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ASYMPTOTIC STUDY TO STRONG SOLUTION OF A 3D REGULARIZATION TO BOUSSINESQ SYSTEM IN SOBOLEV SPACES

Abstract. A regularized periodic three-dimensional Boussinesq system is studied. The existence, uniqueness and continuous dependance with respect to the initial data of weak and strong solutions are proved under the minimum regularity requirements. The main novelty is that these solutions are global in time. Also, convergence results of the unique weak solution and the unique strong solution of the three-dimensional regularized Boussinesq system to solutions of the three-dimensional Boussinesq system are established as the regularizing parameter α vanishes. We overcome the main difficulty caused by the singular dependance of the energy estimates on the regularizing parameter; as if it vanishes, the energy estimates blow up. The proofs use energy methods and compactness arguments.

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რეზიუმე. შესწავლილია რეგულარიზებული პერიოდული ბუსინესკის სამგანზომილებიანი სისტემა. დამტკიცებულია სუსტი და ძლიერი ამონახსნების არსებობა, ერთადერთობა და უწყვეტი დამოკიდებულება საწყის პირობებზე რეგულარობის მინიმალური მოთხოვნის პირობებში. მთავარი სიახლე ის არის, რომ ეს ამონახსნები დროში გლობალურია. ასევე, დადგენილია ბუსინესკის სამგანზომილებიანი რეგულარიზებული სისტემის ერთადერთი სუსტი ამონახსნისა და ერთადერთი ძლიერი ამონახსნის ბუსინესკის სამგანზომილებიანი სისტემის ამონახსნებთან კრებადობის თვისებები, როცა მარეგულირებელი α პარამეტრი ნული ხდება. გადალახულია მთავარი სიძნელე, რომელიც გამოწვეულია ენერგეტიკული შეფასებების რეგულარიზებად პარამეტრზე სინგულარული დამოკიდებულებით, რადგან ამ დროს ენერგეტიკული შეფასებები ფეთქდება, როცა ეს პარამეტრი ხდება ნული. დამტკიცებისას გამოიყენება ენერგეტიკული მეთოდები და კომპაქტურობის არგუმენტები.

1 Introduction

We consider the regularized Boussinesq system (Bq_α) given by

$$\begin{gathered}\partial_t v - \nu\Delta v + (v\cdot\nabla)u = -\nabla p + \theta e_3,\ \ (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ \partial_t\theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = 0,\ \ (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ v = u-\alpha^2\Delta u,\ \ (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ \operatorname{div} u = \operatorname{div} v = 0,\ \ (t,x)\in\mathbb{R}_+\times\mathbb{T}^3,\\ (u,\theta)|_{t=0} = (u^0,\theta^0),\ \ x\in\mathbb{T}^3,\end{gathered}$$

where the unknown velocity, the unknown pressure and the unknown temperature are, respectively, the three-dimensional vector u, the scalar p and the scalar θ. The parameters $\nu,\kappa>0$ denote, respectively, the viscosity and the thermal conductivity of the fluid, $\mathbb{T}^3=(\mathbb{R}/2\pi\mathbb{Z})^3$ is the three-dimensional torus and u^0, θ^0 are the given initial data. The vector $e_3=(0,0,1)^T$.

In [6], we dealt with the weak solution of (Bq_α) in the inhomogeneous sobolev spaces. Mainly, we proved that there exists a weak solution to (Bq_α), $\alpha>0$. This solution depends continuously on the initial data and converges to a weak solution of $(Bq_{\alpha=0})$ as the regularizing parameter α vanishes. Such solution is given by the following

Theorem 1.1. *Let $\theta^0\in L^2(\mathbb{T}^3)$ and let $u^0\in H^1(\mathbb{T}^3)$ be a divergence-free vector field. Then there exists a unique weak solution (u_α,θ_α) of system (Bq_α) such that u_α belongs to $C(\mathbb{R}_+,H^1(\mathbb{T}^3))\cap L^2(\mathbb{R}_+,H^2(\mathbb{T}^3))$ and θ_α belongs to $C(\mathbb{R}_+,L^2(\mathbb{T}^3))\cap L^2(\mathbb{R}_+,H^1(\mathbb{T}^3))$. Moreover, this solution satisfies the energy estimate*

$$\|\theta_\alpha\|^2_{L^2}+\|u_\alpha\|^2_{L^2}+\alpha^2\|\nabla u_\alpha\|^2_{L^2}+2\int_0^t\|\nabla\theta_\alpha\|^2_{L^2(\mathbb{T}^3)}\,d\tau+2\int_0^t\big(\|\nabla u_\alpha\|^2_{L^2}+\alpha^2\|\Delta u_\alpha\|^2_{L^2}\big)\,d\tau$$
$$\le\|\theta^0\|^2_{L^2}+\|u^0\|^2_{L^2}+\alpha^2\|\nabla u^0\|^2_{L^2}+\sigma_\alpha(t),\quad(1.1)$$

where

$$\sigma_\alpha(t)=(e^{2t}-1)\big(\|\theta^0\|^2_{L^2}+\|u^0\|^2_{L^2}+\alpha^2\|\nabla u^0\|^2_{L^2}\big).$$

If the initial velocity is mean free, then the solution is continuously dependent on the initial data on any bounded interval $[0,T]$. In particular, it is unique.

