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INTEGRAL COMPARISON CRITERIA FOR NONOSCILLATION OF HALF-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

Dedicated to Professor Kusano Takaŝi
on the occasion of his 90th birthday

Abstract. A new comparison theorem of the Hille–Wintner type is established for a pair of half-linear differential equations of the second order. It is formulated in terms of solutions continuable to infinity of the generalized Riccati equation associated with a known nonoscillatory second-order half-linear equation.

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რეზიუმე. დადგენილია ახალი ჰილ-ვინტნერის ტიპის შედარების თეორემა მეორე რიგის ნახევრადწრფივ დიფერენციალურ განტოლებათა წყვილისთვის. იგი ჩამოყალიბებულია ცნობილ არაოსცილაციურ მეორე რიგის ნახევრადწრფივ განტოლებასთან დაკავშირებულ რიკატის განზოგადებული განტოლების უსასრულობაში გაგრძელებადი ამონახსნების ტერმინებში.

1 Introduction

We consider nonlinear differential equations of the form

$$\big(p(t)\varphi_\alpha(y')\big)' + q(t)\varphi_\alpha(y) = 0, \;\; t \ge a, \tag{E}$$

where $\alpha > 0$ is a given constant, $p : [a, \infty) \to (0, \infty)$ and $q : [a, \infty) \to \mathbb{R}$ are continuous functions and φ_α denotes the signed α-power function defined by

$$\varphi_\alpha(z) := |z|^\alpha \operatorname{sgn} z \;\text{ if }\; z \neq 0 \;\text{ and }\; \varphi_\alpha(0) = 0,$$

which gives equations of the form (E) called "half-linear" because of its homogeneity (but not additivity) property.

Here, by a solution of (E) we understand a real-valued function y which is continuously differentiable on $[t_y, \infty)$ for some $t_y \ge a$ together with $p\varphi_\alpha(y')$ and satisfies (E) on $[t_y, \infty)$. As usual, a solution of (E) which is not identically zero in any neighborhood of infinity is said to be oscillatory if it has arbitrarily large zeros in $[t_y, \infty)$; otherwise it is called nonoscillatory.

In the early stage of the development of the theory of half-linear differential equations, the authors were motivated by the observation that the basic qualitative properties of solutions of equations of the form (E) were very similar to those of the corresponding second order linear equations

$$(p(t)y')' + q(t)y = 0, \tag{L}$$

where $p : [a, \infty) \to (0, \infty)$ and $q : [a, \infty) \to \mathbb{R}$, and focused their efforts on the generalizations and extensions of classical results known for (L) to equations of the form (E).

One of the important milestones in the history of half-linear ODEs of the second order was the extension of Sturmian comparison and separation theorems to (E) (see [4] and [21]). In particular, it has been shown that the zeros of two linearly independent solutions of (E) separate each other, so that solutions of (E) are similarly as in the linear case either all oscillatory or all nonoscillatory. Thus Eq. (E) itself may be called oscillatory (resp. nonoscillatory) if one (and so every) solution of (E) is oscillatory (resp. nonoscillatory).

Another linear result that extends naturally from (L) to (E) is the Hille–Wintner comparison theorem which in its simplest form says that if a continuous function $q_1 : [a, \infty) \to \mathbb{R}$ is such that $\int_a^\infty q_1(t)\,dt < \infty$ and

$$\Big|\int_t^\infty q(s)\,ds\Big| \le \int_t^\infty q_1(s)\,ds \tag{1.1}$$

for all sufficiently large t, then nonoscillation of the equation

$$z'' + q_1(t)z = 0, \;\; t \ge a, \tag{1.2}$$

implies nonoscillation of

$$y'' + q(t)y = 0, \;\; t \ge a, \tag{1.3}$$

or, equivalently, oscillation of (1.3) implies oscillation of (1.2) (see Hille [6] and Wintner [25, 26]).

This result was extended to a pair of half-linear equations (E) and

$$\big(p_1(t)\varphi_\alpha(z')\big)' + q_1(t)\varphi_\alpha(z) = 0, \tag{E$_1$}$$

where $p(t) = p_1(t) \equiv 1$ and both coefficients $q(t)$ and $q_1(t)$ were assumed to be positive on $[a, \infty)$ with the help of a fixed point technique in Kusano et al. [17] and in the more general case, where $p_1(t) \equiv p(t)$ was assumed to satisfy the condition

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt = \infty \tag{1.4}$$

and no sign restriction was imposed on $q(t)$ in [7] (see also Li and Yeh [19]).

