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# WELL-POSEDNESS AND EXPONENTIAL DECAY FOR PIEZOELECTRIC BEAMS WITH DISTRIBUTED DELAY TERM

**Abstract.** In this paper, we consider a one-dimensional system of piezoelectric beams with distributed delay in the mechanical equation. We first prove the well-posedness of the system by using the semigroup theory. Next, we find the energy expression related to this system, and by using the technique of Lyapunov functional, we demonstrate that this system is exponentially stable and is independent of any coefficient of the system.

**2020 Mathematics Subject Classification.** 35B40, 35B35.

**Key words and phrases.** Piezoelectric beams, distributed delay, semigroup theory, energy expression, Lyapunov functional, exponential stability.

**რეზიუმე.** ნაშრომში განვიხილავთ პიეზოელექტრული სხივების ერთგანზომილებიან სისტემას მექანიკურ განტოლებაში განაწილებული დაგვიანებით. თავდაპირველად, ნახევარჯგუფთა თეორიის გამოყენებით ჩვენ ვამტკიცებთ სისტემის კორექტულობას. შემდეგ ვპოულობთ სისტემასთან დაკავშირებულ ენერგიის გამოსახულებას და ლიაპუნოვის ფუნქციონალების ტექნიკის გამოყენებით ვაჩვენებთ, რომ ეს სისტემა ექსპონენციალურად სტაბილურია და არ არის დამოკიდებული სისტემის კოეფიციენტებზე.

# 1 Introduction

Piezoelectric materials have the property of converting from mechanical energy to electro-magnetic energy, or of generating an internal electrical charge from applied mechanical pressure (see [19]). The brothers, Pierre and Jacques Curie, first demonstrated the direct piezoelectric effect in 1880 [18], where a single crystal quartz was the first material used in early experiments with piezoelectricity. The brothers expanded their working knowledge of crystal fittings and thermoelectric materials (materials that generate an electric charge in response to temperature changes), by measuring the surface charges, where some materials as quartz, Rochelle salt and barium titanate are shown the highest piezoelectric effects, these same materials when exposed to electricity produce a relative tension, this phenomenon is known as the reverse piezoelectric effect, was discovered by Gabriel Lippmann in 1881 [18, 19]. These piezoelectric materials are used in various industries, including manufacturing, medical device industry, telecommunications and information technology.

In [8], Morris and Özer used a variational approach to derive the differential equations and boundary conditions that model a single piezoelectric beam with magnetic effects. Applying a Legendre transformation, they obtained

$$\widetilde{L} = \int_0^T [K - (P + E) + B + W]\, dt,$$

where $K$, $P + E$, $B$ and $W$ denote the (mechanical) kinetic energy, total stored energy, magnetic energy (electrical kinetic energy) of the beam and the work done by the external forces, respectively.

For a beam of length $L$ and thickness $h$, they found

$$P + E = \frac{h}{2}\int_0^L \Big[\alpha(v_x^2 + \frac{h^2}{12}\, w_{xx}^2) - 2\gamma\beta v_x p_x + \beta p_x^2\Big]\, dx,$$

$$B = \frac{\mu h}{2}\int_0^L p_t^2\, dx, \quad K = \frac{\rho h}{2}\int_0^L \Big(v_t^2 + \frac{h^2}{12}\, w_{xt}^2 + w_t^2\Big)\, dx,$$

$$W = \int_0^L -p_x V(t)\, dx,$$

where $V(t)$ denotes the voltage applied to the electrodes. Application of Hamilton's principle and setting the variation of admissible displacements $\{v, w, p\}$ of $L$ to zero yields two sets of equations, one for stretching and one for bending with the associated boundary conditions.

As the applied voltage $V(t)$ affects only the stretching motion, they neglected the equation of bending and studied the stretching equations

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \end{cases}$$

with the boundary and initial conditions

$$\begin{cases} v(0) = p(0) = \alpha v_x(L) - \gamma\beta p_x(L) = 0, \;\; \beta p_x(L) - \gamma\beta v_x(L) = -\dfrac{V(t)}{h}\,, \\ (v, p, v_t, p_t) = (v^0, p^0, v^1, p^1). \end{cases}$$

Finally, by using only an electrical feedback controller (the current flowing through the electrodes), they showed that the closed-loop system is strongly stable in the energy space.

Ramos et al. [16] studied the well-posedness of a solution for piezoelectric beams with magnetic effect

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 & \text{in } (0, L) \times (0, T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 & \text{in } (0, L) \times (0, T), \end{cases} \tag{1.1}$$

with the following conditions:

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) + \xi_1 \dfrac{v_t(L,t)}{h} = 0, & 0 < t < T, \\ p(0,t) = \beta p_x(L,t) - \gamma\beta v_x(L,t) + \xi_2 \dfrac{p_t(L,t)}{h} = 0, & 0 < t < T, \\ v(x,0) = v_0(x), \ \ v_t(x,0) = v_1(x) & \forall\, x \in (0,L), \\ p(x,0) = p_0(x), \ \ p_t(x,0) = p_1(x) & \forall\, x \in (0,L). \end{cases} \tag{1.2}$$

In the case $\xi_1 = \xi_2 = 0$ in system (1.1), (1.2), they obtained the following one-dimensional conservative system:

$$\begin{cases} \rho u_{tt} - \alpha u_{xx} + \gamma\beta z_{xx} = 0 & \text{in } (0,L) \times (0,\infty), \\ \mu z_{tt} - \beta z_{xx} + \gamma\beta u_{xx} = 0 & \text{in } (0,L) \times (0,\infty), \end{cases}$$

with the boundary and initial conditions

$$\begin{cases} u(0,t) = \alpha u_x(L,t) - \gamma\beta z_x(L,t) = 0 & \forall\, t > 0, \\ z(0,t) = z_x(L,t) - \gamma u_x(L,t) = 0 & \forall\, t > 0, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) & \forall\, x \in (0,L), \\ z(x,0) = z_0(x), \ \ z_t(x,0) = z_1(x) & \forall\, x \in (0,L), \end{cases}$$

and using multiplicative techniques, they obtained an observability inequality of a conservative system. Also,using the auxiliary problem given by

