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LIMIT CYCLES FOR A CLASS OF KUKLES TYPE DIFFERENTIAL SYSTEMS

Abstract. In this work, we study the number of limit cycles which can bifurcate from periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$ of generalized polynomial Kukles systems of the form

$$\dot{x} = -y + l(x)\, y^{2\alpha}, \quad \dot{y} = x - f(x)\, y^{2\alpha} - g(x)\, y^{2\alpha+1} - h(x)\, y^{2\alpha+2} - d_0\, y^{2\alpha+3},$$

where

$$l(x) = \epsilon l_1(x) + \epsilon^2 l_2(x), \quad f(x) = \epsilon f_1(x) + \epsilon^2 f_2(x),$$
$$g(x) = \epsilon g_1(x) + \epsilon^2 g_2(x), \;\; h(x) = \epsilon h_1(x) + \epsilon^2 h_2(x) \;\text{ and }\; d_0 = \epsilon d_0^1 + \epsilon^2 d_0^2.$$

$l_k(x)$, $f_k(x)$, $g_k(x)$ and $h_k(x)$ have degree m, n_1, n_2 and n_3, respectively, $d_0^k \neq 0$ is a real number for each $k = 1, 2$, α is a positive integer and ϵ is a small parameter. The main tool of this study is the averaging method of the first and second order.

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რეზიუმე. ნაშრომში შესწავლილია ზღვრული ციკლების რაოდენობა, რომლებიც შეიძლება განშტოვდეს წრფივი $\dot{x} = -y$, $\dot{y} = x$ ცენტრის პერიოდული ორბიტებიდან შემდეგი სახის განზოგადებული პოლინომიალური კუკლესის სისტემისთვის

$$\dot{x} = -y + l(x)\, y^{2\alpha}, \quad \dot{y} = x - f(x)\, y^{2\alpha} - g(x)\, y^{2\alpha+1} - (x)\, y^{2\alpha+2} - d_0\, y^{2\alpha+3},$$

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$$l(x) = \epsilon l_1(x) + \epsilon^2 l_2(x), \quad f(x) = \epsilon f_1(x) + \epsilon^2 f_2(x),$$
$$g(x) = \epsilon g_1(x) + \epsilon^2 g_2(x), \;\; h(x) = \epsilon h_1(x) + \epsilon^2 h_2(x) \;\text{ და }\; d_0 = \epsilon d_0^1 + \epsilon^2 d_0^2,$$

$l_k(x)$, $f_k(x)$, $g_k(x)$ და $h_k(x)$ არის m, n_1, n_2 და n_3 რიგის პოლინომები, შესაბამისად, $d_0^k \neq 0$ ნამდვილი რიცხვია ყოველი $k = 1, 2$-თვის, α დადებითი მთელი რიცხვია და ϵ – მცირე პარამეტრი. კვლევის მთავარ ინსტრუმენტს წარმოადგენს პირველი და მეორე რიგის გასაშუალოების მეთოდი.

1 Introduction and main results

The main open problem in the qualitative theory of real planar differential systems is the determination of limit cycles which is related to the second part of the 16th Hilbert problem [8]. Recall that a limit cycle is an isolated closed orbit defined by Poincaré [16, 17]. The knowledge of the existence or the absence of periodic solutions is very important for understanding the dynamics of the differential systems. A classical way to obtain limit cycles is perturbing a system which has a center. In this case, the perturbed system displays limit cycles that can bifurcate either from the center, or from some of the periodic orbits around the center [18]. The number of limit cycles which bifurcate from periodic orbits of a differential system with a center (linear or not) has been studied by using many distinct tools, the first method is the inverse integral factor [11], the second method is the Lyapunov constants [12], the third method is the Melnikov function [9, 21], and the last method is the averaging theory (see, e.g., [3, 4, 6] and the references therein).

In different works, the limit cycles problem and the center problem are concentrated on specific classes of systems; for instance, we refer to the classical Kukles system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + a_0 y + a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3. \end{cases} \tag{1.1}$$

In [10], Kukles examined the conditions under which the origin of system (1.1) is a center. In [5], it was shown that at most five limit cycles bifurcate from the origin of system (1.1). In [19], Sadovskii solved the center focus problem for systems (1.1) when $a_2 a_7 \neq 0$ and proved that the system can have seven limit cycles.

In [14], Llibre et al., using the averaging theory of the first and second order, studied the generalized Kukles system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - f(x) - g(x)y - h(x)y^2 - d_0 y^3. \end{cases}$$

Another class of generalized Kukles system which can be written as

$$\begin{cases} \dot{x} = -y + l(x), \\ \dot{y} = x - f(x) - g(x)y - h(x)y^2 - d_0 y^3, \end{cases} \tag{1.2}$$

where $l(x)$, $f(x)$, $g(x)$ and $h(x)$ have degrees m, n_1, n_2 and n_3, respectively, $d_0 \neq 0$ is a real number, was studied in [15].

Limit cycles of several generalisations of the Kukles system have been also obtained using averaging theory (see, e.g., [1, 2] and the references therein).

In the present paper, using the averaging theory, we study the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center $\dot{x} = -y$, $\dot{y} = x$ of the more generalized Kukles system

$$\begin{cases} \dot{x} = -y + l(x)\, y^{2\alpha}, \\ \dot{y} = x - f(x)\, y^{2\alpha} - g(x)\, y^{2\alpha+1} - h(x)\, y^{2\alpha+2} - d_0\, y^{2\alpha+3}, \end{cases} \tag{1.3}$$

where

$$f(x) = \varepsilon f_1(x) + \varepsilon^2 f_2(x), \quad g(x) = \varepsilon g_1(x) + \varepsilon^2 g_2(x),$$
$$h(x) = \varepsilon h_1(x) + \varepsilon^2 h_2(x), \;\; l(x) = \varepsilon l_1(x) + \varepsilon^2 l_2(x) \;\text{ and }\; d_0 = \epsilon d_0^1 + \epsilon^2 d_0^2,$$

$f_k(x)$, $g_k(x)$, $h_k(x)$ and $l_k(x)$ have degree n_1, n_2, n_3 and m, respectively, $d_0^k \neq 0$ is a real number for each $k = 1, 2$, $\alpha \in \mathbb{N}$ and ε is a small parameter. Note that system (1.3) was studied in [15] when $\alpha = 0$.

The main result of this paper is the following statement.