Another significant result in this direction obtained by Kusano et al. [15] was the Hille–Wintner type comparison criterion which deduced nonoscillation of Eq. (E) from that of the "integrally majorant" Eq. (E_1) in the case, where both $q(t)$ and $q_1(t)$ were positive, $p_1(t) \equiv p(t)$ and

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty.$$

More precisely, if we denote $\pi(t) := \int_t^\infty p(s)^{-\frac{1}{\alpha}}\,ds$ and assume integrability of $\pi(t)^{\alpha+1}q(t)$ and $\pi(t)^{\alpha+1}q_1(t)$ on $[a,\infty)$, then the comparison inequality which ensures the nonoscillation of (E) in this case is

$$\int_t^\infty \pi(s)^{\alpha+1}q(s)\,ds \le \int_t^\infty \pi(s)^{\alpha+1}q_1(s)\,ds,$$

on some half-line $[t_0,\infty)$, $t_0 \ge a$.

In [19], this result was slightly generalized by removing the restriction $p_1(t) \equiv p(t)$ and replacing it by $p(t) \ge p_1(t)$ for t large enough.

For more recent results concerning the Hille–Wintner comparison theorem see [2,3,8,13,14,20,24,27,28].

Our primary interest in this paper is to consider equations of the form (E) with coefficients $q(t)$ which may change sing in any neighborhhod of infinity and are conditionally integrable on $[a,\infty)$, i.e.,

$$\lim_{T\to\infty}\int_a^T q(t)\,dt \ \text{ exists as a finite number.} \tag{1.5}$$

For such a coefficient we define the function $\rho(t)$ by

$$\rho(t) = \int_t^\infty q(s)\,ds, \quad t \ge a. \tag{1.6}$$

In the case where condition (1.4) hol ds, we also introduce the function

$$P(t) = \int_a^t p(s)^{-\frac{1}{\alpha}}\,ds, \quad t \ge a.$$

Under the conditions (1.4) and (1.5), the above integral test (1.1) is adequate to provide some basic nonoscillation criteria for (E) such as Hille's condition

$$P(t)^\alpha \big|\rho(t)\big| \le \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^\alpha$$

for all sufficiently large t, but it is not powerful enough to give some other known criteria for nonoscillation of (E) such as the Hille–Nehari criterion

$$-\frac{2\alpha+1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^\alpha \le P(t)^\alpha\rho(t) \le \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^\alpha$$

for all t sufficiently large, or the more general Moore type nonoscillation test

$$-k^{\frac{\alpha}{\alpha+1}} - k \le P(t)^\alpha\rho(t) \le k^{\frac{\alpha}{\alpha+1}} - k,$$

where $k > 0$ is a constant, which is again assumed to hold for all t large enough, say, for $t \ge T \ge a$.

Thus, our purpose here is to refine and extend the integral comparison criterion (1.1) so that the refinement would not only cover special cases mentioned above, but also produce new nonoscillation tests for (E).

Our result is formulated in terms of solutions v defined in some neighborhood of infinity of the generalized Riccati equation

$$v' + \alpha p_1(t)^{-\frac{1}{\alpha}}|v|^{1+\frac{1}{\alpha}} + q_1(t) = 0 \tag{R1}$$

associated with the (nonoscillatory) comparison equation (E_1).

2 Main results

The key tool in establishing our extension of the Hille–Wintner theorem is the following auxiliary result which characterizes nonoscillation of (E) through the solvability of certain Riccati-like inequality associated with (E). It was proved in 1980 by Skhalyakho [23] and re-discovered fifteen years later by Li and Yeh [18].

Lemma 2.1. *Eq.* (E) *is nonoscillatory if and only if there exists a function* $u \in C^1([t_1,\infty),\mathbb{R})$ *for some* $t_1 \ge a$ *such that*

$$u'(t) + \alpha p(t)^{-\frac{1}{\alpha}}|u(t)|^{1+\frac{1}{\alpha}} + q(t) \le 0$$

for $t \ge t_1$.

Our main result now follows.