In the above theorem, it is clear that the right-hand side of (1.1) depends on time and therefore the solution belongs to $L^\infty_{loc}(\mathbb{R}_+,H^1(\mathbb{T}^3))$ and it will blow up if $T\to\infty$. Also, we studied strong solutions to the Bardina alpha model in [2], the Lagrangian alpha model in [5] and our solutions were defined only on a time interval $[0,T]$, where T is a finite time.

Here, we investigate weak and strong solutions to (Bq_α) in the homogeneous Sobolev spaces, as they are more flexible when closing the estimates, especially when dealing with the buoyancy force $\theta e_3=(0,0,\theta)^T$. Moreover, we consider a mean free initial data to obtain *a global in time weak and strong solution*. Such solutions belong to Sobolev spaces which are energy spaces and hence physically meaningful. Moreover, we prove that these solutions depend continuously on the initial data and they are thus unique. Such dependance is very interesting in computational mathematics, as it ensures the stability of numerical scheme, for example. Finally, we prove that the weak and strong solutions converge, respectively, to a global in time Leray type weak solution of the three-dimensional Boussinesq system and to a local in time strong solution of the three-dimensional Boussinesq system as the regularizing parameter $\alpha\to0$. This convergence result is one of the main targets of the regularisation. In fact, on one hand, it allows in practical situations to consider systems with $\alpha>0$ as small as needed and fully profit from the uniqueness and continuous dependance, while keeping very closed to a weak solution (respectively, strong solution) of the three-dimensional Boussinesq system for which uniqueness of a weak solution is still an open problem and the existence of a strong unique solution is only local in time. On the other hand, it is, in fact, another different method to prove the existence of a weak solution and a strong solution to the three-dimensional Bousssinesq system.

Let us recall that the homogeneous Sobolev spaces are given by

$$\dot{H}^s(\mathbb{T}^3) = \Big\{ \widehat{u} \in \mathcal{S}'(\mathbb{T}^3),\ \int_{\mathbb{T}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi < \infty \Big\}$$

and endowed with the natural norm

$$\|u\|_{\dot{H}^s(\mathbb{T}^3)} = \Big(\int_{\mathbb{T}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi \Big)^{\frac{1}{2}},$$

where s is a real number and $\mathcal{S}'(\mathbb{T}^3)$ is the Schwartz space. Also, we recall that a function ϕ is mean free if

$$\frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} \phi(x) dx = 0$$

or, equivalently.

$$\frac{1}{8\pi^3} \int_{[0,2\pi]^3} \phi(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = 0.$$

If ϕ is expended in Fourier series, that is,

$$\phi(x) = \sum_{k \in \mathbb{Z}^3} C_k e^{ik \cdot x},$$

where the complex Fourier coefficient is

$$C_k(\phi) = \widehat{\phi}(k) = \frac{1}{8\pi^3} \int_{[0,2\pi]^3} \phi(x) e^{-ik\cdot x} \, dx,$$

then the mean free condition reads as $C_0(\phi) = 0$. A three-dimensional vector field is mean free if all its components are mean free.

2 Existence of a weak solution

For $n \in \mathbb{N}$, let P_n denote the projection into the Fourier modes of order up to n, that is,

$$P_n \Big(\sum_{k \in \mathbb{Z}^3} \widehat{\phi}(k) e^{ik\cdot x} \Big) = \sum_{|k| \le n} \widehat{\phi}(k) e^{ik \cdot x}.$$

Let $u_n = P_n u$, $\theta_n = P_n \theta$ and $p_n = P_n p$. One approximates the continuous problem (Bq_α) by the following ordinary differential system denoted by $(Bq_\alpha)_n$:

$$\partial_t v_n - \Delta v_n + P_n \operatorname{div}(v_n u_n) = -\nabla p_n + \theta_n e_3, \tag{2.1}$$

$$\partial_t \theta_n - \Delta \theta_n + P_n \operatorname{div}(\theta_n u_n) = 0, \tag{2.2}$$

$$v_n = u_n - \alpha^2 \Delta u_n, \tag{2.3}$$

$$\operatorname{div} u_n = \operatorname{div} v_n = 0, \tag{2.4}$$

$$(u_n, \theta_n)_{t=0} = (u_n^0, \theta_n^0) = (P_n u^0, P_n \theta^0). \tag{2.5}$$

The ordinary differential equation theory implies that there exists some maximal $\tau_n^* > 0$ and a unique local solution $u_n \in C^\infty([0, \tau_n^*) \times \mathbb{T}^3)$ to $(Bq_\alpha)_n$. Taking the inner product in $L^2(\mathbb{T}^3)$ of equations (2.2) with θ_n and (2.1) with u_n, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\theta_n\|^2_{L^2(\mathbb{T}^3)} + \kappa \|\nabla \theta_n\|^2_{L^2(\mathbb{T}^3)} = 0, \tag{2.6}$$

$$\frac{1}{2} \frac{d}{dt} \Big(\|u_n\|^2_{L^2} + \alpha^2 \|\nabla u_n\|^2_{L^2}) + \nu(\|\nabla u_n\|^2_{L^2} + \alpha^2 \|\Delta u_n\|^2_{L^2} \Big) = \langle \theta_n e_3, u_n \rangle_{L^2}.$$

