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GLOBAL EXISTENCE AND GENERAL DECAY OF SOLUTION FOR A NONLINEAR WAVE EQUATION WITH VARIABLE EXPONENTS AND MEMORY TERM

Abstract. In this paper, we consider the following wave equation:

$$u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u.$$

First, we prove that the equation has a unique local solution for a suitable conditions by using Faedo–Galerkin methods, and we also prove that the local solution is global in time. Finally, we demonstrate that the solution with positive initial energy decays exponentially.

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$$u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u.$$

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1 Introduction

We consider the following boundary value problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \int\limits_0^t g(t-s)\Delta u(s)\,ds + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u \ \text{ in } Q, \\ u(x,t) = 0, \ \ x \in \partial\Omega, \ \ t \in (0,T), \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \ \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $Q = \Omega \times (0,T)$ and Ω is a bounded domain in $\mathbb{R}^n, n \geq 2$, with a smooth boundary $\partial\Omega$. $p(\cdot)$ and $m(\cdot)$ are the given measurable functions on Ω satisfying

$$2 \leq \theta^- \leq \theta(x) \leq \theta^+ \leq \theta^*, \tag{1.2}$$
$$\theta^- := \operatorname*{ess\,inf}_{x\in\Omega} \theta(x), \quad \theta^+ := \operatorname*{ess\,sup}_{x\in\Omega} \theta(x)$$

and

$$\theta^* = \begin{cases} \infty, & \text{if } n = 2, \\ \dfrac{2n}{n-2}, & \text{if } n \geq 3. \end{cases} \tag{1.3}$$

We also assume that $p(\cdot)$ and $m(\cdot)$ satisfy the log-Hölder continuity condition

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|} \ \text{ for a.e. } x, y \in \Omega, \ \text{ with } |x-y| < \delta, \ \ A > 0, \ \ 0 < \delta < 1. \tag{1.4}$$

Equation (1.1) can be viewed as a generalization of the evolutional equation

$$u_{tt} - \Delta u - \Delta u_{tt} + \int\limits_0^t g(t-s)\Delta u(s)\,ds + \omega|u_t|^{m-2}u_t = b|u|^{r-2}u \ \text{ in } \Omega \times (0,T)$$

with the constant exponent of nonlinearity, $m, r \in (2,\infty)$, which appears in various physical contexts.

In the case $p(x) = p$ and $m(x) = m$, equation (1.1) proved the existence and blow up of solutions. The results have been established by many authors (see [1–3, 5, 11, 12, 18, 23]).

Recently, many authors have been treated the problem with variable exponents (see [2, 10, 14, 16, 19]). The study of these equations is based on the use of the Lebesgue and Sobolev spaces with variable exponents (see, e.g., [6–9, 13]).

Messaoudi et al. [17] studied the solution of the equation

$$u_{tt} - \Delta u + |u_t|^{p(x)-2}u_t = b|u|^{q(x)-2}u \ \text{ in } \Omega \times (0,T)$$

and used the Faedo–Galerkin method to establish the existence of a unique weak solution. They also proved that the solutions with negative initial energy blow up in a finite time. Messaoudi and Talahmeh [16] studied the blow-up in solutions of a quasilinear wave equation with variable exponent nonlinearities:

$$u_{tt} - div(|\nabla u|^{r(x)-2}\nabla u) + a|u_t|^{p(x)-2}u_t = b|u|^{q(x)-2}u \ \text{ in } \Omega \times (0,T).$$

They obtained the blow-up result for the solutions with negative initial energy and for certain solutions with positive energy.

The outline of this paper is as follows. In Section 2, we state some results about the variable exponent, Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. In Section 3, we prove the local existence. In Section 4, we show that the local solution is global in time, and the exponential decay results are proved.

2 Preliminaries and assumptions

In this section, we present some Lemmas about the Lebesgue and Sobolev space with variable components (see [6–9, 13]). Let $p : \Omega \to [1, +\infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$.

We define the Lebesgue space with a variale exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) := \big\{ v : \Omega \to \mathbb{R} : \text{ measurable in } \Omega, \ \varrho_{p(\cdot)}(\lambda v) < +\infty \text{ for some } \lambda > 0 \big\},$$

where

$$\varrho_{p(\cdot)}(v) = \int_\Omega |v(x)|^{p(x)}\, dx.$$

The set $L^{p(\cdot)}(\Omega)$ is equipped with the norm (Luxemburg's norm)

$$\|v\|_{p(\cdot)} := \inf \Big\{ \lambda > 0 : \int_\Omega \Big| \frac{v(x)}{\lambda} \Big|^{p(x)} dx \le 1 \Big\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space [6].

Next, we define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega) := \Big\{ v \in L^{p(\cdot)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{p(\cdot)}(\Omega) \Big\}.$$

This is a Banach space with respect to the norm $\|v\|_{W^{1,p(\cdot)}(\Omega)} = \|v\|_{p(\cdot)} + \|\nabla v\|_{p(\cdot)}$.

Furthermore, we set $W_0^{1,p(\cdot)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p(\cdot)}(\Omega)$. Note that the space $W^{1,p(\cdot)}(\Omega)$ has a different definition in the case of variable exponents.

However, under condition (1.4) both definitions are equivalent (see [6]). The space $W^{-1,p'(\cdot)}(\Omega)$, dual of $W_0^{1,p(\cdot)}(\Omega)$, is defined in the same way as the classical Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Lemma 2.1 (Poincaré's Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose that $p(\cdot)$ satisfies (1.4), then*

$$\|v\|_{p(\cdot)} \le c \|\nabla v\|_{p(\cdot)} \quad \textit{for all } v \in W_0^{1,p(\cdot)}(\Omega),$$

where $c > 0$ is a constant which depends on p^-, p^+, and Ω only. In particular, $\|\nabla v\|_{p(\cdot)}$ define an equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$.