Theorem 2.1. *Let equation* (E_1) *be nonoscillatory. Suppose furthermore that* $p_1(t) \le p(t)$ *on* $[a,\infty)$ *and*

$$\left| v(t) - \int_t^\infty q_1(s)\,ds + \int_t^\infty q(s)\,ds \right| \le |v(t)| \tag{2.1}$$

holds for all sufficiently large t*, where* v *is a solution of the generalized Riccati equation*

$$v' + \alpha p_1(t)^{-\frac{1}{\alpha}}|v|^{1+\frac{1}{\alpha}} + q_1(t) = 0 \tag{R1}$$

defined in some neighborhood of infinity, say, for $t \ge T \ge a$*. Then equation* (E) *is nonoscillatory, too.*

Proof. Define

$$u(t) = v(t) - \int_t^\infty q_1(s)\,ds + \int_t^\infty q(s)\,ds, \quad t \ge T \ge a.$$

Then

$$\begin{aligned}
&u'(t) + \alpha p(t)^{-\frac{1}{\alpha}}|u(t)|^{1+\frac{1}{\alpha}} + q(t) \\
&\quad = v'(t) + q_1(t) - q(t) + \alpha p(t)^{-\frac{1}{\alpha}}\left| v(t) - \int_t^\infty q_1(s)\,ds + \int_t^\infty q(s)\,ds \right|^{1+\frac{1}{\alpha}} + q(t) \\
&\quad = -\alpha p_1(t)^{-\frac{1}{\alpha}}|v(t)|^{1+\frac{1}{\alpha}} + \alpha p(t)^{-\frac{1}{\alpha}}\left| v(t) - \int_t^\infty q_1(s)\,ds + \int_t^\infty q(s)\,ds \right|^{1+\frac{1}{\alpha}} \\
&\quad \le \alpha p_1(t)^{-\frac{1}{\alpha}}\left[\left| v(t) - \int_t^\infty q_1(s)\,ds + \int_t^\infty q(s)\,ds \right|^{1+\frac{1}{\alpha}} - |v(t)|^{1+\frac{1}{\alpha}} \right] \le 0
\end{aligned}$$

for all large t because of (2.1) and the assertion follows from Lemma 2.1. □

As the first application of Theorem 2.1, we obtain the following sufficient condition for nonoscillation of (E). Under the additional restriction $q(t) > 0$ on $[a,\infty)$, this was proved in [7] by applying the Lebesgue monotone convergence theorem to the sequence of functions $\{v_n(t)\}$, $t \ge T$, defined by

$$v_0(t) = P(t)^\alpha \rho(t)$$

and

$$v_n(t) = P(t)^\alpha \rho(t) + \alpha P(t)^\alpha \int_t^\infty p(s)^{-\frac{1}{\alpha}} P(s)^{-\alpha-1} |v_{n-1}(s)|^{1+\frac{1}{\alpha}}\, ds$$

for $n = 1, 2, \dots$ and $t \ge T$.

Corollary 2.1. *Equation* (E) *is nonoscillatory if*

$$P(t)^\alpha |\rho(t)| \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$$

for all sufficiently large t.

Compare with Corollary 2.3 below. Needless to emphasize that the scope of application of Corollary 2.1 is wider than that of Theorem 2.2 in [7] as we allow $q(t)$ to oscillate (with the lower bound for $P(t)^\alpha \rho(t)$ given by $-\alpha^\alpha/(\alpha+1)^{\alpha+1}$), but, on other hand, it could be obtained also from the "classical" half-linear version of the Hille–Wintner criterion without the use of the improved condition (2.1) from Theorem 2.1.

Corollary 2.2. *Let* (1.4) *hold. If*

$$-2 \le P(t)^\alpha \rho(t) \le 0$$

for all large t*, then Eq.* (E) *is nonoscillatory.*

Proof. In Theorem 2.1, it suffices to compare (E) with the equation

$$\big(p(t)\varphi_\alpha(z')\big)' = 0$$

for which the Riccati equation (R1) is a solvable equation

$$v' + \alpha p(t)^{-\frac{1}{\alpha}} |v|^{1+\frac{1}{\alpha}} = 0$$

with the solution

$$v(t) = P(t)^{-\alpha}, \quad t > a.$$

□

The next application in which we compare (E) with the (nonoscillatory) generalized Euler equation

$$\big(p(t)\varphi_\alpha(z')\big)' + \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} p(t)^{-\frac{1}{\alpha}} P(t)^{-\alpha-1} \varphi_\alpha(z) = 0$$

for which the known solution of the corresponding Riccati equation is

$$v(t) = \Big(\frac{\alpha}{\alpha+1}\Big)^\alpha P(t)^{-\alpha}, \quad t \ge T > a,$$

shows that our extension of the Hille–Wintner theorem significantly widens the class of equations (with oscillatory coefficients q) which can be treated by means of comparison with other half-linear equations whose nonoscillatory character is known.

