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MIXED TYPE BOUNDARY-TRANSMISSION PROBLEMS WITH INTERIOR CRACKS OF THE THERMO-PIEZO-ELECTRICITY THEORY WITHOUT ENERGY DISSIPATION

Abstract. In the paper, we study mixed type interaction problem of pseudo-oscillations between thermo-elastic and thermo-piezo-elastic bodies with interior cracks. The model under consideration is based on the Green–Naghdi theory of thermo-piezo-electricity without energy dissipation. This theory permits propagation of thermal waves only with a finite speed. Using the potential theory and boundary pseudodifferential equations method, we prove the existence and uniqueness of solutions and analyze their smoothness.

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რეზიუმე. სტატიაში შევისწავლით შერეული ტიპის თერმო-დრეკადი და თერმო-ელექტრო-დრეკადი სხეულების ფსევდორხევის ურთიერთქმედების ტრანსმისიის ამოცანას შიგა ბზარებით. განხილული თერმო-ელექტრო-დრეკადი მოდელი ეფუძნება გრინ-ნახდის თეორიას ენერგიის დისიპაციის გარეშე. ამ თეორიაში დასაშვებია თერმული ტალღების გავრცელება სასრული სიჩქარით. პოტენციალთა და სასაზღვრო ფსევდოდიფერენციალურ განტოლებათა მეთოდის გამოყენებით მტკიცდება ამოცანის ამონახსნთა არსებობისა და ერთადერთობის თეორემები და შესწავლილია მათი სიგლუვე.

1 Introduction

In this paper, we investigate boundary-transmission problem, i.e., mixed type interaction problem of pseudo-oscillations between thermo-elastic and thermo-piezo-elastic bodies. The model under consideration is based on the Green–Naghdi theory of thermo-piezo-electricity without energy dissipation. This theory permits propagation of thermal waves only with a finite speed.

Other models of thermo-piezo-electricity, in particular, Foigt's and Mindlin's model, are well known. The model under consideration is refined taking into account microrotation and microstraght of the particle.

Almost complete historical and bibliographical notes in this direction can be found in [15], where the dynamical equations of the thermopiezo-electricity without energy dissipation are derived on the basis of the Green–Naghdi theory established in [11, 12] and Eringen's results obtained in [8, 9]. In the present paper, we consider the pseudo-oscillation equations obtained by the Laplace transform from the dynamical equations derived by Ieşan in [15] for homogeneous isotropic solids possessing thermo-piezo-electricity properties without energy dissipation.

Using the potential theory and the method of boundary pseudodifferential equations, we prove the existence and uniqueness theorems of solutions in appropriate function spaces. We prove regularity results of the mixed type boundary-transmission problem with interior cracks. Further, we analyze regularity of solutions of the mixed type boundary-transmission problem near the exceptional curves, where different type boundary conditions collide, and near the crack edges. The regularity of solutions near the crack edges is $C^{\frac{1}{2}}$, whereas for the temperature of elasticity body is $C^{\frac{3}{2}}$. The regularity of solutions near the curve, where different type boundary conditions meet, depends on the material constants and does not depend on the geometry of the exceptional curve. If these constants meet certain conditions, then the smoothness of solutions is $C^{\frac{1}{2}}$ (cf. [2–6]).

2 Thermo-elastic field equations and thermo-piezo-elastic field equations without energy dissipation

The model under consideration is based on the Green-Naghdi theory of thermo-piezo-electricity without energy dissipation.

Consider disjoint bounded domains Ω_1 and Ω_2 in the Euclidean space $\mathbb{R}^3$ with C^∞-smooth boundaries $\partial\Omega_1 = S_1$ and $\partial\Omega_2 = S_1 \cup S_2$ $(S_1 \cap S_2 = \varnothing)$. $S_2 = \overline{S}_2^{(D)} \cup \overline{S}_2^{(N)}$, $S_2^{(D)} \cap S_2^{(N)} = \varnothing$, $\ell_m = \partial S_2^{(D)} = \partial S_2^{(N)} \in C^\infty$. We assume that the solids under consideration contain interior cracks. We identify the crack surfaces as two-dimensional, two-sided manifolds Σ_k, $k = 1, 2$, with the crack edges $\ell_c^{(k)} := \partial\Sigma_k$, $k = 1, 2$. We assume that Σ_k, $k = 1, 2$, are proper parts of closed surfaces $S_0^{(k)} \subset \Omega_k$, $k = 1, 2$, surrounding domains $\overline{\Omega}_0^{(k)} \subset \Omega_k$ and that Σ_k, $S_0^{(k)}$ and $\ell_c^{(k)}$, $k = 1, 2$, are C^∞-smooth. Denote $\Omega_{\Sigma_k} := \Omega_k \setminus \overline{\Sigma}_k$, $k = 1, 2$.

Throughout the paper, $n = (n_1, n_2, n_3)$ stands for the exterior unit normal vector to $\partial\Omega_1 = S_1$ and $S_0^{(1)} = \partial\Omega_0^{(1)}$. A vector $\nu = (\nu_1, \nu_2, \nu_3)$ is an exterior unit normal vector to $\partial\Omega_2 = S_1 \cup S_2$ and $S_0^{(2)} = \partial\Omega_0^{(2)}$.

Suppose the domain Ω_1 is filled with a homogeneous thermo-elastic material, then the system of governing differential equations of pseudo-oscillations with respect to the sought vector function $U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top$, where $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top$ is the displacement vector and $\vartheta^{(1)}$ is the temperature, has the following form (see [16]):

$$(\mu^{(1)} + \varkappa^{(1)})\Delta u^{(1)} + (\lambda^{(1)} + \mu^{(1)}) \operatorname{grad} \operatorname{div} u^{(1)} - \tau^2 \rho_1 u^{(1)} - \tau\beta_0^{(1)} \operatorname{grad} \vartheta^{(1)} = (\mathcal{F}_1^{(1)}, \mathcal{F}_2^{(1)}, \mathcal{F}_3^{(1)})^\top, \quad (2.1)$$

$$k^{(1)}\Delta\vartheta^{(1)} - \tau^2 a^{(1)}\vartheta^{(1)} - \tau\beta_0^{(1)} \operatorname{div} u^{(1)} = \mathcal{F}_4^{(1)}, \quad (2.2)$$

where $(\mathcal{F}_1^{(1)}, \mathcal{F}_2^{(1)}, \mathcal{F}_3^{(1)})^\top$ is a mass force density, $\mathcal{F}_4^{(1)}$ is a heat source density, ρ_1 is the mass density,

$\mu^{(1)}, \varkappa^{(1)}, \lambda^{(1)}, \beta_0^{(1)}, k^{(1)}$ and $a^{(1)}$ are the thermo-elastic constants and satisfy the conditions

$$\varkappa^{(1)} > 0, \ \ \varkappa^{(1)} + 2\mu^{(1)} > 0, \ \ \varkappa^{(1)} + 2\mu^{(1)} + 3\lambda^{(1)} > 0, \ \ k^{(1)} > 0, \ \ \rho_1 > 0, \ \ a^{(1)} > 0,$$
$$\beta_0^{(1)} > 0, \ \ \tau = \sigma + i\omega, \ \ \sigma > \sigma_0 > 0, \ \ \omega \in \mathbb{R}.$$

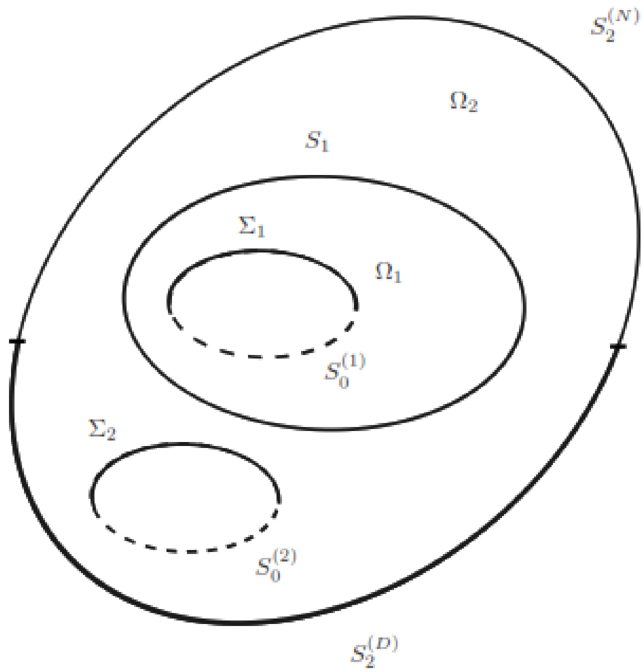

Figure 1

The stress operator for a homogeneous isotropic system of equations is defined as follows:

$$T^{(1)} = T^{(1)}(\partial_x, n, \tau) = \big[T_{ij}^{(1)}(\partial_x, n, \tau)\big]_{4\times 4}$$
$$:= \begin{bmatrix} \big[\lambda^{(1)} n_i \partial_j + \mu^{(1)} n_j \partial_i + \delta_{ij}(\mu^{(1)} + \varkappa^{(1)}) n_k \partial_k\big]_{3\times 3}, & [-\tau\beta_0^{(1)} n]_{3\times 1} \\ [0]_{1\times 3} & k^{(1)} n_l \partial_l \end{bmatrix}_{4\times 4}.$$

We can write the above system (2.1), (2.2) of equations for pseudo-oscillations of the theory of homogeneous isotropic thermo-elasticity in the following matrix form:

$$A^{(1)}(\partial_x, \tau) U^{(1)} = \mathcal{F}^{(1)},$$

where $U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top$, $\mathcal{F}^{(1)} = (\mathcal{F}_1^{(1)}, \mathcal{F}_2^{(1)}, \mathcal{F}_3^{(1)}, \mathcal{F}_4^{(1)})^\top$, and $A^{(1)}(\partial_x, \tau)$ is the 4-dimensional matrix differential operator of generalized thermo-elasticity:

$$A^{(1)}(\partial_x, \tau) = \big[A_{ij}^{(1)}(\partial_x, \tau)\big]_{4\times 4}$$
$$:= \begin{bmatrix} \big[\delta_{ij}(\mu^{(1)} + \varkappa^{(1)})\Delta + (\lambda^{(1)} + \mu^{(1)})\partial_i \partial_j - \tau^2 \rho_1 \delta_{ij}\big]_{3\times 3}, & -\tau\beta_0^{(1)}[\partial_i]_{3\times 1} \\ -\tau\beta_0^{(1)}[\partial_j]_{1\times 3} & -\tau^2 a^{(1)} + k^{(1)}\Delta \end{bmatrix}_{4\times 4},$$

where δ_{ij} is the Kronecker delta.

The domain Ω_2 is filled with a thermo-electro-elastic material. The corresponding system of differential equations of pseudo-oscillations with respect to the sought vector function $U^{(2)}$ has the following form (see [15]):

$$(\mu^{(2)}+\varkappa^{(2)})\partial_j\partial_j u_i^{(2)}+(\lambda^{(2)}+\mu^{(2)})\partial_i\partial_j u_j^{(2)}-\rho_2\tau^2 u_i^{(2)}+\varkappa^{(2)}\varepsilon_{ijk}\partial_j\phi_k^{(2)} \\ +\lambda_0^{(2)}\partial_i\varphi^{(2)}-\tau\beta_0^{(2)}\partial_i\vartheta^{(2)}=-\rho_2 f_i,\ \ i=1,2,3, \tag{2.3}$$

$$k^{(2)}\partial_j\partial_j\vartheta^{(2)}-\tau^2 a^{(2)}\vartheta^{(2)}-\tau\beta_0^{(2)}\partial_j u_j^{(2)}-\tau c_0^{(2)}\varphi^{(2)}+\nu_1^{(2)}\partial_j\partial_j\varphi^{(2)}-\nu_3^{(2)}\partial_j\partial_j\psi^{(2)}=-\frac{1}{T_0}\,\rho_2 Q, \tag{2.4}$$

$$\gamma^{(2)}\partial_j\partial_j\phi_i^{(2)}+(\alpha^{(2)}+\beta^{(2)})\partial_j\partial_i\phi_j^{(2)}-\tau^2 I_0^{(2)}\phi_i^{(2)} \\ +\varkappa^{(2)}\varepsilon_{ijk}\partial_j u_k^{(2)}-2\varkappa^{(2)}\phi_i^{(2)}=-\rho_2 X_i,\ \ i=1,2,3, \tag{2.5}$$

$$(a_0^{(2)}\partial_j\partial_j-\xi_0^{(2)})\varphi^{(2)}-j_0^{(2)}\tau^2\varphi^{(2)}-\lambda_2^{(2)}\partial_j\partial_j\psi^{(2)}+\nu_1^{(2)}\partial_j\partial_j\vartheta^{(2)}+c_0^{(2)}\tau\vartheta^{(2)}-\lambda_0^{(2)}\partial_j u_j^{(2)}=-\rho_2 F, \tag{2.6}$$

$$\lambda_0^{(2)}\partial_j\partial_j\varphi^{(2)}+\chi^{(2)}\partial_j\partial_j\psi^{(2)}+\nu_3^{(2)}\partial_j\partial_j\vartheta^{(2)}=-f, \tag{2.7}$$

where $U^{(2)}=(u_1^{(2)},u_2^{(2)},u_3^{(2)},\vartheta^{(2)},\phi_1^{(2)},\phi_2^{(2)},\phi_3^{(2)},\varphi^{(2)},\psi^{(2)})^\top$, $u^{(2)}=(u_1^{(2)},u_2^{(2)},u_3^{(2)})^\top$ is the displacement vector, $\vartheta^{(2)}$ is the temperature, $\phi^{(2)}=(\phi_1^{(2)},\phi_2^{(2)},\phi_3^{(2)})^\top$ is the vector of microrotation, $\varphi^{(2)}$ is the microstretch, $\psi^{(2)}$ is the electric field potential, and (f_1,f_2,f_3) is the external body force per unit mass, Q is the external rate of supply of heat per unit mass, X_i is the external body couple per unit mass, F is the microstretch body force, f is the density of free charge, T_0 is the initial reference temperature.