Lemma 2.2 (Hölder's Inequality)**.** *Suppose that p, q, $s \ge 1$ are measurable functions defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \quad \textit{for a.e. } y \in \Omega.$$

If $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, then $uv \in L^{s(\cdot)}(\Omega)$ with

$$\|uv\|_{s(\cdot)} \le 2 \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}.$$

Lemma 2.3 (Lars et al. [6])**.** *If p is a measurable function on Ω satisfying (1.2), then we have*

$$\min \Big\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \Big\} \le \varrho_{p(\cdot)}(u) \le \max \Big\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \Big\}$$

for any $u \in L^{p(\cdot)}(\Omega)$.

Lemma 2.4 (Lars et al. [6])**.** *If p is a measurable function on Ω satisfying (1.2) and (1.3), then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.*

From Lemma 2.4, there exists the positive constant B satisfying

$$\|u\|_{p(\cdot)} \le B \|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega).$$

We denote the total energy related to problem (1.1) as

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx, \quad (2.1)$$

where

$$(g\circ\nabla u)(t)=\int_0^t\int_\Omega g(t-s)|\nabla u(t)-\nabla u(s)|^2\,dx\,ds.$$

We also introduce the following functionals:

$$\widetilde{E}(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p^-}\int_\Omega |u|^{p(x)}\,dx, \quad (2.2)$$

$$\widetilde{\widetilde{E}}(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p^+}\int_\Omega |u|^{p(x)}\,dx, \quad (2.3)$$

$$I(t)=\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)-\int_\Omega |u|^{p(x)}\,dx, \quad (2.4)$$

and

$$J(t)=\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p^-}\int_\Omega |u|^{p(x)}\,dx.$$

We show that

$$\widetilde{E}(t)\le E(t)\le \widetilde{\widetilde{E}}(t). \quad (2.5)$$

Let us introduce the assumptions:

(A_1) $g:\mathbb{R}^+\to\mathbb{R}^+_*$ is a bounded C^1 function satisfying

$$1-\int_0^\infty g(s)\,ds=l>0 \ \text{ and } \ g'(t)\le -g(t). \quad (2.6)$$

(A_2) Assume that

$$I(0)>0$$

and

$$\operatorname{Max}\left(\frac{B^{p^-}}{l}\left(\frac{2p^-}{l(p^--2)}E(0)\right)^{\frac{p^--2}{2}},\frac{B^{p^+}}{l}\left(\frac{2p^-}{l(p^--2)}E(0)\right)^{\frac{p^+-2}{2}}\right)=\lambda<1.$$

Theorem 2.1. *Suppose that $m(\cdot)$, $p(\cdot)\in C(\overline{\Omega})$ and (1.4) holds with*

$$2\le p^-\le p(x)\le p^+\le 2\,\frac{n-1}{n-2} \ \text{ if } n\ge 3,$$
$$p(x)\ge 2 \ \text{ if } n=2,$$

and

$$2\le m^-\le m(x)\le m^+\le 2\,\frac{n-1}{n-2} \ \text{ if } n\ge 3,$$
$$m(x)\ge 2 \ \text{ if } n=2.$$

Then for any $(u_0,u_1)\in H_0^1(\Omega)\times H_0^1(\Omega)$, problem (1.1) has a unique weak local solution

$$u\in L^\infty([0,T);H_0^1(\Omega)),$$
$$u_t\in L^\infty([0,T);H_0^1(\Omega))\cap L^{m(\cdot)}(\Omega\times[0,T)),$$
$$u_{tt}\in L^2([0,T);H_0^1(\Omega)).$$

3 Existence of weak solutions

In this section, we are going to obtain the existence of weak solutions to problem (1.1). We will use Faedo–Galerkin's method of approximation. Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^1(\Omega)$ which constructs a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k = \operatorname{span}\{v_1, v_2, \dots, v_k\}$ the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. By the normalization, we have $\|v_l\| = 1$, and for any given integer k, we consider the approximation solution

$$u_k(t) = \sum_{l=1}^{k} u_{lk}(t) v_l,$$

where u_k are the solutions to the following Cauchy problem:

$$(u_k''(t), v_l) - (\Delta u_k(t), v_l) - (\Delta u_k''(t), v_l) - \int_0^t g(t-s)(\Delta u_k(s), v_l)\, ds + \big(|u_k'(t)|^{m(x)-2} u_k'(t), v_l\big) = \big(|u_k(t)|^{p(x)-2} u_k(t), v_l\big), \quad l = 1, 2, \dots, k, \tag{3.1}$$

$$u_k(0) = u_{0k} = \sum_{i=1}^{k} (u_k(0), v_i) v_i \to u_0 \ \text{ in } H_0^1(\Omega), \tag{3.2}$$

$$u_k'(0) = u_{1k} = \sum_{l=1}^{k} (u_k'(0), v_l) v_l \to u_1 \ \text{ in } H_0^1(\Omega). \tag{3.3}$$

Note that, system (3.1)–(3.3) can be solved by the Picard iteration method in ordinary differential equations. Hence there exists a solution in $[0, T_*)$ for some $T_* > 0$, and we can extend this solution to the whole interval $[0, T]$ for any given $T > 0$ by making use of a priori estimates below.

