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DARBOUX TYPE PROBLEM FOR A CLASS OF FOURTH-ORDER NONLINEAR HYPERBOLIC EQUATIONS

Abstract. Darboux type problem for a class of fourth-order nonlinear hyperbolic equations is considered. The theorems on existence, uniqueness and nonexistence of solutions of this problem are proved.

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1 Statement of the problem

On the plane of variables x and t, we consider the fourth-order hyperbolic equation of the following form:

$$\square^2 u + f(\square u) + g(u) = F(x,t), \tag{1.1}$$

where $\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$; f, g and F are given functions, while u is an unknown function.

Denote by $D_T: \ 0 < x < t, \ t < T$, an angular domain bounded by a characteristic segment $\gamma_{1,T}: \ x = t, \ 0 \le t \le T$, and by time and spatial orientation segments $\gamma_{2,T}: \ x = 0, \ 0 \le t \le T$, and $\gamma_{3,T}: \ t = T, \ 0 \le x \le T$, respectively; for $T = \infty$, we have $D_\infty: \ t > |x|, \ x > 0$, and

$$\gamma_{1,\infty}: \ \ x = t, \ \ 0 \le t < \infty; \quad \gamma_{2,T}: \ \ x = 0, \ \ 0 \le t < \infty.$$

For equation (1.1) in the domain D_T, consider the following boundary value problem: find in D_T a solution $u = u(x,t)$ to equation (1.1) which on the parts $\gamma_{1,T}$ and $\gamma_{2,T}$ of the boundary satisfies the following conditions:

$$u\big|_{\gamma_{1,T}} = u(t,t) = \mu_1(t), \quad \frac{\partial u}{\partial \nu}\Big|_{\gamma_{1,T}} = \frac{\partial u}{\partial \nu}\,(t,t) = \mu_2(t), \ \ 0 \le t \le T, \tag{1.2}$$

$$u\big|_{\gamma_{2,T}} = u(0,t) = \mu_3(t), \quad \frac{\partial^2 u}{\partial x^2}\Big|_{\gamma_{2,T}} = \frac{\partial^2 u}{\partial x^2}\,(0,t) = \mu_4(t), \ \ 0 \le t \le T, \tag{1.3}$$

where μ_i, $i = 1,\dots,4$, are the given scalar functions and the functions μ_1 and μ_2 at a common point $O = O(0,0)$ of the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ satisfy the condition of agreement $\mu_1(0) = \mu_3(0)$, $\nu = (\nu_x, \nu_t)$ is a unit vector of outer normal to the boundary ∂D_T.

It is noteworthy that the Darboux problems for the second order hyperbolic equation

$$\square u + f(x,t,u) = F(x,t)$$

in angular domain D_T with the Dirichlet or Neumann boundary conditions on the boundary segments $\gamma_{1,T}$ and $\gamma_{2,T}$ were studied by many authors [1–14, 16–22, 26–29, 31, 32, 34]. Some boundary value problems for equation (1.1) in spatial multidimensional case when $\square := \frac{\partial^2}{\partial t^2} - \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n > 1$, $f = 0$, were studied in [15, 23–25].

Remark 1.1. Let $f, g \in C(R)$, $F \in C(\overline{D}_T)$. If u, where $u, \square u \in C^2\overline{D}_T$, represents a classical solution to problem (1.1)–(1.3), then introducing a function $v = \square u$ this problem can be reduced to the following boundary value problem with respect to unknown functions u and v:

$$L_1(u,v) := \square u - v = 0, \ \ (x,t) \in D_T, \tag{1.4}$$

$$L_2(u,v) := \square v + f(v) + g(u) = F(x,t), \ \ (x,t) \in D_T, \tag{1.5}$$

$$u\big|_{\gamma_{1,T}} = u(t,t) = \mu_1(t), \ \ u\big|_{\gamma_{2,T}} = u(0,t) = \mu_3(t), \ \ 0 \le t \le T, \tag{1.6}$$

$$v\big|_{\gamma_{1,T}} = v(t,t) = -\sqrt{2}\,\mu_2'(t), \ \ v\big|_{\gamma_{2,T}} = v(0,t) = \mu_3''(t) - \mu_4(t), \ \ 0 \le t \le T. \tag{1.7}$$

Here, in receiving the first equality of (1.7), we took into account that

$$\frac{d}{dt}\,w(t,t) = \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big)\,w\big|_{t=x}, \quad \frac{\partial}{\partial \nu}\Big|_{\gamma_{1,T}} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\Big),$$

therefore,

$$v\big|_{\gamma_{1,T}} = \square u\big|_{\gamma_{1,T}} = \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\Big)u\big|_{\gamma_{1,T}} = -\sqrt{2}\,\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big)\,\frac{\partial u}{\partial \nu}\Big|_{\gamma_{1,T}} = -\sqrt{2}\,\mu_2'(t),$$

while in receiving the second equality of (1.7), we took into account (1.2), (1.3), and

$$v = \Box\, u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\,,$$

therefore,

$$v\big|_{\gamma_{2,T}} = v(0,t) = \frac{\partial^2 u}{\partial t^2}\,(0,t) - \frac{\partial^2 u}{\partial x^2}\,(0,t) = \mu_3''(t) - \mu_4(t).$$

Vice versa, if $u, v \in C^2(\overline{D}_T)$ represents a classical solution to problem (1.4)–(1.7), where $\mu_1, \mu_4 \in C^2([0,T])$, $\mu_2 \in C^3([0,T])$, $\mu_3 \in C^4([0,T])$, then the function u will be a classical solution to problem (1.1)–(1.3).

Definition 1.1. Let $f, g \in C(R)$, $F \in C(\overline{D}_T)$ and, for simplicity, $\mu_i = 0$, $i = 1, \dots, 4$. The system of functions u and v is called a generalized solution of problem (1.4)–(1.7) of the class C if $u, v \in C(\overline{D}_T)$ and there exist the sequences

$$u_n, v_n \in \overset{\circ}{C}{}^2(\overline{D}_T) := \Big\{ w\in C^2(\overline{D}_T):\ \ w\big|_{\gamma_{i,T}} = 0,\ \ i = 1,2 \Big\} \tag{1.8}$$

such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|v_n - v\|_{C(\overline{D}_T)} = 0, \tag{1.9}$$

$$\lim_{n\to\infty} \|L_1(u_n, v_n)\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|L_2(u_n, v_n) - F\|_{C(\overline{D}_T)} = 0. \tag{1.10}$$

Remark 1.2. It is clear that the classical solution $u, v \in C^2(\overline{D}_T)$ of problem (1.4)–(1.7) represents a generalized solution of class C of this problem.

2 A priori estimate of a solution of the problem (1.4)–(1.7)

Lemma 2.1. *Let $f, g \in C(R)$, $F \in C(\overline{D}_T)$, $\mu_i = 0$, $i = 1, \dots, 4$. Then for any solution u, v of problem (1.4)–(1.7) of class C the following inequality is valid:*

$$|u(x,t)| \le te^t \|v\|_{L_2(D_t)}, \ \ (x,t) \in D_T. \tag{2.1}$$

Proof. Let u, v be the generalized solution of class C of problem (1.4)–(1.7), then there exist the sequences u_n, v_n which satisfy conditions (1.8)–(1.10).

Consider a function $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T)$ as a classical solution to the following boundary value problem:

$$L_1(u_n, v_n) := \Box\, u_n - v_n = G_n(x,t), \ \ (x,t) \in D_T, \tag{2.2}$$

$$u_n\big|_{\gamma_{1,T}} = u_n(t,t) = 0, \quad u_n\big|_{\gamma_{2,T}} = u_n(0,t) = 0, \ \ 0 \le t \le T, \tag{2.3}$$

where the function

$$G_n := L_1(u_n, v_n) \tag{2.4}$$

due to (1.10) satisfies the condition

$$\lim_{n\to\infty} \|G_n\|_{C(\overline{D}_T)} = 0. \tag{2.5}$$

Multiplying both sides of equation (2.2) by the function $\frac{\partial u_n}{\partial t}$ and integrating over the domain $D_\tau := \{(x,t) \in D_T :\ t < \tau\}$, where $0 < \tau \le T$, we get

$$\frac{1}{2} \int\limits_{D_\tau} \frac{\partial}{\partial t} \Big(\frac{\partial u_n}{\partial t}\Big)^2 \,dx\,dt - \int\limits_{D_\tau} \frac{\partial^2 u_n}{\partial x^2}\, \frac{\partial u_n}{\partial t}\,dx\,dt - \int\limits_{D_\tau} v_n\, \frac{\partial u_n}{\partial t}\,dx\,dt = \int\limits_{D_\tau} G_n\, \frac{\partial u_n}{\partial t}\,dx\,dt. \tag{2.6}$$

Using integration by parts and the Green formula, we obtain

$$\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt=\frac{1}{2}\int\limits_{\partial D_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\nu_t\,ds, \tag{2.7}$$

$$\begin{aligned}-\int\limits_{D_\tau}\frac{\partial^2 u_n}{\partial x^2}\cdot\frac{\partial u_n}{\partial t}\,dx\,dt&=-\int\limits_{\partial D_\tau}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds+\int\limits_{D_\tau}\frac{\partial u_n}{\partial x}\cdot\frac{\partial^2 u_n}{\partial t\partial x}\,dx\,dt\\&=-\int\limits_{\partial D_\tau}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds+\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial x}\Big)^2\,dx\,dt\\&=-\int\limits_{\partial D_\tau}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds+\frac{1}{2}\int\limits_{\partial D_\tau}\Big(\frac{\partial u_n}{\partial x}\Big)^2\nu_t\,ds,\end{aligned} \tag{2.8}$$

where $\nu=(\nu_x,\nu_t)$ is a unit vector of the outer normal to the boundary ∂D_τ.