Integrating (2.2) with respect to x, we deduce that the first Fourier coefficient of θ is conserved during time, that is, $C_0(\theta(t)) = C_0(\theta^0)$, $\forall\, t > 0$. Since θ^0 is mean free, we have $C_n(\theta^0) = 0$. So,

$$\langle \theta_n e_3, u_n\rangle_{L^2} = \sum_{k\neq(0,0,0)} \widehat{\theta}_n(k)\widehat{u}_n^3(k). \tag{2.7}$$

Applying respectively the Cauchy–Schwarz inequality and the Young inequality for products to (2.7), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2\right) + \nu\left(\|\nabla u_n\|_{L^2}^2 + \alpha^2\|\Delta u_n\|_{L^2}^2\right) \le \frac{1}{2\nu}\|\theta_n\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u_n\|_{L^2}^2. \tag{2.8}$$

Summing up (2.6) and (2.8) and integrating with respect to time, we obtain

$$\begin{aligned}\|\theta_n(t)\|_{L^2}^2 + \|u_n(t)\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2 + 2\kappa\int_0^t \|\nabla\theta_n(\tau)\|_{L^2}^2\,d\tau + \nu\int_0^t \|\nabla u_n(\tau)\|_{L^2}^2\,d\tau \\ + 2\nu\alpha^2\int_0^t \|\Delta u_n\|_{L^2}^2\,d\tau \le \|\theta_n^0\|_{L^2}^2 + \|u_n^0\|_{L^2}^2 + \alpha^2\|\nabla u_n^0\|_{L^2}^2 + \frac{1}{\nu}\int_0^t \|\theta_n\|_{L^2}^2\,d\tau.\end{aligned}$$

Using respectively the Poincaré inequality and (2.6), we deduce that

$$\int_0^t \|\theta_n\|_{L^2}^2\,d\tau \le \int_0^t \|\nabla\theta_n\|_{L^2}^2\,d\tau \le \|\theta^0\|_{L^2}^2.$$

Thus, we reach our target in closing the estimates by a constant which is independent of time:

$$\begin{aligned}&\|\theta_n(t)\|_{L^2}^2 + \|u_n(t)\|_{L^2}^2 + \alpha^2\|\nabla u_n\|_{L^2}^2 \\ &\quad + 2\kappa\int_0^t \|\nabla\theta_n(\tau)\|_{L^2(\mathbb{T}^3)}^2\,d\tau + \nu\int_0^t \|\nabla u_n(\tau)\|_{L^2}^2\,d\tau + 2\nu\alpha^2\int_0^t \|\Delta u_n\|_{L^2}^2\,d\tau \le C(\alpha,\nu,\kappa,u^0,\theta^0),\end{aligned}$$

where $C(\alpha,\nu,\kappa,u^0,\theta^0) = \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + (1+\frac{1}{2\nu\kappa})\|\theta^0\|_{L^2}^2$. A standard compactness argument (see. e.g., [4]) allows to prove that there exists a weak continuous and global in time solution such that u_α belongs to $C(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{T}^3))$ and θ_α belongs to $C(\mathbb{R}_+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$. As a conclusion, we have the following

Theorem 2.1. *Let $\theta^0 \in L^2(\mathbb{T}^3)$ be mean free and let $u^0 \in \dot{H}^1(\mathbb{T}^3)$ be a divergence-free vector field. Then there exists a global in time weak solution $(u_\alpha, \theta_\alpha)$ of system (Bq_α) such that u_α belongs to $C(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{T}^3))$ and θ_α belongs to $C(\mathbb{R}_+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$. Moreover, this solution satisfies the energy estimate*

$$\begin{aligned}\|\theta_\alpha(t)\|_{L^2}^2 + \|u_\alpha(t)\|_{L^2}^2 + \alpha^2\|\nabla u_\alpha\|_{L^2}^2 + 2\kappa\int_0^t \|\nabla\theta_\alpha(\tau)\|_{L^2(\mathbb{T}^3)}^2\,d\tau \\ + \nu\int_0^t \|\nabla u_\alpha(\tau)\|_{L^2}^2\,d\tau + 2\nu\alpha^2\int_0^t \|\Delta u_\alpha\|_{L^2}^2\,d\tau \le C(\alpha,\nu,\kappa,u^0,\theta^0),\end{aligned} \tag{2.9}$$

where

$$C(\alpha,\nu,\kappa,u^0,\theta^0) = \|u^0\|_{L^2}^2 + \alpha^2\|\nabla u^0\|_{L^2}^2 + \left(1 + \frac{1}{2\nu\kappa}\right)\|\theta^0\|_{L^2}^2.$$

