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MARTINGALE HARDY SPACES AND PARTIAL SUMS AND FEJÉR MEANS WITH RESPECT TO THE ONE-DIMENSIONAL WALSH–FOURIER SERIES

Abstract. In this paper, we prove and discuss some new (H_p, L_p) type inequalities for partial Sums and Fejér means with respect to the Walsh system. It is also proved that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

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1 Preliminaries

It is well-known that (for details see, e.g., [34, 54] and [44]) for every $p > 1$, there exists an absolute constant c_p, depending only on p such that

$$\|S_n f\|_p \le c_p \|f\|_p, \text{ when } p > 1 \text{ and } f \in H_1(G).$$

Moreover, Watari [89] (see also Gosselin [35] and Young [96]) proved that there exists an absolute constant c such that for $n = 1, 2, \dots,$

$$\lambda \mu \big(|S_n f| > \lambda\big) \le c \, \|f\|_1, \quad f \in L_1(G), \quad \lambda > 0.$$

On the other hand, it is also well-known that (for details see, e.g., [1, 54] and [81]) a Walsh system is not Schauder's basis in $L_1(G)$ space. Moreover, there exists function $f \in H_1(G)$ such that partial sums with respect to the Walsh system are not uniformly bounded in $L_1(G)$.

Applying Lebesgue constants

$$L(n) := \|D_n\|_1,$$

we easily obtain that (for details see, e.g., [2] and [54]) subsequences of partial sums $S_{n_k} f$ with respect to the Walsh system converge to f in L_1 norm if and only if

$$\sup_{k \in \mathbb{N}} L(n_k) \le c < \infty. \tag{1.1}$$

Since the n-th Lebesgue constant with respect to the Walsh system, where

$$n = \sum_{j=0}^{\infty} n_j \, 2^j \quad (n_j \in Z_2),$$

can be estimated by the variation of natural number

$$V(n) = n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|,$$

and it is also well known that (for details see, e.g., [8] and [54]) the following two-sided estimate

$$\frac{1}{8} V(n) \le L(n) \le V(n)$$

is true, to obtain the convergence of subsequences of partial sums $S_{n_k} f$ with respect to the Walsh system of $f \in L_1$ in $f \in L_1$-norm, condition (1.1) can be replaced by

$$\sup_{k \in \mathbb{N}} V(n_k) \le c < \infty.$$

It follows that (for details see, e.g., [54] and [90]) a subsequence of partial sums S_{2^n} is bounded from $H_p(G)$ to $H_p(G)$ for every $p > 0$, whence we obtain

$$\|S_{2^n} f - f\|_{H_p(G)} \to 0 \text{ as } n \to \infty. \tag{1.2}$$

On the other hand (see, e.g., [68]), there exists a martingale $f \in H_p(G)$ $(0 < p < 1)$ such that

$$\sup_{n \in \mathbb{N}} \|S_{2^n+1} f\|_{\text{weak-}L_p(G)} = \infty.$$

The main reason of divergence of the subsequence $S_{2^n+1} f$ of partial sums is that (for details see [69]) the Fourier coefficients of $f \in H_p(G)$ are not uniformly bounded for $0 < p < 1$.

When $0 < p < 1$, in [9] and [82], the boundedness of subsequences of partial sums with respect to the Walsh system from $H_p(G)$ to $H_p(G)$ was investigated. In particular, the following result is true.

Theorem T1. *Let $0<p<1$ and $f\in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_{m_k}f\|_{H_p(G)}\leq c_p\|f\|_{H_p(G)}$$

if and only if the following condition holds:

$$\sup_{k\in\mathbb{N}} d(m_k)<c<\infty, \tag{1.3}$$

where

$$d(m_k):=|m_k|-\langle m_k\rangle.$$

In particular, from Theorem T1 immediately follows

Theorem T2. *Let $p>0$ and $f\in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_{2^n}f\|_{H_p(G)}\leq c_p\|f\|_{H_p(G)}$$

and

$$\|S_{2^n+2^{n-1}}f\|_{H_p(G)}\leq c_p\|f\|_{H_p(G)}.$$

On the other hand, we have the following result.

Theorem T3. *Let $p>0$. Then there exists a martingale $f\in H_p(G)$ such that*

$$\sup_{n\in\mathbb{N}}\|S_{2^n+1}f\|_{H_p(G)}=\infty.$$

Taking into account these results, it is interesting to find behaviour of a rate of divergence of subsequences of partials sums with respect to the Walsh system of martingales $f\in H_p(G)$ in the martingale Hardy spaces $H_p(G)$.

In Section 2 (see also [70]), we investigate the above-mentioned problem. For $0<p<1$, we have the following result.

Theorem 2.1. *Let $f\in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that the following inequality is true:*

$$\|S_nf\|_{H_p(G)}\leq c_p\,2^{d(n)(1/p-1)}\|f\|_{H_p(G)}. \tag{1.4}$$

On the other hand, if $0<p<1$, $\{m_k:\ k\geq 0\}$ is an increasing subsequence of natural numbers such that

$$\sup_{k\in\mathbb{N}} d(m_k)=\infty \tag{1.5}$$

and $\Phi:\mathbb{N}_+\to[1,\infty)$ is a non-decreasing function satisfying the condition

$$\overline{\lim_{k\to\infty}}\frac{2^{d(m_k)(1/p-1)}}{\Phi(m_k)}=\infty,$$

then there exists a martingale $f\in H_p(G)$ such that

$$\sup_{k\in\mathbb{N}}\Big\|\frac{S_{m_k}f}{\Phi(m_k)}\Big\|_{\text{weak-}L_p(G)}=\infty.$$

Theorem 2.1 easily implies the following

Corollary 2.1. *Let $0<p<1$ and $f\in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_nf\|_{H_p(G)}\leq c_p\big(n\mu\{\text{supp}(D_n)\}\big)^{1/p-1}\|f\|_{H_p(G)}.$$

On the other hand, if $0<p<1$, $\{m_k:\ k\geq 0\}$ *is an increasing sequence of natural numbers such that*

$$\sup_{k\in\mathbb{N}} m_k\mu\big\{\operatorname{supp}(D_{m_k})\big\}=\infty$$

and $\Phi:\mathbb{N}_+\to[1,\infty)$ *is a non-decreasing function satisfying the condition*

$$\overline{\lim_{k\to\infty}}\frac{(m_k\mu\{\operatorname{supp}(D_{m_k})\})^{1/p-1}}{\Phi(m_k)}=\infty,$$

then there exists a martingale $f\in H_p(G)$ *such that*

$$\sup_{k\in\mathbb{N}}\Big\|\frac{S_{m_k}f}{\Phi(m_k)}\Big\|_{\text{weak-}L_p(G)}=\infty.$$

In particular, we also get the proofs of Theorem T1 and Theorem T2.

In Section 2, we also investigate the case $p=1$. In this case, the following result is true.

Theorem 2.2. *Let* $n\in\mathbb{N}_+$ *and* $f\in H_1(G)$. *Then there exists an absolute constant c such that*

$$\|S_nf\|_{H_1(G)}\leq c\,V(n)\|f\|_{H_1(G)}. \tag{1.6}$$

Moreover, if $\{m_k:\ k\geq 0\}$ *is an increasing sequence of natural numbers* $\mathbb{N}_+$ *such that*

$$\sup_{k\in\mathbb{N}}V(m_k)=\infty$$

and $\Phi:\mathbb{N}_+\to[1,\infty)$ *is a non-decreasing function satisfying the condition*

$$\overline{\lim_{k\to\infty}}\frac{V(m_k)}{\Phi(m_k)}=\infty.$$

then there exists a martingale $f\in H_1(G)$ *such that*

$$\sup_{k\in\mathbb{N}}\Big\|\frac{S_{m_k}f}{\Phi(m_k)}\Big\|_1=\infty.$$

When $0<p<1$, in [82] the boundedness of maximal operators of subsequences of partial sums from $H_p(G)$ to $L_p(G)$ was proved. In particular, the following theorem is true.

Theorem T4. *Let* $0<p<1$ *and* $f\in H_p(G)$. *Then the maximal operator*

$$\sup_{k\in\mathbb{N}}|S_{m_k}f|$$

is bounded from $H_p(G)$ *to* $L_p(G)$ *if and only if condition* (1.3) *is fulfilled.*

In the special cases we find that the following theorem is true.

Theorem T5. *Let* $p>0$ *and* $f\in H_p(G)$. *Then there exists an absolute constant* c_p, *depending only on p, such that*

$$\Big\|\sup_{n\in\mathbb{N}}|S_{2^n}f|\Big\|_p\leq c_p\|f\|_{H_p(G)} \tag{1.7}$$

and

$$\Big\|\sup_{n\in\mathbb{N}}|S_{2^n+2^{n-1}}f|\Big\|_p\leq c_p\|f\|_{H_p(G)}.$$

On the other hand, we have the following result.

Theorem T6. *Let* $p>0$. *Then there exists a martingale* $f\in H_p(G)$ *such that*

$$\Big\|\sup_{n\in\mathbb{N}}|S_{2^n+1}f|\Big\|_p=\infty.$$

The above-mentioned condition (1.3) is sufficient for the case $p = 1$ as well, but there exist subsequences which do not satisfy this condition, but maximal operators of these subsequences of partial sums with respect to the Walsh system are not bounded from $H_1(G)$ to $L_1(G)$.

Such necessary and sufficient conditions that provide the boundedness of maximal operators of subsequences of partial sums with respect to the Walsh system from $H_1(G)$ to $L_1(G)$ remains still an open problem.

In [69] and [82], the boundedness of weighted maximal operators from $H_p(G)$ to $L_p(G)$, when $0 < p \leq 1$, was investigated.

Theorem T7. *Let $0 < p \leq 1$. Then the weighted maximal operator*

$$\widetilde{S}_p^* f := \sup_{n\in\mathbb{N}_+} \frac{|S_n f|}{(n+1)^{1/p-1}\log^{[p]}(n+1)}$$

is bounded from $H_p(G)$ to $L_p(G)$, where $[p]$ denotes an integer part of p.

Moreover, for any non-decreasing function $\varphi : \mathbb{N}_+ \to [1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-1}\log^{[p]}(n+1)}{\varphi(n+1)} = +\infty,$$

there exists a martingale $f \in H_p(G)$ $(0 < p \leq 1)$ such that

$$\sup_{n\in\mathbb{N}} \Big\| \frac{S_n f}{\varphi(n)} \Big\|_p = \infty.$$

According to the sharpness of result, for the weighted maximal operator of partial sums of Walsh–Fourier series, we immediately get the following result.

Theorem S1. *There exists a martingale $f \in H_p(G)$ $(0 < p \leq 1)$ such that*

$$\sup_{n\in\mathbb{N}} \|S_n f\|_p = \infty.$$

On the other hand, the boundedness of weighted maximal operators immediately leads to the following estimation.

Theorem S2. *Let $0 < p \leq 1$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_n f\|_p \leq c_p (n+1)^{1/p-1}\log^{[p]}(n+1)\|f\|_{H_p(G)} \ \textit{for}\ 0 < p \leq 1,$$

where $[p]$ denotes an integer part of p.

Applying this inequality (see [67]), we find the necessary and sufficient conditions for the martingale $f \in H_p(G)$ for which partial sums with respect to the Walsh system of martingales $f \in H_p(G)$ converge in $H_p(G)$ norm.

Theorem T8. *Let $0 < p \leq 1$, $[p]$ denote an integer part of p, $f \in H_p(G)$ and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^N}, f\Big) = o\Big(\frac{1}{2^{N(1/p-1)}N^{[p]}}\Big) \ \textit{as}\ N \to \infty.$$

Then

$$\|S_n f - f\|_p \to 0 \ \text{ as } \ n \to \infty.$$

Moreover, there exists a martingale $f \in H_p(G)$, where $0 < p < 1$, such that

$$\omega_{H_p(G)}\Big(\frac{1}{2^N}, f\Big) = O\Big(\frac{1}{2^{N(1/p-1)}N^{[p]}}\Big) \ \textit{as}\ N \to \infty$$

and

$$\|S_n f - f\|_{\text{weak-}L_p(G)} \nrightarrow 0 \ \text{ as } \ n \to \infty.$$

Taking these results into account, it is interesting to find the necessary and sufficient conditions for modulus of continuity such that the subsequences of partial sums with respect to the Walsh system of martingales $f\in H_p(G)$ converge in $H_p(G)$ norm.

In Section 2 (see also [70]), we investigate this problem. Combining inequalities (1.4) and (1.6), we get the following

Theorem 2.3. *Let $2^k<n\leq 2^{k+1}$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_nf-f\|_{H_p(G)}\leq c_p\,2^{d(n)(1/p-1)}\omega_{H_p(G)}\Big(\frac{1}{2^k},f\Big)\ \ (0<p<1) \tag{1.8}$$

and

$$\|S_nf-f\|_{H_1(G)}\leq c_1V(n)\omega_{H_1(G)}\Big(\frac{1}{2^k},f\Big). \tag{1.9}$$

By applying inequality (1.8), in Section 2, the following result is proved.

Theorem 2.4. *Let $0<p<1$, $f\in H_p(G)$ and $\{m_k:\ k\geq 0\}$ be an increasing sequence of natural numbers satisfying the condition*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|m_k|}},f\Big)=o\Big(\frac{1}{2^{d(m_k)(1/p-1)}}\Big)\ \ \textit{as}\ \ k\to\infty.$$

Then

$$\|S_{m_k}f-f\|_{H_p(G)}\to 0\ \ \text{as}\ \ k\to\infty. \tag{1.10}$$

On the other hand, if $\{m_k:\ k\geq 0\}$ is an increasing sequence of natural numbers satisfying condition (1.5)*, then there exists a martingale $f\in H_p(G)$ and a subsequence $\{\alpha_k:\ k\geq 0\}\subset\{m_k:\ k\geq 0\}$ for which*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big)=O\Big(\frac{1}{2^{d(\alpha_k)(1/p-1)}}\Big)\ \ \textit{as}\ \ k\to\infty$$

and

$$\limsup_{k\to\infty}\|S_{\alpha_k}f-f\|_{\text{weak-}L_p(G)}>c_p>0\ \ \textit{as}\ \ k\to\infty, \tag{1.11}$$

where c_p is an absolute constant depending only on p.

According to this theorem, we immediately get that the following result is true.

Corollary 2.5. *Let $0<p<1$, $f\in H_p(G)$ and $\{m_k:\ k\geq 0\}$ be an increasing sequence of natural numbers satisfying the condition*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|m_k|}},f\Big)=o\Big(\frac{1}{(m_k\mu(\operatorname{supp}D_{m_k}))^{1/p-1}}\Big)\ \ \textit{as}\ \ k\to\infty.$$

Then (1.10) *holds.*

On the other hand, if $\{m_k:\ k\geq 0\}$ is an increasing sequence of natural numbers satisfying the condition

$$\overline{\lim_{k\to\infty}}\frac{(m_k\mu\{\operatorname{supp}(D_{m_k})\})^{1/p-1}}{\Phi(m_k)}=\infty,$$

then there exist a martingale $f\in H_p(G)$ and a subsequence $\{\alpha_k:\ k\geq 0\}\subset\{m_k:\ k\geq 0\}$ such that

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big)=O\Big(\frac{1}{(\alpha_k\mu(\operatorname{supp}D_{\alpha_k}))^{1/p-1}}\Big)\ \ \textit{as}\ \ k\to\infty$$

and (1.11) *holds.*

Applying (1.9), we prove that the following result is true.

Theorem 2.5. *Let $f\in H_1(G)$ and $\{m_k:\ k\geq 0\}$ be an increasing sequence of natural numbers satisfying the condition*

$$\omega_{H_1(G)}\Big(\frac{1}{2^{|m_k|}},f\Big)=o\Big(\frac{1}{V(m_k)}\Big)\ \ \textit{as}\ \ k\to\infty.$$

Then

$$\|S_{m_k}f-f\|_{H_1(G)}\to 0 \ \ as \ \ k\to\infty.$$

Moreover, if $\{m_k: \ \ k\geq 0\}$ *is an increasing sequence of natural numbers satisfying condition* (1.5), *then there exist a martingale* $f\in H_1(G)$ *and a subsequence* $\{\alpha_k: \ k\geq 0\}\subset\{m_k: \ k\geq 0\}$ *for which*

$$\omega_{H_1(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big)=O\Big(\frac{1}{V(\alpha_k)}\Big) \ \ as \ \ k\to\infty$$

and

$$\limsup_{k\to\infty}\|S_{\alpha_k}f-f\|_1>c>0 \ \ as \ \ k\to\infty,$$

where c *is an absolute constant.*

Applying Theorems 2.4 and 2.5, we immediately get the proof of Theorem T8.

Weisz [91] considered the convergence in a norm of Fejér means of the one-dimensional Walsh–Fourier series and proved the following

Theorem We1. *Let* $p>1/2$ *and* $f\in H_p(G)$. *Then there exists an absolute constant* c_p, *depending only on* p, *such that*

$$\|\sigma_k f\|_{H_p(G)}\leq c_p\|f\|_{H_p(G)}.$$

Weisz (for details see, e.g., [90]) also considered the boundedness of subsequences of Fejér means σ_{2^n} of the one-dimensional Walsh–Fourier series from $H_p(G)$ to $H_p(G)$ when $p>0$.

Theorem We2. *Let* $p>0$ *and* $f\in H_p(G)$. *Then*

$$\|\sigma_{2^k}f-f\|_{H_p(G)}\to 0 \ \ as \ \ k\to\infty. \tag{1.12}$$

On the other hand, in [63], the following result was proved.

Theorem T9. *There exists a martingale* $f\in H_p(G)$ $(0<p\leq 1/2)$ *such that*

$$\sup_{n\in\mathbb{N}}\|\sigma_{2^n+1}f\|_{H_p(G)}=\infty.$$

Goginava [29] (see also [51]) proved that the following result is true.

Theorem Gog1. *Let* $0<p\leq 1$. *Then the sequence of operators* $|\sigma_{2^n}f|$ *is not bounded from* $H_p(G)$ *to* $H_p(G)$.

If $0<p<1/2$, then in [52] it was proved the boundedness of subsequences of Fejér means of the one-dimensional Walsh–Fourier from $H_p(G)$ to $H_p(G)$. In particular, the following statement is true.

Theorem T10. *Let* $0<p<1/2$ *and* $f\in H_p(G)$. *Then there exists an absolute constant* c_p, *depending only on* p, *such that the estimation*

$$\|\sigma_{m_k}f\|_{H_p(G)}\leq c_p\|f\|_{H_p(G)}$$

holds if and only if condition (1.3) *is fulfilled.*

Theorem T10 immediately follows from theorem of Weisz (see Theorem We2) and we get the interesting results.

Theorem T11. *Let* $p>0$ *and* $f\in H_p(G)$. *Then there exists an absolute constant* c_p, *depending only on* p, *such that*

$$\|\sigma_{2^n}f\|_{H_p(G)}\leq c_p\|f\|_{H_p(G)}$$

and

$$\|\sigma_{2^n+2^{n-1}}f\|_{H_p(G)}\leq c_p\|f\|_{H_p(G)}.$$

On the other hand, we have the following result.

Theorem T12. *Let $p>0$. Then there exists a martingale $f\in H_p(G)$ such that*

$$\sup_{n\in\mathbb{N}}\|\sigma_{2^n+1}f\|_{H_p(G)}=\infty.$$

According to the above-mentioned results, it is interesting to find a rate of divergence of subsequences $\sigma_{n_k}f$ of Fejér means of the one-dimensional Walsh–Fourier series in the Hardy spaces $H_p(G)$.

In Section 3 (see also [71]), we find a rate of divergence of subsequences of Fejér means of the one-dimensional Walsh–Fourier series on the martingale Hardy spaces $H_p(G)$, when $0<p\le 1/2$.

First, we consider the case $p=1/2$.

Theorem 3.1. *Let $n\in\mathbb{N}_+$ and $f\in H_{1/2}(G)$. Then there exists an absolute constant c such that*

$$\|\sigma_n f\|_{H_{1/2}(G)}\le c\,V^2(n)\|f\|_{H_{1/2}(G)}. \tag{1.13}$$

Moreover, if $\{m_k:\ k\ge 0\}$ is an increasing sequence of natural numbers such that

$$\sup_{k\in\mathbb{N}}V(m_k)=\infty,$$

and $\Phi:\mathbb{N}_+\to[1,\infty]$ is a non-decreasing function satisfying the condition

$$\overline{\lim_{k\to\infty}}\frac{V^2(m_k)}{\Phi(m_k)}=\infty,$$

then there exists a martingale $f\in H_{1/2}(G)$ such that

$$\sup_{k\in\mathbb{N}}\left\|\frac{\sigma_{m_k}f}{\Phi(m_k)}\right\|_{1/2}=\infty.$$

The case $0<p<1/2$ was also been considered and it was proved that the following statement is true.

Theorem 3.2. *Let $0<p<1/2$ and $f\in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_n f\|_{H_p(G)}\le c_p\,2^{d(n)(1/p-2)}\|f\|_{H_p(G)}. \tag{1.14}$$

On the other hand, if $0<p<1/2$, $\{m_k:\ k\ge 0\}$ is an increasing sequence of natural numbers satisfying condition (1.5) *and $\Phi:\mathbb{N}_+\to[1,\infty)$ is a non-decreasing function such that*

$$\overline{\lim_{k\to\infty}}\frac{2^{d(m_k)(1/p-2)}}{\Phi(m_k)}=\infty,$$

then there exists a martingale $f\in H_p(G)$ such that

$$\sup_{k\in\mathbb{N}}\left\|\frac{\sigma_{m_k}f}{\Phi(m_k)}\right\|_{\text{weak-}L_p(G)}=\infty.$$

From these results also follows the proof of Theorem We2.