Theorem 1.1. *For $|\epsilon| \neq 0$ sufficiently small, the maximum number of limit cycles of the generalized Kukles polynomial differential system* (1.3) *bifurcating from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$ using the averaging theory*

(a) *of the first order is*

$$\lambda_1 = \max\left\{\left[\frac{m-1}{2}\right], \left[\frac{n_2}{2}\right], 1\right\},$$

(b) *of the second order is*

$$\lambda = \max\{\Lambda_1, \Lambda_2 + 1, \Lambda_3 + 2\},$$

where

$$\Lambda_1 = \max\left\{\left[\frac{n_1}{2}\right] + \left[\frac{n_2-1}{2}\right] + \alpha, \left[\frac{n_1}{2}\right] + \left[\frac{m}{2}\right] - 1 + \alpha, \left[\frac{n_3}{2}\right] + \left[\frac{m}{2}\right] + \alpha, \left[\frac{n_2}{2}\right], \left[\frac{m-1}{2}\right], \left[\frac{n_1-1}{2}\right] + \mu + \alpha, 1\right\},$$

$$\Lambda_2 = \max\left\{\left[\frac{n_1-1}{2}\right] + \alpha, \left[\frac{n_2-1}{2}\right] + \left[\frac{n_3}{2}\right] + \alpha, \left[\frac{n_3-1}{2}\right] + \mu + \alpha\right\},$$

$$\Lambda_3 = \left[\frac{n_3-1}{2}\right] + \alpha,$$

and

$$\mu = \min\left\{\left[\frac{m-1}{2}\right], \left[\frac{n_2}{2}\right]\right\}.$$

The proofs of statements (a) and (b) of Theorem 1.1 are given in Sections 3 and 4, respectively.

In [15], it was shown that there exists a generalized Kukles system (1.3) with $\alpha = 0$ having at least

$$\lambda_2 = \max\left\{\left[\frac{n_1}{2}\right] + \left[\frac{n_2-1}{2}\right], \left[\frac{n_1}{2}\right] + \left[\frac{m}{2}\right] - 1, \left[\frac{n_1+1}{2}\right], \left[\frac{n_3+3}{2}\right], \left[\frac{n_3}{2}\right] + \left[\frac{m}{2}\right], \left[\frac{n_2+1}{2}\right] + \left[\frac{n_3}{2}\right], \left[\frac{n_2}{2}\right], \left[\frac{m-1}{2}\right], \left[\frac{n_1-1}{2}\right] + \mu, \left[\frac{n_3+1}{2}\right] + \mu, 1\right\}$$

limit cycles. The result in Theorem 1.1 improves this lower estimate ($\lambda > \lambda_2$ for all $n_1 \geq 1$, $n_2 \geq 2$, $n_3 \geq 1$, $m \geq 2$ and $\alpha \geq 1$). For each fixed $n_1 \geq 1$, $n_2 \geq 2$, $n_3 \geq 1$ and $m \geq 2$, there exists $\alpha_0 \geq 1$ such that $\lambda > \lambda_2$ for all $\alpha \geq \alpha_0$.

In Section 2, we introduce the averaging theory of the first and second order.

2 Averaging theory of the first and second order

In this section, we summarize the main results of the averaging theory of the first and second order for computing limit cycles of the generalized Kukles polynomial differential system (1.3).

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{2.1}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}$. Assume that the following conditions hold:

(i) $F_1(t,x) \in C^2(D)$, $F_2(t,x) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R, $D_x F_1$ are locally Lipschitz with respect to x and R is twice differentiable with respect to ε. We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)\,ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \left[D_x F_1(s,z) y_1(s,z) + F_2(s,z)\right] ds,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)\,dt,$$

(ii) For $V \subset D$, an open and bounded set, and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that

$$F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0 \text{ and } d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0.$$

Then for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $\varphi(t, \varepsilon)$ of system (2.1) such that $\varphi(0, \varepsilon) \to a_\varepsilon$ as $\varepsilon \to 0$. The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} : V \to \mathbb{R}$ at the fixed point a_ε is not zero. A sufficient condition for this inequality to be true is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}$ are mainly the zeros of F_{10} for ϵ sufficiently small. In this case, the previous result provides the averaging theory of the first order.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case, the previous result provides the averaging theory of the second order.

For a general introduction to the averaging theory, see [13, 20].

3 Proof of statement (a) of Theorem 1.1

In order to apply the first order averaging method, we write system (1.3) with $k = 1$ in polar coordinates (θ, r), where $x = r\cos\theta$, $y = r\sin\theta$ and $r > 0$. In this context, we take

$$f_1(x) = \sum_{i=0}^{n_1} a_i x^i, \quad g_1(x) = \sum_{i=0}^{n_2} b_i x^i, \quad h_1(x) = \sum_{i=0}^{n_3} c_i x^i \text{ and } l_1(x) = \sum_{i=0}^{m} e_i x^i, \tag{3.1}$$

then system (1.3) can be written as

$$\begin{cases} \dot{r} = \epsilon\Big(r^{2\alpha}\cos\theta\sin^{2\alpha}\theta \displaystyle\sum_{i=0}^{m} e_i B_{i,0}(\theta)\, r^i - r^{2\alpha}\sin^{2\alpha+1}\theta\, P_1(\theta, r)\Big), \\ \dot{\theta} = 1 - \dfrac{\epsilon}{r}\Big(r^{2\alpha}\sin^{2\alpha+1}\theta \displaystyle\sum_{i=0}^{m} e_i B_{i,0}(\theta)\, r^i + r^{2\alpha}\sin^{2\alpha}\theta\cos\theta\, P_1(\theta, r)\Big), \end{cases} \tag{3.2}$$

where

$$P_1(\theta, r) = \sum_{i=0}^{n_1} a_i B_{i,0}(\theta)\, r^i + \sum_{i=0}^{n_2} b_i B_{i,1}(\theta)\, r^{i+1} + \sum_{i=0}^{n_3} c_i B_{i,2}(\theta) r^{i+2} + d_0^1\, r^3 B_{0,3}(\theta)$$

and

$$B_{i,j}(\theta) = \cos^i\theta \sin^j\theta.$$

Now, taking θ as the new independent variable, system (3.2) becomes

$$\begin{aligned} \frac{dr}{d\theta} &= \epsilon\Big(r^{2\alpha}\cos\theta\sin^{2\alpha}\theta \sum_{i=0}^{m} e_i B_{i,0}(\theta)\, r^i - r^{2\alpha}\sin^{2\alpha+1}\theta\, P_1(\theta, r)\Big) + O(\epsilon^2) \\ &= \epsilon F_1(\theta, r) + O(\epsilon^2). \end{aligned}$$

Therefore, based on Section 2 we must study the simple positive zeros of the function

$$\begin{aligned} 2\pi F_{10}(r) = \sum_{i=0}^{m} e_i A_{i+1,2\alpha}(2\pi)\, r^{i+2\alpha} - \sum_{i=0}^{n_1} a_i A_{i,2\alpha+1}(2\pi)\, r^{i+2\alpha} - \sum_{i=0}^{n_2} b_i A_{i,2\alpha+2}(2\pi) r^{i+2\alpha+1} \\ - \sum_{i=0}^{n_3} c_i A_{i,2\alpha+3}(2\pi) r^{i+2\alpha+2} - d_0^1\, r^{2\alpha+3} A_{0,2\alpha+4}(2\pi), \end{aligned} \tag{3.3}$$

where

$$A_{i,j}(2\pi) = \int_0^{2\pi} B_{i,j}(\theta)\, d\theta.$$

In the next proposition, we obtain the exact expression of $F_{10}(r)$.