The coefficients $\lambda^{(2)}$, $\mu^{(2)}$, $\varkappa^{(2)}$, $\lambda_0^{(2)}$, $\beta_0^{(2)}$, $\alpha^{(2)}$, $\beta^{(2)}$, $\gamma^{(2)}$, $\lambda_1^{(2)}$, $\nu_1^{(2)}$, $a_0^{(2)}$, $\lambda_2^{(2)}$, $\nu_2,\xi_0^{(2)}$, $c_0^{(2)}$, $a^{(2)}$, $k^{(2)}$, $\nu_3^{(2)}$, $b_0^{(2)}$, $\chi^{(2)}$ are constitutive constants, and $I_0^{(2)}$ is the coefficient of inertia, $j_0^{(2)}$ is the microstretch inertia, ε_{ijk} is the Levi–Civita symbol.

Due to the positiveness of internal energy, the coefficients of system (2.3)–(2.7) must satisfy the following conditions:

$$\varkappa^{(2)}>0,\ \ \varkappa^{(2)}+2\mu^{(2)}>0,\ \ \varkappa^{(2)}+2\mu^{(2)}+3\lambda^{(2)}>0,$$
$$\xi_0^{(2)}(\varkappa^{(2)}+2\mu^{(2)}+3\lambda^{(2)})>3(\lambda_0^{(2)})^2,\ \ \gamma^{(2)}>|\beta^{(2)}|,\ \ a_0^{(2)}k^{(2)}-(\nu_1^{(2)})^2>0,$$
$$\beta^{(2)}+\gamma^{(2)}+3\alpha^{(2)}>0,\ \ \chi^{(2)}>0,\ \ a^{(2)}>0,\ \ k^{(2)}>0,\ \ a_0^{(2)}>0,\ \ a_0^{(2)}(\gamma^{(2)}-\beta^{(2)})>2(b_0^{(2)})^2,$$
$$(\gamma^{(2)}-\beta^{(2)})\big[a_0^{(2)}k^{(2)}-(\nu_1^{(2)})^2\big]+4b_0^{(2)}\nu_1^{(2)}\nu_2^{(2)}-2a_0^{(2)}(\nu_2^{(2)})^2-2k^{(2)}(b_0^{(2)})^2>0,$$
$$\rho_2>0,\ \ I_0^{(2)}>0,\ \ j_0^{(2)}>0,\ \ \beta_0^{(2)}>0,$$

where ρ_2 is the mass density.

Denote by

$$A^{(2)}(\partial_x,\tau)=[A_{ij}^{(2)}(\partial_x,\tau)]_{9\times 9}$$

the matrix differential operator generated by the left-hand side expressions in (2.3)–(2.7),

$$A_{ij}^{(2)}(\partial_x,\tau)=\delta_{ij}(\mu^{(2)}+\varkappa^{(2)})\partial_l\partial_l+(\lambda^{(2)}+\mu^{(2)})\partial_i\partial_j-\tau^2\rho_2\delta_{ij},$$
$$A_{i4}^{(2)}(\partial_x,\tau)=-\tau\beta_0^{(2)}\partial_i,\quad A_{i,j+4}^{(2)}(\partial_x,\tau)=-\varkappa^{(2)}\varepsilon_{ijl}\partial_l,\quad A_{i8}^{(2)}(\partial_x,\tau)=\lambda_0^{(2)}\partial_i,\quad A_{i9}^{(2)}(\partial_x,\tau)=0,$$
$$A_{4j}^{(2)}(\partial_x,\tau)=-\tau\beta_0^{(2)}\partial_j,\quad A_{44}^{(2)}(\partial_x,\tau)=k^{(2)}\partial_l\partial_l-\tau^2 a^{(2)},\quad A_{4,j+4}^{(2)}(\partial_x,\tau)=0,$$
$$A_{48}^{(2)}(\partial_x,\tau)=\nu_1^{(2)}\partial_l\partial_l-\tau c_0^{(2)},\quad A_{49}^{(2)}(\partial_x,\tau)=-\nu_3^{(2)}\partial_l\partial_l,\quad A_{i+4,j}^{(2)}(\partial_x,\tau)=-\varkappa^{(2)}\varepsilon_{ijl}\partial_l,$$
$$A_{i+4,4}^{(2)}(\partial_x,\tau)=0,\quad A_{i+4,j+4}^{(2)}(\partial_x,\tau)=\delta_{ij}\gamma^{(2)}\partial_l\partial_l+(\alpha^{(2)}+\beta^{(2)})\partial_i\partial_j-(2\varkappa^{(2)}+\tau^2 I_0^{(2)})\delta_{ij},$$
$$A_{i+4,8}^{(2)}(\partial_x,\tau)=0,\quad A_{i+4,9}^{(2)}(\partial_x,\tau)=0,\quad A_{8j}^{(2)}(\partial_x,\tau)=-\lambda_0^{(2)}\partial_j,$$
$$A_{84}^{(2)}(\partial_x,\tau)=\nu_1^{(2)}\partial_l\partial_l+\tau c_0^{(2)},\quad A_{8,j+4}^{(2)}(\partial_x,\tau)=0,\quad A_{88}^{(2)}(\partial_x,\tau)=a_0^{(2)}\partial_l\partial_l-(\xi_0^{(2)}+\tau^2 j_0^{(2)}),$$
$$A_{89}^{(2)}(\partial_x,\tau)=-\lambda_2^{(2)}\partial_l\partial_l,\quad A_{9j}^{(2)}(\partial_x,\tau)=0,\quad A_{94}^{(2)}(\partial_x,\tau)=\nu_3^{(2)}\partial_l\partial_l,$$
$$A_{9,j+4}^{(2)}(\partial_x,\tau)=0\quad A_{98}^{(2)}(\partial_x,\tau)=\lambda_2^{(2)}\partial_l\partial_l,\quad A_{99}(\partial_x,\tau)=\chi^{(2)}\partial_l\partial_l,\ \ i,j=1,2,3.$$

The stress differential operator of thermo-electro-elasticity is defined as follows:

$$T^{(2)} = T^{(2)}(\partial_x, \nu, \tau) := [T^{(2)}_{ij}(\partial_x, \nu, \tau)]_{9\times 9},$$

where

$$\begin{gathered}
T^{(2)}_{ij}(\partial_x,\nu,\tau) = \lambda^{(2)}\nu_i\partial_j + \mu^{(2)}\nu_j\partial_i + \delta_{ij}(\mu^{(2)} + \varkappa^{(2)})\nu_k\partial_k, \quad T^{(2)}_{i4}(\partial_x,\nu,\tau) = -\tau\beta^{(2)}_0\nu_i, \\
T^{(2)}_{i,j+4}(\partial_x,\nu,\tau) = -\varkappa^{(2)}\varepsilon_{ijk}\nu_k, \quad T^{(2)}_{i8}(\partial_x,\nu,\tau) = \lambda^{(2)}_0\nu_i, \quad T^{(2)}_{i9}(\partial_x,\nu,\tau) = 0, \\
T^{(2)}_{4j}(\partial_x,\nu,\tau) = 0, \quad T^{(2)}_{44}(\partial_x,\nu,\tau) = k^{(2)}\nu_l\partial_l, \quad T^{(2)}_{4,j+4}(\partial_x,\nu,\tau) = -\nu^{(2)}_2\varepsilon_{ljk}\nu_l\partial_k, \\
T^{(2)}_{48}(\partial_x,\nu,\tau) = \nu^{(2)}_1\nu_k\partial_k, \quad T^{(2)}_{49}(\partial_x,\nu,\tau) = -\nu^{(2)}_3\nu_k\partial_k, \quad T^{(2)}_{i+4,j}(\partial_x,\nu,\tau) = 0, \\
T^{(2)}_{i+4,4}(\partial_x,\nu,\tau) = \nu^{(2)}_2\varepsilon_{lik}\nu_l\partial_k, \quad T^{(2)}_{i+4,j+4}(\partial_x,\nu,\tau) = \alpha^{(2)}\nu_i\partial_j + \beta^{(2)}\nu_j\partial_i + \delta_{ij}\gamma^{(2)}\nu_k\partial_k, \\
T^{(2)}_{i+4,8}(\partial_x,\nu,\tau) = b^{(2)}_0\varepsilon_{lik}\nu_l\partial_k, \quad T^{(2)}_{i+4,9}(\partial_x,\nu,\tau) = \lambda^{(2)}_1\varepsilon_{lik}\nu_l\partial_k, \quad T^{(2)}_{8j}(\partial x,\nu,\tau) = 0, \\
T^{(2)}_{84}(\partial_x,\nu,\tau) = \nu^{(2)}_1\nu_k\partial_k, \quad T^{(2)}_{8,j+4}(\partial_x,\nu,\tau) = -b^{(2)}_0\varepsilon_{lik}\nu_l\partial_k, \quad T^{(2)}_{88}(\partial_x,\nu,\tau) = a^{(2)}_0\nu_k\partial_k, \\
T^{(2)}_{89}(\partial_x,\nu,\tau) = -\lambda^{(2)}_2\nu_k\partial_k, \quad T^{(2)}_{9j}(\partial x,\nu,\tau) = 0, \quad T^{(2)}_{94}(\partial_x,\nu,\tau) = \nu^{(2)}_3\nu_k\partial_k, \\
T^{(2)}_{9,j+4}(\partial_x,\nu,\tau) = -\lambda^{(2)}_1\varepsilon_{ljk}\nu_l\partial_k, \quad T^{(2)}_{98}(\partial_x,\nu,\tau) = \lambda^{(2)}_2\nu_k\partial_k, \quad T^{(2)}_{99}(\partial_x,\nu,\tau) = \chi^{(2)}\nu_k\partial_k, \quad i,j = 1,2,3.
\end{gathered}$$

Introduce the boundary operator $\widetilde{T}^{(2)}(\partial_x,\nu,\tau) = [\widetilde{T}^{(2)}_{ij}(\partial_x,\nu,\tau)]_{9\times 9}$ associated with the formally adjoint differential operator $(A^{(2)})^*(\partial_x,\tau)$:

$$\begin{gathered}
\widetilde{\mathcal{T}}^{(2)}_{ij}(\partial_x,\nu,\tau) = \lambda^{(2)}\nu_i\partial_j + \mu^{(2)}\nu_j\partial_i + \delta_{ij}(\mu^{(2)} + \varkappa^{(2)})\nu_k\partial_k, \quad \widetilde{\mathcal{T}}^{(2)}_{i4}(\partial_x,\nu,\tau) = \overline{\tau}\beta^{(2)}_0\nu_i, \\
\widetilde{\mathcal{T}}^{(2)}_{i,j+4}(\partial_x,\nu,\tau) = -\varkappa^{(2)}\varepsilon_{ijk}\nu_k, \quad \widetilde{\mathcal{T}}^{(2)}_{i8}(\partial_x,\nu,\tau) = \lambda^{(2)}_0\nu_i, \quad \widetilde{\mathcal{T}}^{(2)}_{i9}(\partial_x,\nu,\tau) = 0, \quad \widetilde{\mathcal{T}}^{(2)}_{4j}(\partial_x,\nu,\tau) = 0, \\
\widetilde{\mathcal{T}}^{(2)}_{44}(\partial_x,\nu,\tau) = k^{(2)}\nu_l\partial_l, \quad \widetilde{\mathcal{T}}_{4,j+4}(\partial_x,\nu,\tau) = 0, \quad \widetilde{\mathcal{T}}^{(2)}_{48}(\partial_x,\nu,\tau) = \nu^{(2)}_1\nu_k\partial_k, \\
\widetilde{\mathcal{T}}^{(2)}_{49}(\partial_x,\nu,\tau) = \nu^{(2)}_3\nu_k\partial_k, \quad \widetilde{\mathcal{T}}^{(2)}_{i+4,j}(\partial_x,\nu,\tau) = 0, \quad \widetilde{\mathcal{T}}^{(2)}_{i+4,4}(\partial_x,\nu,\tau) = \nu^{(2)}_2\varepsilon_{ilk}\nu_l\partial_k, \\
\widetilde{\mathcal{T}}^{(2)}_{i+4,j+4}(\partial_x,\nu,\tau) = \alpha^{(2)}\nu_i\partial_j + \beta^{(2)}\nu_j\partial_i + \delta_{ij}\gamma^{(2)}\nu_k\partial_k, \quad \widetilde{\mathcal{T}}^{(2)}_{i+4,8}(\partial_x,\nu,\tau) = b^{(2)}_0\varepsilon_{ilk}\nu_l\partial_k, \\
\widetilde{\mathcal{T}}^{(2)}_{i+4,9}(\partial_x,\nu,\tau) = \lambda^{(2)}_1\varepsilon_{lik}\nu_l\partial_k, \quad \widetilde{\mathcal{T}}^{(2)}_{8j}(\partial_x,\nu,\tau) = 0, \quad \widetilde{\mathcal{T}}^{(2)}_{84}(\partial_x,\nu,\tau) = \nu^{(2)}_1\nu_k\partial_k, \\
\widetilde{\mathcal{T}}^{(2)}_{8,j+4}(\partial_x,\nu,\tau) = 0, \quad \widetilde{\mathcal{T}}^{(2)}_{88}(\partial_x,\nu,\tau) = a^{(2)}_0\nu_k\partial_k, \quad \widetilde{\mathcal{T}}^{(2)}_{89}(\partial_x,\nu,\tau) = \lambda^{(2)}_2\nu_k\partial_k, \\
\widetilde{\mathcal{T}}^{(2)}_{9j}(\partial_x,\nu,\tau) = 0, \quad \widetilde{\mathcal{T}}^{(2)}_{94}(\partial_x,\nu,\tau) = -\nu^{(2)}_3\nu_k\partial_k, \quad \widetilde{\mathcal{T}}^{(2)}_{9,j+4}(\partial_x,\nu,\tau) = 0, \\
\widetilde{\mathcal{T}}^{(2)}_{98}(\partial_x,\nu,\tau) = -\lambda^{(2)}_2\nu_k\partial_k, \quad \widetilde{\mathcal{T}}^{(2)}_{99}(\partial_x,\nu,\tau) = \chi^{(2)}\nu_k\partial_k, \quad i,j = 1,2,3.
\end{gathered}$$

The system of equations (2.3)–(2.7) can be written in the matrix form

$$A^{(2)}(\partial_x,\tau)U^{(2)} = \mathcal{F}^{(2)},$$

where

$$\begin{aligned}
U^{(2)} &= \big(u^{(2)}_1, u^{(2)}_2, u^{(2)}_3, \vartheta^{(2)}, \phi^{(2)}_1, \phi^{(2)}_2, \phi^{(2)}_3, \varphi^{(2)}, \psi^{(2)}\big)^\top, \\
\mathcal{F}^{(2)} &= -\Big(\rho_2 f_1, \rho_2 f_2, \rho_2 f_3, \frac{1}{T_0}\,\rho_2 Q, \rho_2 X_1, \rho_2 X_2, \rho_2 X_3, \rho_2 F, f\Big)^\top
\end{aligned}$$

and $A^{(2)}(\partial_x,\tau)$ is the 9-dimensional matrix differential operator corresponding to system (2.3)–(2.7).