In 1975, Schipp [53] (see also [2] and [97]) proved that the maximal operator of Fejér means σ^* is of type *weak* $-(1,1)$:

$$\mu(\sigma^*f>\lambda)\le\frac{c}{\lambda}\,\|f\|_1\ \ (\lambda>0).$$

Using the Marcinkiewicz interpolation theorem, it follows that σ^* is of strong type-(p,p), when $p>1$:

$$\|\sigma^*f\|_p\le c\,\|f\|_p\ \ (p>1).$$

The boundedness does not hold for $p=1$, but Fujii [20] (see also [95]) proved that the maximal operator of Fejér means is bounded from $H_1(G)$ to $L_1(G)$. Weisz in [92] generalized the result of

Fujii and proved that the maximal operator of Fejér means is bounded from $H_p(G)$ to $L_p(G)$, when $p > 1/2$. Simon [55] constructed the counterexample showing that the boundedness does not hold when $0 < p < 1/2$. Goginava [25] (see also [14] and [15]) generalized this result for $0 < p \leq 1/2$ and proved that the following theorem is true.

Theorem Gog2. *There exists a martingale $f \in H_p(G)$ $(0 < p \leq 1/2)$ such that*

$$\sup_{n\in\mathbb{N}} \|\sigma_n f\|_p = \infty.$$

Weisz [93] (see also Goginava [27]) proved that the following theorem is true.

Theorem We3. *Let $f \in H_{1/2}(G)$. Then there exists an absolute constant c such that*

$$\|\sigma^* f\|_{weak-L_{1/2}(G)} \leq c\,\|f\|_{H_{1/2}(G)}.$$

In [52], the boundedness of maximal operators of subsequences of Fejér means of the one-dimensional Walsh–Fourier series from $H_p(G)$ to $L_p(G)$ for $0 < p < 1/2$ was considered. In particular, the following result is true.

Theorem T13. *Let $0 < p < 1/2$ and $f \in H_p(G)$. Then the maximal operator*

$$\widetilde{\sigma}^* f := \sup_{k\in\mathbb{N}} |\sigma_{m_k} f|$$

is bounded from $H_p(G)$ to $L_p(G)$ if and only if condition (1.3) *is fulfilled.*

As consequences, the following results are true.

Theorem T14. *Let $p > 0$ and $f \in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\left\| \sup_{n\in\mathbb{N}} |\sigma_{2^n} f| \right\|_p \leq c_p \|f\|_{H_p(G)} \tag{1.15}$$

and

$$\left\| \sup_{n\in\mathbb{N}} |\sigma_{2^n+2^{n-1}} f| \right\|_p \leq c_p \|f\|_{H_p(G)}.$$

On the other hand, we have the following negative result.

Theorem T15. *Let $0 < p < 1/2$. Then there exists a martingale $f \in H_p(G)$ such that*

$$\left\| \sup_{n\in\mathbb{N}} |\sigma_{2^n+1} f| \right\|_p = \infty.$$

The above-mentioned condition is sufficient for the case $p = 1/2$ too, but there exist the subsequences that do not satisfy condition (1.3) and the maximal operator of subsequences of Fejér means of the one-dimensional Walsh–Fourier series are bounded from $H_{1/2}(G)$ to $L_{1/2}(G)$.

However, the problem of finding the necessary and sufficient conditions on the indices, which provide the boundedness of maximal operator of subsequences of Fejér means of the one-dimensional Walsh–Fourier series from $H_{1/2}(G)$ to $L_{1/2}(G)$ is still open.

In [26] and [63] (see also [30, 50, 62, 65]), it is proved

Theorem GT1. *Let $0 < p \leq 1/2$ and $f \in H_p(G)$. Then the maximal operator*

$$\widetilde{\sigma}_p^* f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from $H_p(G)$ to $L_p(G)$.

Moreover, for any nondecreasing function $\varphi : \mathbb{N}_+ \to [1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}{\varphi(n)} = +\infty,$$

there exists a martingale $f \in H_p(G)$ $(0 < p < 1/2)$ *such that*

$$\sup_{n\in\mathbb{N}} \left\| \frac{\sigma_n f}{\varphi(n)} \right\|_p = \infty.$$

From the divergence of weighted maximal operators we immediately get that there exists a martingale $f \in H_p(G)$ $(0 < p \leq 1/2)$ such that

$$\sup_{n\in\mathbb{N}} \|\sigma_n f\|_p = \infty,$$

and from the boundedness results of weighted maximal operators we immediately get that for any $f \in H_p(G)$ there exists an absolute constant c_p such that the inequality

$$\|\sigma_n f\|_p \leq c_p n^{1/p-2} \log^{2[1/2+p]}(n+1) \|f\|_{H_p(G)} \text{ as } 0 < p \leq \frac{1}{2} \tag{1.16}$$

holds true. Applying inequality (1.16) in [67], the necessary and sufficient conditions were found for the modulus of continuity of a martingale $f \in H_p(G)$, for which Fejér means of the one-dimensional Walsh–Fourier series converge in $H_p(G)$ norm.

Theorem T16. *Let* $0 < p \leq 1/2$, $f \in H_p(G)$ *and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^N}, f\Big) = o\Big(\frac{1}{2^{N(1/p-2)} N^{2[1/2+p]}}\Big) \text{ as } N \to \infty.$$

Then

$$\|\sigma_n f - f\|_p \to 0 \text{ as } n \to \infty.$$

Moreover, there exists a martingale $f \in H_p(G)$, *for which*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^N}, f\Big) = O\Big(\frac{1}{2^{N(1/p-2)} N^{2[1/2+p]}}\Big) \text{ as } N \to \infty$$

and

$$\|\sigma_n f - f\|_p \nrightarrow 0 \text{ as } n \to \infty.$$

According to the above-mentioned results, it is of interest to find the necessary and sufficient conditions for the modulus of continuity, for which subsequences $\sigma_{n_k} f$ of Fejér means of the one-dimensional Walsh–Fourier series converge in $H_p(G)$ norm.

In Section 3, we find the necessary and sufficient conditions for the modulus of continuity, for which subsequences $\sigma_{n_k} f$ of Fejér means of the one-dimensional Walsh–Fourier series converge in $H_p(G)$ norm (see also [71]).

Applying inequality (1.13) to the case $p = 1/2$, the following necessary and sufficient conditions are found.

Theorem 3.3. *Let* $f \in H_{1/2}(G)$ *and* $\{m_k : k \geq 0\}$ *be an increasing sequence of natural numbers such that*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^{|m_k|}}, f\Big) = o\Big(\frac{1}{V^2(m_k)}\Big) \text{ as } k \to \infty.$$

Then

$$\|\sigma_{m_k} f - f\|_{H_{1/2}(G)} \to 0 \text{ as } k \to \infty.$$

Moreover, if $\{m_k : k \geq 0\}$ *is an increasing sequence of natural numbers such that* (1.5) *holds true, then there exist a martingale* $f \in H_{1/2}(G)$ *and a subsequence* $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$ *such that*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^{|\alpha_k|}}, f\Big) = O\Big(\frac{1}{V^2(\alpha_k)}\Big) \text{ as } k \to \infty$$

and

$$\limsup_{k\to\infty} \|\sigma_{\alpha_k} f - f\|_{1/2} > c > 0 \text{ as } k \to \infty,$$

where c is an absolute constant.

Applying inequality (1.14), we investigate the case $0<p<1/2$. In Section 3, we prove that the following theorem is true.

Theorem 3.4. *Let $0<p<1/2$, $f\in H_p(G)$ and $\{m_k:\ k\geq 0\}$ be an increasing sequence of natural numbers such that*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|m_k|}},f\Big)=o\Big(\frac{1}{2^{d(m_k)(1/p-2)}}\Big)\ \ as\ \ k\to\infty.$$

Then

$$\|\sigma_{m_k}f-f\|_{H_p(G)}\to 0\ \ as\ \ k\to\infty.$$

On the other hand, if $\{m_k:\ k\geq 0\}$ is an increasing sequence of natural numbers satisfying condition (1.5)*, then there exist a martingale $f\in H_p(G)$ and a subsequence $\{\alpha_k:\ k\geq 0\}\subset\{m_k:\ k\geq 0\}$, for which*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big)=O\Big(\frac{1}{2^{d(\alpha_k)(1/p-2)}}\Big)\ \ as\ \ k\to\infty$$

and

$$\limsup_{k\to\infty}\|\sigma_{\alpha_k}f-f\|_{\text{weak-}L_p(G)}>c_p>0\ \ as\ \ k\to\infty,$$

where c_p is constant depending only on p.

However, Simon in [56] and [58] (see also [18, 57, 59]) considered strong convergence theorems of the one-dimensional Walsh–Fourier series and proved the following

Theorem Si1. *Let $0<p\leq 1$ and $f\in H_1(G)$. Then there exists an absolute constant c_p, depending only on p, such that the following inequality is true:*

$$\frac{1}{\log^{[p]}n}\sum_{k=1}^{n}\frac{\|S_kf\|_{H_p(G)}}{k^{2-p}}\leq c_p\|f\|_{H_p(G)},$$

Analogous result for trigonometric system was proved in [60], and for unbounded Walsh systems in [22].

In [64], it was proved that the following theorem is true.

Theorem T17. *For any $0<p<1$ and non-decreasing function $\varphi:\mathbb{N}_+\to[1,\ \infty)$ satisfying the condition*

$$\overline{\lim_{n\to\infty}}\frac{n^{2-p}}{\varphi(n)}=+\infty,$$

there exists a martingale $f\in H_p(G)$ such that

$$\sum_{k=1}^{\infty}\frac{\|S_kf\|^p_{\text{weak-}L_p(G)}}{\varphi(k)}=\infty\ \ (0<p<1).$$

From Theorem Si1 it follows that if $f\in H_1(G)$, then the following equalities are true:

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf-f\|_1}{k}=0$$

and

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf\|_1}{k}=\|f\|_{H_1(G)}.$$

When $0<p<1$ and $f\in H_p(G)$, then from Theorem Si1 follows that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{n^{1/2-p/2}}\sum_{k=1}^{n}\frac{\|S_kf\|^p_{H_p(G)}}{k^{3/2-p/2}}\leq c_p\|f\|^p_{H_p(G)}.$$

Moreover,

$$\frac{1}{n^{1/2-p/2}}\sum_{k=1}^{n}\frac{\|S_kf-f\|_{H_p(G)}^p}{k^{3/2-p}}=0.$$

We have the equality

$$\frac{1}{n^{1/2-p/2}}\sum_{k=1}^{n}\frac{\|S_kf\|_{H_p(G)}^p}{k^{3/2-p/2}}=\|f\|_{H_p(G)}^p.$$

In Section 3, we consider strong convergence results of Fejér means of the one-dimensional Walsh–Fourier series. According to Theorem We1 and Theorem Gog2, we only have to consider the case $0<p\leq 1/2$ (for details see [66] and also [8,10–13]):

Theorem 3.5. *Let $0<p\leq 1/2$ and $f\in H_p(G)$. Then there exists an absolute constant c_p depending only on p, such that*

$$\frac{1}{\log^{[1/2+p]}n}\sum_{m=1}^{n}\frac{\|\sigma_mf\|_{H_p(G)}^p}{m^{2-2p}}\leq c_p\|f\|_{H_p(G)}^p.$$

Moreover, let $0<p<1/2$ and $\Phi:\mathbb{N}_+\to[1,\infty)$ be a non-decreasing, non-negative function such that $\Phi(n)\uparrow\infty$ and

$$\overline{\lim_{k\to\infty}}\frac{k^{2-2p}}{\Phi(k)}=\infty.$$

Then there exists a martingale $f\in H_p(G)$ such that

$$\sum_{m=1}^{\infty}\frac{\|\sigma_mf\|_{\text{weak-}L_p(G)}^p}{\Phi(m)}=\infty.$$

When $p=1/2$, it was also proved that the following theorem is true.

Theorem 3.6. *Let $f\in H_{1/2}(G)$. Then*

$$\sup_{n\in\mathbb{N}_+}\sup_{\|f\|_{H_p(G)}\leq 1}\frac{1}{n}\sum_{m=1}^{n}\|\sigma_mf\|_{1/2}^{1/2}=\infty.$$

Theorem 3.5 implies that if $f\in H_{1/2}(G)$, then the following equalities are true:

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\sigma_kf-f\|_{H_{1/2}(G)}^{1/2}}{k}=0$$

and

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\sigma_kf\|_{H_{1/2}(G)}^{1/2}}{k}=\|f\|_{H_{1/2}(G)}^{1/2}.$$

When $0<p<1/2$ and $f\in H_p(G)$, then Theorem 3.5 impliles that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{n^{1/2-p}}\sum_{k=1}^{n}\frac{\|\sigma_kf\|_{H_p(G)}^p}{k^{3/2-p}}\leq c_p\|f\|_{H_p(G)}^p.$$

Moreover,

$$\frac{1}{n^{1/2-p}}\sum_{k=1}^{n}\frac{\|\sigma_kf-f\|_{H_p(G)}^p}{k^{3/2-p}}=0.$$

Thus we have

$$\frac{1}{n^{1/2-p}}\sum_{k=1}^{n}\frac{\|\sigma_kf\|_{H_p(G)}^p}{k^{3/2-p}}=\|f\|_{H_p(G)}^p.$$

2 Partial sums with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

2.1 Basic notations

Denote by $\mathbb{N}_+$ the set of positive integers and by $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ the set of non-negative integers. Denote by Z_2 an additive group of integers modulo-2, which contains only two elements $Z_2 := \{0, 1\}$, group operation is modulo-2 sum and all sets are open.

Define the group G as the complete direct product of the groups Z_2 with the product of the discrete topologies Z_2. The direct product μ of measures $\mu_n(\{j\}) := 1/2$ $(j \in Z_2)$ is the Haar measure on G with $\mu(G) = 1$.

The elements of G are represented by the sequences

$$x := (x_0, x_1, \dots, x_j, \dots) \ \ (x_k = 0, 1).$$

It is easy to give a base for the neighbourhood of G,

$$I_0(x) = G,$$
$$I_n(x) := \left\{y \in G \mid \ y_0 = x_0, \dots, y_{n-1} = x_{n-1}\right\} \ \ (x \in G, \ \ n \in \mathbb{N}).$$

Set $I_n := I_n(0)$ for any $n \in \mathbb{N}$ and $\overline{I}_n := G \setminus I_n$.

It is evident that

$$\overline{I}_M = \Big(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l)\Big) \cup \Big(\bigcup_{k=0}^{M-1} I_M(e_k)\Big) = \bigcup_{k=0}^{M-1} I_k \setminus I_{k+1}. \tag{2.1}$$

If $n \in \mathbb{N}$, then it can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j\, 2^j$, where $n_j \in Z_2$ $(j \in \mathbb{N})$, and only a finite number of n_js differs from zero. Set

$$\langle n \rangle := \min\left\{j \in \mathbb{N}, \ n_j \neq 0\right\} \ \text{ and } \ |n| := \max\left\{j \in \mathbb{N}, \ n_j \neq 0\right\},$$

It is evident that $2^{|n|} \leq n \leq 2^{|n|+1}$. Let

$$d(n) := |n| - \langle n \rangle \ \text{ for any } \ n \in \mathbb{N}.$$

Denote by $V(n)$ the variation of natural number $n \in \mathbb{N}$,

$$V(n) = n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|.$$

Define k-th Rademacher functions by

$$r_k(x) := (-1)^{x_k} \ \ (x \in G, \ \ k \in \mathbb{N}).$$

By using Rademacher functions, we define the Walsh system $w := (w_n : \ n \in \mathbb{N})$ G as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \ \ (n \in \mathbb{N}).$$

The norm (quasi-norm) of the space $L_p(G)$ and weak-$L_p(G)$ for $(0 < p < \infty)$ are respectively defined as

$$\|f\|_p^p := \Big(\int_G |f(x)|^p \, d\mu(x)\Big), \quad \|f\|^p_{\text{weak-}L_p(G)} := \sup_{\lambda > 0} \lambda^p \mu\big(x \in G : \ |f| > \lambda\big).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [54]).

For any $f \in L_1(G)$, the numbers

$$\widehat{f}(n) := \int_G f(x) w_n(x)\, d\mu(x)$$

are called n-th Walsh–Fourier coefficient of f.

The n-th partial sum is denoted by

$$S_n(f;x) := \sum_{i=0}^{n-1} \widehat{f}(i) w_i(x).$$

The Dirichlet kernels are defined by

$$D_n(x) := \sum_{i=0}^{n-1} w_i(x).$$

We also define the following maximal operators:

$$S^* f = \sup_{n\in\mathbb{N}} |S_n f|, \quad \widetilde{S}^*_{\#} f = \sup_{n\in\mathbb{N}} |S_{2^n} f|.$$

The σ-algebra generated by the intervals $I_n(x)$ with measure 2^{-n} is denoted by $\digamma_n (n \in \mathbb{N})$. The conditional exponential operator with respect to $\digamma_n$ $(n \in \mathbb{N})$ is denoted by E_n and it is given by

$$E_n f(x) = S_{2^n} f(x) = \sum_{k=0}^{2^n-1} \widehat{f}(k) w_k(x) = \frac{1}{|I_n(x)|} \int_{I_n(x)} f(x)\, d\mu(x),$$

where $|I_n(x)| = 2^{-n}$ denotes length of the set $I_n(x)$.

The sequence $f = (f_n,\ n \in \mathbb{N})$ of functions $f_n \in L_1(G)$ is called a dyadic martingale (for details see [43, 54]) if

(i) f_n is measurable with respect to σ-algebras $\digamma_n$ for any $n \in \mathbb{N}$,

(ii) $E_n f_m = f_n$ for any $n \le m$.

The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbb{N}} |f_n|.$$

In case $f \in L_1(G)$, the maximal functions are also given by

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|.$$

For $0 < p < \infty$, the Hardy martingale space $H_p(G)$ consists of all martingales, for which

$$\|f\|_{H_p(G)} := \|f^*\|_p < \infty.$$

A bounded measurable function a is said to be a p-atom if there exists a dyadic interval I such that

$$\int_I a\, d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

It is easy to show that for a martingale $f = (f_n,\ n \in \mathbb{N})$ and for any $k \in \mathbb{N}$, there exists a limit

$$\widehat{f}(k) := \lim_{n\to\infty} \int_G f_n(x) w_k(x)\, d\mu(x)$$

and it is called the k-th Walsh–Fourier coefficients of f.

If $f_0 \in L_1(G)$ and $f := (E_n f_0 : n \in \mathbb{N})$ is a regular martingale, then

$$\widehat{f}(k) = \int_G f(x) w_k(x)\, d\mu(x) = \widehat{f}_0(k), \quad k \in \mathbb{N}.$$

The modulus of continuity in the space $H_p(G)$ is defined by

$$\omega_{H_p(G)}\Big(\frac{1}{2^n}, f\Big) := \|f - S_{2^n} f\|_{H_p(G)}.$$

It is important to describe how one can understand the difference $f - S_{2^n} f$, where f is a martingale and $S_{2^n} f$ is a function:

Remark 2.1. Let $0 < p \le 1$. Since

$$S_{2^n} f = f_n \in L_1(G), \quad \text{where} \ \ f = (f_n : n \in \mathbb{N}) \in H_p(G),$$

and

$$\begin{aligned}(S_{2^k} f_n : k \in \mathbb{N}) &= (S_{2^k} S_{2^n}, \ k \in \mathbb{N}) \\ &= (S_{2^0} f, \dots, S_{2^{n-1}} f, S_{2^n} f, S_{2^n} f, \dots) = (f_0, \dots, f_{n-1}, f_n, f_n, \dots),\end{aligned}$$

under the difference $f - S_{2^n} f$ we mean the following martingale:

$$f := \big((f - S_{2^n} f)_k, \ k \in \mathbb{N}\big),$$

where

$$(f - S_{2^n} f)_k = \begin{cases} 0, & k = 0, \dots, n, \\ f_k - f_n, & k \ge n+1. \end{cases}$$

Consequently, the norm $\|f - S_{2^n} f\|_{H_p(G)}$ is understood as H_p-norm of

$$f - S_{2^n} f = \big((f - S_{2^n} f)_k, \ k \in \mathbb{N}\big).$$

Watari [88] showed that there are strong connections between

$$\omega_p\Big(\frac{1}{2^n}, f\Big), \quad E_{2^n}(L_p, f) \ \ \text{and} \ \ \|f - S_{2^n} f\|_p, \quad p \ge 1, \quad n \in \mathbb{N}.$$

In particular,

$$\frac{1}{2}\,\omega_p\Big(\frac{1}{2^n}, f\Big) \le \|f - S_{2^n} f\|_p \le \omega_p\Big(\frac{1}{2^n}, f\Big)$$

and

$$\frac{1}{2}\,\|f - S_{2^n} f\|_p \le E_{2^n}(L_p, f) \le \|f - S_{2^n} f\|_p.$$

2.2 Auxiliary lemmas

First, we present and prove equalities and estimations of the Dirichlet kernel and Lebesgue constants with respect to the one-dimensional Walsh–Fourier systems (see Lemmas 2.1–3.5).

The first equality of the following Lemma is proved in [54] and the second identity is proved in [23].

Lemma 2.1. *Let $j, n \in \mathbb{N}$. Then*

$$D_{j+2^n} = D_{2^n} + w_{2^n} D_j, \quad \textit{when} \ \ j \le 2^n,$$

and

$$D_{2^n - j} = D_{2^n} - \psi_{2^n - 1} D_j, \quad \textit{when} \ \ j < 2^n.$$

The following estimation of the Dirichlet kernel with respect to the one-dimensional Walsh–Fourier systems is proved in [54].