$$\begin{cases} \rho \phi_{tt} - \alpha \phi_{xx} + \gamma\beta \psi_{xx} = 0 & \text{in } (0,L) \times (0,T), \\ \mu \psi_{tt} - \beta \psi_{xx} + \gamma\beta \phi_{xx} = 0 & \text{in } (0,L) \times (0,T), \end{cases}$$

where $p = \psi + z$, $v = \phi + u$ and $(u,z)$ is a solution of conservative problem, with the boundary and initial conditions

$$\begin{cases} \phi(0,t) = \alpha \phi_x(L,t) - \gamma\beta \psi_x(L,t) + \xi_1 \dfrac{v_t(L,t)}{h} = 0 & \forall\, t > 0, \\ \psi(0,t) = \beta \psi_x(L,t) - \gamma\beta \phi_x(L,t) + \xi_2 \dfrac{p_t(L,t)}{h} = 0 & \forall\, t > 0, \\ \phi(x,0) = \phi_0(x) = \phi_t(x,0) = \phi_1(x) = 0 & \forall\, x \in (0,L), \\ \psi(x,0) = \psi_0(x) = \psi_t(x,0) = \psi_1(x) = 0 & \forall\, x \in (0,L), \end{cases}$$

and by using some lemmas, they proved the equivalence between stabilization and observability.

In [15], Ramos et al. proved exponential stability for piezoelectric beams with magnetic effect

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t = 0 & \text{in } (0,L) \times (0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 & \text{in } (0,L) \times (0,T), \end{cases} \tag{1.3}$$

with the following conditions:

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & 0 \le t \le T, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, & 0 \le t \le T, \\ v(x,0) = v_0(x), \ \ v_t(x,0) = v_1(x), & 0 \le x \le L, \\ p(x,0) = p_0(x), \ \ p_t(x,0) = p_1(x), & 0 \le x \le L, \end{cases}$$

where

$$\alpha = \alpha_1 + \gamma^2 \beta.$$

Using the finite differences method, they found numerical energy related to system (1.3), where specific values $L$, $\rho$, $\mu$, $\gamma$, $\beta$, $\delta$ were used for the numerical simulations.

Recently, Ramos et al. [17] proved the exponential stability for the system of piezoelectric beams with delays

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \xi_1 v_t + \xi_2 v_t(x, t-\tau) = 0 & \text{in } (0,L)\times(0,+\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 & \text{in } (0,L)\times(0,+\infty), \end{cases}$$

where $\xi_2 v_t(x, t-\tau)$ is the time of delay on vertical displacement, $\tau > 0$ is the respective retardation time. They proved this stability under the conditions $\xi_1 > \xi_2$.

We refer the reader to [5, 9, 10, 13, 20] and the references therein for more results related to piezoelectric systems.

In [4], Foughali et al. studied the well-posedness by using the semigroup theory for a porous-thermoelastic system with second sound and a distributed delay term and heat flux given by Cattaneo's law, they also proved the exponential stability.

For more results related to distributed delay term in different dimensions see [1–3, 6, 7, 11, 12] and the references therein.

Motivated by the above works, in the present paper, we consider the following problem:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \mu_1 v_t + \int\limits_{\tau_1}^{\tau_2} \mu_2(s) v_t(x, t-s)\, ds = 0 & \text{in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 & \text{in } (0,L)\times(0,\infty), \\ v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & t \geq 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, & t \geq 0, \\ v(x,0) = v_0(x), \;\; v_t(x,0) = v_1(x), & x \in (0,L), \\ p(x,0) = p_0(x), \;\; p_t(x,0) = p_1(x), & x \in (0,L), \\ v_t(x,-t) = f_0(x,t), & (x,t) \in (0,L)\times(0,\tau_2), \end{cases} \tag{1.4}$$

where the parameters $\rho$, $\alpha$, $\gamma$, $\mu$, $\beta$ and $L$ represent, respectively, the mass density, elastic stiffness, piezoelectric coefficient, magnetic permeability, water resistance coefficient of the beam and the length of the beam; $\tau_1$, $\tau_2$, $\mu_1$ are positive numbers, $\tau_2 \geq \tau_1$, and $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function verifying the following assumption:

$$\int\limits_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \leq \mu_1. \tag{1.5}$$

In addition, we consider the following condition:

$$\alpha_1 = \alpha - \gamma^2\beta > 0. \tag{1.6}$$

The paper is organized as follows. In Section 2, using the Hille–Yosida theorem (see [14, 21]) we prove the well-posedness of system (1.4). In Section 3, we construct the Lyapunov functionals and exploiting conditions (1.6), (1.5), we establish an exponential stability of system (1.4).

# 2 Well-posedness

In this section, we prove the existence and uniqueness of solutions for (1.4) by using semigroup theory.

We introduce as in [12] the new variable

$$z(x,\rho,t,s) = v_t(x, t-\rho s), \;\; x \in (0,L), \;\; \rho \in (0,1), \;\; s \in (\tau_1,\tau_2), \;\; t \geq 0,$$

then we obtain

$$s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \;\; x \in (0,L), \;\; \rho \in (0,1), \;\; s \in (\tau_1,\tau_2), \;\; t \geq 0.$$

Problem (1.4) takes the form

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \mu_1 v_t + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds = 0 \ \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \ \text{ in } (0,L)\times(0,\infty), \\ s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad x\in(0,L), \quad \rho\in(0,1), \quad s\in(\tau_1,\tau_2), \quad t\geq 0, \end{cases} \tag{2.1}$$

with the following initial and boundary conditions:

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & t\geq 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, & t\geq 0, \\ v(x,0) = v_0(x), \ \ v_t(x,0) = v_1(x) & \forall\, x\in(0,L), \\ p(x,0) = p_0(x), \ \ p_t(x,0) = p_1(x) & \forall\, x\in(0,L), \\ z(x,\rho,0,s) = f_0(x,\rho,s), & x\in(0,L), \ \ \rho\in(0,1), \ \ s\in(0,\tau_2). \end{cases}$$