Step 1. Multiplying equation (3.1) by $u_{lk}'(t)$ and summing over l from 1 to k, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|u_k'\|_2^2 + \frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\|\nabla u_k\|_2^2 + \frac{1}{2}(g \circ \nabla u_k)(t) - \int_\Omega \frac{1}{p(x)} |u_k|^{p(x)}\, dx\right) = -\int_\Omega |u_k'|^{m(x)}\, dx + \frac{1}{2}(g' \circ \nabla u_k)(t) - \frac{1}{2} g(t)\|\nabla u_k\|_2^2. \tag{3.4}$$

Then, by virtue of (2.1), assumption (A_1) and definition of the expression $(g' \circ \nabla u_k)(t)$, we have

$$E'(u_k(t)) = -\int_\Omega |u_k'|^{m(x)}\, dx + \frac{1}{2}(g' \circ \nabla u_k)(t) - \frac{1}{2} g(t)\|\nabla u_k\|_2^2 \le 0.$$

Integrating (3.4) over $(0, t)$, we obtain the estimate

$$\frac{1}{2}\|u_k'\|_2^2 + \frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\|\nabla u_k\|_2^2 + \frac{1}{2}(g \circ \nabla u_k)(t) - \int_\Omega \frac{1}{p(x)} |u_k|^{p(x)}\, dx + \int_0^t \int_\Omega |u_k'|^{m(x)}\, dx\, ds - \frac{1}{2}\int_0^t (g' \circ \nabla u_k)(s)\, ds + \frac{1}{2}\int_0^t g(s)\|\nabla u_k\|_2^2\, ds \le E(0). \tag{3.5}$$

Since $I(0) > 0$, by the continuity there exists $T_* < T$ such that $I(t) \ge 0$ for all $t \in [0, T_*]$. From (2.3) and (2.4) we get

$$J(u_k(t)) = \frac{p^- - 2}{2p^-}\left(\left(1 - \int_0^t g(s)\, ds\right)\|\nabla u_k\|_2^2 + (g \circ \nabla u_k)(t)\right) + \frac{1}{p^-} I(t).$$

Then

$$J(u_k(t)) \geq \frac{p^- - 2}{2p^-}\left(\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u_k\|_2^2 + (g \circ \nabla u_k)(t)\right).$$

Hence we have

$$\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u_k\|_2^2 \leq \frac{2p^-}{p^- - 2}\, J(u_k(t)).$$

From (2.1), (2.2) and (2.4), we obviously have $\forall\, t \in [0, T_*]$, $J(u_k(t)) \leq \widetilde{E}(u_k(t)) \leq E(u_k(t)) \leq E(0)$. Thus we obtain

$$\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u_k\|_2^2 \leq \frac{2p^-}{p^- - 2}\, E(0). \tag{3.6}$$

Before continuing the proof, we need the following

Lemma 3.1. *Suppose that* (1.2) *and assumptions* (A_1), (A_2) *hold, then*

$$\varrho_{p(\,\cdot\,)}(u_k) \leq l\,\|\nabla u_k\|_2^2, \tag{3.7}$$

where l is defined in (2.6).

Proof. By Lemmas 2.3 and 2.4, we have

$$\varrho_{p(\,\cdot\,)}(u_k) \leq \max\left\{\|u_k\|_{p(\,\cdot\,)}^{p^-}, \|u_k\|_{p(\,\cdot\,)}^{p^+}\right\} \leq \max\left\{B^{p^-}\|\nabla u_k\|_2^{p^-}, B^{p^+}\|\nabla u_k\|_2^{p^+}\right\},$$

and from assumptions (A_1), (A_2) and (3.6), we get

$$\begin{aligned}
\varrho_{p(\,\cdot\,)}(u_k) &\leq \max\left\{B^{p^-}\|\nabla u_k\|_2^2 \times \|\nabla u_k\|_2^{p^- - 2}, B^{p^+}\|\nabla u_k\|_2^2 \times \|\nabla u_k\|_2^{p^+ - 2}\right\}\\
&\leq \max\left(l\|\nabla u_k\|_2^2 \times \frac{B^{p^-}}{l}\left(\frac{2p^-}{l(p^- - 2)}E(0)\right)^{\frac{p^- - 2}{2}}, l\|\nabla u_k\|_2^2 \times \frac{B^{p^+}}{l}\left(\frac{2p^-}{l(p^- - 2)}E(0)\right)^{\frac{p^+ - 2}{2}}\right)\\
&\leq l\|\nabla u_k\|_2^2.
\end{aligned}$$

Due to (3.7), inequality (3.5) becomes

$$\begin{aligned}
&\frac{1}{2}\|u_k'\|_2^2 + \frac{1}{2}\|\nabla u_k'\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right)\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u_k\|_2^2 + \frac{1}{2}(g \circ \nabla u_k)(t)\\
&\qquad + \int_0^t \int_\Omega |u_k'|^{m(x)}\,dx\,ds - \frac{1}{2}\int_0^t (g' \circ \nabla u_k)(s)\,ds + \frac{1}{2}\int_0^t g(s)\|\nabla u_k\|_2^2\,ds \leq E(0).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\sup_{t\in(0,T_*)}\|u_k'\|_2^2 + \frac{1}{2}\sup_{t\in(0,T_*)}\|\nabla u_k'\|_2^2\\
&\quad + \left(\frac{1}{2} - \frac{1}{p^-}\right)\left(1 - \int_0^t g(s)\,ds\right)\sup_{t\in(0,T_*)}\|\nabla u_k\|_2^2 + \frac{1}{2}(g \circ \nabla u_k)(t) + \int_0^t \int_\Omega |u_k'|^{m(x)}\,dx\,ds\\
&\qquad - \frac{1}{2}\int_0^t (g' \circ \nabla u_k)(s)\,ds + \frac{1}{2}\int_0^t g(s)\|\nabla u_k\|_2^2\,ds \leq E(0).
\end{aligned} \tag{3.8}$$

From (3.8), we conclude that

$$\begin{cases} u_k \ \text{ is uniformly bounded in } L^\infty([0,T), H_0^1(\Omega)),\\ u_k' \ \text{ is uniformly bounded in } L^\infty([0,T), H_0^1(\Omega)) \cap L^{m(\,\cdot\,)}(\Omega \times [0,T)). \end{cases} \tag{3.9}$$

Furthermore, from Lemma 2.4 and (3.9) we have

$$\begin{aligned}&\{|u_k|^{p(x)-2}u_k\} \text{ is uniformly bounded in } L^\infty([0,T),L^2(\Omega)),\\&\{|u_k'|^{m(x)-2}u_k'\} \text{ is uniformly bounded in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times[0,T)).\end{aligned}\tag{3.10}$$

