vartheta-1}\Big|_{t=t_2}\Big) - \Big(\mathcal{I}_{a^+}^{p,\varrho;\Xi}(\Xi(t)-\Xi(a))^{\vartheta-1}\Big|_{t=t_1}\Big)\Big\}\|\mathbb{Q}\|_{\mathcal{P}C_{1-\vartheta;\Xi}(J,\mathbb{R})} \\
&\leq \frac{\Gamma(\vartheta)}{\vartheta^p\Gamma(\vartheta+p)}\Big\{(\Xi(t_2)-\Xi(a))^{1-\vartheta+p} - (\Xi(t_1)-\Xi(a))^{1-\vartheta+p}\Big\}\|\mathbb{Q}\|_{\mathcal{P}C_{1-\vartheta;\Xi}(J,\mathbb{R})}.
\end{aligned}$$

Hence

$$\big|(\mathcal{H}\mathcal{A}(t_2)) - (\mathcal{H}\mathcal{A}(t_1))\big| \to 0 \;\text{ as }\; t_2 \to t_1.$$

This shows that $\mathcal{H}$ is equicontinuous on $(t_k, t_{k+1}]$. Therefore, $\mathcal{H}$ is relatively compact on B_r. By the $\mathcal{P}C_{1-\vartheta;\Xi}$ type Arzelà–Ascoli theorem, $\mathcal{H}$ is compact on B_k. Since all the assumptions of Krasnoselskii's fixed point theorem are satisfied, the operator equation $\mathcal{A} = \mathcal{G}\mathcal{A} + \mathcal{H}\mathcal{A}$ has a fixed point $\widetilde{\mathcal{A}} \in \mathcal{P}C_{1-\vartheta;\Xi}(J,\mathbb{R})$, which is the the solution of our proposed problem. □

Theorem 4.2. *Let $0 < p < 1$, $0 \leq q \leq 1$ and $\vartheta = p + q(1-p)$. Suppose that the assumptions (H_1) and (H_2) hold for $\mathbb{Q} : (a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, then the problem has a unique solution in the space $\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})$.*

Proof. Consider the set B_r and define the operator $\mathcal{T}$ on B_r by

$$(\mathcal{T}\mathcal{A})(t) = \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(\delta + \sum_{a<t_k<t}\mu_k\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big), \quad t \in J.$$

To prove that $\mathcal{A} = \mathcal{T}\mathcal{A}$ has a fixed point, we show that $\mathcal{T}B_r \subset B_r$. To do this, we take any $\mathcal{A} \in B_r$. Then, by (H_2) for any $t \in J$, we have

$$\begin{aligned}
&|\mathcal{T}\mathcal{A}(t)| = \left|\eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(\delta + \sum_{a<t_k<t}\mu_k\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\right| \\
&\leq \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{1}{\vartheta^p\Gamma(p)}\int_a^{t}\mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\Big|\mathbb{Q}\Big(\sigma, \mathcal{A}(\sigma), \mathcal{A}\Big(\frac{\sigma}{\eta}\Big)\Big)\Big|\,d\sigma \\
&\leq \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{1}{\vartheta^p\Gamma(p)}\int_a^{t}\mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\Big|\mathbb{Q}\Big(\sigma, \mathcal{A}(\sigma), \mathcal{A}\Big(\frac{\sigma}{\eta}\Big)\Big) - \mathbb{Q}(\sigma,0,0)\Big|\,d\sigma \\
&\quad + \frac{1}{\vartheta^p\Gamma(p)}\int_a^{t}\mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)|\mathbb{Q}(\sigma,0,0)|\,d\sigma \\
&\leq \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{L\Gamma(\vartheta)}{\Gamma(p+\vartheta)}(\Xi(t)-\Xi(a))^p\|\overline{\mathcal{A}}\|_{\mathcal{P}C_{1-\vartheta}(J,\mathbb{R})} + \frac{\mathcal{M}(\Xi(t)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)} \\
&\leq \eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(|\delta| + \sum_{a<t_k<t}|\mu_k|\Big) + \frac{L\Gamma(\vartheta)}{\Gamma(p+\vartheta)}(\Xi(T)-\Xi(a))^p r + \frac{\mathcal{M}(\Xi(T)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)}.
\end{aligned}$$

