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COUPLED SYSTEM OF SECOND-ORDER STOCHASTIC DIFFERENTIAL INCLUSIONS DRIVEN BY LÉVY NOISE

Abstract. In this paper, we prove the existence of mild solutions for a second-order impulsive semilinear stochastic differential inclusion with an infinite-dimensional standard cylindrical Wiener process and Lévy noise. We consider the non convex-valued cases.

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რეზიუმე. ნაშრომში დამტკიცებულია სუსტი ამონახსნების არსებობა მეორე რიგის იმპულსური ნახევრადწრფივი სტოქასტური დიფერენციალური ჩართვისთვის უსასრულო განზომილებიანი სტანდარტული ცილინდრული ვინერის პროცესით და ლევის ხმაურით. განხილულია არაამოზნექილ მნიშვნელობიანი შემთხვევები.

1 Introduction

There has recently been increasing interest in the theory of stochastic equations with the Lévy noise that have discontinuous jumps. In this case, the noise is obtained from the Lévy process via its Lévy-Itô decomposition into Brownian motion (continuous part) and independent Poisson random measure (jump part), and it has attracted particular interest (see, e.g., Applebaum [3]). Note that the Poisson noise is a special non-Gaussian Lévy noise.

Recently, stochastic differential and partial differential inclusions have been extensively studied. For instance, in [1, 5], the authors investigated the existence of solutions of nonlinear stochastic differential inclusions by means of a Banach fixed point theorem and a semigroup approach. Balasubramaniam [4] obtained the existence of solutions of functional stochastic differential inclusions by Kakutani's fixed point theorem, the authors in [5] initiated the study of the existence of solutions of semilinear stochastic evolution inclusions in a Hilbert space by using the nonlinear alternative of Leray–Schauder type. In [18], some existence results for impulsive neutral stochastic evolution inclusions in the Hilbert Space, where a class of second-order evolution inclusions with a convex and nonconvex cases are considered. In [22], Henriquez studied the existence of solutions of non-autonomous second order functional differential equations with infinite delay by using Leray–Schauder's Alternative fixed point theorem.

That is why in the recent years they have been the objective of many investigations. We refer to the monographs by Benchohra et al. [6], amongst others, to see several studies on the properties of their solutions. The reader can also find a detailed and extensive bibliography in the previously mentioned books (see also Da Prato and Zabczyk [11], Gard [15], Gikhman and Skorokhod [16], Sobzyk [34]). As a motivating example, let us refer to a stochastic model for drug distribution in a biological system which was described by Tsokos and Padgett [36] as a closed system with a simplified heart, one organ or capillary bed, and recirculation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential inclusions, see the monographs of Bharucha–Reid [8], Mao [25], Øksendal [29], Tsokos and Padgett [36], Sobczyk [34] and Da Prato and Zabczyk [11].

In many realistic cases, it is advantageous to treat the second-order stochastic differential directly rather than to convert them to first order differential (see Henriquez [22]). For instance, it is useful for engineers to model mechanical vibrations or charge on a capacitor or condenser subjected to white noise excitation through a second-order stochastic differential (see Da Prato and Zabczyk [12]).

In this paper, we consider the following second-order system of stochastic impulsive differential inclusions and Lévy noise of the following type:

$$\begin{cases} d(x'(t) - g^1(t, x(t), y(t))) \in (Ax(t) + F^1(t, x(t), y(t))\, dt + f^1(t, x(t), y(t))\, dW(t) \\ \qquad\qquad + \displaystyle\int_Z k^1(t, x(t-), y(t-), z)\, \widetilde{N}(dt, dz), \quad t \in [0, b], \quad \text{a.e.}, \quad t \neq t_k, \\ d(y'(t) - g^2(t, x(t), y(t))) \in (Ax(t) + F^2(t, x(t), y(t)))\, dt + f^2(t, x(t), y(t))\, dW(t) \\ \qquad\qquad + \displaystyle\int_Z k^2(t, x(t-), y(t-), z)\, \widetilde{N}(dt, dz), \quad t \in [0, b], \quad \text{a.e.}, \quad t \neq t_k, \\ \Delta x(t) = I_k(x(t_k), y(t_k)), \quad \Delta x'(t) = I_k^1(x(t_k), y(t_k)), \quad t = t_k, \quad k = 1, 2, \ldots, m, \\ \Delta y(t) = \overline{I}_k(x(t_k), y(t_k)), \quad \Delta y'(t) = \overline{I}_k^2(x(t_k), y(t_k)), \quad t = t_k, \quad k = 1, 2, \ldots, m, \\ x(0) = x_0, \quad y(0) = y_0, \\ x'(0) = x_1, \quad y'(0) = y_1, \end{cases} \tag{1.1}$$

where the state $(x(\cdot), y(\cdot))$ takes values in a separable real Hilbert space X with inner product $\langle \cdot, \cdot \rangle$ induced by the norm $\| \cdot \|$, where A is the infinitesimal generator of a strongly continuous cosine family of linear operators $\{C(t), t \geq 0\}$, $\{W(t) : t \in [0, b]\}$ is a standard cylindrical Wiener process on Y – an arbitrary separable Hilbert space, and $F^i : [0, b] \times X \times X \to \mathcal{P}(X)$ and k^i :

$[0,b] \times X \times X \times (Z - \{0\}) \to X$ are given set-valued functions, where $\mathcal{P}(X)$ denotes the family of nonempty subsets of X, $\overline{I}_k, I_k, I_k^1, \overline{I}_k^2 \in C(X \times X, X)$ $(k = 1, 2, \dots, m)$, $L_Q(Y,X)$ denotes the space of all Q-Hilbert–Schmidt operators from Y into X, $f^i : J \times X \times X \to L_Q(Y,X)$. Here, $L_Q(Y,X) = L^0(Y,X) = L_2(Q^{1/2}Y, X)$ is a separable Hilbert space with respect to the Hilbert–Schmidt norm $\|\cdot\|_{L^0}$ and Q-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ with a linear bounded covariance operator Q such that $trQ < \infty$. Let $\{W(t), t \in \mathbb{R}\}$ be a standard cylindrical Wiener process with values in Y and, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, be a complete probability space. Suppose $\{p(t), t > 0\}$ is a σ-finite stationary $\mathcal{F}_t$-adapted Poisson point process taking values in a measurable space $(U, \mathcal{B}(U))$, which will also be defined in the next section. Moreover, the fixed times t_k satisfy $0 < t_1 < t_2 < \cdots < t_m < b$, $y(t_k^-)$ and $y(t_k^+)$ denote the left and right limits of $y(t)$ at $t = t_k$, respectively.

Several significant results concerning SDEs with Lévy noise have been presented in the existing literature [7, 19, 32, 33]. The existence of solutions to SDEs driven by time and space Poisson random measure were considered by, amongst others, Mueller [27], Hausenblas [20], Kallianpur and Xiong [23], Wu, Zhang [2] and Mytnik [28].

The rest of this paper is organized as follows. In Section 2, we briefly present some basic notation and preliminary information. In Section 3, we introduce all the background material used, such as stochastic calculus, some properties of generalized Banach spaces and the Covitz and Nadler fixed point theorem for contraction multi-valued maps in a generalized metric space (in the nonconvex case).