Using the notation

$$v_t = u, \ p_t = q \ \text{ and } \ U = (v,u,p,q,z)^T,$$
$$\partial_t U = (v_t, u_t, p_t, q_t, z_t)^T,$$

problem (2.1) can be rewritten as

$$\begin{cases} \partial_t U = AU, \\ U(0) = U_0 = (v_0, v_1, p_0, p_1, f_0), \end{cases} \tag{2.2}$$

where the operator $A : D(A) \subset H \to H$ is defined by

$$AU := \begin{pmatrix} v_t \\ \dfrac{\alpha}{\rho} v_{xx} - \dfrac{\mu_1}{\rho} v_t - \dfrac{\gamma\beta}{\rho} p_{xx} - \dfrac{1}{\rho}\displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds \\ p_t \\ -\dfrac{\gamma\beta}{\mu} v_{xx} + \dfrac{\beta}{\mu} p_{xx} \\ -\dfrac{1}{s} z_\rho \end{pmatrix}.$$

We consider the following spaces:

$$\widehat{H}^1(0,L) = \{v\in H^1(0,L): \ \ v(0) = 0\}, \quad \widehat{H}^2(0,L) = H^2(0,L)\cap \widehat{H}^1(0,L),$$

and define the previous Hilbert space $H$ as

$$H := \widehat{H}^1(0,L)\times L^2(0,L)\times \widehat{H}^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)).$$

The inner product on $H$ is

$$\langle U, \widetilde{U}\rangle_H = \rho\int_0^L v_t\widetilde{v}_t\, dx + \mu\int_0^L p_t\widetilde{p}_t\, dx + \alpha_1\int_0^L v_x\widetilde{v}_x\, dx + \beta\int_0^L (\gamma v_x - p_x)(\gamma\widetilde{v}_x - \widetilde{p}_x)\, dx$$
$$+ \int_0^L\int_{\tau_1}^{\tau_2} s|\mu_2(s)| \int_0^1 z(x,\rho,t,s)\widetilde{z}(x,\rho,t,s)\, d\rho\, ds\, dx,$$

$$= \rho \int_0^L v_t \widetilde{v}_t \, dx + \mu \int_0^L p_t \widetilde{p}_t \, dx - \gamma\beta \int_0^L v_x \widetilde{p}_x \, dx - \gamma\beta \int_0^L \widetilde{v}_x p_x \, dx + \alpha \int_0^L v_x \widetilde{v}_x \, dx$$
$$+ \beta \int_0^L p_x \widetilde{p}_x \, dx + \int_0^L \int_{\tau_1}^{\tau_2} s|\mu_2(s)| \int_0^1 z(x,\rho,t,s)\widetilde{z}(x,\rho,t,s) \, d\rho \, ds \, dx.$$

Now, we define the previous domain of operator $A$ as

$$D(A) := \Big\{ U = (v, v_t, p, p_t, z) \in \widehat{H}^2(0,L) \times \widehat{H}^1(0,L) \times \widehat{H}^2(0,L) \times \widehat{H}^1(0,L) \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)) : z(x,0,t,s) = u;\;\; v_x(L) = p_x(L) = 0 \Big\}.$$

Clearly, $D(A)$ is dense in $H$.

**Theorem 2.1.** *Let $U_0 \in H$, then problem (2.2) possesses a unique solution $U \in C(\mathbb{R}^+, H)$. Moreover, if $U_0 \in D(A)$, then $U \in C(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, H)$.*

*Proof.* First, we prove that the operator $A$ is dissipative.

Let $U = (v, v_t, p, p_t, z)^T \in D(A)$. Using the previous inner product, we obtain

$$\langle AU, U\rangle_H = \left\langle \begin{pmatrix} v_t \\ \dfrac{\alpha}{\rho} v_{xx} - \dfrac{\mu_1}{\rho} v_t - \dfrac{\gamma\beta}{\rho} p_{xx} - \dfrac{1}{\rho}\displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds \\ p_t \\ -\dfrac{\gamma\beta}{\mu} v_{xx} + \dfrac{\beta}{\mu} p_{xx} \\ -\dfrac{1}{s} z_\rho \end{pmatrix}, \begin{pmatrix} v \\ v_t \\ p \\ p_t \\ z \end{pmatrix} \right\rangle_H .$$

Integrating by parts together with the boundary conditions, we have

$$\langle AU, U\rangle_H = \rho \int_0^L \Big( \frac{\alpha}{\rho} v_{xx} - \frac{\mu_1}{\rho} v_t - \frac{\gamma\beta}{} \rho p_{xx} - \frac{1}{\rho} \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds \Big) v_t \, dx$$
$$+ \mu \int_0^L \Big( -\frac{\gamma\beta}{\mu} v_{xx} + \frac{\beta}{\mu} p_{xx} \Big) p_t \, dx - \gamma\beta \int_0^L v_{tx} p_x \, dx - \gamma\beta \int_0^L p_{tx} v_x \, dx$$
$$+ \alpha \int_0^L v_{tx} v_x \, dx + \beta \int_0^L p_{tx} p_x \, dx - \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 z_\rho(x,\rho,t,s) z(x,\rho,t,s) \, d\rho \, ds \, dx$$
$$= -\mu_1 \int_0^L v_t^2 \, dx - \int_0^L v_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) \, ds \, dx$$
$$- \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 z_\rho(x,\rho,t,s) z(x,\rho,t,s) \, d\rho \, ds \, dx,$$

also, by integration with respect to $\rho$, we obtain

$$\int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 z_\rho(x,\rho,t,s) z(x,\rho,t,s)\, d\rho\, ds\, dx$$
$$= \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^L v_t^2\, dx, \tag{2.3}$$

and using Young's inequality, we get

$$-\int_0^L v_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds\, dx \le \frac{1}{2} \int_0^L v_t^2\, dx \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx. \tag{2.4}$$

By virtue of (2.3), (2.4) and condition (1.5), we obtain

$$\langle AU, U \rangle_H \le -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) \int_0^L v_t^2\, dx,$$

we find that $A$ is a dissipative operator.

Next, we prove that the operator $(I - A)$ is surjective.