3 Formulation of the mixed type boundary-transmission problem with interior cracks $(TM)_{c,\tau}$ of pseudo-oscillations

By H^s with $s \in \mathbb{R}$, we denote the Sobolev–Slobodetsky space. Let $\mathcal{M}_0$ be a smooth surface without boundary. For a proper sub-manifold $\mathcal{M} \subset \mathcal{M}_0$, we denote by $\widetilde{H}^s(\mathcal{M})$ the subspace of $H^s(\mathcal{M}_0)$,

$$\widetilde{H}^s(\mathcal{M}) = \big\{g :\ g \in H^s(\mathcal{M}_0),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\big\},$$

while $H^s(\mathcal{M})$ stands for the space of restriction on $\mathcal{M}$ of functions from $H^s(\mathcal{M}_0)$.

We are looking for a solution

$$U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top = (u^{(1)}, u_4^{(1)})^\top \in [H^1(\Omega_{\Sigma_1})]^4,$$
$$U^{(2)} = \big(u^{(2)}, \vartheta^{(2)}, \phi^{(2)}, \varphi^{(2)}, \psi^{(2)}\big)^\top = \big(u^{(2)}, u_4^{(2)}, u_5^{(2)}, \dots, u_9^{(2)}\big)^\top \in [H^1(\Omega_{\Sigma_2})]^9$$

of pseudo-oscillation equations

$$A^{(1)}(\partial_x, \tau)U^{(1)} = 0 \ \text{ in } \Omega_{\Sigma_1}, \tag{3.1}$$
$$A^{(2)}(\partial_x, \tau)U^{(2)} = 0 \ \text{ in } \Omega_{\Sigma_2}, \tag{3.2}$$

which satisfy the following transmission conditions on the surface S_1:

$$\{u_j^{(1)}\}^+ - \{u_j^{(2)}\}^+ = f_j^{(1)} \ \text{ on } S_1, \ \ j = \overline{1,4}, \tag{3.3}$$
$$\big\{T^{(1)}(\partial_x, n, \tau)U^{(1)}\big\}_j^+ + \{T^{(2)}(\partial_x, \nu, \tau)U^{(2)}\}_j^+ = f_j^{(2)} \ \text{ on } S_1, \ \ j = \overline{1,4}, \ \ \nu = -n, \tag{3.4}$$

boundary conditions on the surface S_1:

$$\{u_j^{(2)}\}^+ = Q_j^{(2)} \ \text{ on } S_1, \ \ j = \overline{5,9}, \tag{3.5}$$

mixed boundary conditions on the surface S_2:

$$\{U^{(2)}\}^+ = p_2^{(D)} \ \text{ on } S_2^{(D)}, \tag{3.6}$$
$$\big\{T^{(2)}(\partial_x, \nu, \tau)U^{(2)}\big\}^+ = q_2^{(N)} \ \text{ on } S_2^{(N)}, \tag{3.7}$$

crack boundary conditions on Σ_1:

$$\big\{T^{(1)}(\partial_x, n, \tau)U^{(1)}\big\}_j^\pm = F_j^{(1),\pm} \ \text{ on } \Sigma_1, \ \ j = 1,2,3, \tag{3.8}$$
$$\{u_4^{(1)}\}^+ - \{u_4^{(1)}\}^- = G_4^{(1)} \ \text{ on } \Sigma_1, \tag{3.9}$$
$$\big\{T^{(1)}(\partial_x, n, \tau)U^{(1)}\big\}_4^+ - \{T^{(1)}(\partial_x, n, \tau)U^{(1)}\}_4^- = F_4^{(1)} \ \text{ on } \Sigma_1, \tag{3.10}$$

crack boundary conditions on Σ_2:

$$\big\{T^{(2)}(\partial_x, \nu, \tau)U^{(2)}\big\}_j^\pm = F_j^{(2),\pm} \ \text{ on } \Sigma_2, \ \ j = 1,2,3,5,6,7, \tag{3.11}$$
$$\{u_j^{(2)}\}^+ - \{u_j^{(2)}\}^- = G_j^{(2)} \ \text{ on } \Sigma_2, \ \ j = 4,8,9, \tag{3.12}$$
$$\big\{T^{(2)}(\partial_x, \nu, \tau)U^{(2)}\big\}_j^+ - \{T^{(2)}(\partial_x, \nu, \tau)U^{(2)}\}_j^- = F_j^{(2)} \ \text{ on } \Sigma_2, \ \ j = 4,8,9, \tag{3.13}$$

where

$$f_j^{(1)} \in H^{\frac12}(S_1), \quad f_j^{(2)} \in H^{-\frac12}(S_1), \ \ j = \overline{1,4}, \quad Q_j^{(2)} \in H^{\frac12}(S_1), \ \ j = \overline{5,9},$$
$$p_2^{(D)} \in [H^{\frac12}(S_2^{(D)}]^9, \quad q_2^{(N)} \in [H^{-\frac12}(S_2^{(N)}]^9, \quad F_j^{(1),\pm} \in H^{-\frac12}(\Sigma_1), \ \ j = 1,2,3,$$
$$G_4^{(1)} \in \widetilde{H}^{\frac12}(\Sigma_1), \ \ F_4^{(1)} \in \widetilde{H}^{-\frac12}(\Sigma_1), \quad F_j^{(2),\pm} \in H^{-\frac12}(\Sigma_2), \ \ j = 1,2,3,5,6,7,$$
$$G_j^{(2)} \in \widetilde{H}^{\frac12}(\Sigma_2), \ \ j = 4,8,9, \quad F_j^{(2)} \in \widetilde{H}^{-\frac12}(\Sigma_2), \ \ j = 4,8,9,$$

and the compatibility conditions

$$F_j^{(1),+} - F_j^{(1),-} \in \widetilde{H}^{-\frac12}(\Sigma_1), \ \ j = 1,2,3,$$
$$F_j^{(2),+} - F_j^{(2),-} \in \widetilde{H}^{-\frac12}(\Sigma_2), \ \ j = 1,2,3,5,6,7,$$

are satisfied.

For the mixed type boundary-transmission problem $(TM)_{c,\tau}$, the following uniqueness theorem holds.

Theorem 3.1. *The mixed type boundary-transmission problem $(TM)_{c,\tau}$ cannot have two different solutions in the Sobolev space $[H^1(\Omega_{\Sigma_1})]^4 \times [H^1(\Omega_{\Sigma_2})]^9$.*

Proof. It is sufficient to show that the homogeneous problem $(TM)_{c,\tau}$ has only the trivial solution.

Indeed, suppose $(U^{(1)}, U^{(2)})$ is a solution to the homogeneous problem $(TM)_{c,\tau}$. Let us add together two Green's formulas in the domains $\Omega_k \setminus \overline{\Omega}_0^{(k)}$ and $\Omega_0^{(k)}$, $k=1,2$, where $\Omega_0^{(k)}$ is the above introduced auxiliary domain $\overline{\Omega}_0^{(k)} \subset \Omega_k$, $k=1,2$. We recall that the crack surface Σ_k is a proper part of the boundary $S_0^{(k)} = \partial\Omega_0^{(k)} \subset \Omega_0$. Any solutions to the homogeneous differential equations $A^{(k)}(\partial_x,\tau)=0$, $k=1,2$, of the classes $[H^1(\Omega_{\Sigma_1})]^4$ and $[H^1(\Omega_{\Sigma_2})]^9$, respectively, and their derivatives are continuous across the surfaces $S_0^{(k)} \setminus \overline{\Sigma}_k$, $k=1,2$:

$$\int\limits_{\Omega_{\Sigma_1}} E_\tau^{(1)}(U^{(1)}, \overline{U}^{\,(1)})\,dx = \big\langle \{T^{(1)}U^{(1)}\}^+, \{U^{(1)}\}^+ \big\rangle_{S_1}, \tag{3.14}$$

$$\int\limits_{\Omega_{\Sigma_2}} E_\tau^{(2)}(U^{(2)}, \overline{U}^{\,(2)})\,dx = \big\langle \{T^{(2)}U^{(2)}\}^+, \{U^{(2)}\}^+ \big\rangle_{S_1\cup S_2}, \tag{3.15}$$

where the symbols $\langle\,\cdot\,,\,\cdot\,\rangle_{S_1}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{S_1\cup S_2}$ denote the duality between the function spaces $[H^{-\frac12}(S_1))]^4$ and $[H^{\frac12}(S_1))]^4$, and the function spaces $[H^{-\frac12}(S_1\cup S_2)]^9$ and $[H^{\frac12}(S_1\cup S_2)]^9$, respectively, and

$$U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top, \;\; U^{(2)} = (u^{(2)}, \vartheta^{(2)}, \phi^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top,$$

$$\begin{aligned} E_\tau^{(1)}(U^{(1)}, \overline{U}^{\,(1)}) &= \mathcal{E}(u^{(1)}, \overline{u}^{\,(1)}) + \rho_1\tau^2|u^{(1)}|^2 - \tau\beta_0^{(1)}\vartheta^{(1)}\operatorname{div}\overline{u}^{\,(1)} \\ &\qquad + k^{(1)}|\operatorname{grad}\vartheta^{(1)}|^2 + \tau\beta_0^{(1)}\operatorname{div}u^{(1)}\vartheta^{(1)} + \tau^2 a^{(1)}|\vartheta^{(1)}|^2, \\ \mathcal{E}(u^{(1)}, \overline{u}^{\,(1)}) &= (\mu^{(1)} + \varkappa^{(1)})|\operatorname{grad}u^{(1)}|^2 + (\lambda^{(1)} + \mu^{(1)})|\operatorname{div}u^{(1)}|^2. \end{aligned}$$

Here and in what follows, $a\cdot b$ denotes the scalar product of two, in general, complex valued vectors

$$a\cdot b = \sum_{k=1}^N a_k\overline{b}_k, \quad a,b\in\mathbb{C}^N.$$

Obviously, $\mathcal{E}(u^{(1)}, \overline{u}^{\,(1)}) > 0 \;\; \forall\, u^{(1)} \neq 0$,

$$\begin{aligned} E_\tau^{(2)}(U^{(2)}, \overline{U}^{(2)}) &= B(v^{(2)}, \overline{v}^{(2)}) + 2i\lambda_1^{(2)}\varepsilon_{ijk}\operatorname{Im}(\partial_k\psi^{(2)}\partial_i\overline{\phi}_j^{(2)}) \\ &\quad + 2i\lambda_2^{(2)}\operatorname{Im}(\partial_j\varphi^{(2)}\overline{\psi}^{\,(2)}) + 2i\nu_3^{(2)}\operatorname{Im}(\partial_j\vartheta^{(2)}\partial_j\overline{\psi}^{\,(2)}) + 2i\tau\beta_0^{(2)}\operatorname{Im}(\partial_j u_j^{(2)}\vartheta^{(2)}) \\ &\qquad + 2i\tau c_0^{(2)}\operatorname{Im}(\varphi^{(2)}\vartheta^{(2)}) + \tau^2\Big(\rho_2|u^{(2)}|^2 + I_0^{(2)}|\phi^{(2)}|^2 + j_0^{(2)}|\varphi^{(2)}|^2 + a^{(2)}|\vartheta^{(2)}|^2\Big); \end{aligned}$$

here, $B(v^{(2)}, \overline{v}^{\,(2)}) > 0 \;\forall\, v^{(2)} \neq 0$ (for the definition of this form (see [6, formula (2.19)]).

Adding Green's formulas (3.14) and (3.15), and taking into account that $(U^{(1)}, U^{(2)})$ is a solution to the homogeneous transmission problem $(TM)_{c,\tau}$, we get

$$\begin{aligned} &\int\limits_{\Omega_{\Sigma_1}} E_\tau^{(1)}(U^{(1)}, \overline{U}^{\,(1)})\,dx + \int\limits_{\Omega_{\Sigma_2}} E_\tau^{(2)}(U^{(2)}, \overline{U}^{\,(2)})\,dx \\ &\qquad = \sum_{j=1}^4 \big\langle \{T^{(1)}U^{(1)}\}_j^+, \{U^{(1)}\}_j^+ \big\rangle_{S_1} + \sum_{j=1}^9 \big\langle \{T^{(2)}U^{(2)}\}_j^+, \{U^{(2)}\}_j^+ \big\rangle_{S_1} \\ &\qquad\qquad = \sum_{j=1}^4 \Big\langle \{T^{(1)}U^{(1)}\}_j^+ + \{T^{(2)}U^{(2)}\}_j^+, \{U^{(2)}\}_j^+ \Big\rangle_{S_1} = 0. \end{aligned}$$

Therefore, we obtain

$$\int\limits_{\Omega_{\Sigma_1}} E_\tau^{(1)}(U^{(1)}, \overline{U}^{\,(1)})\,dx + \int\limits_{\Omega_{\Sigma_2}} E_\tau^{(2)}(U^{(2)}, \overline{U}^{\,(2)})\,dx = 0.$$

Similarly, we get (see [6])

$$\int_{\Omega_{\Sigma_1}} E_\tau^{(1)}(U^{(1)}, \overline{U}^{\,(1)})\,dx + \int_{\Omega_{\Sigma_2}} \widetilde{E}_\tau^{(2)}(U^{(2)}, \overline{U}^{\,(2)})\,dx = 0,$$

where

$$\begin{aligned}\widetilde{E}_\tau^{(2)}(U^{(2)}, \overline{U}^{\,(2)}) &:= B^{(2)}(v^{(2)}, \overline{v}^{\,(2)}) + 2i\tau\beta_0^{(2)}\,\mathrm{Im}(\partial_j u_j^{(2)}\overline{\vartheta}^{\,(2)})\\ &\quad + 2i\tau c_0^{(2)}\,\mathrm{Im}\left(\varphi^{(2)}\overline{\vartheta}^{\,(2)}\right) + \tau^2\Big(\rho_2|u^{(2)}|^2 + I_0^{(2)}|\phi^{(2)}|^{(2)} + j_0^{(2)}|\varphi^{(2)}|^{(2)} + a^{(2)}|\vartheta^{(2)}|^2\Big).\end{aligned}$$

Now if we repeat the reasoning in Theorem 4.1 and Theorem 4.2 of [7] (see also [6, Section 5], we get

$$\begin{gathered}u^{(1)} = 0, \ \ \vartheta^{(1)} = 0 \ \text{ in } \Omega_{\Sigma_1},\\ u^{(2)} = 0, \ \ \phi^{(2)} = 0, \ \ \varphi^{(2)} = 0, \ \ \vartheta^{(2)} = 0, \ \ \psi^{(2)} = b \ \text{ in } \Omega_{\Sigma_2},\end{gathered}$$

where b is an arbitrary constant.

