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NONLOCAL BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER LINEAR HYPERBOLIC EQUATIONS WITH TWO INDEPENDENT VARIABLES

Dedicated to the 90th birthday of Professor Takaŝi Kusano

Abstract. For linear hyperbolic equations of higher order nonlocal boundary value problems in a characteristic rectangle are investigated. Necessary and sufficient conditions of solvability and well-posedness of problems under consideration are established.

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რეზიუმე. მაღალი რიგის წრფივი ჰიპერბოლური განტოლებებისთვის შესწავლილია არალოკალური სასაზღვრო ამოცანები მახასიათებელ სწორკუთხედში. დადგენილია ამ ამოცანათა ამოხსნადობისა და კორექტულობის აუცილებელი და საკმარისი პირობები.

1 Formulation of the main results

In the rectangle $\Omega = [0,\omega_1]\times[0,\omega_2]$ consider the boundary value problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{1.1}$$

$$\ell_j(u(\,\cdot\,,x_2)) = \varphi_j(x_2) \quad (j=1,\dots,m_1), \quad h_k\big(u^{(m_1,0)}(x_1,\,\cdot\,)\big) = \psi_k^{(m_1)}(x_1) \ \ (k=1,\dots,m_2), \tag{1.2}$$

where $\mathbf{x}=(x_1,x_2)$, $\mathbf{m}=(m_1,m_2)$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1+\alpha_2}u(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\,,$$

$p_{\boldsymbol{\alpha}} \in C(\Omega)$ ($\boldsymbol{\alpha} < \mathbf{2}$), $q\in C(\Omega)$, $\varphi_j \in C^{m_2}([0,\omega_2])$ ($j=1,\dots,m_1$), $\psi_k \in C^{m_1}([0,\omega_1])$, and $\ell_j : C^{m_1-1}([0,\omega_1]) \to \mathbb{R}$ ($j=1,\dots,m_1$) and $h_k : C^{m_2-1}([0,\omega_2]) \to \mathbb{R}$ ($k=1,\dots,m_2$) are bounded linear functionals such that

$$\ell_j\circ h_k = h_k\circ \ell_j \qquad (j=1,\dots m_1;\ k=1,\dots,m_2). \tag{1.3}$$

Throughout the paper the following notations will be used:

$\mathbf{m}=(m_1,m_2)$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$.

$\mathbf{0}=(0,0)$, $\mathbf{1}=(1,1)$, $\mathbf{1}_1=(1,0)$, $\mathbf{1}_2=(0,1)$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2)<\boldsymbol{\beta}=(\beta_1,\beta_2) \iff \alpha_i\le\beta_i$ ($i=1,2$) and $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2)\le\boldsymbol{\beta}=(\beta_1,\beta_2) \iff \boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\|\boldsymbol{\alpha}\|=\alpha_1+\alpha_2$, $\boldsymbol{O}_{\mathbf{m}}=\big\{\boldsymbol{\alpha}<\mathbf{m}:\ \|\boldsymbol{\alpha}\| \text{ is odd}\big\}$.

$\mathbf{x}_{\boldsymbol{\alpha}}=(\chi(\alpha_1)x_1,\chi(\alpha_2)x_2)$, where $\chi(\alpha)=0$ if $\alpha=0$, and $\chi(\alpha)=1$ if $\alpha>0$.

$\mathbf{x}_{(j,k)}=(\chi(j)x_1,\chi(k)x_2)$.

$\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}=\mathbf{x}-\mathbf{x}_{\boldsymbol{\alpha}}$, $\widehat{\mathbf{x}}_{(j,k)}=\mathbf{x}-\mathbf{x}_{\boldsymbol{\alpha}}$. If $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ and $\alpha_1\,\alpha_2>0$, then $\mathbf{x}_{\boldsymbol{\alpha}}=\mathbf{x}$ and $\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}=\mathbf{0}$.

If $\boldsymbol{\alpha}=(\alpha_1,0)$, ($\boldsymbol{\alpha}=(0,\alpha_2)$), then $\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with x_1 g(with x_2).

By $C^{\mathbf{m}}(\Omega)$ denote the Banach space of functions $u:\Omega\to\mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ ($\boldsymbol{\alpha}\le\mathbf{m}$), endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

If $\ell_j(z) = z^{(j-1)}(x_0)$ ($j=1,\dots,m_1$), where $x_0\in[0,\omega_1]$, then conditions (1.2) turn into the initial-boundary conditions

$$u^{(j-1)}(x_0,x_2)=\varphi_j(x_2) \ \ (j=1,\dots,m_1), \quad h_k(u^{(m_1,0)}(x_1,\,\cdot\,)) = \psi_k^{(m_1)}(x_1) \ \ (k=1,\dots,m_2).,\tag{1.4}$$

In the present paper we will only briefly touch on the initial-boundary value problem (1.1), (1.4), since its more general version, where $h_k : C^{m_2-1}([0,\omega_2]) \to C([0,\omega_1])$ ($k=1,\dots,m_2$) are bounded linear operators, was studied in detail in [12, 13].

In [12] there were established necessary and sufficient conditions of well-posedness of problem (1.1), (1.4).

A complete description of uniquely solvable ill-posed problems (1.1), (1.4) was given in [13]. In particular, necessary conditions of solvability of problem (1.1), (1.4) (compatibility conditions) and sharp a priori estimates for its solutions were established.

In [19] there were established necessary and sufficient conditions of strong well-posedness of initial-boundary value problems for higher order nonlinear hyperbolic equations with two independent variables.

Initial-boundary value problems, as well as problems on periodic and bounded solutions for second order linear hyperbolic systems were studied in detail in [4].

Several special cases of initial-boundary value problem for nonlinear hyperbolic equations and systems were investigated in [14, 15].

Dirichlet type boundary value problems for fourth and higher order linear hyperbolic equations were studied in [6, 7, 10, 11, 21].

Several special cases of nonlocal boundary value problems for linear and quasi-linear hyperbolic equations of higher order were investigated in [3, 18].

One of the most important special cases of conditions (1.2) are the periodic boundary conditions, i.e. the case, where

$$l_j(z) = z^{(j-1)}(0) - z^{(j-1)}(\omega_1) \quad (j = 1, \dots, m_1),$$
$$h_k(z) = z^{(k-1)}(0) - z^{(k-1)}(\omega_2) \quad (k = 1, \dots, m_2).$$

As it follows from Theorem 1.1 below, the nonhomogeneous periodic problem is not well-posed in the sense of Definition 1.1 below. On the other hand, it is natural to study the periodic problem with homogeneous boundary conditions and periodic coefficients. In other words, it makes sense to study a problem on periodic solutions for equations with periodic coefficients.

Problems on doubly-periodic solutions for second order linear hyperbolic systems were studied in [5].

Problems on doubly-periodic solutions for nonlinear hyperbolic equations were studied in [8, 16].

Multidimensional periodic problems for higher order linear hyperbolic equations were studied in detail in [20].

One may think that the boundary conditions

$$\ell_j(u(\cdot\,, x_2)) = \varphi_j(x_2) \ \ (j = 1, \dots, m_1), \quad h_k(u(x_1, \cdot\,)) = \Psi_k(x_1) \ \ (k = 1, \dots, m_2), \tag{$\widetilde{1.2}$}$$

are more natural than conditions (1.2). All the more so, conditions ($\widetilde{1.2}$) obviously imply conditions (1.2).

The main reason for studying problem (1.1), (1.2) instead of problem (1.1), ($\widetilde{1.2}$) is that problem (1.1), ($\widetilde{1.2}$) is ill-posed, since functions φ_j and ψ_k should satisfy certain compatibility conditions. Indeed if $u \in C^{m_1, m_2}(\Omega)$ is an arbitrary function satisfying conditions ($\widetilde{1.2}$) then, in view of (1.3), we have

$$\ell_j(\psi_k) = \ell_j \circ h_k(u) = h_k \circ \ell_j(u) = h_k(\varphi_j).$$

By a solution of problem (1.1), (1.2) we understand a *classical* solution, i.e., a function $u \in C^{\mathbf{m}}(\Omega)$ satisfying equation (1.1) and boundary conditions (1.2) everywhere in Ω.

Along with problem (1.1), (1.2) consider its corresponding homogeneous problem

$$u^{(\mathbf{m})} = \sum_{\alpha < \mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{1.1_0}$$

$$\ell_j(u(\cdot\,, x_2)) = 0 \ \ (j = 1, \dots, m_1), \quad h_k(u^{(m_1, 0)}(x_1, \cdot\,)) = 0 \ \ (k = 1, \dots, m_2), \tag{1.2_0}$$

as well as the problems

$$v^{(m_1)} = \sum_{j=0}^{m_1} p_{j\,m_2}(x_1, x_2^*) v^{(j)}, \tag{1.1_1}$$

$$\ell_j(v) = 0 \ \ (j = 1, \dots, m_1) \tag{1.2_1}$$

and

$$v^{(m_2)} = \sum_{k=0}^{m_2} p_{m_1\,k}(x_1^*, x_2) v^{(k)}, \tag{1.1_2}$$

$$h_k(v) = 0 \ \ (k = 1, \dots, m_2). \tag{1.2_2}$$

Problems (1.1_1), (1.2_1) are (1.1_2), (1.2_2) called ***associated problems*** of problem (1.1), (1.2). Notice that problem (1.1_1), (1.2_1) (problem (1.1_1), (1.2_1)) is a boundary value problem for a linear ordinary differential equation depending on a parameter x_2^* (a parameter x_1^*).

The concept of $\boldsymbol{\sigma}$-associated problems for n-dimensional periodic problems was introduced in [20], and for two-dimensional Dirichlet type problems in [21].

1.1 Necessary conditions of solvability

Theorem 1.1. *Let problem (1.1), (1.2) be solvable for arbitrary $\varphi_j \in C^{m_2}([0,\omega_2])$ and $\psi_k \in C([0,\omega_1])$ $(j = 1,\dots,m_1;\ k = 1,\dots,m_2)$. Then the problem*

$$z^{(m_1)} = 0, \quad \ell_j(z) = 0 \ \ (j = 1,\dots,m_1) \tag{1.5}$$

has only the trivial solution.

Remark 1.1. If problem (1.5) has on the trivial solution, then problem $(1.1_0), (1.2_0)$ is equivalent to the homogeneous problem

$$u^{(\mathbf{m})} = \sum_{\alpha<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})},$$

$$\ell_j(u(\,\cdot\,,x_2)) = 0 \ \ (j = 1,\dots,m_1), \quad h_k(u(x_1,\,\cdot\,)) = 0 \ \ (k = 1,\dots,m_2). \tag{$\widetilde{1.2_0}$}$$

Theorem 1.2. *Let all of the coefficients of equation (1.1) be constants. Furthermore, let the associate problem $(1.1_1), (1.2_1)$ have a nontrivial solution, and let*

$$p_{j\,k} + p_{j\,m_2}p_{m_1\,k} = 0 \ \ \text{for} \ \ 0 < j < m_1, \ \ 0 < k < m_2. \tag{1.6}$$

Then for solvability of problem (1.1), (1.2) it is necessary that for every $k \in \{1,\dots,m_2\}$ the problem

$$v^{(m_1)} = \sum_{j=0}^{m_1-1} p_{j\,m_2}\, v^{(j)} + (p_{0\,0} + p_{m_1 0}\, p_{0\,m_2})\Psi_k(x_1) + h_k(q(x_1,\,\cdot\,)), \tag{1.7}$$

$$\ell_j(v) = h_k\Big(\varphi_j^{(m_2)} - \sum_{k=0}^{m_2-1} p_{m_1\,k}\varphi_j^{(k)}\Big) \ \ (j = 1,\dots,m_1), \tag{1.8}$$

where Ψ_k is a solution of the problem

$$z^{(m_1)} = \psi_k(x_1), \quad \ell_j(z) = h_k(\varphi_j) \ \ (j = 1,\dots,m_1) \tag{1.9}$$

is solvable.