2 Preliminaries

In this section, we introduce some notations, recall some definitions and preliminary facts which are used throughout this paper. Actually, we borrow them from [9, 26]. Although we could simply refer to this paper whenever we need it, we prefer to include this summary here in order to make our paper as much self-contained as possible and to make easier its reading.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). For a stochastic process $x(\cdot,\cdot) : [0,T] \times \Omega \to X$, we write $x(t)$ (or, simply, x when no confusion is possible) instead of $x(t,\omega)$. Let W be a Q-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ with the linear bounded covariance operator Q such that $Tr(Q) < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}$ in Y, a bounded sequence of nonnegative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda e$, $n = 1, 2, \dots$, and a sequence $\{\beta_n\}$ of independent Wiener processes such that

$$\langle W(t), e\rangle = \sum \lambda_n \langle e, e_n\rangle \beta_n(t), \;\; e \in Y.$$

Let $\phi \in L(X,Y)$ and define

$$|\phi|_Q^2 = Tr(\phi\, Q\, \phi^*) = \sum_{n=1}^{\infty} \left| \sqrt{\lambda_n}\, \phi\, e_n \right|^2.$$

If $|\phi|_Q < +\infty$, then ϕ is called a Q-Hilbert–Schmidt operator. Let $L_Q(X,Y)$ denote the space of all Q-Hilbert–Schmidt operators $\phi : Y \to X$. The completion $L_Q(Y,X)$ of $L(K,H)$ with respect to the topology is induced by the norm $|\cdot|_Q$, where $|\phi|_Q^2 = \langle \phi, \phi\rangle$ is a Hilbert space with the above norm topology. We assume that the filtration is generated by the Q-Weiner process $W(\cdot)$. The random measure N_p defined by $N_p((0,t] \times \Lambda) := \sum_{s \in (0,t]} 1_\Lambda(p(s))$ for $\Lambda \in \mathcal{B}(U)$ is called the Poisson random measure induced by $p(\cdot)$; thus we can define the measure $\widetilde{N}$ by

$$\widetilde{N}(dt, dz) = N_p(dt, dz) - \nu\,(dz)\, dt,$$

where ν is the characteristic measure of N_p, which is called the compensated Poisson random measure for a Borel set $Z \in \mathcal{B}(U - \{0\})$, that is,

$$\mathcal{F}_t = \sigma\{W(t);\; s \leq t\} \vee \sigma\big\{N_p((0,s] \times \Lambda);\; s \leq t,\; \Lambda \in \mathcal{B}(U)\big\} \vee \mathcal{N}, \;\; t \in J,$$

where $\mathcal{N}$ is the class of $\mathbb{P}$-null sets.

The next result is known as the Burholder–Davis–Gundy inequalities.

Theorem 2.1 ([31]). *For each $p > 0$, there exist the constants $c_p, C_p \in (0,\infty)$ such that for any process x with the property that for some $t \in [0,\infty)$, $\int_0^t |X|^2(s)\,ds < \infty$ a.s., we have*

$$c_p\mathbb{E}\left(\int_0^t |X|^2(s)\,ds\right)^{\frac{p}{2}} \le \mathbb{E}\left|\sup_{s\in[0,t]}\int_0^t X(s)\,dW(s)\right|^p \le C_p\mathbb{E}\left(\int_0^t |X|^2(s)\,ds\right)^{\frac{p}{2}}. \tag{2.1}$$

In order to prove the main theorem of this paper, we also need a key lemma which is stated below and whose proof can be found in [35].

Lemma 2.1. *Let $p = 1$ or 2. Assume that the function $\sigma : \mathbb{R}^+ \times U \to X$ is a progressively measurable function satisfying*

$$\int_0^t \mathbb{E}\left(\int_Z |\sigma(s,x)|^p \nu\,(dx)\,ds\right) < \infty.$$

Then, for $c > 0$, the Burkholder inequality holds:

$$\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s\int_Z \sigma(s,x)\,\widetilde{N}(ds,dx)\right|^p\right) \le c\mathbb{E}\int_0^t\int_Z |\sigma(s,x)|^p\nu\,(dx)\,ds. \tag{2.2}$$

2.1 Some results on fixed point theorems and set-valued analysis

We note that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 2.1. A generalized metric space (X,d), where $d(x,y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_n(x,y)\end{pmatrix}$, is complete if (X,d_i) is a complete metric space for every $i = 1,\dots,n$.

Definition 2.2. A square matrix of real numbers M_{trix} is said to be convergent to zero if its spectral radius $\rho(M_{trix})$ is strictly less than 1. In other words, this means that all the eigenvalues of M_{trix} are in the open unit disc (i.e., $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M_{trix} - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$).

Denote

$$\begin{aligned}
\mathcal{P}_{cl}(X) &= \big\{y \in \mathcal{P}(X) :\ y \text{ closed}\big\},\\
\mathcal{P}_{b}(X) &= \big\{y \in \mathcal{P}(X) :\ y \text{ bounded}\big\},\\
\mathcal{P}_{c}(X) &= \big\{y \in \mathcal{P}(X) :\ y \text{ convex}\big\},\\
\mathcal{P}_{cp}(X) &= \big\{y \in \mathcal{P}(X) :\ y \text{ compact}\big\}.
\end{aligned}$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^n_+ \cup \{\infty\}$ defined by

$$H_d(A,B) := \begin{pmatrix} H_{d_1}(A,B) \\ \dots\dots\dots\dots \\ H_{d_n}(A,B)\end{pmatrix}.$$

Let (X,d) be a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \dots\dots\dots \\ d_n(x,y)\end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1, \dots, n$, are metrics on X,

$$H_d(A, B) = \max\Big\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\Big\},$$

where

$$d(A, b) = \inf_{a \in A} d(a, b), \quad d(a, B) = \inf_{b \in B} d(a, b).$$

Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

A multi-valued map $F : X \to \mathcal{P}(X)$ is convex (closed) valued if $F(y)$ is convex (closed) for all $y \in X$, F is bounded on bounded sets if $F(B) = \bigcup_{y \in B} F(y)$ is bounded in X for all $B \in \mathcal{P}_b(X)$. F is called upper semi-continuous (u.s.c., for short) on X if for each $y_0 \in X$, the set $F(y_0)$ is a nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(y_0)$, there exists an open neighborhood $\mathcal{V}$ of y_0 such that $F(\mathcal{V}) \in \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$.

If the multi-valued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

A multi-valued map $F : J \to \mathcal{P}_{cp,c}$ is said to be measurable if for each $y \in X$, the mean-square distance between y and $F(t)$ is measurable.

Lemma 2.2 ([24])**.** *Let I be a compact interval and X be a Hilbert space. Let F be an L^2-Carathéodory multi-valued map with $S_{F,y} \neq \varnothing$, and let Γ be a linear continuous mapping from $L^2(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S_F : C(I, X) \to \mathcal{P}_{cp,c}(L^2([0, T], X)), \quad y \mapsto (\Gamma \circ S_F)(y) = \Gamma(S_F, y),$$

is a closed graph operator in $C(I, X) \times C(I, X)$, where $S_{F,y}$ is known as the selectors set from F and is given by

$$S_{F,y} = \big\{ f \in L^2([0, T], X) : \ \ f(t) \in F(t, y) \ \text{ for a.e. } \ t \in [0, T]\big\}.$$

We denote the graph of G to be the set

$$gr(G) = \big\{(x, y) \in X \times Y, \ y \in G(x)\big\}.$$

Lemma 2.3 ([13])**.** *If $G : X \to P_{cl}(Y)$ is u.s.c., then $gr(G)$ is a closed subset of $X \times Y$. Conversely, if G is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Definition 2.3. A set-valued operator $G : J \to \mathcal{P}_{cl}(X)$ is said to be a contraction if there exists $0 \le \gamma < 1$ such that

$$H_d(G(x), G(y)) \le \gamma d(x, y) \ \text{ for all } \ x, y \in X.$$

Now, we present a second result of problem (1.1) with a nonconvex valued right-hand side. Our considerations are based on a fixed point theorem for contraction multi-valued operators given by Covitz and Nadler in 1970 (see also Deimling [13, Theorem 11.1]). In this section, we provide a multi-valued version of Perov's fixed point theorem (see [30]).