Corollary 2.3 (Hille–Nehari)**.** *If*

$$-\frac{2\alpha+1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^\alpha \le P(t)^\alpha \int_t^\infty q(s)\, ds \le \frac{1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^\alpha$$

for all large t*, then Eq.* (E) *is nonoscillatory.*

In the case of linear Euler equation with the "critical" constant 1/4, i.e., for the equation

$$(p(t)z')' + \frac{1}{4}\,p(t)^{-1}P(t)^{-2}z = 0, \quad t \ge t_0 > 0,$$

as a particular solution of the associated Riccati equation we can take

$$v(t) = \frac{1}{2P(t)} + \frac{1}{p(t)\,\log P(t)}\,, \quad t \ge t_0,$$

and get the following slight improvement of the classical linear Hille–Nehari theorem (see Kamenev [9, 11]).

Corollary 2.4. *Suppose that*

$$\int\limits_a^\infty \frac{1}{p(t)}\,dt = \infty, \qquad \int\limits_a^\infty q(t)\,dt < \infty$$

and the condition

$$-\frac{3}{4} - \frac{2}{\log P(t)} \le P(t)\rho(t) \le \frac{1}{4}$$

holds for all sufficiently large t. *Then the equation*

$$(p(t)y')' + q(t)y = 0 \tag{L}$$

is nonoscillatory.

To obtain our next result, we need the following lemma. For the proof see [17].

Lemma 2.2. *Let* (1.4) *hold. Equation* (E) *is nonoscillatory if and only if there exists a continuous solution of the integral equation*

$$u(t) = \int\limits_t^\infty q(s)\,ds + \alpha \int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}|u(s)|^{1+\frac{1}{\alpha}}\,ds$$

defined in some neighborhood of infinity.

Substituting the integral expression for the solution v of the Riccati equation into (2.1) leads to the following result.

Theorem 2.2. *Suppose that* (1.4) *holds and Eq.* (E_1) *is nonoscillatory. If* q *is integrable on* $[a,\infty)$ *(possibly only conditionally),* $p_1(t) \le p(t)$ *and*

$$\left|\alpha \int\limits_t^\infty p_1(s)^{-\frac{1}{\alpha}}|v(s)|^{1+\frac{1}{\alpha}}\,ds + \int\limits_a^\infty q(s)\,ds\right| \le \left|\alpha \int\limits_t^\infty p_1(s)^{-\frac{1}{\alpha}}|v(s)|^{1+\frac{1}{\alpha}}\,ds + \int\limits_t^\infty q_1(s)\,ds\right| \tag{2.2}$$

for $t \ge T \ge a$, *where* v *is the solution of* (R1) *on* $[T,\infty)$, *then Eq.* (E) *is also nonoscillatory.*

Proof. Since condition (1.4) holds, by Lemma 2.2, we can express the solution v of (R1) as

$$v(t) = \alpha \int\limits_t^\infty p_1(s)^{-\frac{1}{\alpha}}|v(s)|^{1+\frac{1}{\alpha}}\,ds + \int\limits_t^\infty q_1(s)\,ds, \quad t \ge T \ge a. \tag{2.3}$$

Inserting (2.3) into the left-hand side of (2.1) and using (2.2), we conclude that the conditions of Theorem 2.1 are satisfied and so, equation (E) is nonoscillatory, as claimed. □

Theorem 2.3. *Suppose that q is (conditionally) integrable on $[a,\infty)$ and there exists a function $f \in C^1([a,\infty), R)$ such that*

$$\lim_{t\to\infty} f(t) = 0, \quad \int_a^\infty p(t)^{-\frac{1}{\alpha}} |f(t)|^{1+\frac{1}{\alpha}}\, dt < \infty$$

and

$$\left| \alpha \int_t^\infty p(s)^{-\frac{1}{\alpha}} |f(s)|^{1+\frac{1}{\alpha}}\, ds + \int_t^\infty q(s)\, ds \right| \le f(t)$$

for all large t. Then Eq. (E) *is nonoscillatory.*

Proof. In Theorem 2.2, we use

$$\big(p(t)\varphi_\alpha(z')\big)' + \Big(-\alpha p(t)^{-\frac{1}{\alpha}} |f(t)|^{1+\frac{1}{\alpha}} - f'(t)\Big)\varphi_\alpha(z) = 0 \tag{2.4}$$

as the comparison equation and observe that $v = f$ is a particular solution of the Riccati equation associated with (2.4). □

Remark 2.1. In Theorem 2.2, we do not assume *a priori* that the coefficient p satisfies condition (1.4). But if we do so, then the existence of a function f with the properties stated in the theorem is also a necessary condition for the nonoscillation of (E).

