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ASYMPTOTIC ANALYSIS OF SOLUTIONS OF SECOND ORDER QUASILINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE EXPONENTS OF NONLINEARITY

Dedicated to Professor Kusano Takaŝi
on the occasion of his 90-th birthday

Abstract. This paper deals with the nonoscillatory (eventually positive) solutions of second order quasilinear differential equations with variable exponents of nonlinearity. We classify the set of all positive solutions into three disjoint classes according to their asymptotic behavior at infinity, and obtain criteria for the existence of solutions belonging to each of those classes.

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1 Introduction

We are concerned with the nonoscillatory behavior of a quasilinear differential equation of the type

$$(p(t)\varphi_{\alpha(t)}(x'))' + q(t)\varphi_{\beta(t)}(x) = 0, \quad t \geqq a, \tag{A}$$

where $\alpha(t)$, $\beta(t)$, $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$, $a \geqq 1$, and the use is made of the notation

$$\varphi_{\gamma(t)}(\xi) = |\xi|^{\gamma(t)-1}\xi = |\xi|^{\gamma(t)} \operatorname{sgn}\xi, \quad \xi \in \mathbb{R}, \quad \gamma \in C[1,\infty).$$

By a solution of (A) we mean a function $x \in C^1[T_x,\infty)$, $T_x \geqq a$, which has the property $p(t)\varphi_{\alpha(t)}(x') \in C^1[T_x,\infty)$ and satisfies the equation at all points $t \geqq T_x$. A nontrivial solution $x(t)$ of (A) is said to be nonoscillatory if $x(t) \neq 0$ for all large t, and oscillatory otherwise. In this paper, we restrict our attention to its eventually positive solutions.

Since the appearance of the pioneering works of Elbert [14] and Mirzov [39], the qualitative behavior of differential equation of the form

$$\big(p(t)\varphi_{\alpha}(x')\big)' + q(t)\varphi_{\beta}(x) = 0, \quad t \geqq a \quad (\alpha > 0, \quad \beta > 0, \quad \alpha = \beta \ \text{ or } \ \alpha \neq \beta) \tag{1.1}$$

has been intensively studied by numerous authors including Agarwal, Došlá, Došlý, Elbert, Kusano, Jaroš, Manojlović, Marić, Mirzov, Naito, Řehák, Tanigawa, Usami and Yoshida with, for instance, their monographs [1, 13, 40] and papers [5–7, 12, 19–25, 27, 28, 30–38, 41]. Consequently, a qualitative theory for (1.1) has been established at present. Moreover, in the special case $\alpha(t) = 1$, $\beta(t) = \beta > 0$, (A) reduces to the Emden–Fowler type equation describing gas dynamics in astrophysics

$$(p(t)x')' + q(t)\varphi_{\beta}(x) = 0, \quad t \geqq a,$$

whose qualitative behavior has been studied in great detail by many authors; see, for example, the monographs [26, 40] and the papers [8, 11, 29].

In recent years, there has been well analyzed the oscillatory and nonoscillatory behavior of the equation with $p(t)$-Laplacian

$$\big(a(t)\varphi_{p(t)-1}(x')\big)' \pm b(t)\varphi_{q(t)-1}(x) = 0, \quad t \geqq a \quad (p(t) = q(t) \ \text{ or } \ p(t) \neq q(t)),$$

which is of the same type as (A), but written in a different representation of $p(t) = a(t)$, $q(t) = b(t)$, $\alpha(t) = p(t) - 1$ ($p(t) > 1$) and $\beta(t) = q(t) - 1$ ($q(t) > 1$) in equation (A) (see [2, 9, 10, 15–17]). We note that $p(t)$-Laplacian is the one-dimensional polar form of the $p(x)$-Laplace operator (the so-called $p(x)$-Laplacian) $\Delta_{p(x)}u = \operatorname{div}(\|\nabla u\|^{p(x)-2}\nabla u)$, where ∇ is the Hamilton nabla operator and div is the usual divergence operator. It is known that the study of differential equations involving $p(t)$-Laplacian or $p(x)$-Laplacian occurs from some mathematical models of nonlinear elasticity theory, image processing and electrorheological fluids (see [3, 4, 18, 42–49]).