Definition 2.4. Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero a matrix M such that

$$d(N(x), N(y)) \le M_{trix} d(x, y) \ \text{ for all } \ x, y \in X.$$

Theorem 2.2 ([30])**.** *Let (X, d) be a complete generalized metric space and $F : X \to \mathcal{P}_{cl,b}(X)$ be a contractive multi-valued operator with Lipschitz matrix M. Then N has at least one fixed point.*

Now, let us recall some facts about cosine families of operators.

Definition 2.5. The one-parameter family $\{C(t) : \ t \in \mathbb{R}\}$ of operators from X to X satisfying the conditions

(i) $C(0) = I$,

(ii) $C(t)x$ is continuous in t on $\mathbb{R}$, for all $x \in X$,

(iii) $C(t+s) - C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$

is called a strongly continuous cosine family.

The corresponding strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ associated to the given strongly continuous cosine family $\{C(t) :\ t \in \mathbb{R}\}$ is defined by

$$S(t)x = \int_0^t C(s)x\, ds \ \text{ for all } x \in X, \quad t \in \mathbb{R}.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) :\ t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2} C(t)x\big|_{t=0} \ \text{ for all } \ x \in D(A),$$

where

$$D(A) = \big\{x \in X :\ C(\,\cdot\,)x \in C^2(\mathbb{R}; X)\big\}.$$

It is well known that the infinitesimal generator A is a closed, densely defined operator on X. Such cosine and its corresponding sine families and their generators satisfy the following properties appearing, for instance, in Fattorini [14].

Proposition 2.1. *Suppose that A is the infinitesimal generator of a cosine family of operators $\{C(t) :\ t \in \mathbb{R}\}$. Then:*

(i) *There exist $M^* \geq 1$ and $\alpha \geq 0$ such that $\|C(t)\| \leq M^* e^{\alpha|t|}$ and hence $\|S(t)\| \leq M^* e^{\alpha|t|}$.*

(ii) $A \int_s^r S(u)x\, du = (C(r) - C(s))x$ *for all* $0 \leq s \leq r < \infty$.

(iii) *There exists $N^* \geq 1$ such that*

$$\|S(s) - S(r)\| \leq N^* \int_s^r e^{\alpha|s|}\, ds \ \text{ for all } \ 0 \leq s \leq r < \infty.$$

The uniform boundedness principle, together with Proposition 2.1 part (i), imply that both $\{C(t) : t \in [0, b]\}$ and $\{S(t) :\ t \in [0, b]\}$ are uniformly bounded by a positive constant M.

2.2 Main result

Let $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$. In order to define a solution for problem (1.1), consider the following space of picewise continuous functions.

Let us introduce the spaces

$$\begin{aligned} &H_2([0, b]; L^2(\Omega, X)) = \Big\{x :\ \ J \to L^2(\Omega, X), \\ &\quad x\big|_{(t_k, t_{k+1}]} \in C((t_k, t_{k+1}], L^2(\Omega, X)),\ k = 1, 2, \ldots, m \ \text{ and there exist } \ x(t_k^+) \text{ for } k = 1, 2, \ldots, m\Big\}, \end{aligned}$$

and

$$\begin{aligned} &H_2'([0, b]; L^2(\Omega, X)) = \Big\{x :\ \ J \to L^2(\Omega, X), \\ &\quad x\big|_{(t_k, t_{k+1}]} \in C^1((t_k, t_{k+1}], L^2(\Omega, X)),\ k = 1, 2, \ldots, m \ \text{ and there exist } \ x(t_k^+) \text{ for } k = 1, 2, \ldots, m\Big\}. \end{aligned}$$

It is clear that $H_2([0,b];L^2(\Omega,X))$ endowed with the norm

$$\|x\|_{H_2} = \mathbb{E}\Big(\sup_{s\in[0,b]} \|x(s,\,\cdot\,)\|^2\Big)^{\frac{1}{2}}.$$

It is easy to see that H_2' is a Banach space with the norm $\|x\|_{H_2'} = \|x\|_{H_2} + \|x'\|_{H_2}$, let the space

$$\mathcal{M}_2 = \Big\{x:\ \ [0,b]\to L^2(\Omega,X) \ \text{ and } \ x\big|_J \in H_2',\ \ \mathbb{E}\Big(\sup_{t\in[0,b]} \|x(t,\,\cdot\,)\|^2\Big) < \infty \ \text{ almost surely}\Big\},$$

be endowed with the norm

$$\|x\|_{\mathcal{M}_2} = \mathbb{E}\Big(\sup_{s\in[0,b]} \|x(s,\,\cdot\,)\|^2\Big)^{\frac{1}{2}}.$$

It is not difficult to check that $\mathcal{M}_2$ is a Banach space with the norm $\|\cdot\|_{\mathcal{M}_2}$.

Now, we first define the concept of a mild solution to our problem. It is worth mentioning that we can use several types of solutions (weak, strong, etc.), each of them needs a different set of assumptions like absolutely continuous, differentiable absolutely continuous. We refer the reader to the book [21] for a detailed explanation of different types of solutions as well as to some results concerning their relationships. Our interest in this paper is the so-called mild solution which will be defined below. However, it is also possible to consider a similar analysis for the classical concept of a solution. An important feature is that, in the deterministic case, every classical solution becomes a mild solution (see, e.g., [21, p. 436 ff.]), so it is expected that the same result holds in this stochastic framework. But it is not our objective in this paper to prove this kind of results, and that is why we will prove directly the existence and eventual uniqueness of a mild solution to our problem.

Definition 2.6. An $X-$valued stochastic process $u=(x,y)\in\mathcal{M}_2\times\mathcal{M}_2$ is said to be a solution of (1.1) with respect to the probability space $(\Omega,\mathcal{F},\mathbb{P})$ if:

(1) $u(t)$ is $\mathcal{F}_t$-adapted for all $t\in J_k=(t_k,t_{k+1}]$, $k=1,2,\dots,m$;

(2) $u(t)$ is right continuous and has limit on the left, and there exist selections f^i, $i=1,2$, such that $f^i_*(t)\in F^i(t,u(t))$ a.e. $t\in J$ with

$$\mathbb{P}\bigg(\int_0^t\int_Z \big|k^i(s,x(s-),y(s-),z)\big|_X^2 \nu\,(dz)\,ds < \infty\bigg) = 1;$$