Taking into account that $\partial D_\tau=\gamma_{1,\tau}\cup\gamma_{2,\tau}\cup\omega_\tau$, where $\gamma_{i,\tau}=\gamma_{i,T}\cap\{t\le\tau\}$, $i=1,2$, and $\omega_\tau=\partial D_\tau\cap\{t=\tau\}=\{t=\tau,\ 0\le x\le\tau\}$, we have

$$(\nu_x,\nu_t)\big|_{\gamma_{1,T}}=\Big(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\Big), \tag{2.9}$$

$$(\nu_x,\nu_t)\big|_{\gamma_{2,T}}=(-1,0),\quad (\nu_x,\nu_t)|_{\omega_\tau}=(0,1), \tag{2.10}$$

$$({\nu_x}^2-{\nu_t}^2)\big|_{\gamma_{1,T}}=0. \tag{2.11}$$

Taking into account (2.9)–(2.11), since $u_n|_{\gamma_{2,T}}=0$ (see (2.3)) and, therefore, $\frac{\partial u_n}{\partial t}|_{\gamma_{2,T}}=0$, from (2.7) and (2.8) we get

$$\begin{aligned}\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt&=\frac{1}{2}\int\limits_{\partial D_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\nu_t\,ds\\&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,ds+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\frac{\partial u_n}{\partial t}\Big)^2\nu_t\,ds+\frac{1}{2}\int\limits_{\gamma_{2,\tau}}\Big(\frac{\partial u_n}{\partial t}\Big)^2\nu_t\,ds\\&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\frac{\partial u_n}{\partial t}\Big)^2\nu_t\,ds,\end{aligned} \tag{2.12}$$

$$\begin{aligned}-\int\limits_{D_\tau}\frac{\partial^2 u_n}{\partial x^2}\cdot\frac{\partial u_n}{\partial t}\,dx\,dt&=-\int\limits_{\partial D_\tau}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds+\frac{1}{2}\int\limits_{\partial D_\tau}\Big(\frac{\partial u_n}{\partial x}\Big)^2\nu_t\,ds\\&=-\int\limits_{\omega_\tau}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds-\int\limits_{\gamma_{1,\tau}}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds-\int\limits_{\gamma_{2,\tau}}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds\\&\quad+\frac{1}{2}\int\limits_{\omega_\tau}\Big(\frac{\partial u_n}{\partial x}\Big)^2\nu_t\,ds+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\frac{\partial u_n}{\partial x}\Big)^2\nu_t\,ds+\frac{1}{2}\int\limits_{\gamma_{2,\tau}}\Big(\frac{\partial u_n}{\partial x}\Big)^2\nu_t\,ds\\&=0-\int\limits_{\gamma_{1,\tau}}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds-0+\frac{1}{2}\int\limits_{\omega_\tau}\Big(\frac{\partial u_n}{\partial x}\Big)^2\cdot 1\,dx\\&\quad+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\frac{\partial u_n}{\partial x}\Big)^2\nu_t\,ds+0\\&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\frac{\partial u_n}{\partial x}\Big)^2\,dx+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\frac{\partial u_n}{\partial x}\Big)^2\nu_t\,ds-\int\limits_{\gamma_{1,\tau}}\frac{\partial u_n}{\partial x}\cdot\frac{\partial u_n}{\partial t}\,\nu_x\,ds.\end{aligned} \tag{2.13}$$

From (2.12) and (2.13), in view of (2.11), we have

$$\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt-\int\limits_{D_\tau}\frac{\partial^2 u_n}{\partial x^2}\cdot\frac{\partial u_n}{\partial t}\,dx\,dt$$
$$=\frac{1}{2}\int\limits_{\omega_\tau}\Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx+\int\limits_{\gamma_{1,\tau}}\frac{1}{2\nu_t}\Big[\Big(\frac{\partial u_n}{\partial x}\,\nu_t-\frac{\partial u_n}{\partial t}\,\nu_x\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2(\nu_t^2-\nu_x^2)\Big]\,ds$$
$$=\frac{1}{2}\int\limits_{\omega_\tau}\Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx+\int\limits_{\gamma_{1,\tau}}\frac{1}{2\nu_t}\Big(\frac{\partial u_n}{\partial x}\,\nu_t-\frac{\partial u_n}{\partial t}\,\nu_x\Big)^2\,ds. \tag{2.14}$$

Taking into account that $(v_t\frac{\partial}{\partial x}-v_x\frac{\partial}{\partial t})$ represents the derivative in a tangent direction, i.e., an inner differential on the curve $\gamma_{1,T}$, due to the equality $u_n|_{\gamma_{1,T}}=0$, we have

$$\frac{\partial u_n}{\partial x}\,\nu_t-\frac{\partial u_n}{\partial t}\,\nu_x=0,$$

and from (2.14) we obtain

$$\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt-\int\limits_{D_\tau}\frac{\partial^2 u_n}{\partial x^2}\cdot\frac{\partial u_n}{\partial t}\,dx\,dt=\frac{1}{2}\int\limits_{\omega_\tau}\Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx. \tag{2.15}$$

From (2.6) and (2.15) it follows that

$$\int\limits_{\omega_\tau}\Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx=2\int\limits_{D_\tau}v_n\,\frac{\partial u_n}{\partial t}\,dx\,dt+2\int\limits_{D_\tau}G_n\,\frac{\partial u_n}{\partial t}\,dx\,dt. \tag{2.16}$$

Using a simple inequality $2ab\le a^2+b^2$, from (2.16) we obtain

$$\int\limits_{\omega_\tau}\Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx\le\int\limits_{D_\tau}\Big[v_n^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx\,dt+\int\limits_{D_\tau}\Big[G_n^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx\,dt$$
$$=2\int\limits_{D_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt+\int\limits_{D_\tau}\big[v_n^2+G_n^2\big]\,dx\,dt. \tag{2.17}$$

If we introduce the notation

$$w(\tau)=\int\limits_{\omega_\tau}\Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx$$

and take into account that

$$\int\limits_{D_\tau}\Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx\,dt=\int\limits_0^\tau w(\sigma)\,d\sigma,$$

then from (2.17) we have

$$w(\tau)\le 2\int\limits_{D_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt+\int\limits_{D_\tau}\big[v_n^2+G_n^2\big]\,dx\,dt$$
$$\le 2\int\limits_{D_\tau}\Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big]\,dx\,dt+\int\limits_{D_\tau}\big[v_n^2+G_n^2\big]\,dx\,dt$$
$$=2\int\limits_0^\tau w(\sigma)\,d\sigma+\int\limits_{D_\tau}v_n^2\,dx\,dt+\int\limits_{D_\tau}G_n^2\,dx\,dt$$
$$=2\int\limits_0^\tau w(\sigma)\,d\sigma+\|v_n\|^2_{L_2(D_\tau)}+\|G_n\|^2_{L_2(D_\tau)},\quad 0<\tau\le T. \tag{2.18}$$

According to the Gronwall lemma, from (2.18) we obtain

$$w(\tau) \le \Big(\|v_n\|^2_{L_2(D_\tau)} + \|G_n\|^2_{L_2(D_\tau)}\Big)e^{2\tau}, \ \ 0 < \tau \le T. \tag{2.19}$$

Since $u_n(0,t) = 0$, $0 \le t \le T$, we have

$$u_n(x,t) = \int\limits_0^x \frac{\partial u_n}{\partial x}(\xi,t)\,d\xi, \ \ (x,t) \in D_T,$$

whence due to the Cauchy inequality, we have

$$\begin{aligned} u_n^2(x,t) &\le \int\limits_0^x 1^2\,d\xi \int\limits_0^t \Big(\frac{\partial u_n}{\partial x}\Big)^2(\xi,t)\,d\xi \le x\int\limits_0^t \Big(\frac{\partial u_n}{\partial x}\Big)^2(\xi,t)\,d\xi \\ &\le t\int\limits_0^t \Big(\frac{\partial u_n}{\partial x}\Big)^2(\xi,t)\,d\xi \le t\int\limits_0^t \Big[\Big(\frac{\partial u_n}{\partial x}\Big)^2 + \Big(\frac{\partial u_n}{\partial t}\Big)^2\Big](\xi,t)\,d\xi = tw(t), \ \ (x,t) \in D_T. \end{aligned} \tag{2.20}$$

Here, we take into account that if $(x,t) \in D_T$, then $x < t$.
From (2.19) and (2.20) follows

$$\begin{aligned} |u_n(x,t)| &\le t^{\frac12} w^{\frac12}(t) \le t^{\frac12}\Big(\|v_n\|^2_{L_2(D_t)} + \|G_n\|^2_{L_2(D_t)}\Big)^{\frac12} e^t \\ &\le t^{\frac12}\Big(\|v_n\|_{L_2(D_t)} + \|G_n\|_{L_2(D_t)}\Big)e^t, \ \ (x,t) \in D_T. \end{aligned} \tag{2.21}$$