Lemma 2.2. *Let $n \in \mathbb{N}$. Then*

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n, \end{cases}$$

and

$$D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}} - D_{2^k}) \text{ for } n = \sum_{i=0}^{\infty} n_i 2^i.$$

The following two-sided estimations of the Lebesgue constants with respect to the one-dimensional Walsh–Fourier systems is proved in [54] and the second equality is proved in [19].

Lemma 2.3. *Let $n \in \mathbb{N}$. Then*

$$\frac{1}{8} V(n) \leq \|D_n\|_1 \leq V(n)$$

and

$$\frac{1}{n \log n} \sum_{k=1}^{n} V(k) = \frac{1}{4 \log 2} + o(1).$$

The Hardy martingale space $H_p(G)$ for any $0 < p \leq 1$ can be characterized by simple functions which are called p-atoms. The following lemma is true (for details see [57, 90, 94]).

Lemma 2.4. *A martingale $f = (f_n,\ n \in \mathbb{N})$ belongs to $H_p(G)$ $(0 < p \leq 1)$ if and only if there exist a sequence of p-atoms of $(a_k,\ k \in \mathbb{N})$ and a sequence of real numbers $(\mu_k,\ k \in \mathbb{N})$ such that*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f_n \text{ for all } n \in \mathbb{N} \tag{2.2}$$

and

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p(G)} \backsim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p},$$

where the infimum is taken over all decomposition of f of form (2.2).

The next five Examples of martingales will be used frequently to prove the sharpness of our main results. Such counterexamples appeared first in the paper by Goginava [28] (see also [24, 27]). Such constructions of martingales are also used in the papers [3–7, 16, 17, 31–33, 36–42, 45–49, 61, 66, 70–80, 82–87]. So, we leave out the details of proof.

Example 2.1. Let $0 < p \leq 1$, $\{\lambda_k : k \in \mathbb{N}\}$ be a sequence of real numbers

$$\sum_{k=0}^{\infty} |\lambda_k|^p \leq c_p < \infty \tag{2.3}$$

and $\{a_k : k \in \mathbb{N}\}$ be a sequence of p-atoms given by

$$a_k(x) := 2^{|\alpha_k|(1/p-1)} \big(D_{2^{|\alpha_k|+1}}(x) - D_{2^{|\alpha_k|}}(x) \big),$$

where $|\alpha_k| := \max\{j \in \mathbb{N} : (\alpha_k)_j \neq 0\}$ and $(\alpha_k)_j$ denotes j-th binary coefficients of real number of $\alpha_k \in \mathbb{N}_+$. Then $f = (f_n : n \in \mathbb{N})$, where

$$f_n(x) := \sum_{\{k:\ |\alpha_k| < n\}} \lambda_k a_k(x)$$

is a martingale, which belongs to $H_p(G)$ for any $0<p\le 1$.

It is easy to show that

$$\widehat{f}(j)=\begin{cases}\lambda_k\,2^{(1/p-1)|\alpha_k|}, & j\in\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\\ 0, & j\notin\bigcup\limits_{k=1}^{\infty}\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\end{cases}\quad k\in\mathbb{N}_+. \tag{2.4}$$

Let $2^{|\alpha_{l-1}|+1}\le j\le 2^{|\alpha_l|}$, $l\in\mathbb{N}_+$. Then

$$S_jf=S_{2^{|\alpha_{l-1}|+1}}=\sum_{\eta=0}^{l-1}\lambda_\eta\,2^{|\alpha_\eta|(1/p-1)}(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|}}). \tag{2.5}$$

Let $2^{|\alpha_l|}\le j<2^{|\alpha_l|+1}$, $l\in\mathbb{N}_+$. Then

$$\begin{aligned}S_jf&=S_{2^{|\alpha_l|}}+\lambda_l\,2^{(1/p-1)|\alpha_l|}w_{2^{|\alpha_l|}}D_{j-2^{|\alpha_l|}}\\&=\sum_{\eta=0}^{l-1}\lambda_\eta\,2^{(1/p-1)|\alpha_\eta|}(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|}})+\lambda_l\,2^{(1/p-1)|\alpha_l|}w_{2^{|\alpha_l|}}D_{j-2^{|\alpha_l|}}.\end{aligned} \tag{2.6}$$

Moreover, for the modulus of continuity for $0<p\le 1$, we have the following estimation:

$$\omega_{H_p}\Big(\frac{1}{2^n},f\Big)=O\Big(\sum_{\{k:\ |\alpha_k|\ge n\}}^{\infty}|\lambda_k|^p\Big)^{1/p}\quad\text{as}\ \ n\to\infty. \tag{2.7}$$

Applying Lemma 2.4, we easily obtain that the following lemma is true (see [94]).

Lemma 2.5. *Let $0<p\le 1$ and T be a σ-sub-linear operator such that for any p-atom a,*

$$\int\limits_G|Ta(x)|^p\,d\mu(x)\le c_p<\infty.$$

Then

$$\|Tf\|_p\le c_p\|f\|_{H_p(G)}. \tag{2.8}$$

In addition, if T is bounded from $L_\infty(G)$ to $L_\infty(G)$, then to prove (2.8) it suffices to show that

$$\int\limits_{\overline{I}}|Ta(x)|^p\,d\mu(x)\le c_p<\infty$$

for every p-atom a, where I denotes a support of the atom a.

In concrete cases, the norm of Hardy martingale spaces can be calculated by simpler formulas (for details see [57, 90, 91]).

Lemma 2.6. *If $g\in L_1(G)$ and $f:=(E_ng:\ n\in\mathbb{N})$ is a regular martingale, then for $0<p\le 1$, $H_p(G)$ norm can be calculated by*

$$\|f\|_{H_p(G)}=\Big\|\sup_{n\in\mathbb{N}}|S_{2^n}g|\Big\|_p.$$

The following lemmas are proved in [66, 70, 71].

Lemma 2.7. *Let $0<p\le 1$, $2^k\le n<2^{k+1}$ and S_nf be the n-th partial sum with respect to the one-dimensional Walsh–Fourier series, where $f\in H_p(G)$. Then for any fixed $n\in\mathbb{N}$,*

$$\|S_nf\|_{H_p(G)}^p\le\Big\|\sup_{0\le l\le k}|S_{2^l}f|\Big\|_p^p+\|S_nf\|_p^p\le\|\widetilde{S}^*_{\#}f\|_p^p+\|S_nf\|_p^p.$$

Proof. Let us consider the following martingales:

$$f_{\#} := (S_{2^k}S_nf,\ k\in\mathbb{N}_+) = (S_{2^0}, S_{2^k}f, S_nf, \dots, S_nf, \dots).$$

Hence from Lemma 2.6 immediately follows

$$\|S_nf\|_{H_p(G)}^p \le \Big\| \sup_{0\le l\le k} |S_{2^l}f| \Big\|_p^p + \|S_nf\|_p^p \le \|\widetilde{S}_{\#}^*f\|_p^p + \|S_nf\|_p^p.$$

Lemma is proved. □

2.3 Boundedness of subsequences of partial sums with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

In this section, we consider the boundedness of subsequences of partial sums with respect to the one-dimensional Walsh–Fourier series in the martingale Hardy spaces (for details see [70]).

Theorem 2.1.

(a) *Let $0<p<1$ and $f\in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_nf\|_{H_p(G)} \le c_p\, 2^{d(n)(1/p-1)}\|f\|_{H_p(G)}.$$

(b) *Let $0<p<1$, $\{m_k:\ k\in\mathbb{N}_+\}$ be a non-negative, increasing sequence of natural numbers such that*

$$\sup_{k\in\mathbb{N}} d(m_k) = \infty \tag{2.9}$$

and let $\Phi:\mathbb{N}_+\to[1,\infty)$ be a non-decreasing function satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{2^{d(m_k)(1/p-1)}}{\Phi(m_k)} = \infty. \tag{2.10}$$

Then there exists a martingale $f\in H_p(G)$ such that

$$\sup_{k\in\mathbb{N}} \Big\| \frac{S_{m_k}f}{\Phi(m_k)} \Big\|_{\text{weak-}L_p(G)} = \infty.$$

Proof. Suppose that

$$\big\|2^{(1-1/p)d(n)}S_nf\big\|_p \le c_p\|f\|_{H_p(G)}. \tag{2.11}$$

Combining Lemma 2.7 and inequalities (1.7) and (2.11), since $2^{(1-1/p)d(n)}\le c_p$, we obtain

$$\begin{aligned}&\big\|2^{(1-1/p)d(n)}S_nf\big\|_{H_p(G)}^p \\ &\quad\le \big\|2^{(1-1/p)d(n)}S_nf\big\|_p^p + 2^{(1-1/p)d(n)}\|\widetilde{S}_{\#}^*f\|_p^p \le c_p\|f\|_{H_p(G)}^p + c_p\|\widetilde{S}_{\#}^*f\|_p^p \le c_p\|f\|_{H_p(G)}^p.\end{aligned} \tag{2.12}$$

Combining Lemma 2.5 and (2.12), it suffices to show that

$$\int_G \big|2^{(1-1/p)d(n)}S_na\big|^p\,d\mu \le c_p < \infty \tag{2.13}$$

for every p-atom a, with support I, such that $\mu(I)=2^{-M}$.

Without loss of generality, we may assume that a p-atom a has support $I=I_M$. Then it is easy to see that $S_na=0$, where $2^M\ge n$. So, we may assume that $2^M<n$. Since $\|a\|_\infty\le 2^{M/p}$, we can conclude that

$$\begin{aligned}&\big|2^{(1-1/p)d(n)}S_na(x)\big| \\ &\quad\le 2^{(1-1/p)d(n)}\|a\|_\infty \int_{I_M} |D_n(x+t)|\,d\mu(t) \le 2^{M/p}\,2^{(1-1/p)d(n)} \int_{I_M} |D_n(x+t)|\,d\mu(t).\end{aligned} \tag{2.14}$$

Let $x \in I_M$. Since $V(n) \le 2d(n)$, using the first estimations of Lemma 2.3, we can conclude that

$$\left|2^{(1-1/p)d(n)} S_n a\right| \le 2^{M/p}\, 2^{(1-1/p)d(n)} V(n) \le 2^{M/p} d(n)\, 2^{(1-1/p)d(n)}$$

and

$$\int_{I_M} \left|2^{(1-1/p)d(n)} S_n a\right|^p d\mu \le d(n)\, 2^{(1-1/p)d(n)} < c_p < \infty. \tag{2.15}$$

Let $t \in I_M$ and $x \in I_s \setminus I_{s+1}$, where $0 \le s \le M-1 < \langle n\rangle$ or $0 \le s < \langle n\rangle \le M-1$. Then $x+t \in I_s \setminus I_{s+1}$ and if we use both equalities of Lemma 2.2, we get $D_n(x+t) = 0$ and thus

$$\left|2^{(1-1/p)d(n)} S_n a(x)\right| = 0. \tag{2.16}$$

Let $x \in I_s \setminus I_{s+1}$, $\langle n\rangle \le s \le M-1$. Then $x+t \in I_s \setminus I_{s+1}$, where $t \in I_M$. Then by using again both equality of Lemma 2.2 we have that

$$|D_n(x+t)| \le \sum_{j=0}^{s} n_j\, 2^j \le c\, 2^s.$$

If we apply again (2.14), we can conclude that

$$\begin{aligned}\left|2^{(1-1/p)d(n)} S_n a(x)\right| &\le 2^{(1-1/p)d(n)}\, 2^{M/p}\, \frac{2^s}{2^M} \\ &\le 2^{\langle n\rangle(1/p-1)}\, 2^{M(1/p-1)}\, \frac{2^s}{2^{|n|(1/p-1)}} \le 2^{\langle n\rangle(1/p-1)}\, 2^s. \end{aligned}\tag{2.17}$$

By identity (2.1) and inequalities (2.16) and (2.17), we find that

$$\int_{\overline{I}_M} \left|2^{(1-1/p)d(n)} S_n a(x)\right|^p d\mu(x) = \sum_{s=\langle n\rangle}^{M-1} \int_{I_s\setminus I_{s+1}} \left|2^{\langle n\rangle(1/p-1)}\, 2^s\right|^p d\mu(x) \le c \sum_{s=\langle n\rangle}^{M-1} \frac{2^{\langle n\rangle(1-p)}}{2^{s(1-p)}} \le c_p < \infty.$$

Now, we prove part b) of Theorem 2.1. Using condition (2.10), there exists the sequence of natural numbers $\{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\}$ such that

$$\sum_{\eta=0}^{\infty} \frac{\Phi^{p/2}(\alpha_\eta)}{2^{d(\alpha_\eta)(1-p)/2}} < \infty. \tag{2.18}$$

Let $f = (f_n,\ n \in \mathbb{N}_+) \in H_p(G)$ be a martingale from Example 2.1, where

$$\lambda_k = \frac{\Phi^{1/2}(\alpha_k)}{2^{d(\alpha_k)(1/p-1)/2}}\,. \tag{2.19}$$

Then if we use (2.18), we find that condition (2.3) is fulfilled, and hence $f = (f_n,\ n \in \mathbb{N}_+) \in H_p(G)$. If we apply (2.4) when λ_k are given by formula (2.19), then we get

$$\widehat{f}(j) = \begin{cases} \Phi^{1/2}(\alpha_k)\, 2^{(|\alpha_k|+\langle\alpha_k\rangle)(1/p-1)/2}, & \text{if } j \in \left\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1}-1\right\}, \\ 0, & \text{if } j \notin \bigcup\limits_{k=0}^{\infty} \left\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1}-1\right\}, \end{cases} \quad k \in \mathbb{N}_+. \tag{2.20}$$

In view of (2.6), when λ_k are given by (2.19), we get

$$\begin{aligned}\frac{S_{\alpha_k} f}{\Phi(\alpha_k)} &= \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{k-1} \Phi^{1/2}(\alpha_\eta)\, 2^{(|\alpha_\eta|+\langle\alpha_\eta\rangle)(1/p-1)/2} \left(D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}\right) \\ &\quad + \frac{2^{(|\alpha_k|+\langle\alpha_k\rangle)(1/p-1)/2} w_{2^{|\alpha_k|}} D_{\alpha_k - 2^{|\alpha_k|}}}{\Phi^{1/2}(\alpha_k)} := I + II. \end{aligned}\tag{2.21}$$

Using (2.18), for I we have

$$\|I\|^p_{\text{weak-}L_p(G)} \le \frac{1}{\Phi^p(\alpha_k)} \sum_{\eta=0}^{k-1} \frac{\Phi^{p/2}(\alpha_\eta)}{2^{d(\alpha_\eta)(1-p)/2}} \left\|2^{|\alpha_\eta|(1/p-1)}(D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}})\right\|^p_{\text{weak-}L_p(G)}$$
$$\le \frac{1}{\Phi^p(\alpha_k)} \sum_{\eta=0}^{\infty} \frac{\Phi^{p/2}(\alpha_\eta)}{2^{d(\alpha_\eta)(1-p)/2}} \le c < \infty. \tag{2.22}$$

Let $x \in I_{\langle\alpha_k\rangle} \setminus I_{\langle\alpha_k\rangle+1}$. Since $|\alpha_k| \neq \langle\alpha_k\rangle$ and $\langle\alpha_k - 2^{|\alpha_k|}\rangle = \langle\alpha_k\rangle$, using both inequalities of Lemma 2.2, we get

$$|D_{\alpha_k - 2^{|\alpha_k|}}(x)| = \Big|\big(D_{2^{\langle\alpha_k\rangle+1}}(x) - D_{2^{\langle\alpha_k\rangle}}(x)\big)$$
$$+ \sum_{j=\langle\alpha_k\rangle+1}^{|\alpha_k|-1} (\alpha_k)_j (D_{2^{i+1}}(x) - D_{2^i}(x))\Big| = |-D_{2^{\langle\alpha_k\rangle}}(x)| = 2^{\langle\alpha_k\rangle} \tag{2.23}$$

and

$$|II| = \frac{2^{(|\alpha_k|+\langle\alpha_k\rangle)(1/p-1)/2}}{\Phi^{1/2}(\alpha_k)} |D_{\alpha_k - 2^{|\alpha_k|}}(x)| = \frac{2^{|\alpha_k|(1/p-1)/2}\, 2^{\langle\alpha_k\rangle(1/p+1)/2}}{\Phi^{1/2}(\alpha_k)}. \tag{2.24}$$

Combining (2.22) and (2.24), we obtain

$$\left\|\frac{S_{\alpha_k} f}{\Phi(\alpha_k)}\right\|^p_{\text{weak-}L_p(G)} \ge \|II\|^p_{\text{weak-}L_p(G)} - \|I\|^p_{\text{weak-}L_p(G)}$$
$$\ge \frac{2^{(|\alpha_k|)(1/p-1)/2}\, 2^{\langle\alpha_k\rangle(1/p+1)/2}}{\Phi^{1/2}(\alpha_k)} \mu\left\{x \in G:\ |II| \ge \frac{2^{(|\alpha_k|)(1/p-1)/2}\, 2^{\langle\alpha_k\rangle(1/p+1)/2}}{\Phi^{1/2}(\alpha_k)}\right\}^{1/p}$$
$$\ge \frac{2^{(|\alpha_k|)(1/p-1)/2}\, 2^{\langle\alpha_k\rangle(1/p+1)/2}}{\Phi^{1/2}(\alpha_k)} \big(\mu\{I_{\langle\alpha_k\rangle} \setminus I_{\langle\alpha_k\rangle+1}\}\big)^{1/p} \ge c\,\frac{2^{d(\alpha_k)(1/p-1)/2}}{\Phi^{1/2}(\alpha_k)} \to \infty \ \text{ as } \ k \to \infty.$$

The proof of Theorem 2.1 is complete. □

Corollary 2.1.

(a) *Let $n \in \mathbb{N}_+$, $0 < p < 1$ and $f \in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_n f\|_{H_p(G)} \le c_p \big(n\mu\{\text{supp}(D_n)\}\big)^{1/p-1} \|f\|_{H_p(G)}.$$

(b) *Let $0 < p < 1$, $\{m_k:\ k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers such that*

$$\sup_{k\in\mathbb{N}} m_k \mu\{\text{supp}(D_{m_k})\} = \infty \tag{2.25}$$

and let $\Phi : \mathbb{N}_+ \to [1,\infty)$ be a non-decreasing function satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{(m_k\mu\{\text{supp}(D_{m_k})\})^{1/p-1}}{\Phi(m_k)} = \infty. \tag{2.26}$$

Then there exists a martingale $f \in H_p(G)$ such that

$$\sup_{k\in\mathbb{N}} \left\|\frac{S_{m_k} f}{\Phi(m_k)}\right\|_{\text{weak-}L_p(G)} = \infty.$$

Proof. Applying both inequalities of Lemma 2.2, we get

$$I_{\langle n\rangle} \setminus I_{\langle n\rangle+1} \subset \text{supp}\{D_n\} \subset I_{\langle n\rangle} \ \text{ and } \ 2^{-\langle n\rangle-1} \le \mu\{\text{supp}(D_n)\} \le 2^{-\langle n\rangle}.$$

Hence

$$\frac{2^{d(n)(1/p-1)}}{4} \le \big(n\mu\{\text{supp}(D_n)\}\big)^{1/p-1} \le 2^{d(n)(1/p-1)}.$$

Corollary 2.1 is proved. □

Theorem 2.2.

(a) *Let $n \in \mathbb{N}_+$ and $f \in H_1(G)$. Then there exists an absolute constant c such that*

$$\|S_n f\|_{H_1(G)} \le c\,V(n)\|f\|_{H_1(G)}.$$

(b) *Let $\{m_k:\ k \in \mathbb{N}_+\}$ be a non-negative increasing sequence of natural numbers such that*

$$\sup_{k\in\mathbb{N}} V(m_k) = \infty \tag{2.27}$$

and let $\Phi : \mathbb{N}_+ \to [1,\infty)$ be a non-decreasing function satisfying the condition

$$\overline{\lim_{k\to\infty}}\, \frac{V(m_k)}{\Phi(m_k)} = \infty. \tag{2.28}$$

Then there exists a martingale $f \in H_1(G)$ such that

$$\sup_{k\in\mathbb{N}} \Big\| \frac{S_{m_k} f}{\Phi(m_k)} \Big\|_1 = \infty.$$

Proof. Since

$$\Big\| \frac{S_n f}{V(n)} \Big\|_1 \le \|f\|_1 \le \|f\|_{H_1(G)}, \tag{2.29}$$

combining Lemmas 2.7 and (2.29), we can conclude that

$$\Big\| \frac{S_n f}{V(n)} \Big\|_{H_1(G)} \le \Big\| \frac{S_n f}{V(n)} \Big\|_1 + \frac{1}{V(n)}\, \|\widetilde{S}^*_{\#} f\|_1 \le c\,\|f\|_{H_1(G)} + c\,\|\widetilde{S}^*_{\#} f\|_1 \le c\,\|f\|_{H_1(G)}. \tag{2.30}$$

Now, we prove the second part of Theorem 2.2. Let $\{m_k:\ k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers and the function $\Phi : \mathbb{N}_+ \to [1,\infty)$ satisfy conditions (2.27) and (2.28). Then there exists a non-negative, increasing sequence $\{\alpha_k:\ k \in \mathbb{N}_+\} \subset \{m_k:\ k \in \mathbb{N}_+\}$ such that

$$\sum_{k=1}^{\infty} \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} \le \beta < \infty. \tag{2.31}$$

Let $f = (f_n,\ n \in \mathbb{N}_+)$ be a martingale from Example 2.1, where

$$\lambda_k = \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)}\,. \tag{2.32}$$

Applying condition (2.31), we can conclude that condition (2.3) is fulfilled and it follows that $f = (f_n,\ n \in \mathbb{N}_+) \in H_1(G)$.