To the best of the authors' knowledge, details are unknown about nonoscillatory behavior of (A), and so, in this paper, we make an attempt to investigate in detail the existence and asymptotic behavior of eventually positive solutions of (A).

This paper is divided into three sections. In Section 2, we classify the set of positive solutions $x(t)$ of (A) according to their asymptotic behavior as $t \to \infty$, showing that $x(t)$ falls into one of the three types listed below:

I. $\quad \lim_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = \mathit{const} > 0, \qquad \lim_{t\to\infty} x(t) = \infty;$

II. $\quad \lim_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = 0, \qquad \lim_{t\to\infty} x(t) = \infty;$

III. $\quad \lim_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = 0, \qquad \lim_{t\to\infty} x(t) = \mathit{const} > 0.$

As for the solutions of two types I and III, we can establish the necessary and sufficient conditions for their existence on the basis of the integral equations characterizing these types of solutions obtained. On the other hand, regarding the solution of type II, it is impossible to characterize the existence of such a solution, and so we have to content with presenting sufficient conditions for (A) to have solutions of type II. Finally, some examples illustrating our main results for (A) are be given in Section 3.

2 Existence of positive solutions

In this paper, we make the following assumptions without further mentioning:

$$\int_a^\infty \Big[\frac{k}{p(t)}\Big]^{\frac{1}{\alpha(t)}} dt = \infty \tag{2.1}$$

for every constant $k > 0$, and employ the notation

$$P_{\alpha(t),k}(t) = \int_T^t \Big[\frac{k}{p(s)}\Big]^{\frac{1}{\alpha(s)}} ds, \quad t \geqq T \geqq a. \tag{2.2}$$

From (2.1) and (2.2) it is obvious that

$$P_{\alpha(T),k}(T) = 0, \quad \lim_{t\to\infty} P_{\alpha(t),k}(t) = \infty \ \text{ for every } \ k > 0,$$
$$P_{\alpha(t),k}(t) > P_{\alpha(t),l}(t), \ \ t > T \ \text{ for } \ k > l > 0 \ \text{ and } \ \lim_{k\to 0} P_{\alpha(t),k}(t) = 0 \ \text{ for each } \ t \geqq T.$$

As mentioned in the previous section, we begin by classifying all possible positive solutions of equation (A) according to their asymptotic behavior as $t \to \infty$.

Lemma 2.1. *One and only one of the following cases occurs for each positive solution $x(t)$ of* (A)*:*

I. $\lim_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = const > 0$, $\lim_{t\to\infty} x(t) = \infty$;

II. $\lim_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = 0$, $\lim_{t\to\infty} x(t) = \infty$;

III. $\lim_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = 0$, $\lim_{t\to\infty} x(t) = const > 0$.

Proof. Let $x(t)$ be a positive solution of (A), so that $x(t) > 0$, $t \geqq T$. Since

$$\big(p(t)\varphi_{\alpha(t)}(x'(t))\big)' = -q(t)\varphi_{\beta(t)}(x(t)) < 0, \quad t \geqq T,$$

the function $p(t)\varphi_{\alpha(t)}(x'(t))$ is a monotone decreasing for $t \geqq T$. We claim that $p(t)\varphi_{\alpha(t)}(x'(t)) > 0$, $t \geqq T$, so that the finite limit $\lim_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) \geqq 0$ exists. In fact, if $p(T_1)\varphi_{\alpha(T_1)}(x'(T_1)) = -k < 0$ for some $T_1 \geqq T$ and $k > 0$, then

$$p(t)\varphi_{\alpha(t)}(x'(t)) \leqq -k \ \text{ for } \ t \geqq T_1, \ \text{ or } \ x'(t) \leqq -\Big[\frac{k}{p(t)}\Big]^{\frac{1}{\alpha(t)}}, \ \ t \geqq T_1.$$

Integrating the last inequality from T_1 to t and letting $t \to \infty$, we see that in view of (2.1), $x(t) \to -\infty$ as $t \to \infty$, which contradicts the assumed positivity of $x(t)$. Therefore, $p(t)\varphi_{\alpha(t)}(x'(t)) > 0$ for $t \geqq T$ as claimed. A consequence of this observation is that $x'(t) > 0$ for $t \geqq T$, i.e., the function $x(t)$ is strictly increasing.