By (3.9) and (3.10), we infer that there exist a subsequence $u_n$of u_k and a function u such that

$$\begin{cases}u_k \rightharpoonup u \text{ weakly star in } L^\infty([0,T),H_0^1(\Omega)),\\ u_k' \rightharpoonup u' \text{ weakly star in } L^\infty([0,T),H_0^1(\Omega)),\\ |u_k'|^{m(x)-2}u_k' \rightharpoonup \psi \text{ weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times[0,T)).\end{cases}\tag{3.11}$$

By the Aubin–Lions compactness Lemma [15], from (3.11) we conclude that

$$u_k \rightharpoonup u \text{ strongly in } C([0,T),H_0^1(\Omega))$$

which implies

$$u_k \rightharpoonup u \text{ everywhere in } [0,T]\times\Omega.\tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\begin{cases}|u_k|^{p(x)-2}u_k \rightharpoonup |u|^{p(x)-2}u \text{ weakly in } L^\infty([0,T),L^2(\Omega)),\\ |u_k'|^{m(x)-2}u_k' \rightharpoonup |u'|^{m(x)-2}u' \text{ weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times[0,T)).\end{cases}\tag{3.13}$$

Next, multiplying equation (3.1) by $u_{lk}''(t)$ and summing over l from 1 to k, we get

$$\begin{aligned}&\|u_k''\|_2^2+\|\nabla u_k''\|_2^2+\frac{d}{dt}\Big(\int_\Omega \frac{1}{m(x)}|u_k'|^{m(x)}\,dx\Big)\\&\quad=-\int_\Omega \nabla u_k\nabla u_k''\,dx+\int_0^t g(t-\tau)\int_\Omega \nabla u_k(\tau)\nabla u_k''(t)\,dx\,d\tau+\int_\Omega |u_k|^{p(x)-2}u_ku_k''\,dx.\end{aligned}\tag{3.14}$$

From Young's inequality, we have

$$\Big|-\int_\Omega \nabla u_k\nabla u_k''\,dx\Big|\le \delta\|\nabla u_k''\|_2^2+\frac{1}{4\delta}\|\nabla u_k\|_2^2,\tag{3.15}$$

$$\begin{aligned}\Big|\int_0^t g(t-\tau)\int_\Omega \nabla u_k(\tau)\nabla u_k''(t)\,dx\,d\tau\Big|&\le \delta\|\nabla u_k''\|_2^2+\frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(t-\tau)\nabla u_k(\tau)\,d\tau\Big)^2dx\\&\le \delta\|\nabla u_k''\|_2^2+\frac{1}{4\delta}\int_0^t g(s)\,ds\int_0^t g(t-\tau)\int_\Omega |\nabla u_k(\tau)|^2\,dx\,d\tau\\&\le \delta\|\nabla u_k''\|_2^2+\frac{(1-l)g(0)}{4\delta}\int_0^t \|\nabla u_k(\tau)\|^2d\tau,\end{aligned}\tag{3.16}$$

and

$$\Big|\int_\Omega |u_k|^{p(x)-2}u_ku_k''\,dx\Big|\le \delta\|u_k''\|_2^2+\frac{1}{4\delta}\int_\Omega |u_k|^{2p(x)-2}\,dx.\tag{3.17}$$

From (3.14)–(3.17), inequality (3.14) becomes

$$(1-\delta)\|u_k''\|_2^2 + (1-2\delta)\|\nabla u_k''\|_2^2 + \frac{d}{dt}\left(\int\limits_\Omega \frac{1}{m(x)}\,|u_k'|^{m(x)}\,dx\right) \le \frac{1}{4\delta}\,\|\nabla u_k\|_2^2 + \frac{(1-l)g(0)}{4\delta}\int\limits_0^t \|\nabla u_k(\tau)\|^2\,d\tau + \frac{1}{4\delta}\int\limits_\Omega |u_k|^{2(p(x)-1)}\,dx.$$

We have $u_k \in L^\infty([0,T), H_0^1(\Omega))$, then

$$\int\limits_\Omega |u_k|^{2p(x)-2}\,dx \le \int\limits_\Omega |u_k|^{2p^- -2}\,dx + \int\limits_\Omega |u_k|^{2p^+ -2}\,dx < +\infty,$$

since

$$2(p^- - 1) \le 2(p(x)-1) \le 2(p^+ - 1) \le \frac{2n}{n-2}\,.$$

We chose δ small enough to find a positive constant λ such that

$$\int\limits_0^t \|u_k''\|_2^2\,ds + \lambda\int\limits_0^t \|\nabla u_k''\|_2^2\,ds + \int\limits_\Omega \frac{1}{m(x)}\,|u_k'|^{m(x)}\,dx \le C.$$

Then

$$u_k'' \text{ is bounded in } L^2([0,T), H_0^1(\Omega)).$$

Similarly, we have

$$u_k'' \rightharpoonup u'' \text{ weakly star in } L^2([0,T), H_0^1(\Omega)). \tag{3.18}$$

Setting up $k \longrightarrow \infty$ and passing to the limit in (3.1), we obtain

$$(u''(t), v_l) - (\Delta u(t), v_l) - (\Delta u''(t), v_l) - \int\limits_0^t g(t-s)(\Delta u(s), v_l)\,ds + (|u'(t)|^{m(x)-2}u'(t), v_l) = \big(|u(t)|^{p(x)-2}u(t), v_l\big),\quad l = 1,2,\dots,k.$$

Since $\{v_l\}_{l=1}^\infty$ is a basis of $H_0^1(\Omega)$, we deduce that u satisfies the equation of (1.1). From (3.11), (3.13), (3.18) and Lemma 3.1.7 in [22] with $B = H_0^1(\Omega)$ in the both cases, we infer that

$$\begin{cases} u_k(0) \rightharpoonup u(0) \text{ weakly in } H_0^1(\Omega), \\ u_k'(0) \rightharpoonup u'(0) \text{ weakly in } H_0^1(\Omega). \end{cases} \tag{3.19}$$

We get from (3.2) and (3.19) that $u(0) = u_0$, $u'(0) = u_1$.