Thus

$$\begin{aligned}&\left|(\Xi(t)-\Xi(a))^{1-\vartheta}\mathcal{T}\mathcal{A}(t)\right|\\&\qquad\le\frac{1}{\Gamma(\vartheta)}\Big(|\delta|+\sum_{a<t_k<t}|\mu_k|\Big)+\frac{L\Gamma(\vartheta)}{\Gamma(p+\vartheta)}(\Xi(T)-\Xi(a))^p r+\frac{\mathcal{M}(\Xi(T)-\Xi(a))^{1-\vartheta+p}}{\vartheta^p\Gamma(p+1)}.\end{aligned}$$

By the choice of the constants r and L, it can be easily verified that

$$\|\mathcal{T}\mathcal{A}\|_{\mathcal{P}C_{1-\vartheta;\Xi}(J,\mathbb{R})}\le r.$$

This proves that $\mathcal{T}B_r\subset B_r$.

Now, we prove that the operator $\mathcal{T}$ is a contraction on B_r. Then, by the assumption (H_2), for any $t\in J$,

$$\begin{aligned}&\left|(\Xi(t)-\Xi(a))^{1-\vartheta}\big((\mathcal{T}\mathcal{A}_1)(t)-(\mathcal{T}\mathcal{A}_2)(t)\big)\right|\\&\quad=\Bigg|(\Xi(t)-\Xi(a))^{1-\vartheta}\Bigg(\eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(\delta+\sum_{a<t_k<t}\mu_k\Big)+\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}_1(t),\mathcal{A}_1\Big(\frac{t}{\eta}\Big)\Big)\\&\qquad\qquad-\eth_{\Xi}^{\varrho,\vartheta}(t,a)\Big(\delta+\sum_{a<t_k<t}\mu_k\Big)+\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}_2(t),\mathcal{A}_2\Big(\frac{t}{\eta}\Big)\Big)\Bigg)\Bigg|\\&\quad=\Bigg|(\Xi(t)-\Xi(a))^{1-\vartheta}\Bigg(\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}_1(t),\mathcal{A}_1\Big(\frac{t}{\eta}\Big)\Big)-\mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t,\mathcal{A}_2(t),\mathcal{A}_2\Big(\frac{t}{\eta}\Big)\Big)\Bigg)\Bigg|\\&\quad\le\frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t\mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)\Big|\mathbb{Q}\Big(\sigma,\mathcal{A}_1(\sigma),\mathcal{A}_1\Big(\frac{\sigma}{\eta}\Big)\Big)-\mathbb{Q}\Big(\sigma,\mathcal{A}_2(\sigma),\mathcal{A}_2\Big(\frac{\sigma}{\eta}\Big)\Big)\Big|\,d\sigma\\&\quad\le\frac{L(\Xi(t)-\Xi(a))^{1-\vartheta}}{\vartheta^p\Gamma(p)}\int_a^t\mathcal{L}_{\Xi}^{p,\varrho}(t,\sigma)|\mathcal{A}_1(\sigma)-\mathcal{A}_2(\sigma)|\,d\sigma\\&\quad\le\frac{L\Gamma(\vartheta)}{\Gamma(p+\vartheta)}(\Xi(T)-\Xi(a))^p\|\mathcal{A}_1-\mathcal{A}_2\|_{\mathcal{P}C_{1-\vartheta;\Xi}(J,\mathbb{R})}.\end{aligned}$$

From the choice of constant L, it follows that

$$\|\mathcal{T}\mathcal{A}_1-\mathcal{T}\mathcal{A}_2\|_{\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})}\le\frac{1}{2}\|\mathcal{A}_1-\mathcal{A}_2\|_{\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})}.$$

Thus $\mathcal{T}$ is a contraction and, by the Banach contraction principle, it has a unique fixed point in $B_r\subseteq\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})$, which is the unique solution of our proposed problem. □

5 Stability theory

Now, we consider the Ulam–Hyers(U–H) stability for the problem. Let $\mathcal{A}\in\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R}),\epsilon>0,\tau>0$ and $\nu:(a,b]\to[0,\infty)$ be a continuous function.