3 Continuous dependence with respect to initial data and uniqueness of the weak solution

We turn to the continuous dependence of solutions with respect to the initial data. Consider two solutions (u,θ) and $(\overline{u},\overline{\theta})$ of (Bq_α) corresponding, respectively, to (u^0,θ^0) and $(\overline{u}^0,\overline{\theta}^0)$. Denote $\delta u = u-\overline{u}$, $\delta v = v-\overline{v}$, $\delta\theta = \theta-\overline{\theta}$ and $\delta p = p-\overline{p}$. Then

$$\begin{gathered}
\partial_t\delta\theta - \Delta\delta\theta + (\delta u\cdot\nabla)\theta + (\overline{u}\cdot\nabla)\delta\theta = 0,\\
\partial_t\delta v - \Delta\delta v + (\delta v\cdot\nabla)u + (\overline{v}\cdot\nabla)\delta u = -\nabla\delta p + \delta\theta e_3,\\
\delta v = \delta u - \alpha^2\Delta\delta u, \quad \operatorname{div}\delta u = \operatorname{div}\delta v = 0, \quad (\delta u,\delta\theta)_{t=0} = (u^0-\overline{u}^0, \theta^0-\overline{\theta}^0).
\end{gathered}$$

Using (2.9), we deduce that $\frac{d}{dt}\,\delta\theta \in L^2([0,T],\dot H^{-1})$, $\delta\theta\in L^2([0,T],\dot H^1)$, $\frac{d}{dt}\,\delta v\in L^2([0,T],\dot H^{-2})$ and $\delta u\in L^2([0,T],\dot H^2)$. The duality action gives

$$\begin{gathered}
\Big\langle \frac{d}{dt}\,\delta\theta,\delta\theta\Big\rangle_{\dot H^{-1}} + \|\nabla\delta\theta\|_{L^2}^2 + \big\langle \delta u\cdot\nabla\theta,\delta\theta\big\rangle_{\dot H^{-1}} = 0,\\
\Big\langle \frac{d}{dt}\,\delta v,\delta u\Big\rangle_{\dot H^{-2}} + \big(\|\nabla\delta u\|_{L^2}^2 + \alpha^2\|\Delta\delta u\|_{L^2}^2\big) + \big\langle \delta v\cdot\nabla u,\delta u\big\rangle_{\dot H^{-2}} = \langle\delta\theta,\delta u\rangle_{\dot H^{-1}}
\end{gathered}$$

for almost every time t in $\mathbb{R}$. Applying the Lions-Magenes lemma concerning the derivatives of a function with value in Banach spaces (see, e.g., [7, Chaper 3, p. 169]), one has

$$\begin{aligned}
\Big\langle \frac{d}{dt}\,\delta\theta,\delta\theta\Big\rangle_{\dot H^{-1}(\mathbb{T}^3)} &= \frac12\,\frac{d}{dt}\,\|\delta\theta\|_{L^2(\mathbb{T}^3)}^2,\\
\Big\langle \frac{d}{dt}\,\delta v,\delta u\Big\rangle_{\dot H^{-2}(\mathbb{T}^3)} &= \frac12\,\frac{d}{dt}\,\big(\|\delta u\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\nabla\delta u\|_{L^2(\mathbb{T}^3)}^2\big).
\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
\frac12\,\frac{d}{dt}\,\big(\|\delta u\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\nabla\delta u\|_{L^2(\mathbb{T}^3)}^2 + \|\delta\theta\|_{L^2(\mathbb{T}^3)}^2\big) + \big(\|\nabla\delta u\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\Delta\delta u\|_{L^2(\mathbb{T}^3)}^2\big) + \|\nabla\delta\theta\|_{L^2(\mathbb{T}^3)}^2\\
= \langle\delta\theta,\delta u\rangle_{\dot H^{-1}(\mathbb{T}^3)} \underbrace{-\langle\delta v\cdot\nabla u,\delta u\rangle_{\dot H^{-2}(\mathbb{T}^3)}}_{I_2} \underbrace{-\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}}_{I_3}.
\end{aligned}$$

Since the temperature and the velocity are both mean free, we use, respectively, the Cauchy–Schwarz inequality, the Poincaré inequality, and then the Young inequality for the products to obtain

$$\big|\langle\delta\theta,\delta u\rangle_{\dot H^{-1}(\mathbb{T}^3)}\big| \le \frac12\,\big(\|\nabla\delta u\|_{L^2}^2 + \|\nabla\delta\theta\|_{L^2}^2\big). \tag{3.1}$$

To deal with I_2, we first use the Hölder inequality to obtain

$$\big|\langle\delta v\cdot\nabla u,\delta u\rangle_{\dot H^{-2}}\big| \le \|\delta u\|_{L^\infty}\|\nabla u\|_{L^2}\|\delta v\|_{L^2}.$$

For every $\vartheta\in\dot H^2(\mathbb{T}^3)$, the Agmon inequality [1] reads as

$$\|\vartheta\|_{L^\infty(\mathbb{T}^3)} \le c\|\vartheta\|_{\dot H^1(\mathbb{T}^3)}^{1/2}\|\vartheta\|_{\dot H^2(\mathbb{T}^3)}^{1/2}.$$