Theorem 1.3. *Let all of the coefficients of equation (1.1) be constants and let conditions (1.5) and (1.6) hold. Furthermore, let the associate problem $(1.1_2), (1.2_2)$ have a nontrivial solution. Then for solvability of problem (1.1), (1.2) it is necessary that for every $j \in \{1,\dots,m_1\}$ the problem*

$$v^{(m_2)} = \sum_{k=0}^{m_2-1} p_{m_1\,k}\, v^{(k)} + (p_{0\,0} + p_{m_1\,0}\, p_{0\,m_2})\varphi_j(x_2) + \ell_j(q(\,\cdot\,,x_2)), \tag{1.10}$$

$$h_k(v) = \ell_j\Big(\psi_k - \sum_{j=0}^{m_1-1} p_{j\,m_2}\Psi_k^{(j)}\Big) \ \ (k = 1,\dots,m_2), \tag{1.11}$$

where Ψ_k is a solution of the problem (1.9), is solvable.

Remark 1.2. Solvability of ill-posed nonhomogenous associated problem (1.10), (1.11) actually means additional compatibility conditions between the boundary values φ_j and ψ_k, coefficients $p_{\boldsymbol{\alpha}}$ and q. Indeed, consider the problem

$$u^{(2,2)} = -u^{(2,0)} + p_0 u + p_1 u^{(0,1)} + p_2 u^{(0,2)} + q(x_1,x_2), \tag{1.12}$$

$$u^{(j-1,0)}(0,x_2) = \varphi_j(x_2) \ \ (j = 1,2), \quad u^{(m,0)}(x_1,0) = 0, \ \ u^{(m,0)}(x_2,\pi) = 0, \tag{1.13}$$

where p_1 and p_2 are positive constants and $q \in C^{m,0}(\Omega)$. By Corollary 1.2 from [13] problem (1.12), (1.13) is solvable if and only if

$$\int_0^{\pi} \Big(\sum_{k=0}^{2} p_k\, \varphi_1^{(k)}(0) + q(0,t)\Big) \sin t\, dt = 0. \tag{1.14}$$

Thus, for problem (1.12), (1.13), solvability of ill-posed nonhomogenous associated problem (1.10), (1.11) is equivalent to the compatibility condition (1.14).

Remark 1.3. Solvability of the ill-posed nonhomogenous associated problem (1.10), (1.11) is necessary for solvability of problem (1.1), (1.2) and is in no case sufficient, even if the homogeneous problem (1.1_0), (1.2_0) has only the trivial solution. Indeed, consider the problem

$$u^{(2,1)} = \cos^2 x_1 u - q(x_1), \tag{1.15}$$

$$u(0,x_2) = u(\pi,x_2) = 0, \quad u^{(2,0)}(x_1,\pi) = u^{(2,0)}(x_1,0), \tag{1.16}$$

where q is a continuous function such that $q(\pi) = q(0) = 0$. Problem (1.15), (1.16) is ill-posed, and its corresponding homogeneous problem has only the trivial solution. Furthermore, for problem (1.15), (1.16) all compatibility conditions hold. Therefore, due to uniqueness, the only possible solution of problem (1.15), (1.16) should be

$$u(x_1) = \frac{q(x_1)}{\cos^2 x_1}\,.$$

On the other hand, it is clear that problem (1.15), (1.16) has a solution if and only if

$$q(x_1) = \cos^2 x_1\, \widetilde{q}(x_1),$$

where $\widetilde{q} \in C^1([0,\pi])$. In particular, if $q(x_1) \equiv 1$, then problem (1.15), (1.16) has no solution despite the fact that all coefficients of equation (1.15) and boundary data are analytic functions.

1.2 Necessary and sufficient conditions of well-posedness

Theorem 1.4. *Let the following conditions hold:*

(A_0) *problem* (1.5) *has only the trivial solution;*

(A_1) *problem* (1.1_1), (1.2_1) *has only the trivial solution for every* $x_2^* \in [0,\omega_2]$*;*

(A_2) *problem* (1.1_2), (1.2_2) *have only the trivial solution for every* $x_1^* \in [0,\omega_1]$.

Then problem (1.1), (1.2) *has the Fredholm property, i.e. the following assertions hold:*

(i) *problem* (1.1_0), (1.2_0) *has a finite dimensional space of solutions;*

(ii) *if problem* (1.1_0), (1.2_0) *has only the trivial solution, then problem* (1.1), (1.2) *is uniquely solvable, and its solution* u *and admits the estimate*

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M\Big(\|q\|_{C(\Omega)} + \sum_{j=1}^{m_1} \|\varphi_j\|_{C^{m_2}([0,\omega_2])} + \sum_{k=1}^{m_2} \|\psi_k\|_{C([0,\omega_1])}\Big), \tag{1.17}$$

where M *is a positive constant independent of* φ_j, ψ_k *and* q.

Definition 1.1. Problem (1.1), (1.2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi_j \in C^{m_2}([0,\omega_2])$ $(j = 1,\dots,m_1)$, $\psi_k \in C([0,\omega_1])$ $(k = 1,\dots,m_2)$ and $q \in C(\Omega)$, and its solution u admits the estimate (1.17), where M is a positive constant independent of φ_j, ψ_k and q.

Theorem 1.5. *Let problem* (1.1), (1.2) *be well-posed. Then conditions* (A_1) *and* (A_2) *of Theorem 1.4 hold.*

Remark 1.4. Consider the problem

$$\begin{aligned}u^{(2,2)} = p_1(x_1,x_2)u^{(2,0)} - p^2(x_2)u^{(0,2)}\\ -4p(x_2)p'(x_2)u^{(0,1)} + \Big(p_1(x_1,x_2)p^2(x_2) - 2{p'}^2(x_2)p''(x_2) - 2p(x_2)p''(x_2)\Big)u\\ +q''(x_2) - p_1(x_1,x_2)p^2(x_2),\end{aligned} \tag{1.18}$$

$$u(0,x_2) = \varphi_1(x_2), \quad u(\pi,x_2) = \varphi_2(x_2), \quad u^{(2,0)}(x_1,0) = 0, \quad u^{(2,0)}(x_1,\pi) = 0, \tag{1.19}$$

where $p_1 \in C^\infty(\Omega)$ is an arbitrary *nonnegative* function, $p \in C^\infty([0,\pi])$ is such that

$$0 < p(x_2) \le 1 \ \text{ for } \ x_2 \in [0,\omega_2]$$

and φ_1, φ_2 and $q \in C^\infty([0,\pi])$ are such that

$$\varphi_j(0) = \varphi_j(\pi) = 0 \ \ (j = 1,2), \quad q(0) = q(\pi) = 0. \tag{1.20}$$

Let $u \in C^{2,2}(\Omega)$ satisfy (1.19). Then equalities (1.20) imply

$$u(x_1, 0) = u(x_1, \pi) = 0 \ \text{ for } \ x_2 \in [0,\pi].$$

It is easy to see that equation (1.18) is equivalent to the equation

$$\big(u^{(2,0)} + p^2(x_2)\, u - q(x_2)\big)^{(0,2)} = p_1(x_1,x_2)\big(u^{(2,0)} + p^2(x_2)\, u - q(x_2)\big). \tag{1.21}$$

Let u be a solution of problem (1.18), (1.19). Then, in view of (1.19), (1.20) and (1.21), u is a solution of the problem

$$u^{(2,0)} + p^2(x_2)\, u - q(x_2) = 0, \tag{1.22}$$

$$u(0,x_2) = \varphi_1(x_2), \quad u(\pi,x_2) = \varphi_2(x_2). \tag{1.23}$$

Set:

$$I_p = \big\{x_2 \in [0,\pi] : \ p(x_2) = 1\big\}.$$

If $x_2 \notin I_p$, then problem (1.22), (1.23) has a unique solution

$$\begin{aligned} u(x_1,x_2) = {} & \frac{\sin(p(x_2)(\pi - x_1))}{p(x_2)}\, \varphi_1(x_2) \\ & + \frac{\sin(p(x_2)(x_1))}{p(x_2)}\, \varphi_2(x_2) + \frac{1}{p^2(x_2)} \Big(1 - \frac{\cos(p(x_2)(x_1)(x_1 - \frac{\pi}{2}))}{\cos(p(x_2)\,\frac{\pi}{2})}\Big)\, q(x_2). \end{aligned} \tag{1.24}$$

From (1.24) it is clear that if $I_p \cap (0,\pi) \ne \varnothing$ and $|\varphi(x_2^*)| + |\varphi(x_2^*)| + |q(x_2^*)| > 0$ for some $x_2^* \in I_p \cap (0,\pi) \ne \varnothing$, then problem (1.18), (1.19) *has no classical solutions* despite the fact that all coefficients of equation (1.18) and the boundary data of (1.19) are C^∞ functions.

Let there exist $\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{q} \in C([0,\omega_2])$ such that

$$\varphi_1(x_2) = (p(x_2) - 1)\widetilde{\varphi}_1(x_2); \quad \varphi_1(x_2) = (p(x_2) - 1)\widetilde{\varphi}_2(x_2); \quad q(x_2) = (p(x_2) - 1)\widetilde{q}(x_2).$$

Then:

(i) problem (1.18), (1.19) is well-posed if and only if $I_p = \varnothing$;

(ii) if $\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{q} \in L^\infty([0,\omega_2])$, then problem (1.18), (1.19) has a unique weak solution if and only if $\operatorname{mes} I_p = 0$, and has an *infinite dimensional set* of *nonclassical* weak solutions otherwise. If $\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{q} \in C([0,\omega_2])$ and $\operatorname{mes} I_p = 0$, then that unique weak solution is a classical solution; ;

(iii) if $\widetilde{\varphi_1}$, $\widetilde{\varphi_2}$, $\widetilde{q} \in C([0,\omega_2])$, then problem (1.18), (1.19) has a *unique classical* solution if and only if I_p is nowhere dense in $[0,\omega_2]$, and has an *infinite dimensional set* of classical solutions otherwise;

(iv) if $\widetilde{\varphi_1}$, $\widetilde{\varphi_2}$, $\widetilde{q} \in C([0,\omega_2])$, then problem (1.18), (1.19) has a *unique classical* solution and an *infinite dimensional set* of *weak* solutions if I_p is a nowhere dense set of a positive measure.

Theorem 1.6. *Let conditions (A_0), (A_1), (A_2) of Theorem 1.4 hold, and let $p_{j\,m_2} \in C^{0,m_2}(\Omega)$ $(j = 0,\dots,m_1-1)$ be such that*

$$h_k(v) = 0 \ \ (k = 1,\dots,m_2) \implies h_k\big(p_{j\,m_2}(\,\cdot\,,x_2)v(\,\cdot\,)\big) = 0 \ \textit{ for } \ x_2 \in [0,\omega_2] \ \ (k = 1,\dots,m_2) \tag{1.25}$$

for every function $v \in C^{m_2-1}([0,\omega_2])$. *Then there exists* $\varepsilon > 0$ *such that if*

$$\Big| p_{jk}(\mathbf{x}) + \sum_{i=k}^{m_2} \frac{i!}{k!(i-k)!}\, p_{m_1 i}(\mathbf{x})\, p_{j\,m_2}^{(0,i-k)}(\mathbf{x}) - \frac{m_2!}{k!(m_2-k)!}\, p_{j\,m_2}^{(0,m_2-k)}(\mathbf{x}) \Big| \le \varepsilon \tag{1.26}$$
$$\text{for } \mathbf{x} \in \Omega \ \ (j = 0,\dots,m_1-1; \ \ k = 0,\dots m_2-1),$$

then problem (1.1), (1.2) *is well-posed. In particular, if*

$$p_{jk}(\mathbf{x}) + \sum_{i=k}^{m_2} \frac{i!}{k!(i-k)!}\, p_{m_1 i}(\mathbf{x})\, p_{j\,m_2}^{(0,i-k)}(\mathbf{x}) - \frac{m_2!}{k!(m_2-k)!}\, p_{j\,m_2}^{(0,m_2-k)}(\mathbf{x}) \equiv 0 \tag{1.27}$$
$$(j = 0,\dots,m_1-1; \ k = 0,\dots m_2-1),$$

then the solution of problem (1.1), (1.2_0) *admits the representation*

$$u(x_1,x_2) = \int_0^{\omega_1}\int_0^{\omega_2} g_1(x_1,s_1,x_2) g_2(x_2,s_2,s_1)\, q(s_1,s_2)\, ds_2\, ds_1, \tag{1.28}$$

where g_j *is Green's function of problem* $(1.1_j), (1.2_j)$ $(j = 1,2)$.