If we pass to the limit in inequality (2.21) as $n \to \infty$, then in view of (1.9), (1.10) and (2.2), we obtain

$$|u(x,t)| \le t^{\frac12} e^t \|v\|_{L_2(D_t)}, \ \ (x,t) \in D_T.$$

□

Consider the conditions imposed on the functions f and g:

$$\int\limits_0^s f(\tau)\,d\tau \ge -M_1 - M_2 s^2 \ \ \forall\, s \in R, \ \ M_i = const \ge 0, \ \ i = 1,2, \tag{2.22}$$

$$|g(s)| \le N_1 + N_2|s| \ \ \forall\, s \in R, \ \ N_i = const \ge 0, \ \ i = 1,2. \tag{2.23}$$

Lemma 2.2. *Let $f,g \in C(R)$, $F \in C(\overline{D})$, $\mu_i = 0$, $i = 1,\dots,4$, and the functions f and g satisfy conditions (2.22) and (2.23). Then for any generalized solution u, v of problem (1.4)–(1.7) of class C, the following a priori estimates are valid:*

$$|u(x,t)| \le C_1\|F\|_{L_2(D_t)} + C_2, \ \ (x,t) \in D_T, \tag{2.24}$$

$$|v(x,t)| \le C_3\|F\|_{L_2(D_t)} + C_4, \ \ (x,t) \in D_T, \tag{2.25}$$

where the values $C_i = C_i(t) \ge 0$, $i = 1,\dots,4$, do not depend on the functions u, v and F.

Proof. Let u, v be a generalized solution of problem (1.4)–(1.7) of class C, then there exist the sequences u_n, v_n which satisfy conditions (1.8)–(1.10).

Consider the function $v_n \in \overset{\circ}{C}{}^2(\overline{D}_T)$ as a classical solution of the following boundary value problem:

$$L_2(u_n,v_n) := \Box\, v_n + f(v_n) + g(u_n) = Q_n(x,t), \ \ (x,t) \in D_T, \tag{2.26}$$

$$v_n\big|_{\gamma_{1,T}} = v_n(t,t) = 0, \quad v_n\big|_{\gamma_{2,T}} = v_n(0,t) = 0, \ \ 0 \le t \le T, \tag{2.27}$$

where the function

$$Q_n := L_2(u_n,v_n) \tag{2.28}$$

due to (1.10) satisfies the condition

$$\lim_{n\to\infty} \|Q_n - F\|_{C(\overline{D}_T)} = 0. \tag{2.29}$$

Multiplying both sides of equation (2.26) by the function $\frac{\partial v_n}{\partial t}$ and integrating over the domain $D_\tau := \{(x,t) \in D_T :\ t < \tau\}$, where $0 < \tau \leq T$, we obtain

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial v_n}{\partial t}\Big)^2\,dx\,dt - \int\limits_{D_\tau} \frac{\partial^2 v_n}{\partial x^2}\cdot\frac{\partial v_n}{\partial t}\,dx\,dt \\ + \int\limits_{D_\tau} f(v_n)\,\frac{\partial v_n}{\partial t}\,dx\,dt + \int\limits_{D_\tau} g(u_n)\,\frac{\partial v_n}{\partial t}\,dx\,dt = \int\limits_{D_\tau} Q_n\,\frac{\partial v_n}{\partial t}\,dx\,dt. \tag{2.30}$$

Analogously as we obtained (2.16) from (2.6) when proving Lemma 2.1, from (2.30) we have the following equality:

$$\int\limits_{\omega_\tau} \Big[\Big(\frac{\partial v_n}{\partial x}\Big)^2 + \Big(\frac{\partial v_n}{\partial t}\Big)^2\Big]\,dx \\ = -2\int\limits_{D_\tau} f(v_n)\,\frac{\partial v_n}{\partial t}\,dx\,dt - 2\int\limits_{D_\tau} g(u_n)\,\frac{\partial v_n}{\partial t}\,dx\,dt + 2\int\limits_{D_\tau} Q_n\,\frac{\partial v_n}{\partial t}\,dx\,dt. \tag{2.31}$$

Using the notation

$$I(s) = \int\limits_0^s f(\tau)\,d\tau, \tag{2.32}$$

we have

$$\frac{\partial I(v_n)}{\partial t} = f(v_n)\,\frac{\partial v_n}{\partial t}\,.$$

Taking into account that $I(0) = 0$, $v_n|_{\gamma_{i,T}} = 0$, $i = 1,2$, and, therefore $I(v_n)|_{\gamma_{i,T}} = 0$, $i = 1,2$, due to (2.10) and the Green formula, we obtain

$$-2\int\limits_{D_\tau} f(v_n)\,\frac{\partial v_n}{\partial t}\,dx\,dt = -2\int\limits_{D_\tau} \frac{\partial I(v_n)}{\partial t}\,dx\,dt = -2\int\limits_{\partial D_\tau} I(v_n)\nu_t\,ds \\ = -2\int\limits_{\omega_\tau} I(v_n)\cdot 1\,ds - 2\int\limits_{\gamma_{1,\tau}} I(v_n)\nu_t\,ds - 2\int\limits_{\gamma_{2,\tau}} I(v_n)\nu_t\,ds = -2\int\limits_{\omega_\tau} I(v_n)\,dx. \tag{2.33}$$

In view of (2.22), from (2.32) and (2.33) we get

$$-2\int\limits_{D_\tau} f(v_n)\,\frac{\partial v_n}{\partial t}\,dx\,dt \leq 2\int\limits_{\omega_\tau} (M_1 + M_2 v_n^2)\,dx \leq 2M_1\tau + 2M_2\int\limits_{\omega_\tau} v_n^2\,dx. \tag{2.34}$$

According to condition (2.23), we have

$$-2\int\limits_{D_\tau} g(u_n)\,\frac{\partial v_n}{\partial t}\,dx\,dt \leq \int\limits_{D_\tau} \Big(g^2(u_n) + \Big(\frac{\partial v_n}{\partial t}\Big)^2\Big)\,dx\,dt \\ \leq \int\limits_{D_\tau} \big(N_1 + N_2|u_n|\big)^2\,dx\,dt + \int\limits_{D_\tau} \Big(\frac{\partial v_n}{\partial t}\Big)^2\,dx\,dt \\ \leq \int\limits_{D_\tau} (2N_1^2 + 2N_2^2 u_n^2)\,dx\,dt + \int\limits_{D_\tau} \Big(\frac{\partial v_n}{\partial t}\Big)^2\,dx\,dt \\ = \tau^2 N_1^2 + 2N_2^2\int\limits_{D_\tau} u_n^2(x,t)\,dx\,dt + \int\limits_{D_\tau} \Big(\frac{\partial v_n}{\partial t}\Big)^2\,dx\,dt, \tag{2.35}$$

where we use the simple inequalities $2ab \le a^2+b^2$, $(a+b)^2 \le 2a^2+2b^2$ and the equality

$$\int\limits_{D_\tau} 1\cdot\, dx\, d\tau = \frac{1}{2}\,\tau^2.$$

For $(x,t)\in D_\tau$, from (2.21) we have

$$u_n^2(x,t) \le t\Big(\|v_n\|_{L_2(D_\tau)} + \|G_n\|_{L_2(D_\tau)}\Big)^2 e^{2t} \\ \le 2te^{2t}\Big(\|v_n\|^2_{L_2(D_\tau)} + \|G_n\|^2_{L_2(D_\tau)}\Big) \le 2\tau e^{2\tau}\Big(\|v_n\|^2_{L_2(D_\tau)} + \|G_n\|^2_{L_2(D_\tau)}\Big),$$

whence we obtain

$$\int\limits_{D_\tau} u_n^2(x,t)\, dx\, dt \le 2\tau e^{2\tau}\Big(\|v_n\|^2_{L_2(D_\tau)} + \|G_n\|^2_{L_2(D_\tau)}\Big)\int\limits_{D_\tau} 1\cdot\, dx\, d\tau \\ = \tau^3 e^{2\tau}\|v_n\|^2_{L_2(D_\tau)} + \tau^3 e^{2\tau}\|G_n\|^2_{L_2(D_\tau)} = \tau^3 e^{2\tau}\int\limits_{D_\tau} v_n^2\, dx\, dt + \tau^3 e^{2\tau}\|G_n\|^2_{L_2(D_\tau)}. \tag{2.36}$$

Due to (2.34), (2.35) and (2.36), from (2.31) we obtain

$$\int\limits_{\omega_\tau}\Big[\Big(\frac{\partial v_n}{\partial x}\Big)^2 + \Big(\frac{\partial v_n}{\partial t}\Big)^2\Big]\, dx \\ \le 2M_1\tau + 2M_2\int\limits_{\omega_\tau} v_n^2\, dx + \tau^2 N_1^2 + 2N_2^2\Big[\tau^3 e^{2\tau}\int\limits_{D_\tau} v_n^2\, dx\, dt + \tau^3 e^{2\tau}\|G_n\|^2_{L_2(D_\tau)}\Big] \\ + \int\limits_{D_\tau}\Big(\frac{\partial v_n}{\partial t}\Big)^2\, dx\, dt + \int\limits_{D_\tau}\Big(\frac{\partial v_n}{\partial t}\Big)^2\, dx\, dt + \int\limits_{D_\tau} Q_n^2\,\frac{\partial v_n}{\partial t}\, dx\, dt \\ \le (2M_2 + 2N_2^2\tau^3 e^{2\tau} + 2)\int\limits_{D_\tau}\Big[v_n^2 + \Big(\frac{\partial v_n}{\partial x}\Big)^2 + \Big(\frac{\partial v_n}{\partial t}\Big)^2\Big]\, dx\, dt \\ + 2M_1\tau + \tau^2 N_1^2 + 2N_2^2\tau^3 e^{2\tau}\|G_n\|^2_{L_2(D_\tau)} + \int\limits_{D_\tau} Q_n^2\, dx\, dt. \tag{2.37}$$