Thus the proof of the existence is complete.

Now, it remains to prove the uniqueness. Let u^1, u^2 be two solutions in the class described in the statement of this theorem, and $w = u^1 - u^2$.

Then w satisfies

$$w_{tt} - \Delta w - \Delta w_{tt} + \int\limits_0^t g(t-s)\Delta w(s)\,ds + \omega\big(|u_t^1|^{m(x)-2}u_t^1 - |u_t^2|^{m(x)-2}u_t^2\big) = |u^1|^{p(x)-2}u^1 - |u^2|^{p(x)-2}u^2 \tag{3.20}$$

and

$$w(x,0) = w_0(x), w_t(x,0) = w_1(x).$$

Multiplying (3.20) by w_t, then integrating with respect to x, we get

$$\frac{1}{2}\int_\Omega |w_t|^2\,dx + \frac{1}{2}\int_\Omega |\nabla w_t|^2\,dx + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla w\|_2^2$$
$$+\frac{1}{2}(g\circ\nabla w)(t) - \frac{1}{2}\int_0^t (g'\circ\nabla w)(s)\,ds + \frac{1}{2}\int_0^t g(s)\|\nabla w\|_2^2\,ds$$
$$+\omega\int_0^t\int_\Omega\left(|u_t^1|^{m(x)-2}u_t^1 - |u_t^2|^{m(x)-2}u_t^2\right)w_t\,dx\,ds = \int_0^t\int_\Omega\left(|u^1|^{p(x)-2}u^1 - |u^2|^{p(x)-2}u^2\right)w_t\,dx\,ds.$$

By using the inequality

$$\left(|a|^{m(x)-2}a - |b|^{m(x)-2}b\right)(a-b)\geq 0$$

for all $a, b\in\mathbb{R}$ and a.e. $x\in\Omega$, this implies that

$$\frac{1}{2}\int_\Omega |w_t|^2\,dx + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla w\|_2^2 \leq C\int_0^t\int_\Omega\left(|u^1|^{p(x)-2}u^1 - |u^2|^{p(x)-2}u^2\right)w_t\,dx\,ds.$$

Repeating the estimate as in [17], we arive at

$$\int_\Omega |w_t|^2\,dx + \|\nabla w\|_2^2 \leq C\int_0^t\left(\int_\Omega |w_t|^2\,dx + \|\nabla w\|_2^2\right)ds.$$

Gronwall's inequality yields

$$\int_\Omega |w_t|^2\,dx + \|\nabla w\|_2^2 = 0.$$

Thus $w = 0$. The shows the uniqueness. □

4 Global existence and energy decay

Theorem 4.1. *Suppose that the assumptions of Theorem 2.1 and (A_1) and (A_2) hold. If $(u_0, u_1)\in H_0^1(\Omega)\times H_0^1(\Omega)$, then the solution of (1.1) is bounded and global in time.*

Proof. It suffices to show that $\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2$ is bounded independently of t. To obtain this, we observe that

$$\begin{aligned} E(0) &\geq E(t)\geq \widetilde{E}(t)\\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{p^- - 2}{2p^-}\left(\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right) + \frac{1}{p^-}I(t)\\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{p^- - 2}{2p^-}\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right), \end{aligned} \tag{4.1}$$

since $I(t) > 0$, $(g\circ\nabla u)(t)$ are positives. Therefore,

$$\|\nabla u\|_2^2 + \|u_t\|_2^2 \leq CE(0),$$

where C is a positive constant, depends only on p^- and l and is independent of t. This infer that the solution of (1.1) is bounded and global in time. □

Lemma 4.1. *Under the assumptions of Theorem 2.1, we have*

$$\int_\Omega |u|^{2p(x)-2}\,dx \le c\|\nabla u\|_2^2, \qquad \int_\Omega |u_t|^{2m(x)-2}\,dx \le c\|\nabla u_t\|_2^2.$$

Proof. By Lemma 2.3, we have

$$\int_\Omega |u|^{2(p(x)-1)}\,dx \le \max\Big\{\|u\|_{2(p(\cdot)-1)}^{2(p^- -1)}, \|u\|_{2(p(\cdot)-1)}^{2(p^+ -1)}\Big\}.$$

On the other hand, by Lemma 2.4, we have

$$\begin{aligned}\int_\Omega |u|^{2(p(x)-1)}\,dx &\le \max\Big\{B^{2(p^- -1)}\|\nabla u\|_2^{2(p^- -1)}, B^{2(p^+ -1)}\|\nabla u\|_2^{2(p^+ -1)}\Big\}\\ &\le \max\Big\{B^{2(p^- -1)}\|\nabla u\|_2^{2(p^- -2)}, B^{2(p^+ -1)}\|\nabla u\|_2^{2(p^+ -2)}\Big\}\|\nabla u\|_2^2,\end{aligned}$$

since

$$2(p^- - 1) \le 2(p(x)-1) \le 2(p^+ - 1) \le \frac{2n}{n-2}.$$

Using (4.1), we obtain

$$\begin{aligned}\int_\Omega |u|^{2(p(x)-1)}\,dx &\le \max\Big\{B^{2(p^- -1)}\Big(\frac{2p^-}{l(p^- -2)}E(0)\Big)^{p^- -2}, B^{2(p^+ -1)}\Big(\frac{2p^-}{l(p^- -2)}E(0)\Big)^{p^+ -2}\Big\}\|\nabla u\|_2^2\\ &\le c\|\nabla u\|_2^2.\end{aligned}$$

Similarly, we get

$$\int_\Omega |u_t|^{2m(x)-2}\,dx \le c\|\nabla u_t\|_2^2. \qquad \square$$

Now, we define

$$G(t) = ME(t) + \epsilon\Phi(t) + \Psi(t), \tag{4.2}$$

where M and ϵ are positive constants which specified later and

$$\Phi(t) = \int_\Omega u_t u\,dx + \int_\Omega \nabla u_t(t)\nabla u(t)\,dx, \tag{4.3}$$

$$\Psi(t) = \int_\Omega (\Delta u_t - u_t)\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx. \tag{4.4}$$

Before we prove our result, we need the following lemmas.