1.3 Initial-boundary value problems with nonlocal boundary conditions

Theorems 1.3 and 1.4 imply

Theorem 1.7. *Problem* (1.1), (1.4) *is well-posed if and only if the associated problem* $(1.1_2), (1.2_2)$ *has only the trivial solution for every* $x_1^* \in [0,\omega_1]$.

Notice that Theorem 1.7 is a particular case of Theorem 1.1 from [12].

Consider the initial-boundary value problems with integral boundary conditions

$$u^{(m,1)} = \sum_{(j,k)<(m,1)} p_{jk}(\mathbf{x}) u^{(j,k)} + q(\mathbf{x}), \tag{1.29}$$

$$u^{(j-1,0)}(0,x_2) = \varphi_j(x_2) \ \ (j = 1,\dots,m), \qquad \int_0^{\omega_1} H(t)\, u^{(m,0)}(x,t)\, dt = \psi(x_1), \tag{1.30}$$

and

$$u^{(m,2)} = \sum_{(j,k)<(m,2)} p_{jk}(\mathbf{x}) u^{(j,k)} + q(\mathbf{x}), \tag{1.31}$$

$$u^{(j-1,0)}(0,x_2) = \varphi_j(x_2) \ \ (j = 1,\dots,m), \qquad \int_0^{\omega_1} H_k(t)\, u^{(m,k-1)}(x,t)\, dt = \psi_k(x_1) \ \ (k = 1,2), \tag{1.32}$$

where $H(x_2)$, $H_1(x_2)$ and $H_2(x_2)$ are *not identically* zero functions.

Corollary 1.1. *Let*

$$H(x_2) \ge 0 \ \ \text{for} \ \ x_2 \in [0,\omega_2].$$

Then problem (1.29), (1.30) *is well-posed.*

Corollary 1.2. *Let*

$$H_k(x_2) \ge 0 \ \ \text{for} \ \ x_2 \in [0,\omega_2] \ \ (k = 1,2) \tag{1.33}$$

and let

$$p_{m0}(x_1,x_2) \ge 0 \ \ \text{for} \ \ (x_1,x_2) \in \Omega. \tag{1.34}$$

Then problem (1.31), (1.32) *is well-posed.*

1.4 Dirichlet type problems

For the following equations even and odd orders

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})u^{(2\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\in O_{2\mathbf{m}}} p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \text{ [1]} \tag{1.35}$$

$$u^{(2\mathbf{m}+\mathbf{1}_1)} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\alpha})u^{(2\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\in O_{2\mathbf{m}+\mathbf{1}_1}} p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\alpha})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{1.36}$$

and

$$u^{(2\mathbf{m}+\mathbf{1}_1)} = p_{\mathbf{0}}(\mathbf{x})u + q(\mathbf{x}) \tag{1.37}$$

consider the boundary conditions

$$\begin{gathered} u^{(j-1,0)}(0,x_2) = \varphi_{1j}(x_2), \quad u^{(j-1,0)}(\omega_1,x_2) = \varphi_{2j}(x_2) \ \ (j=1,\dots,m_1), \\ u^{(m_1,k-1)}(x_1,0) = \psi_{1k}(x_1), \quad u^{(m_1,k-1)}(x_1,\omega_2) = \psi_{2k}(x_1) \ \ (k=1,\dots,m_2), \end{gathered} \tag{1.38}$$

$$\begin{gathered} u^{(j-1,0)}(0,x_2) = \varphi_{1j}(x_2) \ \ (j=1,\dots,m_1+1), \quad u^{(j-1,0)}(\omega_1,x_2) = \varphi_{2j}(x_2) \ \ (j=1,\dots,m_1), \\ u^{(m_1,k-1)}(x_1,0) = \psi_{1k}(x_1), \quad u^{(m_1,k-1)}(x_1,\omega_2) = \psi_{2k}(x_1) \ \ (k=1,\dots,m_2), \end{gathered} \tag{1.39}$$

and

$$\begin{gathered} u^{(2(j-1),0)}(0,x_2) = \varphi_{1j}(x_2), \quad u^{(2(j-1),0)}(\omega_1,x_2) = \varphi_{2j}(x_2) \ \ (j=1,\dots,m_1), \\ u^{(m_1,2(k-1))}(x_1,0) = \psi_{1k}(x_1), \quad u^{(m_1,2(k-1))}(x_1,\omega_2) = \psi_{2k}(x_1) \ \ (k=1,\dots,m_2). \end{gathered} \tag{1.40}$$

Corollary 1.3. *Let there exist nonnegative numbers c_{jk} such that the inequalities*

$$(-1)^{\|\boldsymbol{m}\|} p_{00}(\boldsymbol{x}) \le c_{00}, \tag{1.41}$$

$$(-1)^{\|\boldsymbol{m}\|+j} p_{2j0}(x_2) \le c_{j0} \ \ (j=1,\dots,m_1), \tag{1.42}$$

$$(-1)^{\|\boldsymbol{m}\|+k} p_{02k}(x_1) \le c_{0k} \ \ (k=1,\dots,m_2), \tag{1.43}$$

$$(-1)^{\|\boldsymbol{m}\|+j+k} p_{2j\,2k} \le c_{jk} \ \ (j=1,\dots,m_1-1; \ \ k=1,\dots,m_2-1) \tag{1.44}$$

and

$$c_{00}\,\frac{\omega_1^{2m_1}\omega_2^{2m_2}}{\pi^{2\|\mathbf{m}\|}} + \sum_{j=1}^{m_1} c_{jm_2}\,\frac{\omega_1^{2(m_1-j)}}{\pi^{2(\|\mathbf{m}\|-j)}} + \sum_{k=1}^{m_2} c_{m_1k}\,\frac{\omega_2^{2(m_2-k)}}{\pi^{2(\|\mathbf{m}\|-k)}} + \sum_{j=1}^{m_1-1}\sum_{k=1}^{m_2-1} c_{jk}\,\frac{\omega_1^{2(m_1-j)}\omega_2^{2(m_2-k)}}{\pi^{2(\|\mathbf{m}\|-j-k)}} < 1 \tag{1.45}$$

hold. Then problem (1.35), (1.38) is well-posed.

Corollary 1.4. *Let the inequalities*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) \ge 0 \ \ (\boldsymbol{\alpha}<\mathbf{m}) \tag{1.46}$$

hold. Then problem (1.35), (1.38) is well-posed.

Corollary 1.5. *Let the inequalities*

$$\begin{gathered} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) \ge 0 \ \ (\boldsymbol{\alpha}\le\mathbf{m}), \\ (-1)^{\|\boldsymbol{m}\|+\|\mathbf{m}_1\|+\|\boldsymbol{\alpha}\|-1} p_{2\mathbf{m}_1+\mathbf{1}_1+2\boldsymbol{\alpha}}(\mathbf{x}_{\mathbf{1}_1+\boldsymbol{\alpha}}) \ge 0 \ \ (\boldsymbol{\alpha}<\mathbf{m_2}) \end{gathered} \tag{1.47}$$

hold, and let

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{2\boldsymbol{\beta}+2\mathbf{m}_2}(\widehat{\mathbf{x}}_{\beta+\mathbf{m}_2}) > 0 \tag{1.48}$$

for some $\boldsymbol{\beta}\le\mathbf{m}_1$. Then problem (1.36), (1.39) is well-posed.

[1] If $\alpha_1 > 0$ and $\alpha_2 > 0$, then $f(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})$ means that f is a constant function.

Corollary 1.6. *Let the inequality*

$$(-1)^{\|\boldsymbol{m}\|-1} p_{\mathbf{0}}(\mathbf{x}) \geq 0 \tag{1.49}$$

hold. Then problem (1.37), (1.39) *is well-posed.*

Corollary 1.7. *Let there exist nonnegative numbers* c_{jk} *such that the inequalities* (1.41)–(1.45) *hold and let*

$$p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\alpha}) \equiv 0 \ \text{ for } \ \boldsymbol{\alpha} \in O_{\mathbf{m}}. \tag{1.50}$$

Then problem (1.35), (1.40) *is well-posed.*

Corollary 1.8. *Let conditions* (1.46) *and* (1.50) *hold. Then problem* (1.35), (1.40) *is well-posed.*

1.5 Periodic Type Boundary Value Problems

For the equations

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) u^{(2\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{1.51}$$

and

$$u^{(2\mathbf{m})} = p_{\mathbf{0}}(\mathbf{x}) u + q(\mathbf{x}) \tag{1.52}$$

consider the boundary conditions

$$\begin{aligned} u^{(j-1,0)}(0,x_2) - a_j u^{(j-1,0)}(\omega_1,x_2) &= \varphi_{1j}(x_2) \ \ (j=1,\dots,m_1), \\ u^{(m_1,k-1)}(x_1,0) - b_k u^{(m_1,k-1)}(x_1,\omega_2) &= \psi_{1k}(x_1) \ \ (k=1,\dots,m_2). \end{aligned} \tag{1.53}$$

Corollary 1.9. *Let along with inequalities* (1.46) *the following conditions hold:*

$$a_j \neq 1 \ \ (j=1,\dots,2m_1), \quad b_k \neq 1 \ \ (k=1,\dots,2m_2), \tag{1.54}$$

$$a_j\, a_{2n+1-j} = 1 \ \ (j=1,\dots,n; \ \ n=1,\dots,m_1), \tag{1.55}$$

$$b_k\, b_{2i+n-k} = 1 \ \ (k=1,\dots,n; \ \ n=1,\dots,m_2) \tag{1.56}$$

Then problem (1.51), (1.53) *is well-posed.*

Corollary 1.10. *Let along with inequalities* (1.54) *the following conditions*

$$(-1)^{\|\boldsymbol{m}\|-1} p_{\mathbf{0}}(\mathbf{x}) \geq 0,$$

$$a_j\, a_{2m_1+1-j} = 1 \ \ (j=1,\dots,2m_1), \tag{1.57}$$

$$b_k\, b_{2m_2+1-k} = 1 \ \ (k=1,\dots,2m_2), \tag{1.58}$$

hold. Then problem (1.52), (1.53) *is well-posed.*

Remark 1.5. Conditions (1.55) and (1.56) are equivalent to the conditions

$$a_j = a^{(-1)^j} \ \ (j=1,\dots,2m_1), \quad b_k = b^{(-1)^k} \ \ (k=1,\dots,2m_2)$$

for some $a \neq 0$ and $b \neq 0$. Conditions (1.55) and (1.56) guarantee that every function $u \in C^{\mathbf{m}}(\Omega)$ satisfying conditions

$$\begin{aligned} u^{(j-1,0)}(0,x_2) &= a_j u^{(j-1,0)}(\omega_1,x_2) \ \ (j=1,\dots,m_1), \\ u^{(m_1,k-1)}(x_1,0) &= b_k u^{(m_1,k-1)}(x_1,\omega_2) \ \ (k=1,\dots,m_2), \end{aligned} \tag{1.53$_0$}$$

satisfies the equality

$$\iint\limits_{\Omega} u^{(2\boldsymbol{\alpha})}(\mathbf{x})\, u(\mathbf{x})\, d\mathbf{x} = (-1)^{\|\boldsymbol{\alpha}\|} \iint\limits_{\Omega} \left| u^{(\boldsymbol{\alpha})}(\mathbf{x}) \right|^2 d\mathbf{x} \tag{1.59}$$

for every $\boldsymbol{\alpha} \leq \mathbf{m}$.

In Corollary 1.10 conditions (1.55) and (1.56) are replaced by more relaxed conditions (1.57) and (1.58). Conditions (1.57) and (1.58) guarantee that every function $u \in C^{\mathbf{m}}(\Omega)$ satisfying conditions (1.53_0) satisfies equality (1.59) for $\boldsymbol{\alpha} = \mathbf{m}$ only.