Corollary 2.5. *Suppose that* (1.5) *holds and define $\rho(t)$ by* (1.6)*. If*

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}} |\rho(t)|^{\frac{\alpha+1}{\alpha}}\, dt < \infty$$

and for all sufficiently large t the inequality

$$\int_t^\infty p(s)^{-\frac{1}{\alpha}} |\rho(s)|^{1+\frac{1}{\alpha}}\, ds \le \frac{(\alpha+1)^{-\frac{\alpha+1}{\alpha}}}{\alpha} \big[(\alpha+1)|\rho(t)| - \rho(t)\big] \tag{2.5}$$

holds, then Eq. (E) *is nonoscillatory.*

Proof. Follows from Theorem 2.3, where $f(t) = (\alpha+1)|\rho(t)|$. □

Remark 2.2. Compare with Corollary 8 in Yang and Lo [28].

By using an iterative technique, we can further improve the sufficient condition (2.5) for nonoscillation of (E) as follows. Let $\rho(t) := \int_t^\infty q(s)\, ds$ and define the sequence $\{\omega_k(t)\}$ by

$$\omega_0(t) = q(t), \quad \omega_k(t) = \left| \rho(t) + \alpha \int_t^\infty p(s)^{-\frac{1}{\alpha}} \omega_{k-1}(s)\, ds \right|^{1+\frac{1}{\alpha}},$$

$k = 1, 2, \dots$ and $t \ge T$.

Corollary 2.6. *Suppose that there exists a nonnegative integer n such that the functions $\omega_0(t), \dots, \omega_n(t)$ are well defined, the integral $\int_a^\infty p(t)^{-\frac{1}{\alpha}} \omega_n(t)\, dt$ converges and the inequality*

$$\left| \rho(t) + \alpha \int_t^\infty p(s)^{-\frac{1}{\alpha}} \omega_n(s)\, ds \right| \le |\omega_n(t)|^{\frac{\alpha}{\alpha+1}} \tag{2.6}$$

holds for all sufficiently large t. Then Eq. (E) *is nonoscillatory.*

Proof. Follows from Theorem 2.3, where we choose $f(t) = |\omega_n(t)|^{\frac{\alpha}{\alpha+1}}$. □

Remark 2.3. If $n = 0$ and $p(t)^{-\frac{1}{\alpha}} q(t)$ is integrable on $[a, \infty)]$, then (2.6) reduces to

$$\left| \rho(t) + \alpha \int_t^\infty p(s)^{-\frac{1}{\alpha}} q(s)\, ds \right| \le |q(t)|^{\frac{\alpha}{\alpha+1}},$$

which is a half-linear generalization of the linear Wintner's criterion (with $\alpha = 1$, $p(t) \equiv 1$ and nonnegative $q(t)$)

$$[\rho(t)]^2 \le \frac{1}{4} q(t)$$

(see [25]).

If $n = 1$ and $p(t) \equiv 1$, then (2.6) becomes the Opial type nonoscillation criterion (2.5).

3 Further generalizations and extensions

An inspection of the proof of Theorem 2.1 indicates that instead of a (known) solution v of the Riccati equation (R1) associated with the nonoscillatory majorant equation (E_1) we can take any continuously differentiable function g from $[a, \infty)$ to $\mathbb{R}$ such that

$$g'(t) \le -q_1(t) \tag{3.1}$$

for all sufficiently large t. We reflect this fact in the following theorem.