(3) $u(t)$ satisfies for each $t\in J$, a.e. $\omega\in\Omega$,

$$\begin{aligned}
x(t) &= C(t)x_0 + S(t)(x_1 - g^1(0,x_0,y_0)\\
&\quad + \int_0^t C(t-s)g^1(s,x(s),y(s))\,ds + \int_0^t S(t-s)\int_Z k^1(t,x(s),y(s),z)\,d\widetilde{N}(ds,dz)\\
&\quad + \int_0^t S(t-s)f^1_*(s)\,ds + \int_0^t S(t-s)f^1(t,x(s),y(s))\,dW(s)\\
&\quad + \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k),y(t_k)) + \sum_{0<t_k<t} S(t-t_k)I^1_k(x(t_k),y(t_k)),\ \ \text{a.e. } t\in J,
\end{aligned}$$

and

$$\begin{aligned}
y(t) &= C(t)y_0 + S(t)(y_1 - g^2(0,x_0,x_0)\\
&\quad + \int_0^t C(t-s)g^2(s,x(s),y(s))\,ds + \int_0^t S(t-s)\int_Z k^2(t,x(s),y(s),z)\,\widetilde{N}(ds,dz)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t S(t-s)f_*^2(s)\,ds+\int_0^t S(t-s)f^2(t,x(s),y(s))\,dW(s)\\
&+\sum_{0<t_k<t} C(t-t_k)\overline{I}_k(x(t_k),y(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^2(x(t_k),y(t_k)),\ \text{ a.e. } t\in J.
\end{aligned}
$$

In this paper, we will work under the following assumptions which will be imposed in our main theorem. In this section, we assume that there exists $M>0$ such that the cosine family of operators $\{C(t):\ t\in[0,b]\}$ on X and the corresponding sine family $\{S(t):\ t\in[0,b]\}$ satisfy

$$
\|S(t)\|\le M,\quad \|C(t)\|\le M \ \text{ for every } \ t\in[0,b],
$$

and $S(t)$, $C(t)$ are compact in X for $t\ge 0$.

(H_1) $F^i: J\times X\times X\to \mathcal{P}_{cp}(X); (t,y)\to F^i(t,x,y)$ is measurable for each $(x,y)\in X\times X$.

(H_2) There exist the constants $a_i,\ b_i\in\mathbb{R}^+$ such that

$$
H_{d_i}^2\big(F^i(t,x,y),F^i(t,\overline{x},\overline{y})\big)\le a_i|x-\overline{x}|_X^2+b_i|y-\overline{y}|_X^2
$$

for all $x,y,\overline{x},\overline{y}\in X$ for each $i=1,2$.

(H_3) There exist the constants $a_{f_i},b_{f_i},\overline{a}_{g_i},\overline{b}_{g_i},\alpha_{k_i},\beta_{k_i}\in\mathbb{R}^+$ such that

$$
\big|f^i(t,x,y)-f^i(t,\overline{x},\overline{y})\big|_X^2\le a_{f_i}|x-\overline{x}|_X^2+b_{f_i}|y-\overline{y}|_X^2
$$

and

$$
\begin{aligned}
\big|g^i(t,x,y)-g^i(t,\overline{x},\overline{y})\big|_X^2&\le \overline{a}_{g_i}|x-\overline{x}|_X^2+\overline{b}_{g_i}|y-\overline{y}|_X^2,\\
\int_Z \big|k^i(t,x,y,z)-k^i(t,\overline{x},\overline{y},z)\big|_X^2\,\nu(dz)&\le \alpha_{k_i}|x-\overline{x}|_X^2+\beta_{k_i}|y-\overline{y}|_X^2
\end{aligned}
$$

for all $x,y,\overline{x},\overline{y}\in X$.

(H_4) There exist the constants $d_k^i\ge 0$ and $\overline{d}_k^{\,i}\ge 0$, $k=1,\dots,m$ and $i=1,2$ such that

$$
\big|I_k^1(x,y)-I_k^1(\overline{x},\overline{y})\big|_X^2\vee\big|I_k(x,y)-I_k(\overline{x},\overline{y})\big|_X^2\le d_k^1|x-\overline{x}|_X^2+d_k^2|x-\overline{x}|_X^2
$$

and

$$
\big|I_k^2(x,y)-I_k^2(\overline{x},\overline{y})\big|_X^2\vee\big|\overline{I}_k(x,y)-\overline{I}_k(\overline{x},\overline{y})\big|^2\le \overline{d}_k^{\,1}|x-\overline{x}|_X^2+\overline{d}_k^{\,2}|y-\overline{y}|_X^2,
$$

for all $x,y,\overline{x},\overline{y}\in X$.

Consider the multi-valued operator $N:\mathcal{M}_2\times\mathcal{M}_2\to\mathcal{P}(\mathcal{M}_2\times\mathcal{M}_2)$ defined by

$$
N(x,y)=(N_1(x,y),N_2(x,y)),\ \ (x,y)\in\mathcal{M}_2\times\mathcal{M}_2
$$

and given by

$$
N(x,y)=\big\{(h_1,h_2)\in\mathcal{M}_2\times\mathcal{M}_2\big\},
$$

where

$$
\begin{aligned}
h_1(t)&=C(t)x_0+S(t)\big(x_1-g^1(0,x_0,y_0)\big)\\
&\quad+\int_0^t C(t-s)g^1(s,x(s),y(s))\,ds+\int_0^t S(t-s)\int_Z k^1(t,x(s),y(s),z)\,\widetilde{N}(ds,dz)
\end{aligned}
$$

$$+\int_0^t S(t-s)f_*^1(s)\,ds+\int_0^t S(t-s)f^1(t,x(s),y(s))\,dW(s)$$

$$+\sum_{0<t_k<t} C(t-t_k)I_k(x(t_k),y(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^1(x(t_k),y(t_k)) \text{ if } t\in[0,b],$$

$$h_2(t)=C(t)y_0+S(t)\big(y_1-g^2(0,x_0,y_0)\big)$$

$$+\int_0^t C(t-s)g^2(s,x(s),y(s))\,ds+\int_0^t S(t-s)\int_Z k^2(t,x(s),y(s),z)\,\widetilde{N}(ds,dz)$$

$$+\int_0^t S(t-s)f_*^2(s)\,ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)f^2(t,x(s),y(s))\,dW(s)$$

$$+\sum_{0<t_k<t} C(t-t_k)\overline{I}_k(x(t_k),y(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^2(x(t_k),y(t_k)) \text{ if } t\in[0,b];$$

here,

$$f_*^i\in S_{F^i,u}=\Big\{f_*^i\in L^2(J,X): f_*^i(t)\in F^i(t,x,y) \text{ for each } t\in J,\ \ x,y\in\mathcal{M}_2\Big\},\quad i=1,2.$$

Theorem 2.3. *Assume that hypotheses $(H_1)-(H_4)$ are satisfied. If the matrix*

$$M_{trix}=\begin{pmatrix} B_1 & B_2\\ B_3 & B_4\end{pmatrix},$$

where

$$B_1=M\sqrt{6\overline{a}_{g_1}+6cb\alpha_{k_1}+6a_1+a_{f_1}C_2+12m\sum_{0<t_k<t}d_k^1},$$

$$B_2=M\sqrt{6\overline{b}_{g_1}+6cb\beta_{k_1}+6b_1+6M^2b_{f_1}C_2+12m\sum_{0<t_k<t}d_k^2},$$

and

$$B_3=M\sqrt{6\overline{a}_{g_2}+6cb\alpha_{k_2}+6M^2a_2+a_{f_2}C_2+12m\sum_{0<t_k<t}\overline{d}_k^{\,1}},$$

$$B_4=M\sqrt{6\overline{b}_{g_2}+6cb\beta_{k_2}+6b_2+6b_{f_2}C_2+12m\sum_{0<t_k<t}\overline{d}_k^{\,2}},$$

converges to zero, then problem (1.1) has at least one mild solution.