We consider the following inequality:

$$\begin{aligned}&\Big|\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{A}(t)-\mathbb{Q}\Big(t,\mathcal{A}(t),\mathcal{A}\Big(\frac{t}{\eta}\Big)\Big)\Big|\le\epsilon\nu(t),\quad t\in J,\\&\big|\triangle\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k)-\mu_k\big|=\epsilon\tau,\quad k=1,2,3,4,\dots,m.\end{aligned}\tag{5.1}$$

Definition 5.1 ([19]). Our proposed problem is U–H–R stable with respect to (ν,τ) if there exists a real number $\alpha_{\mathbb{Q}}>0$ such that for each $\epsilon>0$ and for each solution $\mathcal{A}\in\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})$ of inequality (5.1) there exists a solution $\overline{\mathcal{A}}\in\mathcal{P}C_{1-\vartheta,\Xi}(J,\mathbb{R})$ of our proposed problem with

$$|\mathcal{A}(t)-\overline{\mathcal{A}}(t)|\le\epsilon\alpha_{\mathbb{Q},\nu}(\nu(t)+\tau),\quad t\in(a,b].$$

Remark 5.1. A function $\mathcal{A} \in \mathcal{PC}_{1-\vartheta,\Xi}(J,\mathbb{R})$ is a solution of inequality (5.1) if and only if there exists a function $\lambda \in \mathcal{PC}_{1-\vartheta,\Xi}(J,\mathbb{R})$ such that

- $|\lambda(t)| < \epsilon\nu(t)$ and $|\lambda_k| \leq \epsilon\tau$, $t \in J$.
- $\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{A}(t) = \mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\dfrac{t}{\eta}\Big)\Big) + \lambda(t)$, $t \in J$.
- $\triangle\mathcal{I}_{a^+}^{p,q,\varrho;\Xi}\mathcal{A}(t_k) = \mu_k + \lambda_k$, $k = 1,2,3,4,\ldots,m$.

Theorem 5.1. *Let us assume that (H_1) and (H_2) and the following hypotheses hold:*

(H_3) *There exist an increasing function $\nu \in \mathcal{PC}_{1-\vartheta;\Xi}(J,\mathbb{R})$ and $\zeta_\nu > 0$ such that for each $t \in (a,b]$, we have*
$$(\mathcal{I}_{a^+}^{p,\varrho;\Xi}\nu) \leq \zeta_\nu \nu(t).$$

(H_4) *There exists a continuous function $\psi : [a,b] \to [0,\infty)$ such that for each $t \in J_k$, $k = 0,1,2,\ldots,m$. we have*
$$p(t) \leq \psi(t)\nu(t).$$

Then our proposed problem (1.1) is U–H–R stable with respect to (ν,τ).

Proof. Let $\mathcal{A} \in \mathcal{PC}_{1-\vartheta,\Xi}(J,\mathbb{R})$ be a function which satisfies inequality (5.1) and let $\overline{\mathcal{A}} \in \mathcal{PC}_{1-\vartheta,\Xi}(J,\mathbb{R})$ be the unique solution of the problem
$$\begin{gathered}
\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{A}(t) = \mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big),\\
\triangle\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}(t_k) = \mu_k \in \mathbb{R}, \quad k = 1,2,3,\ldots,m,\\
\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}(a) = \delta \in \mathbb{R},
\end{gathered}$$
where $0 < p < 1$ and $0 \leq q \leq 1$.