In view of (2.4), when λ_k are given by (2.32), we get

$$\widehat{f}(j) = \begin{cases} \dfrac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)}, & \text{if } j \in \big\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1} - 1\big\}, \\ 0, & \text{if } j \notin \displaystyle\bigcup_{k=0}^{\infty} \big\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1} - 1\big\}, \end{cases} \qquad k = 0, 1, \dots. \tag{2.33}$$

Analogously to (2.21), if we apply (2.6), when λ_k are given by (2.32) we get

$$S_{\alpha_k} f = \sum_{\eta=0}^{k-1} \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)}\,\big(D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}\big) + \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)}\, w_{2^{|\alpha_k|}} D_{\alpha_k - 2^{|\alpha_k|}}.$$

Applying first the estimation of Lemma 2.3 and (2.31), we can conclude that

$$\left\|\frac{S_{\alpha_k}f}{\Phi(\alpha_k)}\right\|_1 \geq \frac{\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V^{1/2}(\alpha_k)}\|D_{\alpha_k-2^{|\alpha_k|}}\|_1 - \frac{1}{\Phi(\alpha_k)}\sum_{\eta=0}^{k-1}\frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)}\|D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}\|_1$$

$$\geq \frac{V(\alpha_k - 2^{|\alpha_k|})\Phi^{1/2}(\alpha_k)}{8\Phi(\alpha_k)V^{1/2}(\alpha_k)} - \frac{1}{\Phi(\alpha_k)}\sum_{\eta=0}^{\infty}\frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \geq \frac{c\,V^{1/2}(\alpha_k)}{\Phi^{1/2}(\alpha_k)} \to \infty \text{ as } k \to \infty.$$

Thus Theorem 2.2 is proved. □

Corollary 2.2. *Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $f \in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_{2^n}f\|_{H_p(G)} \leq c_p\|f\|_{H_p(G)}. \tag{2.34}$$

Proof. To prove Theorem 2.2, we have only to show that

$$|2^n| = n, \quad \langle 2^n\rangle = n-1 \text{ and } d(2^n) = 0.$$

Applying the first part of Theorem 2.1, we immediately obtain (2.34) for any $0 < p \leq 1$ and thus Corollary 2.2 is proved. □

Corollary 2.3. *Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $f \in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_{2^n+2^{n-1}}f\|_{H_p(G)} \leq c_p\|f\|_{H_p(G)}. \tag{2.35}$$

Proof. Since

$$|2^n+2^{n-1}| = n, \quad \langle 2^n+2^{n-1}\rangle = n-1 \text{ and } d(2^n+2^{n-1}) = 1,$$

by the first part of Theorem 2.1 we get that (2.35) holds for any $0 < p \leq 1$ and the proof of Corollary 2.3 is complete. □

Corollary 2.4. *Let $n \in \mathbb{N}$ and $0 < p < 1$. Then there exists a martingale $f \in H_p(G)$ such that*

$$\sup_{n\in\mathbb{N}}\|S_{2^n+1}f\|_{\text{weak-}L_p(G)} = \infty. \tag{2.36}$$

On the other hand, there exists an absolute constant c, such that

$$\|S_{2^n+1}f\|_{H_1(G)} \leq c\,\|f\|_{H_1(G)}. \tag{2.37}$$

Proof. Since

$$|2^n+1| = n, \quad \langle 2^n+1\rangle = 0 \text{ and } d(2^n+1) = n, \tag{2.38}$$

applying the second part of Theorem 2.1, we get that there exists a martingale $f = (f_n,\ n \in \mathbb{N}_+) \in H_p(G)$ for $0 < p < 1$ such that (2.36) holds.

On the other hand, the proof of (2.37) leads to a simple observation that

$$V(2^n+1) = 4 < \infty.$$

Thus Corollary 2.4 is proved. □

2.4 Modulus of continuity and convergence in norm of subsequences of partial sums with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

In this section, we apply Theorems 2.1 and 2.2 to find the necessary and sufficient conditions for the modulus of continuity, for which subsequences of partial sums with respect to the one-dimensional Walsh–Fourier series are bounded in the martingale Hardy spaces.

First, we prove the following estimation.

Theorem 2.3. *Let $n \in \mathbb{N}_+$ and $2^k < n \leq 2^{k+1}$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|S_n f - f\|_{H_p(G)} \leq c_p\, 2^{d(n)(1/p-1)} \omega_{H_p(G)}\Big(\frac{1}{2^k}, f\Big) \ \ (f \in H_p(G)) \ \ (0 < p < 1) \tag{2.39}$$

and

$$\|S_n f - f\|_{H_1(G)} \leq c_1 V(n) \omega_{H_1(G)}\Big(\frac{1}{2^k}, f\Big) \ \ (f \in H_1(G)). \tag{2.40}$$

Proof. Let $0 < p < 1$ and $2^k < n \leq 2^{k+1}$. Applying the first part of Theorem 2.1, we get

$$\begin{aligned}
\|S_n f - f\|^p_{H_p(G)} &\leq c_p \|S_n f - S_{2^k} f\|^p_{H_p(G)} + c_p \|S_{2^k} f - f\|^p_{H_p(G)} \\
&= c_p \big\|S_n(S_{2^k} f - f)\big\|^p_{H_p(G)} + c_p \|S_{2^k} f - f\|^p_{H_p(G)} \\
&\leq c_p \big(1 + 2^{d(n)(1-p)}\big) \omega^p_{H_p(G)}\Big(\frac{1}{2^k}, f\Big) \leq c_p\, 2^{d(n)(1-p)} \omega^p_{H_p(G)}\Big(\frac{1}{2^k}, f\Big).
\end{aligned} \tag{2.41}$$

The proof of (2.40) is analogous to that of (2.39). So, we leave out the details. Theorem 2.3 is proved. □

Theorem 2.4.

(a) *Let $k \in \mathbb{N}_+$, $0 < p < 1$, $f \in H_p(G)$ and $\{m_k : \ k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers such that*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|m_k|}}, f\Big) = o\Big(\frac{1}{2^{d(m_k)(1/p-1)}}\Big) \ \ \text{as} \ \ k \to \infty. \tag{2.42}$$

Then

$$\|S_{m_k} f - f\|_{H_p(G)} \to 0 \ \ \text{as} \ \ k \to \infty. \tag{2.43}$$

(b) *Let $\{m_k : \ k \in \mathbb{N}_+\}$ be an increasing sequence of natural numberssuch that condition (2.9) is fulfilled. Then there exist a martingale $f \in H_p(G)$ and an increasing sequence of natural numbers $\{\alpha_k : \ k \in \mathbb{N}_+\} \subset \{m_k : \ k \in \mathbb{N}_+\}$ such that*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}}, f\Big) = O\Big(\frac{1}{2^{d(\alpha_k)(1/p-1)}}\Big) \ \ \text{as} \ \ k \to \infty$$

and

$$\limsup_{k\to\infty} \|S_{\alpha_k} f - f\|_{\text{weak-}L_p(G)} > c_p > 0 \ \ \text{as} \ \ k \to \infty, \tag{2.44}$$

where c_p is an absolute constant, depending only on p.

Proof. Let $0 < p < 1$, $f \in H_p(G)$ and $\{m_k : \ k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers such that condition (2.42) is fulfilled. Combining Theorem 2.3 and estimation (2.39), we get that (2.43) holds true.

Now, we prove the second part of Theorem 2.4. In view of (2.9), we simply get that there exists a sequence $\{\alpha_k : \ k \in \mathbb{N}_+\} \subset \{m_k : \ k \in \mathbb{N}_+\}$ such that

$$2^{d(\alpha_k)} \uparrow \infty \ \ \text{as} \ \ k \to \infty, \quad 2^{2(1/p-1)d(\alpha_k)} \leq 2^{(1/p-1)d(\alpha_{k+1})}. \tag{2.45}$$

Let $f=(f_n,\ n\in\mathbb{N})$ be a martingale from Example 2.1 such that

$$\lambda_i = 2^{-(1/p-1)d(\alpha_i)}. \tag{2.46}$$

Applying (2.45), we obtain that condition (2.3) is fulfilled, and hence $f\in H_p(G)$.
Applying (2.4), when λ_k are given by (2.46), we have

$$\widehat{f}(j) = \begin{cases} 2^{(1/p-1)\langle\alpha_k\rangle}, & \text{if } \ j\in\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\}, \\ 0, & \text{if } \ j\notin \bigcup\limits_{k=0}^{\infty}\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\}, \end{cases} \quad k\in\mathbb{N}_+ \tag{2.47}$$

Combining (2.45) and (2.7), we have

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big) \le \sum_{i=k}^{\infty}\frac{1}{2^{(1/p-1)d(\alpha_i)}} = O\Big(\frac{1}{2^{(1/p-1)d(\alpha_k)}}\Big) \ \text{ as } \ k\to\infty. \tag{2.48}$$

Using (2.23), we get

$$|D_{\alpha_k-2^{\langle\alpha_k\rangle}}| \ge 2^{\langle\alpha_k\rangle}, \ \text{ where } \ I_{\langle\alpha_k\rangle}\setminus I_{\langle\alpha_k\rangle+1}.$$

In view of (2.6), we can conclude that

$$S_{\alpha_k}f = S_{2^{|\alpha_k|}}f + 2^{(1/p-1)\langle\alpha_k\rangle}w_{2^{|\alpha_k|}}D_{\alpha_k-2^{|\alpha_k|}}.$$

Since

$$\begin{aligned}\|D_{\alpha_k}\|_{\text{weak-}L_p(G)} &\ge 2^{\langle\alpha_k\rangle}\mu\Big\{x\in I_{\langle\alpha_k\rangle}\setminus I_{\langle\alpha_k\rangle+1}:\ |D_{\alpha_k}|\ge 2^{\langle\alpha_k\rangle}\Big\}^{1/p} \\ &\ge 2^{\langle\alpha_k\rangle}\big(\mu\big\{I_{\langle\alpha_k\rangle}\setminus I_{\langle\alpha_k\rangle+1}\big\}\big)^{1/p} \ge 2^{\langle\alpha_k\rangle(1-1/p)},\end{aligned}$$

if we apply (1.2) (see also Theorem T2), we obtain

$$\begin{aligned}\|f-S_{\alpha_k}f\|^p_{\text{weak-}L_p(G)} &\ge 2^{(1-p)\langle\alpha_k\rangle}\big\|w_{2^{|\alpha_k|}}D_{\alpha_k-2^{|\alpha_k|}}\big\|^p_{\text{weak-}L_p(G)} \\ &\quad -\big\|f-S_{2^{|\alpha_k|}}f\big\|^p_{\text{weak-}L_p(G)} \ge c-o(1) > c > 0 \ \text{ as } \ k\to\infty.\end{aligned}$$

The proof of Theorem 2.4 is complete. □

Corollary 2.5.

(a) *Let $0<p<1$, $f\in H_p(G)$ and $\{m_k:\ k\in\mathbb{N}_+\}$ be an increasing sequence of natural numbers such that*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|m_k|}},f\Big) = o\Big(\frac{1}{(m_k\mu(\operatorname{supp}D_{m_k}))^{1/p-1}}\Big) \ \text{ as } \ k\to\infty. \tag{2.49}$$

Then (2.43) holds.

(b) *Let $\{m_k:\ k\in\mathbb{N}_+\}$ be an increasing sequence of natural numbers such that*

$$\sup_{k\in\mathbb{N}_+} m_k\mu\big\{\operatorname{supp}(D_{m_k})\big\} = \infty. \tag{2.50}$$

Then there exist a martingale $f\in H_p(G)$ and a sequence $\{\alpha_k:\ k\in\mathbb{N}_+\}\subset\{m_k:\ k\in\mathbb{N}_+\}$ such that

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big) = O\Big(\frac{1}{(\alpha_k\mu(\operatorname{supp}D_{\alpha_k}))^{1/p-1}}\Big) \ \text{ as } \ k\to\infty$$

and (2.44) holds.

Theorem 2.5.

(a) *Let $f \in H_1(G)$ and $\{m_k : k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers such that*

$$\omega_{H_1(G)}\Big(\frac{1}{2^{|m_k|}}, f\Big) = o\Big(\frac{1}{V(m_k)}\Big) \quad \text{as} \quad k \to \infty. \tag{2.51}$$

Then

$$\|S_{m_k} f - f\|_{H_1(G)} \to 0 \quad \text{as} \quad k \to \infty. \tag{2.52}$$

(b) *Let $\{m_k : k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers such that condition (2.27) is fulfilled. Then there exist a martingale $f \in H_1(G)$ and an increasing sequence of natural numbers $\{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\}$ such that*

$$\omega_{H_1(G)}\Big(\frac{1}{2^{|\alpha_k|}}, f\Big) = O\Big(\frac{1}{V(\alpha_k)}\Big) \quad \text{as} \quad k \to \infty$$

and

$$\limsup_{k\to\infty} \|S_{\alpha_k} f - f\|_1 > c > 0 \quad \text{as} \quad k \to \infty, \tag{2.53}$$

where c is an absolute constant.

Proof. Let $f \in H_1(G)$ and $\{m_k : k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers such that (2.51). Applying Theorem 2.3, we get that condition (2.52) is fulfilled.

Now, we prove the second part of Theorem 2.5. Due to (2.27), we conclude that there exists a sequence $\{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\}$ such that

$$V(\alpha_k) \uparrow \infty \ \text{ as } \ k \to \infty \ \text{ and } \ V^2(\alpha_k) \le V(\alpha_{k+1}), \ \ k \in \mathbb{N}_+. \tag{2.54}$$

Let $f = (f_n,\ n \in \mathbb{N}_+)$ be a martingale from Example 2.1, where

$$\lambda_k = \frac{1}{V(\alpha_k)}\,.$$

Applying (2.54), we conclude that (2.3) is fulfilled and thus $f = (f_n,\ n \in \mathbb{N}_+) \in H_1(G)$.

In view of (2.4), we have

$$\widehat{f}(j) = \begin{cases} \dfrac{1}{V(\alpha_k)}, & \text{if } j \in \big\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1} - 1\big\}, \\ 0, & \text{if } j \notin \bigcup\limits_{k=0}^{\infty} \big\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1} - 1\big\}, \end{cases} \qquad k = 0, 1, \dots . \tag{2.55}$$

According to (2.7), we get

$$w_{H_1(G)}\Big(\frac{1}{2^n}, f\Big) = \|f - S_{2^n} f\|_{H_1(G)} \le \sum_{i=n+1}^{\infty} \frac{1}{V(\alpha_i)} = O\Big(\frac{1}{V(\alpha_n)}\Big) \quad \text{as} \quad n \to \infty.$$

Applying (2.6), we can conclude that

$$S_{\alpha_k} f = S_{2^{|\alpha_k|}} f + \frac{w_{2^{|\alpha_k|}} D_{\alpha_k - 2^{|\alpha_k|}}}{V(\alpha_k)},$$

If we use (1.2) and Theorem T2, we get

$$\|f - S_{\alpha_k} f\|_1 \ge \Big\|\frac{w_{2^{|\alpha_k|}} D_{\alpha_k - 2^{|\alpha_k|}}}{V(\alpha_k)}\Big\|_1 - \|f - S_{2^{|\alpha_k|}} f\|_1 \ge \frac{V(\alpha_k - 2^{|\alpha_k|})}{8V(\alpha_k)} - o(1) > c > 0 \ \text{ as } \ k \to \infty.$$

The proof of Theorem 2.5 is complete. □

Theorem 3.4 implies the following corollaries from [68].

Corollary 2.6.

(a) *Let* $0 < p < 1$, $f \in H_p(G)$ *and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^k}, f\Big) = o\Big(\frac{1}{2^{k(1/p-1)}}\Big) \quad as \quad k \to \infty.$$

Then

$$\|S_k f - f\|_{H_p(G)} \to 0 \quad as \quad k \to \infty.$$

(b) *There exists a martingale* $f \in H_p(G)$ $(0 < p < 1)$ *such that*

$$\omega_{H_p(G)}\Big(\frac{1}{2^k}, f\Big) = O\Big(\frac{1}{2^{k(1/p-1)}}\Big) \quad as \quad k \to \infty$$

and

$$\|S_k f - f\|_{\text{weak-}L_p(G)} \nrightarrow 0 \quad as \quad k \to \infty.$$

Corollary 2.7.

(a) *Let* $f \in H_1(G)$ *and*

$$\omega_{H_1(G)}\Big(\frac{1}{2^k}, f\Big) = o\Big(\frac{1}{k}\Big) \quad as \quad k \to \infty.$$

Then

$$\|S_k f - f\|_{H_1(G)} \to 0 \quad as \quad k \to \infty.$$

(b) *There exists a martingale* $f \in H_1(G)$ *such that*

$$\omega_{H_1(G)}\Big(\frac{1}{2^k}, f\Big) = O\Big(\frac{1}{k}\Big) \quad as \quad k \to \infty$$

and

$$\|S_k f - f\|_1 \nrightarrow 0 \quad as \quad k \to \infty.$$

3 Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

3.1 Basic notations

For the one-dimensional case, the Fejér means with respect to the one-dimensional Walsh–Fourier series σ_n are defined by

$$\sigma_n f(x) := \frac{1}{n}\sum_{k=1}^{n} S_k f(x) \quad (n \in \mathbb{N}_+).$$

The following equality is true (for details see [2] and [54]):

$$\sigma_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1}(D_k * f)(x) = (f * K_n)(x) = \int_G f(t)K_n(x-t)\,d\mu(t),$$

where

$$K_n(x) := \frac{1}{n}\sum_{k=1}^{n} D_k(x) \quad (n \in \mathbb{N}_+).$$

In the literature K_n is called an n-th Fejér kernel.

We also define the following maximal operators:

$$\sigma^* f = \sup_{n\in\mathbb{N}} |\sigma_n f|,$$
$$\widetilde{\sigma}_{\#}^* f = \sup_{n\in\mathbb{N}} |\sigma_{2^n} f|.$$

For any natural number $n \in \mathbb{N}$, we need the following expression:

$$n = \sum_{i=1}^{s} 2^{n_i}, \quad n_1 < n_2 < \cdots < n_s.$$

Set

$$n^{(i)} := 2^{n_1} + \cdots + 2^{n_{i-1}}, \quad i = 2, \ldots, s,$$

and

$$\mathbb{A}_{0,2} := \Big\{ n \in \mathbb{N} : \ n = 2^0 + 2^2 + \sum_{i=3}^{s_n} 2^{n_i} \Big\}.$$

Then for any natural number $n \in \mathbb{N}$, there exist the numbers

$$0 \le l_1 \le m_1 \le l_2 - 2 < l_2 \le m_2 \le \cdots \le l_s - 2 < l_s \le m_s$$

such that it can be written as

$$n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k,$$

where s is depending on n.

It is evident that

$$s \le V(n) \le 2s + 1.$$

3.2 Auxiliary lemmas

The following equality and estimation of Fejér kernels with respect to the one-dimensional Walsh–Fourier series are proved in [54].

Lemma 3.1. *Let $n \in \mathbb{N}$ and $n = \sum_{i=1}^{s} 2^{n_i}$, $n_1 < n_2 < \cdots < n_s$. Then*

$$nK_n = \sum_{r=1}^{s} \Big(\prod_{j=r+1}^{s} w_{2^{n_j}} \Big) 2^{n_r} K_{2^{n_r}} + \sum_{t=2}^{s} \Big(\sum_{j=t+1}^{s} w_{2^{n_j}} \Big) n^{(t)} D_{2^{n_t}}$$

and

$$\sup_{n\in\mathbb{N}} \int_G |K_n(x)| \, d\mu(x) \le c < \infty,$$

where c is an absolute constant.

The following equality is proved in [54] (see also [21]).

Lemma 3.2. *Let $n > t$ and $t, n \in \mathbb{N}$. Then for the 2^n-th Fejér kernels with respect to the one-dimensional Walsh–Fourier series, we have the following expression:*

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, & \text{if } x \in I_n(e_t), \\ \dfrac{2^n + 1}{2}, & \text{if } x \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

The following estimation has been proved by Goginava [26].

Lemma 3.3. *Let $x \in I_{l+1}(e_k + e_l)$, $k = 0, \ldots, M-2$, $l = 0, \ldots, M-1$. Then*

$$\int\limits_{I_M} |K_n(x+t)|\, d\mu(t) \le \frac{c\, 2^{l+k}}{n\, 2^M}, \quad \text{where } n > 2^M.$$

Let $x \in I_M(e_k), m = 0, \ldots, M-1$. Then

$$\int\limits_{I_M} |K_n(x+t)|\, d\mu(t) \le \frac{c\, 2^k}{2^M} \quad \text{for } n > 2^M,$$

where c is an absolute constant.

The following estimations of Fejér kernels with respect to the one-dimensional Walsh–Fourier series are proved in [71]:

Lemma 3.4. *Let*

$$n = \sum_{i=1}^{r} \sum_{k=l_i}^{m_i} 2^k,$$

where

$$m_1 \ge l_1 > l_1 - 2 \ge m_2 \ge l_2 > l_2 - 2 > \cdots > m_s \ge l_s \ge 0.$$

Then

$$|nK_n| \le c \sum_{A=1}^{r} \Big(2^{l_A} |K_{2^{l_A}}| + 2^{m_A} |K_{2^{m_A}}| + 2^{l_A} \sum_{k=l_A}^{m_A} D_{2^k} \Big) + c\, V(n),$$

where c is an absolute constant.