Proof. In order to transform problem (1.1) into a fixed point one, show that N satisfies the assumptions of Theorem 2.3. Note that (H_2) implies that F^i for each $i=1,2$ has at most linear growth, i.e.m.,

$$\mathbb{E}|F^i(t,x,y)|_X^2\le a_i\mathbb{E}|x|_X^2+b_i\mathbb{E}|y|_X^2$$

for a.e. $t\in J$ and all $x,y\in X$.

(a) $N(x,y)\in\mathcal{P}_{cl}(\mathcal{M}_2\times\mathcal{M}_2)$ for each $(x,y)\in\mathcal{M}_2\times\mathcal{M}_2$. Let $u_n=(x_n,y_n)\to z_*=(x_*,y_*)$, $(h_{1n},h_{2n})\in N(u_n)$ and $(h_{1n},h_{2n})\to(h_{1*},h_{2*})$ as $n\to\infty$. We prove that $h_*\in N_1(u_*)$. The facts that $h_{1n}\in N_1(u_n)$ and $h_{2n}\in N_2(u_n)$ means that there exists $f_{*,n}^i\in S_{F^i,u_n}$ for each $i=1,2$ such that

$$
\begin{aligned}
h_{1n} &= C(t)x_0 + S(t)\big(x_1 - g^1(0, x_0, y_0)\big) \\
&\quad + \int_0^t C(t-s)g^1(s, x_n(s), y_n(s))\, ds + \int_0^t S(t-s) \int_Z k^1(t, x(s), y(s), z)\, \widetilde{N}(ds, dz) \\
&\quad + \int_0^t S(t-s)f^1_{*,n}(s)\, ds + \int_0^t S(t-s)f^1(t, x_n(s), y_n(s))\, dW(s) \\
&\quad + \sum_{0<t_k<t} C(t-t_k)I_k(x_n(t_k), y_n(t_k)) + \sum_{0<t_k<t} S(t-t_k)I^1_k(x_n(t_k), y_n(t_k)).
\end{aligned}
$$

First, notice that as $n \to \infty$,

$$
\begin{aligned}
&\Bigg\| h_{1n} - C(t)x_0 - S(t)(x_1 - g^1(0, x_0, y_0)) - \int_0^t C(t-s)g^1(s, x_n(s), y_n(s))\, ds \\
&\quad - \int_0^t S(t-s) \int_Z k^1(s, x_n(s), y_n(s), z)\widetilde{N}\,(ds, dz) - \int_0^t S(t-s)f^1(s, x_n(s), y_n(s))\, dW(s) \\
&\quad - \sum_{0<t_k<t} C(t-t_k)I_k(x_n(t_k), y_n(t_k)) - \sum_{0<t_k<t} S(t-t_k)I^1_k(x_n(t_k), y_n(t_k)) \\
&\quad - \big(h_{1*} - C(t)x_0 - S(t)(x_1 - g^1(0, x_0, y_0))\big) - \int_0^t C(t-s)g^1(s, x_*(s), y_*(s))\, ds \\
&\quad - \sum_{l=1}^{\infty} \int_0^t S(t-s) \int_Z k^1(s, x_*(s), y_*(s), z)\, \widetilde{N}(ds, dz) - \int_0^t S(t-s)f^1(t, x_*(s), y_*(s))\, dW(s) \\
&\quad - \sum_{0<t_k<t} C(t-t_k)I_k(x_*(t_k), x_*(t_k)) - \sum_{0<t_k<t} S(t-t_k)I^1_k(x_*(t_k), y_*(t_k)) \Bigg\|_{\mathcal{M}_2} \longrightarrow 0.
\end{aligned}
$$

Now, consider the continuous linear operator $\Gamma : L^2(J, X) \to \mathcal{M}_2$ defined for each $i = 1, 2$ by

$$
\Gamma(v^i)(t) = \int_0^t S(t-s)v^i(s)\, ds.
$$

From the definition of Γ we know that

$$
\begin{aligned}
&h_{1n} - \Bigg(C(t)x_0 + S(t)(x_1 - g^1(0, x_0, y_0)) + \int_0^t C(t-s)g^1(s, x_n(s), y_n(s))\, ds \\
&\quad + \int_0^t S(t-s) \int_Z k^1(s, x_n(s), y_n(s), z)\, \widetilde{N}(ds, dz) + \int_0^t S(t-s)f^1(t, x_n(s), y_n(s))\, dW(s) \\
&\quad + \sum_{0<t_k<t} C(t-t_k)I_k(x_n(t_k), y_n(t_k)) - \sum_{0<t_k<t} S(t-t_k)I^1_k(x_n(t_k), y_n(t_k)) \Bigg) \in \Gamma(S_{F^1, u_n})
\end{aligned}
$$

and

$$
\overline{h}_{1n} - \Bigg(C(t)x_0 + S(t)(y_1 - g^2(0, x_0, y_0)) + \int_0^t C(t-s)g^2(s, x_n(s), y_n(s))\, ds
$$

$$+\int_0^t S(t-s)\int_Z k^2(s,x_n(s),y_n(s),z)\,\widetilde{N}(ds,dz)+\int_0^t S(t-s)f^2(t,x_n(s),y_n(s))\,dW(s)$$
$$+\sum_{0<t_k<t} C(t-t_k)\overline{I}_k(y_n(t_k),x_n(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^2(x_n(t_k),y_n(t_k))\Bigg)\in\Gamma(S_{F^2,u_n}).$$

Since $u_n=(x_n,y_n)\to z_*=(x_*,y_*)$ and $\Gamma\circ S_{F^i}$ is a closed graph operator by Lemma 2.2, there exists $f^i_{**}\in S_{F^i,u_*}$ for each $i=1,2$ such that

$$\begin{aligned}h_{1*}&=C(t)x_0+S(t)(x_1-g^1(0,x_0,y_0))\\&+\int_0^t C(t-s)g^1(s,x_*(s),y_*(s))\,ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\int_Z k^1(t,x_*(s),y_*(s),z)\,\widetilde{N}(ds,dz)\\&+\int_0^t S(t-s)f^1_{**}(s)\,ds+\int_0^t S(t-s)f^1(t,x_*(s),y_*(s))\,dW(s)\\&+\sum_{0<t_k<t} C(t-t_k)I_k(x_*(t_k),y_*(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^1(x_*(t_k),y_*(t_k)).\end{aligned}$$

Similarly,

$$\begin{aligned}h_{2*}&=C(t)y_0+S(t)(y_1-g^2(0,x_0,y_0))\\&+\int_0^t C(t-s)g^2(s,x_*(s),y_*(s))\,ds+\int_0^t S(t-s)\int_Z k^2(s,x_*(s),y_*(s),z)\,\widetilde{N}(ds,dz)\\&+\int_0^t S(t-s)f^2_{**}(s)\,ds+\int_0^t S(t-s)f^2(t,x_*(s),y_*(s))\,dW(s)\\&+\sum_{0<t_k<t} C(t-t_k)\overline{I}_k(x_*(t_k),y_n(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^2(x_*(t_k),y_*(t_k)).\end{aligned}$$

Hence $(h_{1*},h_{2*})\in(N_1(u_*),N_2(u_*))$, proving our claim.