If we take into account conditions (2.27) and use the Newton–Leibniz formula, we get

$$v_n(x,\tau) = v_n(x,x) + \int\limits_x^\tau \frac{\partial v_n}{\partial t}\,(x,t)\, dt = \int\limits_x^\tau \frac{\partial v_n}{\partial t}\,(x,t)\, dt, \quad (x,\tau)\in D_T,$$

and, therefore, using Cauchy's inequality, we get

$$v_n^2(x,\tau) \le \Big[\int\limits_x^\tau 1\cdot\Big|\frac{\partial v_n}{\partial t}(x,t)\Big|\, dt\Big]^2 \le \int\limits_x^\tau 1^2\, dt\cdot\int\limits_x^\tau\Big(\frac{\partial v_n}{\partial t}(x,t)\Big)^2\, dt \\ = (\tau - x)\int\limits_x^\tau\Big(\frac{\partial v_n}{\partial t}(x,t)\Big)^2\, dt \le T\int\limits_x^\tau\Big(\frac{\partial v_n}{\partial t}(x,t)\Big)^2\, dt. \tag{2.38}$$

Integrating equality (2.38), we obtain

$$\int\limits_{\omega_\tau} v_n^2\, dx = \int\limits_0^\tau v_n^2(x,\tau)\, dx \le T\int\limits_0^\tau\Big[\int\limits_x^\tau\Big(\frac{\partial v_n}{\partial t}(x,t)\Big)^2\, dt\Big]\, dx = T\int\limits_{D_\tau}\Big(\frac{\partial v_n}{\partial t}\Big)^2\, dx\, dt. \tag{2.39}$$

If we add inequalities (2.37) and (2.39), we get

$$\int\limits_{\omega_\tau}\Big[v_n^2+\Big(\frac{\partial v_n}{\partial x}\Big)^2+\Big(\frac{\partial v_n}{\partial t}\Big)^2\Big]\,dx \\ \leq (2M_2+2N_2^2\tau^3e^{2\tau}+T+3)\int\limits_{D_\tau}\Big[v_n^2+\Big(\frac{\partial v_n}{\partial x}\Big)^2+\Big(\frac{\partial v_n}{\partial t}\Big)^2\Big]\,dx\,dt \\ +2M_1\tau+\tau^2N_1^2+2N_2^2\tau^3e^{2\tau}\|G_n\|^2_{L_2(D_\tau)}+\|Q_n\|^2_{L_2(D_\tau)}. \tag{2.40}$$

Using the notation

$$w_1(\tau)=\int\limits_{\omega_\tau}\Big[v_n^2+\Big(\frac{\partial v_n}{\partial x}\Big)^2+\Big(\frac{\partial v_n}{\partial t}\Big)^2\Big]\,dx \tag{2.41}$$

and taking into account

$$\int\limits_{D_\tau}\Big[v_n^2+\Big(\frac{\partial v_n}{\partial x}\Big)^2+\Big(\frac{\partial v_n}{\partial t}\Big)^2\Big]\,dx=\int\limits_0^\tau w(\sigma)\,d\sigma,$$

from (2.40) we obtain

$$w_1(\tau)\leq M_3\int\limits_0^\tau w(\sigma)\,d\sigma+\widetilde{M}_4,\ \ 0<\tau\leq T, \tag{2.42}$$

where

$$M_3=2M_2+2N_2^2T^3e^{2T}+T+3, \\ \widetilde{M}_4=2M_1\tau+\tau^2N_1^2+2N_2^2\tau^3e^{2\tau}\|G_n\|^2_{L_2(D_\tau)}+\|Q_n\|^2_{L_2(D_\tau)}. \tag{2.43}$$

According to the Gronwall lemma, from (2.42) we obtain

$$w_1(\tau)\leq\widetilde{M}_4e^{M_3\tau},\ \ 0<\tau\leq T. \tag{2.44}$$

Analogously to how inequality (2.20) was obtained, from (2.41) and (2.44) we get

$$|v_n(x,t)|\leq t^{\frac12}w_1^{\frac12}(t)\leq\widetilde{M}_4^{\frac12}t^{\frac12}e^{\frac12 M_3t},\ \ (x,t)\in D_T, \tag{2.45}$$

where $\tau=t$ in $\widetilde{M}_4$.

If we pass to the limit in (2.45) as $n\to\infty$, due to the limit equalities (1.9), (2.5) and (2.29), we obtain

$$|v(x,t)|\leq M_4^{\frac12}t^{\frac12}e^{\frac12 M_3t},\ \ (x,t)\in D_T, \tag{2.46}$$

where

$$M_4=2M_1t+t^2N_1^2+\|F\|^2_{L_2(D_t)}. \tag{2.47}$$

From (2.1) and (2.46) it follows that

$$|u(x,t)|\leq te^t\|v\|_{L_2(D_t)}=te^t\Big(\int\limits_{D_t}v^2\,dx\,dt\Big)^{\frac12} \\ \leq te^t\Big(\int\limits_{D_t}M_4Te^{M_3T}\,dx\,dt\Big)^{\frac12}=te^t\Big(M_4Te^{M_3T}\int\limits_{D_t}1\,dx\,dt\Big)^{\frac12} \\ =te^t\Big(M_4Te^{M_3T}\,\frac12\,t^2\Big)^{\frac12}=\frac{1}{\sqrt2}\,t^2T^{\frac12}M_4^{\frac12}e^{t+\frac12 M_3T},\ \ (x,t)\in D_T. \tag{2.48}$$

According to the simple inequality

$$\Big(\sum_{i=1}^{m} a_i^2\Big)^{\frac{1}{2}} \le \sum_{i=1}^{m} |a_i|,$$

from (2.47) we have

$$M_4^{\frac{1}{2}} \le (2M_1 t)^{\frac{1}{2}} + tN_1 + \|F\|_{L_2(D_t)} \tag{2.49}$$

and (2.46), (2.48) can be rewritten as follows:

$$|u(x,t)| \le C_1\|F\|_{L_2(D_t)} + C_2,\ \ (x,t)\in D_T,$$
$$|v(x,t)| \le C_3\|F\|_{L_2(D_t)} + C_4,\ \ (x,t)\in D_T,$$

where

$$C_1 = \frac{1}{\sqrt{2}}\, t^2 T^{\frac{1}{2}} e^{t+\frac{1}{2}M_3 T},\quad C_2 = \big[(2M_1 t)^{\frac{1}{2}} + tN_1\big]\frac{1}{\sqrt{2}}\, t^2 T^{\frac{1}{2}} e^{t+\frac{1}{2}M_3 T}, \tag{2.50}$$

$$C_3 = t^{\frac{1}{2}} e^{\frac{1}{2}M_3 t},\quad C_4 = \big[(2M_1 t)^{\frac{1}{2}} + tN_1\big] t^{\frac{1}{2}} e^{\frac{1}{2}M_3 t}. \tag{2.51}$$

This proves Lemma 2.2, where the constants C_i, $i = 1,\dots,4$, from (2.24) and (2.25) are given by formulas (2.50) and (2.51). □

3 The uniqueness of a solution of the problem (1.4)–(1.7)

Definition 3.1. We say that the functions f and g satisfy the Lipchitz local condition if $\forall\, r = const > 0$,

$$|f(s_2) - f(s_1)| \le \Lambda_1(r)|s_2 - s_1|\ \ \forall\, s_1, s_2 \in R:\ \ |s_i| \le r,\ \ i = 1,2, \tag{3.1}$$

and

$$|g(s_2) - g(s_1)| \le \Lambda_2(r)|s_2 - s_1|\ \ \forall\, s_1, s_2 \in R:\ \ |s_i| \le r,\ \ i = 1,2, \tag{3.2}$$

where $\Lambda_i = \Lambda_i(r) = const \ge 0$, $i = 1,2$.

It is obvious that if f (resp. g) $\in C^1(R)$, then condition (3.1) (resp. (3.2)) is valid, where due to the Lagrange theorem $\Lambda_1(r) = \max\limits_{|s|\le r} |f'(s)|$ (resp. $\Lambda_2(r) = \max\limits_{|s|\le r} |g'(s)|$).