Finally for the equation (1.37) consider the following boundary conditions

$$\begin{aligned} u^{(j-1,0)}(0,x_2)-a_j u^{(j-1,0)}(\omega_1,x_2)=\varphi_{1j}(x_2)\ \ (j=1,\dots,m_1+1),\\ u^{(m_1,k-1)}(x_1,0)-b_k u^{(m_1,k-1)}(x_1,\omega_2)=\psi_{1k}(x_1)\ \ (k=1,\dots,m_2). \end{aligned} \tag{1.60}$$

Corollary 1.11. *Let along with equalities (1.58) the following conditions hold:*

$$\begin{gathered} a_j\neq 1\ \ (j=1,\dots,2m_1+1),\quad b_k\neq 1\ \ (k=1,\dots,2m_2),\\ a_j\,a_{2n+2-j}=1\ \ (j=1,\dots,n;\ \ n=1,\dots,m_1),\\ \sigma\,p_{\mathbf{0}}(\mathbf{x})\geq 0, \end{gathered} \tag{1.61}$$

where

$$\sigma=(-1)^{\|\boldsymbol{m}\|-1}(1-a_{m+1}^2)\ \ \text{if}\ \ a_{m+1}\neq -1,\ \ \text{and}\ \ \sigma\in\{-1,1\}\ \ \text{if}\ \ a_{m+1}=-1. \tag{1.62}$$

Moreover, let there exist a point (x_1^*,x_2^*) such that ether

$$\sigma\,p_{\mathbf{0}}(x_1^*,x_2)>0\ \ \text{for}\ \ x_2\in[0,\omega_2], \tag{1.63}$$

or

$$\sigma\,p_{\mathbf{0}}(x_1,x_2^*)>0\ \ \text{for}\ \ x_1\in[0,\omega_1]. \tag{1.64}$$

Then problem (1.37), (1.60) is well-posed.

2 Auxiliary statements

Consider the boundary value problem

$$z^{(m)}=\sum_{k=0}^{m-1}p_k(t)z^{(k)}+q(t), \tag{2.1}$$

$$h_k(z)=c_k\ \ (k=1,\dots,m), \tag{2.2}$$

and its corresponding homogeneous problem

$$z^{(m)}=\sum_{k=0}^{m-1}p_k(t)z^{(k)}, \tag{2.1$_0$}$$

$$h_k(z)=0\ \ (k=1,\dots,m), \tag{2.2$_0$}$$

where $p_k\in C([0,\omega])$ $(k=0,\dots,m-1)$, $q\in C([0,\omega])$, $c_k\in\mathbb{R}$ $(k=1,\dots,m)$, and $h_k: C^{m-1}([0,\omega])\to\mathbb{R}$ $(k=1,\dots,m)$ are bounded linear functionals.

Lemma 2.1. *The following statements are equivalent:*

(i) *problem (2.1), (2.2) is solvable for arbitrary $q\in C(\Omega)$ and $c_k\in\mathbb{R}$ $(k=1,\dots,m)$;*

(ii) *problem (2.1), (2.2$_0$) is solvable for arbitrary $q\in C([0,\omega])$;*

(ii) *problem (2.1$_0$), (2.2) is solvable for arbitrary $c_k\in\mathbb{R}$ $(k=1,\dots,m)$;*

(iv) *problem (2.1$_0$), (2.2$_0$) has only the trivial solution.*

Lemma 2.1 is a well-known fact in the theory of boundary value problems for ordinary differential equations (e.g. see Theorem 1.1 from [2]). If problem (2.1$_0$), (2.2$_0$) has only the trivial solution then a solution of problem (2.1), (2.2) admits the representation

$$z(t)=\Gamma(c_1,\dots,c_m)(t)+\mathcal{G}(q)(t),$$

where $\Gamma : \mathbb{R}^m \to C^m([0,\omega])$ and $\mathcal{G} : C([0,\omega]) \to C^m([0,\omega])$ are bounded linear operators. Moreover, the operator $\mathcal{G}$ admits the representation

$$\mathcal{G}(q)(t) = \int\limits_0^\omega g(t,\tau)q(\tau)\,d\tau,$$

where $g : [0,\omega] \times [0,\omega] \to \mathbb{R}$ is called the **Green's function of problem** $(2.1_0), (2.2_0)$ (for more about Green's functions see [2]).

Lemma 2.2. *Let problem $(2.1_0), (2.2_0)$ have a nontrivial solution. Then for arbitrary $\varepsilon > 0$ there exist bounded linear functionals $\widetilde{h}_k : C^{m-1}([0,\omega]) \to \mathbb{R}$ $(k = 1,\dots,m)$ such that*

$$\|h_k - \widetilde{h}_k\| < \varepsilon \ \ (k = 1,\dots,n)$$

and the problem

$$z^{(m)} = \sum_{k=0}^{m-1} p_k(t) z^{(k)},$$

$$\widetilde{h}_k(z) = 0 \quad (k = 1,\dots,m), \tag{$\widetilde{2.2_0}$}$$

has only the trivial solution.

Proof. By Theorem 1.1 from [2], problem $(2.1_0), (\widetilde{2.2_0})$ has a nontrivial solution if and only if

$$\det(\widetilde{h}_j(z_k))_{j,k=1}^m = 0,$$

where $z_1(t),\dots,z_m(t)$ is an arbitrary fundamental set of solutions of (2.1_0). Set

$$\widetilde{h_k}(z) = (1-\lambda)h_k(z) + \lambda f_k(z) \ \ (k = 1,\dots,m),$$

where $f_k(z) = z^{(k-1)}(0)$ $(k = 1,\dots,m)$. Then $D(\lambda) = \det(\widetilde{h}_j(z_k))_{j,k=1}^m$ is a polynomial (of degree not greater than m) with respect to λ. Moreover, it is a non-identically zero polynomial. Indeed,

$$D(1) = \det\big(\widetilde{h}_j(z_k)\big)_{j,k=1}^m \neq 0 \ \text{ for } \ \lambda = 1,$$

since the initial value problem

$$z^{(m)} = \sum_{k=0}^{m-1} p_k(t) z^{(k)}, \quad z^{(k-1)} = 0 \ \ (k = 1,\dots,m)$$

has only the trivial solution. Hence, $D(\lambda)$ has at most m zeros.

Consequently, there exists $\delta > 0$ such that

$$D(\lambda) \neq 0 \ \text{ for } \ \lambda \in (0,\delta),$$

$$\|h_k - \widetilde{h}_k\| = \|\lambda(f_k - h_k)\| \le \lambda\big(1 + \|h_k\|\big)$$

(notice that $\|f_k\| = 1$ $(k = 1,\dots,m)$). The latter inequality with

$$\lambda < \min\Big\{\delta, \frac{\varepsilon}{(1 + \|h_k\|)}\Big\}$$

implies $\|h_k - \widetilde{h}_k\| < \varepsilon$ $(k = 1,\dots,n)$. □

Definition 2.1. $\mathcal{G} : C([0,\omega]) \to C^m([0,\omega])$ is called the **Green's operator** of problem $(2.1_0), (2.2_0)$.

Definition 2.2. $\Gamma : \mathbb{R}^m \to C^m([0,\omega])$ is called the **Green's boundary operator** of problem $(2.1_0), (2.2_0)$.

Consider the problem

$$v^{(m_1)} = \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(x_1, x_2) v^j, \tag{2.3}$$

$$\ell_j(v) = 0 \ \ (j = 1, \dots, m_1), \tag{2.4}$$

where $\widetilde{p}_{j\,m_2} \in C^{0,m_2}(\Omega)$ $(j = 0, \dots, m_1 - 1)$.

Lemma 2.3. *Let conditions (A_0), (A_1), (A_2) of Theorem 1.4 hold, and let problem (2.3), (2.4) have only the trivial solution for every $x_2 \in [0, \omega_2]$. Then an arbitrary solution u of problem (1.1), (1.2) admits the following representations:*

$$\begin{aligned}
u^{(m_1,0)}(x_1, x_2) = \int_0^{\omega_2} g_2(x_2, s_2; x_1) \Big(&\sum_{j=0}^{m_1-1} p_{j\,m_2}(x_1, s_2) u^{(j,m_2)}(x_1, s_2) \\
&+ \sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} p_{jk}(x_1, s_2) u^{(j,k)}(x_1, s_2) + q(x_1, s_2) \Big)\, ds_2 \\
&+ \Gamma_2\big(\psi_1^{(m_1)}(x_1), \dots, \psi_{m_2}^{(m_1)}(x_1)\big)(x_2);
\end{aligned}$$

$$\begin{aligned}
u^{(0,m_2)}(x_1, x_2) = \int_0^{\omega_1} g_1(x_1, s_1; x_2) \Big(&\sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x}) u^{(m_1,k)}(s_1, x_2) \\
&+ \sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} p_{jk}(s_1, x_2) u^{(j,k)}(s_1, x_2) + q(s_1, x_2) \Big)\, ds_1 \\
&+ \Gamma_1\big(\varphi_1^{(m_2)}(x_2), \dots, \varphi_{m_1}^{(m_2)}(x_2)\big)(x_1);
\end{aligned}$$

$$\begin{aligned}
u(x_1, x_2) = \int_0^{\omega_1} \widetilde{g}_1(x_1, s_1; x_2) \int_0^{\omega_2} g_2(x_2, s_2; s_1) \Big(&\sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} \rho_{jk}(s_1, s_2) u^{(\boldsymbol{\alpha})}(s_1, s_2) \\
&+ \sum_{j=0}^{m_1-1} \big(p_{j\,m_2}(s_1, s_2) - \widetilde{p}_{j\,m_2}(s_1, s_2)\big) u^{(j,m_2)}(s_1, s_2) + q(s_1, s_2) \Big)\, ds_2\, ds_1 \\
&+ \mathcal{P}\big[u; \psi_1^{(m_1)}, \dots, \psi_{m_2}^{(m_1)}\big](x_1, x_2) + \widetilde{\Gamma}_1\big(\varphi_1(x_2), \dots, \varphi_{m_1}(x_2)\big)(x_1),
\end{aligned}$$

where

$$\begin{aligned}
\rho_{jk}(x_1, x_2) = p_{jk}(x_1, x_2) &+ \sum_{i=k}^{m_2-1} \frac{k!}{i!(i-k)!}\, p_{m_1\,i}(\mathbf{x})\, \widetilde{p}_{j\,m_2}^{(0,i-k)}(x_1, x_2) \\
&- \frac{m_2!}{k!(m_2-k)!}\, \widetilde{p}_{j\,m_2}^{(0,m_2-k)}(x_1, x_2) \quad (j = 0, \dots, m_1 - 1; \ \ k = 0, \dots m_2 - 1),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{P}\big[u; \psi_1^{(m_1)}, \dots, \psi_{m_2}^{(m_1)}\big](x_1, x_2) \\
&\quad = \int_0^{\omega_1} \widetilde{g}_1(x_1, s_1; x_2) \Gamma_2 \Big[\psi_1^{(m_1)}(s_1) - \sum_{j=0}^{m_1-1} h_1\Big(\widetilde{p}_{j\,m_2}(s_1, \cdot\,) u^{(j,0)}(s_1, \cdot\,)\Big), \dots, \\
&\qquad\qquad \psi_{m_2}^{(m_1)}(s_1) - \sum_{j=0}^{m_1-1} h_{m_2}\Big(\widetilde{p}_{j\,m_2}(s_1, \cdot\,) u^{(j,0)}(s_1, \cdot\,)\Big) \Big]\, ds_1,
\end{aligned}$$

g_j and Γ_j, respectively, are the Green's function and Green's boundary operator of problem (1.1_j), (1.2_j) *$(j = 1, 2)$, and $\widetilde{g}_1$ and $\widetilde{\Gamma}_1$, respectively,are the Green's function and the Green's boundary operator of problem* (2.3), (2.4).