The limit of $p(t)\varphi_{\alpha(t)}(x'(t))$ as $t \to \infty$ is either positive or zero. In the first case, $x(t)$ is unbounded, since there are positive constants k_1, k_2 $(k_1 < k_2)$ and T such that

$$x(T) + P_{\alpha(t),k_1}(t) \leqq x(t) \leqq x(T) + P_{\alpha(t),k_2}(t), \quad t \geqq T.$$

In the second case, since $x(t)$ is monotone increasing, $x(t)$ tends to a positive limit, finite or infinite, as $t \to \infty$. This completes the proof. $\square$

We want to obtain criteria for the existence of positive solutions of (A) of types I, II and III.

Theorem 2.1. *Suppose that for each fixed $k > 0$ and $T \geqq a$,*

$$\lim_{l\to 0} \frac{P_{\alpha(t),l}(t)}{P_{\alpha(t),k}(t)} = 0 \tag{2.3}$$

uniformly on any interval of the form $[T_1, \infty)$, $T_1 > T$. Then equation (A) *possesses a positive solution of type* I *if and only if*

$$\int_a^\infty q(t)(P_{\alpha(t),k}(t))^{\beta(t)}\,dt < \infty \ \text{ for some constant } \ k > 0. \tag{2.4}$$

Theorem 2.2. *Equation* (A) *possesses a positive solution of type* III *if and only if*

$$\int_a^\infty \left[\frac{1}{p(t)}\int_t^\infty q(s)c^{\beta(s)}\,ds\right]^{\frac{1}{\alpha(t)}} dt < \infty \ \text{ for some constant } \ c > 0. \tag{2.5}$$

Proof of Theorem 2.1. (*The "only if" part*) Suppose that (A) possesses a positive solution of type I of (A), i.e., $x(t) > 0$, $t \geqq T(\geqq a)$. There exist positive constants k_1 and T_1 ($\geqq T$) such that $P_{\alpha(t),k_1}(t) \leqq x(t)$ for $t \geqq T_1$. An integration of (A) over $[t, \infty)$ yields

$$\int_t^\infty q(s)\varphi_{\beta(s)}(x(s))\,ds < \infty,$$

which combined with the above inequality leads to

$$\int_T^\infty q(t)(P_{\alpha(t),k_1}(t))^{\beta(t)}\,dt < \infty.$$

(*The "if" part*) Suppose (2.4) holds for some $k > 0$. Because of (2.3), we can choose $l > 0$ and $T > a$ such that $l < k/2$ and

$$\int_T^\infty q(t)(P_{\alpha(t),2l}(t))^{\beta(t)}\,dt \leqq l. \tag{2.6}$$

We now define the set X to be the set of continuous functions $x(t)$ on $[T, \infty)$ satisfying

$$X = \Big\{x \in C[T,\infty) : \ \ P_{\alpha(t),l}(t) \leqq x(t) \leqq P_{\alpha(t),2l}(t), \ \ t \geqq T\Big\}.$$

It is clear that X is a closed convex subset of the locally convex space $C[T, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T, \infty)$. Define the integral operator

$$\mathcal{F}x(t) = \int_T^t \left[\frac{1}{p(s)}\left\{l + \int_s^\infty q(r)\varphi_{\beta(r)}(x(r))\,dr\right\}\right]^{\frac{1}{\alpha(s)}} ds, \ \ t \geqq T,$$

and let it act on the set X defined above. It can be shown that $\mathcal{F}$ is a self-map on X and sends X continuously on a relatively compact subset of $C[T, \infty)$.

(i) $\mathcal{F}$ maps x into itself. If $x \in X$, then since

$$\mathcal{F}x(t) \geqq \int_T^t \left[\frac{l}{p(s)}\right]^{\frac{1}{\alpha(s)}} ds = P_{\alpha(t),l}(t), \ \ t \geqq T,$$

and by means of (2.6),

$$\mathcal{F}x(t) \leqq \int\limits_T^t \left[\frac{1}{p(s)}\left\{l + \int\limits_s^\infty q(r)(P_{\alpha(r),2l}(r))^{\beta(r)}\,dr\right\}\right]^{\frac{1}{\alpha(s)}} dt$$
$$\leqq \int\limits_T^t \left[\frac{2l}{p(s)}\right]^{\frac{1}{\alpha(s)}} ds = P_{\alpha(t),2l}(t), \ \ t \geqq T,$$

we get $\mathcal{F}x \in X$.