Theorem 3.1. *Let $f, g \in C(R)$, $F \in C(\overline{D}_T)$ and $\mu_i = 0$, $i = 1,\dots,4$. If the functions f and g satisfy the Lipschitz local conditions (3.1) and (3.2), then problem (1.4)–(1.7) cannot have more than one generalized solution of class C.*

Proof. Let problem (1.4)–(1.7) have two generalized solutions u_1, v_1 and u_2, v_2 of class C, i.e., due to the definition, there exist the sequences u_{1n}, v_{1n} and u_{2n}, v_{2n} which belong to the class $\overset{\circ}{C}{}^2(\overline{D}_T)$ defined in (1.8) and satisfy the following limit equalities:

$$\lim_{n\to\infty} \|u_{in} - u_i\|_{C(\overline{D}_T)} = 0,\quad \lim_{n\to\infty} \|v_{in} - v_i\|_{C(\overline{D}_T)} = 0,\ \ i = 1,2, \tag{3.3}$$

$$\lim_{n\to\infty} \|L_1(u_{in}, v_{in})\|_{C(\overline{D}_T)} = 0,\quad \lim_{n\to\infty} \|L_2(u_{in}, v_{in}) - F\|_{C(\overline{D}_T)} = 0,\ \ i = 1,2. \tag{3.4}$$

Introducing the notation

$$\varphi_n = u_{2n} - u_{1n},\quad \psi_n = v_{2n} - v_{1n}, \tag{3.5}$$

and taking into account the definition of operators L_1 and L_2 from (1.4) and (1.5), we have

$$\square\, \varphi_n = \psi_n + A_n(x,t),\ \ (x,t)\in D_T, \tag{3.6}$$
$$\varphi_n\big|_{\gamma_{1,T}} = \varphi_n(t,t) = 0,\quad \varphi_n\big|_{\gamma_{2,T}} = \varphi_n(0,t) = 0,\ \ 0 \le t \le T,$$

$$\square\, \psi_n = -(f(v_{2n}) - f(v_{1n})) - (g(v_{2n}) - g(v_{1n})) + B_n(x,t),\ \ (x,t)\in D_T, \tag{3.7}$$
$$\psi_n\big|_{\gamma_{1,T}} = \psi_n(t,t) = 0,\quad \psi_n\big|_{\gamma_{2,T}} = \psi_n(0,t) = 0,\ \ 0 \le t \le T,$$

where the sequences

$$A_n := L_1(u_{2n}, v_{2n}) - L_1(u_{1n}, v_{1n}),$$
$$B_n := L_2(u_{2n}, v_{2n}) - L_2(u_{1n}, v_{1n}),$$

according to the limit equalities (3.4), satisfy the conditions

$$\lim_{n\to\infty} \|A_n\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|B_n\|_{C(\overline{D}_T)} = 0. \tag{3.8}$$

Multiplying both sides of equation (3.6) by the function $\frac{\partial \varphi_n}{\partial t}$, integrating over the domain $D_\tau := \{(x,t)\in D_T : t < \tau\}$, where $0 < \tau \le T$, and repeating those reasonings which were used for obtaining (2.16) from (2.6), we get

$$\int\limits_{\omega_\tau} \Big[\Big(\frac{\partial \varphi_n}{\partial x}\Big)^2 + \Big(\frac{\partial \varphi_n}{\partial t}\Big)^2\Big]\, dx = 2\int\limits_{D_\tau} \psi_n \frac{\partial \varphi_n}{\partial t}\, dx\, dt + 2\int\limits_{D_\tau} A_n \frac{\partial \varphi_n}{\partial t}\, dx\, dt. \tag{3.9}$$

Similarly, as (2.16) was obtained, from (2.36) and (3.9) we get

$$\int\limits_{D_\tau} \varphi_n^2(x,t)\, dx\, dt \le \tau^3 e^{2\tau} \int\limits_{D_\tau} \psi_n^2\, dx\, dt + \tau^3 e^{2\tau} \|A_n\|^2{}_{L_2(D_T)}. \tag{3.10}$$

Multiplying both sides of (3.7) by the function $\frac{\partial \psi_n}{\partial t}$ and integrating over the domain D_τ by analogy to the equality (2.31), we have

$$\begin{aligned}\int\limits_{\omega_\tau} \Big[\Big(\frac{\partial \psi_n}{\partial x}\Big)^2 + \Big(\frac{\partial \psi_n}{\partial t}\Big)^2\Big]\, dx &= -2\int\limits_{D_\tau} (f(v_{2n}) - f(v_{1n}))\, \frac{\partial \psi_n}{\partial t}\, dx\, dt \\ &\quad - 2\int\limits_{D_\tau} (g(v_{2n}) - g(v_{1n}))\, \frac{\partial \psi_n}{\partial t}\, dx\, dt + 2\int\limits_{D_\tau} B_n \frac{\partial \psi_n}{\partial t}\, dx\, dt. \end{aligned} \tag{3.11}$$

Due to the limit equalities (3.3), since the sequences $\{u_{in}\}$ and $\{v_{in}\}$ converge in the space $C(\overline{D}_T)$, they are bounded in this space. Therefore, there exists $r > 0$ such that

$$\|u_{in}\|_{C(\overline{D}_T)} \le r, \quad \|v_{in}\|_{C(\overline{D}_T)} \le r \ \ \forall\, n \in N, \ \ i = 1, 2. \tag{3.12}$$

In view of (3.1), (3.5) and (3.12), we have

$$\begin{aligned}\Big| -2\int\limits_{D_\tau} (f(v_{2n}) - f(v_{1n}))\, \frac{\partial \psi_n}{\partial t}\, dx\, dt\Big| &\le 2\int\limits_{D_\tau} \Lambda_1(r)|v_{2n} - v_{1n}|\, \Big|\frac{\partial \psi_n}{\partial t}\Big|\, dx\, dt \\ &= \Lambda_1(r)\int\limits_{D_\tau} 2\psi_n\Big|\frac{\partial \psi_n}{\partial t}\Big|\, dx\, dt \le \Lambda_1 \int\limits_{D_\tau} \psi_n^2\, dx\, dt + \Lambda_1 \int\limits_{D_\tau} \Big|\frac{\partial \psi_n}{\partial t}\Big|^2\, dx\, dt. \end{aligned} \tag{3.13}$$

Analogously, from (3.2), (3.5) and (3.12) we obtain

$$\Big| -2\int\limits_{D_\tau} (g(v_{2n}) - g(v_{1n}))\, \frac{\partial \psi_n}{\partial t}\, dx\, dt\Big| \le \Lambda_2 \int\limits_{D_\tau} \varphi_n^2\, dx\, dt + \Lambda_2 \int\limits_{D_\tau} \Big|\frac{\partial \psi_n}{\partial t}\Big|^2\, dx\, dt. \tag{3.14}$$

From (3.11), (3.13) and (3.14) we have

$$\begin{aligned}\int\limits_{\omega_\tau} \Big[\Big(\frac{\partial \psi_n}{\partial x}\Big)^2 + \Big(\frac{\partial \psi_n}{\partial t}\Big)^2\Big]\, dx &\le \Lambda_1 \int\limits_{D_\tau} \psi_n^2\, dx\, dt + \Lambda_1 \int\limits_{D_\tau} \Big|\frac{\partial \psi_n}{\partial t}\Big|^2\, dx\, dt \\ &\quad + \Lambda_2 \int\limits_{D_\tau} \varphi_n^2\, dx\, dt + \Lambda_2 \int\limits_{D_\tau} \Big|\frac{\partial \psi_n}{\partial t}\Big|^2\, dx\, dt + \int\limits_{D_\tau} B_n^2\, dx\, dt + \int\limits_{D_\tau} \Big|\frac{\partial \psi_n}{\partial t}\Big|^2\, dx\, dt \\ &= \Lambda_1 \int\limits_{D_\tau} \psi_n^2\, dx\, dt + \Lambda_2 \int\limits_{D_\tau} \varphi_n^2\, dx\, dt + (\Lambda_1 + \Lambda_2 + 1)\int\limits_{D_\tau} \Big|\frac{\partial \psi_n}{\partial t}\Big|^2\, dx\, dt + \int\limits_{D_\tau} B_n^2\, dx\, dt, \quad 0 < \tau \le T, \end{aligned}$$

whence due to (3.10),

$$\int\limits_{\omega_\tau}\Big[\Big(\frac{\partial\psi_n}{\partial x}\Big)^2+\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\Big]\,dx\le\Lambda_1\int\limits_{D_\tau}\psi_n^2\,dx\,dt+\Lambda_2\tau^3e^{2\tau}\int\limits_{D_\tau}\psi_n^2\,dx\,dt+\Lambda_2\tau^3e^{2\tau}\|A_n\|^2_{L_2(D_T)}$$
$$+(\Lambda_1+\Lambda_2+1)\int\limits_{D_\tau}\Big|\frac{\partial\psi_n}{\partial t}\Big|^2\,dx\,dt+\int\limits_{D_\tau}B_n^2\,dx\,dt\le(\Lambda_1+\Lambda_2T^3e^{2T})\int\limits_{D_\tau}\psi_n^2\,dx\,dt$$
$$+(\Lambda_1+\Lambda_2+1)\int\limits_{D_\tau}\Big|\frac{\partial\psi_n}{\partial t}\Big|^2\,dx\,dt+\Lambda_2T^3e^{2T}\|A_n\|^2_{L_2(D_T)}+\|B_n\|^2_{L_2(D_T)}. \tag{3.15}$$

Note that inequality (2.39) is valid if instead of v_n we take the function ψ_n, i.e.,

$$\int\limits_{\omega_\tau}\psi_n^2\,dx\le T\int\limits_{D_\tau}\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\,dx\,dt. \tag{3.16}$$