It follows that

$$|I_2| \le C\|\delta v\|_{L^2}\|\nabla u\|_{L^2}\|\delta u\|_{\dot H^1}^{1/2}\|\delta u\|_{\dot H^2}^{1/2}.$$

The velocity is mean free, the Poincaré inequality applies, which leads to

$$\|\delta v\|_{L^2} \le (c+\alpha^2)\|\Delta\delta u\|_{L^2}. \tag{3.2}$$

Using (3.2) and the Young inequality for products with the pair $(4,4/3)$, we obtain

$$|I_2| \le C(c+\alpha^2)\|\nabla u\|_{L^2}\|\delta u\|_{\dot H^1}^{1/2}\|\delta u\|_{\dot H^2}^{3/2} \le \frac{C}{\alpha^6}\,(c+\alpha^2)^4\|\nabla u\|_{L^2}^4\|\nabla\delta u\|_{L^2}^2 + \frac{\alpha^2}{2}\,\|\Delta\delta u\|_{L^2}^2. \tag{3.3}$$

To investigate I_3, we use the Hölder inequality to obtain

$$\left|\langle \delta u \cdot \nabla\theta, \delta\theta\rangle_{\dot H^{-1}}\right| \le \|\delta u\|_{L^3}\|\nabla\theta\|_{L^2}\|\delta\theta\|_{L^6}.$$

We recall the following inequalities: for every $\vartheta \in \dot H^1(\mathbb{T}^3)$,

$$\|\vartheta\|_{L^6(\mathbb{T}^3)} \le c\|\vartheta\|_{\dot H^1(\mathbb{T}^3)} \tag{3.4}$$

and

$$\|\vartheta\|_{L^3(\mathbb{T}^3)} \le c\|\vartheta\|^{1/2}_{L^2(\mathbb{T}^3)}\|\vartheta\|^{1/2}_{\dot H^1(\mathbb{T}^3)}. \tag{3.5}$$

Using inequalities (3.4) and (3.5), we obtain

$$\left|\langle \delta u \cdot \nabla\theta, \delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}\right| \le \|\delta u\|^{1/2}_{L^2}\|\delta u\|^{1/2}_{\dot H^1}\|\nabla\theta\|_{L^2}\|\delta\theta\|_{\dot H^1} \le \|\delta u\|^{1/2}_{L^2}\|\nabla\delta u\|^{1/2}_{L^2}\|\nabla\theta\|_{L^2}\|\nabla\delta\theta\|_{L^2}.$$

Using twice the Young inequality for products, we obtain

$$|I_3| \le \frac{1}{4\alpha}\big(\|\delta u\|^2_{L^2} + \alpha^2\|\nabla\delta u\|^2_{L^2}\big)\|\nabla\theta\|^2_{L^2} + \frac{1}{2}\|\nabla\delta\theta\|^2_{L^2}. \tag{3.6}$$

Summing up estimates (3.1), (3.3) and (3.6), we infer that

$$\begin{aligned}\frac{d}{dt}\Big(\|\delta u\|^2_{L^2} + \alpha^2\|\nabla\delta u\|^2_{L^2} + \|\delta\theta\|^2_{L^2}\Big) + \big(\|\nabla\delta u\|^2_{L^2} + \alpha^2\|\Delta\delta u\|^2_{L^2}\big)&\\ \le g(t)\Big(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)} + \|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\Big)&,\end{aligned}$$

where $g(t) = C_1(\alpha)\|\nabla u\|^4_{L^2} + C_2(\alpha)\|\nabla\theta\|^2_{L^2}$. Omitting the dissipative positive term on the left-hand side, we obtain

$$\frac{d}{dt}\Big(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)} + \|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\Big) \le g(t)\Big(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)} + \|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\Big).$$

Since $\theta \in L^2(\mathbb{R}_+, \dot H^1)$ and $u \in L^\infty(\mathbb{R}_+, \dot H^1)$, Grönwall's lemma leads to

$$\begin{aligned}&\Big(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)} + \|\delta\theta\|^2_{L^2(\mathbb{T}^3)}\Big)\\ &\qquad \le \Big(\|\delta u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\delta u^0\|^2_{L^2(\mathbb{T}^3)} + \|\delta\theta^0\|^2_{L^2(\mathbb{T}^3)}\Big)\exp\Big(\int_0^{+\infty} g(s)\,ds\Big).\end{aligned}$$

Hence, we obtain the continuous dependence of the weak solution with respect to the initial data on $[0,T]$, $\forall\, T > 0$. In particular, the solution is unique. Since the solution is continuous in time, uniqueness over $\mathbb{R}_+$ follows. As a conclusion, we have the following

Theorem 3.1. *If the initial velocity is mean free, then the global in time weak solution dealt with in Theorem 2.1 is continuously dependent on the initial data (u^0, θ^0), on $\mathbb{R}_+$. In particular, it is unique.*