(ii) $\mathcal{F}$ is a continuous map. Let $\{x_n\}$ be a sequence of elements of X converging to $x \in X$ as $n \to \infty$ in the topology of $C[T,\infty)$. We have

$$\big|\mathcal{F}x_n(t) - \mathcal{F}x(t)\big| \leqq \int\limits_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\alpha(s)}} F_n(s)\,ds, \ \ t \geqq T,$$

where

$$F_n(t) = \left|\left(l + \int\limits_t^\infty q(s)(x_n(s))^{\beta(s)}\,ds\right)^{\frac{1}{\alpha(t)}} - \left(l + \int\limits_t^\infty q(s)(x(s))^{\beta(s)}\,ds\right)^{\frac{1}{\alpha(t)}}\right|.$$

By means of the inequality $|u^\sigma - v^\sigma| \leqq \sigma \max\{u^{\sigma-1}, v^{\sigma-1}\}|u-v|$, $u, v \in \mathbb{R}^+$, holding for $\sigma \in (0,\infty)$ which is shown by the mean value theorem, and the triangle inequality for integrals, we see that if $\alpha(t) \geqq 1$, then

$$F_n(t) \leqq \frac{1}{\alpha(t)}\left(\int\limits_t^\infty q(s)(P_{\alpha(s),l}(s))^{\beta(s)}\right)^{\frac{1}{\alpha(t)}-1} \int\limits_t^\infty q(s)\big|(x_n(s))^{\beta(s)} - (x(s))^{\beta(s)}\big|\,ds,$$

and if $\alpha(t) < 1$, then

$$F_n(t) \leqq \frac{1}{\alpha(t)}\left(\int\limits_t^\infty q(s)(P_{\alpha(s),2l}(s))^{\beta(s)}\,ds\right)^{\frac{1}{\alpha(t)}-1} \int\limits_t^\infty q(s)\big|(x_n(s))^{\beta(s)} - (x(s))^{\beta(s)}\big|\,ds$$

for any fixed $t \geqq T$. Thus, using $q(t)|(x_n(s))^{\beta(t)} - (x(t))^{\beta(t)}| \to 0$ as $n \to \infty$ at each point $t \in [T,\infty)$ and $q(t)|(x_n(t))^{\beta(t)} - (x(t))^{\beta(t)}| \leqq 2q(t)(P_{\alpha(t),2l}(t))^{\beta(t)}$ for $t \geqq T$, while $q(t)(P_{\alpha(t),2l}(t))^{\beta(t)}$ is integrable on $[T,\infty)$, the uniform convergence $F_n(t) \to 0$ on $[T,\infty)$ follows by the application of the Lebesgue dominated convergence theorem. We conclude that $\mathcal{F}x_n(t) \to \mathcal{F}x(t)$ uniformly on any compact subinterval of $[T,\infty)$ as $n \to \infty$, which proves the continuity of $\mathcal{F}$.

(iii) $\mathcal{F}(X)$ is relatively compact. Since $\mathcal{F}(X) \subset X$, it is clear that $\mathcal{F}(X)$ is locally uniformly bounded on $[T,\infty)$. From the inequality

$$|(\mathcal{F}x)'(t)| = \left|\left[\frac{1}{p(t)}\left\{l + \int\limits_t^\infty q(s)(x(s))^{\beta(s)}\,ds\right\}\right]^{\frac{1}{\alpha(t)}}\right| \leqq \left(\frac{2l}{p(t)}\right)^{\frac{1}{\alpha(t)}}, \ \ t \geqq T,$$

holding for all $x \in X$, it follows that $\mathcal{F}(X)$ is locally equi-continuous on $[T,\infty)$. Then the relative compactness of $\mathcal{F}(X)$ follows from the Arzelà–Ascoli lemma. Thus all the hypotheses of the Schauder–Tychonoff fixed point theorem are fufilled, and so $\mathcal{F}$ has a fixed point $x \in X$, which means that x satisfies the integral equation

$$x(t) = \int\limits_T^t \left[\frac{1}{p(t)}\left\{l + \int\limits_s^\infty q(r)\varphi_{\beta(r)}(x(r))\,dr\right\}\right]^{\frac{1}{\alpha(s)}} ds, \ \ t \geqq T.$$