Summing up inequalities (3.15) and (3.16), we obtain

$$\int\limits_{\omega_\tau}\Big[\psi_n^2+\Big(\frac{\partial\psi_n}{\partial x}\Big)^2+\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\Big]\,dx\le(\Lambda_1+\Lambda_2T^3e^{2T})T\int\limits_{D_\tau}\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\,dx\,dt$$
$$+(\Lambda_1+\Lambda_2+1)\int\limits_{D_\tau}\Big|\frac{\partial\psi_n}{\partial t}\Big|^2\,dx\,dt+\Lambda_2T^3e^{2T}\|A_n\|^2_{L_2(D_T)}+\|B_n\|^2_{L_2(D_T)}$$
$$\le(\Lambda_1T+\Lambda_2T^4e^{2T}+\Lambda_1+\Lambda_2+1)\int\limits_{D_\tau}\Big|\frac{\partial\psi_n}{\partial t}\Big|^2\,dx\,dt$$
$$+\Lambda_2T^3e^{2T}\|A_n\|^2_{L_2(D_T)}+\|B_n\|^2_{L_2(D_T)}$$
$$\le(\Lambda_1T+\Lambda_2T^4e^{2T}+\Lambda_1+\Lambda_2+1)\int\limits_{D_\tau}\Big[\psi_n^2+\Big(\frac{\partial\psi_n}{\partial x}\Big)^2+\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\Big]\,dx\,dt$$
$$+\Lambda_2T^3e^{2T}\|A_n\|^2_{L_2(D_T)}+\|B_n\|^2_{L_2(D_T)}$$
$$\le K_1\int\limits_{D_\tau}\Big[\psi_n^2+\Big(\frac{\partial\psi_n}{\partial x}\Big)^2+\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\Big]\,dx\,dt+K_{2n}, \tag{3.17}$$

where

$$K_1=(\Lambda_1T+\Lambda_2T^4e^{2T}+\Lambda_1+\Lambda_2+1),\quad K_{2n}=\Lambda_2T^3e^{2T}\|A_n\|^2_{L_2(D_T)}+\|B_n\|^2_{L_2(D_T)}. \tag{3.18}$$

Introducing the notation

$$w_3(\tau):=\int\limits_{\omega_\tau}\Big[\psi_n^2+\Big(\frac{\partial\psi_n}{\partial x}\Big)^2+\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\Big]\,dx \tag{3.19}$$

and taking into account the equality

$$\int\limits_{D_\tau}\Big[\psi_n^2+\Big(\frac{\partial\psi_n}{\partial x}\Big)^2+\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\Big]\,dx\,dt=\int\limits_0^\tau w_3(\sigma)\,d\sigma,$$

from (3.17) we obtain

$$w_3(\sigma)\le K_1\int\limits_0^\tau w_3(\sigma)\,d\sigma+K_{2n},\quad 0<\tau\le T, \tag{3.20}$$

and due to the Gronwall lemma, from (3.20) it follows that

$$w_3(\tau)\le K_{2n}e^{K_1\tau},\quad 0<\tau\le T.$$

According to the limit equality (3.8), we have

$$\lim_{n\to\infty}\|A_n\|_{L_2(D_T)}=0,\quad \lim_{n\to\infty}\|B_n\|_{L_2(D_T)}=0$$

Therefore, in view of (3.18), we obtain

$$\lim_{n\to\infty}K_{2n}=0. \tag{3.21}$$

Analogously to (2.20), for the function ψ_n, the inequality

$$\psi_n^2(x,t)\le t\int_0^t\Big[\Big(\frac{\partial\psi_n}{\partial x}\Big)^2+\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\Big](\xi,t)\,d\xi$$

is valid and, therefore, (3.19) implies

$$\psi_n^2(x,t)\le t\int_0^t\Big[\psi_n^2+\Big(\frac{\partial\psi_n}{\partial x}\Big)^2+\Big(\frac{\partial\psi_n}{\partial t}\Big)^2\Big](\xi,t)\,d\xi=tw_3(t)\le tK_{2n}e^{K_1t},\ \ (x,t)\in D_T. \tag{3.22}$$

Passing to the limit in inequality (3.22) as $n\to\infty$, and taking into account the limit equalities (3.3), (3.5) and (3.21), we have

$$|(v_2-v_1)(x,t)|^2=\lim_{n\to\infty}\big|(v_{2n}-v_{1n})(x,t)\big|^2=\lim_{n\to\infty}\psi_n^2(x,t)\le te^{K_1t}\lim_{n\to\infty}K_{2n}=0, \tag{3.23}$$

whence we get $v_2(x,t)=v_1(x,t)$, $(x,t)\in D_T$.

From (3.5), (3.8), (3.10) and (3.23), we obtain

$$\begin{aligned}\int_{D_T}(u_2-u_1)^2\,dx\,dt&=\lim_{n\to\infty}\int_{D_T}(u_{2n}-u_{1n})^2\,dx\,dt=\lim_{n\to\infty}\int_{D_T}\varphi_n^2\,dx\,dt\\&\le T^3e^{2T}\lim_{n\to\infty}\int_{D_T}\psi_n^2\,dx\,dt+T^3e^{2T}\lim_{n\to\infty}\|A_n\|^2_{L_2(D_T)}\le T^3e^{2T}\lim_{n\to\infty}\int_{D_T}TK_{2n}e^{K_1T}\,dx\,dt\\&=T^4e^{2T}e^{K_1T}\int_{D_T}1\,dx\,dt\lim_{n\to\infty}K_n=T^4e^{2T}e^{K_1T}\cdot\frac12T^2\lim_{n\to\infty}K_{2n}=0,\end{aligned}$$

whence we conclude that $u_2=u_1$ in the domain D_T. The theorem is proved. □

4 Equivalent reduction of problem (1.4)–(1.7) to a system of Volterra type integral equations

Let us now consider the equivalent reduction of problem (1.4)–(1.7) to a system of Volterra type integral equations in the class of continuous functions $C(\overline{D}_T)$.

Let the functions u and v represent a generalized solution of the class C to problem (1.4)–(1.7), i.e., there exist the sequences $\{u_n\}$ and $\{v_n\}$ satisfying conditions (1.8), (1.9) and (1.10). As it has been shown, the function u_n is a classical solution of problem (2.2), (2.3), where the function G_n is given by formula (2.4), and it satisfies the limit equality (2.5). Analogously, the function v_n is a classical solution of problem (2.26), (2.27), where the function Q_n is given by formula (2.28), and it satisfies the limit equality (2.29).

Let $P=P(x,t)$ be any point of D_T. Denote by $\Omega_{x,t}$ the characteristic rectangle $PP_1P_2P_3$ with vertices P_1 and P_2, P_3 laying on the curves $\gamma_{2,T}$ and $\gamma_{1,T}$, respectively, i.e.,

$$P_1:=P_1(0,t-x),\quad P_2:=P_2\Big(\frac{t-x}{2},\frac{t-x}{2}\Big),\quad P_3:=P_3\Big(\frac{t+x}{2},\frac{t+x}{2}\Big).$$

Integrating equation (2.2) over the rectangle $\Omega_{x,t}$, conducting integration by parts and taking into account homogeneous boundary conditions (2.3), we obtain [15]

$$u_n(x,t) - \frac{1}{2}\int\limits_{\Omega_{x,t}} v_n(x',t')\,dx'\,dt' = \frac{1}{2}\int\limits_{\Omega_{x,t}} G_n(x',t')\,dx'\,dt',\ \ (x,t)\in D_T. \tag{4.1}$$

By analogous reasoning with respect to problem (2.26), (2.27), we have

$$v_n(x,t) + \frac{1}{2}\int\limits_{\Omega_{x,t}} \big[f(v_n)+g(u_n)\big](x',t')\,dx'\,dt' = \frac{1}{2}\int\limits_{\Omega_{x,t}} Q_n(x',t')\,dx'\,dt',\ \ (x,t)\in D_T. \tag{4.2}$$

Passing to the limit in equalities (4.1) and (4.2) as $n\to\infty$ and due to the limit equalities (1.9), (1.10) and (2.5), (2.29) with respect to the functions u and v, we obtain the following Volterra type system of nonlinear integral equations in the class of continuous functions $C(\overline{D}_T)$:

$$u(x,t) - \frac{1}{2}\int\limits_{\Omega_{x,t}} v(x',t')\,dx'\,dt' = 0,\ \ (x,t)\in D_T, \tag{4.3}$$

$$v(x,t) + \frac{1}{2}\int\limits_{\Omega_{x,t}} \big[f(v)+g(u)\big](x',t')\,dx'\,dt' = \frac{1}{2}\int\limits_{x,t} F(x',t')\,dx'\,dt',\ \ (x,t)\in D_T. \tag{4.4}$$

Remark 4.1. When $f,g\in C^1(R)$, $F\in C^1(\overline{D}_T)$, the reverse proposition is valid: if the functions u and v represent a solution of the class $C(\overline{D}_T)$ to system (4.3), (4.4), then these functions represent a generalized solution of class C to problem (1.4)–(1.7) [1,16].