Proof. Let u be a solution of problem (1.1), (1.2). Set

$$v(x_1,x_2)=u^{(m_1,0)}(x_1,x_2);\quad w(x_1,x_2)=u^{(0,m_2)}(x_1,x_2);$$
$$\widetilde{v}(x_1,x_2)=u^{(m_1,0)}(x_1,x_2)-\sum_{j=0}^{m_1-1}\widetilde{p}_{jn}(x_1,x_2)u^{(j,0)}(x_1,x_2).$$

In order to prove Lemma 2.2, one needs to notice that v, w and $\widetilde{v}$, respectively, are solution of the following boundary value problems:

$$\begin{aligned}
v^{(0,m_2)}&=\sum_{k=0}^{m_2-1}p_{m_1k}(x_1,x_2)v^{(0,k)}+\sum_{j=0}^{m_1-1}p_{jm_2}(x_1,x_2)u^{(j,m_2)}(x_1,x_2)\\
&\qquad+\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1}p_{jk}(x_1,x_2)u^{(j,k)}(x_1,x_2)+q(x_1,x_2),\\
h_k(v(x_1,\cdot\,))&=\psi_k^{(m_1)}(x_1)\ \ (k=1,\dots,m_2);\\
w^{(m_1,0)}&=\sum_{j=0}^{m_1-1}p_{jm_2}(x_1,x_2)(x_1,x_2)w^{(j,0)}+\sum_{k=0}^{m_2-1}p_{m_1k}(x_1,x_2)u^{(m_1,k)}(x_1,x_2)\\
&\qquad+\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1}p_{jk}(x_1,x_2)u^{(j,k)}(x_1,x_2)+q(x_1,x_2),\\
\ell_j(w(\,\cdot\,,x_2))&=\varphi_j^{(m_2)}(x_2)\ \ \ (j=1,\dots,m_1);\\
\widetilde{v}^{(0,m_2)}&=\sum_{k=0}^{m_2-1}p_{m_1k}(x_1,x_2)\widetilde{v}^{(0,k)}+\sum_{j=0}^{m_1-1}\big(p_{jm_2}(x_1,x_2)-\widetilde{p}_{jm_2}(x_1,x_2)\big)u^{(j,m_2)}(x_1,x_2)\\
&\qquad+\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1}\rho_{jk}(x_1,x_2)u^{(j,k)}(x_1,x_2)+q(x_1,x_2),\\
h_k(v(x_1,\cdot\,))&=\psi_k^{(m_1)}(x_1)-h_k\Big(\sum_{j=0}^{m_1-1}\widetilde{p}_{jm_2}(x_1,\cdot\,)u^{(j,m_2)}(x_1,\cdot\,)\Big)\ \ (k=1,\dots,m_2).
\end{aligned}$$

□

Lemma 2.4. *Let conditions* (A_0), (A_1), (A_2) *of Theorem 1.4 hold. Then problem* (1.1), (1.2) *has the Fredholm property.*

Proof. In view of Lemma 2.2, problem (1.1), (1.2) is equivalent to the following system of integral equations

$$v(x_1,x_2)=\mathcal{F}_1(u,w)(x_1,x_2);\tag{2.5}$$
$$w(x_1,x_2)=\mathcal{F}_2(u,v)(x_1,x_2);\tag{2.6}$$
$$u(x_1,x_2)=\mathcal{F}(u,w)(x_1,x_2),\tag{2.7}$$

where

$$\begin{aligned}
\mathcal{F}_1(u,w)(x_1,x_2)&=\int_0^{\omega_2}g_2(x_2,s_2;x_1)\bigg(\sum_{j=0}^{m_1-1}p_{j\,m_2}(x_1,s_2)w^{(j,0)}(x_1,s_2)\\
&\qquad+\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1}p_{jk}(x_1,s_2)u^{(j,k)}(x_1,s_2)+q(x_1,s_2)\bigg)\,ds_2\\
&\qquad+\Gamma_2\big(\psi_1^{(m_1)}(x_1),\dots,\psi_{m_2}^{(m_1)}(x_1)\big)(x_2);\\
\mathcal{F}_2(u,v)(x,y)&=\int_0^{\omega_1}g_1(x_1,s_1;x_2)\bigg(\sum_{k=0}^{m_2-1}p_{m_1\,k}(\mathbf{x})v^{(0,k)}(s_1,x_2)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(x_1,s_2)(s_1,x_2)u^{(j,k)}(s_1,x_2)+q(s_1,x_2)\Big)\,ds_1\\
&+\Gamma_1\big(\varphi_1^{(m_2)}(x_2),\dots,\varphi_{m_1}^{(m_2)}(x_2)\big)(x_1);
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_0(u,w)(x_1,x_2)=\int\limits_0^{\omega_1}\widetilde{g}_1(x_1,s_1;x_2)\int\limits_0^{\omega_2} g_2(x_2,s_2;s_1)\Big(&\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1}\rho_{jk}(s_1,s_2)u^{(j,k)}(s_1,s_2)\\
&+\sum_{j=0}^{m_1-1}\big(p_{j\,m_2}(s_1,s_2)-\widetilde{p}_{j\,m_2}(s_1,s_2)\big)w^{(j,0)}(s_1,s_2)+q(s_1,s_2)\Big)\,ds_2\,ds_1\\
&+\mathcal{P}\big[u;\psi_1^{(m_1)},\dots,\psi_{m_2}^{(m_1)}\big](x_1,x_2)+\widetilde{\Gamma}_1\big(\varphi_1(x_2),\dots,\varphi_{m_1}(x_2)\big)(x_1).
\end{aligned}
$$

Let $\mathcal{F}_1^0(u,w)$, $\mathcal{F}_2^0(u,v)$ and $\mathcal{F}_0^0(u,w)$ be the homogeneous parts of the operators $\mathcal{F}_1(u,w)$, $\mathcal{F}_2(u,v)$ and $\mathcal{F}_0(u,w)$, respectively, and set:

$$
\mathcal{K}(u,v,w)=\big(\mathcal{F}_1^0(u,w),\mathcal{F}_2^0(u,v),\mathcal{F}_0^0(u,w)\big).
$$

It is clear that $\mathcal{K}$ is a bounded linear operator from $C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)$ into $C^{m_1-1,m_2}(\Omega)\times C^{m_1,m_2-1}(\Omega)\times C^{m_1,m_2}(\Omega)$.

Notice that $\mathcal{K}^2=\mathcal{K}\circ\mathcal{K}$ is a compact operator from

$$
C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)
$$

into $C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)$. The latter fact implies that the system of operator equations (2.5)–(2.7) and, consequently, problem (1.1), (1.2) have the Fredholm property. □

Lemma 2.5. *Let conditions (A_0), (A_1), (A_2) of Theorem 1.4 hold, and let $p_{jm_2}\in C^{0,m_2}(\Omega)$ $(j=0,\dots,m_1-1)$. Then problem (1.1), (1.2) is equivalent to the operator equation*

$$
u(\mathbf{x})=\mathcal{F}(u)(\mathbf{x}),
$$

where

$$
\begin{aligned}
\mathcal{F}(u)(\mathbf{x})&=\int\limits_0^{\omega_1} g_1(x_1,s_1;x_2)\int\limits_0^{\omega_2} g_2(x_2,s_2;s_1)\Big(\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1}\rho_{jk}(s_1,s_2)u^{(j,k)}(s_1,s_2)+q(s_1,s_2)\Big)\,ds_2\,ds_1\\
&\quad+\mathcal{P}\big[u;\psi_1^{(m_1)},\dots,\psi_{m_2}^{(m_1)}\big](x_1,x_2)+\Gamma_1\big(\varphi_1(x_2),\dots,\varphi_{m_1}(x_2)\big)(x_1),\\
\rho_{jk}(x_1,x_2)&=p_{jk}(x_1,x_2)+\sum_{i=k}^{m_2-1}\frac{k!}{i!(i-k)!}\,p_{m_1\,i}(x_1,x_2)\,p_{j\,m_2}^{(0,i-k)}(x_1,x_2)\\
&\quad-\frac{m_2!}{k!(m_2-k)!}\,p_{j\,m_2}^{(0,m_2-k)}(x_1,x_2)\ \ (j=0,\dots,m_1-1;\ \ k=0,\dots m_2-1),\\
\mathcal{P}\big[u;\psi_1^{(m_1)},&\dots,\psi_{m_2}^{(m_1)}\big](x_1,x_2)\\
&=\int\limits_0^{\omega_1} g_1(x_1,s_1;x_2)\Gamma_2\Big[\psi_1^{(m_1)}(s_1)-\sum_{j=0}^{m_1-1}h_1\big(p_{j\,m_2}(s_1,\cdot\,)u^{(j,0)}(s_1,\cdot\,)\big),\dots,\\
&\qquad\qquad\psi_{m_2}^{(m_1)}(s_1)-\sum_{j=0}^{m_1-1}h_{m_2}\big(p_{j\ m_2}(s_1,\cdot\,)u^{(j,0)}(s_1,\cdot\,)\big)\Big]\,ds_1,
\end{aligned}
$$

and g_j and Γ_j, respectively, are the Green's function and Green's boundary operator of problem $(1.1_j),(1.2_j)$ $(j=1,2)$.

3 Proofs of the main results

Proof of Theorem 1.1. Let $\psi_k(x_1) \equiv 0$ $(k = 1, \dots, m_2)$, and let

$$\varphi_j(x_2) = c_j \varphi(x_2) \ \ (j = 1, \dots, m_1),$$

where $c_1, \dots, c_{m_1}$ are arbitrary real numbers and $h_1(\varphi) = 1$ (the latter equality is possible, since h_1 is not a zero functional).

Let u be an arbitrary solution of problem (1.1), (1.2). Set $z = h_1(u(x_1, \cdot))$. Then z is a solution of the problem

$$z^{(m_1)} = 0, \tag{3.1}$$

$$\ell_j(z) = c_j \ \ (j = 1, \dots, m_1). \tag{3.2}$$

Consequently, problem (3.1), (3.2) is solvable for arbitrary boundary values $c_1, \dots, c_{m_1}$. By Lemma 2.1, the homogeneous problem (1.4) has only the trivial solution. □

Proof of Theorem 1.2. Let u be a solution of problem (1.1), (1.2). Set:

$$w(x_1, x_2) = u^{(0,m_2)}(x_1, x_2) - \sum_{k=0}^{m_2-1} p_{m_1 k}\, u^{(0,k)}(x_1, x_2),$$

$$v_k(x_1) = h_k(w(x_1, \cdot)) \ \ (k = 1, \dots, m_2).$$

In view of (1.2) and (1.6), w is a solution of the problem

$$w^{(m_1,0)} = \sum_{j=0}^{m_1-1} p_{j m_2}\, w^{(j,0)} + (p_{00} + p_{m_1 0} p_{0 m_2})\, u(x_1, x_2) + q(x_1, x_2), \tag{3.3}$$

$$\begin{aligned} \ell_j\big(w(\cdot, x_2)\big) &= \ell_j\Big(u^{(0,m_2)}(\cdot, x_2) - \sum_{k=0}^{m_2-1} p_{m_1 k}\, u^{(0,k)}(\cdot, x_2)\Big) \\ &= \varphi_j^{(m_2)}(x_2) - \sum_{k=0}^{m_2-1} p_{m_1 k}\, \varphi_j^{(k)}(x_2) \ \ (j = 1, \dots, m_i). \end{aligned} \tag{3.4}$$

After applying the operator h_k to (3.3) and (3.4) and utilizing (1.5), we get:

$$\begin{aligned} &h_k\big(w^{(m_1,0)}(x_1, \cdot)\big) \\ &\quad = v_k^{(m_1)}(x_1) = \sum_{j=0}^{m_1-1} p_{j m_2}\, v_k^{(j)}(x_1) + (p_{00} + p_{m_1 0} p_{0 m_2})\, \Psi_k(x_1) + h_k\big(q(x_1, \cdot)\big) \\ &\qquad\qquad = h_k\Big(\sum_{j=0}^{m_1-1} p_{j m_2}\, w^{(0,k)}(x_1, \cdot) + (p_{00} + p_{m_1 0} p_{0 m_2}) u(x_1, \cdot) + q(x_1, \cdot)\Big) \end{aligned}$$

and

$$h_k\big(\ell_j(w)\big) = \ell_j\big(h_k(w)\big) = \ell_j\big(v_k\big) = h_k\Big(\varphi_j^{(m_2)} - \sum_{k=0}^{m_2-1} p_{m_1 k}\, \varphi_j^{(k)}\Big) \ \ (j = 1, \dots, m_i).$$