It follows that x is a positive solution of equation (A) on $[T,\infty)$ by differentiation of the above integral equation $x(t) = \mathcal{F}x(t)$, $t \geqq T$. It is obvious that $x(t)$ is of type I. This completes the proof. $\square$

Proof of Theorem 2.2. (*The "only if" part*) Suppose that (A) possesses a positive solution $x(t)$ of type III of (A). There are positive constants c_1 and $T(\geqq a)$ such that $x(t) \geqq c_1$ for $t \geqq T$. Integrating (A) from t to ∞, we have

$$p(t)\varphi_{\alpha(t)}(x'(t)) = \int_t^\infty q(s)\varphi_{\beta(s)}(x(s))\,ds, \quad t \geqq T,$$

which implies

$$x'(t) = \left[\frac{1}{p(t)}\int_t^\infty q(s)(x(s))^{\beta(s)}\,ds\right]^{\frac{1}{\alpha(t)}}, \quad t \geqq T.$$

Integrating the above over $[T,\infty)$ and using $x(t) \geqq c_1$, we conclude that

$$\int_a^\infty \left[\frac{1}{p(t)}\int_t^\infty q(s)c_1^{\beta(s)}\,ds\right]^{\frac{1}{\alpha(t)}} dt < \infty.$$

(*The "if" part*) Suppose that (2.5) holds for $c > 0$. Let $T \geqq a$ be such that

$$\int_T^\infty \left[\frac{1}{p(t)}\int_t^\infty q(s)c^{\beta(s)}\,ds\right]^{\frac{1}{\alpha(t)}} dt \leqq \frac{c}{2} \tag{2.7}$$

and define

$$X = \left\{x \in C[T,\infty) : \ \frac{c}{2} \leqq x(t) \leqq c, \ \ t \geqq T\right\} \tag{2.8}$$

and

$$\mathcal{F}x(t) = c - \int_t^\infty \left[\frac{1}{p(t)}\int_s^\infty q(r)\varphi_{\beta(r)}(x(r))\,dr\right]^{\frac{1}{\alpha(s)}} ds, \quad t \geqq T. \tag{2.9}$$

If $x \in X$, then from (2.7), (2.8) and (2.9) it follows that

$$\frac{c}{2} \leqq \mathcal{F}x(t) \leqq c, \quad t \geqq T.$$

This shows that $\mathcal{F}$ maps X into itself. It can be shown in a routine manner that $\mathcal{F}$ is continuous and $\mathcal{F}(X)$ is relatively compact in the topology of $C[T,\infty)$. Therefore, by the Schauder–Tychonoff fixed point theorem, there exists an element $x \in X$ such that $x(t) = \mathcal{F}x(t)$, $t \geqq T$, which gives the desired solution of type III of equation (A). This completes the proof. □

Unlike the solution of types I and III, it is difficult to characterize the type II solution of (A), and so we content ourselves with sufficient conditions for the existence of such solutions of (A).

Theorem 2.3. *Suppose that* (2.3) *holds. Equation* (A) *possesses a positive solution of type* II *if* (2.4) *holds for some constant* $k > 0$ *and*

$$\int_a^\infty \left[\frac{1}{p(t)}\int_t^\infty q(s)d^{\beta(s)}\,ds\right]^{\frac{1}{\alpha(t)}} dt = \infty \tag{2.10}$$

for every constant $d > 0$.