Given $M = (g_1, g_2, g_3, g_4, g_5)^T \in H$, we show that there exists a unique $U = (v, u, p, q, z)^T \in D(A)$ such that

$$(I - A)U = M,$$

i.e.,

$$\begin{pmatrix} v \\ u \\ p \\ q \\ z \end{pmatrix} - \begin{pmatrix} u \\ \frac{\alpha}{\rho} v_{xx} - \frac{\mu_1}{\rho} u - \frac{\gamma\beta}{\rho} p_{xx} - \frac{1}{\rho} \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds \\ q \\ -\frac{\gamma\beta}{\mu} v_{xx} + \frac{\beta}{\mu} p_{xx} \\ -\frac{1}{s} z_\rho \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{pmatrix}. \tag{2.5}$$

Then by (2.5), we get

$$\begin{cases} v - u = g_1, \\ \rho u - \alpha v_{xx} + \mu_1 u + \gamma\beta p_{xx} + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds = \rho g_2, \\ p - q = g_3, \\ \mu q + \gamma\beta v_{xx} - \beta p_{xx} = \mu g_4, \\ z + \frac{1}{s} z_\rho = g_5, \end{cases} \tag{2.6}$$

also, using (2.6), we have

$$\begin{cases} u = v - g_1, \\ q = p - g_3, \end{cases} \tag{2.7}$$

as

$$z(x,0,t,s) = v_t(x,t) = u(x,t) \text{ for } x \in (0,L), \quad s \in (\tau_1,\tau_2), \quad t \ge 0,$$

and by $(2.6)_5$, we get

$$z(x,\rho,t,s)+\frac{1}{s}\,z_\rho(x,\rho,t,s)=g_5(x,\rho,s),\quad x\in(0,L),\quad \rho\in(0,1),\quad s\in(\tau_1,\tau_2), \tag{2.8}$$

that implies

$$z(x,\rho,t,s)=se^{-s\rho}\int_0^\rho g_5(x,\tau,s)e^{s\tau}\,d\tau+ue^{-s\rho}, \tag{2.9}$$

in particular,

$$z(x,1,t,s)=se^{-s}\int_0^1 g_5(x,\tau,s)e^{s\tau}\,d\tau+ue^{-s}. \tag{2.10}$$

Now, using (2.7)–(2.10) in the other equations for (2.6), we obtain

$$\begin{aligned}&\rho(v-g_1)-\alpha v_{xx}+\mu_1(v-g_1)+\gamma\beta p_{xx}\\&\qquad+\int_{\tau_1}^{\tau_2}\mu_2(s)se^{-s}\int_0^1 g_5(x,\tau,s)e^{s\tau}\,d\tau\,ds+(v-g_1)\int_{\tau_1}^{\tau_2}\mu_2(s)e^{-s}\,ds=\rho g_2,\\&\qquad\mu(p-g_3)+\gamma\beta v_{xx}-\beta p_{xx}=\mu g_4,\end{aligned}$$

and we get

$$\begin{cases}-\alpha v_{xx}+\gamma\beta p_{xx}+\varpi_1 v=Q_1\in L^2(0,L),\\ \gamma\beta v_{xx}-\beta p_{xx}+\mu p=Q_2\in L^2(0,L),\end{cases} \tag{2.11}$$

where

$$\begin{aligned}\varpi_1&=(\mu_1+\rho)+\int_{\tau_1}^{\tau_2}\mu_2(s)e^{-s}\,ds,\\ Q_1&=\varpi_1 g_1+\rho g_2-\int_{\tau_1}^{\tau_2}\mu_2(s)se^{-s}\int_0^1 g_5(x,\tau,s)e^{s\tau}\,d\tau\,ds,\\ Q_2&=\mu(g_4+g_3).\end{aligned}$$

Multiplying $(2.11)_1$, $(2.11)_2$, respectively, by $\widetilde{v},\widetilde{p}\in\widehat{H}^1(0,L)$, and integrating by parts together with the boundary conditions, we have

$$\begin{cases}\alpha\int_0^L v_x\widetilde{v}_x\,dx-\gamma\beta\int_0^L p_x\widetilde{v}_x\,dx+\varpi_1\int_0^L v\widetilde{v}\,dx=\int_0^L Q_1\widetilde{v}\,dx,\\ -\gamma\beta\int_0^L v_x\widetilde{p}_x\,dx+\beta\int_0^L p_x\widetilde{p}_x\,dx+\mu\int_0^L p\widetilde{p}\,dx=\int_0^L Q_2\widetilde{p}\,dx.\end{cases} \tag{2.12}$$

Consequently, problem (2.12) is equivalent to the problem

$$a((v,p),(\widetilde{v},\widetilde{p}))=b(\widetilde{v},\widetilde{p}), \tag{2.13}$$

where $a:(\widehat{H}^1(0,L)\times\widehat{H}^1(0,L))^2\to\mathbb{R}$ is the bilinear form given by

$$\begin{aligned}a((v,p),(\widetilde{v},\widetilde{p}))&=\alpha\int_0^L v_x\widetilde{v}_x\,dx+\beta\int_0^L p_x\widetilde{p}_x\,dx\\&\quad-\gamma\beta\int_0^L p_x\widetilde{v}_x\,dx-\gamma\beta\int_0^L v_x\widetilde{p}_x\,dx+\varpi_1\int_0^L v\widetilde{v}\,dx+\mu\int_0^L p\widetilde{p}\,dx,\end{aligned}$$

$b : \widehat{H}^1(0,L) \times \widehat{H}^1(0,L) \to \mathbb{R}$ is the linear form given by

$$b(\widetilde{v},\widetilde{p}) = \int_0^L Q_1\widetilde{v}\,dx + \int_0^L Q_2\widetilde{p}\,dx.$$

Now, define $\widetilde{H} := \widehat{H}^1(0,L) \times \widehat{H}^1(0,L)$ equipped by the norm

$$\|(v,p)\|_{\widetilde{H}} = \left( \left\| \left( v_x - \frac{\gamma\beta}{\alpha} p_x \right) \right\|_2^2 + \|v\|_2^2 + \|p\|_2^2 + \|p_x\|_2^2 \right)^{\frac{1}{2}}.$$

We can easily prove that the bilinear and linear forms $a$ and $b$ continue, and we also have

$$\begin{aligned} a((v,p),(v,p)) &= \alpha \int_0^L \left( v_x - \frac{\gamma\beta}{\alpha} p_x \right)^2 dx \\ &\quad + \left( \beta - \frac{(\gamma\beta)^2}{\alpha} \right) \int_0^L p_x^2\,dx + \varpi_1 \int_0^L v^2\,dx + \mu \int_0^L p^2\,dx \geq \widehat{m}\|(v,p)\|_{\widetilde{H}}^2, \end{aligned}$$

where

$$\widehat{m} = \min\left( \alpha, \left( \beta - \frac{(\gamma\beta)^2}{\alpha} \right), \varpi_1, \mu \right).$$