For the function $\psi^{(2)}$, from the Dirichlet homogeneous condition on the surface $S_D^{(2)}$ it follows that $b = 0$. Therefore, for the homogeneous transmission problem $(TM)_{c,\tau}$, we obtain

$$\begin{aligned}&U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top = 0 \ \text{ in } \Omega_{\Sigma_1},\\ &U^{(2)} = \left(u^{(2)}, \vartheta^{(2)}, \phi^{(2)}, \varphi^{(2)}, \psi^{(2)}\right)^\top = 0 \ \text{ in } \Omega_{\Sigma_2}.\end{aligned}$$

□

4 Properties of potentials and boundary operators

The single and double layer potentials are defined as follows (their properties see [6, 18]):

$$V_{S_1}^{(1)}(g)(x) = \int_{S_1} \Gamma^{(1)}(x-y)g(y)\,d_yS,$$

$$V_{S_1}^{(2)}(f)(x) = \int_{S_1} \Gamma^{(2)}(x-y)f(y)\,d_yS, \quad V_{S_2}^{(2)}(h)(x) = \int_{S_2} \Gamma^{(2)}(x-y)h(y)\,d_yS,$$

$$W_c^{(1)}(\chi^{(1)})(x) = \int_{\Sigma_1} \big[T^{(1)}(\partial_y, n(y), \tau)\Gamma^{(1)}(x-y)\big]^\top \chi^{(1)}(y)\,d_yS,$$

$$W_c^{(2)}(\chi^{(2)})(x) = \int_{\Sigma_2} \big[\widetilde{T}^{(2)}(\partial_y, \nu(y), \tau)[\Gamma^{(2)}(x-y)]^\top\big]^\top \chi^{(2)}(y)\,d_yS,$$

$$V_c^{(1)}(\Psi^{(1)})(x) = \int_{\Sigma_1} \Gamma^{(1)}(x-y)\Psi^{(1)}(y)\,d_yS, \quad V_c^{(2)}(\Psi^{(2)})(x) = \int_{\Sigma_2} \Gamma^{(2)}(x-y)\Psi^{(2)}(y)\,d_yS,$$

where $\Gamma^{(1)}(x-y)$ and $\Gamma^{(2)}(x-y)$ are the fundamental solutions of the differential operators $A^{(1)}(\partial_x, \tau)$ and $A^{(2)}(\partial_x, \tau)$, respectively (see [6, 18]).

The following theorem holds (see [6, 16, 18]).

Theorem 4.1. *Let* $g \in [H^{-\frac{1}{2}}(S_1)]^4$, $f \in [H^{-\frac{1}{2}}(S_1)]^9$, $h \in \big[H^{-\frac{1}{2}}(S_2)\big]^9$, $\chi^{(1)} \in [\widetilde{H}^{\frac{1}{2}}(\Sigma_1)]^4$, $\chi^{(2)} \in [\widetilde{H}^{\frac{1}{2}}(\Sigma_2)]^9$, $\Psi^{(1)} \in [\widetilde{H}^{-\frac{1}{2}}(\Sigma_1)]^4$, $\Psi^{(2)} \in [\widetilde{H}^{-\frac{1}{2}}(\Sigma_2)]^9$, *then the following jump relations hold:*

$$\begin{aligned}&\big\{T^{(1)}(\partial_x, n, \tau)V_{S_1}^{(1)}(g)\big\}^\pm = \Big(\mp\frac{1}{2}\,I_4 + \mathcal{K}_{S_1}^{(1)}\Big)g \ \text{ on } S_1,\\ &\big\{T^{(2)}(\partial_x, \nu, \tau)V_{S_1}^{(2)}(f)\big\}^\pm = \Big(\mp\frac{1}{2}\,I_9 + \mathcal{K}_{S_1}^{(2)}\Big)f \ \text{ on } S_1,\end{aligned}$$

$$\big\{T^{(2)}(\partial_x,\nu,\tau)V^{(2)}_{S_2}(h)\big\}^{\pm}=\Big(\mp\frac{1}{2}\,I_9+\mathcal{K}^{(2)}_{S_2}\Big)h\ \ on\ S_2,$$
$$\big\{T^{(1)}(\partial_x,n,\tau)V^{(1)}_c(\Psi^{(1)})\big\}^{\pm}=\Big(\mp\frac{1}{2}\,I_4+\mathcal{K}^{(1)}_c\Big)\Psi^{(1)}\ \ on\ \Sigma_1,$$
$$\big\{T^{(2)}(\partial_x,\nu,\tau)V^{(2)}_c(\Psi^{(2)})\big\}^{\pm}=\Big(\mp\frac{1}{2}\,I_9+\mathcal{K}^{(2)}_c\Big)\Psi^{(2)}\ \ on\ \Sigma_2,$$
$$\big\{W^{(1)}_c(\chi^{(1)})\big\}^{\pm}=\Big(\pm\frac{1}{2}\,I_4+\mathcal{N}^{(1)}_c\Big)\chi^{(1)}\ \ on\ \Sigma_1,$$
$$\big\{W^{(2)}_c(\chi^{(2)})\big\}^{\pm}=\Big(\pm\frac{1}{2}\,I_9+\mathcal{N}^{(2)}_c\Big)\chi^{(2)}\ \ on\ \Sigma_2,$$

where

$$\mathcal{K}^{(1)}_{S_1}(g)(z)=\int\limits_{S_1}T^{(1)}(\partial_z,n(z),\tau)\Gamma^{(1)}(z-y)g(y)\,d_yS,\ \ z\in S_1,$$
$$\mathcal{K}^{(2)}_{S_1}(f)(z)=\int\limits_{S_1}T^{(2)}(\partial_z,\nu(z),\tau)\Gamma^{(2)}(z-y)f(y)\,d_yS,\ \ z\in S_1,$$
$$\mathcal{K}^{(2)}_{S_2}(h)(z)=\int\limits_{S_2}T^{(2)}(\partial_z,\nu(z),\tau)\Gamma^{(2)}(z-y)h(y)\,d_yS,\ \ z\in S_2,$$
$$\mathcal{K}^{(1)}_{c}(\Psi^{(1)})(z)=\int\limits_{\Sigma_1}T^{(1)}(\partial_z,n(z),\tau)\Gamma^{(1)}(z-y)\Psi^{(1)}(y)\,d_yS,\ \ z\in \Sigma_1,$$
$$\mathcal{K}^{(2)}_{c}(\Psi^{(2)})(z)=\int\limits_{\Sigma_2}T^{(1)}(\partial_z,\nu(z),\tau)\Gamma^{(2)}(z-y)\Psi^{(2)}(y)\,d_yS,\ \ z\in \Sigma_2,$$
$$\mathcal{N}^{(1)}_{c}(\chi^{(1)})(z)=\int\limits_{\Sigma_1}\big[T^{(1)}(\partial_y,n(y),\tau)\Gamma^{(1)}(z-y)\chi^{(1)}(y)\big]^{\top}\,d_yS,\ \ z\in \Sigma_1,$$
$$\mathcal{N}^{(2)}_{c}(\chi^{(2)})(z)=\int\limits_{\Sigma_2}\Big[\widetilde{T}^{(1)}(\partial_y,\nu(y),\tau)\big[\Gamma^{(2)}(z-y)\chi^{(2)}(y)\big]^{\top}\Big]^{\top}\,d_yS,\ \ z\in \Sigma_2,$$

and

$$\mathcal{H}^{(1)}_{S_1}(g)(z)=\{V^{(1)}_{S_1}(g)(z)\}^{+}=\{V^{(1)}_{S_1}(g)(z)\}^{-},\ \ z\in S_1,$$
$$\mathcal{H}^{(2)}_{S_1}(f)(z)=\{V^{(2)}_{S_1}(f)(z)\}^{+}=\{V^{(2)}_{S_1}(f)(z)\}^{-},\ \ z\in S_1,$$
$$\mathcal{H}^{(2)}_{S_2}(h)(z)=\{V^{(2)}_{S_2}(h)(z)\}^{+}=\{V^{(2)}_{S_2}(h)(z)\}^{-},\ \ z\ \ S_2,$$
$$\mathcal{L}^{(1)}_{c}(g)(z)=\big\{T^{(1)}W^{(1)}_c(g)(z)\big\}^{+}=\big\{T^{(1)}W^{(1)}_c(g)(z)\big\}^{-},\ \ z\in \Sigma_1,$$
$$\mathcal{L}^{(2)}_{c}(f)(z)=\big\{T^{(2)}W^{(2)}_c(f)(z)\big\}^{+}=\big\{T^{(2)}W^{(2)}_c(f)(z)\big\}^{-},\ \ z\in \Sigma_2.$$

Here, we collect some theorems describing the mapping properties of potentials and corresponding boundary (pseudodifferential) operators. The proof of these theorems can be found in [1, 6, 10, 13, 14, 17–19].

Theorem 4.2. *Let $s\in\mathbb{R}$. Then the single- and double-layer potentials can be extended to the continuous operators*

$$V^{(1)}_{S_1}:\big[H^s(S_1)\big]^4\to\big[H^{s+\frac{3}{2}}(\Omega_1)\big]^4,$$
$$V^{(2)}_{S_1}:\big[H^s(S_1)\big]^9\to\big[H^{s+\frac{3}{2}}(\Omega_1)\big]^9,\quad V^{(2)}_{S_2}:\big[H^s(S_2)\big]^9\to\big[H^{s+\frac{3}{2}}(\Omega_2)\big]^9,$$
$$V^{(1)}_{c}:\big[\widetilde{H}^s(\Sigma_1)\big]^4\to\big[H^{s+\frac{3}{2}}(\Omega_{\Sigma_1}\big]^4,\quad V^{(2)}_{c}:\big[\widetilde{H}^s(\Sigma_2)\big]^9\to\big[H^{s+\frac{3}{2}}(\Omega_{\Sigma_2}\big]^9,$$
$$W^{(1)}_{c}:\big[\widetilde{H}^s(\Sigma_1)\big]^4\to\big[H^{s+\frac{1}{2}}(\Omega_{\Sigma_1}\big]^4,\quad W^{(2)}_{c}:\big[\widetilde{H}^s(\Sigma_2)\big]^9\to\big[H^{s+\frac{1}{2}}(\Omega_{\Sigma_2}\big]^9.$$

Theorem 4.3. *Let $s \in \mathbb{R}$. Then the pseudodifferential operators of order -1*

$$\mathcal{H}^{(1)}_{S_1} : \big[H^s(S_1)\big]^4 \to \big[H^{s+1}(S_1)\big]^4,$$
$$\mathcal{H}^{(2)}_{S_1} : \big[H^s(S_1)\big]^9 \to \big[H^{s+1}(S_1)\big]^9, \quad \mathcal{H}^{(2)}_{S_2} : \big[H^s(S_2)\big]^9 \to \big[H^{s+1}(S_2)\big]^9$$

are invertible and the pseudodifferential operators of order 1

$$r_{\Sigma_1}\mathcal{L}^{(1)}_c : \big[\widetilde{H}^s(\Sigma_1)\big]^4 \to \big[H^{s-1}(\Sigma_1)\big]^4, \quad r_{\Sigma_2}\mathcal{L}^{(2)}_c : \big[\widetilde{H}^s(\Sigma_2)\big]^9 \to \big[H^{s-1}(\Sigma_2)\big]^9$$

are invertible.

Theorem 4.4. *Let $s \in \mathbb{R}$. Then the singular integral operators*

$$\begin{aligned}
\mathcal{K}^{(1)}_{S_1} &: \big[H^s(S_1)\big]^4 \to \big[H^s(S_1)\big]^4,\\
\mathcal{K}^{(2)}_{S_1} &: \big[H^s(S_1)\big]^9 \to \big[H^s(S_1)\big]^9,\\
\mathcal{K}^{(2)}_{S_2} &: \big[H^s(S_2)\big]^9 \to \big[H^s(S_2)\big]^9,\\
\mathcal{K}^{(1)}_{c} &: \big[\widetilde{H}^s(\Sigma_1)\big]^4 \to \big[H^s(S^{(1)}_0)\big]^4,\\
\mathcal{K}^{(2)}_{c} &: \big[\widetilde{H}^s(\Sigma_2)\big]^9 \to \big[H^s(S^{(2)}_0)\big]^9
\end{aligned}$$

are continuous.