Consequently, $v_k(x_1)$ is a solution of problem (1.7), (1.8). □

Proof of Theorem 1.3. Let u be a solution of problem (1.1), (1.2). Set:

$$w(x_1, x_2) = u^{(m_1,0)}(x_1, x_2) - \sum_{j=0}^{m_1-1} p_{j m_2}\, u^{(j,0)}(x_1, x_2),$$

$$v_j(x_2) = \ell_j(w(\cdot, x_2)) \ \ (j = 1, \dots, m_1).$$

In view of (1.2) and (1.6), w is a solution of the problem

$$w^{(0,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1k}\, w^{(0,k)} + (p_{00} + p_{m_10}p_{0m_2})\, u(x_1,x_2) + q(x_1,x_2), \tag{3.5}$$

$$\begin{aligned} h_k(w(x_1,\,\cdot\,)) &= h_k\Big(u^{(m_1,0)}(x_1,\,\cdot\,) - \sum_{j=0}^{m_1-1} p_{jm_2}\, u^{(j,0)}(x_1,\,\cdot\,)\Big) \\ &= \psi_k(x_1) - \sum_{j=0}^{m_1-1} p_{jm_2}\, \Psi_k^{(j)}(x_1) \ \ (k=1,\dots,m_2). \end{aligned} \tag{3.6}$$

After applying the operator ℓ_j to (3.5) and (3.6) and utilizing (1.5), we get:

$$\begin{aligned} \ell_j\big(w^{(0,m_2)}(\,\cdot\,,x_2)\big) &= v_j^{(m_2)}(x_2) \\ &= \sum_{k=0}^{m_2-1} p_{m_1k}\, v_j^{(k)}(x_2) + (p_{00} + p_{m_10}p_{0m_2})\, \varphi_j(x_2) + \ell_j\big(q(\,\cdot\,,x_2)\big) \\ &= \ell_j\Big(\sum_{k=0}^{m_2-1} p_{m_1k}\, w^{(0,k)}(\,\cdot\,,x_2) + (p_{00} + p_{m_10}p_{0m_2})\, u(\,\cdot\,,x_2) + q(\,\cdot\,,x_2)\Big)\Big) \end{aligned}$$

and

$$\ell_j(h_k(w)) = h_k(\ell_j(w)) = h_k(v_j) = \ell_j\Big(\psi_k - \sum_{j=0}^{m_1-1} p_{jm_2}\, \Psi_k^{(j)}\Big) \ \ (j=1,\dots,m_i).$$

Consequently, $v_j(x_2)$ is a solution of problem (1.10), (1.11). □

Proof of Theorem 1.4. Theorem 1.4 follows from Lemmas 2.3 and 2.4. □

Proof of Theorem 1.5. Let problem (1.1), (1.2) be well-posed. Assume the contrary: either condition (A_1) or (A_2) condition be is not satisfied.

If condition (A_1) is not satisfied, then problem $(1.1_1), (1.2_1)$ has a nontrivial solution $\xi_0(x_1)$ for some $x_2^* \in [0,\omega_2]$. Due to well-posedness of problem (1.1), (1.2) there exist $\delta > 0$ and $\widetilde{p}_{j\,m_2} \in C^{(0,m_2)}(\Omega)$ $(j=0,\dots,m_1-1)$ such that

$$\widetilde{p}_{j\,m_2}(x_1,x_2) = p_{j\,m_2}(x_1,x_2^*) \ \text{ for } \ x_2 \in [x_2^*-\delta, x_2^*+\delta] \cap [0,\omega_2] \ \ (j=0,\dots,m_1-1),$$

and the problem

$$u^{(m_1,m_2)} = \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x}) u^{(j,m_2)} + \sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x}) u^{(m_1,k)} + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(\mathbf{x}) u^{(j,k)} + q(\mathbf{x}), \tag{3.7}$$

$$\ell_j(u(\,\cdot\,,x_2)) = 0 \ \ (j=1,\dots,m_1), \quad h_k(u^{(m_1)}(x_1,\,\cdot\,)) = \psi_k(x_1) \ \ (k=1,\dots,m_2) \tag{3.8}$$

is well-posed. In other words, problem (3.7), (3.8) has a unique solution

$$u(\mathbf{x}) = \mathcal{A}(\psi_1,\dots,\psi_{m_2},q)(\mathbf{x}),$$

where $\mathcal{A} : C([0,\omega_1]) \times \cdots \times C([0,\omega_1]) \times C(\Omega) \to C^{\mathbf{m}}(\Omega)$ is a bounded linear operator.

Consider the problem

$$u^{(m_1,m_2)} = \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x}) u^{(j,m_2)} + \sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x}) u^{(m_1,k)} + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} \widetilde{p}_{jk}(\mathbf{x}) u^{(j,k)}, \tag{3.9}$$

$$\begin{aligned} &\ell_j(u(\,\cdot\,,x_2)) = 0 \ \ (j=1,\dots,m_1), \\ &h_k\Big(u^{(m_1,0)}(x_1,\,\cdot\,) - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(x_1,\,\cdot\,) u^{(j,0)}(x_1,\,\cdot\,)\Big) = 0 \ \ (k=1,\dots,m_2), \end{aligned} \tag{3.10}$$

where

$$\widetilde{p}_{j\,k}(\mathbf{x}) = -\sum_{i=k}^{m_2-1} \frac{k!}{i!(i-k)!}\, p_{m_1\,i}(\mathbf{x})\, \widetilde{p}_{j\,m_2}^{(0,i-k)}(\mathbf{x}) + \frac{m_2!}{k!(m_2-k)!}\, \widetilde{p}_{j\,m_2}^{(0,m_2-k)}(\mathbf{x})$$
$$(j=0,\dots,m_1-1;\quad k=0,\dots m_2-1).$$

Every solution u of problem (3.9), (3.10) is also a solution of the operator equation

$$u(\mathbf{x}) = \widetilde{\mathcal{A}}(u)(\mathbf{x}), \tag{3.11}$$

where

$$\widetilde{\mathcal{A}}(u)(\mathbf{x}) = \mathcal{A}\big(\mathbf{F}_1(u),\dots,\mathbf{F}_{m_2}(u),Q(u)\big)(\mathbf{x}),$$
$$\mathbf{F}_k(u)(x_1) = h_k\Big(\sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(x_1,\,\cdot\,)u^{(j,0)}(x_1,\,\cdot\,)\Big)\ \ (k=1,\dots,m_2),$$
$$Q(u)(\mathbf{x}) = \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} \big(\widetilde{p}_{jk}(\mathbf{x}) - p_{jk}(\mathbf{x})\big)u^{(j,k)}(\mathbf{x}).$$

It is clear that $\widetilde{\mathcal{A}} : C^{\mathbf{m}-\mathbf{1}}(\Omega)\to C^{\mathbf{m}}(\Omega)$ is a bounded linear operator. Consequently, $\widetilde{\mathcal{A}} : C^{\mathbf{m}-\mathbf{1}}(\Omega)\to C^{\mathbf{m}-\mathbf{1}}(\Omega)$ is a compact operator. Therefore, (3.11) has a finite dimensional space of solutions in $C^{\mathbf{m}-\mathbf{1}}(\Omega)$. But then problem (3.9), (3.10) has a finite dimensional space of solutions too.

On the other hand, (3.9) is equivalent to the equation

$$\Big(u^{(m_1,0)} - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}\Big)^{(0,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x})\Big(u^{(m_1,0)} - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}\Big)^{(0,k)}. \tag{3.12}$$

Hence, every solution $u\in C^{\mathbf{m}}(\Omega)$ of the problem

$$u^{(m_1,0)} = \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}, \tag{3.13}$$
$$\ell_j(u(\,\cdot\,,x_2)) = 0\ \ (j=1,\dots,m_1), \tag{3.14}$$

is a solution of problem (3.12), (3.10) and, consequently, of problem (3.9), (3.10).

Let $\gamma\in C^\infty([0,\omega_2])$ be an arbitrary function such that $\operatorname{supp}\gamma\subset[x_2^*-\delta,x_2^*+\delta]\cap[0,\omega_2]$. Then

$$u(\mathbf{x}) = \xi_0(x_1)\gamma(x_2)$$

is a solution of problem (3.13), (3.14) and, consequently, of problem (3.9), (3.10). Thus problem (3.9), (3.10) has an infinite dimensional space of solutions, which contradicts to the fact that equation (3.11) has a finite dimensional space of solutions.

Now let us assume that problem (1.1_2), (1.2_2) has a nontrivial solution $\eta_0(x_2)$ for some $x_1^*\in[0,\omega_1]$. In view of well-posedness of problem (1.1), (1.2) and Lemma 2.2, there exist $\delta>0$ and $\widetilde{p}_{m_2k}\in C^{(m_1,0)}(\Omega)$ $(k=0,\dots,m_2-1)$ and bonded linear functionals $\widetilde{h}_k : C^{m_2-1}([0,\omega_1])$ $(k=1,\dots,m_2)$ such that

$$\widetilde{p}_{m_1k}(x_1,x_2) = p_{m_1k}(x_1^*,x_2)\ \text{ for }\ x_1\in[x_1^*-\delta,x_1^*+\delta]\cap[0,\omega_1]\ \ (k=0,\dots,m_2-1),$$

the problem

$$z^{(m_2)} = 0,\quad \widetilde{h}_k(z) = 0\ \ (k=1,\dots,m_2) \tag{3.15}$$

has only the trivial solution, and the problem

$$u^{(m_1,m_2)} = \sum_{j=0}^{m_1-1} p_{j\,m_2}(\mathbf{x})u^{(j,m_2)} + \sum_{k=0}^{m_2-1} \widetilde{p}_{m_1\,k}(\mathbf{x})u^{(m_1,k)} + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(\mathbf{x})u^{(j,k)} + q(\mathbf{x}), \tag{3.16}$$
$$\ell_j(u(\,\cdot\,,x_2)) = \varphi_j(x_2)\ \ (j=1,\dots,m_1),\quad \widetilde{h}_k(u^{(m_1)}(x_1,\,\cdot\,)) = 0\ \ (k=1,\dots,m_2) \tag{3.17}$$

is well-posed. In other words, problem (3.16), (3.17) has a unique solution

$$u(\mathbf{x}) = \mathcal{B}(\varphi_1, \dots, \varphi_{m_1}, q)(\mathbf{x}),$$

where $\mathcal{B} : C([0,\omega_2]) \times \cdots \times C([0,\omega_2]) \times C(\Omega) \to C^{\mathbf{m}}(\Omega)$ is a bounded linear operator.

Consider the problem

$$u^{(m_1,m_2)} = \sum_{j=0}^{m_1-1} p_{j\,m_2}(\mathbf{x})u^{(j,m_2)} + \sum_{k=0}^{m_2-1} \widetilde{p}_{m_1\,k}(\mathbf{x})u^{(m_1,k)} + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} \widetilde{p}_{jk}(\mathbf{x})u^{(j,k)}, \tag{3.18}$$

$$\begin{gathered} \ell_j\Bigg(u(x_1,\,\cdot\,) - \int\limits_0^{\omega_2} g_0(x_2,t)\Big(\sum_{k=0}^{m_2-1} \widetilde{p}_{m_1k}(\,\cdot\,,t)u^{(0,k)}(\,\cdot\,,t)\Big)\,dt\Bigg) = 0 \ \ (j=1,\dots,m_1), \\ \widetilde{h}_k\big(u^{(m_1,0)}(x_1,\,\cdot\,)\big) = 0 \ \ (k=1,\dots,m_2), \end{gathered} \tag{3.19}$$

where

$$\widetilde{p}_{j\,k}(\mathbf{x}) = -\sum_{i=j}^{m_1-1} \frac{j!}{i!(i-j)!}\, p_{i\,m_2}(\mathbf{x})\,\widetilde{p}_{m_1k}^{(i-j,0)}(\mathbf{x}) + \frac{m_1!}{j!(m_1-j)!}\,\widetilde{p}_{m_1k}^{(m_1-j,0)}(\mathbf{x})$$

$$(j=0,\dots,m_1-1;\ \ k=0,\dots m_2-1)$$

and g_0 is Green's function of the problem (3.15).