Let us introduce the notation $U:=(u,v)$ and rewrite the system of integral equations (4.3), (4.4) in a vectorial form

$$U(x,t) + (KU)(x,t) = \Phi(x,t),\ \ (x,t)\in D_T, \tag{4.5}$$

where

$$K=(K_1,K_2);\quad (K_1U)(x,t) = -(K_0v)(x,t),$$

$$(K_2U)(x,t) = (K_0(f(v)+g(u)))(x,t), \tag{4.6}$$

$$(K_0w)(x,t) = \frac{1}{2}\int\limits_{x,t} w(x',t')\,dx'\,dt', \tag{4.7}$$

$$\Phi(x,t) = (0,(K_0F)(x,t)). \tag{4.8}$$

5 The smoothness of a solution of problem (1.4)–(1.7). Global solvability of problem (1.4)–(1.7) in the class of continuous functions. The existence of a global solution in the domain D_∞

Remark 5.1. As is known, the operator K_0 defined by formula (4.7) satisfies the following conditions of smoothness: if $w\in C^k(\overline{D}_T)$, then $K_0w\in C^{k+1}(\overline{D}_T)$, $k=0,1,\dots$. Therefore, when $f,g\in C^1(R)$, $F\in C^1(\overline{D}_T)$, the continuous solution $U=(u,v)$ of system (4.5) satisfies the following conditions of smoothness: $u,v\in C^2(\overline{D}_T)$ and represents a classical solution of problem (1.4)–(1.7).

Remark 5.2. As is known, the space $C^1(\overline{D}_T)$ is compactly embedded into the space $C(\overline{D}_T)$. Therefore, if we take into account Remark 5.1 and consider K as an operator acting from the space $C(\overline{D}_T)$ to the space $C(\overline{D}_T)$, then due to formula (4.5), we find that the operator

$$K : C(\overline{D}_T)\to C(\overline{D}_T)$$

is continuous and compact. Therefore, the operator $L : C(\overline{D}_T) \to C(\overline{D}_T)$ acting by the rule

$$(LU)(x,t) = -(KU)(x,t) + \Phi(x,t), \ \ (x,t) \in \overline{D}_T, \tag{5.1}$$

will also be continuous and compact, and equation (4.5) in the space $C(\overline{D}_T)$ can be rewritten as follows:

$$U = LU. \tag{5.2}$$

Remark 5.3. It follows from the above reasoning that if $f, g \in C^1(R)$, $F \in C^1(\overline{D}_T)$, then $U := (u,v) \in C(\overline{D}_T)$ is a generalized solution of class C to problem (1.4)–(1.7) if and only if U is a solution of problem (5.2) of class $C(\overline{D}_T)$. Hence it follows from Lemma 2.2 that when conditions (2.22), (2.23) are fulfilled, the solution of equation (5.2) of class $C(\overline{D}_T)$ satisfies a priori estimates (2.24) and (2.25). From equation (5.2) and the structure of constants C_i, $i = 1, \dots, 4$, and from a priori estimates (2.24) and (2.25), it follows that the solution of the equation $U = \tau LU$ of class $C(\overline{D}_T)$, where the parameter $\tau \in [0,1]$, satisfies the same a priori estimates (2.24) and (2.25), where the constants C_i, $i = 1, \dots, 4$, in view of (2.22), (2.23), (2.43), (2.50) and (2.51), do not depend on the function F and the parameter τ. Therefore, since the operator $L : C(\overline{D}_T) \to C(\overline{D}_T)$ from equation (5.2) is continuous and compact, according to the Leray–Schauder theorem [33], equation (5.2) has at least one solution in the space $C(\overline{D}_T)$ which, as it was noted above, is also a generalized solution of problem (1.4)–(1.7) of class C.

Thus, according to Theorem 3.1 and Remark 5.1, the following statement is valid.

Theorem 5.1. *Let $f, g \in C^1(R)$, $F \in C^1(\overline{D}_T)$, $\mu_i = 0$, $i = 1, \dots, 4$, and the functions f and g satisfy conditions (2.22) and (2.23). Then problem (1.4)–(1.7) has a unique generalized solution of the class C which is also a classical solution of the same problem in the domain D_T.*

From Theorems 3.1 and 5.1 follows

Theorem 5.2. *Let $f, g \in C^1(R)$, $F \in C^1(\overline{D}_\infty)$, $\mu_i = 0$, $i = 1, \dots, 4$, and the functions f and g satisfy conditions (2.22) and (2.23), then problem (1.4)–(1.7) for $T = \infty$ has a unique global classical solution in the domain D_∞.*

Proof. From Theorem 5.1, it follows that there exists a unique classical solution u_k, v_k of problem (1.4)–(1.7) in the domain D_T, where $T = k \in N$. Since $u_{k+1}|_{D_k}$ is also a classical solution of problem (1.4)–(1.7) in the domain D_k, because of the uniqueness of the solution, we have $u_{k+1}|_{D_k} = u_k$, $v_{k+1}|_{D_k} = v_k$. Therefore, the functions u and v constructed by the rule $u(x,t) = u_k(x,t)$, $v(x,t) = v_k(x,t)$, when $k = [t] + 1$, where $[t]$ is an entire part of number t and $(x,t) \in D_\infty$, represent a unique global solution of problem (1.4)–(1.7) in the domain D_∞. The theorem is proved. □

Definition 5.1. Let $f, g \in C(R)$, $F \in C(\overline{D}_\infty)$, $\mu_i = 0$, $i = 1, \dots, 4$. Problem (1.4)–(1.7) is called globally solvable in the class C if for any positive T, this problem has at least one generalized solution of class C in the domain D_T in the sense of Definition 1.1.

Remark 5.4. It is obvious that if problem (1.4)–(1.7) is not globally solvable in the class C in the sense of Definition 3.1, then it does not have a global classical solution in the domain D_∞. Besides, if the conditions of Theorem 5.2 are fulfilled, then problem (1.4)–(1.7) has a global classical solution in the domain D_∞ and, therefore, it is also globally solvable in the class C.

6 Nonexistence of solutions of problem (1.4)–(1.7)

Below, we show that if conditions (2.22) and (2.23) are violated, then problem (1.4)–(1.7) may not be globally solvable in the sense of Definition 3.1.

Theorem 6.1. *Let $f = 0$, $g \in C^1(R)$, $F_0 \in C^1(\overline{D}_T)$, $F_0|_{D_T} > 0$ and $F = \lambda F_0$, $\lambda = const > 0$, $\mu_i = 0$, $i = 1, \dots, 4$. Then if $g(u) \le -|u|^\alpha$, $\alpha = const > 1$, there exists a number $\lambda_0 = \lambda_0(F_0, \alpha) > 0$ such that for $\lambda > \lambda_0$, problem (1.4)–(1.7) does not have a generalized solution of class C in the domain D_T.*

Proof. Let u, v represent a generalized solution of problem (1.4)–(1.7) of class C. Since $f = 0$, $g \in C^1(R)$ and $F \in C^1(\overline{D}_T)$, according to Remarks 4.1 and 5.1, this solution will be a classical solution of problem (1.4)–(1.7). Therefore, the function u satisfies equation (1.1) in the domain D_T, i.e.,

$$\square^2 u + g(u) = F(x,t), \ \ (x,t) \in D_T, \tag{6.1}$$

and $g(u), \square^2 u \in C(\overline{D}_T)$.

Let us consider a test function

$$\varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T) := \Big\{ \psi \in C^4(\overline{D}_T): \ \ \psi\big|_{D_T} \geq 0, \ \ \psi|_{\partial D_T} = \frac{\partial^i \psi}{\partial \nu^i}\Big|_{\partial D_T} = 0, \ \ i = 1,2,3 \Big\},$$

where $\nu = (\nu_x, \nu_t)$ is a unit vector of the outer norm to the boundary ∂D_T. Let us multiply by it both sides of equation (6.1) and integrate over the domain D_T. By integration by parts and taking into account that $\psi|_{\partial D_T} = \frac{\partial^i \psi}{\partial \nu^i}|_{\partial D_T} = 0$, $i = 1,2,3$, we obtain

$$\int\limits_{D_T} u \square^2 \varphi \, dx\, dt = -\int\limits_{D_T} g(u)\varphi \, dx\, dt + \lambda \int\limits_{D_T} F_0 \varphi \, dx\, dt \ \ \forall \, \varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T). \tag{6.2}$$

According to the conditions $g(u) \leq -|u|^\alpha$ and $\varphi \geq 0$, from (6.2) it follows

$$\int\limits_{D_T} |u|^\alpha \varphi \, dx\, dt \leq \int\limits_{D_T} u \square^2 \varphi \, dx\, dt - \lambda \int\limits_{D_T} F_0 \varphi \, dx\, dt \ \ \forall \, \varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T). \tag{6.3}$$

Below, we use the method of test functions [30]. Consider the test function $\varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T)$ such that $\varphi|_{D_T} > 0$. If in the Young inequality with parameter $\varepsilon > 0$

$$ab \leq \frac{\varepsilon}{\alpha}\, a^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}}\, b^{\alpha'}, \ \ a,b \geq 0, \ \ \alpha' = \frac{\alpha}{\alpha - 1}\,,$$

we take $a = |u|\varphi^{\frac{1}{\alpha}}$ and $b = \frac{|\square^2 \varphi|}{\varphi^{\frac{1}{\alpha}}}$, then due to $\frac{\alpha'}{\alpha} = \alpha - 1$, we obtain

$$|u \square^2 \varphi| = |u| \varphi^{\frac{1}{\alpha}}\, \frac{|\square^2 \varphi|}{\varphi^{\frac{1}{\alpha}}} \leq \frac{\varepsilon}{\alpha}\, |u|^\alpha \varphi + \frac{1}{\alpha' \varepsilon^{\alpha'-1}}\, \frac{|\square^2 \varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,.$$