4 Existence and uniqueness of the strong solution

Taking L^2-inner product of (2.1) with $-\Delta u_n$ and (2.2) with $-\Delta\theta_n$ and summing up, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\theta_n\|^2_{L^2(\mathbb{T}^3)} + \|\nabla u_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u_n\|^2_{L^2(\mathbb{T}^3)}\Big)&\\ + \kappa\|\Delta\theta_n\|^2_{L^2(\mathbb{T}^3)} + \nu\big(\|\Delta u_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\Delta u_n\|^2_{L^2(\mathbb{T}^3)}\big)&\\ \le \big|\big((u_n\cdot\nabla)\theta_n, \Delta\theta_n\big)\big| + \big|\big((v_n\cdot\nabla)u_n, \Delta u_n\big)\big| + \big|(\theta_n e_3, \Delta u_n)\big|&.\end{aligned} \tag{4.1}$$

Note that time derivatives of $\|\nabla\theta_n\|^2_{L^2(\mathbb{T}^3)}$ and $\|\nabla u_n\|^2_{L^2(\mathbb{T}^3)}$ require initial data $(\theta^0, u^0) \in \dot{H}^1 \times \dot{H}^2$. Using in this order the Hölder inequality, inequalities (3.4) and (3.5), and the Young inequality for products with the pair $(4/3, 4)$, we obtain

$$\begin{aligned}\big|((u_n\cdot\nabla)\theta_n, \Delta\theta_n)\big| &\le \|u_n\|_{L^6(\mathbb{T}^3)}\|\nabla\theta_n\|_{L^3(\mathbb{T}^3)}\|\Delta\theta_n\|_{L^2(\mathbb{T}^3)}\\ &\le c\|u_n\|_{\dot{H}^1}\|\nabla\theta_n\|^{1/2}_{L^2}\|\nabla\theta_n\|^{1/2}_{\dot{H}^1}\|\Delta\theta_n\|_{L^2} \le C\|\nabla\theta_n\|^2_{L^2(\mathbb{T}^3)}\|u_n\|^4_{\dot{H}^1(\mathbb{T}^3)} + \frac{\kappa}{2}\|\Delta\theta_n\|^2_{L^2(\mathbb{T}^3)}.\end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\big|(\theta_n e_3, -\Delta u_n)\big| \le \|\nabla\theta_n\|^2_{L^2} + \|\nabla u_n\|^2_{L^2}.$$

First we use symmetry, then the classical properties of the bilinear term (see, for example, the first statement of assertion (iii) of Lemma 1 in [3]), after that the Poincaré inequality, and finally the Young inequality for products with the pair $(4/3, 4)$. We obtain

$$\begin{aligned}\big|(v_n\cdot\nabla)u_n, \Delta u_n)\big| = \big|\langle(v_n\cdot\nabla)\Delta u_n, u_n\rangle_{\dot{H}^{-2}}\big| &\le \|v_n\|^{1/2}_{L^2(\mathbb{T}^3)}\|v_n\|^{1/2}_{\dot{H}^1(\mathbb{T}^3)}\|\Delta^{3/2}u_n\|_{L^2(\mathbb{T}^3)}\|u_n\|_{\dot{H}^1(\mathbb{T}^3)}\\ &\le (C+\alpha^2)\|\Delta u_n\|^{1/2}_{L^2(\mathbb{T}^3)}\|\Delta^{3/2}u_n\|^{3/2}_{L^2(\mathbb{T}^3)}\|u_n\|_{\dot{H}^1(\mathbb{T}^3)}\\ &\le (C+\alpha^2)^4\alpha^{-6}\|\Delta u_n\|^2_{L^2(\mathbb{T}^3)}\|u_n\|^4_{\dot{H}^1(\mathbb{T}^3)} + \frac{1}{2}\nu\alpha^2\|\Delta^{3/2}u_n\|^2_{L^2(\mathbb{T}^3)}.\end{aligned}$$

Integrating with respect to time, we get

$$\begin{aligned}&\|\nabla\theta\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \alpha^2\|\Delta u\|^2_{L^2} + \int_0^t \big[\kappa\|\Delta\theta\|^2_{L^2} + \nu\big(\|\Delta u\|^2_{L^2} + \alpha^2\|\nabla\Delta u\|^2_{L^2}\big)\big]\,d\tau\\ &\quad\le C\int_0^t \|u\|^4_{\dot{H}^1}\Big((C+\alpha^2)^4\alpha^{-6}\|\Delta u\|^2_{L^2} + \|\nabla\theta\|^2_{L^2}\Big)\,d\tau\\ &\qquad + \int_0^t \big(\|\nabla\theta\|^2_{L^2} + \|\nabla u\|^2_{L^2}\big)\,d\tau + \|\nabla\theta^0_n\|^2_{L^2(\mathbb{T}^3)} + \|\nabla u^0_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u^0_n\|^2_{L^2(\mathbb{T}^3)}.\end{aligned}$$

Since the data $(\theta^0, u^0) \in \dot{H}^1 \times \dot{H}^2$ satisfy the hypothesis of Theorem 2.1, the energy estimate (2.9) is applicable, which leads to

$$\begin{aligned}&\|\nabla\theta_n\|^2_{L^2(\mathbb{T}^3)} + \|\nabla u_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u_n\|^2_{L^2(\mathbb{T}^3)}\\ &+ \int_0^t \Big[\kappa\|\Delta\theta_n\|^2_{L^2(\mathbb{T}^3)} + \nu\big(\|\Delta u_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\Delta u_n\|^2_{L^2(\mathbb{T}^3)}\big)\Big]\,d\tau \le C(\nu, \kappa, u^0_n, \theta^0_n)\,\frac{(C+\alpha^2)^4}{\alpha^{10}}.\end{aligned} \tag{4.2}$$

A standard compactness argument (see, e.g., [4]) completes the proof. As a conclusion, we have the following existence and uniqueness theorem.