By Theorem 2.3, we get
$$\overline{\mathcal{A}}(t) = \eth_\Xi^\vartheta(t,a)\Big(\delta + \sum_{a<t_k<t}\mu_k\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \overline{\mathcal{A}}(t), \overline{\mathcal{A}}\Big(\frac{t}{\eta}\Big)\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\lambda(t), \quad t \in J.$$
Since $\mathcal{A}$ is the solution of inequality (5.1), by Remark 5.1, we have
$$\begin{aligned}
|\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{A}(t)| &\leq \mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big) + \lambda(t), \quad t \in J,\\
|\triangle\mathcal{I}_{a^+}^{1-\vartheta,\varrho;\Xi}\mathcal{A}(t_k)| &= \mu_k + \lambda_k, \quad k = 1,2,3,4,\ldots,m.
\end{aligned} \tag{5.2}$$
Clearly, the solution of (5.2) is given by
$$\mathcal{A}(t) = \eth_\Xi^\vartheta(t,a)\Big(\delta + \sum_{a<t_k<t}\mu_k\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big) + \mathcal{I}_{a^+}^{p,\varrho;\Xi}\lambda(t), \quad t \in J,$$
where $g : (a,b] \to E$ is a function satisfying the functional equation
$$g(t) = \mathbb{Q}\Big(t, g(t), g\Big(\frac{t}{\eta}\Big)\Big).$$
We have
$$\triangle\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}(t_k) = \mu_k + \sum_{a<t_k<t}\lambda_k, \quad k = 1,2,3,\ldots,m.$$
Hence for each $t \in (a,b]$,
$$\begin{aligned}
|\mathcal{A}(t) - \overline{\mathcal{A}}(t)| &\leq \Big|\mathcal{I}_{a^+}^{p,\varrho;\Xi}\Big(\mathbb{Q}\Big(s, \overline{\mathcal{A}}(s), \overline{\mathcal{A}}\Big(\frac{s}{\eta}\Big)\Big) - \mathbb{Q}\Big(s, \mathcal{A}(s), \mathcal{A}\Big(\frac{s}{\eta}\Big)\Big)\Big)\Big| + \mathcal{I}_{a^+}^{p,\varrho;\Xi}|\lambda(s)|\\
&\leq \epsilon\zeta_\nu\nu(t) + 2\psi^*\mathcal{I}_{a^+}^{p,\varrho;\Xi}\nu(t) \leq (\epsilon + 2\psi^*)\zeta_\nu\nu(t) \leq \Big(1 + \frac{2\psi^*}{\epsilon}\Big)\zeta_\nu\epsilon(\tau + \nu(t)) \leq \alpha_{\mathbb{Q},\nu}\epsilon(\tau + \nu(t)),
\end{aligned}$$

where $\psi^* = \sup_{t\in J} \psi(t)$ and

$$\alpha_{\mathbb{Q},\nu} = \Big(1 + \frac{2\psi^*}{\epsilon}\Big)\zeta_\nu.$$

This completes the proof of the theorem. □

6 Examples

Example 6.1. Consider the following impulsive type Ambartsumian equation of the Ξ-Hilfer generalized PFD:

$$\begin{aligned} \mathcal{D}_{a^+}^{\frac{1}{8},\frac{1}{5},\frac{1}{3},\Xi}\mathcal{A}(t) &= \frac{1}{80}\mathcal{A}\Big(\frac{t}{80}\Big) - \mathcal{A}(t), \quad t \in [0,1], \\ \mathcal{I}_{a^+}^{1-\vartheta,\frac{1}{3},\Xi}\mathcal{A}(0) &= 0, \\ \triangle_{a^+}^{1-\vartheta,\frac{1}{3},\Xi}\mathcal{A}\Big(\frac{1}{2}\Big) &= \frac{2}{3} \in \mathbb{R}. \end{aligned} \tag{6.1}$$