Proof. Let

$$n = \sum_{i=1}^{r} 2^{n_i}, n_1 > n_2 > \cdots > n_r \ge 0.$$

Using Lemma 3.1 for the n-th Fejér kernels, we can conclude that

$$nK_n = \sum_{A=1}^{r} \Big(\prod_{j=1}^{A-1} w_{2^{n_j}} \Big) \big(2^{n_A} K_{2^{n_A}} + (2^{n_A} - 1) D_{2^{n_A}} \big) - \sum_{A=1}^{r} \Big(\sum_{j=1}^{A-1} w_{2^{n_j}} \Big) (2^{n_A} - 1 - n^{(A)}) D_{2^{n_A}} = I_1 - I_2.$$

For I_1, we have the following equality:

$$\begin{aligned} I_1 &= \sum_{v=1}^{r} \Big(\prod_{j=1}^{v-1} \prod_{i=l_j}^{m_j} w_{2^i} \Big) \Big(\sum_{k=l_v}^{m_v} \Big(\prod_{j=k+1}^{m_v} w_{2^j} \Big) (2^k K_{2^k} - (2^k - 1) D_{2^k}) \Big) \\ &= \sum_{v=1}^{r} \Big(\prod_{j=1}^{v-1} \prod_{i=l_j}^{m_j} w_{2^i} \Big) \Big(\sum_{k=0}^{m_v} - \sum_{k=0}^{l_v-1} \Big) \Big(\prod_{j=k+1}^{m_v} w_{2^j} \Big) (2^k K_{2^k} - (2^k - 1) D_{2^k}) \\ &= \sum_{v=1}^{r} \Big(\prod_{j=1}^{v-1} \prod_{i=l_j}^{m_j} w_{2^i} \Big) \Big(\sum_{k=0}^{m_v} \Big(\prod_{j=k+1}^{m_v} w_{2^j} \Big) (2^k K_{2^k} - (2^k - 1) D_{2^k}) \Big) \\ &\quad - \sum_{v=1}^{r} \Big(\prod_{j=1}^{v} \prod_{i=l_j}^{m_j} w_{2^i} \Big) \Big(\sum_{k=0}^{l_v-1} \Big(\prod_{j=k+1}^{l_v-1} w_{2^j} \Big) (2^k K_{2^k} - (2^k - 1) D_{2^k}) \Big). \end{aligned}$$

Since

$$2^n - 1 = \sum_{k=0}^{n-1} 2^k$$

and

$$(2^n - 1)K_{2^n-1} = \sum_{k=0}^{n-1} \Big(\prod_{j=k+1}^{n-1} w_{2^j} \Big) \big(2^k K_{2^k} - (2^k - 1) D_{2^k} \big),$$

we obtain

$$I_1 = \sum_{v=1}^{r} \Big(\prod_{j=1}^{v-1} \prod_{i=l_j}^{m_j} w_{2^i} \Big) (2^{m_v+1} - 1) K_{2^{m_v+1}-1} - \sum_{v=1}^{r} \Big(\prod_{j=1}^{v} \prod_{i=l_j}^{m_j} w_{2^i} \Big) (2^{l_v} - 1) K_{2^{l_v}-1}.$$

If we apply estimations

$$|K_{2^n}| \le c\,|K_{2^{n-1}}|$$

and

$$|K_{2^n-1}| \le c\,|K_{2^n}| + c,$$

we get

$$|I_1| \le c \sum_{v=1}^{r} \big(2^{l_v} |K_{2^{l_v}}| + 2^{m_v} |K_{2^{m_v}}| + cr \big). \tag{3.1}$$

Let $l_j < n_A \le m_j$, where $j = 1, \dots, s$. Then

$$n^{(A)} \ge \sum_{v=l_j}^{n_A-1} 2^v \ge 2^{n_A} - 2^{l_j}$$

and

$$2^{n_A} - 1 - n^{(A)} \le 2^{l_j}.$$

If $l_j = n_A$, where $j = 1, \dots, s$, then

$$n^{(A)} \le 2^{m_{j-1}+1} < 2^{l_j}.$$

Using these estimations, we can conclude that

$$|I_2| \le c \sum_{v=1}^{r} 2^{l_v} \sum_{k=l_v}^{m_v} D_{2^k}. \tag{3.2}$$

Combining (3.1), (3.2), we obtain the proof of Lemma 3.4. □

The following estimations of Fejér kernels with respect to the one-dimensional Walsh–Fourier series are proved in [71].

Lemma 3.5. *Let*

$$n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k,$$

where

$$0 \le l_1 \le m_1 \le l_2 - 2 < l_2 \le m_2 \le \dots \le l_s - 2 < l_s \le m_s.$$

Then

$$n|K_n(x)| \ge \frac{2^{2l_i}}{16} \quad \textit{for} \;\; x \in I_{l_i+1}(e_{l_i-1} + e_{l_i}).$$

Proof. If we apply Lemma 3.1 for $n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k$, we can write

$$nK_n = \sum_{r=1}^{s} \sum_{k=l_r}^{m_r} \Big(\prod_{j=r+1}^{s} \prod_{q=l_j}^{m_j} w_{2^q} \prod_{j=k+1}^{m_r} w_{2^j} \Big) 2^k K_{2^k} + \sum_{r=1}^{s} \sum_{k=l_r}^{m_r} \Big(\prod_{j=r+1}^{s} \prod_{q=l_j}^{m_j} w_{2^q} \prod_{j=k+1}^{m_r} w_{2^j} \Big) \Big(\sum_{t=1}^{r-1} \sum_{q=l_t}^{m_t} 2^q + \sum_{q=l_r}^{k-1} 2^q \Big) D_{2^k}.$$

Let $x \in I_{l_i+1}(e_{l_i-1} + e_{l_i})$. Then

$$n|K_n| \geq |2^{l_i} K_{2^{l_i}}| - \sum_{r=1}^{i-1} \sum_{k=l_r}^{m_r} |2^k K_{2^k}| - \sum_{r=1}^{i-1} \sum_{k=l_r}^{m_r} |2^k D_{2^k}| = I - II - III.$$

From Lemma 3.2 it follows that

$$I = |2^{l_i} K_{2^{l_i}}(x)| = \frac{2^{2l_i}}{4}\,. \tag{3.3}$$

Since $m_{i-1} \leq l_i - 2$, we easily obtain that the estimation

$$II \leq \sum_{n=0}^{l_i-2} |2^n K_{2^n}(x)| \leq \sum_{n=0}^{l_i-2} 2^n \frac{(2^n+1)}{2} \leq \frac{2^{2l_i}}{24} + \frac{2^{l_i}}{4} - \frac{2}{3} \tag{3.4}$$

is true.

For III, we get

$$III \leq \sum_{k=0}^{l_i-2} |2^k D_{2^k}(x)| \leq \sum_{k=0}^{l_i-2} 4^k = \frac{2^{2l_i}}{12} - \frac{1}{3}\,. \tag{3.5}$$

Combining (3.3)–(3.5), we can conclude that

$$n|K_n(x)| \geq I - II - III \geq \frac{2^{2l_i}}{8} - \frac{2^{l_i}}{4} + 1. \tag{3.6}$$

Suppose that $l_i \geq 2$. Then

$$n|K_n(x)| \geq \frac{2^{2l_i}}{8} - \frac{2^{2l_i}}{16} \geq \frac{2^{2l_i}}{16}\,.$$

If $l_i = 0$ or $l_i = 1$, then applying (3.6), we get

$$n|K_n(x)| \geq \frac{7}{8} \geq \frac{2^{2l_i}}{16}\,.$$

Lemma is proved. □

The following estimations of Fejér kernels with respect to the one-dimensional Walsh–Fourier series are proved in [71] (see also [82]).

Lemma 3.6. *Let $0 < p \leq 1$, $2^k \leq n < 2^{k+1}$ and $\sigma_n f$ be Fejér means with respect to the one-dimensional Walsh–Fourier series, where $f \in H_p(G)$. Then for any fixed $n \in \mathbb{N}$,*

$$\|\sigma_n f\|_{H_p(G)} \leq \Big\| \sup_{0\leq l\leq k} |\sigma_{2^l} f| \Big\|_p + \Big\| \sup_{0\leq l\leq k} |S_{2^l} f| \Big\|_p + \|\sigma_n f\|_p \leq \|\widetilde{\sigma}^*_{\#} f\|_p + \|\widetilde{S}^*_{\#} f\|_p + \|\sigma_n f\|_p.$$

Proof. Let us consider the martingale

$$\begin{aligned} f_{\#} &= (S_{2^k}\sigma_n f,\ k \in \mathbb{N}) \\ &= \Big(\frac{2^0 \sigma_{2^0}}{n} + \frac{(n-2^0) S_{2^0} f}{n}, \ldots, \frac{2^k \sigma_{2^k} f}{n} + \frac{(n-2^k) S_{2^k} f}{n}, \sigma_n f, \ldots, \sigma_n f, \ldots \Big). \end{aligned}$$

Using Lemma 2.6, we immediately get

$$\|\sigma_n f\|^p_{H_p(G^2)} \leq \Big\| \sup_{0\leq l\leq k} |\sigma_{2^l} f| \Big\|^p_p + \Big\| \sup_{0\leq l\leq k} |S_{2^l} f| \|^p_p + \Big\| S_n f \|^p_p \leq \|\widetilde{\sigma}^*_{\#} f\|^p_p + \|\widetilde{S}^*_{\#} f\|^p_p + \|\sigma_n f\|^p_p.$$

Thus the lemma is proved. □

3.3 Boundedness of subsequences of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

In this subsection, we study the boundedness of subsequences of Fejér means with respect to the one-dimensional Walsh–Fourier series in the martingale Hardy spaces (for details see [71]).

First, we consider the case $p = 1/2$. The following estimation is true.

Theorem 3.1.

(a) *Let $f \in H_{1/2}(G)$. Then there exists an absolute constant c such that*

$$\|\sigma_n f\|_{H_{1/2}(G)} \leq c\, V^2(n) \|f\|_{H_{1/2}(G)}.$$

(b) *Let $\{n_k : k \in \mathbb{N}_+\}$ be an increasing sequence of natural numbers such that* $\sup_{k\in\mathbb{N}_+} V(n_k) = \infty$ *and let $\Phi : \mathbb{N}_+ \to [1,\infty)$ be a non-decreasing function satisfying the conditions $\Phi(n) \uparrow \infty$ and*

$$\overline{\lim_{k\to\infty}} \frac{V^2(n_k)}{\Phi(n_k)} = \infty. \tag{3.7}$$

Then there exists a martingale $f \in H_{1/2}(G)$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{\sigma_{n_k} f}{\Phi(n_k)} \right\|_{1/2} = \infty.$$

Proof. Suppose that

$$\left\| \frac{\sigma_n f}{V^2(n)} \right\|_{1/2} \leq c\, \|f\|_{H_{1/2}(G)}. \tag{3.8}$$

Combining estimations (1.7), (1.15) and Lemma 3.6, we can conclude that

$$\begin{aligned}\left\| \frac{\sigma_n f}{V^2(n)} \right\|_{H_{1/2}(G)}^{1/2} &\leq \left\| \frac{\sigma_n f}{V^2(n)} \right\|_{1/2}^{1/2} + \frac{1}{V^2(n)} \|\sigma^*_{\#} f\|_{1/2}^{1/2} + \frac{1}{V^2(n)} \|\widetilde{S}^*_{\#}\|_{1/2}^{1/2} \\ &\leq \left\| \frac{\widetilde{\sigma}_n f}{V^2(n)} \right\|_{1/2}^{1/2} + \|\widetilde{\sigma}^*_{\#} f\|_{1/2}^{1/2} + \|\widetilde{S}^*_{\#} f\|_{1/2}^{1/2} \leq c\, \|f\|_{H_{1/2}(G)}^{1/2}.\end{aligned} \tag{3.9}$$

Combining Lemma 2.5 and (3.9), Theorem 3.1 will be proved if we show that

$$\int_{\overline{I}_M} \left(\frac{|\sigma_n a|}{V^2(n)} \right)^{1/2} d\mu \leq c < \infty$$

for any 1/2-atom a.

Without loss of generality, we may assume that a is 1/2-atom, with support I, for which $\mu(I) = 2^{-M}$, $I = I_M$. It is easy to check that $\sigma_n(a) = 0$, when $n \leq 2^M$. Therefore, we may assume that $n > 2^M$. Set

$$II^1_{\alpha_A}(x) := 2^M \int_{I_M} 2^{\alpha_A} |K_{2^{\alpha_A}}(x+t)|\, d\mu(t),$$

$$II^2_{l_A}(x) = 2^M \int_{I_M} 2^{l_A} \sum_{k=l_A}^{m_A} D_{2^k}(x+t)\, d\mu(t).$$

Let $x \in I_M$. Since σ_n is bounded from $L_\infty(G)$ to $L_\infty(G)$, for $n > 2^M$ and $\|a\|_\infty \leq 2^{2M}$, using Lemma 3.3, we can conclude that

$$\frac{|\sigma_n a(x)|}{V^2(n)} \le \frac{c}{V^2(n)} \int\limits_{I_M} |a(x)|\, |K_n(x+t)|\, d\mu(t)$$
$$\le \frac{c\,\|a\|_\infty}{V^2(n)} \int\limits_{I_M} |K_n(x+t)|\, d\mu(t) \le \frac{c\,2^{2M}}{V^2(n)} \int\limits_{I_M} |K_n(x+t)|\, d\mu(t)$$
$$\le \frac{c\,2^M}{V^2(n)} \Big\{ \sum_{A=1}^{s} \int\limits_{I_M} 2^{l_A} |K_{2^{l_A}}(x+t)|\, d\mu(t) + \int\limits_{I_M} 2^{m_A} |K_{2^{m_A}}(x+t)|\, d\mu(t) \Big\}$$
$$+ \frac{c\,2^M}{V^2(n)} \sum_{A=1}^{s} \int\limits_{I_M} 2^{l_A} \sum_{k=l_A}^{m_A} D_{2^k}(x+t)\, d\mu(t) + \frac{c\,2^M}{V^2(n)} \int\limits_{I_M} V(n)\, d\mu(t)$$
$$= \frac{c}{V^2(n)} \sum_{A=1}^{s} \big(II^1_{l_A}(x) + II^1_{m_A}(x) + II^2_{l_A}(x) \big) + c.$$

Hence

$$\int\limits_{\overline{I}_M} \Big| \frac{\sigma_n a()}{V^2(n)} \Big|^{1/2} d\mu(x)$$
$$\le \frac{c}{V(n)} \Bigg(\sum_{A=1}^{s} \int\limits_{\overline{I}_M} |II^1_{l_A}(x)|^{1/2}\, d\mu(x) + \int\limits_{\overline{I}_M} |II^1_{m_A}(x)|^{1/2}\, d\mu(x) + \int\limits_{\overline{I}_M} |II^2_{l_A}(x)|^{1/2}\, d\mu(x) \Bigg) + c.$$

Since $s \le 4V(n)$, we obtain that Theorem 3.1 will be proved if we show that

$$\int\limits_{\overline{I}_M} |II^1_{\alpha_A}(x)|^{1/2}\, d\mu(x) \le c < \infty, \quad \int\limits_{\overline{I}_M} |II^2_{l_A}(x)|^{1/2}\, d\mu(x) \le c < \infty, \tag{3.10}$$

where $\alpha_A = l_A$ or $\alpha_A = m_A$, $A = 1, \ldots, s$.

Let $t \in I_M$ and $x \in I_{l+1}(e_k + e_l)$, $0 \le k < l < \alpha_A \le M$ or $0 \le k < l \le M \le \alpha_A$. Since $x + t \in I_{l+1}(e_k + e_l)$, applying Lemma 3.2, we can conclude that

$$K_{2^{\alpha_A}}(x+t) = 0 \ \text{ and } \ II^1_{\alpha_A}(x) = 0. \tag{3.11}$$

Let $x \in I_{l+1}(e_k + e_l)$, $0 \le k < \alpha_A \le l \le M$. Then $x + t \in I_{l+1}(e_k + e_l)$, where $t \in I_M$, and if we apply again Lemma 3.2, we get

$$2^{\alpha_A} |K_{2^{\alpha_A}}(x+t)| \le 2^{\alpha_A + k} \ \text{ and } \ II^1_{\alpha_A}(x) \le 2^{\alpha_A + k}. \tag{3.12}$$

Analogously to (3.12), for $0 \le \alpha_A \le k < l \le M$, we can prove that

$$2^{\alpha_A} |K_{2^{\alpha_A}}(x+t)| \le 2^{2\alpha_A}, \quad II^1_{\alpha_A}(x) \le 2^{2\alpha_A}, \ \ t \in I_M, \ \ x \in I_{l+1}(e_k + e_l). \tag{3.13}$$

Let $0 \le \alpha_A \le M - 1$, where $A = 1, \ldots, s$. According to (2.1) and (3.11)–(3.13), we find that

$$\int\limits_{\overline{I}_M} |II^1_{\alpha_A}(x)|^{1/2}\, d\mu(x)$$
$$= \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \int\limits_{I_{l+1}(e_k+e_l)} |II^1_{\alpha_A}(x)|^{1/2}\, d\mu(x) + \sum_{k=0}^{M-1} \int\limits_{I_M(e_k)} |II^1_{\alpha_A}(x)|^{1/2}\, d\mu(x)$$
$$\le c \sum_{k=0}^{\alpha_A - 1} \sum_{l=\alpha_A+1}^{M-1} \int\limits_{I_{l+1}(e_k+e_l)} 2^{(\alpha_A+k)/2}\, d\mu(x) + c \sum_{k=\alpha_A}^{M-2} \sum_{l=k+1}^{M-1} \int\limits_{I_{l+1}(e_k+e_l)} 2^{\alpha_A}\, d\mu(x)$$

$$+c\sum_{k=0}^{\alpha_A-1}\int\limits_{I_M(e_k)}2^{(\alpha_A+k)/2}\,d\mu(x)+c\sum_{k=\alpha_A}^{M-1}\int\limits_{I_M(e_k)}2^{\alpha_A}\,d\mu(x)$$

$$\leq c\sum_{k=0}^{\alpha_A-1}\sum_{l=\alpha_A+1}^{M-1}\frac{2^{(\alpha_A+k)/2}}{2^l}+c\sum_{k=\alpha_A}^{M-2}\sum_{l=k+1}^{M-1}\frac{2^{\alpha_A}}{2^l}+c\sum_{k=0}^{\alpha_A-1}\frac{2^{(\alpha_A+k)/2}}{2^M}+c\sum_{k=\alpha_A}^{M-1}\frac{2^{\alpha_A}}{2^M}\leq c<\infty.$$

Let $\alpha_A\geq M$. Analogously to $II^1_{\alpha_A}(x)$, we can prove (3.10) for $A=1,\dots,s$.

Now, we prove the boundedness of $II^2_{l_A}$. Let $t\in I_M$ and $x\in I_i\setminus I_{i+1}$, $i\leq l_A-1$. Since $x+t\in I_i\setminus I_{i+1}$, if we apply the first equality of Lemma 2.2, we get

$$II^2_{l_A}(x)=0. \tag{3.14}$$

Let $x\in I_i\setminus I_{i+1}$, $l_A\leq i\leq m_A$. Since $n\geq 2^M$ and $t\in I_M$, if we apply the first equality of Lemma 2.2, we get

$$II^2_{l_A}(x)\leq 2^M\int\limits_{I_M}2^{l_A}\sum_{k=l_A}^{i}D_{2^k}(x+t)\,d\mu(t)\leq c\,2^{l_A+i}. \tag{3.15}$$

Let $x\in I_i\setminus I_{i+1}$, $m_A<i\leq M-1$. Then $x+t\in I_i\setminus I_{i+1}$ for any $t\in I_M$, and by the first equality of Lemma 2.2, we have

$$II^2_{l_A}(x)\leq c\,2^M\int\limits_{I_M}2^{l_A+m_A}\leq c\,2^{l_A+m_A}. \tag{3.16}$$

Let $0\leq l_A\leq m_A\leq M$. Then, in view of (2.1) and (3.14)–(3.16)) we can conclude that

$$\int\limits_{\overline{I}_M}|II^2_{l_A}(x)|^{1/2}\,d\mu(x)=\Big(\sum_{i=0}^{l_A-1}+\sum_{i=l_A}^{m_A}+\sum_{i=m_A+1}^{M-1}\Big)\int\limits_{I_i\setminus I_{i+1}}|II^2_{l_A}(x)|^{1/2}\,d\mu(x)$$

$$\leq c\sum_{i=l_A}^{m_A}\int\limits_{I_i\setminus I_{i+1}}2^{(l_A+i)/2}\,d\mu(x)+c\sum_{i=m_A+1}^{M-1}\int\limits_{I_i\setminus I_{i+1}}2^{(l_A+m_A)/2}\,d\mu(x)$$

$$\leq c\sum_{i=l_A}^{m_A}2^{(l_A+i)/2}\,\frac{1}{2^i}+c\sum_{i=m_A+1}^{M-1}2^{(l_A+m_A)/2}\,\frac{1}{2^i}\leq c<\infty.$$

Analogously, we can prove same estimations for the cases $0\leq l_A\leq M<m_A$ and $M\leq l_A\leq m_A$.