Theorem 4.1. *Let $\theta^0 \in \dot{H}^1(\mathbb{T}^3)$ be mean free and let $u^0 \in \dot{H}^2(\mathbb{T}^3)$ be a divergence-free and mean free vector field. Then there exists a global in time unique strong solution $(u_\alpha, \theta_\alpha)$ of system (Bq_α) such that u_α belongs to $C(\mathbb{R}_+, \dot{H}^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^3(\mathbb{T}^3))$ and θ_α belongs to $C(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{T}^3))$. Moreover, this solution satisfies the energy estimate* (4.2).

Remark. As a strong solution is also weak, the uniqueness of the weak solution in Theorem 3.1 implies that of the strong solution in Theorem 4.1.

5 Convergence results

To address the convergence problem, as the regularizing parameter α goes to zero, the first step is to perform uniform estimates with respect to α. It is clear that neither the energy estimates satisfied by the weak solution nor the one satisfied by the strong solution fulfill this condition. However, as for the weak solution, the dependence of the right-hand side on α is of polynomial type, it is sufficient to suppose that there exists a fixed value of α denoted by α_0 such that $0 < \alpha \leq \alpha_0$ and to replace $\alpha = \alpha_0$ in the energy estimates, in order to obtain a uniform bound with respect to α. This can be done because α is destined to vanishe.

In contrast to the weak solution, the strong solution is problematic. In fact, the right-hand side of estimates (4.2) depends in a singular way on the parameter α and it blows up as α goes to zero. In the following, we overcome this difficulty.

Above we proved that

$$\big|\langle u_\alpha \nabla\theta_\alpha, -\Delta\theta_\alpha\rangle\big| \leq c\|u_\alpha\|_{\dot{H}^1}\|\nabla\theta_\alpha\|_{L^2}^{1/2}\|\Delta\theta_\alpha\|_{L^2}^{3/2}.$$

Using twice the Young inequality for products, respectively, with the pairs $(4, 4/3)$ and $(3/2, 3)$, we infer that

$$\big|\langle u_\alpha \nabla\theta_\alpha, -\Delta\theta_\alpha\rangle\big| \leq c\|u_\alpha\|_{\dot{H}^1}^4\|\nabla\theta_\alpha\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta\theta_\alpha\|_{L^2}^2 \leq c\big(\|u_\alpha\|_{\dot{H}^1}^6 + \|\nabla\theta_\alpha\|_{L^2}^6\big) + \frac{\kappa}{2}\|\Delta\theta_\alpha\|_{L^2}^2.$$

Also, we proved that

$$\big|\langle v_\alpha \nabla u_\alpha, -\Delta u_\alpha\rangle\big| \leq (C+\alpha^2)\|\Delta u_n\|_{L^2(\mathbb{T}^3)}^{1/2}\|\Delta^{3/2}u_n\|_{L^2(\mathbb{T}^3)}^{3/2}\|u_n\|_{\dot{H}^1(\mathbb{T}^3)}.$$

Using twice the Young inequality for products, respectively, with the pairs $(4, 4/3)$ and $(3/2, 3)$, we infer that

$$\begin{aligned}\big|\langle v_\alpha \nabla u_\alpha, -\Delta u_\alpha\rangle\big| &\leq \alpha^{-6}(C+\alpha^2)^4\|u_n\|_{\dot{H}^1(\mathbb{T}^3)}^4\|\Delta u_n\|_{L^2(\mathbb{T}^3)}^2 + \frac{\nu}{2}\,\alpha^2\|\Delta^{3/2}u_n\|_{L^2(\mathbb{T}^3)}^2\\ &\leq \|u_n\|_{\dot{H}^1(\mathbb{T}^3)}^4 + (\alpha^{-6}(C+\alpha^2)^4)^3\|\Delta u_n\|_{L^2(\mathbb{T}^3)}^6 + \frac{\nu}{2}\alpha^2\,\|\Delta^{3/2}u_n\|_{L^2(\mathbb{T}^3)}^2.\end{aligned}$$

Regarding the buoyancy force, we use the Cauchy–Schwarz inequality and the Poincaré inequality, respectively, to get

$$\big|\langle \theta_\alpha e_3, -\Delta u_\alpha\rangle\big| \leq \|\nabla\theta_\alpha\|_{L^2}^2 + C\|\Delta u_\alpha\|_{L^2}^2.$$