(b) There exists a matrix M_{trix} with $\|M_{trix}\|<1$ such that

$$\mathbb{E}H_d(N(x,y),N(\overline{x},\overline{y}))\le M_{trix}\begin{pmatrix}\|x-\overline{x}\|_{\mathcal{M}_2}\\ \|y-\overline{y}\|_{\mathcal{M}_2}\end{pmatrix}$$

for any $x,y,\overline{x},\overline{y}\in\mathcal{M}_2$ and $h_i\in N_i(x,y)$. Then there exists $f^i_*(\cdot)\in S_{F^i,x,y}$ (f^i_* is a measurable selection) such that for each $i=1,2$ and $t\in[0,b]$, we have

$$\begin{aligned}h_1(t)&=C(t)x_0+S(t)\big(x_1-g^1(0,x_0,y_0)\big)\\&+\int_0^t C(t-s)g^1(s,x(s),y(s))\,ds+\int_0^t S(t-s)\int_Z k^1(t,x(s),y(s),z)\,\widetilde{N}(ds,dz)\\&+\int_0^t S(t-s)f^1_*(s)\,ds+\int_0^t S(t-s)f^1(t,x(s),y(s))\,dW(s)\\&+\sum_{0<t_k<t} C(t-t_k)I_k(x(t_k),y(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^1(x(t_k),y(t_k))\end{aligned}$$

and

$$
\begin{aligned}
h_2(t) &= C(t)y_0 + S(t)\big(y_1 - g^2(0, x_0, y_0)\big) \\
&\quad + \int_0^t C(t-s)g^2(s, x(s), y(s))\,ds + \int_0^t S(t-s)\int_Z k^2(t, x(s), y(s), z)\,\widetilde{N}(ds, dz) \\
&\qquad + \int_0^t S(t-s)f_*^2(s)\,ds + \int_0^t S(t-s)f^2(t, x(s), y(s))\,dW(s) \\
&\qquad\quad + \sum_{0<t_k<t} C(t-t_k)\overline{I}_k(x(t_k), y(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x(t_k), y(t_k)).
\end{aligned}
$$

From (H_1) and (H_2),

$$
\mathbb{E}H_{d_i}^2\big(F^i(t,x,y), F^i(t,\overline{x},\overline{y})\big) \le a_i\mathbb{E}|x-\overline{x}|_X^2 + b_i\mathbb{E}|y-\overline{y}|_X^2, \ \ \text{a.e.} t \in J.
$$

Hence there is $(w, \overline{w}) \in F^1(t, \overline{x}(t), \overline{y}(t)) \times F^2(t, \overline{x}(t), \overline{y}(t))$ such that

$$
\mathbb{E}|f_*^1(t) - w|_X^2 \le a_1\mathbb{E}|x-\overline{x}|_X^2 + b_1\mathbb{E}|y-\overline{y}|_X^2, \ \ t \in [0,b],
$$

and

$$
\mathbb{E}|f_*^2(t) - \overline{w}|^2 \le a_2\mathbb{E}|x-\overline{x}|_X^2 + b_2\mathbb{E}|y-\overline{y}|_X^2, \ \ t \in [0,b].
$$

Consider the multi-valued maps $U_i : [0,b] \to \mathcal{P}(X)$, $i = 1,2$, defined by

$$
U_1(t) = \Big\{w \in F^1(t, \overline{x}(t), \overline{y}(t)) : \ \ \mathbb{E}|f^1(t) - w|^2 \le a_1\mathbb{E}|x-\overline{x}|_X^2 + b_1\mathbb{E}|y-\overline{y}|_X^2 \ \text{ a.e. } \ t \in [0,b]\Big\},
$$

and similarly,

$$
U_2(t) = \Big\{\overline{w} \in F^2(t, \overline{x}(t), \overline{y}(t)) : \ \ \mathbb{E}|f^2(t) - \overline{w}|^2 \le a_2\mathbb{E}|x-\overline{x}|_X^2 + b_2\mathbb{E}|y-\overline{y}|_X^2, \ \text{ a.e. } \ t \in [0,b]\Big\},
$$

that is,

$$
U_1 = \overline{B}\Big(f_*^1(t), a_1E|x-\overline{x}|_X^2 + b_1\mathbb{E}|y-\overline{y}|_X^2\Big)
$$

and

$$
U_2 = \overline{B}\Big(f_*^2(t), a_2\mathbb{E}|x-\overline{x}|_X^2 + b_2\mathbb{E}|y-\overline{y}|_X^2\Big).
$$

Since f^i, a_i, b_i, x, y, $\overline{x}$, $\overline{y}$ are measurable for each $i = 1,2$, Theorem III.4.1 in [10] tells us that the closed ball U_i is measurable. In addition, (H_1) and (H_2) imply that for each $(x,y) \in \mathcal{M}_2 \times \mathcal{M}_2$, $F^i(t, x(t), y(t))$ is measurable. Finally, the set $V_i(\,\cdot\,) = U_i(\,\cdot\,) \cap F^i(\,\cdot\,, \overline{x}(\,\cdot\,), \overline{y}(\,\cdot\,))$ is nonempty, since it contains w, $\overline{w}$. Therefore, the intersection multi-valued operator V_i is measurable with nonempty, closed values [17], there exists a function $\overline{f}^i(t)$ which is a measurable selection for $V_i(\,\cdot\,)$. Thus $\overline{f}_*^i(t) \in F^i(t, \overline{x}(t), \overline{y}(t))$ and

$$
\mathbb{E}\big|f_*^1(t) - \overline{f}_*^1(t)\big|_X^2 \le a_1\mathbb{E}|x-\overline{x}|_X^2 + b_1\mathbb{E}|y-\overline{y}|_X^2 \ \text{ for a.e. } \ t \in J
$$

and

$$
\mathbb{E}\big|f_*^2(t) - \overline{f}_*^2(t)\big|_X^2 \le a_2\mathbb{E}|x-\overline{x}|_X^2 + b_2\mathbb{E}|y-\overline{y}|_X^2, \ \text{ for a.e. } \ t \in J.
$$

Let $(\overline{h}_1, \overline{h}_2)$ be defined by

$$
\begin{aligned}
\overline{h}_1(t) &= C(t)x_0 + S(t)(x_1 - g^1(0, x_0, y_0)) \\
&\quad + \int_0^t C(t-s)g^1(s, \overline{x}(s), \overline{y}(s))\,ds + \int_0^t S(t-s)\int_Z k^1(t, x(s), y(s), z)\,\widetilde{N}(ds, dz)
\end{aligned}
$$

$$+\int_0^t S(t-s)\overline{f}_*^1(s)\,ds+\int_0^t S(t-s)f^1(t,\overline{x}(s),\overline{y}(s))\,dW(s)$$
$$+\sum_{0<t_k<t} C(t-t_k)I_k(\overline{x}(t_k),\overline{y}(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^1(\overline{x}(t_k),\overline{y}(t_k))$$

and

$$\begin{aligned}\overline{h}_2(t)&=C(t)y_0+S(t)(y_1-g^2(0,x_0,y_0))\\&+\int_0^t C(t-s)g^2(s,\overline{x}(s),\overline{y}(s))\,ds+\int_0^t S(t-s)\int_Z k^2(t,x(s),y(s),z)\,\widetilde{N}(ds,dz)\\&+\int_0^t S(t-s)\overline{f}_*^2(s)\,ds+\int_0^t S(t-s)f^1(t,\overline{x}(s),\overline{y}(s))\,dW(s)\\&+\sum_{0<t_k<t} C(t-t_k)\overline{I}_k(\overline{x}(t_k),\overline{y}(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^2(\overline{y}x(t_k),\overline{y}(t_k)).\end{aligned}$$