5 Existence of a solution to the mixed type boundary-transmission problem $(TM)_{c,\tau}$ of pseudo-oscillations

We are looking for a solution to the mixed boundary-transmission problem $(TM)_{c,\tau}$ in the form of the following single layer potentials:

$$\begin{aligned}
U^{(1)} &= V^{(1)}_{S_1}(\mathcal{H}^{(1)}_{S_1})^{-1}g^{(1)} + W^{(1)}_c\chi^{(1)} + V^{(1)}_c\Psi^{(1)} \ \text{ in } \Omega_{\Sigma_1},\\
U^{(2)} &= V^{(2)}_{S_1}(\mathcal{H}^{(2)}_{S_1})^{-1}g^{(2)} + V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}h^{(2)} + W^{(2)}_c\chi^{(2)} + V^{(2)}_c\Psi^{(2)} \ \text{ in } \Omega_{\Sigma_2},
\end{aligned}$$

where the unknown densities $g^{(1)}$, $g^{(2)}$, $h^{(2)}$, $\chi^{(1)}$, $\chi^{(2)}$, $\Psi^{(1)}$ and $\Psi^{(2)}$ belong to the following Sobolev spaces:

$$\begin{gathered}
g^{(1)} = (g^{(1)}_1, \dots, g^{(1)}_4)^\top \in [H^{\frac12}(S_1)]^4, \quad g^{(2)} = (g^{(2)}_1, \dots, g^{(2)}_9)^\top \in [H^{\frac12}(S_1)]^9,\\
h^{(2)} = (h^{(2)}_1, \dots, h^{(2)}_9)^\top \in [H^{\frac12}(S_2)]^9, \quad \chi^{(1)} = (\chi^{(1)}_1, \dots, \chi^{(1)}_4)^\top \in [\widetilde{H}^{\frac12}(\Sigma_1)]^4,\\
\chi^{(2)} = (\chi^{(2)}_1, \dots, \chi^{(2)}_9)^\top \in [\widetilde{H}^{\frac12}(\Sigma_2)]^9, \quad \Psi^{(1)} = (\Psi^{(1)}_1, \dots, \Psi^{(1)}_4)^\top \in [\widetilde{H}^{-\frac12}(\Sigma_1)]^4,\\
\Psi^{(2)} = (\Psi^{(2)}_1, \dots, \Psi^{(2)}_9)^\top \in [\widetilde{H}^{-\frac12}(\Sigma_2)]^9.
\end{gathered}$$

Let us note that the boundary conditions on crack faces Σ_k, $k = 1, 2$, (3.8) and (3.11) can be transformed equivalently as (see [5, 6])

$$\begin{aligned}
\{T^{(1)}U^{(1)}\}^+_j - \{T^{(1)}U^{(1)}\}^-_j &= F^{(1),+}_j - F^{(1),-}_j \ \text{ on } \Sigma_1, \ \ j = \overline{1,3},\\
\{T^{(1)}U^{(1)}\}^+_j + \{T^{(1)}U^{(1)}\}^-_j &= F^{(1),+}_j + F^{(1),-}_j \ \text{ on } \Sigma_1, \ \ j = \overline{1,3},\\
\{T^{(2)}U^{(2)}\}^+_j - \{T^{(2)}U^{(2)}\}^-_j &= F^{(2),+}_j - F^{(2),-}_j \ \text{ on } \Sigma_2, \ \ j = 1,2,3,5,6,7,\\
\{T^{(2)}U^{(2)}\}^+_j + \{T^{(2)}U^{(2)}\}^-_j &= F^{(2),+}_j + F^{(2),-}_j \ \text{ on } \Sigma_2, \ \ j = 1,2,3,5,6,7.
\end{aligned}$$

Therefore, the boundary and boundary-transmission conditions (3.1)–(3.13) of the problem $(TM)_{c,\tau}$ can be rewritten as the transmission conditions on the surface S_1:

$$\{u^{(1)}_j\}^+ - \{u^{(2)}_j\}^+ = f^{(1)}_j \ \text{ on } S_1, \ \ j = \overline{1,4},$$

$$\{T^{(1)}U^{(1)}\}_j^+ + \{T^{(2)}U^{(2)}\}_j^+ = f_j^{(2)} \ \text{ on } S_1, \ \ j=\overline{1,4}, \ \ \nu=-n,$$

boundary conditions on the surface S_1:

$$\{u_j^{(2)}\}^+ = Q_j^{(2)} \ \text{ on } S_1, \ \ j=\overline{5,9},$$

mixed boundary conditions on the surface S_2:

$$\begin{aligned}\{U^{(2)}\}^+ &= p_2^{(D)} \ \text{ on } S_2^{(D)},\\ \{T^{(2)}U^{(2)}\}^+ &= q_2^{(N)} \ \text{ on } S_2^{(N)},\end{aligned}$$

crack boundary conditions on Σ_1:

$$\begin{aligned}\{T^{(1)}U^{(1)}\}_j^+ - \{T^{(1)}U^{(1)}\}_j^- &= F_j^{(1),+} - F_j^{(1),-} \ \text{ on } \Sigma_1, \ \ j=\overline{1,3},\\ \{T^{(1)}U^{(1)}\}_j^+ + \{T^{(1)}U^{(1)}\}_j^- &= F_j^{(1),+} + F_j^{(1),-} \ \text{ on } \Sigma_1, \ \ j=\overline{1,3},\\ \{u_4^{(1)}\}^+ - \{u_4^{(1)}\}^- &= G_4^{(1)} \ \text{ on } \Sigma_1,\\ \{T^{(1)}U^{(1)}\}_4^+ - \{T^{(1)}U^{(1)}\}_4^- &= F_4^{(1)} \ \text{ on } \Sigma_1,\end{aligned}$$

crack boundary conditions on Σ_2:

$$\begin{aligned}\{T^{(2)}U^{(2)}\}_j^+ - \{T^{(2)}U^{(2)}\}_j^- &= F_j^{(2),+} - F_j^{(2),-} \ \text{ on } \Sigma_2, \ \ j=1,2,3,5,6,7,\\ \{T^{(2)}U^{(2)}\}_j^+ + \{T^{(2)}U^{(2)}\}_j^- &= F_j^{(2),+} + F_j^{(2),-} \ \text{ on } \Sigma_2, \ \ j=1,2,3,5,6,7,\\ \{u_j^{(2)}\}^+ - \{u_j^{(2)}\}^- &= G_j^{(2)} \ \text{ on } \Sigma_2, \ \ j=4,8,9,\\ \{T^{(2)}U^{(2)}\}_j^+ - \{T^{(2)}U^{(2)}\}_j^- &= F_j^{(2)} \ \text{ on } \Sigma_2, \ \ j=4,8,9.\end{aligned}$$

Taking into account the boundary and boundary-transmission conditions of the mixed type problem $(TM)_{c,\tau}$, we obtain the following system of equations with respect to the vector functions $g^{(1)}$, $g^{(2)}$, $h^{(2)}$, $\chi^{(1)}$, $\chi^{(2)}$, $\Psi^{(1)}$ and $\Psi^{(2)}$:

$$g_j^{(2)} + r_{S_1}\big[V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}h^{(2)}\big]_j + r_{S_1}[W_c^{(2)}\chi^{(2)}]_j + r_{S_1}[V_c^{(2)}\Psi^{(2)}]_j = Q_j^{(2)} \ \text{ on } S_1, \ \ j=\overline{5,9}, \tag{5.1}$$

$$\begin{aligned}g_j^{(1)} - g_j^{(2)} &- r_{S_1}[V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j + r_{S_1}[W_c^{(1)}\chi^{(1)}]_j + r_{S_1}[V_c^{(1)}\Psi^{(1)}]_j\\ &- r_{S_1}[W_c^{(2)}\chi^{(2)}]_j - r_{S_1}[V_c^{(2)}\Psi^{(2)}]_j = f_j^{(1)} \ \text{ on } S_1, \ \ j=\overline{1,4},\end{aligned} \tag{5.2}$$

$$\begin{aligned}\Big[\Big(-\frac{1}{2}\,I_4 + \mathcal{K}_{S_1}^{(1)}\Big)(\mathcal{H}_{S_1}^{(1)})^{-1}g^{(1)}\Big]_j &+ \Big[\Big(-\frac{1}{2}\,I_9 + \mathcal{K}_{S_1}^{(2)}\Big)(\mathcal{H}_{S_1}^{(2)})^{-1}g^{(2)}\Big]_j\\ &+ r_{S_1}[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j + r_{S_1}[T^{(1)}W_c^{(1)}\chi^{(1)}]_j + r_{S_1}[T^{(1)}V_c^{(1)}\Psi^{(1)}]_j\\ &+ r_{S_1}\big[T^{(2)}W_c^{(2)}\chi^{(2)}\big]_j + r_{S_1}\big[T^{(2)}V_c^{(2)}\Psi^{(2)}\big]_j = f_j^{(2)} \ \text{ on } S_1, \ \ j=\overline{1,4},\end{aligned} \tag{5.3}$$

$$r_{S_2^{(D)}}\big[V_{S_1}^{(2)}(\mathcal{H}_{S_1}^{(2)})^{-1}g^{(2)}\big] + r_{S_2^{(D)}}h^{(2)} + r_{S_2^{(D)}}[W_c^{(2)}\chi^{(2)}] + r_{S_2^{(D)}}[V_c^{(2)}\Psi^{(2)}] = p_2^{(D)} \ \text{ on } S_2^{(D)}, \tag{5.4}$$

$$\begin{aligned}r_{S_2^{(N)}}\big[T^{(2)}V_{S_1}^{(2)}(\mathcal{H}_{S_1}^{(2)})^{-1}g^{(2)}\big] &+ r_{S_2^{(N)}}\Big(-\frac{1}{2}\,I_9 + \mathcal{K}_{S_2}^{(2)}\Big)(\mathcal{H}_{S_2}^{(2)})^{-1}h^{(2)}\\ &+ r_{S_2^{(N)}}[T^{(2)}W_c^{(2)}\chi^{(2)}] + r_{S_2^{(N)}}[T^{(2)}V_c^{(2)}\Psi^{(2)}] = q_2^{(N)} \ \text{ on } S_2^{(N)},\end{aligned} \tag{5.5}$$

$$\begin{aligned}r_{\Sigma_1}\big[T^{(1)}V_{S_1}^{(1)}(\mathcal{H}_{S_1}^{(1)})^{-1}g^{(1)}\big]_j &+ r_{\Sigma_1}[\mathcal{L}_c^{(1)}\chi^{(1)}]_j + r_{\Sigma_1}\big[\mathcal{K}_c^{(1)}\Psi^{(1)}\big]_j\\ &= 2^{-1}(F_j^{(1),+} + F_j^{(1),-}) \ \text{ on } \Sigma_1, \ \ j=\overline{1,3},\end{aligned} \tag{5.6}$$

$$\begin{aligned}r_{\Sigma_2}\big[T^{(2)}V_{S_1}^{(2)}(\mathcal{H}_{S_1}^{(2)})^{-1}g^{(2)}\big]_j &+ r_{\Sigma_2}\big[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}h^{(2)}\big]_j + r_{\Sigma_2}[\mathcal{L}_c^{(2)}\chi^{(2)}]_j + r_{\Sigma_2}\big[\mathcal{K}_c^{(2)}\Psi^{(2)}\big]_j\\ &= 2^{-1}(F_j^{(2),+} + F_j^{(2),-}) \ \text{ on } \Sigma_2, \ \ j=1,2,3,5,6,7,\end{aligned} \tag{5.7}$$

where

$$\Psi_j^{(1)} = -(F_j^{(1),+} - F_j^{(1),-}) \ \text{ on } \Sigma_1, \ \ j=\overline{1,3}, \tag{5.8}$$

$$\Psi_4^{(1)} = -F_4^{(1)}, \quad \chi_4^{(1)} = G_4^{(1)} \ \text{ on } \Sigma_1, \tag{5.9}$$
$$\Psi_j^{(2)} = -(F_j^{(2),+} - F_j^{(2),-}) \ \text{ on } \Sigma_2, \ \ j = 1,2,3,5,6,7, \tag{5.10}$$
$$\Psi_j^{(2)} = -F_j^{(2)}, \quad \chi_j^{(2)} = G_j^{(2)} \ \text{ on } \Sigma_2, \ \ j = 4,8,9. \tag{5.11}$$

We introduce the notation

$$\chi_0^{(1)} := (\chi_1^{(1)}, \chi_2^{(1)}, \chi_3^{(1)})^\top, \quad \chi_0^{(2)} := (\chi_1^{(2)}, \chi_2^{(2)}, \chi_3^{(2)}, \chi_5^{(2)}, \chi_6^{(2)}, \chi_7^{(2)})^\top,$$
$$\overline{\chi}_0^{(1)} := (0,0,0,\chi_4^{(1)})^\top, \quad \overline{\chi}_0^{(2)} := (0,0,0,\chi_4^{(2)},0,0,0,\chi_8^{(2)},\chi_9^{(2)})^\top.$$

As we see, the sought for densities $\Psi^{(1)}$, $\Psi^{(2)}$, the forth component of the vector $\chi^{(1)}$, and the forth, eighth and ninth components of the vector $\chi^{(2)}$ are determined explicitly by the data of the transmission problem. Hence it remains to find the densities $g^{(1)}$, $g^{(2)}$, $h^{(2)}$ and $\chi_0^{(1)}$, $\chi_0^{(2)}$.

Equation (5.4) can be rewritten as follows:

$$r_{S_2} V_{S_1} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)} + h^{(2)} + r_{S_2}[W_c^{(2)} \chi^{(2)}] + r_{S_2}[V_c^{(2)} \Psi^{(2)}] = \Phi_0^{(2)} + h_0^{(2)} \ \text{ on } S_2, \tag{5.12}$$

where $\Phi_0^{(2)} \in [H^{\frac12}(S_2)]^9$ is a fixed extension of the Dirichlet condition, the vector function $p_2^{(D)} \in [H^{\frac12}(S_2^{(D)})]^9$ over the entire surface S_2, and $h_0^{(2)} \in [\widetilde{H}^{\frac12}(S_2^{(N)})]^9$, $\operatorname{supp} h_0^{(2)} \subset \overline{S}_2^{(N)}$. Let us determine $h^{(2)}$ from equations (5.12) in the following way:

$$h^{(2)} = \Phi_0^{(2)} + h_0^{(2)} - r_{S_2}\big[V_{S_1}^{(2)} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\big] - r_{S_2}[W_c^{(2)} \chi^{(2)}] - r_{S_2}[V_c^{(2)} \Psi^{(2)}]$$

and insert it into equations (5.1), (5.2), (5.3), (5.5) and (5.7) of system (5.1)–(5.7). At the same time, we change the places of equations (5.1) and (5.2), and multiply equation (5.1) by -1. In this case, we get the following equivalent system of equations with respect to the vector functions $g^{(1)}$, $g^{(2)}$, $h_0^{(2)}$, $\widehat{\chi}^{(1)} := (\chi_0^{(1)}, 0)^\top$ and $\widehat{\chi}^{(2)} := (\chi_1^{(2)}, \chi_2^{(2)}, \chi_3^{(2)}, 0, \chi_5^{(2)}, \chi_6^{(2)}, \chi_7^{(2)}, 0, 0)^\top$:

$$\begin{aligned} & g_j^{(1)} - g_j^{(2)} + r_{S_1}\Big[V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} \big(r_{S_2} V_{S_1}^{(2)} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\big)\Big]_j - r_{S_1}\big[V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} h_0^{(2)}\big]_j + r_{S_1}\big[W_c^{(1)} \widehat{\chi}^{(1)}\big]_j \\ & \qquad - r_{S_1}\big[W_c^{(2)} \widehat{\chi}^{(2)}\big]_j + r_{S_1}\Big[V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} \big(r_{S_2}[W_c^{(2)} \widehat{\chi}^{(2)}]\big)\Big]_j = \widetilde{f}_j^{(1)} \ \text{ on } S_1, \ \ j = \overline{1,4}, \end{aligned} \tag{5.13}$$