Proposition 3.1. *$F_{10}(r)$ is a polynomial in r given by*

$$F_{10}(r) = \frac{r^{2\alpha+1}}{4}\Bigg(\sum_{i=0}^{[\frac{m-1}{2}]} \frac{2i+1}{i+\alpha+1}\, e_{2i+1} r^{2i}\sigma_{2i,2\alpha} - \sum_{i=0}^{[\frac{n_2}{2}]} \frac{2\alpha+1}{i+\alpha+1}\, b_{2i} r^{2i}\sigma_{2i,2\alpha} - d_0\,\frac{(2\alpha+3)(2\alpha+1)}{(2\alpha+4)(\alpha+1)}\, r^2\sigma_{0,2\alpha}\Bigg), \quad (3.4)$$

where

$$\sigma_{2i,2\alpha} = \frac{(2\alpha-1)(2\alpha-3)\cdots 1}{(2\alpha+2i)(2\alpha+2i-4)\cdots(2i+2)}\left(\frac{1}{2^{2i-1}}\right)\binom{2i}{i}.$$

Proof. From (3.3) and for simplifying the expression of $F_{10}(r)$, we use the following integrals

$$A_{i,j}(2\pi) = \begin{cases} 0 & \text{if } i \text{ is odd or } j \text{ is odd,} \\ \dfrac{(j-1)(j-3)\cdots 1}{(j+i)(j+i-2)\cdots(i+2)}\,\dfrac{1}{2^{i-1}}\dbinom{i}{i/2}\pi & \text{if } i \text{ and } j \text{ are even} \end{cases} \quad (3.5)$$

and

$$\begin{aligned} \int_0^\theta \cos^i t \sin^\alpha t\, dt &= -\frac{\cos^{i-1}\theta \sin^{\alpha+1}\theta}{i+\alpha} + \frac{i-1}{i+\alpha}\int_0^\theta \cos^{i-2} t \sin^\alpha t\, dt \\ &= -\frac{\cos^{i+1}\theta \sin^{\alpha-1}\theta}{i+\alpha} + \frac{\alpha-1}{i+\alpha}\int_0^\theta \cos^{i} t \sin^{\alpha-2} t\, dt \end{aligned} \quad (3.6)$$

(for more details see [7]). We obtain

$$\begin{aligned} F_{10}(r) &= \frac{r^{2\alpha+1}}{2\pi}\Bigg(\sum_{i=0}^{[\frac{m-1}{2}]} e_{2i+1} r^{2i} A_{2i+2,2\alpha}(2\pi) - \sum_{i=0}^{[\frac{n_2}{2}]} b_{2i} r^{2i} A_{2i,2\alpha+2}(2\pi) - d_0\, r^2 A_{0,2\alpha+4}(2\pi)\Bigg) \\ &= \frac{r^{2\alpha+1}}{4\pi}\Bigg(\sum_{i=0}^{[\frac{m-1}{2}]} \frac{2i+1}{i+\alpha+1}\, e_{2i+1} r^{2i} A_{2i,2\alpha}(2\pi) \\ &\qquad - \sum_{i=0}^{[\frac{n_2}{2}]} \frac{2\alpha+1}{i+\alpha+1}\, b_{2i} r^{2i} A_{2i,2\alpha}(2\pi) - d_0\,\frac{(2\alpha+3)(2\alpha+1)}{(2\alpha+4)(\alpha+1)}\, r^2 A_{0,2\alpha}(2\pi)\Bigg). \end{aligned}$$

This completes the proof of Proposition 3.1. □

From Proposition 3.1, the polynomial $F_{10}(r)$ has at most $\{[\frac{m-1}{2}], [\frac{n_2}{2}], 1\}$ positive roots. Hence statement (a) of Theorem 1.1 is proved.

4 Proof of statement (b) of Theorem 1.1

We write $f_1(x)$, $g_1(x)$, $h_1(x)$ and $l_1(x)$ as in (3.1), and

$$f_2(x) = \sum_{i=0}^{n_1} p_i x^i, \quad g_2(x) = \sum_{i=0}^{n_2} q_i x^i, \quad h_2(x) = \sum_{i=0}^{n_3} s_i x^i, \quad l_2(x) = \sum_{i=0}^{m} v_i x^i.$$

System (1.3) in the polar coordinates (r,θ) with $r>0$ becomes

$$\begin{cases} \dot r = \epsilon\Big(r^{2\alpha}\cos\theta\sin^{2\alpha}\theta\displaystyle\sum_{i=0}^{m} e_iB_{i,0}(\theta)\,r^i - r^{2\alpha}\sin^{2\alpha+1}\theta\,P_1(\theta,r)\Big) \\ \qquad\qquad +\epsilon^2\Big(r^{2\alpha}\cos\theta\sin^{2\alpha}\theta\displaystyle\sum_{i=0}^{m} v_iB_{i,0}(\theta)\,r^i - r^{2\alpha}\sin^{2\alpha+1}\theta\,P_2(\theta,r)\Big), \\ \dot\theta = 1-\dfrac{\epsilon}{r}\Big(r^{2\alpha}\sin^{2\alpha+1}\theta\displaystyle\sum_{i=0}^{m} e_iB_{i,0}(\theta)\,r^i + r^{2\alpha}\sin^{2\alpha}\theta\cos\theta\,P_1(\theta,r)\Big) \\ \qquad\qquad -\dfrac{\epsilon^2}{r}\Big(r^{2\alpha}\sin^{2\alpha+1}\theta\displaystyle\sum_{i=0}^{m} v_iB_{i,0}(\theta)\,r^i + r^{2\alpha}\sin^{2\alpha}\theta\cos\theta\,P_2(\theta,r)\Big), \end{cases} \tag{4.1}$$

where

$$P_2(\theta,r)=\sum_{i=0}^{n_1} p_iB_{i,0}(\theta)\,r^i+\sum_{i=0}^{n_2} q_iB_{i,1}(\theta)\,r^{i+1}+\sum_{i=0}^{n_3} s_iB_{i,2}(\theta)r^{i+2}+d_0^2\,r^3B_{0,3}(\theta).$$