Lemma 4.2. *Let $u \in L^\infty([0,T]; H_0^1(\Omega))$, then we have*

$$\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))\,ds\Big)^2 dx \le (1-l)c^2(g\circ\nabla u)(t),$$

where c is Sobolev–Poincaré constant.

Proof. By the Hölder inequality, we get

$$\begin{aligned}\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))\,ds\Big)^2 dx &\le \int_\Omega \Big(\int_0^t g(t-s)\,ds\Big)\Big(\int_0^t g(t-s)|u(t)-u(s)|^2\,ds\Big)dx\\ &\le (1-l)c^2\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2\,ds \le (1-l)c^2(g\circ\nabla u)(t). \qquad \square\end{aligned}$$

Lemma 4.3. *Let u be a solution of (1.1), then there exist two positive constants B_1 and B_2 such that*

$$B_1E(t) \leq G(t) \leq B_2E(t).$$

Proof. By Young's inequality, we have

$$\left|\int_\Omega u_t u\,dx\right| \leq \delta\|u_t\|_2^2 + \frac{1}{4\delta}\,\|u\|_2^2 \leq \delta\|u_t\|_2^2 + \frac{c}{4\delta}\,\|\nabla u\|_2^2 \tag{4.5}$$

and

$$\left|\int_\Omega \nabla u_t \nabla u\,dx\right| \leq \delta\|\nabla u_t\|_2^2 + \frac{1}{4\delta}\,\|\nabla u\|_2^2. \tag{4.6}$$

It follows from (4.4) that

$$\Psi(t) = -\int_\Omega \nabla u_t \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx - \int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx. \tag{4.7}$$

By Young's inequality and Hölder's inequality, the first term on the right-hand side of (4.7) can be estimated as

$$\begin{aligned}
&\left| -\int_\Omega \nabla u_t \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx\right| \\
&\qquad \leq \frac{1}{2}\,\|\nabla u_t\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)^2 dx \\
&\qquad \leq \frac{1}{2}\,\|\nabla u_t\|_2^2 + \frac{1-l}{2}\,(g\circ\nabla u)(t).
\end{aligned} \tag{4.8}$$

Applying similar arguments as in deriving (4.8) and then using Lemma 4.2, we have

$$\begin{aligned}
&\left| -\int_\Omega u_t \int_0^t g(t-s)(u(t) - u(s))\,ds\,dx\right| \\
&\qquad \leq \frac{1}{2}\,\|u_t\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\,ds\right)^2 dx \\
&\qquad \leq \frac{1}{2}\,\|u_t\|_2^2 + \frac{1-l}{2}\,c^2(g\circ\nabla u)(t).
\end{aligned} \tag{4.9}$$

Hence, by using (4.5)–(4.9), from (4.2) we have the following inequalities:

$$\begin{aligned}
G(t) &\leq ME(t) + \epsilon\Phi(t) + \Psi(t) \\
&\leq ME(t) + \lambda_1\|u_t\|_2^2 + \lambda_2\|\nabla u_t\|_2^2 + \lambda_3\|\nabla u\|_2^2 + \lambda_4(g\circ\nabla u)(t) \\
&\leq ME(t) + \lambda_5\Big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t)\Big),
\end{aligned}$$

where

$$\lambda_1 = \frac{1}{2} + \epsilon\delta, \quad \lambda_2 = \frac{1}{2} + \epsilon\delta, \quad \lambda_3 = \frac{1+c}{4\delta}, \quad \lambda_4 = \frac{1-l}{2}\,(1+c^2).$$

On the other hand, we have

$$G(t) \geq ME(t) - \lambda_5\Big(\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t)\Big),$$

where $\lambda_5 = \max(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Thus from the definition of $E(t)$ and (4.1), choosing M sufficiently large and ϵ small enough, there exist two positive constants B_1 and B_2 such that

$$B_1E(t) \leq G(t) \leq B_2E(t). \qquad \square$$

Theorem 4.2. *Given* $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$*, suppose that* (A_1) *and* (A_2) *hold. Then for* $t \geq t_0$ *the energy of the solution of* (1.1) *satisfies*

$$E(t) \leq ke^{-\xi(t-t_0)}, \quad t \geq t_0,$$

where ζ *is a positive constant.*

Proof. In order to obtain the decay result of $E(t)$, we need to estimate the derivative of $G(t)$. From (4.3) and the first equation of (1.1) it follows that

$$\Phi'(t) = \|u_t\|_2^2 + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 - \int_\Omega |u_t|^{m(x)-2} u_t u \, dx + \int_\Omega |u|^{p(x)} \, dx + \int_\Omega \nabla u \int_0^t g(t-s) \nabla u(s) \, ds \, dx. \tag{4.10}$$

The last term on the right-hand side of (4.10) can be estimated as

$$\left| \int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s) \, ds \, dx \right| \leq \int_\Omega \left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)| \, ds \right) dx + \int_0^t g(s) \, ds \|\nabla u\|_2^2$$
$$\leq (1+\eta) \int_0^t g(s) \, ds \|\nabla u\|_2^2 + \frac{1}{4\eta} (g \circ \nabla u)(t) \leq (1+\eta)(1-l)\|\nabla u\|_2^2 + \frac{1}{4\eta} (g \circ \nabla u)(t) \ \text{ for } \ \eta > 0. \tag{4.11}$$