$$\begin{aligned} & -g_j^{(2)} + r_{S_1}\Big[V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} \big(r_{S_2} V_{S_1}^{(2)} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\big)\Big]_j - r_{S_1}\big[V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} h_0^{(2)}\big]_j - r_{S_1}[W_c^{(2)} \widehat{\chi}^{(2)}]_j \\ & \qquad + r_{S_1}\Big[V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} \big(r_{S_2}[W_c^{(2)} \widehat{\chi}^{(2)}]\big)\Big]_j = \widetilde{Q}_j^{(2)} \ \text{ on } S_1, \ \ j = \overline{5,9}, \end{aligned} \tag{5.14}$$

$$\begin{aligned} & \Big[\Big(-\frac12 I_4 + \mathcal{K}_{S_1}^{(1)}\Big)(\mathcal{H}_{S_1}^{(1)})^{-1} g^{(1)}\Big]_j + \Big[\Big(-\frac12 I_9 + \mathcal{K}_{S_1}^{(2)}\Big)(\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\Big]_j \\ & \qquad - r_{S_1}\Big[T^{(2)} V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} \big(r_{S_2} V_{S_1}^{(2)} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\big)\Big]_j + r_{S_1}\big[T^{(2)} V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} h_0^{(2)}\big]_j \\ & \qquad + r_{S_1}\big[T^{(1)} W_c^{(1)} \widehat{\chi}^{(1)}\big]_j + r_{S_1}\big[T^{(2)} W_c^{(2)} \widehat{\chi}^{(2)}\big]_j \\ & \qquad - r_{S_1}\big[T^{(2)} V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} r_{S_2} (W_c^{(2)} \widehat{\chi}^{(2)})\big]_j = \widetilde{f}_j^{(2)} \ \text{ on } S_1, \ \ j = \overline{1,4}, \end{aligned} \tag{5.15}$$

$$\begin{aligned} & r_{S_2^{(N)}}\big[T^{(2)} V_{S_1}^{(2)} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\big] - r_{S_2^{(N)}}\Big[\Big(-\frac12 I_9 + \mathcal{K}_{S_2}^{(2)}\Big)(\mathcal{H}_{S_2}^{(2)})^{-1} \big(r_{S_2} V_{S_1}^{(2)} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\big)\Big] \\ & \qquad + r_{S_2^{(N)}}\Big(-\frac12 I_9 + \mathcal{K}_{S_2}^{(2)}\Big)(\mathcal{H}_{S_2}^{(2)})^{-1} h_0^{(2)} - r_{S_2^{(N)}}\Big[\Big(-\frac12 I_9 + \mathcal{K}_{S_2}^{(2)}\Big)(\mathcal{H}_{S_2}^{(2)})^{-1} \big(r_{S_2}[W_c^{(2)} \widehat{\chi}^{(2)}]\big)\Big] \\ & \qquad + r_{S_2^{(N)}}\big[T^{(2)} W_c^{(2)} \widehat{\chi}^{(2)}\big] = \widetilde{q}_2^{(N)} \ \text{ on } S_2^{(N)}, \end{aligned} \tag{5.16}$$

$$r_{\Sigma_1}\big[T^{(1)} V_{S_1}^{(1)} (\mathcal{H}_{S_1}^{(1)})^{-1} g^{(1)}\big]_j + r_{\Sigma_1}\big[\mathcal{L}_c^{(1)} \widehat{\chi}^{(1)}\big]_j = \widetilde{F}_j^{(1)} \ \text{ on } \Sigma_1, \ \ j = \overline{1,3}, \tag{5.17}$$

$$\begin{aligned} & r_{\Sigma_2}\big[T^{(2)} V_{S_1}^{(2)} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\big]_j - r_{\Sigma_2}\Big[T^{(2)} V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} \big(r_{S_2} V_{S_1}^{(2)} (\mathcal{H}_{S_1}^{(2)})^{-1} g^{(2)}\big)\Big]_j \\ & \qquad + r_{\Sigma_2}\big[T^{(2)} V_{S_2}^{(2)} (\mathcal{H}_{S_2}^{(2)})^{-1} h_0^{(2)}\big]_j + r_{\Sigma_2}\big[\mathcal{L}_c^{(2)} \widehat{\chi}^{(2)}\big]_j \end{aligned}$$

$$-\,r_{\Sigma_2}\Big[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}(r_{S_2}W_c^{(2)}\widehat{\chi}^{(2)})\Big]_j=\widetilde{F}_j^{(2)}\ \text{ on } \Sigma_2,\ \ j=1,2,3,5,6,7, \tag{5.18}$$

where

$$\begin{aligned}
\widetilde{f}_j^{(1)} &:= f_j^{(1)} + r_{S_1}\big[V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}\Phi_0^{(2)}\big]_j - r_{S_1}[V_c^{(1)}\Psi^{(1)}]_j + r_{S_1}\big[V_c^{(2)}\Psi^{(2)}\big]_j - r_{S_1}[W_c^{(1)}\overline{\chi}_0^{(1)}]_j\\
&\qquad + r_{S_1}\big[W_c^{(2)}\overline{\chi}_0^{(2)}\big]_j - r_{S_1}\big[V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}(r_{S_2}V_c^{(2)}\Psi^{(2)})\big]_j\\
&\qquad - r_{S_1}\big[V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}(r_{S_2}W_c^{(2)}\overline{\chi}_0^{(2)})\big]_j,\ \ j=\overline{1,4},\\
\widetilde{Q}_j^{(2)} &:= -Q_j^{(2)} + r_{S_1}\big[V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}\Phi_0^{(2)}\big]_j + r_{S_1}[V_c^{(2)}\Psi^{(2)}]_j - r_{S_1}\big[V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}(r_{S_2}V_c^{(2)}\Psi^{(2)})\big]_j\\
&\qquad + r_{S_1}[W_c^{(2)}\overline{\chi}_0^{(2)}]_j - r_{S_1}\big[V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}(r_{S_2}W_c^{(2)}\overline{\chi}_0^{(2)})\big]_j,\ \ j=\overline{5,9},\\
\widetilde{f}_j^{(2)} &:= f_j^{(2)} - r_{S_1}\big[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}\Phi_0^{(2)}\big]_j + r_{S_1}\Big[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}\big([W_c^{(2)}\overline{\chi}_0^{(2)}]\big)\Big]_j\\
&\qquad + r_{S_1}\Big[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}([V_c^{(2)}\Psi^{(2)}]_j) - r_{S_1}[T^{(1)}W_c^{(1)}\overline{\chi}_0^{(1)}]\Big]_j\\
&\qquad - r_{S_1}\big[T^{(1)}V_c^{(1)}\Psi^{(1)}\big]_j - r_{S_1}\big[T^{(2)}W_c^{(2)}\overline{\chi}_0^{(2)}\big]_j - r_{S_1}\big[T^{(2)}V_c^{(2)}\Psi^{(2)}\big]_j,\ \ j=\overline{1,4},\\
\widetilde{q}_2^{(N)} &:= q_2^{(N)} - r_{S_2^{(N)}}\Big(-\frac12 I_9+\mathcal{K}_{S_2}^{(2)}\Big)(\mathcal{H}_{S_2}^{(2)})^{-1}\Phi_0^{(2)} + r_{S_2^{(N)}}\Big[\Big(-\frac12 I_9+\mathcal{K}_{S_2}^{(2)}\Big)(\mathcal{H}_{S_2}^{(2)})^{-1}r_{S_2}\big[W_c^{(2)}\overline{\chi}_0^{(2)}\big]\Big]\\
&\qquad + r_{S_2^{(N)}}\Big[\Big(-\frac12 I_9+\mathcal{K}_{S_2}^{(2)}\Big)(\mathcal{H}_{S_2}^{(2)})^{-1}r_{S_2}[V_c^{(2)}\Psi^{(2)}]\Big]\\
&\qquad - r_{S_2^{(N)}}\big[T^{(2)}W_c^{(2)}\overline{\chi}_0^{(2)}\big] - r_{S_2^{(N)}}\big[T^{(2)}V_c^{(2)}\Psi^{(2)}\big],\\
\widetilde{F}_j^{(1)} &:= 2^{-1}(F_j^{(1),+}+F_j^{(1),-}) - r_{\Sigma_1}\big[\mathcal{K}_c^{(1)}\Psi^{(1)}\big]_j - r_{\Sigma_1}\big[\mathcal{L}_c^{(1)}\overline{\chi}_0^{(1)}\big],\ \ j=\overline{1,3},\\
\widetilde{F}_j^{(2)} &:= 2^{-1}(F_j^{(2),+}+F_j^{(2),-}) - r_{\Sigma_2}\big[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}\Phi_0^{(2)}\big]_j\\
&\qquad + r_{\Sigma_2}\Big[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}\big(r_{S_2}[W_c^{(2)}\overline{\chi}_0^{(2)}]\big)\Big]_j - r_{\Sigma_2}\big[\mathcal{L}_c^{(2)}\overline{\chi}_0^{(2)}\big]\\
&\qquad - r_{\Sigma_2}\Big[T^{(2)}V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}(r_{S_2}[V_{S_2}\Psi^{(2)}])\Big]_j - r_{\Sigma_2}\big[\mathcal{K}_c^{(2)}\Psi^{(2)}\big]_j,\ \ j=1,2,3,5,6,7.
\end{aligned}$$

System (5.13)–(5.18) can be rewritten in the following matrix form:

$$\mathcal{P}_c(g^{(1)},g^{(2)},h_0^{(2)},\chi_0^{(1)},\chi_0^{(2)})^\top=\mathcal{F}, \tag{5.19}$$

where

$$\mathcal{F}:=\Big(\widetilde{f}_1^{(1)},\dots,\widetilde{f}_4^{(1)},\widetilde{Q}_5^{(2)},\dots,\widetilde{Q}_9^{(2)},\widetilde{f}_1^{(2)},\dots,\widetilde{f}_4^{(2)},\widetilde{q}_2^{(N)},\\
\widetilde{F}_1^{(1)},\widetilde{F}_2^{(1)},\widetilde{F}_3^{(1)},\widetilde{F}_1^{(2)},\widetilde{F}_2^{(2)},\widetilde{F}_3^{(2)},\widetilde{F}_5^{(2)},\widetilde{F}_6^{(2)},\widetilde{F}_7^{(2)}\Big)^\top,$$

$$\mathcal{P}_c:=\begin{bmatrix}
I_{9\times4} & -I_9+\mathcal{B} & \mathcal{T} & \mathcal{B}_{\Sigma_1} & \mathcal{T}_{\Sigma_2}\\
\mathcal{A}_{S_1}^{(1)} & [\mathcal{A}_{S_1}^{(2)}]_{4\times9}+\mathcal{C} & \mathcal{R} & \mathcal{D}_{\Sigma_1} & \mathcal{R}_{\Sigma_2}\\
[0]_{9\times4} & \mathcal{D} & r_{S_2^{(N)}}\mathcal{A}_{S_2}^{(2)} & [0]_{9\times3} & \mathcal{N}_{\Sigma_2}\\
\mathcal{M}_{\Sigma_1} & [0]_{3\times9} & [0]_{3\times9} & r_{\Sigma_1}[\widetilde{\mathcal{L}}_c^{(1)}]_{3\times3} & [0]_{3\times6}\\
[0]_{6\times4} & \mathcal{M}_{\Sigma_2} & \mathcal{D}_{\Sigma_2} & [0]_{6\times3} & r_{\Sigma_2}[\widetilde{\mathcal{L}}_c^{(2)}]_{6\times6}+\mathcal{C}_{\Sigma_2}
\end{bmatrix}_{31\times31},$$

here,

$$I_{9\times4}:=\begin{bmatrix}
1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0
\end{bmatrix}^\top,$$

the operators

$$\mathcal{A}_{S_1}^{(1)}:=\Big(-\frac12 I_4+\mathcal{K}_{S_1}^{(1)}\Big)(\mathcal{H}_{S_1}^{(1)})^{-1},\ \ \mathcal{A}_{S_1}^{(2)}:=\Big(-\frac12 I_9+\mathcal{K}_{S_1}^{(2)}\Big)(\mathcal{H}_{S_1}^{(2)})^{-1},\ \ \mathcal{A}_{S_2}^{(2)}:=\Big(-\frac12 I_9+\mathcal{K}_{S_2}^{(2)}\Big)(\mathcal{H}_{S_2}^{(2)})^{-1}$$

are the Poincaré–Steklov type operators (see [6, 17, 18]), which are strongly elliptic pseudodifferential operators of order 1,

$$\mathcal{B} := r_{S_1}\Big[V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}(r_{S_2}V^{(2)}_{S_1}(\mathcal{H}^{(2)}_{S_1})^{-1})\Big]_{9\times 9},\ \ \mathcal{C} := -r_{S_1}\Big[T^{(2)}V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}(r_{S_2}V^{(2)}_{S_1}(\mathcal{H}^{(2)}_{S_1})^{-1})\Big]_{4\times 9},$$

$$\mathcal{D} := r_{S_2^{(N)}}\big[T^{(2)}V^{(2)}_{S_1}(\mathcal{H}^{(2)}_{S_1})^{-1}\big]_{9\times 9} - r_{S_2^{(N)}}\big[\mathcal{A}^{(2)}_{S_2}(r_{S_2}V^{(2)}_{S_1}(\mathcal{H}^{(2)}_{S_1})^{-1})\big]_{9\times 9},$$

$$\mathcal{T} := -r_{S_1}\big[V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}\big]_{9\times 9},\quad \mathcal{R} := r_{S_1}\big[T^{(2)}V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}\big]_{4\times 9},\quad \mathcal{B}_{\Sigma_1} := \begin{bmatrix} r_{S_1}[W^{(1)}_c]_{4\times 3} \\ [0]_{5\times 3}\end{bmatrix}_{9\times 3},$$

$$\mathcal{D}_{\Sigma_1} := r_{S_1}[T^{(1)}W^{(1)}_c]_{4\times 3},\quad \mathcal{T}_{\Sigma_2} := r_{S_1}\Big[V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}r_{S_2}[W^{(2)}_c]\Big]_{9\times 6} - r_{S_1}[W^{(2)}_c]_{9\times 6},$$

$$\mathcal{R}_{\Sigma_2} := -r_{S_1}\Big[T^{(2)}V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}r_{S_2}[W^{(2)}_c]\Big]_{4\times 6} + r_{S_1}[T^{(2)}W^{(2)}_c]_{4\times 6},$$

$$\mathcal{N}_{\Sigma_2} := -r_{S_2^{(N)}}\Big[\mathcal{A}^{(2)}_{S_2}(r_{S_2}[W^{(2)}_c])\Big]_{9\times 6} + r_{S_2^{(N)}}[W^{(2)}_c]_{9\times 6},$$

$$\mathcal{C}_{\Sigma_2} := r_{\Sigma_2}\big[T^{(2)}V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}r_{S_2}[W^{(2)}_c]\big]_{6\times 6},\quad \mathcal{D}_{\Sigma_2} := r_{\Sigma_2}\big[T^{(2)}V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}\big]_{6\times 9},$$

$$\mathcal{M}_{\Sigma_1} := r_{\Sigma_1}\big[T^{(1)}V^{(1)}_{S_1}(\mathcal{H}^{(1)}_{S_1})^{-1}\big]_{3\times 4},$$

$$\mathcal{M}_{\Sigma_2} := r_{\Sigma_2}\big[T^{(2)}V^{(2)}_{S_1}(\mathcal{H}^{(2)}_{S_1})^{-1}\big]_{6\times 9} - r_{\Sigma_2}\big[T^{(2)}V^{(2)}_{S_2}(\mathcal{H}^{(2)}_{S_2})^{-1}r_{S_2}(V^{(2)}_{S_1}(\mathcal{H}^{(2)}_{S_1})^{-1})\big]_{6\times 9},$$

$$[\widetilde{\mathcal{L}}^{(2)}_c]_{6\times 6} := \big[[\mathcal{L}^{(2)}_c]_{jk}\big]_{6\times 6},\quad j,k = 1,2,3,5,6,7,$$

$$[\widetilde{\mathcal{L}}^{(1)}_c]_{3\times 3} := \big[[\mathcal{L}^{(1)}_c]_{jk}\big]_{3\times 3},\quad j,k = 1,2,3.$$