This implies that

$$\begin{aligned}\mathbb{E}\big|h_1(t)-\overline{h}_1(t)\big|_X^2&\le 6\mathbb{E}\Big|\int_0^t C(t-s)(g^1(s,x(s),y(s))-g^1(s,\overline{x}(s),\overline{y}(s)))\,ds\Big|_X^2\\&+6\mathbb{E}\Big|\int_0^t\int_Z S(t-s)\big(k^1(t,x(s),y(s),z)-k^1(s,\overline{x}(s),\overline{y}(s),z)\big)\,\widetilde{N}(ds,dz)\Big|_X^2\\&+6\mathbb{E}\Big|\int_0^t S(t-s)(f_*^1(s)-\overline{f}_*^1(s))\,ds\Big|_X^2+6\mathbb{E}\Big|\int_0^t S(t-s)\big(f^1(t,x(s),y(s))-f^1(t,\overline{x}(s),\overline{y}(s))\big)\,dW(s)\Big|_X^2\\&+6\mathbb{E}\Big|\sum_{0<t_k<t} C(t-t_k)\big(I_k(x(t_k),y(t_k))-I_k(\overline{x}(t_k),\overline{y}(t_k))\big)\Big|_X^2\\&+6\mathbb{E}\Big|\sum_{0<t_k<t} S(t-t_k)\big(I_k^1(x(t_k),y(t_k))-I_k^1(\overline{x}(t_k),\overline{y}(t_k))\big)\Big|_X^2.\end{aligned}$$

Firstly, applying inequalities (2.1), (2.2) and (H_2)–(H_4), we get

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in J}|h_1(t)-\overline{h}_1(t)|_X^2\Big)&\le 6M^2\overline{a}_{g_1}\int_0^t\mathbb{E}|x(s)-\overline{x}(s)|_X^2+\overline{b}_{g_1}6M^2\int_0^t E|y(s)-\overline{y}(s)|_X^2\\&+6M^2cb\alpha_{k_1}\int_0^t\mathbb{E}|x(s)-\overline{x}(s)|_X^2+6M^2cb\beta_{k_1}\int_0^t\mathbb{E}|y(s)-\overline{y}(s)|_X^2\,ds\\&+6M^2a_1\int_0^t\mathbb{E}|x(s)-\overline{x}(s)|_X^2\,ds+6M^2b_1\int_0^t\mathbb{E}|y(s)-\overline{y}(s)|_X^2\,ds\\&+6M^2a_{f_1}C_2\int_0^t\mathbb{E}|x(s)-\overline{x}(s)|_X^2\,ds+6M^2b_{f_1}C_2\int_0^t\mathbb{E}|y(s)-\overline{y}(s)|_X^2\,ds\\&+12M^2m\sum_{0<t_k<t} d_k^1\mathbb{E}|x(t_k)-\overline{x}(t_k)|_X^2+12M^2m\sum_{0<t_k<t} d_k^2E|y(t_k)-\overline{y}(t_k)|_X^2.\end{aligned}$$

Taking the supremum, we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in J}|h_1(t)-\overline{h}_1(t)|_X^2\Big)\\
&\quad\le 6M^2\overline{a}_{g_1}\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|x(\theta)-\overline{x}(\theta)|_X^2\Big)\,ds+6\overline{b}_{g_1}M^2\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|y(\theta)-\overline{y}(\theta)|_X^2\Big)\,ds\\
&\quad+6M^2cb\alpha_{k_1}\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|x(\theta)-\overline{x}(\theta)|_X^2\Big)\,ds+6M^2cb\beta_{k_1}\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|y(\theta)-\overline{y}(\theta)|_X^2\Big)\,ds\\
&\quad+6M^2a_1\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|x(\theta)-\overline{x}(\theta)|_X^2\Big)\,ds+6M^2b_1\int_0^t E\Big(\sup_{0\le\theta\le s}|y(\theta)-\overline{y}(\theta)|_X^2\Big)\,ds\\
&\quad+6M^2a_{f_1}C_2\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|x(\theta)-\overline{x}(\theta)|_X^2\Big)\,ds+6M^2b_{f_1}C_2\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|y(\theta)-\overline{y}(\theta)|_X^2\Big)\,ds\\
&\quad+12M^2m\sum_{0<t_k<t}d_k^1\mathbb{E}\Big(\sup_{0\le\theta\le t}|x(\theta)-\overline{x}(\theta)|_X^2\Big)+12M^2m\sum_{0<t_k<t}d_k^2\mathbb{E}\Big(\sup_{0\le\theta\le t}|y(\theta)-\overline{y}(\theta)|_X^2\Big).
\end{aligned}$$

It is easy to get the estimations for h_2:

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in J}|h_2(t)-\overline{h}_2(t)|_X^2\Big)\\
&\quad\le 6M^2\overline{a}_{g_2}\int_0^t E\Big(\sup_{0\le\theta\le s}|x(\theta)-\overline{x}(\theta)|_X^2\Big)\,ds+\overline{b}_{g_2}6M^2\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|y(\theta)-\overline{y}(\theta)|_X^2\Big)\,ds\\
&\quad+6M^2cb\alpha_{k_2}\int_0^t E\Big(\sup_{0\le\theta\le s}|x(\theta)-\overline{x}(\theta)|_X^2\Big)\,ds+6M^2cb\beta_{k_2}\int_0^t E\Big(\sup_{0\le\theta\le s}\Big|y(\theta)-\overline{y}(\theta)\Big|_X^2\Big)\,ds\\
&\quad+6M^2a_2\int_0^t E\Big(\sup_{0\le\theta\le s}|x(\theta)-\overline{x}(\theta)|_X^2\Big)\,ds+6M^2b_2\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|y(\theta)-\overline{y}(\theta)|_X^2\Big)\,ds\\
&\quad+6M^2a_{f_2}C_2\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}|x(\theta)-\overline{x}(\theta)|_X^2)\,ds+6M^2b_{f_2}C_2\int_0^t E\Big(\sup_{0\le\theta\le s}|y(\theta)-\overline{y}(\theta)|_X^2\Big)\,ds\\
&\quad+12M^2m\sum_{0<t_k<t}\overline{d}_k^1\mathbb{E}\Big(\sup_{0\le\theta\le t}|x(\theta)-\overline{x}(\theta)|_X^2\Big)+12M^2m\sum_{0<t_k<t}\overline{d}_k^2\mathbb{E}\Big(\sup_{0\le\theta\le t}|y(\theta)-\overline{y}(\theta)|_X^2\Big).
\end{aligned}$$

So, we prove that

$$\mathbb{E}\Big(\sup_{t\in J}|h_1(t)-\overline{h}_1(t)|_X\Big)\le B_1\|x-\overline{x}\|_{\mathcal{M}_2}+B_2\|y-\overline{y}\|_{\mathcal{M}_2}.$$

On the other hand,

$$\mathbb{E}\Big(\sup_{t\in J}|h_2(t)-\overline{h}_2(t)|_X\Big)\le B_3\|x-\overline{x}\|_{\mathcal{M}_2}+B_4\|y-\overline{y}\|_{\mathcal{M}_2}.$$