Now comparing equation (6.1) and our proposed problem (1.1), we get

$$p = \frac{1}{8}, \quad q = \frac{1}{5}, \quad \varrho = \frac{1}{3}, \quad \vartheta = \frac{3}{10}, \quad a = 0, \quad b = 1,$$

Also, $\mathbb{Q} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big) = \frac{1}{80}\mathcal{A}\Big(\frac{t}{80}\Big) - \mathcal{A}(t), \quad t \in [0,1].$$

Clearly, $\mathbb{Q}$ is continuous and

$$\Big|\mathbb{Q}\Big(t, \mathcal{A}_1(t), \mathcal{A}_1\Big(\frac{t}{\eta}\Big)\Big) - \mathbb{Q}\Big(t, \mathcal{A}_2(t), \mathcal{A}_2\Big(\frac{t}{\eta}\Big)\Big)\Big| \le \frac{1}{80}\left(|u - \overline{u}| + |v - \overline{v}|\right),$$

where $u, \overline{u}, v, \overline{v} \in \mathbb{R}$. Hence the hypotheses (H_1), (H_2) hold with $k = \frac{1}{80} < 1$.

Thus by Theorem 4.1 and Theorem 4.2, the considered problem has at least one solution which is unique on J.

An existing solution of (6.1) satisfies all the conditions of Theorem 5.1. Hence the solution is U–H–R Stable.

Example 6.2. Let us consider the impulsive type Ambartsumian equation with the Ξ-Hilfer generalized PFD:

$$\begin{aligned} \mathcal{D}_{0^+}^{\frac{1}{3},\frac{1}{7},\frac{2}{3},\Xi}\mathcal{A}(t) &= \frac{1}{70}\mathcal{A}\Big(\frac{t+3}{70}\Big) - \mathcal{A}(t+3), \quad t \in [0,1], \\ \mathcal{I}_{0^+}^{1-\vartheta,\frac{2}{3},\Xi}\mathcal{A}(0) &= 0, \\ \triangle_{0^+}^{1-\vartheta,\frac{2}{3},\Xi}\mathcal{A}\Big(\frac{1}{7}\Big) &= \frac{2}{5} \in \mathbb{R}. \end{aligned} \tag{6.2}$$

Now, comparing equation (6.2) and our proposed problem (1.1), we get

$$p = \frac{1}{3}, \quad q = \frac{1}{7}, \quad \varrho = \frac{2}{3}, \quad \vartheta = \frac{3}{7}, \quad a = 0, \quad b = 1 \in [0,1].$$

Also, $\mathbb{Q} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$\mathbb{Q}\Big(t, \mathcal{A}(t), \mathcal{A}\Big(\frac{t}{\eta}\Big)\Big) = \frac{1}{70}\mathcal{A}\Big(\frac{t+3}{70}\Big) - \mathcal{A}(t+3), \quad t \in [0,1].$$

Clearly, $\mathbb{Q}$ is continuous and

$$\Big|\mathbb{Q}\Big(t, \mathcal{A}_1(t), \mathcal{A}_1\Big(\frac{t}{\eta}\Big)\Big) - \mathbb{Q}\Big(t, \mathcal{A}_2(t), \mathcal{A}_2\Big(\frac{t}{\eta}\Big)\Big)\Big| \le \frac{1}{70}\left(|u - \overline{u}| + |v - \overline{v}|\right),$$

where $u, \overline{u}, v, \overline{v} \in \mathbb{R}$. Hence the hypotheses $(H_1), (H_2)$ hold with $k = \frac{1}{70}$.

We have $k < 1$, which means that the assumption (H_4) is also satisfied. Hence, by Theorem 4.1 and Theorem 4.2, the considered problem has at least one solution which is unique on J.

An existing solution of (6.2) satisfies all the conditions of Theorem 5.1. Hence the solution is U–H–R stable.

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