For all $\varpi_1 \geq 0$, thus $a$ is coercive, consequently, by the Lax–Milgram theorem, system (2.13) has a unique solution

$$(v,p) \in \widehat{H}^1(0,L) \times \widehat{H}^1(0,L).$$

Substituting $v$, $p$ into (2.7), we obtain

$$(u,q) \in \widehat{H}^1(0,L) \times \widehat{H}^1(0,L),$$

also, substituting $u$ into (2.9) and $(2.6)_5$, we get

$$z, z_\rho \in L^2((0,L) \times (0,1) \times (\tau_1,\tau_2)),$$

and by (2.11), we have

$$v_{xx} = \frac{\varpi_1}{\alpha_1} v + \frac{\gamma\mu}{\alpha_1} p - \frac{1}{\alpha_1} Q_1 - \frac{\gamma}{\alpha_1} Q_2 \in L^2(0,L) \Longrightarrow v \in H^2(0,L) \Longrightarrow p \in H^2(0,L) \tag{2.14}$$

Also, $(2.12)_1$ implies

$$-\alpha v_{xx} + \gamma\beta p_{xx} + \varpi_1 v = Q_1, \text{ in the distribution sense.} \tag{2.15}$$

Multiplying (2.15) by $\widetilde{v} \in \widehat{H}^1(0,L)$ and using integration by parts and $(2.12)_1$ again, we get

$$-\alpha v_x(L)\widetilde{v}(L) + \gamma\beta p_x(L)\widetilde{v}(L) = 0 \quad \forall \widetilde{v} \in \widehat{H}^1(0,L).$$

We choose

$$\widetilde{v}(x) = \frac{x}{L},$$

then we obtain

$$\gamma\beta p_x(L) = \alpha v_x(L). \tag{2.16}$$

Also, $(2.12)_2$ implies

$$\gamma\beta v_{xx} - \beta p_{xx} + \mu p = Q_2, \text{ in the distribution sense.} \tag{2.17}$$

Multiplying (2.17) by $\widetilde{p} \in \widehat{H}^1(0,L)$ and using integration by parts and $(2.12)_2$ again, we get

$$\gamma\beta v_x(L)\widetilde{p}(L) - \beta p_x(L)\widetilde{p}(L) = 0 \quad \forall \widetilde{p} \in \widehat{H}^1(0,L).$$

We choose

$$\widetilde{p}(x) = \frac{x}{L}\,,$$

then we obtain

$$\gamma\beta v_x(L) - \beta p_x(L) = 0. \tag{2.18}$$

Using (2.16) in (2.18), we get

$$v_x(L) = p_x(L) = 0, \tag{2.19}$$

then, by (2.14) and (2.19), we obtain

$$v, p \in \widehat{H}^2(0,L) : p_x(L) = v_x(L) = 0.$$

Thus the operator $(I - A)$ is surjective.

Therefore, $A$ is a maximal dissipative operator, then by Hille–Yousida theorem [14,21], we get the well-posedness of solution for problem (2.2). □

# 3 Exponential stability

In this section, we state and prove technical lemmas needed for proving our stability result.

**Lemma 3.1.** *Let $(v,p,z)$ be a solution of (2.1), then the expression of energy $E(t)$ is defined as follows:*

$$E(t) = \frac{1}{2}\int_0^L \left(\rho v_t^2 + \mu p_t^2 + \alpha_1 v_x^2 + \beta(\gamma v_x - p_x)^2 + \int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\right)dx,$$

*and satisfies*

$$\frac{d}{dt}E(t) \le -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds\right)\int_0^L v_t^2\,dx.$$

*Proof.* Multiplying $(2.1)_1$ by $v_t$, $(2.1)_2$ by $p_t$ and integrating over $(0,L)$ with respect to $x$, we obtain

$$\begin{aligned}\rho\frac{d}{2dt}\int_0^L v_t^2\,dx + \mu\frac{d}{2dt}\int_0^L p_t^2\,dx + \alpha_1\frac{d}{2dt}\int_0^L v_x^2\,dx + \gamma\beta\int_0^L(\gamma v_x - p_x)v_{xt}\,dx \\ - \beta\int_0^L(\gamma v_x - p_x)p_{xt}\,dx + \mu_1\int_0^L v_t^2\,dx + \int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds\,dx = 0,\end{aligned} \tag{3.1}$$

and by (3.1), we get

$$\begin{aligned}\rho\frac{d}{2dt}\int_0^L v_t^2\,dx + \mu\frac{d}{2dt}\int_0^L p_t^2\,dx + \frac{1}{2}\frac{d}{dt}\int_0^L \beta(\gamma v_x - p_x)^2\,dx \\ + \alpha_1\frac{d}{2dt}\int_0^L v_x^2\,dx + \mu_1\int_0^L v_t^2\,dx + \int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds\,dx = 0.\end{aligned}$$

Next, multiplying (2.1)$_3$ by $|\mu_2(s)|z(x,\rho,t,s)$ and integrating over $(0,L)\times(0,1)\times(\tau_1,\tau_2)$ with respect to $x$, $\rho$ and $s$, we obtain

$$\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z(x,\rho,t,s)z_t(x,\rho,t,s)\,ds\,d\rho\,dx+\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,\rho,t,s)z_\rho(x,\rho,t,s)\,ds\,d\rho\,dx=0.$$

Thus we have

$$\frac{d}{2dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx+\frac{1}{2}\int_0^L\int_0^1\frac{d}{d\rho}\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx=0,$$

as

$$\begin{aligned}\frac{1}{2}\int_0^L\int_0^1\frac{d}{d\rho}\int_{\tau_1}^{\tau_2}&|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx\\&=\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|v_t^2 ds\,dx,\end{aligned}$$

then we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L&\left(\rho v_t^2+\mu p_t^2+\alpha_1 v_x^2+\beta(\gamma v_x-p_x)^2+\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\right)dx\\&+\mu_1\int_0^L v_t^2\,dx+\int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds\,dx\\&+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|v_t^2\,ds\,dx=0.\end{aligned}$$