Every solution u of problem (3.18), (3.19) is also a solution of the operator equation

$$u(\mathbf{x}) = \widetilde{\mathcal{B}}(u)(\mathbf{x}), \tag{3.20}$$

where

$$\begin{aligned} \widetilde{\mathcal{B}}(u)(\mathbf{x}) &= \mathcal{B}\big(\mathbf{P}_1(u),\dots,\mathbf{P}_{m_1}(u),Q(u)\big)(\mathbf{x}), \\ \mathbf{P}_j(u)(x_2) &= \ell_j\Bigg(\int\limits_0^{\omega_2} g_0(x_2,t)\Big(\sum_{k=0}^{m_2-1} \widetilde{p}_{m_1k}(\,\cdot\,,t)u^{(0,k)}(\,\cdot\,,t)\Big)\,dt\Bigg) \ \ (j=1,\dots,m_1), \\ Q(u)(\mathbf{x}) &= \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} \big(\widetilde{p}_{jk}(\mathbf{x}) - p_{jk}(\mathbf{x})\big)u^{(j,k)}(\mathbf{x}). \end{aligned}$$

It is clear that $\widetilde{\mathcal{B}} : C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m}}(\Omega)$ is a bounded linear operator. Consequently, $\widetilde{\mathcal{B}} : C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m-1}}(\Omega)$ is a compact operator. Therefore, (3.20) has a finite dimensional space of solutions in $C^{\mathbf{m-1}}(\Omega)$. But then problem (3.18), (3.19) has a finite dimensional space of solutions too.

On the other hand, (3.18) is equivalent to the equation

$$\Big(u^{(0,m_2)} - \sum_{k=0}^{m_2-1} \widetilde{p}_{m_1k}(\mathbf{x})u^{(0,k)}\Big)^{(m_1,0)} = \sum_{j=0}^{m_1-1} p_{jm_2}(\mathbf{x})\Big(u^{(0,m_2)} - \sum_{k=0}^{m_2-1} \widetilde{p}_{m_1k}(\mathbf{x})u^{(0,k)}\Big)^{(0,k)}. \tag{3.21}$$

Hence, every solution $u \in C^{\mathbf{m}}(\Omega)$ of the problem

$$u^{(0,m_2)} = \sum_{k=0}^{m_2-1} \widetilde{p}_{m_1k}(\mathbf{x}) \tag{3.22}$$

$$\widetilde{h}_k(u(x_1,\,\cdot\,)) = 0 \ \ (k=1,\dots,m_2), \tag{3.23}$$

is a solution of problem (3.21), (3.19) and, consequently, of problem (3.18), (3.19).

Let $\gamma \in C^\infty([0,\omega_1])$ be an arbitrary function such that $\operatorname{supp}\gamma \subset [x_1^* - \delta, x_1^* + \delta] \cap [0,\omega_1]$. Then

$$u(\mathbf{x}) = \eta_0(x_2)\gamma(x_1)$$

is a solution of problem (3.22), (3.23) and, consequently, of problem (3.18), (3.19). Thus problem (3.18), (3.19) has an infinite dimensional space of solutions, which contradicts to the fact that equation (3.20) has a finite dimensional space of solutions. The obtained contradiction completes the proof of the theorem. □

Proof of Theorem 1.6. In view of Lemma 2.5, problem (1.1), (1.2_0) is equivalent to the operator equation

$$u(\mathbf{x}) = \mathcal{F}(u)(\mathbf{x}) \tag{3.24}$$

in the space $C^{\mathbf{m-1}}(\Omega)$, where

$$\mathcal{F}(u)(\mathbf{x}) = \int_0^{\omega_1} g_1(x_1, s_1; x_2) \int_0^{\omega_2} g_2(x_2, s_2; s_1) \bigg(\sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} \rho_{jk}(s_1, s_2) u^{(j,k)}(s_1, s_2) + q(s_1, s_2) \bigg) \, ds_2 \, ds_1,$$

and

$$\begin{aligned} \rho_{jk}(x_1, x_2) &= p_{jk}(x_1, x_2) \\ &+ \sum_{i=k}^{m_2-1} \frac{k!}{i!(i-k)!} \, p_{m_1\, i}(x_1, x_2) \, p_{j\, m_2}^{(0,i-k)}(x_1, x_2) - \frac{m_2!}{k!(m_2-k)!} \, p_{j\, m_2}^{(0,m_2-k)}(x_1, x_2) \\ &\qquad (j = 0, \dots, m_1 - 1; \ \ k = 0, \dots m_2 - 1). \end{aligned}$$

Hence, it is obvious that if conditions (1.26) hold, then for $\varepsilon > 0$ sufficiently small, $\mathcal{F}$ is an operator of contraction. Therefore, equation (3.24) is uniquely solvable and, thus, problem (1.1), (1.2) is well-posed. Moreover, if equalities (1.27) hold, then the unique solution u of equation (3.24), as well as of problem (1.1), (1.2), admits representation (1.28). □

Theorem 1.7 is a particular case of Theorem 1.1 from [12].

Proof of Corollary 1.1. Let $x_1^* \in [0, \omega_1]$ and let $v \in C^1([0, \omega_2])$ be a solution of the problem

$$v' = p_{m0}(x_1^*, x_2) v, \tag{3.25}$$

$$\int_0^{\omega_1} H(t) \, v(t) \, dt = 0 \tag{3.26}$$

has only the trivial solution for every $x_1 \in [0, \omega_1]$. In view of inequality (1.26) an arbitrary function $v \in C([0, \omega_2])$ satisfying the condition (3.26) necessarily changes its sign and, consequently, has at least one zero in $[0, \omega_2]$. But then, by the existence and uniqueness theorem, every solution of problem (3.25), (3.26) has only the trivial solution for every $x_1 \in [0, \omega_1]$. Hence, by Theorem 1.7, problem (1.22), (1.23) is well-posed. □

Proof of Corollary 1.2. Let $x_1^* \in [0, \omega_1]$ and let $v \in C^2([0, \omega_2])$ be a solution of the problem

$$v'' = p_{m0}(x_1^*, x_2) v + p_{m1}(x_1^*, x_2) v', \tag{3.27}$$

$$\int_0^{\omega_1} H_k(t) \, v^{(k-1)}(t) \, dt = 0 \ \ (k = 1, 2).$$

In view of inequalities (1.27) there exist numbers a and $b \in [0, \omega_2]$ such that

$$v(a) = 0, \quad v'(b) = 0. \tag{3.28}$$

If $a = b$, then, by the existence and uniqueness theorem, $v(x_2) \equiv 0$. After multiplying (3.27) by $v(x_2) e^{-\int_a^{x_2} p_{m1}(x_1^*, \tau) \, d\tau}$ and integrating over the $[a, b]$ interval ($[b, a]$ interval, if $b < a$), we get

$$\int_a^b e^{-\int_a^t p_{m1}(x_1^*, \tau) \, d\tau} \big(v'^2(t) + p_{m0}(x_1^*, t) v^2(t) \big) \, dt = 0.$$

The latter equality, along with (1.28) and (3.28), immediately implies $v(x_2) \equiv 0$ for $x_2 \in [a, b]$. By the existence and uniqueness theorem, $v(x_2) \equiv 0$ on the entire interval $[0, \omega_2]$. Hence, by Theorem 1.7, problem (1.24), (1.25) is well-posed. □

Proof of Corollary 1.3. Firstly notice that condition (A_0) of Theorem 1.4 holds, since the problem

$$v^{(2m_1)} = 0, \quad v^{(j-1)}(0) = 0, \quad v^{(j-1)}(\omega_1) = 0 \ \ (j = 1, \dots, m_1)$$

has only the trivial solution.

Consider the associated problems of problem (1.29), (1.31):

$$v^{(2m_1)} = p_{0\,2m_2}(x_1)v + \sum_{j=1}^{2m_1-1} p_{j\,2m_2} v^{(j)}, \tag{3.29}$$

$$v^{(j-1)}(0) = 0, \quad v^{(j-1)}(\omega_1) = 0 \ \ (j = 1, \dots, m_1) \tag{3.30}$$

and

$$v^{(2m_2)} = p_{2m_1\,0}(x_2)v + \sum_{k=1}^{2m_2-1} p_{2m_1\,k} v^{(k)}, \tag{3.31}$$

$$v^{(k-1)}(0) = 0, \quad v^{(k-1)}(\omega_1) = 0 \ \ (k = 1, \dots, m_2). \tag{3.32}$$

Let v be an arbitrary solution of problem (3.31), (3.32). After multiplying (3.31) by $v(x_2)$, integrating over $[0, \omega_2]$ and taking into account conditions (3.32), we get

$$\int_0^{\omega_2} {v^{(m_2)}}^2(t) = \int_0^{\omega_2} \Big((-1)^{m_2-1} p_{2m_1 0}(t) v^2(t) + \sum_{k=1}^{m_2-1} (-1)^{m_2+k} p_{2m_1\,2k}\, {v^{(k)}}^2(t) \Big)\, dt.$$

In view of inequalities (1.42)–(1.45), by Wirtinger's inequality (see inequality (2.57) in [1]), we get

$$\int_0^{\omega_2} {v^{(m_2)}}^2(t) \le \int_0^{\omega_2} \sum_{k=0}^{m_2-1} c_{m_1 k}\, {v^{(k)}}^2(t)\, dt \le \sum_{k=0}^{m_2-1} c_{m_1 k} \frac{\omega_2^{2(m_2-k)}}{\pi^{2(\|\mathbf{m}\|-k)}} \int_0^{\omega_2} {v^{(m_2)}}^2(t) < \int_0^{\omega_2} {v^{(m_2)}}^2(t).$$

The latter equality along with (3.32) implies $v(x_1) \equiv 0$.

Similarly one can show that problem (3.29), (3.30) has only the trivial solution.

In view of Theorem 1.4, it remains to show that the homogeneous problem

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) u^{(2\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\in O_{2\mathbf{m}}} p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) u^{(\boldsymbol{\alpha})}, \tag{3.33}$$

$$\begin{aligned} u^{(j-1,0)}(0, x_2) = 0, \quad u^{(j-1,0)}(\omega_1, x_2) = \varphi_{2j}(x_2) \ \ (j = 1, \dots, m_1), \\ u^{(m_1,k-1)}(x_1, 0) = 0, \quad u^{(m_1,k-1)}(x_1, \omega_2) = \psi_{2k}(x_1) \ \ (k = 1, \dots, m_2), \end{aligned} \tag{3.34}$$

has only the trivial solution. Let u be an arbitrary solution of problem (3.33), (3.34). Multiply equation (3.33) by u and integrate over Ω. After integrating by parts multiple times and taking into account conditions (3.34), we arrive at the equality

$$\iint_\Omega \big|u^{(\mathbf{m})}(\mathbf{x})\big|^2\, d\mathbf{x} = \iint_\Omega \Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}} (-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) |u^{(\boldsymbol{\alpha})}(\mathbf{x})|^2 \Big)\, d\mathbf{x},$$

whence, in view of inequalities (1.41)–(1.45) and Wirtinger's inequality, we get

$$\begin{aligned} \iint_\Omega \big|u^{(\mathbf{m})}(\mathbf{x})\big|^2\, d\mathbf{x} &\le \iint_\Omega \Big(\sum_{(j,k)<\mathbf{m}} c_{jk} |u^{(j,k)}(\mathbf{x})|^2 \Big)\, d\mathbf{x} \\ &\le \sum_{(j,k)<\mathbf{m}} c_{jk} \frac{\omega_1^{2(m_1-j)} \omega_2^{2(m_2-k)}}{\pi^{2(\|\mathbf{m}\|-j-k)}} \iint_\Omega \big|u^{(\mathbf{m})}(\mathbf{x})\big|^2\, d\mathbf{x} < \iint_\Omega \big|u^{(\mathbf{m})}(\mathbf{x})\big|^2\, d\mathbf{x} \end{aligned}$$

and, consequently,

$$u^{(\mathbf{m})}(\mathbf{x}) = 0. \tag{3.35}$$

$u(\mathbf{x}) \equiv 0$ immediately follows from (3.35) and (3.34). □

Corollary 1.4 is particular case of Corollary 1.3.

Proof of Corollary 1.5. Firstly notice that condition (A_0) of Theorem 1.4 holds, since the problem

$$v^{(2m_1+1)} = 0, \quad v^{(j-1)}(0) = 0, \quad v^{(j-1)}(\omega_1) = 0 \ \ (j = 1, \dots, 2m_1 + 1) \tag{3.36}$$

has only the trivial solution.