Proof. Let $a > 0$ be an arbitrary fixed constant, and choose $l > 0$ small enough and $T > 0$ large enough so that $a + P_{\alpha(t),l}(t) \leqq P_{\alpha(t),k}(t)$ for $t \geqq T$ and

$$\int_T^\infty q(t)(a + P_{\alpha(t),l}(t))^{\beta(t)}\,dt \leqq l.$$

As in the proof of Theorem 2.1, this is possible because of (2.3) and the fact that $\lim_{t\to\infty} P_{\alpha(t),k}(t) = \infty$. Then, applying the Schauder–Tychonoff fixed point theorem, we can show that the mapping $\mathcal{F}$ defined by

$$\mathcal{F}x(t) = a + \int_T^t \left[\frac{1}{p(s)} \int_s^\infty q(r)\varphi_{\beta(r)}(x(r))\,dr\right]^{\frac{1}{\alpha(s)}} ds, \quad t \geqq T,$$

possesses a fixed element x in the set X given by

$$X = \Big\{x \in C[T,\infty): \ \ a \leqq x(t) \leqq a + P_{\alpha(t),l}(t), \ \ t \geqq T\Big\}.$$

From the integral equation for x, i.e.,

$$x(t) = a + \int_T^t \left[\frac{1}{p(s)} \int_s^\infty q(r)\varphi_{\beta(r)}(x(r))\,dr\right]^{\frac{1}{\alpha(s)}} ds, \quad t \geqq T,$$

it follows that

$$p(t)\varphi_{\alpha(t)}(x'(t)) = \int_t^\infty q(s)\varphi_{\beta(s)}(x(s))\,ds \to 0 \ \text{ as } \ t\to\infty$$

and by (2.10),

$$x(t) \geqq a + \int_T^t \left[\frac{1}{p(s)} \int_s^\infty q(r)a^{\beta(r)}\,dr\right]^{\frac{1}{\alpha(s)}} ds \to \infty \ \text{ as } \ t\to\infty.$$

Thus $x(t)$ is a positive type II solution of (A). This completes the proof. □

3 Examples

We now present some examples illustrating our main results and it is posssible to depict the complete clear picture of the overall structure of the set of positive solutions of equation (A) under assumption (2.1).

Example 3.1. Consider the equations with variable exponents of nonlinearity

$$\big(e^{-(t^2-1)}\varphi_t(x')\big)' + q_1(t)\varphi_t(x) = 0, \ \ t \geqq e, \tag{E$_1$}$$

$$\big(e^{-(t^2-1)}\varphi_t(x')\big)' + q_2(t)\varphi_{\frac{1}{t}}(x) = 0, \ \ t \geqq e, \tag{E$_2$}$$

and

$$\big(e^{-(1-\frac{1}{t^2})}\varphi_{\frac{1}{t}}(x')\big)' + q_3(t)\varphi_{\frac{1}{t}}(x) = 0, \ \ t \geqq e, \tag{E$_3$}$$

where the functions $q_i(t)$, $i = 1,2,3$, are

$$q_1(t) = e^{-(t^2-1)}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\},$$

$$q_2(t) = e^{-(1-\frac{1}{t^2})}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}$$

and

$$q_3(t) = e^{-(1-\frac{1}{t^2})}\frac{1}{t^2}\Big(1+\frac{1}{t^2}\Big)^{\frac{1}{t}}\Big\{\frac{2}{t^2+1} + \log\Big(1+\frac{1}{t^2}\Big)\Big\},$$

respectively. They are special cases of (A) with $\alpha(t) = t$ in (E$_i$), $i = 1,2$, $\alpha(t) = 1/t$ in (E$_3$), $\beta(t) = t$ in (E$_1$), $\beta(t) = 1/t$ in (E$_i$), $i = 2,3$, $p(t) = e^{-(t^2-1)}$ in (E$_i$), $i = 1,2$, $p(t) = e^{-(1-\frac{1}{t^2})}$ in (E$_3$) and