Using condition (1.6), we get

$$\begin{aligned}\frac{d}{dt}E(t)=&-\mu_1\int_0^L v_t^2\,dx-\int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds\,dx\\&-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx+\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^L v_t^2\,dx,\end{aligned}\tag{3.2}$$

and using Young's inequality, we obtain

$$\begin{aligned}-\int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds\,dx&\le\int_0^L\int_{\tau_1}^{\tau_2}|v_t||\mu_2(s)|^{\frac12}|\mu_2(s)|^{\frac12}|z(x,1,t,s)|\,ds\,dx\\&\le\frac{1}{2}\int_0^L v_t^2\,dx\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx.\end{aligned}\tag{3.3}$$

Then using (3.2), (3.3), we have

$$\frac{d}{dt}E(t)\le-\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_0^L v_t^2\,dx,$$

also, using (1.5), we obtain
$$\frac{d}{dt}E(t) \le 0. \qquad \square$$

**Lemma 3.2.** *Let $(v, p, z)$ be a solution of system (2.1), then the functional*
$$I_1(t) = \rho \int_0^L v_t v\, dx + \gamma\mu \int_0^L p_t v\, dx + \frac{\mu_1}{2}\int_0^L v^2\, dx \quad \forall\, t \ge 0,$$
*for some positive constant $\varepsilon_1$, satisfies*
$$I_1'(t) \le -\frac{\alpha_1}{2}\int_0^L v_x^2\, dx + \Big(\rho + \frac{(\gamma\mu)^2}{4\varepsilon_1}\Big)\int_0^L v_t^2\, dx + \varepsilon_1 \int_0^L p_t^2\, dx + \frac{c_0\mu_1}{2\alpha_1}\int_0^L\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx. \tag{3.4}$$

*Proof.* Multiplying equation $(2.1)_1$ by $v$ and integrating with respect to $x$ in $(0, L)$, we get the following equation:
$$\frac{d}{dt}\rho\int_0^L v_t v\, dx - \rho\int_0^L v_t^2\, dx + \alpha_1 \int_0^L v_x^2\, dx + \gamma\int_0^L (\beta p_{xx} - \gamma\beta v_{xx}) v\, dx + \frac{d}{dt}\frac{\mu_1}{2}\int_0^L v^2\, dx + \int_0^L v \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds = 0,$$
also, using equation $(2.1)_2$, we get
$$\frac{d}{dt}\rho\int_0^L v_t v\, dx - \rho\int_0^L v_t^2\, dx + \alpha_1 \int_0^L v_x^2\, dx + \gamma\mu\int_0^L p_{tt} v\, dx + \frac{d}{dt}\frac{\mu_1}{2}\int_0^L v^2\, dx + \int_0^L v \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds = 0. \tag{3.5}$$

Hence
$$\frac{d}{dt}\Big(\rho\int_0^L v_t v\, dx + \gamma\mu\int_0^L p_t v\, dx + \frac{\mu_1}{2}\int_0^L v^2\, dx\Big) = \rho\int_0^L v_t^2\, dx - \alpha_1\int_0^L v_x^2\, dx + \gamma\mu\int_0^L p_t v_t\, dx - \int_0^L v\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds\, dx. \tag{3.6}$$

By using Young's, Poincaré's and Cauchy–Schwartz inequalities, for any $\varepsilon_1 > 0$, we get
$$\gamma\mu\int_0^L p_t v_t\, dx \le \varepsilon_1\int_0^L p_t^2\, dx + \frac{(\gamma\mu)^2}{4\varepsilon_1}\int_0^L v_t^2\, dx, \tag{3.7}$$
$$-\int_0^L v\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\, ds\, dx \le \frac{\alpha_1}{2}\int_0^L v_x^2\, dx + \frac{c_0\mu_1}{2\alpha_1}\int_0^L\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx, \tag{3.8}$$
and using (3.7), (3.8) in (3.6), we get (3.4). $\square$

**Lemma 3.3.** *Let $(v,p,z)$ be a solution of system (2.1), then the functional*

$$I_2(t)=\mu\int_0^L p_t p\,dx+\rho\int_0^L v_t v\,dx$$

*satisfies*

$$I_2'(t)\leq -\beta\int_0^L(\gamma v_x-p_x)^2\,dx-\frac{\alpha_1}{4}\int_0^L v_x^2\,dx$$
$$+\Big(\rho+\frac{c_0\mu_1^2}{2\alpha_1}\Big)\int_0^L v_t^2\,dx+\mu\int_0^L p_t^2\,dx+\frac{c_0\mu_1}{\alpha_1}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx. \tag{3.9}$$

*Proof.* Differentiating $I_2(t)$ and using $(2.1)_1$, $(2.1)_2$, we have

$$\begin{aligned}I_2'(t)&=\mu\int_0^L p_t^2\,dx+\mu\int_0^L p_{tt}p\,dx+\rho\int_0^L v_t^2\,dx+\rho\int_0^L v_{tt}v\,dx\\&=\mu\int_0^L p_t^2\,dx-\beta\int_0^L(\gamma v_x-p_x)^2\,dx+\rho\int_0^L v_t^2\,dx-\alpha_1\int_0^L v_x^2\,dx\\&\quad-\mu_1\int_0^L v_t v\,dx-\int_0^L v\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds\,dx.\end{aligned}\tag{3.10}$$

Using Young's, Poincaré's and Cauchy–Schwartz inequalities, we get

$$-\mu_1\int_0^L v_t v\,dx\leq\frac{\alpha_1}{2}\int_0^L v_x^2\,dx+\frac{c_0\mu_1^2}{2\alpha_1}\int_0^L v_t^2\,dx \tag{3.11}$$

and

$$\int_0^L v\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds\,dx\leq\frac{\alpha_1}{4}\int_0^L v_x^2\,dx+\frac{c_0\mu_1}{\alpha_1}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx. \tag{3.12}$$

Using (3.11), (3.12) in (3.10), we get (3.9). □

**Lemma 3.4.** *Let $(v,p,z)$ be a solution of system (2.1), then the functional*

$$I_3(t)=\rho\int_0^L v_t(\gamma v-p)\,dx+\gamma\mu\int_0^L p_t(\gamma v-p)\,dx,$$