The operator $\mathcal{P}_c : \mathbf{X} \to \mathbf{Y}$ is bounded, where

$$\mathbf{X} := [H^{\frac12}(S_1)]^{13} \times [\widetilde{H}^{\frac12}(S_2^{(N)})]^9 \times [\widetilde{H}^{\frac12}(\Sigma_1)]^3 \times [\widetilde{H}^{\frac12}(\Sigma_2)]^6,$$

$$\mathbf{Y} := [H^{\frac12}(S_1)]^9 \times [H^{-\frac12}(S_1)]^4 \times [H^{-\frac12}(S_2^{(N)})]^9 \times [\widetilde{H}^{-\frac12}(\Sigma_1)]^3 \times [\widetilde{H}^{-\frac12}(\Sigma_2)]^6.$$

The following theorem holds.

Theorem 5.1. *The operator $\mathcal{P}_c : \mathbf{X} \to \mathbf{Y}$ is invertible.*

Proof. First, we show that the operator $\mathcal{P}_c : \mathbf{X} \to \mathbf{Y}$ is Fredholm with zero index.

Indeed, obviously, the operators

$$\mathcal{B} : [H^{\frac12}(S_1)]^9 \to [H^{\frac12}(S_1)]^9,\quad \mathcal{T}_{\Sigma_2} : [\widetilde{H}^{\frac12}(\Sigma_2]^6 \to [H^{\frac12}(S_1)]^9,$$

$$\mathcal{C} : [H^{\frac12}(S_1)]^9 \to [H^{-\frac12}(S_1)]^4,\quad \mathcal{R}_{\Sigma_2} : [\widetilde{H}^{\frac12}(\Sigma_2]^6 \to [H^{-\frac12}(S_1)]^4,$$

$$\mathcal{D} : [H^{\frac12}(S_1)]^9 \to [H^{-\frac12}(S_2^{(N)})]^9,\quad \mathcal{R} : [\widetilde{H}^{\frac12}(S_2^{(N)})]^9 \to [H^{-\frac12}(S_1)]^4,$$

$$\mathcal{T} : [\widetilde{H}^{\frac12}(S_2^{(N)})]^9 \to [H^{\frac12}(S_1)]^9,\quad \mathcal{B}_{\Sigma_1} : [\widetilde{H}^{\frac12}(\Sigma_1]^3 \to [H^{\frac12}(S_1)]^9,$$

$$\mathcal{D}_{\Sigma_1} : [\widetilde{H}^{\frac12}(\Sigma_1]^3 \to [H^{-\frac12}(S_1)]^4,\quad \mathcal{M}_{\Sigma_1} : [H^{\frac12}(S_1]^4 \to [\widetilde{H}^{-\frac12}(\Sigma_1)]^3,$$

$$\mathcal{N}_{\Sigma_2} : [\widetilde{H}^{\frac12}(\Sigma_2]^6 \to [H^{-\frac12}(S_2^{(N)})]^9,\quad \mathcal{M}_{\Sigma_2} : [H^{\frac12}(S_1]^9 \to [\widetilde{H}^{-\frac12}(\Sigma_2)]^6,$$

$$\mathcal{C}_{\Sigma_2} : [\widetilde{H}^{\frac12}(\Sigma_2]^6 \to [\widetilde{H}^{-\frac12}(\Sigma_2)]^6,\quad \mathcal{D}_{\Sigma_2} : [\widetilde{H}^{\frac12}(\Sigma_2]^6 \to [\widetilde{H}^{-\frac12}(\Sigma_2)]^9$$

are compact, since $S_1 \cap S_2 = \varnothing$ and $S_j \cap \Sigma_k = \varnothing$, $j,k = 1,2$.

Now, we consider the operator

$$\mathcal{P}^{(1)}_c := \begin{bmatrix} I_{9\times 4} & -I_9 & [0]_{9\times 9} & [0]_{9\times 3} & [0]_{9\times 6} \\ \mathcal{A}^{(1)}_{S_1} & [\mathcal{A}^{(2)}_{S_1}]_{4\times 9} & [0]_{4\times 9} & [0]_{4\times 3} & [0]_{4\times 6} \\ [0]_{9\times 4} & [0]_{9\times 9} & r_{S_2^{(N)}}\mathcal{A}^{(2)}_{S_2} & [0]_{9\times 3} & [0]_{9\times 6} \\ [0]_{3\times 4} & [0]_{3\times 9} & [0]_{3\times 9} & r_{\Sigma_1}[\widetilde{\mathcal{L}}^{(1)}_c]_{3\times 3} & [0]_{3\times 6} \\ [0]_{6\times 4} & [0]_{6\times 9} & [0]_{6\times 9} & [0]_{6\times 3} & r_{\Sigma_2}[\widetilde{\mathcal{L}}^{(2)}_c]_{6\times 6}, \end{bmatrix}_{31\times 31},$$

where the operator $\mathcal{P}_c - \mathcal{P}_c^{(1)} : \mathbf{X} \to \mathbf{Y}$ is compact. If we show that the operator $\mathcal{P}_c^{(1)} : \mathbf{X} \to \mathbf{Y}$ is Fredholm with zero index, then the operator $\mathcal{P}_c : \mathbf{X} \to \mathbf{Y}$ will be Fredholm with zero index.

Write the system corresponding to the operator $\mathcal{P}_c^{(1)}$ as follows:

$$\widetilde{g}_j^{(1)} - \widetilde{g}_j^{(2)} = \widetilde{f}_j^{(1)} \ \text{ on } S_1, \ \ j = \overline{1,4}, \tag{5.20}$$

$$-\widetilde{g}_j^{(2)} = \widetilde{F}_j \ \text{ on } S_1, \ \ j = \overline{5,9}, \tag{5.21}$$

$$[\mathcal{A}_{S_1}^{(1)} \widetilde{g}^{\,(1)}]_j + [\mathcal{A}_{S_1}^{(2)} \widetilde{g}^{\,(2)}]_j = \widetilde{f}_j^{(2)} \ \text{ on } S_1, \ \ j = \overline{1,4}, \tag{5.22}$$

$$r_{S_2^{(N)}} \mathcal{A}_{S_2}^{(2)} \widetilde{h}_0 = \widetilde{q}_2^{(N)} \ \text{ on } S_2^{(N)}, \tag{5.23}$$

$$r_{\Sigma_1} \widetilde{\mathcal{L}}^{(1)} \widetilde{\chi}_0^{(1)} = \widetilde{F}^{(1)} \ \text{ on } \Sigma_1, \tag{5.24}$$

$$r_{\Sigma_2} \widetilde{\mathcal{L}}^{(2)} \widetilde{\chi}_0^{(1)} = \widetilde{F}^{(2)} \ \text{ on } \Sigma_2, \tag{5.25}$$

System (5.20)–(5.25) is equivalent to the following system:

$$\widetilde{g}_j^{(1)} - \widetilde{g}_j^{(2)} = \widetilde{f}_j^{(1)} \ \text{ on } S_1, \ \ j = \overline{1,4}, \tag{5.26}$$

$$-\widetilde{g}_j^{(2)} = \widetilde{F}_j \ \text{ on } S_1, \ \ j = \overline{5,9}, \tag{5.27}$$

$$(\mathcal{A}_{S_1}^{(1)} + \overline{\mathcal{A}}_{S_1}^{(2)}) \widetilde{g}^{\,(1)} = \Psi \ \text{ on } S_1, \tag{5.28}$$

$$r_{S_2^{(N)}} \mathcal{A}_{S_2}^{(2)} \widetilde{h}_0^{(2)} = \widetilde{q}_2^{(N)} \ \text{ on } S_2^{(N)}, \tag{5.29}$$

$$r_{\Sigma_1} \widetilde{\mathcal{L}}^{(1)} \widetilde{\chi}_0^{(1)} = \widetilde{F}^{(1)} \ \text{ on } \Sigma_1, \tag{5.30}$$

$$r_{\Sigma_2} \widetilde{\mathcal{L}}^{(2)} \widetilde{\chi}_0^{(1)} = \widetilde{F}^{(2)} \ \text{ on } \Sigma_2, \tag{5.31}$$

where

$$\overline{\mathcal{A}}_{S_1}^{(2)} := \big[\mathcal{A}_{S_1,ji}^{(2)}\big]_{4\times 4}, \ \ j,i = \overline{1,4}, \quad \Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T,$$

$$\psi_j := \widetilde{f}_j^{(2)} + \sum_{i=1}^{4} \mathcal{A}_{S_1,ji}^{(2)} \widetilde{f}_i^{(1)} + \sum_{i=5}^{9} \mathcal{A}_{S_1,ji}^{(2)} \widetilde{F}_i, \ \ j = \overline{1,4}.$$

The operator corresponding to system (5.26)–(5.31) has the form

$$\mathcal{P}_c^{(2)} := \begin{bmatrix} I_{9\times 4} & -I_9 & [0]_{9\times 9} & [0]_{9\times 3} & [0]_{9\times 6} \\ \mathcal{A}_{S_1}^{(1)} + \overline{\mathcal{A}}_{S_1}^{(2)} & [0]_{4\times 9} & [0]_{4\times 9} & [0]_{4\times 3} & [0]_{4\times 6} \\ [0]_{9\times 4} & [0]_{9\times 9} & r_{S_2^{(N)}} \mathcal{A}_{S_2}^{(2)} & [0]_{9\times 3} & [0]_{9\times 6} \\ [0]_{3\times 4} & [0]_{3\times 9} & [0]_{3\times 9} & r_{\Sigma_1} [\widetilde{\mathcal{L}}_c^{(1)}]_{3\times 3} & [0]_{3\times 6} \\ [0]_{6\times 4} & [0]_{6\times 9} & [0]_{6\times 9} & [0]_{6\times 3} & r_{\Sigma_2} [\widetilde{\mathcal{L}}_c^{(2)}]_{6\times 6}. \end{bmatrix}_{31\times 31},$$

Evidently, the operator $\mathcal{P}_c^{(2)} : \mathbf{X} \to \mathbf{Y}$ is bounded.

Consider the composition $\mathcal{P}_c^{(3)} := \mathcal{P}_c^{(2)} \circ \mathcal{Q}$, where

$$\mathcal{Q} := \begin{bmatrix} [0]_{4\times 9} & -I_4 & [0]_{4\times 9} & [0]_{4\times 3} & [0]_{4\times 6} \\ I_9 & [0]_{9\times 4} & [0]_{9\times 9} & [0]_{9\times 3} & [0]_{9\times 6} \\ [0]_{9\times 9} & [0]_{9\times 4} & I_9 & [0]_{9\times 3} & [0]_{9\times 6} \\ [0]_{3\times 4} & [0]_{3\times 9} & [0]_{3\times 9} & I_3 & [0]_{3\times 6} \\ [0]_{6\times 4} & [0]_{6\times 9} & [0]_{6\times 9} & [0]_{6\times 3} & I_6, \end{bmatrix}_{31\times 31}.$$

Obviously, the operator $\mathcal{Q} : \mathbf{X} \to \mathbf{X}$ is invertible.

The operator $\mathcal{P}_c^{(3)}$ has the following form:

$$\mathcal{P}_c^{(3)} := \begin{bmatrix} -I_9 & I_{9\times4} & [0]_{9\times9} & [0]_{9\times3} & [0]_{9\times6} \\ [0]_{4\times9} & \mathcal{A}_{S_1}^{(1)} + \overline{\mathcal{A}}_{S_1}^{(2)} & [0]_{4\times9} & [0]_{4\times3} & [0]_{4\times6} \\ [0]_{9\times9} & [0]_{9\times4} & r_{S_2^{(N)}}\mathcal{A}_{S_2}^{(2)} & [0]_{9\times3} & [0]_{9\times6} \\ [0]_{3\times9} & [0]_{3\times4} & [0]_{3\times9} & r_{\Sigma_1}[\widetilde{\mathcal{L}}_c^{(1)}]_{3\times3} & [0]_{3\times6} \\ [0]_{6\times9} & [0]_{6\times4} & [0]_{6\times9} & [0]_{6\times3} & r_{\Sigma_2}[\widetilde{\mathcal{L}}_c^{(2)}]_{6\times6}, \end{bmatrix}_{31\times31},$$

To show that the operator $\mathcal{P}_c : \mathbf{X} \to \mathbf{Y}$ is Fredholm with zero index, it is sufficient to show that the operator $\mathcal{P}_c^{(3)} : \mathbf{X} \to \mathbf{Y}$ is Fredholm with zero index.

Since the operator $\mathcal{P}_c^{(3)}$ is triangular diagonal, it is sufficient to show that the operators standing on the diagonal are Fredholm with zero index.