Theorem 3.1. *Let condition (1.4) be satisfied. Assume that equation* (E_1) *with* $p_1(t) \le p(t)$ *on* $[a, \infty)$ *is nonoscillatory and there exists a function* $g \in C^1([a, \infty), \mathbb{R})$ *such that (3.1) is satisfied for* $t \ge T \ge a$ *and*

$$\left| g(t) - \int_t^\infty q_1(s)\, ds + \int_t^\infty q(s)\, ds \right| \le (\alpha^{-\alpha} p(t))^{\frac{1}{\alpha+1}} \big[-g'(t) - q_1(t) \big]^{\frac{\alpha}{\alpha+1}} \tag{3.2}$$

or, equivalently,

$$- (\alpha^{-\alpha} p(t))^{\frac{1}{\alpha+1}} \big[-g'(t) - q_1(t) \big]^{\frac{\alpha}{\alpha+1}} - g(t) + \int_t^\infty q_1(s)\, ds \\ \le \int_t^\infty q(s)\, ds \le (\alpha^{-\alpha} p(t))^{\frac{1}{\alpha+1}} \big[-g'(t) - q_1(t) \big]^{\frac{\alpha}{\alpha+1}} - g(t) + \int_t^\infty q_1(s)\, ds$$

on $[T, \infty)$. *Then equation* (E) *is nonoscillatory, too.*

Proof. As in the proof of Theorem 2.1, we define

$$u(t) = g(t) - \int_t^\infty q_1(s)\, ds + \int_t^\infty q(s)\, ds, \quad t \ge T \ge a,$$

and verify that

$$u'(t) + \alpha p(t)^{-\frac{1}{\alpha}} |u(t)|^{1+\frac{1}{\alpha}} + q(t) = g'(t) + q_1(t) + \alpha p(t)^{-\frac{1}{\alpha}} \left| g(t) - \int_t^\infty q_1(s)\, ds + \int_t^\infty q(s)\, ds \right|^{1+\frac{1}{\alpha}},$$

which is nonnegative on $[T, \infty)$ because of (3.2). Thus, by Lemma 2.1, equation (E) is nonoscillatory and the proof is complete. □

Corollary 3.1. *If there exists a continuously differentiable function $g : [a,\infty) \to R$ such that $g'(t) \le 0$ on $[T,\infty)$ for some $T \ge a$ and*

$$-\Big(\frac{1}{\alpha}\,|g'(t)|\Big)^{\frac{\alpha}{\alpha+1}} + p(t)^{-\frac{1}{\alpha+1}} g(t) \le p(t)^{-\frac{1}{\alpha+1}} \int_t^\infty q(s)\,ds \le \Big(\frac{1}{\alpha}\,|g'(t)|\Big)^{\frac{\alpha}{\alpha+1}} + p(t)^{-\frac{1}{\alpha+1}} g(t)$$

holds for all $t \ge T$, then Eq. (E) *is nonoscillatory.*

The special choice

$$g(t) = k\Big(\int_a^t p(s)^{-\frac{1}{\alpha}}\,ds\Big)^{-\alpha}$$

in Corollary 3.1, where $k > 0$ is constant, leads to the Moore type criterion

$$-k^{\frac{\alpha}{\alpha+1}} - k \le \Big(\int_a^t p(s)^{-\frac{1}{\alpha}}\,ds\Big)^{\alpha} \int_t^\infty q(s)\,ds \le k^{\frac{\alpha}{\alpha+1}} - k \ \text{ for } \ t \ge T \tag{3.3}$$

mentioned in the introduction (cf. with (1.19)). (See also [20].)

If in the above result we take $k = (\alpha/(\alpha+1))^{\alpha+1}$, then we again obtain the half-linear generalization of the classical linear Hille–Nehari nonoscillation criterion $-3/4 \le t \int_t^\infty q(s)\,ds \le 1/4$, namely,

$$-\frac{2\alpha+1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha} \le \Big(\int_a^t p(s)^{-\frac{1}{\alpha}}\,ds\Big)^{\alpha} \int_t^\infty q(s)\,ds \le \frac{1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}$$

for sufficiently large t, given in [1] (see also [23]).

Remark 3.1. The left-hand side of (3.3) as the function of k tends to $-\infty$ as $t \to \infty$, and so the lower bound for $P(t)^\alpha \rho(t)$ is allowed to be arbitrarily small negative number. However, the right-hand side reaches its maximum $\alpha^\alpha/(\alpha+1)^{\alpha+1}$ for $k = (\alpha/(\alpha+1))^{\alpha+1}$ and becomes negative for $k > 1$. Because of this "interplay" between the LHS and RHS of (3.3), the reasonable choice of $k > 0$ in the Moore type criterion (3.3) for an equation with sign-changing $q(t)$ is a delicate question.

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