Taking θ as the new independent variable, system (4.1) becomes

$$\frac{dr}{d\theta}=\epsilon F_1(\theta,r)+\epsilon^2F_2(r,\theta)+O(\epsilon^3),$$

where

$$F_1(\theta,r)=r^{2\alpha}\cos\theta\,\sin^{2\alpha}\theta\sum_{i=0}^{m} e_iB_{i,0}(\theta)\,r^i-r^{2\alpha}\sin^{2\alpha+1}\theta\,P_1(\theta,r), \tag{4.2}$$

$$F_2(r,\theta)=I(r,\theta)+II(r,\theta), \tag{4.3}$$

with

$$I(r,\theta)=r^{2\alpha}\cos\theta\,\sin^{2\alpha}\theta\sum_{i=0}^{m} v_iB_{i,0}(\theta)\,r^i-r^{2\alpha}\sin^{2\alpha+1}\theta\,P_2(\theta,r), \tag{4.4}$$

$$\begin{aligned} II(r,\theta)=\frac{1}{r}&\Big(r^{2\alpha}\cos\theta\,\sin^{2\alpha}\theta\sum_{i=0}^{m} e_iB_{i,0}(\theta)\,r^i-r^{2\alpha}\sin^{2\alpha+1}\theta\,P_1(\theta,r)\Big)\\ &\times\Big(r^{2\alpha}\sin^{2\alpha+1}\theta\sum_{i=0}^{m} e_iB_{i,0}(\theta)\,r^i+r^{2\alpha}\sin^{2\alpha}\theta\cos\theta\,P_1(\theta,r)\Big). \end{aligned} \tag{4.5}$$

In order to compute $F_{20}(r)$, we need $F_{10}(r)$ to be identically zero. From (3.4), $F_{10}\equiv 0$ if and only if

$$\begin{cases} e_{2i+1}=\dfrac{2\alpha+1}{2i+1}\,b_{2i}, & 0\le i\le\mu\ \text{ and }\ i\ne 1,\\ b_2=\dfrac{3}{2\alpha+1}\,e_3-d_0(2\alpha+3), & i=1,\\ b_{2i}=e_{2i+1}=0, & \mu+1\le i\le\lambda_1, \end{cases} \tag{4.6}$$

where

$$\mu=\min\Big\{\Big[\frac{m-1}{2}\Big],\Big[\frac{n_2}{2}\Big]\Big\},\quad \lambda_1=\max\Big\{\Big[\frac{m-1}{2}\Big],\Big[\frac{n_2}{2}\Big],1\Big\}.$$

First, using (4.6) and by substituting in (4.2), we have

$$
\begin{aligned}
F_1(\theta,r) = & \sum_{i=0}^{[\frac{m}{2}]} e_{2i}\, r^{2i+2\alpha} B_{2i+1,2\alpha}(\theta) + e_3 r^{2\alpha+3} B_{4,2\alpha}(\theta) \\
& - \sum_{i=0}^{n_1} a_i B_{i,2\alpha+1}(\theta)\, r^{i+2\alpha} - \sum_{i=0}^{[\frac{n_2-1}{2}]} b_{2i+1}\, r^{2i+2\alpha+2} B_{2i+1,2\alpha+2}(\theta) \\
& - \sum_{i=0}^{n_3} c_i r^{i+2\alpha+2} B_{i,2\alpha+3}(\theta) - b_2\, r^{2\alpha+3} B_{2,2\alpha+2}(\theta) - d_0^1\, r^{2\alpha+3} B_{0,2\alpha+4}(\theta) \\
& - \sum_{i=0}^{\mu} b_{2i}\, r^{2i+2\alpha+1} B_{2i,2\alpha}(\theta) + \sum_{i=0}^{\mu} \frac{2i+2\alpha+2}{2i+1}\, b_{2i}\, r^{2i+2\alpha+1} B_{2i+2,2\alpha}(\theta).
\end{aligned} \tag{4.7}
$$

Then

$$
\begin{aligned}
\frac{dF_1(r,\theta)}{dr} = & \sum_{i=0}^{[\frac{m}{2}]} (2i+2\alpha) e_{2i}\, r^{2i+2\alpha-1} B_{2i+1,2\alpha}(\theta) + (2\alpha+3) e_3 r^{2\alpha+2} B_{4,2\alpha}(\theta) \\
& - \sum_{i=0}^{n_1} (i+2\alpha) a_i\, r^{i+2\alpha-1} B_{i,2\alpha+1}(\theta) - \sum_{i=0}^{[\frac{n_2-1}{2}]} (2i+2\alpha+2) b_{2i+1}\, r^{2i+2\alpha+1} B_{2i+1,2\alpha+2}(\theta) \\
& - \sum_{i=0}^{n_3} (i+2\alpha+2) c_i r^{i+2\alpha+1} B_{i,2\alpha+3}(\theta) - (2\alpha+3) b_2 r^{2\alpha+2} B_{2,2\alpha+2}(\theta) - (2\alpha+3) d_0^1\, r^{2\alpha+2} B_{0,2\alpha+4}(\theta) \\
& - \sum_{i=0}^{\mu} \frac{2i+2\alpha+1}{2i+1}\, b_{2i} r^{2i+2\alpha} \big((2i+1) B_{2i,2\alpha+2}(\theta) - (2\alpha+1) B_{2i+2,2\alpha}(\theta)\big).
\end{aligned} \tag{4.8}
$$

In the next proposition we obtain

$$
y(r,\theta) = \int_0^\theta F_1(s,r)\, ds.
$$

Proposition 4.1.

$$
\begin{aligned}
y(r,\theta) = & \sum_{i=0}^{[\frac{m}{2}]} e_{2i}\, r^{2i+2\alpha} A_{2i+1,2\alpha}(\theta) - \sum_{i=0}^{n_1} a_i A_{i,2\alpha+1}(\theta)\, r^{i+2\alpha} \\
& - \sum_{i=0}^{[\frac{n_2-1}{2}]} b_{2i+1} r^{2i+2\alpha+2} A_{2i+1,2\alpha+2}(\theta) - \sum_{i=0}^{n_3} c_i r^{i+2\alpha+2} A_{i,2\alpha+3}(\theta) \\
& - \sum_{i=0}^{\mu} b_{2i} r^{2i+2\alpha+1} T_i(\theta) + e_3 r^{2\alpha+3} S(\theta) + d_0^1 r^{2\alpha+3} R(\theta),
\end{aligned} \tag{4.9}
$$