Also, by Hölder's and Young's inequalities, we get

$$\left| \int_\Omega |u_t|^{m(x)-2} u_t u \, dx \right| \leq \eta \|u\|_2^2 + \frac{1}{4\eta} \int_\Omega |u_t|^{2m(x)-2} \, dx. \tag{4.12}$$

Substitution of (4.11) and (4.12) into (4.10) yields

$$\Phi'(t) \leq \|u_t\|_2^2 + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 + (1+\eta)(1-l)\|\nabla u\|_2^2 + \frac{1}{4\eta}(g \circ \nabla u)(t) + \eta \|u\|_2^2 + \frac{1}{4\eta} \int_\Omega |u_t|^{2m(x)-2} + \int_\Omega |u|^{p(x)} \, dx. \tag{4.13}$$

Next, we would like to estimate $\Psi'(t)$. Taking the derivative of $\Psi(t)$ in (4.4) and using the first equation of (1.1), we get

$$\begin{aligned}
\Psi'(t) = {} & \int_\Omega \nabla u(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
& - \int_\Omega \left(\int_0^t g(t-s) \nabla u(s) \, ds \right) \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \right) dx \\
& + \int_\Omega |u_t|^{m(x)-2} u_t \int_0^t g(t-s)(u(t) - u(s)) \, ds \, dx - \int_\Omega |u|^{p(x)-2} u \int_0^t g(t-s)(u(t)-u(s)) \, ds \, dx \\
& - \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s)) \, ds \, dx - \left(\int_0^t g(s) \, ds \right) \|\nabla u_t\|_2^2 \\
& - \left(\int_0^t g(s) \, ds \right) \|u_t\|_2^2 - \int_\Omega \nabla u_t \int_0^t g'(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx.
\end{aligned} \tag{4.14}$$

Similar to (4.13), in what follows, we estimate the right-hand side of (4.14),

$$\begin{aligned}&\Big|\int_\Omega \nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\Big|\\&\quad\le \delta\|\nabla u\|_2^2+\frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\Big)^2dx\le \delta\|\nabla u\|_2^2+\frac{1-l}{4\delta}(g\circ\nabla u)(t). \qquad (4.15)\end{aligned}$$

and

$$\Big|\int_\Omega\Big(\int_0^t g(t-s)\nabla u(s)\,ds\Big)\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\Big)dx\Big|\le \delta I_1+\frac{1}{4\delta}I_2, \qquad (4.16)$$

where

$$I_1=\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s)|\,ds\Big)^2dx,$$

$$I_2=\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|\,ds\Big)^2dx.$$

By Hölder's and Young's inequalities, for $\eta>0$, we obtain

$$\begin{aligned}I_1&\le\int_\Omega\Big(\int_0^t g(t-s)\big(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|\big)\,ds\Big)^2dx\\&\le\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|\,ds\Big)^2dx+\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)|\,ds\Big)^2dx\\&\qquad+2\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|\,ds\Big)\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)|\,ds\Big)dx\\&\le\Big(\int_0^t g(s)\,ds\Big)^2\|\nabla u\|_2^2+\int_\Omega\Big(\int_0^t g(t-s)\,ds\Big)\Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2\,ds\Big)dx\\&\qquad+\eta\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)|\,ds\Big)^2dx+\frac{1}{\eta}\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|\,ds\Big)^2dx\\&\le(1+\eta)(1-l)^2\|\nabla u\|_2^2+\Big(1+\frac{1}{\eta}\Big)(1-l)(g\circ\nabla u)(t) \qquad (4.17)\end{aligned}$$

and

$$I_2=\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|\,ds\Big)^2dx\le(1-l)(g\circ\nabla u)(t). \qquad (4.18)$$

Taking $\eta=\frac{l}{1-l}$ in (4.17) and using (4.18), from (4.16) we get

$$\begin{aligned}&\Big|-\int_\Omega\Big(\int_0^t g(t-s)\nabla u(s)\,ds\Big)\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\Big)dx\Big|\\&\qquad\le(1-l)\Big(\delta\|\nabla u\|_2^2+\Big(\frac{\delta}{l}+\frac{1}{4\delta}\Big)(1-l)(g\circ\nabla u)(t)\Big). \qquad (4.19)\end{aligned}$$

By Hölder's inequality, Young's inequality and Poincaré's inequality, we have

$$\Big|\int_\Omega |u_t|^{m(x)-2}u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\Big| \le \delta\int_\Omega |u_t|^{2m(x)-2}\,dx+\frac{(1-l)c^2}{4\delta}\,(g\circ\nabla u)(t) \tag{4.20}$$

and

$$\Big|\int_\Omega |u|^{p(x)-2}u\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\Big| \le \delta\int_\Omega |u|^{2p(x)-2}\,dx+\frac{(1-l)c^2}{4\delta}\,(g\circ\nabla u)(t). \tag{4.21}$$

Using Young's inequality and (A_1) to deal with the last term of (4.14), we have

$$\Big|-\int_\Omega \nabla u_t\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\Big| \le \delta\|\nabla u_t\|_2^2-\frac{g(0)}{4\delta}\,(g'\circ\nabla u)(t). \tag{4.22}$$

Exploiting again Young's inequality and (A_1) to estimate the fiveth term, we get

$$\begin{aligned}\Big|-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx\Big|\\ \le \delta\|u_t\|_2^2+\frac{1}{4\delta}\int_\Omega\int_0^t g'(t-s)|u(t)-u(s)|^2\,ds\,dx \le \delta\|u_t\|_2^2-\frac{g(0)c^2}{4\delta}\,(g'\circ\nabla u)(t).\end{aligned} \tag{4.23}$$