Now, we prove part (b) of Theorem 3.1. According to (3.7), there exists an increasing sequence $\{\alpha_k:\ k\in\mathbb{N}_+\}\subset\{n_k:\ k\in\mathbb{N}_+\}$ of natural numbers such that

$$\sum_{k=1}^{\infty}\frac{\Phi^{1/4}(\alpha_k)}{V^{1/2}(\alpha_k)}\leq c<\infty. \tag{3.17}$$

Let $f=(f_n,\ n\in\mathbb{N}_+)$ be a martingale from Example 2.1, where

$$\lambda_k:=\frac{\Phi^{1/2}(\alpha_k)}{V(\alpha_k)}\,.$$

According to (3.17), we get that condition (2.3) is fulfilled and it follows that $f=(f_n,\ n\in\mathbb{N}_+)$. Applying (2.4), we get

$$\widehat{f}(j)=\begin{cases}\dfrac{2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{V(\alpha_k)}, & j\in\big\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\big\},\\ 0, & j\notin\bigcup\limits_{k=0}^{\infty}\big\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\big\},\end{cases}\qquad k\in\mathbb{N}_+ \tag{3.18}$$

Let $2^{|\alpha_k|} < j < \alpha_k$. If we apply (2.6), we get

$$S_j f = S_{2^{|\alpha_k|}} f + \frac{w_{2^{|\alpha_k|}} D_{j-2^{|\alpha_k|}} \Phi^{1/2}(\alpha_k)}{V(\alpha_k)}\,. \tag{3.19}$$

Hence

$$\begin{aligned}\frac{\sigma_{\alpha_k} f}{\Phi(\alpha_k)} &= \frac{1}{\Phi(\alpha_k)\alpha_k}\sum_{j=1}^{2^{|\alpha_k|}} S_j f + \frac{1}{\Phi(\alpha_k)\alpha_k}\sum_{j=2^{|\alpha_k|}+1}^{\alpha_k} S_j f \\ &= \frac{\sigma_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)\alpha_k} + \frac{(\alpha_k - 2^{|\alpha_k|}) S_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)\alpha_k} + \frac{w_{2^{|\alpha_k|}} 2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V(\alpha_k)\alpha_k}\sum_{j=2^{|\alpha_k|}+1}^{\alpha_k} D_{j-2^{|\alpha_k|}} \\ &= III_1 + III_2 + III_3. \end{aligned} \tag{3.20}$$

For III_3, we can conclude that

$$\begin{aligned}|III_3| &= \frac{2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V(\alpha_k)\alpha_k}\Big|\sum_{j=1}^{\alpha_k - 2^{|\alpha_k|}} D_j\Big| \\ &= \frac{2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V(\alpha_k)\alpha_k}\,(\alpha_k - 2^{|\alpha_k|})|K_{\alpha_k - 2^{|\alpha_k|}}| \geq \frac{c(\alpha_k - 2^{|\alpha_k|})|K_{\alpha_k - 2^{|\alpha_k|}}|}{\Phi^{1/2}(\alpha_k)V(\alpha_k)}\,. \end{aligned} \tag{3.21}$$

Let

$$\alpha_k = \sum_{i=1}^{r_k}\sum_{k=l_i^k}^{m_i^k} 2^k,$$

where

$$m_1^k \geq l_1^k > l_1^k - 2 \geq m_2^k \geq l_2^k > l_2^k - 2 \geq \cdots \geq m_s^k \geq l_s^k \geq 0.$$

Since (see Theorems 2.1 and 3.1)

$$\|III_1\|_{1/2} \leq c, \|III_2\|_{1/2} \leq c$$

and

$$\mu\{E_{l_i^k}\} \geq \frac{1}{2^{l_i^k - 1}}\,,$$

combining (3.20), (3.21) and Lemma 3.5, we get

$$\begin{aligned}\int_G \Big|\frac{\sigma_{\alpha_k} f(x)}{\Phi(\alpha_k)}\Big|^{1/2} d\mu(x) &\geq \|III_3\|_{1/2}^{1/2} - \|III_2\|_{1/2}^{1/2} - \|III_1\|_{1/2}^{1/2} \\ &\geq c\sum_{i=2}^{r_k - 2}\int_{E_{l_i^k}} \Big|\frac{2^{2l_i^k}}{\Phi^{1/2}(\alpha_k)V(\alpha_k)}\Big|^{1/2} d\mu(x) - 2c \geq c\sum_{i=2}^{r_k-2}\frac{1}{V^{1/2}(\alpha_k)\Phi^{1/4}(\alpha_k)} - 2c \\ &\geq \frac{c\, r_k}{V^{1/2}(\alpha_k)\Phi^{1/4}(\alpha_k)} \geq \frac{c\, V^{1/2}(\alpha_k)}{\Phi^{1/4}(\alpha_k)} \to \infty \ \text{ as } \ k \to \infty. \end{aligned}$$

Thus Theorem 3.1 is proved. □

Theorem 3.2.

(a) *Let $0 < p < 1/2$, $f \in H_p(G)$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_n f\|_{H_p(G)} \leq c_p 2^{d(n)(1/p-2)}\|f\|_{H_p(G)}.$$

(b) *Let* $0<p<1/2$ *and* $\Phi(n):\mathbb{N}_+\to[1,\infty)$ *be a non-decreasing function such that*

$$\sup_{k\in\mathbb{N}_+} d(n_k)=\infty, \quad \overline{\lim_{k\to\infty}} \frac{2^n d(n_k)(1/p-2)}{\Phi(n_k)}=\infty. \tag{3.22}$$

Then there exists a martingale $f\in H_p(G)$ *such that*

$$\sup_{k\in\mathbb{N}_+} \left\| \frac{\sigma_{n_k} f}{\Phi(n_k)} \right\|_{\text{weak-}L_p(G)}=\infty.$$

Proof. Let $n\in\mathbb{N}$. Analogously to (3.9), it is sufficient to prove that

$$\int_{\overline{I}_M} \left(2^{d(n)(2-1/p)}|\sigma_n(a)|\right)^p d\mu\le c_p<\infty$$

for every p-atom a, where I denotes the support of the atom.

Analogously to Theorem 3.1, we may assume that a is p-atom with the support $I=I_M$, $\mu(I_M)=2^{-M}$ and $n>2^M$. Since $\|a\|_\infty\le 2^{M/p}$, we can conclude that

$$2^{d(n)(2-1/p)}|\sigma_n a|\le 2^{d(n)(2-1/p)}\|a\|_\infty \int_{I_M}|K_n(x+t)|\,d\mu(t)\le 2^{d(n)(2-1/p)}2^{M/p}\int_{I_M}|K_n(x+t)|\,d\mu(t).$$

Let $x\in I_{l+1}(e_k+e_l)$, $0\le k,l\le[n]\le M$. Then, applying Lemma 3.2, we get $K_n(x+t)=0$, where $t\in I_M$ and hence

$$2^{d(n)(2-1/p)}|\sigma_n a|=0. \tag{3.23}$$

Let $x\in I_{l+1}(e_k+e_l)$, $[n]\le k,l\le M$ or $k\le[n]\le l\le M$. Then Lemma 3.4 results in

$$2^{d(n)(2-1/p)}|\sigma_n a|\le 2^{d(n)(2-1/p)}\,2^{M(1/p-2)+k+l}\le c_p\,2^{[n](1/p-2)+k+l}. \tag{3.24}$$

Combining (2.1), (3.23) and (3.24), we can conclude that

$$\begin{aligned}
&\int_{\overline{I}_M}\left|2^{d(n)(2-1/p)}\sigma_n a(x)\right|^p d\mu(x)\\
&\le\left(\sum_{k=0}^{[n]-2}\sum_{l=k+1}^{[n]-1}+\sum_{k=0}^{[n]-1}\sum_{l=[n]}^{M-1}+\sum_{k=[n]}^{M-2}\sum_{l=k+1}^{M-1}\right)\int_{I_{l+1}(e_k+e_l)}\left|2^{d(n)(2-1/p)}\sigma_n a(x)\right|^p d\mu(x)\\
&\quad+\sum_{k=0}^{M-1}\int_{I_M(e_k)}\left|2^{d(n)(2-1/p)}\sigma_n a(x)\right|^p d\mu(x)\\
&\le c_p\sum_{k=[n]}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}\,2^{[n](2p-1)}\,2^{p(k+l)}+c_p\sum_{k=0}^{[n]}\sum_{l=[n]+1}^{M-1}\frac{1}{2^l}\,2^{[n](2p-1)}2^{p(k+l)}\\
&\quad+\frac{c_p\,2^{[n](2p-1)}}{2^M}\sum_{k=0}^{[n]}2^{p(k+M)}<c_p<\infty.
\end{aligned}$$

Now, we prove part b) of Theorem 3.2. According to (3.22), there exists an increasing sequence of natural numbers $\{\alpha_k:\ k\in\mathbb{N}_+\}\subset\{n_k:\ k\in\mathbb{N}_+\}$ such that $\alpha_0\ge 3$ and

$$\sum_{\eta=0}^{\infty}u^{-p}(\alpha_\eta)<c_p<\infty, \quad u(\alpha_k)=\frac{2^{d(\alpha_k)(1/p-2)/2}}{\Phi^{1/2}(\alpha_k)}\,. \tag{3.25}$$

Let f be a martingale from Example 2.1, where

$$\lambda_k=u^{-1}(\alpha_k).$$

If we apply (3.25), we get that (2.3) is fulfilled and it follows that $f \in H_p(G)$. According to (2.4), we have

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{|\alpha_k|(1/p-1)}}{u(\alpha_k)}, & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \\ 0, & j \notin \bigcup\limits_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \end{cases} \qquad k \in \mathbb{N}_+. \tag{3.26}$$

Let $2^{|\alpha_k|} < j < \alpha_k$. Then, analogously to (3.19) and (3.20), if we apply (3.26), we get

$$\frac{\sigma_{\alpha_k} f}{\Phi(\alpha_k)} = \frac{\sigma_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)\alpha_k} + \frac{(\alpha_k - 2^{|\alpha_k|}) S_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)\alpha_k} + \frac{2^{|\alpha_k|(1/p-1)}}{\Phi(\alpha_k) u(\alpha_k)\alpha_k} \sum_{j=2^{|\alpha_k|}}^{\alpha_k - 1} (D_j - D_{2^{|\alpha_k|}}) = IV_1 + IV_2 + IV_3.$$

Let $\alpha_k \in \mathbb{N}$ and $E_{[\alpha_k]} := I_{[\alpha_k]+1}(e_{[\alpha_k]-1} + e_{[\alpha_k]})$. Since $[\alpha_k - 2^{|\alpha_k|}] = [\alpha_k]$, analogously to (3.21), if we apply Lemma 3.5, for IV_3 we have the following estimation:

$$\begin{aligned} |IV_3| &= \frac{2^{|\alpha_k|(1/p-1)}}{\Phi(\alpha_k) u(\alpha_k)\alpha_k} (\alpha_k - 2^{|\alpha_k|}) |K_{\alpha_k - 2^{|\alpha_k|}}| \\ &= \frac{2^{|\alpha_k|(1/p-1)}}{\Phi(\alpha_k) u(\alpha_k)\alpha_k} |2^{[\alpha_k]} K_{[\alpha_k]}| \geq \frac{2^{|\alpha_k|(1/p-2)}\, 2^{2[\alpha_k]-4}}{\Phi(\alpha_k) u(\alpha_k)} \geq \frac{2^{|\alpha_k|(1/p-2)/2}\, 2^{2[\alpha_k]-4}}{\Phi^{1/2}(\alpha_k)}. \end{aligned}$$

Hence

$$\begin{aligned} \|IV_3\|^p_{\text{weak-}L_p(G)} &\geq \Big(\frac{2^{|\alpha_k|(1/p-2)/2}\, 2^{2[\alpha_k]-4}}{\Phi^{1/2}(\alpha_k)} \Big)^p \mu\Big\{ x \in G : \ |IV_3| \geq \frac{2^{|\alpha_k|(1/p-2)/2}\, 2^{2[\alpha_k]-4}}{\Phi^{1/2}(\alpha_k)} \Big\} \\ &\geq c_p \Big(\frac{2^{2[\alpha_k]+|\alpha_k|(1/p-2)/2}}{\Phi^{1/2}(\alpha_k)} \Big)^p \mu(E_{[\alpha_k]}) \geq c_p \Big(\frac{2^{(|\alpha_k|-[\alpha_k])(1/p-2)}}{\Phi(\alpha_k)} \Big)^{p/2} \\ &= c_p \Big(\frac{2^{d(\alpha_k)(1/p-2)}}{\Phi(\alpha_k)} \Big)^{p/2} \to \infty \ \text{ as } \ k \to \infty. \end{aligned}$$

Combining Corollary 2.2 and the first part of Theorem 3.2, we find that

$$\|IV_1\|_{\text{weak-}L_p(G)} \leq c_p < \infty, \quad \|IV_2\|_{\text{weak-}L_p(G)} \leq c_p < \infty.$$

On the other hand, for sufficiently large n, we can conclude that

$$\begin{aligned} \|\sigma_{\alpha_k} f\|^p_{\text{weak-}L_p(G)} &\geq \|IV_3\|^p_{\text{weak-}L_p(G)} - \|IV_2\|^p_{\text{weak-}L_p(G)} - \|IV_1\|^p_{\text{weak-}L_p(G)} \\ &\geq \frac{1}{2} \|IV_3\|^p_{\text{weak-}L_p(G)} \to \infty \ \text{ as } \ k \to \infty. \end{aligned}$$

Theorem 3.2 is proved. □

The proofs of Corollaries 3.1-3.3 are similar to those of Corollaries 2.2–2.4. So, we leave out the details and just present these results.

Corollary 3.1. *Let $p > 0$ and $f \in H_p(G)$. Then*

$$\|\sigma_{2^k} f - f\|_{H_p(G)} \to 0 \ \text{ as } \ k \to \infty.$$

Corollary 3.2. *Let $p > 0$ and $f \in H_p(G)$. Then*

$$\|\sigma_{2^k + 2^{k-1}} f - f\|_{H_p(G)} \to 0 \ \text{ as } \ k \to \infty.$$

Corollary 3.3. *Let $0 < p < 1/2$. Then there exists a martingale $f \in H_p(G)$ such that*

$$\|\sigma_{2^k+1} f - f\|_{\text{weak-}L_p(G)} \nrightarrow 0 \ \text{ as } \ k \to \infty.$$

On the other hand, for any $f \in H_{1/2}(G)$, the following is true:

$$\|\sigma_{2^k+1} f - f\|_{H_{1/2}(G)} \to 0 \ \text{ as } \ k \to \infty.$$

3.4 Modulus of continuity and convergence in norm of subsequences of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

In this section, we apply Theorems 3.1 and 3.2 to find the necessary and sufficient conditions for the modulus of continuity of a martingale $f \in H_p$, for which subsequences of Fejér means with respect to the one-dimensional Walsh–Fourier series converge in H_p-norm.

First, we prove

Theorem 3.3.

(a) *Let* $f \in H_{1/2}(G)$, $\sup_{k\in\mathbb{N}_+} V(n_k) = \infty$ *and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|n_k|}}, f\Big) = o\Big(\frac{1}{V^2(n_k)}\Big) \ \ as \ \ k \to \infty. \tag{3.27}$$

Then

$$\|\sigma_{n_k} f - f\|_{H_{1/2}(G)} \to 0 \ \ as \ \ k \to \infty.$$

(b) *Let* $\sup_{k\in\mathbb{N}_+} V(n_k) = \infty$. *Then there exists a martingale* $f \in H_{1/2}(G)$ *such that*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^{|n_k|}}, f\Big) = O\Big(\frac{1}{V^2(n_k)}\Big) \ \ as \ \ k \to \infty \tag{3.28}$$

and

$$\|\sigma_{n_k} f - f\|_{H_{1/2}(G)} \nrightarrow 0 \ \ as \ \ k \to \infty. \tag{3.29}$$

Proof. Let $f \in H_{1/2}(G)$ and $2^k < n \le 2^{k+1}$. Then

$$\begin{aligned}\|\sigma_n f - f\|_{H_{1/2}(G)}^{1/2} &\le \big\|\sigma_n f - \sigma_n S_{2^k} f\big\|_{H_{1/2}(G)}^{1/2} + \big\|\sigma_n S_{2^k} f - S_{2^k} f\big\|_{H_{1/2}(G)}^{1/2} + \|S_{2^k} f - f\|_{H_{1/2}(G)}^{1/2}\\ &= \big\|\sigma_n (S_{2^k} f - f)\big\|_{H_{1/2}(G)}^{1/2} + \|S_{2^k} f - f\|_{H_{1/2}(G)}^{1/2} + \big\|\sigma_n S_{2^k} f - S_{2^k} f\big\|_{H_{1/2}(G)}^{1/2}\\ &\le c\,(V(n)+1)\omega_{H_{1/2}(G)}^{1/2}\Big(\frac{1}{2^k}, f\Big) + \big\|\sigma_n S_{2^k} f - S_{2^k} f\big\|_{H_{1/2}(G)}^{1/2}.\end{aligned}$$

It is evident that

$$\sigma_n S_{2^k} f - S_{2^k} f = \frac{2^k}{n}\,(S_{2^k}\sigma_{2^k} f - S_{2^k} f) = \frac{2^k}{n}\, S_{2^k}(\sigma_{2^k} f - f).$$

Let $p > 0$. Combining Corollaries 2.2 and 3.1, we can conclude that

$$\big\|\sigma_n S_{2^k} f - S_{2^k} f\big\|_{H_{1/2}(G)}^{1/2} \le \frac{2^{k/2}}{n^{1/2}}\,\big\|S_{2^k}(\sigma_{2^k} f - f)\big\|_{H_{1/2}(G)}^{1/2} \le \|\sigma_{2^k} f - f\|_{H_{1/2}(G)}^{1/2} \to 0 \ \text{ as } \ k \to \infty.$$

Now, we prove part b) of Theorem 3.3. Since $\sup_{k\in\mathbb{N}_+} V(\alpha_k) = \infty$, there exists a martingale $\{\alpha_k : k \in \mathbb{N}_+\} \subset \{n_k : k \in \mathbb{N}_+\}$ such that $V(\alpha_k) \uparrow \infty$ as $k \to \infty$ and

$$V^2(\alpha_k) \le V(\alpha_{k+1}). \tag{3.30}$$

Let f be a martingale from Example 2.1, where

$$\lambda_k = V^{-2}(\alpha_k).$$

If we apply (3.30), we get that condition (2.3) is fulfilled and it follows that $f \in H_p(G)$. Using (2.4), we find that

$$\widehat{f}(j) = \begin{cases} \dfrac{2^{|\alpha_k|}}{V^2(\alpha_k)}, & j \in \big\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1} - 1\big\},\\ 0, & j \notin \bigcup\limits_{k=0}^{\infty} \big\{2^{|\alpha_k|}, \dots, 2^{|\alpha_k|+1} - 1\big\}, \end{cases} \qquad k \in \mathbb{N}_+. \tag{3.31}$$

Combining (2.7) and (3.30), we can conclude that

$$w_{H_{1/2}(G)}\Big(\frac{1}{2^n},f\Big)=\|f-S_{2^n}f\|_{H_{1/2}(G)}\le\sum_{i=n+1}^{\infty}\frac{1}{V^2(\alpha_i)}=O\Big(\frac{1}{V^2(\alpha_n)}\Big)\ \text{ as }\ n\to\infty. \tag{3.32}$$

Let $2^{|\alpha_k|}<j\le\alpha_k$. Using (2.6), we get

$$S_jf=S_{2^{|\alpha_k|}}f+\frac{2^{|\alpha_k|}w_{2^{|\alpha_k|}}D_{j-2^{|\alpha_k|}}}{V^2(\alpha_k)}\,.$$

Hence

$$\sigma_{\alpha_k}f-f=\frac{2^{|\alpha_k|}}{\alpha_k}\,(\sigma_{2^{|\alpha_k|}}f-f)+\frac{\alpha_k-2^{|\alpha_k|}}{\alpha_k}\,(S_{2^{|\alpha_k|}}f-f)+\frac{2^{|\alpha_k|}w_{2^{|\alpha_k|}}(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}}{\alpha_kV^2(\alpha_k)}\,. \tag{3.33}$$

According to (1.2), (1.12) and (3.33), we have

$$\begin{aligned}\|\sigma_{\alpha_k}f-f\|_{1/2}^{1/2}\ge{}&\frac{c}{V(\alpha_k)}\,\big\|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}\big\|_{1/2}^{1/2}\\&-\Big(\frac{2^{|\alpha_k|}}{\alpha_k}\Big)^{1/2}\big\|\sigma_{2^{|\alpha_k|}}f-f\big\|_{1/2}^{1/2}-\Big(\frac{\alpha_k-2^{|\alpha_k|}}{\alpha_k}\Big)^{1/2}\big\|S_{2^{|\alpha_k|}}f-f\big\|_{1/2}^{1/2}.\end{aligned} \tag{3.34}$$

Let

$$\alpha_k=\sum_{i=1}^{r_k}\sum_{k=l_i^k}^{m_i^k}2^k,$$

where

$$m_1^k\ge l_1^k>l_1^k-2\ge m_2^k\ge l_2^k>l_2^k-2>\cdots>m_s^k\ge l_s^k\ge 0$$

and

$$E_{l_i^k}:=I_{l_i^k+1}(e_{l_i^k-1}+e_{l_i^k}).$$

By Lemma 3.5, we get

$$\begin{aligned}&\int_G\big|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}(x)\big|^{1/2}\,d\mu\\&\quad\ge\frac{1}{16}\sum_{i=2}^{r_k-2}\int_{E_{l_i^k}}\big|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}(x)\big|^{1/2}\,d\mu(x)\ge\frac{1}{16}\sum_{i=2}^{r_k-2}\frac{1}{2^{l_i^k}}\,2^{l_i^k}\ge c\,r_k\ge c\,V(\alpha_k).\end{aligned} \tag{3.35}$$

Combining estimations (3.34), (3.35), Corollaries 2.2 and 3.1, we find that (3.29) holds true and Theorem 3.3 is proved. □

Theorem 3.4.