where

$$
\begin{aligned}
T_i(\theta) = & -\frac{1}{2\alpha+2i} \Big(B_{2i+1,2\alpha-1}(\theta) + \sum_{l=1}^{\alpha-1} \gamma_{2i,2\alpha} B_{2i+1,2\alpha-2l-1}(\theta) \Big) \\
& + \frac{\delta_{2i,2\alpha}}{2i} \Big(B_{2i-1,1}(\theta) + \sum_{l=1}^{i-1} \eta_{2i,0} B_{2i-2l-1,1}(\theta) \Big) \\
& + \frac{1}{2i+1} \Big(B_{2i+3,2\alpha-1}(\theta) + \sum_{l=1}^{\alpha-1} \gamma_{2i+2,2\alpha} B_{2i+3,2\alpha-2l-1}(\theta) \Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\alpha+i+1}{(2i+1)(i+1)}\, \delta_{2i+2,2\alpha} \Big(B_{2i+1,1}(\theta) + \sum_{l=1}^{i} \eta_{2i+2,0} B_{2i-2l+1,1}(\theta) \Big), \\
S(\theta) = {}& \frac{3}{(2\alpha+1)(2\alpha+4)} \Big(B_{3,2\alpha-1}(\theta) + \sum_{l=1}^{\alpha} \gamma_{2,2\alpha+2} B_{3,2\alpha-2l-1}(\theta) \Big) \\
&- \frac{3}{2(2\alpha+1)}\, B_{1,1}(\theta) - \frac{1}{(2\alpha+4)} \Big(B_{5,2\alpha-1}(\theta) + \sum_{l=1}^{\alpha-1} \gamma_{4,2\alpha} B_{5,2\alpha-2l-1}(\theta) \Big) \\
&+ \frac{\delta_{4,2\alpha}}{2} \Big(B_{1,1}(\theta) + \frac{1}{4} \left(B_{3,1}(\theta) - B_{1,3}(\theta) \right) \Big), \\
R(\theta) = {}& (2\alpha+3) \Big[- \frac{1}{(2\alpha+4)} \Big(B_{3,2\alpha-1}(\theta) + \sum_{l=1}^{\alpha} \gamma_{2,2\alpha+2} B_{3,2\alpha-2l-1}(\theta) \Big) + \frac{\delta_{2,2\alpha+2}}{2} B_{1,1}(\theta) \Big] \\
&- \frac{1}{(2\alpha+4)} \Big(B_{1,2\alpha+3}(\theta) + \sum_{l=1}^{\alpha+1} \gamma_{0,2\alpha+4} B_{1,2\alpha-2l+3}(\theta) \Big),
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_{i,2\alpha} &= \frac{(2\alpha-1)(2\alpha-3)\cdots(2\alpha-2l+1)}{(2\alpha+i-2)(2\alpha+i-4)\cdots(2\alpha+i-2l)}, \\
\delta_{i,2\alpha} &= \frac{(2\alpha-1)(2\alpha-3)\cdots 1}{(2\alpha+i)(2\alpha+i-2)\cdots(i+2)}, \\
\eta_{2i,0} &= \frac{(2i-1)(2i-3)\cdots(2i-2l+1)}{2^l(i-1)(i-2)\cdots(i-l)}.
\end{aligned}
$$

Proof. We have

$$
T_i(\theta) = A_{2i,2\alpha}(\theta) - \frac{2i+2\alpha+2}{2i+1}\, A_{2i+2,2\alpha}(\theta), \quad S(\theta) = A_{4,2\alpha}(\theta) - \frac{3}{2\alpha+1}\, A_{2,2\alpha+2}(\theta)
$$

and

$$
R(\theta) = (2\alpha+3) A_{2,2\alpha+2}(\theta) - A_{0,2\alpha+4}(\theta).
$$

Take into account that

$$
\delta_{2i,2\alpha}\, \frac{1}{2^i} \binom{2i}{i} \theta - \frac{2i+2\alpha+2}{2i+1}\, \delta_{2i+2,2\alpha} = 0, \quad -\frac{3}{2\alpha+3}\, \delta_{2,2\alpha+2} \left(\frac{1}{2}\, \theta \right) + \delta_{4,2\alpha} \left(\frac{1}{2}\, \theta \right) = 0
$$

and

$$
\frac{2\alpha+3}{2}\, \delta_{2,2\alpha+2} \theta - \frac{(2\alpha+3)!!}{2^{\alpha+2}(\alpha+2)!}\, \theta = 0.
$$

Thus Proposition 4.1 follows. □

Now, we determine the corresponding function

$$
F_{20}(r) = F_{20}^1(r) + F_{20}^2(r)
$$

with

$$
F_{20}^1(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r}\, F_1(\theta, r) y(\theta, r)\, d\theta \ \text{ and } \ F_{20}^2(r) = \frac{1}{2\pi} \int_0^{2\pi} F_2(\theta, r)\, d\theta.
$$

In order to obtain $F_{20}^1(r)$, it is necessary to evaluate the integrals of the form

$$
\int_0^{2\pi} B_{p,q}(\theta) A_{i,j}(\theta)\, d\theta.
$$

In the following lemma, we compute these integrals.

Lemma 4.1. *Let*

$$\Phi^{i,j}_{p,q}(2\pi) = \int_0^{2\pi} B_{p,q}(\theta)A_{i,j}(\theta)\,d\theta.$$

Then the following equalities hold:

(a) *The integral* $\Phi^{2j+1,2\alpha}_{p,q}(2\pi)$ *is zero if* p *is odd or* q *is even and it is equal to*

$$-\frac{1}{2\alpha+2j+1}\Big(A_{p+2j+2,q+2\alpha-1}(2\pi)+\sum_{l=1}^{\alpha-1}\gamma_{2j+1,2\alpha}A_{p+2j+2,q+2\alpha-2l-1}(2\pi)\Big)+\delta_{2j+1,2\alpha}\Phi^{2j+1,0}_{p,q}(2\pi),$$

where

$$\Phi^{2j+1,0}_{p,q}(2\pi) = \frac{1}{2j+1}\Big(A_{p+2j,q+1}(2\pi)+\sum_{l=1}^{j-1}\frac{2^{l+1}j(j-1)\cdots(j-l)}{(2j-1)(2j-3)\cdots(2j-2l-1)}A_{p+2j-2l-2,q+1}(2\pi)\Big),$$

if p *is even and* q *is odd.*

(b) *The integral* $\Phi^{2j,2\alpha+1}_{p,q}(2\pi)$ *is zero if* p *is even or* q *is odd and it is equal to*

$$-\frac{1}{2\alpha+2j+1}\Big(A_{p+2j+1,q+2\alpha}(2\pi)+\sum_{l=1}^{\alpha-1}\beta_{2j,2\alpha+1}A_{p+2j+1,q+2\alpha-2l}(2\pi)\Big),$$

where

$$\beta_{2j,2\alpha+1} = \frac{2^l\alpha(\alpha-1)\cdots(\alpha-l+1)}{(2\alpha+2j-1)(2\alpha+2j-3)\cdots(2\alpha+2j-2l+1)},$$