Further, combining estimates (4.15)–(4.23), (4.14) becomes

$$\begin{aligned}\Psi'(t) &\le \delta\|u_t\|_2^2+\delta\|\nabla u_t\|_2^2+(1-l)\delta\|\nabla u\|_2^2+\delta\int_\Omega |u_t|^{2m(x)-2}\,dx\\ &+\delta\int_\Omega |u|^{2p(x)-2}\,dx+\delta\|\nabla u\|_2^2+\frac{(1-l)}{4\delta}\,(g\circ\nabla u)(t)+\Big(\frac{\delta}{l}+\frac{1}{4\delta}\Big)(1-l)^2(g\circ\nabla u)(t)\\ &+\frac{(1-l)}{4\delta}\,c^2(g\circ\nabla u)(t)+\frac{(1-l)}{4\delta}\,c^2(g\circ\nabla u)(t)-\frac{g(0)}{4\delta}\,(g'\circ\nabla u)(t)\\ &-\frac{g(0)c^2}{4\delta}\,(g'\circ\nabla u)(t)-\Big(\int_0^t g(s)\,ds\Big)\|\nabla u_t\|_2^2-\Big(\int_0^t g(s)\,ds\Big)\|u_t\|_2^2.\end{aligned} \tag{4.24}$$

By (4.24) and Lemma 4.1, we obtain

$$\Psi'(t)\le c_1\|u_t\|_2^2+c_2\|\nabla u_t\|_2^2+c_3\|\nabla u\|_2^2+c_4(g\circ\nabla u)(t)-c_5(g'\circ\nabla u)(t), \tag{4.25}$$

where

$$c_1=\Big(\delta-\int_0^t g(s)\,ds\Big),\quad c_2=\Big(\delta+c\delta-\int_0^t g(s)\,ds\Big),$$

$$c_3=((1-l)\delta+\delta+c\delta),\quad c_4=\Big(\Big(\frac{\delta}{l}+\frac{1}{4\delta}\Big)(1-l)^2+\frac{(1-l)}{4\delta}+\frac{2(1-l)}{4\delta}\,c^2\Big)$$

and

$$c_5=\Big(\frac{g(0)}{4\delta}+\frac{g(0)c^2}{4\delta}\Big).$$

Since $g(t)$ is positive and continuous, for any $t_0 > 0$, there exist g_1, g_0 such that

$$g(t) \geq g_1 \quad \text{and} \quad \int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds = g_0, \quad \forall\, t \geq t_0. \tag{4.26}$$

Hence we conclude from (4.2), (4.13), (4.25) and (4.26) that for any $t \geq t_0 > 0$,

$$\begin{aligned} G'(t) &= ME'(t) + \epsilon\Phi'(t) + \Psi'(t) \\ &\leq \Big(\frac{M}{2} - c_5\Big)(g' \circ \nabla u)(t) + (\epsilon + c_1)\|u_t\|_2^2 + \Big(\epsilon + c_2 + \frac{\epsilon c}{4\eta}\Big)\|\nabla u_t\|_2^2 \\ &+ \Big(-\frac{M}{2}\, g_1 + c_3 - \epsilon + \epsilon c\eta + (1-\eta)(1-l)\Big)\|\nabla u\|_2^2 + \Big(c_4 + \frac{\epsilon}{4\eta}\Big)(g \circ \nabla u)(t) + \epsilon \int_\Omega |u|^{p(x)}\,dx. \end{aligned}$$

However, $g'(t) \leq -g(t)$ by (A_1), thus we can see that

$$\begin{aligned} G'(t) &\leq -(-\epsilon - c_1)\|u_t\|_2^2 \\ &- \Big(-\epsilon - c_2 - \frac{\epsilon c}{4\eta}\Big)\|\nabla u_t\|_2^2 - \Big(\frac{M}{2} g_1 - c_3 + \epsilon - \epsilon c\eta - (1-\eta)(1-l)\Big)\|\nabla u\|_2^2 \\ &- \Big(\frac{M}{2} - c_4 - c_5 - \frac{\epsilon}{4\eta}\Big)(g \circ \nabla u)(t) + \epsilon \int_\Omega |u|^{p(x)}\,dx. \end{aligned}$$

At this point, we take $\delta = \epsilon$, $\eta = \sqrt{\delta}$ and choose ϵ small enough such that $g_0 > (c+2)\epsilon + c\sqrt{\epsilon}$. Once ϵ is fixed, we pick M sufficiently large so that

$$\Big(\frac{M}{2} - c_4 - c_5 - \frac{\epsilon}{4\eta}\Big) > 0 \quad \text{and} \quad \Big(\frac{M}{2} g_1 - c_3 + \epsilon - \epsilon c\eta - (1-\eta)(1-l)\Big) > 0.$$

Therefore, for any $t \geq t_0$, we have

$$G'(t) \leq -\Big(c_6\|u_t\|_2^2 + c_7\|\nabla u_t\|_2^2 + c_8\|\nabla u\|_2^2 + c_9(g \circ \nabla u)(t) - \epsilon \int_\Omega |u|^{p(x)}\,dx\Big),$$

where

$$c_6 = (-\epsilon - c_1), \quad c_7 = \Big(-\epsilon - c_2 - \frac{\epsilon c}{4\eta}\Big), \quad c_8 = \Big(\frac{M}{2}\, g_1 - c_3 + \epsilon - \epsilon c\eta - (1-\eta)(1-l)\Big),$$

and

$$c_9 = \Big(\frac{M}{2} - c_4 - c_5 - \frac{\epsilon}{4\eta}\Big).$$

Combining Lemma 4.3 with (4.1) and (2.5), we get

$$G'(t) \leq -c_{10}E(t) \leq -\frac{c_{10}}{B_2}\, G(t), \tag{4.27}$$

for some positive constant $c_{10} > 0$. The integration of (4.27) over (t_0, t) gives

$$G(t) \leq G(t_0)e^{-\frac{c_{10}}{B_2}(t-t_0)}, \quad t \geq t_0.$$

Again, by virtue of Lemma 4.3,

$$E(t) \leq \frac{G(t_0)}{B_1}\, e^{-\frac{c_{10}}{B_2}(t-t_0)}, \quad t \geq t_0.$$

This completes the proof. □

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