As we know, from Lemma 5.2 in [7] it follows that the operator

$$\mathcal{A}_{S_1}^{(1)} + \overline{\mathcal{A}}_{S_1}^{(2)} : [H^{\frac{1}{2}}(S_1)]^4 \to [H^{-\frac{1}{2}}(S_1)]^4$$

is Fredholm with zero index, while the strongly elliptic pseudodifferential operators

$$r_{S_2^{(N)}}\mathcal{A}_{S_2}^{(2)} : [\widetilde{H}^{\frac{1}{2}}(S_2^{(N)})]^9 \to [H^{-\frac{1}{2}}(S_2^{(N)})]^9,$$
$$r_{\Sigma_1}[\widetilde{\mathcal{L}}_c^{(1)}]_{3\times3} : [\widetilde{H}^{\frac{1}{2}}(\Sigma_1)]^3 \to [H^{-\frac{1}{2}}(\Sigma_1)]^3, \quad r_{\Sigma_2}[\widetilde{\mathcal{L}}_c^{(2)}]_{6\times6} : [\widetilde{H}^{\frac{1}{2}}(\Sigma_2)]^6 \to [H^{-\frac{1}{2}}(\Sigma_2)]^6$$

are invertible (see [6, Theorem 7.7, Theorem 7.6], and [18, Theorem 14]). Hence the operator $\mathcal{P}_c^{(3)}$: $\mathbf{X} \to \mathbf{Y}$ is Fredholm with zero index. Then the operators $\mathcal{P}_c^{(1)}, \mathcal{P}_c^{(2)} : \mathbf{X} \to \mathbf{Y}$ will also be Fredholm with zero index. Therefore, the operator $\mathcal{P}_c : \mathbf{X} \to \mathbf{Y}$ is Fredholm with zero index.

Now we show that the operator $\mathcal{P}_c : \mathbf{X} \to \mathbf{Y}$ is invertible.

The invertibility of the operator $\mathcal{P}_c$ is derived from the uniqueness of the solution of the boundary-transmission problem $(TM)_{c,\tau}$.

Indeed, let $(g^{(1)}, g^{(2)}, h_0^{(2)}, \chi_0^{(1)}, \chi_0^{(2)})^\top \in \mathbf{X}$ be a solution of the homogeneous equation

$$\mathcal{P}_c\big(g^{(1)}, g^{(2)}, h_0^{(2)}, \chi_0^{(1)}, \chi_0^{(2)}\big)^\top = 0. \tag{5.32}$$

We construct the following potentials:

$$U^{(1)} = V_{S_1}^{(1)}(\mathcal{H}_{S_1}^{(1)})^{-1}g^{(1)} + W_c^{(1)}\widehat{\chi}^{(1)}, \tag{5.33}$$
$$U^{(2)} = V_{S_1}^{(2)}(\mathcal{H}_{S_1}^{(2)})^{-1}g^{(2)} + V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}h_0^{(2)} + W_c^{(2)}\widehat{\chi}^{(2)}. \tag{5.34}$$

Since $(g^{(1)}, g^{(2)}, h_0^{(2)}, \chi_0^{(1)}, \chi_0^{(2)})^\top$ is a solution of the homogeneous equation (5.32), i.e., of the homogeneous system (5.13)–(5.18), it is clear that $(U^{(1)}, U^{(2)})$ will be a solution of the homogeneous boundary-transmission problem $(TM)_{c,\tau}$. Then from the uniqueness theorem of the problem $(TM)_{c,\tau}$ it follows that

$$U^{(1)} \equiv 0 \ \text{ in } \Omega_{\Sigma_1}, \tag{5.35}$$
$$U^{(2)} \equiv 0 \ \text{ in } \Omega_{\Sigma_2}. \tag{5.36}$$

Since the single layer potentials are continuous in space $\mathbb{R}^3$, we have

$$\{U^{(1)}\}^+ = \{U^{(1)}\}^- = 0 \ \text{ on } S_1,$$
$$\{U^{(2)}\}^+ = \{U^{(2)}\}^- = 0 \ \text{ on } S_1 \cup S_2.$$

Hence the vector functions $U^{(1)}$ and $U^{(2)}$ satisfy the following Dirichlet problems:

$$\begin{cases} A^{(1)}(\partial_x, \tau)U^{(1)} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}_1, \\ \{U^{(1)}\}^- = 0 & \text{on } S_1, \end{cases}$$

and

$$\begin{cases} A^{(2)}(\partial_x,\tau)U^{(2)}=0 & \text{in } \mathbb{R}^3\setminus\overline{\Omega}_2,\\ \{U^{(2)}\}^-=0 & \text{on } S_1\cup S_2. \end{cases}$$

From the uniqueness of solutions of the Dirichlet problem it follows that these problems have only trivial solution, i.e.,

$$U^{(1)}\equiv 0 \ \text{ in } \mathbb{R}^3\setminus\overline{\Omega}_1,$$
$$U^{(2)}\equiv 0 \ \text{ in } \mathbb{R}^3\setminus\overline{\Omega}_2.$$

Hence from (5.35) and (5.36) we get

$$U^{(1)}\equiv 0 \ \text{ in } \mathbb{R}^3\setminus\overline{\Sigma}_1,$$
$$U^{(2)}\equiv 0 \ \text{ in } \mathbb{R}^3\setminus\overline{\Sigma}_2.$$

Then, applying the jump formulas of potentials (5.33) and (5.34), we get

$$\begin{aligned} \{T^{(1)}U^{(1)}\}^- - \{T^{(1)}U^{(1)}\}^+ &= g^{(1)}=0 \ \text{ on } S_1,\\ \{T^{(2)}U^{(2)}\}^- - \{T^{(2)}U^{(2)}\}^+ &= g^{(2)}=0 \ \text{ on } S_1,\\ \{T^{(2)}U^{(2)}\}^- - \{T^{(2)}U^{(2)}\}^+ &= h_0^{(2)}=0 \ \text{ on } S_2,\\ \{U^{(1)}\}^+ - \{U^{(1)}\}^- &= \widehat{\chi}^{(1)}=0 \ \text{ on } \Sigma_1,\\ \{U^{(2)}\}^+ - \{U^{(2)}\}^- &= \widehat{\chi}^{(2)}=0 \ \text{ on } \Sigma_2. \end{aligned}$$

Therefore, we obtain $\operatorname{Ker}\mathcal{P}_c=\{0\}$ and, since $\operatorname{ind}\mathcal{P}=0$, we have $\operatorname{Ker}\mathcal{P}_c^*=\{0\}$. Thus the operator $\mathcal{P}_c:\mathbf{X}\to\mathbf{Y}$ is invertible, and Theorem 5.1 is proved. □

The invertibility of the operator $\mathcal{P}_c$ implies the unique solvability of equation (5.19), i.e., of systems (5.13)–(5.18) and (5.1)–(5.11). Consequently, we obtain the unique solvability of the mixed type boundary-transmission problem $(TM)_{c,\tau}$.

Thus we obtain the existence and uniqueness theorem of the mixed type boundary-transmission problem $(TM)_{c,\tau}$.

Theorem 5.2. *Let $S_1,S_2,\overline{\Sigma}_1,\overline{\Sigma}_2\in C^\infty$, $\tau=\sigma+i\omega$, $\sigma>\sigma_0>0$, $\omega\in\mathbb{R}$, and*

$$f_j^{(1)}\in H^{\frac12}(S_1),\quad f_j^{(2)}\in H^{-\frac12}(S_2),\ \ j=\overline{1,4},\quad Q_j^{(2)}\in H^{\frac12}(S_1),\ \ j=\overline{5,9},$$
$$p_2^{(D)}\in[H^{\frac12}(S_2^{(D)})]^9,\quad q_2^{(N)}\in[H^{-\frac12}(S_2^{(N)})]^9,\quad F_j^{(1),\pm}\in H^{-\frac12}(\Sigma_1),\ \ j=1,2,3,$$
$$G_4^{(1)}\in\widetilde{H}^{\frac12}(\Sigma_1),\quad F_4^{(1)}\in\widetilde{H}^{-\frac12}(\Sigma_1),\quad F_j^{(2),\pm}\in H^{-\frac12}(\Sigma_2),\ \ j=1,2,3,5,6,7,$$
$$G_j^{(2)}\in\widetilde{H}^{\frac12}(\Sigma_2),\ \ j=4,8,9,\quad F_j^{(2)}\in\widetilde{H}^{-\frac12}(\Sigma_2),\ \ j=4,8,9,$$

and the compatibility conditions

$$F_j^{(1),+}-F_j^{(1),-}\in\widetilde{H}^{-\frac12}(\Sigma_1),\ \ j=1,2,3,$$
$$F_j^{(2),+}-F_j^{(2),-}\in\widetilde{H}^{-\frac12}(\Sigma_2),\ \ j=1,2,3,5,6,7,$$

are satisfied.

Then the mixed boundary-transmission problem $(TM)_{c,\tau}$ has a unique solution

$$(U^{(1)},U^{(2)})\in[H^1(\Omega_{\Sigma_1})]^4\times[H^1(\Omega_{\Sigma_2})]^9,$$

which is presented in the following form:

$$U^{(1)}=V_{S_1}^{(1)}(\mathcal{H}_{S_1}^{(1)})^{-1}g^{(1)}+W_c^{(1)}\chi^{(1)}+V_c^{(1)}\Psi^{(1)} \ \ \text{in } \Omega_{\Sigma_1},$$
$$U^{(2)}=V_{S_1}^{(2)}(\mathcal{H}_{S_1}^{(2)})^{-1}g^{(2)}+V_{S_2}^{(2)}(\mathcal{H}_{S_2}^{(2)})^{-1}h^{(2)}+W_c^{(2)}\chi^{(2)}+V_c^{(2)}\Psi^{(2)} \ \ \text{in } \Omega_{\Sigma_2},$$

where $g^{(1)}$, $g^{(2)}$, $h^{(2)}$, $\chi^{(1)}$, $\chi^{(2)}$, $\Psi^{(1)}$, $\Psi^{(2)}$ are the unique solutions of system (5.1)–(5.11).

Let us introduce the notation

$$d := \frac{c\,b_0^{(2)} + p\,\lambda_1^{(2)} + q\,\nu_2^{(2)}}{2\,\gamma^{(2)}}\,,$$

where

$$c := \frac{1}{2}\,\big(b_0^{(2)}b_{11} + \lambda_1^{(2)}b_{21} + \nu_2^{(2)}b_{31}\big), \quad p := \frac{1}{2}\,\big(b_0^{(2)}b_{12} + \lambda_1^{(2)}b_{22} + \nu_2^{(2)}b_{32}\big),$$

$$q := \frac{1}{2}\,\big(b_0^{(2)}b_{13} + \lambda_1^{(2)}b_{23} + \nu_2^{(2)}b_{33}\big), \quad [b_{jk}]_{3\times 3} := \begin{bmatrix} a_0^{(2)} & -\lambda_2^{(2)} & \nu_1^{(2)} \\ \lambda_2^{(2)} & \chi^{(2)} & \nu_3^{(2)} \\ \nu_1^{(2)} & -\nu_3^{(2)} & k^{(2)} \end{bmatrix}^{-1}.$$

Denote by $C_0^\infty(\overline{\Sigma}_k)$, $k = 1, 2$, the space of functions vanishing along with all tangential (to Σ_k) derivatives at $\ell_c^{(k)} = \partial\Sigma_k$, $k = 1, 2$.

The following regularity theorem holds.

Theorem 5.3. *Suppose $S_1, S_2 \in C^\infty$ and*

$$f_j^{(1)} \in C^\infty(S_1), \quad f_j^{(2)} \in C^\infty(S_1), \quad j = \overline{1,4}, \quad Q_j^{(2)} \in C^\infty(S_1), \quad j = \overline{5,9},$$
$$p_2^{(D)} \in [C^\infty(\overline{S}_2^{(D)})]^9, \quad q_2^{(N)} \in [C^\infty(\overline{S}_2^{(N)})]^9, \quad F_j^{(1),\pm} \in C^\infty(\overline{\Sigma}_1), \quad j = 1, 2, 3,$$
$$G_4^{(1)} \in C^\infty(\overline{\Sigma}_1), \quad F_4^{(1)} \in C^\infty(\overline{\Sigma}_1), \quad F_j^{(2),\pm} \in C^\infty(\Sigma_2), \quad j = 1, 2, 3, 5, 6, 7,$$
$$G_j^{(2)} \in C^\infty(\overline{\Sigma}_2), \quad F_j^{(2)} \in C^\infty(\overline{\Sigma}_2), \quad j = 4, 8, 9,$$
$$F_j^{(1),+} - F_j^{(1),-} \in C_0^\infty(\overline{\Sigma}_1), \quad j = 1, 2, 3, \quad F_j^{(2),+} - F_j^{(2),-} \in C_0^\infty(\overline{\Sigma}_2), \quad j = 1, 2, 3, 5, 6, 7.$$

Let $(U^{(1)}, U^{(2)})$ be the unique solution to the mixed type boundary-transmission problem $(TM)_{c,\tau}$.

Then the components of $u^{(1)}$ and $U^{(2)}$ have $C^{\frac{1}{2}}$-Hölder smoothness in one-sided interior and exterior neighborhoods of the surfaces $S_0^{(1)}$ and $S_0^{(2)}$, respectively, and $\vartheta^{(1)}$ has the $C^{\frac{3}{2}}$-smoothness in one-sided interior and exterior neighborhoods of the surface $S_0^{(1)}$. While

(1) *If $d < 0$, then the vector $U^{(2)}$ belongs to the $[C^{\gamma_1}]^9$-Hölder class in a neighborhood of the line $\ell_m = \partial S_2^{(D)} = \partial S_2^{(N)}$, where $\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\,\mathrm{arctg}\,2\sqrt{-d}$, γ_1 depends on the material constants, does not depend on the geometry of the exceptional line l_m and may take any values from the interval $(0, \frac{1}{2})$;*

(2) *If $d \geq 0$, then the vector $U^{(2)}$ belongs to the $[C^{\frac{1}{2}}]^9$-Hölder class in a neighborhood of the line ℓ_m.*

Proof of this theorem follows from the work [6, Section 9], where the asymptotic properties and the smoothness of solutions of mixed and crack type problems are studied near the change of the boundary conditions, i.e., near the line ℓ_m and crack edges $\ell_c^{(k)} := \partial\Sigma_k$, $k = 1, 2$ (cf. [3, 5]). Note that the smoothness of the vector functions $U^{(1)}$ and $U^{(2)}$ is finite in the neighborhoods of $\ell_c^{(1)}$ and $\ell_c^{(2)}$, ℓ_m, respectively, but taking into account the data conditions of Theorem 5.3, outside these neighborhoods $U^{(1)}$ and $U^{(2)}$ are infinitely differentiable.

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