$q(t) = q_i(t)$, $i = 1, 2, 3$, in the above. The functions $p(t) = e^{-(t^2-1)}$ and $p(t) = e^{-(1-\frac{1}{t^2})}$ satisfy (2.1) with $k = 1$ and, in addition, the function $P_{\alpha(t),1}(t)$ associated with (E_i), $i = 1, 2, 3$, is

$$P_{\alpha(t),1}(t) = \int_e^t \Big[\frac{1}{p(s)}\Big]^{\frac{1}{\alpha(s)}} ds = \int_e^t e^{s-\frac{1}{s}}\, ds \ \sim\ e^{t-\frac{1}{t}} \ \text{ as } \ t \to \infty$$

by (2.2), where the symbol $\sim$ is used to denote the asymptotic equivalence

$$f(t) \sim g(t) \ \text{ as } \ t \to \infty \iff \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Since

$$\begin{aligned}\int_e^\infty q_1(t)(P_{t,1}(t))^t\, dt &= \int_e^\infty e^{-(t^2-1)}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}(e^{t-\frac{1}{t}})^t\, dt\\ &= \int_e^\infty \Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty,\\ \int_e^\infty q_2(t)(P_{t,1}(t))^{\frac{1}{t}}\, dt &= \int_e^\infty \Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty\end{aligned}$$

and

$$\int_e^\infty q_3(t)(P_{\frac{1}{t},1}(t))^{\frac{1}{t}}\, dt = \int_e^\infty \frac{1}{t^2}\Big(1+\frac{1}{t^2}\Big)^{\frac{1}{t}}\Big\{\frac{2}{t^2+1} + \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty,$$

we can apply Theorem 2.1 to conclude that there exists a positive solution of type I such that $x(t) = e^{t-\frac{1}{t}}$, which satisfies

$$\lim_{t\to\infty} p(t)\varphi_t(x'(t)) = \lim_{t\to\infty}\Big(1+\frac{1}{t^2}\Big)^t = 1, \ \ \lim_{t\to\infty} x(t) = \infty$$

for (E_i), $i = 1, 2$, and that for (E_3),

$$\lim_{t\to\infty} p(t)\varphi_{\frac{1}{t}}(x'(t)) = \lim_{t\to\infty}\Big(1+\frac{1}{t^2}\Big)^{\frac{1}{t}} = 1, \ \ \lim_{t\to\infty} x(t) = \infty.$$

Example 3.2. Consider the equations

$$\big(e^{1-t}\varphi_t(x')\big)' + q_4(t)\varphi_{\frac{1}{t}}(x) = 0, \ \ t \geqq e, \tag{E_4}$$

and

$$\big(e^{-(1-\frac{1}{t})^2}\varphi_{1-\frac{1}{t}}(x')\big)' + q_5(t)\varphi_{\frac{1}{t}}(x) = 0, \ \ t \geqq e, \tag{E_5}$$

which are special cases of (A) with $p(t) = e^{1-t}$ in (E_4), $p(t) = e^{-(1-\frac{1}{t})^2}$ in (E_5),

$$q_4(t) = 2e^{-\frac{1}{t}(1-\frac{1}{t})}t^{-2t}(\log t + 1) \ \text{ and } \ q_5(t) = 2e^{-\frac{1}{t}(1-\frac{1}{t})}t^{\frac{2}{t}}(\log t + t - 1).$$

It is easy to see that condition (2.1) with $k = 1$ is satisfied for (E_4) and (E_5), respectively, and that $d^{\frac{1}{t}}$ is decreasing for $0 < d < 1$ and is increasing for $d \geqq 1$. Taking these facts into account, we find that

$$\int_e^\infty \Big[\frac{1}{p(t)}\int_t^\infty q_4(s)\, d^{\frac{1}{s}}\, ds\Big]^{\frac{1}{t}} dt = \int_e^\infty \Big[2e^{t-1}\int_t^\infty e^{-\frac{1}{s}(1-\frac{1}{s})}(\log s + 1)s^{-2s}\, d^{\frac{1}{s}}\, ds\Big]^{\frac{1}{t}} dt < \infty$$

and

$$\int_e^\infty \left[\frac{1}{p(t)}\int_t^\infty q_5(s)d^{\frac{1}{s}}\,ds\right]^{\frac{t}{t-1}} dt = \int_e^\infty \left[2e^{(1-\frac{1}{t})^2}\int_t^\infty e^{-\frac{1}{s}(1-\frac{1}{s})}(\log s + s - 1)s^{\frac{2}{s}}d^{\frac{1}{s}}\,ds\right]^{\frac{t}{t-1}} dt < \infty.$$