if p *is odd and* q *is even.*

(c) *The integral* $\Phi^{2j+1,2\alpha+1}_{p,q}(2\pi)$ *is zero if* p *is even or* q *is odd and it is equal to*

$$-\frac{1}{2\alpha+2j+2}\Big(A_{p+2j+2,q+2\alpha}(2\pi)+\sum_{l=1}^{\alpha-1}\beta_{2j+1,2\alpha+1}A_{p+2j+2,q+2\alpha+q-2l}(2\pi)\Big),$$

where

$$\beta_{2j+1,2\alpha+1} = \frac{2^l\alpha(\alpha-1)\cdots(\alpha-l+1)}{(2\alpha+2j)(2\alpha+2j-2)\cdots(2\alpha+2j-2l+2)}\,,$$

if p *and* q *are both even.*

(f) *The integral*

$$S^{j,\alpha}_{p,q}(2\pi) = \int_0^{2\pi} B_{p,q}(\theta)T_j(\theta)\,d\theta = \Phi^{2j,2\alpha}_{p,q}(2\pi) - \frac{2j+2\alpha+2}{2j+1}\,\Phi^{2j+2,2\alpha}_{p,q}(2\pi)$$

is zero if either p *or* q *is even and it is equal to*

$$-\frac{1}{2\alpha+2j}\Big(A_{p+2j+1,q+2\alpha-1}(2\pi)+\sum_{l=1}^{\alpha-1}\gamma_{2j,2\alpha}A_{p+2j+1,q+2\alpha-2l-1}(2\pi)\Big)$$
$$+\frac{\delta_{2j,2\alpha}}{2j}\Big(A_{p+2j-1,q+1}(2\pi)+\sum_{l=1}^{j-1}\eta_{2j,0}A_{p+2j-2l-1,q+1}(2\pi)\Big)$$

$$+\frac{1}{2j+1}\Big(A_{p+2j+3,q+2\alpha-1}(2\pi)+\sum_{l=1}^{\alpha-1}\gamma_{2j+2,2\alpha}A_{p+2j+3,q+2\alpha-2l-1}(2\pi)\Big)$$
$$-\frac{\alpha+j+1}{(2j+1)(j+1)}\delta_{2j+2,2\alpha}\Big(A_{p+2j+1,q+1}(2\pi)+\sum_{l=1}^{j}\eta_{2j+2,0}A_{p+2j-2l-1,q+1}(2\pi)\Big)$$

if p and q are both odd.

(g) *The integral*

$$T^{\alpha}_{p,q}(2\pi)=\int_0^{2\pi}B_{p,q}(\theta)S(\theta)\,d\theta=\Phi^{4,2\alpha}_{p,q}(2\pi)-\frac{3}{2\alpha+1}\,\Phi^{2\alpha+2,2\alpha}_{p,q}(2\pi)$$

is zero if either p or q is even and it is equal to

$$\frac{3}{(2\alpha+1)(2\alpha+4)}\Big(A_{p+3,q+2\alpha-1}(2\pi)+\sum_{l=1}^{\alpha}\gamma_{2,2\alpha+2}A_{p+3,q+2\alpha-2l-1}(2\pi)\Big)$$
$$-\frac{3}{2(2\alpha+1)}A_{p+1,q+1}(2\pi)-\frac{1}{(2\alpha+4)}\Big(A_{p+5,q+2\alpha-1}(2\pi)+\sum_{l=1}^{\alpha-1}\gamma_{4,2\alpha}A_{p+5,q+2\alpha-1}(2\pi)\Big)$$
$$+\frac{\delta_{4,2\alpha}}{2}\Big(A_{p+1,q+1}(2\pi)+\frac{1}{4}\left(A_{p+3,q+1}(2\pi)-A_{p+1,q+3}(2\pi)\right)\Big)$$

if p and q are both odd.

(h) *The integral*

$$U^{\alpha}_{p,q}(2\pi)=\int_0^{2\pi}B_{p,q}(\theta)R(\theta)\,d\theta=(2\alpha+3)\Phi^{2,2\alpha+2}_{p,q}(2\pi)-\Phi^{0,2\alpha+4}_{p,q}(2\pi)$$

is zero if either p or q is even and it is equal to

$$(2\alpha+3)\Big[-\frac{1}{(2\alpha+4)}\Big(A_{p+3,q+2\alpha+1}(2\pi)$$
$$+\sum_{l=1}^{\alpha}\gamma_{2,2\alpha+2}A_{p+3,q+2\alpha-2l+1}(2\pi)\Big)+\frac{\delta_{2,2\alpha+2}}{2}A_{p+1,q+1}(2\pi)\Big]$$
$$-\frac{1}{(2\alpha+4)}\Big(A_{p+1,q+2\alpha+3}(2\pi)+\sum_{l=1}^{\alpha+1}\gamma_{0,2\alpha+4}A_{p+1,q+2\alpha-2l+3}(2\pi)\Big)$$

if p and q are both odd.

Proof. The six equalities are easily deduced by direct calculation (for more details see [7]). □

Lemma 4.2. *The integral $F^1_{20}(r)$ is given by*

$$2\pi F^1_{20}(r)=\sum_{i=0}^{[\frac{n_1}{2}]}\sum_{j=0}^{[\frac{m}{2}]}H^1_{i,j,\alpha}(2\pi)r^{2i+2j+4\alpha-1}+\sum_{i=0}^{[\frac{n_3}{2}]}\sum_{j=0}^{[\frac{m}{2}]}H^2_{i,j,\alpha}(2\pi)r^{2i+2j+4\alpha+1}$$
$$+\sum_{i=0}^{[\frac{n_1-1}{2}]}H^1_{i,\alpha}(2\pi)r^{2i+4\alpha+3}+\sum_{i=0}^{[\frac{n_3-1}{2}]}H^2_{i,\alpha}(2\pi)r^{2i+4\alpha+5}$$
$$+\sum_{i=0}^{[\frac{n_1}{2}]}\sum_{j=0}^{[\frac{n_2-1}{2}]}H^3_{i,j,\alpha}(2\pi)r^{2i+2j+4\alpha+1}+\sum_{i=0}^{[\frac{n_1-1}{2}]}\sum_{j=0}^{\mu}H^4_{i,j,\alpha}(2\pi)r^{2i+2j+4\alpha+1}$$

$$+\sum_{i=0}^{[\frac{n_2-1}{2}]}\sum_{j=0}^{[\frac{n_3}{2}]} H^5_{i,j,\alpha}(2\pi)r^{2i+2j+4\alpha+3}+\sum_{i=1}^{[\frac{n_3-1}{2}]}\sum_{j=0}^{\mu} H^6_{i,j,\alpha}(2\pi)r^{2i+2j+4\alpha+3}, \quad (4.10)$$