From (6.3) and (6.3), we have

$$\Big(1 - \frac{\varepsilon}{\alpha}\Big) \int\limits_{D_T} |u|^\alpha \varphi \, dx\, dt \leq \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|\square^2 \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx\, dt - \lambda \int\limits_{D_T} F_0 \varphi \, dx\, dt,$$

whence for $\varepsilon < \alpha$, we obtain

$$\int\limits_{D_T} |u|^\alpha \varphi \, dx\, dt \leq \frac{1}{(\alpha - \varepsilon)\alpha' \varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|\square^2 \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx\, dt - \frac{\alpha\lambda}{\alpha - \varepsilon} \int\limits_{D_T} F_0 \varphi \, dx\, dt. \tag{6.4}$$

In view of the equalities $\alpha' = \frac{\alpha}{\alpha-1}$, $\alpha = \frac{\alpha'}{\alpha'-1}$ and

$$\min_{0<\varepsilon<\alpha} \frac{\alpha}{(\alpha - \varepsilon)\alpha' \varepsilon^{\alpha'-1}} = 1,$$

which is reached for $\varepsilon = 1$, from (6.4) we have

$$\int\limits_{D_T} |u|^\alpha \varphi \, dx\, dt \leq \int\limits_{D_T} \frac{|\square^2 \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx\, dt - \alpha' \lambda \int\limits_{D_T} F_0 \varphi \, dx\, dt. \tag{6.5}$$

It is easy to show the existence of a function φ for which

$$\varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T), \quad \varphi|_{D_T} > 0, \quad \kappa_0 = \int\limits_{D_T} \frac{|\Box^2 \varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt < +\infty. \tag{6.6}$$

Indeed, the function built by the formula

$$\varphi(x,t) = [x(t-x)(T-t)]^m$$

for a sufficiently large natural m satisfies conditions (6.6).

Since according to the condition $F_0 \in C(\overline{D}_T)$, $F_0|_{D_T} > 0$ and $\varphi|_{D_T} > 0$, we have

$$0 < \kappa_1 = \int\limits_{D_T} F_0 \varphi\, dx\, dt < +\infty. \tag{6.7}$$

Denote by $\chi(\lambda)$ the right-hand side of inequality (6.5) which is linear with respect to the parameter λ. Then from (6.5), (6.6) and (6.7), we have

$$\chi\lambda) < 0, \ \text{ when } \ \lambda > \mu_0 \ \text{ and } \ \chi(\lambda) > 0, \ \text{ when } \ \lambda < \mu_0, \tag{6.8}$$

where

$$\chi(\lambda) = \kappa_0 - \alpha' \lambda \kappa_1, \quad \lambda_0 = \frac{\kappa_0}{\alpha' \kappa_1}\,.$$

According to (6.8), when $\lambda > \lambda_0$, the left-hand side of (6.5) is negative, while the right-hand side is non-negative. This contradiction proves the theorem. □

Note that when $g(u) \leq -|u|^\alpha$, $\alpha = const > 1$, condition (2.23) is violated.

7 Local solvability of problem (1.4)–(1.7) in the class of continuous functions

Definition 7.1. Let $f, g \in C(R)$, $F \in C(\overline{D}_\infty)$, $\mu_i = 0$, $i = 1, \dots, 4$. Problem (1.4)–(1.7) is called locally solvable in the class C if there exists a positive constant $T_0 = T_0(F)$ such that problem (1.4)–(1.7) has at least one generalized solution of class C in the domain D_T, when $T \leq T_0$.

Theorem 7.1. *Let $f, g \in C^1(R)$, $\mu_i = 0$, $i = 1, \dots, 4$. Then for any function $F \in C^1(\overline{D}_\infty)$, problem (1.4)–(1.7) is locally solvable in the class C. Moreover, there exists a positive constant $T_0 = T_0(F)$ such that problem (1.4)–(1.7) has a unique generalized solution of class C in the domain D_T, when $T \leq T_0$, which represents a classical solution of this problem.*

Remark 7.1. In case the conditions of Theorem 6.1 are fulfilled, problem (1.4)–(1.7) for any function $F \in C^1(\overline{D}_\infty)$ may not be globally solvable. Indeed, if $F_0 \in C^1(\overline{D}_\infty)$, $F_0|_{D_\infty} > 0$, and for a fixed positive T we take $F = \lambda F_0$, then this problem does not have a generalized solution of class C in the domain D_T, when $\lambda > \lambda_0$.

Proof of Theorem 7.1. According to Remark 5.3 $U = (u, v) \in C(\overline{D}_T)$ represents a generalized solution of problem (1.4)–(1.7) of class C if and only if U is a solution of equation (5.2) from the space $C(\overline{D}_T)$.

Let us fix the positive constants T_1 and r. Below, we suppose that $|U| = |(u,v)| = |u| + \lceil v \rceil$, $\|U\|_{C(\overline{D}_T)} = \|(u,v)\|_{C(\overline{D}_T)} = \|u\|_{C(\overline{D}_T)} + \|v\|_{C(\overline{D}_T)}$, and denote by $B_r(0)$ a ball of radius r in the space $\overline{D}_T$ of continuous vector functions $U = (u,v)$ with a center in the null element $(0,0)$, i.e.,

$$B_r(0) := \Big\{ U = (u,v) \in C(\overline{D}_T) : \ \ \|(u,v)\|_{C(\overline{D}_T)} \leq r \Big\}.$$

When $U \in B_r(0)$, due to (4.6)–(5.1), if we take into consideration the structure of the operator L from equation (5.2), take $T \le T_1$ and the point $(x,t) \in \overline{D}_T$, we get

$$\begin{aligned}
|(LU)(x,t)| &\le |(KU)(x,t)| + |\Phi(x,t)| \le |(K_1U)(x,t)| + |(K_2U)(x,t)| + |(K_0F)(x,t)| \\
&\le |(K_0v)(x,t)| + |(K_0(f(v)+g(u)))(x,t)| + |(K_0F)(x,t)| \\
&\le \frac{1}{2}\,\|v\|_{C(\overline{D}_t)} \int\limits_{\Omega_{x,t}} 1\,dx\,dt + \frac{1}{2}\Big(\max_{|s|\le r}|f(s)| + \max_{|s|\le r}|g(s)|\Big) \int\limits_{\Omega_{x,t}} 1\,dx\,dt \\
&\qquad + \frac{1}{2}\,\|F\|_{C(\overline{D}_t)} \int\limits_{\Omega_{x,t}} 1\,dx\,dt \\
&\le \frac{1}{2}\Big(\|v\|_{C(\overline{D}_t)} + \max_{|s|\le r}|f(s)| + \max_{|s|\le r}|g(s)| + \|F\|_{C(\overline{D}_t)}\Big)\,\frac{1}{2}\,t^2 \\
&\le \frac{1}{4}\,T^2\Big(\|v\|_{C(\overline{D}_{T_1})} + \max_{|s|\le r}|f(s)| + \max_{|s|\le r}|g(s)| + \|F\|_{C(\overline{D}_{T_1})}\Big),
\end{aligned}$$

whence we obtain

$$\begin{aligned}
\|LU\|_{C(\overline{D}_T)} &\le \frac{1}{4}\,T^2\Big(\|v\|_{C(\overline{D}_{T_1})} + \max_{|s|\le r}|f(s)| + \max_{|s|\le r}|g(s)| + \|F\|_{C(\overline{D}_{T_1})}\Big) \\
&\le \frac{1}{4}\,T^2\Big(r + \|f\|_{C([-r,r])} + \|g\|_{C([-r,r])} + \|F\|_{C(\overline{D}_{T_1})}\Big). \qquad (7.1)
\end{aligned}$$

From (7.1) it follows that if we take T such that $T \le T_0$, where

$$T_0 = \min\Big(T_1\,\frac{4r}{r + \|f\|_{C([-r,r])} + \|g\|_{C([-r,r])} + \|F\|_{C(\overline{D}_{T_1})}}\Big)^{\frac{1}{2}},$$

then

$$\|LU\|_{C(\overline{D}_T)} \le r, \ \ \text{when} \ \ \|U\|_{C(\overline{D}_T)} \le r. \qquad (7.2)$$

From (7.2) it follows that the operator $L : C(\overline{D}_T) \to C(\overline{D}_T)$ maps the ball $B_r(0)$ into itself and since by Remark 5.2 this operator is continuous and compact, according to Schauder's theorem, equation (5.2) has at least one solution U from the space $C(\overline{D}_T)$. Due to Remark 5.3 and Theorem 5.1, this solution is a unique classical solution of problem (1.4)–(1.7) in the domain. The theorem is completely proved. □

Therefore, from the results obtained above it follows that if we do not require from the functions f and g the fulfillment of conditions (2.22) and (2.23) together with smoothness $f, g \in C^1(R)$, then according to Theorem 6.1, problem (1.4)–(1.6) may not be globally solvable and, moreover, it may not have a global solution in the domain D_∞. Nevertheless, in case of conditions (2.22), (2.23) violate, problem (1.4)–(1.7) is locally solvable for any function $F \in C^1(\overline{D}_\infty)$.

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