(a) *Let* $0<p<1/2$, $f\in H_p(G)$, $\sup_{k\in\mathbb{N}_+}d(n_k)=\infty$ *and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|n_k|}},f\Big)=o\Big(\frac{1}{2^{d(n_k)(1/p-2)}}\Big)\ \textit{ as }\ k\to\infty. \tag{3.36}$$

Then

$$\|\sigma_{n_k}f-f\|_{H_p(G)}\to 0\ \textit{ as }\ k\to\infty. \tag{3.37}$$

(b) *Let* $\sup_{k\in\mathbb{N}_+}d(n_k)=\infty$. *Then there exists a martingale* $f\in H_p(G)$ $(0<p<1/2)$ *such that*

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|n_k|}},f\Big)=O\Big(\frac{1}{2^{d(n_k)(1/p-2)}}\Big)\ \textit{ as }\ k\to\infty \tag{3.38}$$

and

$$\|\sigma_{n_k}f-f\|_{\text{weak-}L_p(G)}\nrightarrow 0\ \textit{ as }\ k\to\infty. \tag{3.39}$$

Proof. Let $0<p<1/2$. Then under condition (3.36), if we repeat the steps of the proof of Theorem 3.3, we immediately get that (3.37) holds.

Let us prove part b) of Theorem 3.4. Since $\sup_k d(n_k)=\infty$, there exists $\{\alpha_k:\ k\in\mathbb{N}_+\}\subset\{n_k:\ k\in\mathbb{N}_+\}$ such that $\sup_{k\in\mathbb{N}_+} d(\alpha_k)=\infty$ and

$$2^{2d(\alpha_k)(1/p-2)}\le 2^{d(\alpha_{k+1})(1/p-2)}. \tag{3.40}$$

Let f be a martingale from Lemma 2.1, where

$$\lambda_k=2^{-(1/p-2)d(\alpha_i)}.$$

If we use (3.40), we can conclude that condition (2.3) is fulfilled and it follows that $f\in H_p(G)$. According to (2.4), we get

$$\widehat{f}(j)=\begin{cases}2^{(1/p-2)[\alpha_k]}, & j\in\{2^{|\alpha_k|},\dots,2^{|\alpha_k|+1}-1\},\\ 0, & j\notin\bigcup\limits_{n=0}^{\infty}\{2^{|\alpha_n|},\dots,2^{|\alpha_n|+1}-1\},\end{cases}\qquad k\in\mathbb{N}_+. \tag{3.41}$$

Combining (2.7) and (3.40), we have

$$\omega_{H_p(G)}\Big(\frac{1}{2^{|\alpha_k|}},f\Big)\le\sum_{i=k}^{\infty}\frac{1}{2^{d(\alpha_i)(1/p-2)}}=O\Big(\frac{1}{2^{d(\alpha_k)(1/p-2)}}\Big)\ \text{ as }\ k\to\infty. \tag{3.42}$$

Analogously to the proof of the previous theorem, if we use also Corollaries 2.2 and 3.1, then for the sufficiently large k, we can conclude that

$$\begin{aligned}\|\sigma_{\alpha_k}f-f\|^p_{\text{weak-}L_p(G)}&\ge 2^{(1-2p)[\alpha_k]}\big\|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}\big\|^p_{\text{weak-}L_p(G)}\\ &\quad-\Big(\frac{2^{|\alpha_k|}}{\alpha_k}\Big)^p\big\|\sigma_{2^{|\alpha_k|}}f-f\big\|^p_{\text{weak-}L_p(G)}-\Big(\frac{\alpha_k-2^{|\alpha_k|}}{\alpha_k}\Big)^p\big\|S_{2^{|\alpha_k|}}f-f\big\|^p_{\text{weak-}L_p(G)}\\ &\ge 2^{(1-2p)[\alpha_k]-1}\big\|(\alpha_k-2^{|\alpha_k|})K_{\alpha_k-2^{|\alpha_k|}}\big\|^p_{\text{weak-}L_p(G)}\end{aligned} \tag{3.43}$$

Let $x\in E_{[\alpha_k]}$. From Lemma 3.5 it follows that

$$\mu\Big(x\in G:\ (\alpha_k-2^{|\alpha_k|})|K_{\alpha_k-2^{|\alpha_k|}}|\ge 2^{2[\alpha_k]-4}\Big)\ge\mu(E_{[\alpha_k]})\ge\frac{1}{2^{[\alpha_k]-4}}$$

and

$$2^{2p[\alpha_k]-4}\mu\Big(x\in G:\ (\alpha_k-2^{|\alpha_k|})|K_{\alpha_k-2^{|\alpha_k|}}|\ge 2^{2[\alpha_k]-4}\Big)\ge 2^{(2p-1)[\alpha_k]-4}. \tag{3.44}$$

Hence combining (1.2), (1.12), (3.43) and (3.44), we get

$$\|\sigma_{n_k}f-f\|_{weak-L_p(G)}\nrightarrow 0\ \text{ as }\ k\to\infty.$$

The proof of Theorem 3.4 is complete. □

Using Theorem 3.4, we easily get an important result proved in [67].

Corollary 3.4.

(a) *Let $f\in H_{1/2}(G)$ and*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^k},f\Big)=o\Big(\frac{1}{k^2}\Big)\ \ as\ \ k\to\infty.$$

Then

$$\|\sigma_k f-f\|_{H_{1/2}(G)}\to 0\ \ as\ \ k\to\infty.$$

(b) *There exists a martingale* $f \in H_{1/2}(G)$ *for which*

$$\omega_{H_{1/2}(G)}\Big(\frac{1}{2^k}, f\Big) = O\Big(\frac{1}{k^2}\Big) \ \ as \ \ k \to \infty$$

and

$$\|\sigma_k f - f\|_{1/2} \nrightarrow 0 \ \ as \ \ k \to \infty.$$

Corollary 3.5.

(a) *Let* $0 < p < 1/2$, $f \in H_p(G)$ *and*

$$\omega_{H_p(G)}\Big(\frac{1}{2^k}, f\Big) = o\Big(\frac{1}{2^{k(1/p-2)}}\Big) \ \ as \ \ k \to \infty.$$

Then

$$\|\sigma_k f - f\|_{H_p(G)} \to 0 \ \ as \ \ k \to \infty.$$

(b) *There exists a martingale* $f \in H_p(G)$ $(0 < p < 1/2)$ *for which*

$$\omega_{H_p(G)}\Big(\frac{1}{2^k}, f\Big) = O\Big(\frac{1}{2^{k(1/p-2)}}\Big) \ \ as \ \ k \to \infty$$

and

$$\|\sigma_k f - f\|_{\text{weak-}L_p(G)} \nrightarrow 0 \ \ as \ \ k \to \infty.$$

3.5 Strong convergence of Fejér means with respect to the one-dimensional Walsh–Fourier series on the martingale Hardy spaces

In this section, we consider the strong convergence results of Fejér means with respect to the one-dimensional Walsh–Fourier series in the martingale Hardy spaces, when $0 < p \leq 1/2$ (for details see [66]).

Theorem 3.5.

(a) *Let* $0 < p \leq 1/2$ *and* $f \in H_p(G)$. *Then there exists a constant* c_p, *depending only on* p, *such that*

$$\frac{1}{\log^{[1/2+p]} n} \sum_{m=1}^{n} \frac{\|\sigma_m f\|_{H_p(G)}^p}{m^{2-2p}} \leq c_p \|f\|_{H_p(G)}^p.$$

(b) *Let* $0 < p < 1/2$, $\Phi : \mathbb{N}_+ \to [1, \infty)$ *be a non-decreasing function such that* $\Phi(n) \uparrow \infty$ *and*

$$\overline{\lim_{k\to\infty}} \frac{k^{2-2p}}{\Phi(k)} = \infty.$$

Then there exists a martingale $f \in H_p(G)$ *such that*

$$\sum_{m=1}^{\infty} \frac{\|\sigma_m f\|_{\text{weak-}L_p(G)}^p}{\Phi(m)} = \infty.$$

Proof. Suppose that

$$\frac{1}{\log^{[1/2+p]} n} \sum_{m=1}^{n} \frac{\|\sigma_m f\|_p^p}{m^{2-2p}} \leq c_p \|f\|_{H_p(G)}^p.$$

Combining (1.7), (1.15) and Lemma 3.6, we can conclude that

$$\frac{1}{\log^{[1/2+p]} n} \sum_{m=1}^{n} \frac{\|\sigma_m f\|_{H_p(G)}^p}{m^{2-2p}} \leq \frac{1}{\log^{[1/2+p]} n} \sum_{m=1}^{n} \frac{\|\sigma_m f\|_p^p}{m^{2-2p}} + \|\widetilde{\sigma}_{\#}^* f\|_{H_p(G)} + \|\widetilde{S}_{\#}^* f\|_{H_p(G)} \leq c_p \|f\|_{H_p(G)}^p. \quad (3.45)$$

According to Lemma 2.5 and (3.45), Theorem 3.5 will be proved if we show that

$$\frac{1}{\log^{[1/2+p]} n}\sum_{m=1}^{n}\frac{\|\sigma_m a\|_p^p}{m^{2-2p}}\le c<\infty,\quad m=2,3,\dots,$$

for any p-atom a. We may assume that a is p-atom, with support I, $\mu(I)=2^{-M}$ and $I=I_M$. It is evident that $\sigma_n(a)=0$, when $n\le 2^M$. Therefore, we may assume that $n>2^M$.

Let $x\in I_M$. Since σ_n is bounded from $L_\infty(G)$ to $L_\infty(G)$ (the boundedness follows from the fact that Fejér kernels are uniformly bounded in the space $L_1(G)$, which is proved in Lemma 3.1) and $\|a\|_\infty\le 2^{M/p}$, we can conclude that

$$\int_{I_M}|\sigma_m a(x)|^p\,d\mu(x)\le\frac{\|\sigma_m a\|_\infty^p}{2^M}\le\frac{\|a\|_\infty^p}{2^M}\le c<\infty,\quad 0<p\le\frac{1}{2}\,.$$

Let $0<p\le 1/2$. Then

$$\frac{1}{\log^{[1/2+p]} n}\sum_{m=1}^{n}\frac{\int_{I_M}|\sigma_m a(x)|^p\,d\mu(x)}{m^{2-2p}}\le\frac{c}{\log^{[1/2+p]} n}\sum_{m=1}^{n}\frac{1}{m^{2-2p}}\le c<\infty.$$

It is evident that

$$|\sigma_m a(x)|\le\int_{I_M}|a(t)|\,|K_m(x+t)|\,d\mu(t)\le 2^{M/p}\int_{I_M}|K_m(x+t)|\,d\mu(t).$$

It follows from Lemma 3.2 that

$$|\sigma_m a(x)|\le\frac{c\,2^{k+l}\,2^{M(1/p-1)}}{m},\quad x\in I_{l+1}(e_k+e_l),\quad 0\le k<l<M \tag{3.46}$$

and

$$|\sigma_m a(x)|\le c\,2^{M(1/p-1)}\,2^k,\quad x\in I_M(e_k),\quad 0\le k<M. \tag{3.47}$$

If we use identity (2.1) and (3.46), (3.47) we get that

$$\begin{aligned}\int_{\overline{I}_M}|\sigma_m a(x)|^p\,d\mu(x)&=\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\int_{I_{l+1}(e_k+e_l)}|\sigma_m a(x)|^p\,d\mu(x)+\sum_{k=0}^{M-1}\int_{I_M(e_k)}|\sigma_m a(x)|^p\,d\mu(x)\\&\le c\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\frac{1}{2^l}\,\frac{2^{p(k+l)}2^{M(1-p)}}{m^p}+c\sum_{k=0}^{M-1}\frac{1}{2^M}\,2^{M(1-p)}2^{pk}\\&\le\frac{c\,2^{M(1-p)}}{m^p}\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\frac{2^{p(k+l)}}{2^l}+c\sum_{k=0}^{M-1}\frac{2^{pk}}{2^{pM}}\\&\le\frac{c\,2^{M(1-p)}M^{[1/2+p]}}{m^p}+c.\end{aligned}\tag{3.48}$$

Hence

$$\frac{1}{\log^{[1/2+p]} n}\sum_{m=2^M+1}^{n}\frac{\int_{\overline{I}_M}|\sigma_m a(x)|^p\,d\mu(x)}{m^{2-2p}}$$

$$\le\frac{1}{\log^{[1/2+p]} n}\Big(\sum_{m=2^M+1}^{n}\frac{c\,2^{M(1-p)}M^{[1/2+p]}}{m^{2-p}}+\sum_{m=2^M+1}^{n}\frac{c}{m^{2-2p}}\Big)<c<\infty.$$

The proof of part (a) of Theorem 3.5 is complete.

Now, we prove part (b) of Theorem 3.5. Let $\Phi(n)$ be a non-decreasing function satisfying the condition

$$\lim_{k\to\infty}\frac{2^{(|n_k|+1)(2-2p)}}{\Phi(2^{|n_k|+1})}=\infty. \tag{3.49}$$

According to (3.49), there exists an increasing sequence $\{\alpha_k:\ k\in\mathbb{N}_+\}\subset\{n_k:\ k\in\mathbb{N}_+\}$ such that

$$|\alpha_k|\geq 2,\ \text{ where }\ k\in\mathbb{N}_+, \tag{3.50}$$

and

$$\sum_{\eta=0}^{\infty}\frac{\Phi^{1/2}(2^{|\alpha_\eta|+1})}{2^{|\alpha_\eta|(1-p)}}=2^{1-p}\sum_{\eta=0}^{\infty}\frac{\Phi^{1/2}(2^{|\alpha_\eta|+1})}{2^{(|\alpha_\eta|+1)(1-p)}}<c<\infty. \tag{3.51}$$

Let $f=(f_n,\ n\in\mathbb{N}_+)\in H_p(G)$ be a martingale from Example 2.1, where

$$\lambda_k=\frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{2^{(|\alpha_k|)(1/p-1)}}.$$

Combining (2.3) and (3.51), we get that $f\in H_p(G)$. According to (2.4), we have

$$\widehat{f}(j)=\begin{cases}\Phi^{1/2p}(2^{|\alpha_k|+1}), & \text{if } j\in\left\{2^{|\alpha_k|},\ldots,2^{|\alpha_k|+1}-1\right\},\\ 0, & \text{if } j\notin\bigcup\limits_{k=0}^{\infty}\left\{2^{|\alpha_k|},\ldots,2^{|\alpha_k|+1}-1\right\}.\end{cases}\quad k\in\mathbb{N}_+. \tag{3.52}$$

Let $2^{|\alpha_k|}<n<2^{|\alpha_k|+1}$. Then

$$\sigma_n f=\frac{1}{n}\sum_{j=1}^{2^{|\alpha_k|}}S_jf+\frac{1}{n}\sum_{j=2^{|\alpha_k|}+1}^{n}S_jf=III+IV. \tag{3.53}$$

It is evident that

$$S_jf=0,\ \text{ if }\ 0\leq j\leq 2^{|\alpha_1|}. \tag{3.54}$$

Let $2^{|\alpha_s|}<j\leq 2^{|\alpha_s|+1}$, where $s=1,2,\ldots,k$. If we apply (2.6), we get

$$S_jf=\sum_{\eta=0}^{s-1}\Phi^{1/2p}(2^{|\alpha_\eta|+1})(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|}})+\Phi^{1/2p}(2^{|\alpha_s|+1})w_{2^{|\alpha_s|}}D_{j-2^{|\alpha_s|}}. \tag{3.55}$$

Let $2^{|\alpha_s|+1}\leq j\leq 2^{|\alpha_{s+1}|}$, $s=0,1,\ldots,k-1$. Then if we use (2.5), we can conclude that

$$S_jf=\sum_{\eta=0}^{s}\Phi^{1/2p}(2^{|\alpha_\eta|+1})(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|}}). \tag{3.56}$$

Let $x\in I_2(e_0+e_1)$. Since (see Lemmas 2.2 and 3.2)

$$D_{2^n}(x)=K_{2^n}(x)=0,\ \text{ where }\ n\geq 2, \tag{3.57}$$

combining (3.50) and (3.54)–(3.57), we get

$$\begin{aligned}III&=\frac{1}{n}\sum_{\eta=0}^{k-1}\Phi^{1/2p}(2^{|\alpha_\eta|+1})\sum_{v=2^{|\alpha_\eta|}+1}^{2^{|\alpha_\eta|+1}}D_v(x)\\&=\frac{1}{n}\sum_{\eta=0}^{k-1}\Phi^{1/2p}(2^{|\alpha_\eta|+1})\left(2^{|\alpha_\eta|+1}K_{2^{|\alpha_\eta|+1}}(x)-2^{|\alpha_\eta|}K_{2^{|\alpha_\eta|}}(x)\right)=0.\end{aligned} \tag{3.58}$$

If we use (3.55), when $s=k$, for IV we can write that

$$IV=\frac{n-2^{|\alpha_k|}}{n}\sum_{\eta=0}^{k-1}\Phi^{1/2p}(2^{|\alpha_\eta|+1})(D_{2^{|\alpha_\eta|+1}}-D_{2^{|\alpha_\eta|}})$$

$$+\frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{n}\sum_{j=2^{|\alpha_k|}+1}^{n}w_{2^{|\alpha_k|}}D_{j-2^{|\alpha_k|}}=IV_1+IV_2. \quad (3.59)$$

Combining (3.50) and (3.57), we can conclude that

$$IV_1=0,\ \text{ where }\ x\in I_2(e_0+e_1). \quad (3.60)$$

Let $\alpha_k\in\mathbb{A}_{0,2}$, $2^{|\alpha_k|}<n<2^{|\alpha_k|+1}$ and $x\in I_2(e_0+e_1)$. Since $n-2^{|\alpha_k|}\in\mathbb{A}_{0,2}$, from Lemmas 2.1 and 3.1 and (3.57), it follows that

$$|IV_2|=\frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{n}\Big|\sum_{j=1}^{n-2^{|\alpha_k|}}D_j(x)\Big|$$

$$=\frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{n}\big|(n-2^{|\alpha_k|})K_{n-2^{|\alpha_k|}}(x)\big|\geq\frac{\Phi^{1/2p}(2^{|\alpha_k|+1})}{2^{|\alpha_k|+1}}. \quad (3.61)$$

Let $0<p<1/2$ and $n\in\mathbb{A}_{0,2}$. Combining (3.53)–(3.61), we get

$$\|\sigma_n f\|^p_{\text{weak-}L_p(G)}\geq\frac{c_p\Phi^{1/2}(2^{|\alpha_k|+1})}{2^{p(|\alpha_k|+1)}}\,\mu\Big\{x\in I_2(e_0+e_1):\ |\sigma_n f|\geq\frac{c_p\Phi^{1/2p}(2^{|\alpha_k|+1})}{2^{|\alpha_k|+1}}\Big\}$$

$$\geq\frac{c_p\Phi^{1/2}(2^{|\alpha_k|+1})}{2^{p(|\alpha_k|+1)}}\,\mu\big\{I_2(e_0+e_1)\big\}\geq\frac{c_p\Phi^{1/2}(2^{|\alpha_k|+1})}{2^{p(|\alpha_k|+1)}}. \quad (3.62)$$

Hence

$$\sum_{n=1}^{\infty}\frac{\|\sigma_n f\|^p_{\text{weak-}L_p(G)}}{\Phi(n)}\geq\sum_{\{n\in\mathbb{A}_{0,2}:\ 2^{|\alpha_k|}<n<2^{|\alpha_k|+1}\}}\frac{\|\sigma_n f\|^p_{\text{weak-}L_p(G)}}{\Phi(n)}$$

$$\geq\frac{1}{\Phi^{1/2}(2^{|\alpha_k|+1})}\sum_{\{n\in\mathbb{A}_{0,2}:\ 2^{|\alpha_k|}<n<2^{|\alpha_k|+1}\}}\frac{1}{2^{p(|\alpha_k|+1)}}\geq\frac{c_p2^{(1-p)(|\alpha_k|+1)}}{\Phi^{1/2}(2^{|\alpha_k|+1})}\to\infty\ \text{ as }\ k\to\infty.$$

The proof of Theorem 3.5 is complete. □

Theorem 3.6. *Let $f\in H_{1/2}(G)$. Then*

$$\sup_{n\in\mathbb{N}_+}\sup_{\|f\|_{H_p}\leq1}\frac{1}{n}\sum_{m=1}^{n}\|\sigma_m f\|_{1/2}^{1/2}=\infty.$$

Proof. Let $0<p\leq1$ and

$$f_k(x):=2^k\big(D_{2^{k+1}}(x)-D_{2^k}(x)\big)$$

Since

$$\operatorname{supp}(f_k)=I_k,\qquad\int_{I_k}a_k\,d\mu=0$$

and

$$\|f_k\|_\infty\leq2^{2k}=\big(\mu(\operatorname{supp}f_k)\big)^{-2},$$

we can conclude that f_k is 1/2-atom, for every $k\in\mathbb{N}$.

Moreover, if we use the orthogonality of Walsh functions, we get

$$S_{2^n}(f_k,x)) = \begin{cases} 0, & n=0,\dots,k, \\ \left(D_{2^{k+1}}(x) - D_{2^k}(x)\right), & n \geq k+1, \end{cases}$$

and

$$\sup_{n\in\mathbb{N}} |S_{2^n}(f_k,x)| = \left|\left(D_{2^{k+1}}(x) - D_{2^k}(x)\right)\right|,$$

where $x \in G$.