*satisfies*

$$I_3'(t)\leq-\frac{\gamma\mu}{2}\int_0^L p_t^2\,dx+(\varepsilon_2+\varepsilon_3c_0+\varepsilon_4c_0)\int_0^L(\gamma v_x-p_x)^2\,dx$$
$$+\Big(\frac{\mu_1^2}{4\varepsilon_3}+\rho\gamma+\frac{\varkappa^2}{2\gamma\mu}\Big)\int_0^L v_t^2\,dx+\frac{\alpha_1^2}{4\varepsilon_2}\int_0^L v_x^2\,dx+\frac{\mu_1}{4\varepsilon_4}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx \tag{3.13}$$

*for any $\varepsilon_2,\varepsilon_3,\varepsilon_4>0$.*

*Proof.* Differentiating $I_3(t)$ and using $(2.1)_1$, $(2.1)_2$, we have

$$\begin{aligned} I_3'(t) &= -\gamma\mu \int_0^L p_t^2\,dx - \alpha_1 \int_0^L v_x(\gamma v_x - p_x)\,dx - \mu_1 \int_0^L v_t(\gamma v - p)\,dx \\ &\quad - \int_0^L (\gamma v - p) \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\,ds\,dx + \rho\gamma \int_0^L v_t^2\,dx + \underbrace{(\gamma^2\mu - \rho)}_{\varkappa} \int_0^L v_t p_t\,dx, \end{aligned} \tag{3.14}$$

and using Young's, Poincaré's and Cauchy–Schwartz inequalities, we obtain

$$-\alpha_1 \int_0^L v_x(\gamma v_x - p_x)\,dx \le \varepsilon_2 \int_0^L (\gamma v_x - p_x)^2\,dx + \frac{\alpha_1^2}{4\varepsilon_2} \int_0^L v_x^2\,dx \quad \forall\,\varepsilon_2 > 0, \tag{3.15}$$

and

$$-\mu_1 \int_0^L v_t(\gamma v - p)\,dx \le \varepsilon_3 c_0 \int_0^L (\gamma v_x - p_x)^2\,dx + \frac{\mu_1^2}{4\varepsilon_3} \int_0^L v_t^2\,dx \ \ \forall\,\varepsilon_3 > 0, \tag{3.16}$$

$$\begin{aligned} &- \int_0^L (\gamma v - p) \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\,ds\,dx \\ &\qquad \le \varepsilon_4 c_0 \int_0^L (\gamma v_x - p_x)^2\,dx + \frac{\mu_1}{4\varepsilon_4} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx \quad \forall\,\varepsilon_4 > 0, \end{aligned} \tag{3.17}$$

and

$$\varkappa \int_0^L v_t p_t\,dx \le \frac{\gamma\mu}{2} \int_0^L p_t^2\,dx + \frac{\varkappa^2}{2\gamma\mu} \int_0^L v_t^2\,dx. \tag{3.18}$$

Using (3.15), (3.16), (3.17), (3.18) in (3.14), we get (3.13). □

**Lemma 3.5.** *Let $(v, p, z)$ be a solution of system (2.1), then the functional*

$$I_4(t) := \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx$$

*satisfies*

$$I_4'(t) \le -e^{-\tau_2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx + \mu_1 \int_0^L v_t^2\,dx - e^{-\tau_2} \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx.$$

*Proof.* Differentiating $I_4(t)$ and using $(2.1)_3$, we have

$$\begin{aligned} I_4'(t) &= -2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z(x,\rho,t,s) z_\rho(x,\rho,t,s)\,ds\,d\rho\,dx \\ &= - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \frac{d}{d\rho}(e^{-s\rho} z^2(x,\rho,t,s))\,ds\,d\rho\,dx - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx \end{aligned}$$

$$= -\int_0^L\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx + \int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^L v_t^2\,dx - \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx,$$

Using the relation $e^{-s}\le e^{-s\rho}\le 1$, $0\le\rho\le 1$, we get

$$I_4'(t)\le -\int_0^L\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx + \int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^L v_t^2\,dx - \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s}|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx.$$

Since $(-e^{-s})' = e^{-s}\ge 0$, we conclude that $-e^{-s}\le -e^{-\tau_2}\ \forall\, s\in(\tau_1,\tau_2)$, and we get

$$I_4'(t)\le -e^{-\tau_2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx + \mu_1\int_0^L v_t^2\,dx - e^{-\tau_2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx. \quad \square$$

Now, for $N$ sufficiently large, we define the Lyapunov functional as follows:

$$L(t) = NE(t) + N_1I_1(t) + N_2I_2(t) + N_3I_3(t) + N_4I_4(t),$$

where $N_1$, $N_2$, $N_3$, $N_4$ are the positive constants, to be chosen later.

**Theorem 3.1.** *Let $(v,p,z)$ be a solution of system (2.1), then there exist two positive constants $c_1, c_2 > 0$ satisfying*

$$c_1E(t)\le L(t)\le c_2E(t)\quad\forall\, t\ge 0. \tag{3.19}$$

*Proof.* Let

$$\Im(t) = L(t) - NE(t) = N_1I_1(t) + N_2I_2(t) + N_3I_3(t) + N_4I_4(t),$$

then

$$|\Im(t)| = |L(t) - NE(t)| \le N_1\left(\rho\int_0^L|v_tv|\,dx + \gamma\mu\int_0^L|p_tv|\,dx + \frac{\mu_1}{2}\int_0^L v^2\,dx\right) + N_2\left(\mu\int_0^L|p_tp|\,dx + \rho\int_0^L|v_tv|\,dx\right) + N_3\left(\rho\int_0^L|v_t(\gamma v - p)|\,dx + \gamma\mu\int_0^L|p_t(\gamma v-p)|\,dx\right) + N_4\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx. \tag{3.20}$$