An analogous relation obtained by exchanging the roles of x, y and $\overline{x}$, $\overline{y}$ results in

$$\mathbb{E}H_d\big(N(t,x,y),N(t,\overline{x},\overline{y})\big)=\begin{pmatrix}\mathbb{E}H_{d_1}(N_1((x,y)-N_1(\overline{x},\overline{y}))\\ \mathbb{E}H_{d_2}(N_2(x,y)-N_2(\overline{x},\overline{y}))\end{pmatrix}\le M_{trix}\begin{pmatrix}\|x-\overline{x}\|_{\mathcal{M}_2}\\ \|y-\overline{y}\|_{\mathcal{M}_2}\end{pmatrix},$$

where

$$M_{trix} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

Therefore, combining the above relations, we obtain

$$\mathbb{E}H_d\big(N(t,x,y), N(t,\overline{x},\overline{y})\big) \leq M_{trix} \begin{pmatrix} \|x-\overline{x}\|_{\mathcal{M}_2} \\ \|y-\overline{y}\|_{\mathcal{M}_2} \end{pmatrix}.$$

As M_{trix} converges to zero, by Theorem 2.2, N has a fixed point (x,y), which is a mild solution to (1.1). This completes the proof.

□

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References

[1] N. U. Ahmed, Nonlinear stochastic differential inclusions on Banach space. *Stochastic Anal. Appl.* **12** (1994), no. 1, 1–10.

[2] S. Albeverio, J.-L. Wu and T.-S. Zhang, Parabolic SPDEs driven by Poisson white noise. *Stochastic Process. Appl.* **74** (1998), no. 1, 21–36.

[3] D. Applebaum, *Lévy Processes and Stochastic Calculus.* Second edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, 2009.

[4] P. Balasubramaniam, Existence of solutions of functional stochastic differential inclusions. *Tamkang J. Math.* **33** (2002), no. 1, 35–43.

[5] P. Balasubramaniam, S. K. Ntouyas and D. Vinayagam, Existence of solutions of semilinear stochastic delay evolution inclusions in a Hilbert space. *J. Math. Anal. Appl.* **305** (2005), no. 2, 438–451.

[6] M. Benchohra, J. Henderson and S. Ntouyas, *Impulsive Differential Equations and Inclusions.* Contemporary Mathematics and Its Applications, 2. Hindawi Publishing Corporation, New York, 2006.

[7] J. Bertoin, *Lévy Processes.* Cambridge Tracts in Mathematics. 121. Cambridge Univ. Press., Cambridge, 1998.

[8] A. T. Bharucha-Reid, *Random Integral Equations.* Mathematics in Science and Engineering, Vol. 96. Academic Press, New York–London, 1972.

[9] T. Blouhi, T. Caraballo and A. Ouahab, Existence and stability results for semilinear systems of impulsive stochastic differential equations with fractional Brownian motion. *Stoch. Anal. Appl.* **34** (2016), no. 5, 792–834.

[10] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions.* Lecture Notes in Mathematics, Vol. 580. Springer-Verlag, Berlin-New York, 1977.

[11] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions.* Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.

[12] G. Da Prato and J. Zabczyk, *Second Order Partial Differential Equations in Hilbert Spaces.* London Mathematical Society Lecture Note Series, 293. Cambridge University Press, Cambridge, 2002.

[13] K. Deimling, *Multivalued Differential Equations.* De Gruyter Series in Nonlinear Analysis and Applications, 1. Walter de Gruyter & Co., Berlin, 1992.

[14] H. O. Fattorini, Second Order Linear Differential Equations in Banach Spaces. North-Holland Mathematics Studies, 108. Notas de Matemática [Mathematical Notes], 99. North-Holland Publishing Co., Amsterdam, 1985.

[15] T. C. Gard, *Introduction to Stochastic Differential Equations.* Monographs and Textbooks in Pure and Applied Mathematics, 114. Marcel Dekker, Inc., New York, 1988.

[16] I. I. Gikhman and A. V. Skorohod, *Stochastic Differential Equations.* Translated from the Russian by Kenneth Wickwire. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 72. Springer-Verlag, New York–Heidelberg, 1972.

[17] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings.* Mathematics and its Applications, 495. Kluwer Academic Publishers, Dordrecht, 1999.

[18] A. Granas and J. Dugundji, *Fixed Point Theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[19] Z. H. Guo, H. Jin and J. Chang, Total colorings of planar graphs with maximum degree 6 and without adjacent chordal 6-cycles. (Chinese) *J. Inn. Mong. Norm. Univ. Nat. Sci.* **47** (2018), no. 5, 398–400.

[20] E. Hausenblas, Existence, Uniqueness and regularity of parabolic SPDEs driven by Poisson random measure. *Electron. J. Probab.* **10** (2005), 1496–1546.

[21] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations.* Monographs and Textbooks in Pure and Applied Mathematics, 181. Marcel Dekker, Inc., New York, 1994.

[22] H. R. Henríquez, Existence of solutions of non-autonomous second order functional differential equations with infinite delay. *Nonlinear Anal.* **74** (2011), no. 10, 3333–3352.

[23] G. Kallianpur and J. Xiong, A nuclear space-valued stochastic differential equation driven by Poisson random measures. *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, 135–143, Lect. Notes Control Inf. Sci., 176, *Springer, Berlin*, 1992.

[24] A. Lasota and Z. Opial, An application of the Kakutani—Ky Fan theorem in the theory of ordinary differential equations. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 781–786.

[25] X. Mao, *Stochastic differential equations and Their Applications.* Horwood Publishing Series in Mathematics & Applications. Horwood Publishing Limited, Chichester, 1997.

[26] S. Mekki, T. Blouhi, J. J. Nieto and A. Ouahab, Some existence results for systems of impulsive stochastic differential equations. *Ann. Math. Sil.* **35** (2021), no. 2, 260–281.

[27] C. Mueller, The heat equation with Lévy noise. *Stochastic Process. Appl.* **74** (1998), no. 1, 67–82.

[28] L. Mytnik, Stochastic partial differential equation driven by stable noise. *Probab. Theory Related Fields* **123** (2002), no. 2, 157–201.

[29] B. Øksendal, *Stochastic Differential Equations. An Introduction with Applications.* Fourth edition. Universitext. Springer-Verlag, Berlin, 1995.

[30] A. Ouahab, Some Perov's and Karsnosel'skii type fixed point results and application. *Commun. Appl. Anal.* **19** (2015), 623–641.

[31] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion.* Third edition. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 293. Springer-Verlag, Berlin, 1999.

[32] K.-i. Sato, *Lévy Processes and Infinitely Divisible Distributions.* Translated from the 1990 Japanese original. Revised by the author. Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 1999.

[33] R. Situ, *Theory of Stochastic Differential Equations with Jumps and Applications. Mathematical and Analytical Techniques with Applications to Engineering.* Springer, New York, 2005.

[34] K. Sobczyk, *Stochastic Differential Equations. With Applications to Physics and Engineering.* Mathematics and its Applications (East European Series), 40. Kluwer Academic Publishers Group, Dordrecht, 1991.

[35] T. Taniguchi, The existence and asymptotic behaviour of solutions to non-Lipschitz stochastic functional evolution equations driven by Poisson jumps. *Stochastics* **82** (2010), no. 4, 339–363.

[36] C. P. Tsokos and W. J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*. Mathematics in Science and Engineering, Vol. 108. Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1974.

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