Hence, there exists a positive solution of type III of (E_4) and (E_5) by Theorem 2.2. One such solution is $x(t) = e^{1-\frac{1}{t}}$ which satisfies

$$\lim_{t\to\infty} p(t)\varphi_t(x'(t)) = \lim_{t\to\infty} t^{-2t} = 0, \quad \lim_{t\to\infty} x(t) = 1$$

for (E_4) and

$$\lim_{t\to\infty} p(t)\varphi_{1-\frac{1}{t}}(x'(t)) = \lim_{t\to\infty} t^{\frac{1}{t}-1} = 0, \quad \lim_{t\to\infty} x(t) = 1$$

for (E_5), respectively.

Example 3.3. Consider the equations which are special cases of (A):

$$\left((\gamma t^\gamma)^{-t}\varphi_t(x')\right)' + q_6(t)\varphi_{\frac{1}{t}}(x) = 0, \quad t \geqq e, \tag{E_6}$$

and

$$\left((\gamma t^\gamma)^{-(1-\frac{1}{t})}\varphi_{1-\frac{1}{t}}(x')\right)' + q_7(t)\varphi_{\frac{1}{t}}(x) = 0, \quad t \geqq e, \tag{E_7}$$

where $\gamma > 0$ is a constant,

$$q_6(t) = t^{-(t+\frac{\gamma}{t})}\log(t+1) \;\text{ and }\; q_7(t) = t^{-3-\frac{1}{t}(\gamma-1)}(\log t + t - 1).$$

Since

$$P_{t,1}(t) = P_{1-\frac{1}{t},1}(t) = \int_e^t \gamma s^\gamma\,ds \sim \frac{\gamma}{\gamma+1}t^{\gamma+1}, \quad P_{t,1}(t) = P_{1-\frac{1}{t},1}(t) \to \infty \;\text{ as }\; t\to\infty,$$

$$\int_e^\infty q_6(t)(P_{t,1}(t))^{\frac{1}{t}}\,dt = \int_e^\infty t^{-(t+\frac{\gamma}{t})}\log(t+1)\Big(\frac{\gamma}{\gamma+1}t^{\gamma+1}\Big)^{\frac{1}{t}} dt < \infty,$$

$$\int_e^\infty q_7(t)(P_{1-\frac{1}{t},1}(t))^{\frac{1}{t}}\,dt = \int_e^\infty t^{-3-\frac{1}{t}(\gamma-1)}(\log t + t - 1)\Big(\frac{\gamma}{\gamma+1}t^{\gamma+1}\Big)^{\frac{1}{t}} dt < \infty,$$

$$\int_e^\infty \left[\frac{1}{p(t)}\int_t^\infty q_6(s)d^{\frac{1}{s}}\,ds\right]^{\frac{1}{t}} dt = \int_e^\infty \left[(\gamma t^\gamma)^t\int_t^\infty s^{-(s+\frac{\gamma}{s})}(\log s + 1)d^{\frac{1}{s}}\,ds\right]^{\frac{1}{t}} dt = \infty$$

and

$$\int_e^\infty \left[\frac{1}{p(t)}\int_t^\infty q_7(s)d^{\frac{1}{s}}\,ds\right]^{\frac{1}{t}} dt = \int_e^\infty \left[(\gamma t^\gamma)^{(1-\frac{1}{t})}\int_t^\infty s^{-3-\frac{1}{s}(\gamma-1)}(\log s + s - 1)d^{\frac{1}{s}}\,ds\right]^{\frac{1}{t}} dt = \infty,$$

any of (E_6) and (E_7) possesses a positive solution of type II by Theorem 2.3. One such solution is $x(t) = t^\gamma$, $\gamma > 0$, which satisfies

$$\lim_{t\to\infty} p(t)\varphi_t(x'(t)) = \lim_{t\to\infty} t^{-t} = 0, \quad \lim_{t\to\infty} x(t) = \infty$$

and

$$\lim_{t\to\infty} p(t)\varphi_{1-\frac{1}{t}}(x'(t)) = \lim_{t\to\infty} t^{-(1-\frac{1}{t})} = 0, \quad \lim_{t\to\infty} x(t) = \infty.$$

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