Using Young's and Poincaré's inequalities in (3.20), for any $\varepsilon > 0$, we obtain

$$|\Im(t)| \le \underbrace{\Big(\frac{N_1\rho^2}{4\varepsilon} + N_2\rho^2\varepsilon + \frac{N_3\rho^2}{4\varepsilon}\Big)}_{\theta_1}\int_0^L v_t^2\,dx$$

$$+ \underbrace{\Big(N_1\,\frac{(\gamma\mu)^2}{4\varepsilon} + N_2\,\frac{\mu^2}{4\varepsilon} + N_3\,\frac{(\gamma\mu)^2}{4\varepsilon}\Big)}_{\theta_2}\int_0^L p_t^2\,dx + \underbrace{\Big(N_1\Big(2\varepsilon c_0 + \frac{c_0\mu_1}{2}\Big) + N_2\Big(2\varepsilon\gamma^2 c_0 + \frac{c_0}{4\varepsilon}\Big)\Big)}_{\theta_3}\int_0^L v_x^2\,dx$$

$$+ \underbrace{(2N_2\varepsilon c_0 + 2N_3\varepsilon c_0)}_{\theta_4}\int_0^L (\gamma v_x - p_x)^2\,dx + N_4\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx,$$

thus

$$|\Im(t)| \le CE(t),$$

where

$$C = \max\Big(\frac{2}{\rho}\,\theta_1, \frac{2}{\mu}\,\theta_2, \frac{2}{\alpha_1}\,\theta_3, \frac{2}{\beta}\,\theta_4, 2N_4\Big).$$

Then we obtain

$$\underbrace{(-C+N)}_{c_1}E(t) \le L(t) \le \underbrace{(C+N)}_{c_2}E(t). \qquad \square$$

**Theorem 3.2.** *Let $(v,p,z)$ be a solution of system (2.1), then there exist two positive constants $k$ and $\lambda$ such that*

$$E(t) \le ke^{-\lambda t} \quad \forall\, t \ge 0. \tag{3.21}$$

*Proof.* Using the previous lemmas, we get

$$L'(t) = NE'(t) + N_1I_1'(t) + N_2I_2'(t) + N_3I_3'(t) + N_4I_4'(t).$$

This leads to

$$L'(t) \le -\Bigg(N\Bigg(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\Bigg) - N_1\Big(\rho + \frac{(\gamma\mu)^2}{4\varepsilon_1}\Big) - N_2\Big(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\Big)$$

$$- N_3\Big(\frac{\mu_1^2}{4\varepsilon_3} + \rho\gamma + \frac{\varkappa^2}{2\gamma\mu}\Big) - N_4\mu_1\Bigg)\int_0^L v_t^2\,dx - \Big(\frac{\gamma\mu N_3}{2} - N_1\varepsilon_1 - N_2\mu\Big)\int_0^L p_t^2\,dx$$

$$- \Big(\frac{N_1\alpha_1}{2} + \frac{N_2\alpha_1}{4} - N_3\frac{\alpha_1^2}{4\varepsilon_2}\Big)\int_0^L v_x^2\,dx - \big(N_2\beta - (N_3\varepsilon_2 + N_3\varepsilon_3 c_0 + N_3\varepsilon_4 c_0)\big)\int_0^L (\gamma v_x - p_x)^2\,dx$$

$$- N_4e^{-\tau_2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx$$

$$- \Big(N_4e^{-\tau_2} - \frac{N_3\mu_1}{4\varepsilon_4} - N_2\frac{c_0\mu_1}{\alpha_1} - N_1\frac{c_0\mu_1}{2\alpha_1}\Big)\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx.$$

We choose the values

$$\varepsilon_1 = \frac{1}{N_1}, \quad \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{1}{N_3}$$

and get

$$
\begin{aligned}
L'(t) \le -\Bigg( & N\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\Big) - N_1\Big(\rho + \frac{N_1(\gamma\mu)^2}{4}\Big) - N_2\Big(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\Big) \\
& - N_3\Big(\frac{N_3\mu_1^2}{4} + \rho\gamma + \frac{\varkappa^2}{2\gamma\mu}\Big) - N_4\mu_1\Bigg) \int_0^L v_t^2\, dx - \Big(\frac{\gamma\mu N_3}{2} - 1 - N_2\mu\Big) \int_0^L p_t^2\, dx \\
& - \Big(\frac{N_1\alpha_1}{2} - \frac{\alpha_1^2}{4} N_3^2\Big) \int_0^L v_x^2\, dx - \big(N_2\beta - (1+2c_0)\big) \int_0^L (\gamma v_x - p_x)^2\, dx \\
& - N_4 e^{-\tau_2} \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\, ds\, d\rho\, dx \\
& - \Big(N_4 e^{-\tau_2} - \frac{N_3^2\mu_1}{4} - N_2\,\frac{c_0\mu_1}{\alpha_1} - N_1\,\frac{c_0\mu_1}{2\alpha_1}\Big) \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx. \qquad (3.22)
\end{aligned}
$$

First, in (3.22), we choose $N_2$ until it becomes

$$N_2\beta - (1+2c_0) > 0.$$

We also choose $N_3$ until it becomes

$$\frac{\gamma\mu N_3}{2} - 1 - N_2\mu > 0.$$

Now, we choose $N_1$ large enough so that

$$\frac{N_1\alpha_1}{2} - \frac{\alpha_1^2}{4}\, N_3^2 > 0.$$

We also choose $N_4$ large enough so that

$$N_4 e^{-\tau_2} - \frac{N_3^2\mu_1}{4} - N_2\,\frac{c_0\mu_1}{\alpha_1} - N_1\,\frac{c_0\mu_1}{2\alpha_1} > 0.$$

Finally, we choose a very large $N$ so that

$$
\begin{aligned}
\Bigg( N\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\Big) - N_1\Big(\rho + \frac{(\gamma\mu)^2}{4}\, N_1\Big) & \\
- N_2\Big(\rho + \frac{c_0\mu_1^2}{2\alpha_1}\Big) - N_3\Big(\frac{N_3\mu_1^2}{4} + \rho\gamma + \frac{\varkappa^2}{2\gamma\mu}\Big) - N_4\mu_1\Bigg) & > 0.
\end{aligned}
$$

Since

$$-\Big(N_4 e^{-\tau_2} - \frac{N_3^2\mu_1}{4} - N_2\,\frac{c_0\mu_1}{\alpha_1} - N_1\,\frac{c_0\mu_1}{2\alpha_1}\Big) \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx \le 0,$$

we get

$$L'(t) \le -mE(t),$$

and, by (3.19), we obtain

$$L'(t) \le -\frac{m}{c_2}\, L(t),$$

which implies that

$$L(t) \le L(0) e^{-\frac{m}{c_2} t}.$$

Using (3.19) again, we have (3.21). □

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