where

$$\begin{aligned}
H^1_{i,j,\alpha}(2\pi) &= a_{2i}e_{2j}\Big(-(2i+2\alpha)\phi^{2j+1,2\alpha}_{2i,2\alpha+1}(2\pi)-(2j+2\alpha)\phi^{2i,2\alpha+1}_{2j+1,2\alpha}(2\pi)\Big),\\
H^2_{i,j,\alpha}(2\pi) &= c_{2i}e_{2j}\Big(-(2i+2\alpha+2)\phi^{2j+1,2\alpha}_{2i,2\alpha+3}(2\pi)-(2j+2\alpha)\phi^{2i,2\alpha+3}_{2j+1,2\alpha}(2\pi)\Big),\\
H^1_{i,\alpha}(2\pi) &= a_{2i+1}\Big[(2\alpha+3)\Big(d^1_0\big(\phi^{2i+1,2\alpha+1}_{0,2\alpha+4}(2\pi)-(2\alpha+3)\phi^{2i+1,2\alpha+1}_{2,2\alpha+2}(2\pi)\big)\\
&\qquad +e_3\big(\phi^{2i+1,2\alpha+1}_{4,2\alpha}(2\pi)-\frac{3}{2\alpha+1}\phi^{2i+1,2\alpha+1}_{2,2\alpha+2}(2\pi)\big)\Big)\\
&\qquad +(2i+2\alpha+1)\big(d^1_0U^{\alpha}_{2i+1,2\alpha+1}(2\pi)-e_3T^{\alpha}_{2i+1,2\alpha+1}(2\pi)\big)\Big],\\
H^2_{i,\alpha}(2\pi) &= c_{2i+1}\Big[(2\alpha+3)\Big(d^1_0\big(\phi^{2i+1,2\alpha+3}_{0,2\alpha+4}(2\pi)-(2\alpha+3)\phi^{2i+1,2\alpha+3}_{2,2\alpha+2}(2\pi)\big)\\
&\qquad +e_3\big(\phi^{2i+1,2\alpha+3}_{4,2\alpha}(2\pi)-\frac{3}{2\alpha+1}\phi^{2i+1,2\alpha+3}_{2,2\alpha+2}(2\pi)\big)\Big)\\
&\qquad +(2i+2\alpha+3)\big(d^1_0U^{\alpha}_{2i+1,2\alpha+3}(2\pi)-e_3T^{\alpha}_{2i+1,2\alpha+3}(2\pi)\big)\Big],\\
H^3_{i,j,\alpha}(2\pi) &= a_{2i}b_{2j+1}\Big((2i+2\alpha)\phi^{2j+1,2\alpha+2}_{2i,2\alpha+1}(2\pi)+(2j+2\alpha+2)\phi^{2i,2\alpha+1}_{2j+1,2\alpha+2}(2\pi)\Big),\\
H^4_{i,j,\alpha}(2\pi) &= a_{2i+1}b_{2j}\Big((2i+2\alpha+1)S^{j,\alpha}_{2i+1,2\alpha+1}(2\pi)\\
&\qquad +\frac{2j+2\alpha+1}{2j+1}\big((2j+1)\phi^{2i+1,2\alpha+1}_{2j,2\alpha+2}(2\pi)-(2\alpha+1)\phi^{2i+1,2\alpha+1}_{2j+2,2\alpha}(2\pi)\big)\Big),\\
H^5_{i,j,\alpha}(2\pi) &= b_{2i+1}c_{2j}\Big((2i+2\alpha+2)\phi^{2j,2\alpha+3}_{2i+1,2\alpha+2}(2\pi)+(2j+2\alpha+2)\phi^{2i+1,2\alpha+2}_{2j,2\alpha+3}(2\pi)\Big),\\
H^6_{i,j,\alpha}(2\pi) &= c_{2i+1}b_{2j}\Big((2i+2\alpha+3)S^{j,\alpha}_{2i+1,2\alpha+3}(2\pi)\\
&\qquad +\frac{2j+2\alpha+1}{2j+1}\big((2j+1)\phi^{2i+1,2\alpha+3}_{2j,2\alpha+2}(2\pi)-(2\alpha+1)\phi^{2i+1,2\alpha+3}_{2j+2,2\alpha}(2\pi)\big)\Big),
\end{aligned}$$

Proof. From (4.8) and (4.9), we have

$$2\pi F^1_{20}(r)=M_1(r)+M_2(r)+M_3(r)+M_4(r)+M_5(r)+M_6(r)+M_7(r),$$

where

$$\begin{aligned}
M_1(r) &= \int_0^{2\pi}\sum_{i=0}^{[\frac{m}{2}]}(2i+2\alpha)e_{2i}r^{2i+2\alpha-1}B_{2i+1,2\alpha}(\theta)y(\theta,r)\,d\theta,\\
M_2(r) &= \int_0^{2\pi}(2\alpha+3)e_3r^{2\alpha+2}\Big(B_{4,2\alpha}(\theta)-\frac{3}{2\alpha+1}\Big)B_{2,2\alpha+2}(\theta)y(\theta,r)\,d\theta,\\
M_3(r) &= \int_0^{2\pi}-\sum_{i=0}^{n_1}(i+2\alpha)a_ir^{i+2\alpha-1}B_{i,2\alpha+1}(\theta)y(\theta,r)\,d\theta,\\
M_4(r) &= \int_0^{2\pi}-\sum_{i=0}^{[\frac{n_2-1}{2}]}(2i+2\alpha+2)b_{2i+1}r^{2i+2\alpha+1}B_{2i+1,2\alpha+2}(\theta)y(\theta,r)\,d\theta,\\
M_5(r) &= \int_0^{2\pi}-\sum_{i=0}^{n_3}(i+2\alpha+2)c_ir^{i+2\alpha+1}B_{i,2\alpha+3}(\theta)y(\theta,r)\,d\theta,
\end{aligned}$$

$$M_6(r)=\int_0^{2\pi}-(2\alpha+3)d_0^1r^{2\alpha+2}\big(B_{0,2\alpha+4}(\theta)-(2\alpha+3)B_{2,2\alpha+2}(\theta)\big)y(\theta,r)\,d\theta,$$

and

$$M_7(r)=\int_0^{2\pi}-\sum_{i=0}^{\mu}\frac{2i+2\alpha+1}{2i+1}\,b_{2i}r^{2i+2\alpha}\Big((2i+1)B_{2i,2\alpha+2}(\theta)-(2\alpha+1)B_{2i+2,2\alpha}(\theta)\Big)y(\theta,r)\,d\theta.$$