Combining the first equality of Lemma 2.1 and Lemma 2.6, we obtain

$$\begin{aligned} \|a_k\|_{H_p(G)} &= 2^k \left\| \sup_{n\in\mathbb{N}} \left| S_{2^n}\left(D_{2^{k+1}}(x) - D_{2^k}(x)\right)\right| \right\|_{1/2} \\ &= 2^k \left\|\left(D_{2^{k+1}}(x) - D_{2^k}(x)\right)\right\|_{1/2} = 2^k \|D_{2^k}(x)\|_{1/2} \leq 2^k \cdot 2^{-k} \leq 1. \end{aligned}$$

It is easy to show that

$$\widehat{f}_m(i) = \begin{cases} 2^m & \text{if } i = 2^m, \dots, 2^{m+1}-1, \\ 0 & \text{otherwise} \end{cases} \tag{3.63}$$

and

$$S_i f_m = \begin{cases} 2^m\left(D_i - D_{2^m}\right) & \text{if } i = 2^m+1,\dots,2^{m+1}-1, \\ f_m & \text{if } i \geq 2^{m+1}, \\ 0 & \text{otherwise}. \end{cases} \tag{3.64}$$

Let $0 < n < 2^m$. Using the first equality of Lemma 2.1, we have

$$\begin{aligned} |\sigma_{n+2^m} f_m| &= \frac{1}{n+2^m} \left| \sum_{j=2^m+1}^{n+2^m} S_j f_m \right| = \frac{1}{n+2^m} \left| 2^m \sum_{j=2^m+1}^{n+2^m} (D_j - D_{2^m}) \right| \\ &= \frac{1}{n+2^m} \left| 2^m \sum_{j=1}^{n} (D_{j+2^m} - D_{2^m}) \right| = \frac{1}{n+2^m} \left| 2^m \sum_{j=1}^{n} D_j \right| = \frac{2^m}{n+2^m} n|K_n|. \end{aligned} \tag{3.65}$$

Let

$$n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k,$$

where

$$0 \leq l_1 \leq m_1 \leq l_2 - 2 < l_2 \leq m_2 \leq \cdots \leq l_s - 2 < l_s \leq m_s.$$

Applying Lemma 3.5 and (3.65), we find that

$$|\sigma_{n+2^m} f_m(x)| \geq c\, 2^{2l_i}, \quad \text{where } x \in I_{l_i+1}(e_{l_i-1} + e_{l_i}).$$

Hence

$$\begin{aligned} &\int_G |\sigma_{n+2^m} f_m(x)|^{1/2} \, d\mu(x) \\ &\qquad \geq \sum_{i=0}^{s} \int_{I_{l_i+1}(e_{l_i-1}+e_{l_i})} |\sigma_{n+2^m} f_m(x)|^{1/2} \, d\mu(x) \geq c \sum_{i=0}^{s} \frac{1}{2^{l_i}} 2^{l_i} \geq cs \geq c\, V(n). \end{aligned}$$

According to the second estimation of Lemma 2.3, we can conclude that

$$\begin{aligned} \sup_{n\in\mathbb{N}_+} \sup_{\|f\|_{H_p}\leq 1} \frac{1}{n} \sum_{k=1}^{n} \|\sigma_k f\|_{1/2}^{1/2} &\geq \frac{1}{2^{m+1}} \sum_{k=2^m+1}^{2^{m+1}-1} \|\sigma_k f_m\|_{1/2}^{1/2} \\ &\geq \frac{c}{2^{m+1}} \sum_{k=2^m+1}^{2^{m+1}-1} V(k-2^m) \geq \frac{c}{2^{m+1}} \sum_{k=1}^{2^m-1} V(k) \geq c \log m \to \infty \ \text{ as } \ m \to \infty. \end{aligned}$$

Thus The proof is complete. □

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References

[1] G. N. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli and A. I. Rubinshteĭn, *Multiplicative Systems of Functions and Harmonic Analysis on Zero-Dimensional Groups.* (Russian) "Èlm", Baku, 1981.

[2] N. K. Bary, *Trigonometric Series.* (Russian) Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961.

[3] D. Baramidze, L.-E. Persson, H. Singh and G. Tephnadze, Some new results and inequalities for subsequences of NÖrlund logarithmic means of Walsh–Fourier series. *J. Inequal. Appl.* **2022**, Paper no. 30, 13 pp.

[4] D. Baramidze, Sharp (H_p, L_p) and $(H_p, \text{weak-}L_p)$ type inequalities of weighted maximal operators of T means with respect to Vilenkin systems. *Trans. A. Razmadze Math. Inst.* **176** (2022), no. 2, 173–182.

[5] D. Baramidze, Z. dvalashvili and G. Tutberidze, Convergence of Nörlund means with respect to Vilenkin systems of integrable functions. *Mem. Differ. Equ. Math. Phys.* **86** (2022), 1–14.

[6] D. Baramidze, N. Gogolashvili and N. Nadirashvili, Convergence of T means with respect to Vilenkin systems of integrable functions. *Georgian Math. J.* **29** (2022), no. 4, 481–491.

[7] L. Baramidze, L.-E. Persson, G. Tephnadze and P. Wall, Sharp H_p-L_p type inequalities of weighted maximal operators of Vilenkin–Nörlund means and its applications. *J. Inequal. Appl.* **2016**, Paper no. 242, 20 pp.

[8] I. Blahota, L.-E. Persson and G. Tephnadze, On the Nörlund means of Vilenkin–Fourier series. *Czechoslovak Math. J.* **65(140)** (2015), no. 4, 983–1002.

[9] I. Blahota, L. E. Persson and G. Tephnadze, Two-sided estimates of the Lebesgue constants with respect to Vilenkin systems and applications. *Glasg. Math. J.* **60** (2018), no. 1, 17–34.

[10] I. Blahota and G. Tephnadze, A note on maximal operators of Vilenkin–Nörlund means. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **32** (2016), no. 2, 203–213.

[11] I. Blahota, G. Tephnadze and R. Toledo, Strong convergence theorem of Cesàro means with respect to the Walsh system. *Tohoku Math. J. (2)* **67** (2015), no. 4, 573–584.

[12] I. Blahota and G. Tephnadze, On the (C, α)-means with respect to the Walsh system. *Anal. Math.* **40** (2014), no. 3, 161–174.

[13] I. Blahota and G. Tephnadze, Strong convergence theorem for Vilenkin–Fejér means. *Publ. Math. Debrecen* **85** (2014), no. 1-2, 181–196.

[14] I. Blahota, G. Gát and U. Goginava, Maximal operators of Fejér means of double Vilenkin–Fourier series. *Colloq. Math.* **107** (2007), no. 2, 287–296.

[15] I. Blahota, G. Gát and U. Goginava, Maximal operators of Fejér means of Vilenkin–Fourier series. *JIPAM. J. Inequal. Pure Appl. Math.* **7** (2006), no. 4, Article 149, 7 pp.

[16] I. Blahota, K. Nagy, L.-E. Persson and G. Tephnadze, A sharp boundedness result for restricted maximal operators of Vilenkin–Fourier series on martingale Hardy spaces. *Georgian Math. J.* **26** (2019), no. 3, 351–360.

[17] I. Blahota, K, Nagy and G. Tephnadze, Approximation by Marcinkiewicz Θ-means of double Walsh–Fourier series. *Math. Inequal. Appl.* **22** (2019), no. 3, 837–853.

[18] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* **83** (1977), no. 4, 569–645.

[19] N. J. Fine, On the Walsh functions. *Trans. Amer. Math. Soc.* **65** (1949), 372–414.

[20] N. Fujii, A maximal inequality for H^1-functions on a generalized Walsh–Paley group. *Proc. Amer. Math. Soc.* **77** (1979), no. 1, 111–116.

[21] G. Gàt, Cesàro means of integrable functions with respect to unbounded Vilenkin systems. *J. Approx. Theory* **124** (2003), no. 1, 25–43.

[22] G. Gàt, Investigations of certain operators with respect to the Vilenkin system. *Acta Math. Hungar.* **61** (1993), no. 1-2, 131–149.

[23] G. Gàt and U. Goginava, Uniform and L-convergence of logarithmic means of Walsh–Fourier series. *Acta Math. Sin. (Engl. Ser.)* **22** (2006), no. 2, 497–506.

[24] U. Goginava, The maximal operator of the (C,α) means of the Walsh–Fourier series. *Ann. Univ. Sci. Budapest. Sect. Comput.* **26** (2006), 127–135.

[25] U. Goginava, Maximal operators of Fejér means of double Walsh–Fourier series. *Acta Math. Hungar.* **115** (2007), no. 4, 333–340.

[26] U. Goginava, Maximal operators of Fejér-Walsh means. *Acta Sci. Math. (Szeged)* **74** (2008), no. 3-4, 615–624.

[27] U. Goginava, The maximal operator of the Fejér means of the character system of the p-series field in the Kaczmarz rearrangement. *Publ. Math. Debrecen* **71** (2007), no. 1-2, 43–55.

[28] U. Goginava, The maximal operator of Marcinkiewicz-Fejér means of the d-dimensional Walsh–Fourier series. *East J. Approx.* **12** (2006), no. 3, 295–302.

[29] U. Goginava, A note on the Walsh–Fejér means. *Anal. Theory Appl.* **26** (2010), no. 4, 320–325.

[30] U. Goginava and K. Nagy, On the maximal operator of Walsh–Kaczmarz–Fejér means. *Czechoslovak Math. J.* **61(136)** (2011), no. 3, 673–686.

[31] N. Gogolashvili, K. Nagy and G. Tephnadze, Strong convergence theorem for Walsh–Kaczmarz–Fejér means. *Mediterr. J. Math.* **18** (2021), no. 2, Paper no. 37, 17 pp.

[32] N. Gogolashvili and G. Tephnadze, Maximal operators of T means with respect to Walsh–Kaczmarz system. *Math. Inequal. Appl.* **24** (2021), no. 3, 737–750.

[33] N. Gogolashvili and G. Tephnadze, On the maximal operators of T means with respect to Walsh–Kaczmarz system. *Studia Sci. Math. Hungar.* **58** (2021), no. 1, 119–135.

[34] B. Golubov, A. Efimov and V. Skvortsov, *Walsh Series and Transforms. Theory and Applications.* Mathematics and its Applications (Soviet Series), 64. Kluwer Academic Publishers Group, Dordrecht, 1991.

[35] J. Gosselin, Almost everywhere convergence of Vilenkin–Fourier series. *Trans. Amer. Math. Soc.* **185** (1973), 345–370 (1974).

[36] D. Lukkassen, L. E. Persson, G. Tephnadze and G. Tutberidze, Some inequalities related to strong convergence of Riesz logarithmic means. *J. Inequal. Appl.* **2020**, Paper no. 79, 17 p.

[37] N. Memić, L. E. Persson and G. Tephnadze, A note on the maximal operators of Vilenkin–Nörlund means with non-increasing coefficients. *Studia Sci. Math. Hungar.* **53** (2016), no. 4, 545–556.

[38] N. Memić, I. Simon and G. Tephnadze, Strong convergence of two-dimensional Vilenkin–Fourier series. *Math. Nachr.* **289** (2016), no. 4, 485–500.

[39] K. Nagy and G. Tephnadze, Strong convergence theorem for Walsh–Marcinkiewicz means. *Math. Inequal. Appl.* **19** (2016), no. 1, 185–195.

[40] K. Nagy and G. Tephnadze, Approximation by Walsh–Marcinkiewicz means on the Hardy space $H_{2/3}$. *Kyoto J. Math.* **54** (2014), no. 3, 641–652.

[41] K. Nagy and G. Tephnadze, Walsh–Marcinkiewicz means and Hardy spaces. *Cent. Eur. J. Math.* **12** (2014), no. 8, 1214–1228.

[42] K. Nagy and G. Tephnadze, Approximation by Marcinkiewicz means of Walsh–Kaczmarz–Fourier series in the Hardy space $H_{2/3}$. *Bull. TICMI* **18** (2014), no. 1, 110–121.

[43] J. Neveu, *Discrete-Parameter Martingales.* North-Holland Mathematical Library, Vol. 10. North-Holland Publishing Co., Amsterdam–Oxford; American Elsevier Publishing Co., Inc., New York, 1975.

[44] L.-E. Persson, F. Schipp, G. Tephnadze and F. Weisz, An analogy of the Carleson-Hunt theorem with respect to Vilenkin systems. *J. Fourier Anal. Appl.* **28** (2022), no. 3, Paper no. 48, 29 pp.

[45] L. E. Persson, G. Tephnadze, G. Tutberidze and P. Wall, Some new results on the strong convergence of Fejér means with respect to Vilenkin systems. *Ukrainian Math. J.* **73** (2021), no. 4, 635–648.

[46] L. E. Persson, G. Tephnadze and P. Wall, On the Nörlund logarithmic means with respect to Vilenkin system in the martingale Hardy space H_1. *Acta Math. Hungar.* **154** (2018), no. 2, 289–301.

[47] L. E. Persson, G. Tephnadze and P. Wall, On an approximation of 2-dimensional Walsh–Fourier series in martingale Hardy spaces. *Ann. Funct. Anal.* **9** (2018), no. 1, 137–150.

[48] L.-E. Persson, G. Tephnadze and G. Tutberidze, On the boundedness of subsequences of Vilenkin–Fejér means on the martingale Hardy spaces. *Oper. Matrices* **14** (2020), no. 1, 283–294.

[49] L.-E. Persson, G. Tephnadze and P. Wall, Maximal operators of Vilenkin–Nörlund means. *J. Fourier Anal. Appl.* **21** (2015), no. 1, 76–94.

[50] L.-E. Persson, G. Tephnadze and P. Wall, Some new (H_p, L_p) type inequalities of maximal operators of Vilenkin–Nörlund means with non-decreasing coefficients. *J. Math. Inequal.* **9** (2015), no. 4, 1055–1069.

[51] L.-E. Persson and G. Tephnadze, A note on Vilenkin–Fejér means on the martingale Hardy spaces H_p. *Bull. TICMI* **18** (2014), no. 1, 55–64.

[52] L.-E. Persson and G. Tephnadze, A sharp boundedness result concerning some maximal operators of Vilenkin–Fejér means. *Mediterr. J. Math.* **13** (2016), no. 4, 1841–1853.

[53] F. Schipp, Certain rearrangements of series in the Walsh system. (Russian) *Mat. Zametki* **18** (1975), no. 2, 193–201.

[54] F. Schipp, W. R. Wade and P. Simon, [J. Pál], *Walsh Series. An Introduction to Dyadic Harmonic Analysis*. With the assistance from J. Pál. Adam Hilger, Bristol etc., 1990.

[55] P. Simon, Cesaro summability with respect to two-parameter Walsh systems. *Monatsh. Math.* **131** (2000), no. 4, 321–334.

[56] P. Simon, Strong convergence of certain means with respect to the Walsh–Fourier series. *Acta Math. Hungar.* **49** (1987), no. 3-4, 425–431.

[57] P. Simon, A note on the Sunouchi operator with respect to Vilenkin system. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **43** (2000), 101–116 (2001).

[58] P. Simon, Strong convergence theorem for Vilenkin–Fourier series. *J. Math. Anal. Appl.* **245** (2000), no. 1, 52–68.

[59] P. Simon and F. Weisz, Weak inequalities for Cesàro and Riesz summability of Walsh–Fourier series. *J. Approx. Theory* **151** (2008), no. 1, 1–19.

[60] B. Smith, A strong convergence theorem for $H_1(\mathbf{T})$. *Banach spaces, harmonic analysis, and probability theory (Storrs, Conn., 1980/1981)*, 169–173, Lecture Notes in Math., 995, *Springer, Berlin–New York*, 1983.

[61] G. Tephnadze, Fejér means of Vilenkin–Fourier series. *Studia Sci. Math. Hungar.* **49** (2012), no. 1, 79–90.

[62] G. Tephnadze, On the maximal operators of Vilenkin–Fejér means. *Turkish J. Math.* **37** (2013), no. 2, 308–318.

[63] G. Tephnadze, On the maximal operators of Vilenkin–Fejér means on Hardy spaces. *Math. Inequal. Appl.* **16** (2013), no. 1, 301–312.

[64] G. Tephnadze, A note on the Fourier coefficients and partial sums of Vilenkin–Fourier series. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **28** (2012), no. 2, 167–176.

[65] G. Tephnadze, On the maximal operators of Walsh–Kaczmarz–Fejér means. *Period. Math. Hungar.* **67** (2013), no. 1, 33–45.

[66] G. Tephnadze, Strong convergence theorems for Walsh–Fejér means. *Acta Math. Hungar.* **142** (2014), no. 1, 244–259.

[67] G. Tephnadze, A note on the norm convergence by Vilenkin–Fejér means. *Georgian Math. J.* **21** (2014), no. 4, 511–517.

[68] G. Tephnadze, On the partial sums of Vilenkin–Fourier series. (Russian) *Izv. Nats. Akad. Nauk Armenii Mat.* **49** (2014), no. 1, 60–72; translation in *J. Contemp. Math. Anal.* **49** (2014), no. 1, 23–32.

[69] G. Tephnadze, On the Vilenkin–Fourier coefficients. *Georgian Math. J.* **20** (2013), no. 1, 169–177.

[70] G. Tephnadze, On the partial sums of Walsh–Fourier series. *Colloq. Math.* **141** (2015), no. 2, 227–242.

[71] G. Tephnadze, On the convergence of Fejér means of Walsh–Fourier series in the space H_p. *J. Contemp. Math. Anal., Armen. Acad. Sci.* **51** (2016), 90–102.

[72] G. Tephnadze, A note on the strong convergence of two-dimensional Walsh–Fourier series. *Proc. A. Razmadze Math. Inst.* **162** (2013), 93–97.

[73] G. Tephnadze, Strong convergence of two-dimensional Walsh–Fourier series. *Ukrainian Math. J.* **65** (2013), no. 6, 914–927.

[74] G. Tephnadze, The maximal operators of logarithmic means of one-dimensional Vilenkin–Fourier series. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **27** (2011), no. 2, 245–256.

[75] G. Tephnadze, On the maximal operators of Riesz logarithmic means of Vilenkin–Fourier series. *Studia Sci. Math. Hungar.* **51** (2014), no. 1, 105–120.

[76] G. Tephnadze, Approximation by Walsh–Kaczmarz–Fejér means on the Hardy space. *Acta Math. Sci. Ser. B (Engl. Ed.)* **34** (2014), no. 5, 1593–1602.

[77] G. Tephnadze, A note on the strong summability of two-dimensional Walsh–Fourier series. *Georgian Math. J.* **28** (2021), no. 3, 477–482.

[78] G. Tephnadze, On the maximal operators of Walsh–Kaczmarz–Nörlund means. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **31** (2015), no. 2, 259–271.

[79] G. Tephnadze, On the convergence of partial sums with respect to Vilenkin system on the Martingale Hardy spaces. (Russian) *Izv. Nats. Akad. Nauk Armenii Mat.* **53** (2018), no. 5, 77–94; translation in *J. Contemp. Math. Anal.* **53** (2018), no. 5, 294–306.

[80] G. Tephnadze, Convergence and strong summability of the two-dimensional Vilenkin–Fourier series. *Nonlinear Stud.* **26** (2019), no. 4, 973–989.

[81] G. Tephnadze, Strong convergence of the two-dimensional Walsh–Fourier series and martingale Hardy spaces. *Mediterr. J. Math.* (to appear).

[82] G. Tephnadze, Martingale Hardy spaces and summability of the one dimensional Vilenkin–Fourier series. *PhD thesis, Department of Engineering Sciences and Mathematics*, Luleå University of Technology, Oct. 2015;
`https://www.diva-portal.org/smash/record.jsf?pid=diva2%3A990652&dswid=-633`

[83] G. Tephnadze and G. Tutberidze, A note on the maximal operators of the Nörlund logaritmic means of Vilenkin–Fourier series. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 1, 107–112.

[84] G. Tutberidze, A note on the strong convergence of partial sums with respect to Vilenkin system. *J. Contemp. Math. Anal., Armen. Acad. Sci.* **54** (2019), 365–370.

[85] G. Tutberidze, Modulus of continuity and convergence of subsequences of Vilenkin–Fejér means in martingale Hardy spaces. *Georgian Math. J.* **29** (2022), no. 1, 153–162.

[86] G. Tutberidze, Maximal operators of T means with respect to the Vilenkin system. *Nonlinear Stud.* **27** (2020), no. 4, 1157–1167.

[87] G. Tutberidze, Sharp (H_p, L_p) type inequalities of maximal operators of T means with respect to Vilenkin systems. *Mediterr. J. Math.* **19** (2022), no. 2, Paper no. 81, 23 pp.

[88] Ch. Watari, Best approximation by Walsh polynomials. *Tohoku Math. J. (2)* **15** (1963), 1–5.

[89] Ch. Watari, On generalized Walsh Fourier series. *Tohoku Math. J. (2)* **10** (1958), 211–241.

[90] F. Weisz, *Martingale Hardy Spaces and Their Applications in Fourier Analysis.* Lecture Notes in Mathematics, 1568. Springer-Verlag, Berlin, 1994.

[91] F. Weisz, Hardy spaces and Cesàro means of two-dimensional Fourier series. *Approximation theory and function series (Budapest, 1995)*, 353–367, Bolyai Soc. Math. Stud., 5, *János Bolyai Math. Soc., Budapest*, 1996.

[92] F. Weisz, Cesààro summability of one- and two-dimensional Walsh–Fourier series. *Anal. Math.* **22** (1996), no. 3, 229–242.

[93] F. Weisz, Weak type inequalities for the Walsh and bounded Ciesielski systems. *Anal. Math.* **30** (2004), no. 2, 147–160.

[94] F. Weisz, *Summability of Multi-Dimensional Fourier Series and Hardy Spaces*. Mathematics and its Applications, 541. Kluwer Academic Publishers, Dordrecht, 2002.

[95] S. Yano, Cesàro summability of Walsh–Fourier series. *Tohoku Math. J. (2)* **9** (1957), 267–272.

[96] W. S. Young, Mean convergence of generalized Walsh–Fourier series. *Trans. Amer. Math. Soc.* **218** (1976), 311–320.

[97] A. Zygmund, *Trigonometric Series*. Vols. I, II. Cambridge University Press, New York, 1959.

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