Proceeding as in the existence part, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\nabla\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla u_\alpha\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2(\mathbb{T}^3)}^2\big)\\ &\qquad + \frac{\kappa}{2}\|\Delta\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2 + \frac{\nu}{2}\big(\|\Delta u_\alpha\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\nabla\Delta u_\alpha\|_{L^2(\mathbb{T}^3)}^2\big)\\ &\leq c\Big(\|\nabla\theta_\alpha\|_{L^2}^6 + \|\nabla u_\alpha\|_{L^2}^6 + \alpha^6\|\Delta u_\alpha\|_{L^2}^6 + \|\nabla\theta_\alpha\|_{L^2}^2 + \|\nabla u_\alpha\|_{L^2}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2}^2\Big),\end{aligned}\tag{5.1}$$

where c is a constant independent of the parameter α. Denote

$$f(t) = \|\nabla\theta_\alpha\|_{L^2(\mathbb{T}^3)}^2 + \|\nabla u_\alpha\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\Delta u_\alpha\|_{L^2(\mathbb{T}^3)}^2.$$

Since

$$cf^3 + f \leq (c+1)(f^3+f) \leq C(f+1)^3,$$

one considers h defined by $h(t) = f(t) + 1$. The function f is a non-negative function, so $h(0) \neq 0$. The estimation above implies that

$$\frac{dh}{dt} \leq Ch^3.$$

Integrating the above ordinary differential inequality with respect to time, we deduce that for all $0 \leq t \leq \frac{1}{4Ch^2(0)}$,

$$h(t) \leq \sqrt{2}h(0).$$

Finally, we infer that for all $0 \leq t \leq T^* = \min\big(T, \frac{1}{4C(1+f(0))^2}\big)$,

$$\begin{aligned}&\|\nabla\theta_\alpha\|^2_{L^2(\mathbb{T}^3)} + \|\nabla u_\alpha\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u_\alpha\|^2_{L^2(\mathbb{T}^3)}\\ &\qquad\leq \sqrt{2}\left(\|\nabla\theta^0\|^2_{L^2(\mathbb{T}^3)} + \|\nabla u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u^0\|^2_{L^2(\mathbb{T}^3)}\right) + \sqrt{2} - 1. \quad (5.2)\end{aligned}$$

Integrating (5.1) over $(0, T^*)$ and taking into account (5.2), we obtain

$$\begin{aligned}&\int_0^{T^*} \left[\kappa\|\Delta\theta_\alpha\|^2_{L^2(\mathbb{T}^3)} + \nu\big(\|\Delta u_\alpha\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\Delta u_\alpha\|^2_{L^2(\mathbb{T}^3)}\big)\right] dt\\ &\qquad\leq \big(1 + 4C\sqrt{2}\big)\big(1 + \|\nabla\theta^0\|^2_{L^2(\mathbb{T}^3)} + \|\nabla u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u^0\|^2_{L^2(\mathbb{T}^3)}\big). \quad (5.3)\end{aligned}$$

Since the parameter α is destined to go to zero, then there exists some fixed value of α denoted by α_0 such that $0 < \alpha \leq \alpha_0$. We take $\alpha = \alpha_0$ in (5.2) and (5.3) to obtain a uniform bound with respect to α. This allows to run a classical compactness argument (see, e.g., [2]) and to obtain the following convergence theorem.

Theorem 5.1. *Let $(u_\alpha, \theta_\alpha)$ be the unique strong solution subject of Theorem 4.1. Then there exist a time $T^* > 0$, subsequences u_{α_k}, v_{α_k}, θ_{α_k} and u, θ in $L^\infty([0,T^*], \dot{H}^1(\mathbb{T}^3)) \cap L^2([0,T^*], \dot{H}^2(\mathbb{T}^3))$ such that*

(1) *u_{α_k} converges to u and θ_{α_k} converges to θ weakly in $L^2([0,T^*], \dot{H}^2(\mathbb{T}^3))$ and strongly in $L^2([0,T^*], \dot{H}^1(\mathbb{T}^3))$.*

(2) *v_{α_k} converges to u weakly in $L^2([0,T^*], \dot{H}^1(\mathbb{T}^3))$ and strongly in $L^2([0,T^*], L^2(\mathbb{T}^3))$.*

(3) *u_{α_k} converges to u and θ_{α_k} converges to θ weakly in $\dot{H}^1(\mathbb{T}^3)$ and uniformly over $[0,T^*]$.*

Moreover, (u,θ) is the unique strong solution of the Boussinesq system (Bq) on $[0,T^*]$ associated to the initial data (u^0,θ^0) and satisfies for all $t \in [0,T^*]$ the energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{T}^3)} + \|\theta(t)\|^2_{L^2(\mathbb{T}^3)} + 2\int_0^t \nu\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)} + \kappa\|\nabla\theta(\tau)\|^2_{L^2(\mathbb{T}^3)}\, d\tau \leq \|u^0\|^2_{L^2(\mathbb{T}^3)} + \|\theta^0\|^2_{L^2(\mathbb{T}^3)}.$$

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