Consider the associated problems of problem (1.30), (1.32):

$$v^{(2m_1+1)} = p_{0\,2m_2}(x_1)\, v + \sum_{j=1}^{2m_1} p_{j\,2m_2} v^{(j)}, \tag{3.37}$$

$$v^{(j-1)}(0) = 0 \ \ (j = 1, \dots, m_1 + 1), \quad v^{(j-1)}(\omega_1) = 0 \ \ (j = 1, \dots, m_1) \tag{3.38}$$

and

$$v^{(2m_2)} = p_{2m_1+1\,0}(x_2)v + \sum_{k=1}^{m_2-1} p_{2m_1+1\,2k} v^{(k)}, \tag{3.39}$$

$$v^{(k-1)}(0) = 0, \quad v^{(k-1)}(\omega_1) = 0 \ \ (k = 1, \dots, m_2). \tag{3.40}$$

Let v be an arbitrary solution of problem (3.37), (3.38). After multiplying (3.37) by $v(x_1)$, integrating over $[0, \omega_1]$ and taking into account boundary conditions (3.38), we get

$$\frac{1}{2}\, {v^{(m_1)}}^2(\omega_1) + \int_0^{\omega_1} \Big((-1)^{m_1-1} p_{0\,2m_2}(t)\, v^2(t) + \sum_{j=1}^{m_1} (-1)^{m_1+j-1} p_{2j\,2m_2}\, {v^{(j)}}^2(t) \Big)\, dt = 0.$$

The latter equality along with (1.34), (1.36) and (3.38) implies $v(x_1) \equiv 0$.

In the proof of Corollary 1.3 it was established that under conditions (1.47) problem (3.39), (3.40) has only the trivial solution.

In view of Theorem 1.4, it remains to show that the homogeneous problem

$$u^{(2\mathbf{m}+\mathbf{1}_1)} = \sum_{\boldsymbol{\alpha} \le \mathbf{m}} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_\alpha) u^{(2\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha} \in O_{2\mathbf{m}+1_1}} p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_\alpha) u^{(\boldsymbol{\alpha})}, \tag{3.41}$$

$$\begin{gathered} u^{(j-1,0)}(0, x_2) = 0 \ \ (j = 1, \dots, m_1 + 1), \quad u^{(j-1,0)}(\omega_1, x_2) = 0 \ \ (j = 1, \dots, m_1), \\ u^{(m_1,k-1)}(x_1, 0) = 0, \quad u^{(m_1,k-1)}(x_1, \omega_2) = 0 \ \ (k = 1, \dots, m_2), \end{gathered} \tag{3.42}$$

has only the trivial solution. Indeed, let u be an arbitrary solution of problem (3.41), (3.42). Multiply equation (3.41) by u and integrate over Ω. After integrating by parts multiple times and taking into account conditions (3.42), we get:

$$\begin{aligned} \frac{1}{2} \int_0^{\omega_2} \Big(\big| u^{(\mathbf{m})}(\omega_1, x_2) \big|^2 + (-1)^{\|\mathbf{m}\|+m_1-1} p_{2m_1+1\,0}(x_2) \big| u^{(m_1,0)}(\omega_1, x_2) \big|^2 \\ + \sum_{k=1}^{m_2-1} (-1)^{\|\mathbf{m}\|+m_1+k-1} p_{2m_1+1\,2k} \big| u^{(m_1,k)}(\omega_1, x_2) \big|^2 \Big)\, dx_2 \\ + \iint_\Omega \Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}} (-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|-1} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) |u^{(\boldsymbol{\alpha})}(\mathbf{x})|^2 \Big)\, d\mathbf{x} = 0. \end{aligned} \tag{3.43}$$

From (1.25) and (3.43) we get

$$u^{(\mathbf{m})}(\mathbf{x}) = 0. \tag{3.44}$$

From (3.44) and from (3.42) it follows that $u(\mathbf{x}) \equiv 0$ follows. ☐

Proof of Corollary 1.6. Conditions (A_0), (A_1) and (A_2) of Theorem 1.4 hold, since problem (3.36), as well as the problem

$$v^{(2m_2)} = 0, \quad v^{(k-1)}(0) = 0, \quad v^{(k-1)}(\omega_2) = 0 \;\; (j = 1, \dots, 2m_2)$$

By Theorem 1.4, it remains to show that the homogeneous problem

$$u^{(\mathbf{2m}+\mathbf{1}_1)} = p_{\mathbf{0}}(\mathbf{x})u, \tag{3.45}$$

$$\begin{gathered} u^{(j-1,0)}(0,x_2) = 0 \;\; (j = 1, \dots, m_1 + 1), \quad u^{(j-1,0)}(\omega_1, x_2) = 0 \;\; (j = 1, \dots, m_1), \\ u^{(m_1,k-1)}(x_1, 0) = 0, \quad u^{(m_1,k-1)}(x_1, \omega_2) = 0 \;\; (k = 1, \dots, m_2), \end{gathered} \tag{3.46}$$

has only the trivial solution. Let u be an arbitrary solution of problem (3.45), (3.46). Multiply equation (3.45) by u and integrate over Ω. After integrating by parts multiple times and taking into account conditions (1.49) and (3.46), we get:

$$\frac{1}{2}\int_0^{\omega_2} \left|u^{(\mathbf{m})}(\omega_1, x_2)\right|^2 dx_2 + \iint_\Omega |p_{\mathbf{0}}(\mathbf{x})|\, u^2(\mathbf{x})\, d\mathbf{x} = 0. \tag{3.47}$$

Consequently,

$$u(\omega_1, x_2) \equiv 0 \tag{3.48}$$

and

$$\iint_\Omega |p_{\mathbf{0}}(\mathbf{x})|\, u^2(\mathbf{x})\, d\mathbf{x} = 0. \tag{3.49}$$

Now multiply (3.45) by $u^{(1,0)}$ and integrate over Ω. In view of (3.46) and (3.48), we get

$$\iint_\Omega \left|u^{(m_1+1,m_2)}(\mathbf{x})\right|^2 d\mathbf{x} = (-1)^{\|m\|} \iint_\Omega p_{\mathbf{0}}(\mathbf{x})\, u(\mathbf{x})\, u^{(1,0)}(\mathbf{x})\, d\mathbf{x} \tag{3.50}$$

On the other hand, in view of (3.46), (3.48), (3.49) and Wirtinger's inequality, we have:

$$\begin{aligned} &\iint_\Omega \left|p_{\mathbf{0}}(\mathbf{x})\, u(\mathbf{x})\, u^{(1,0)}(\mathbf{x})\right| d\mathbf{x} \\ &\quad \le \frac{1}{\varepsilon}\iint_\Omega |p_{\mathbf{0}}(\mathbf{x})|\,|u(\mathbf{x})|^2\, d\mathbf{x} + \varepsilon \iint_\Omega |p_{\mathbf{0}}(\mathbf{x})|\,|u^{(1,0)}(\mathbf{x})|^2\, d\mathbf{x} \le \varepsilon \|p_{\mathbf{0}}\|_{C(\Omega)} \iint_\Omega |u^{(1,0)}(\mathbf{x})|^2\, d\mathbf{x} \\ &\quad \le \varepsilon \|p_{\mathbf{0}}\|_{C(\Omega)} \frac{\omega_1^{2m_1}\omega_2^{2m_2}}{\pi^{2\|\mathbf{m}\|}} \iint_\Omega \left|u^{(m_1+1,m_2)}(\mathbf{x})\right|^2 d\mathbf{x} < \iint_\Omega \left|u^{(m_1+1,m_2)}(\mathbf{x})\right|^2 d\mathbf{x} \end{aligned} \tag{3.51}$$

for every positive

$$\varepsilon < \frac{1}{1 + \|p_{\mathbf{0}}\|_{C(\Omega)}} \frac{\pi^{2\|\mathbf{m}\|}}{\omega_1^{2m_1}\,\omega_2^{2m_2}}.$$

(3.50) and (3.51) imply

$$u^{(m_1+1,m_2)}(\mathbf{x}) = 0, \tag{3.52}$$

and (3.52), (3.46) and (3.47) imply $u(\mathbf{x}) \equiv 0$. □

The proof of Corollary 1.7 is similar to the proof of Corollary 1.3.

Corollary 1.8 is a particular case of Corollary 1.6.

The proofs Corollaries 1.9 and 1.10 are similar to the proof of Corollary 1.3 with $c_{jk} = 0$ $(j, k) < \mathbf{m}$.

Proof of Corollary 1.11. It is easy to see that the associated problems of problem (1.48), (1.60)

$$v^{(2m_1+1)} = 0, \quad v^{(j-1)}(0) = a_j\, v^{(j-1)}(\omega_1) \ \ (j = 1, \dots, 2m_1 + 1) \tag{3.53}$$

and

$$v^{(2m_2)} = 0, \quad v^{(k-1)}(0) = b_k\, v^{(k-1)}(\omega_2) \ \ (k = 1, \dots, 2m_2)$$

have only trivial solutions. Condition (A_0) of Theorem 1.4 also holds, since problem (1.4) is identical to problem (3.53).

By Theorem 1.4, it remains to show that the homogeneous problem

$$u^{(\mathbf{2m}+\mathbf{1}_1)} = p_{\mathbf{0}}(\mathbf{x})u, \tag{3.54}$$

$$\begin{gathered} u^{(j-1,0)}(0, x_2) = a_j\, u^{(j-1,0)}(\omega_1, x_2) = 0 \ \ (j = 1, \dots, 2m_1 + 1), \\ u^{(m_1,k-1)}(x_1, 0) = b_k\, u^{(m_1,k-1)}(x_1, \omega_2) = 0 \ \ (k = 1, \dots, 2m_2) \end{gathered} \tag{3.55}$$

has only the trivial solution. Indeed, let u be an arbitrary solution of problem (3.54), (3.55). Multiply equation (3.54) by u and integrate over Ω. After integrating by parts multiple times and taking into account conditions (3.55), we get:

$$\frac{(-1)^{\|\mathbf{m}\|}}{2}\,(1 - a_{m+1}^2) \int\limits_0^{\omega_2} \big|u^{(\mathbf{m})}(\omega_1, x_2)\big|^2\, dx_2 = \iint\limits_{\Omega} p_{\mathbf{0}}(\mathbf{x}) u^2(\mathbf{x})\, d\mathbf{x}.$$

In view of (1.61)–(1.64), there exists $\delta > 0$ sufficiently small such that either

$$u(x_1, x_2) = 0 \ \text{ for } \ x_2 \in [0, \omega_2], \ \ x_1 \in [x_1^* - \delta, x_1^* + \delta] \cap [0, \omega_1],$$

or

$$u_{\mathbf{0}}(x_1, x_2) > 0 \ \text{ for } \ x_1 \in [0, \omega_1], \ \ x_2 \in [x_2^* - \delta, x_2^* + \delta] \cap [0, \omega_2].$$

But then, either

$$u^{(j-1,0)}(x_1^*, x_2) = 0 \ \ (j = 1, \dots, 2m_1 + 1),$$

or

$$u^{(0,k-1)}(x_1, x_2^*) = 0 \ \ (k = 1, \dots, 2m_2).$$

Consequently, u is a solution of equation (3.54) satisfying either the initial-boundary conditions

$$\begin{gathered} u^{(j-1,0)}(x_1^*, x_2) = 0 \ \ (j = 1, \dots, 2m_1 + 1), \\ u^{(m_1,k-1)}(x_1, 0) = b_k\, u^{(m_1,k-1)}(x_1, \omega_2) = 0 \ \ (k = 1, \dots, 2m_2), \end{gathered} \tag{3.56}$$

or the initial boundary conditions

$$\begin{gathered} u^{(j-1,0)}(0, x_2) = a_j\, u^{(j-1,0)}(\omega_1, x_2) = 0 \ \ (j = 1, \dots, 2m_1 + 1), \\ u^{(0,k-1)}(x_1, x_2^*) = 0 \ \ (k = 1, \dots, 2m_2). \end{gathered} \tag{3.57}$$

By Theorem (1.7), both of the initial-boundary value problems (3.54), (3.56) and (3.54), (3.57) have only trivial solutions. □

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