By using Lemma 4.1, we find that from the 49 products between the different sums only 20 will not be zero after the integration with respect to θ between 0 and 2π. So, the sum of all these terms gives us polynomial (4.10). Hence Lemma 4.2 is proved. □

Lemma 4.3. *The integral $F_{20}^2(r)$ is given by*

$$\begin{aligned}2\pi F_{20}^2(r)=&\sum_{i=0}^{[\frac{m-1}{2}]}M_{i,\alpha}^1(2\pi)r^{2i+2\alpha+1}-\sum_{i=0}^{[\frac{n_2}{2}]}M_{i,\alpha}^2(2\pi)r^{2i+2\alpha+1}+M_{\alpha}^3(2\pi)r^{2\alpha+3}\\&+\sum_{i=0}^{[\frac{n_1}{2}]}\sum_{j=0}^{[\frac{m}{2}]}M_{i,j,\alpha}^1(2\pi)r^{2i+2j+4\alpha-1}+\sum_{i=0}^{[\frac{n_3}{2}]}\sum_{j=0}^{[\frac{m}{2}]}M_{i,j,\alpha}^2(2\pi)r^{2i+2j+4\alpha+1}\\&+\sum_{i=0}^{[\frac{n_1-1}{2}]}M_{i,\alpha}^4(2\pi)r^{2i+4\alpha+3}+\sum_{i=0}^{[\frac{n_3-1}{2}]}M_{i,\alpha}^5(2\pi)r^{2i+4\alpha+5}\\&+\sum_{i=0}^{[\frac{n_1}{2}]}\sum_{j=0}^{[\frac{n_2-1}{2}]}M_{i,j,\alpha}^3(2\pi)r^{2i+2j+4\alpha+1}+\sum_{i=0}^{[\frac{n_1-1}{2}]}\sum_{j=0}^{\mu}M_{i,j,\alpha}^4(2\pi)r^{2i+2j+4\alpha+1}\\&+\sum_{i=0}^{[\frac{n_2-1}{2}]}\sum_{j=0}^{[\frac{n_3}{2}]}M_{i,j,\alpha}^5(2\pi)r^{2i+2j+4\alpha+3}+\sum_{i=1}^{[\frac{n_3-1}{2}]}\sum_{j=0}^{\mu}M_{i,j,\alpha}^6(2\pi)r^{2i+2j+4\alpha+3},\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}M_{i,\alpha}^1(2\pi)&=\frac{2i+1}{i+\alpha+1}\,v_{2i+1}A_{2i,2\alpha}(2\pi),\\M_{i,\alpha}^2(2\pi)&=-\frac{2\alpha+1}{i+\alpha+1}\,q_{2i}A_{2i,2\alpha}(2\pi),\\M_{\alpha}^3(2\pi)&=-d_0^2\frac{(2\alpha+3)(2\alpha+1)}{(2\alpha+4)(\alpha+1)}\,A_{0,2\alpha}(2\pi),\\M_{i,j,\alpha}^1(2\pi)&=a_{2i}e_{2j}\frac{i+j-2\alpha}{i+j+2\alpha+1}\,A_{2i+2j,4\alpha}(2\pi)\\M_{i,j,\alpha}^2(2\pi)&=c_{2i}e_{2j}\frac{i+j-2\alpha-1}{i+j+2\alpha+2}\,A_{2i+2j,4\alpha+2}(2\pi),\\M_{i,j,\alpha}^3(2\pi)&=-2a_{2i}b_{2j+1}A_{2i+2j+2,4\alpha+2}(2\pi)\\M_{i,j,\alpha}^4(2\pi)&=a_{2i+1}b_{2j}\Big(\frac{6j\alpha+j+4\alpha^2+4\alpha-2\alpha i-i}{i+j+2\alpha+2}\,A_{2i+2j+2,4\alpha}(2\pi)\Big),\\M_{i,j,\alpha}^5(2\pi)&=-2b_{2i+1}c_{2j}A_{2i+2j+2,4\alpha+4}(2\pi),\\M_{i,\alpha}^4(2\pi)&=a_{2i+1}\Big[2d_0^1\Big(\frac{2i\alpha+3i+\alpha+3}{i+2\alpha+3}\,A_{2i+2,4\alpha+2}(2\pi)\Big)\\&\qquad+e_3\Big(\frac{2i\alpha+i-10\alpha-4\alpha^2-1}{(2\alpha+1)(i+2\alpha+3)}\,A_{2i+4,4\alpha}(2\pi)\Big)\Big],\\M_{i,\alpha}^5(2\pi)&=c_{2i+1}\Big[2d_0^1\Big(\frac{2i\alpha+3i+\alpha+2}{i+2\alpha+4}\,A_{2i+2,4\alpha+4}(2\pi)\Big)\end{aligned}$$

$$+ e_3\Big(\frac{2i\alpha + i - 12\alpha - 4\alpha^2 - 8}{(2\alpha+1)(i+2\alpha+4)}\,A_{2i+4,4\alpha+2}(2\pi)\Big)\Big],$$

$$M^6_{i,j,\alpha}(2\pi) = c_{2i+1}b_{2j}\Big(\frac{6j\alpha + 5j + 4\alpha^2 + 6\alpha - 2\alpha i - i + 3}{i+j+2\alpha+2}\,A_{2i+2j+2,4\alpha+2}(2\pi)\Big).$$

Proof. First, we calculate $\int\limits_0^{2\pi} I(r,\theta)\,d\theta$:

$$\begin{aligned}\int\limits_0^{2\pi} I(r,\theta)\,d\theta &= \sum_{i=0}^{[\frac{m-1}{2}]} v_{2i+1}\,r^{2i+2\alpha+1}\int\limits_0^{2\pi} B_{2i+2,2\alpha}(\theta)\\ &\qquad - \sum_{i=0}^{[\frac{n_2}{2}]} q_{2i}\,r^{2i+2\alpha+1}\int\limits_0^{2\pi} B_{i,2\alpha+2}(\theta) - d_0^2\,r^{2\alpha+3}\int\limits_0^{2\pi} B_{0,2\alpha+4}\,d\theta\\ &= \sum_{i=0}^{[\frac{m-1}{2}]} \frac{2i+1}{2i+2\alpha+2}\,v_{2i+1}r^{2i+2\alpha+1}A_{2i,2\alpha}(2\pi)\\ &\qquad - \sum_{i=0}^{[\frac{n_2}{2}]} \frac{2\alpha+1}{2i+2\alpha+2}\,q_{2i}r^{2i+2\alpha+1}A_{2i,2\alpha}(2\pi) - d_0^2\,\frac{(2\alpha+3)(2\alpha+1)}{(2\alpha+4)(2\alpha+2)}\,r^{2\alpha+3}A_{0,2\alpha}(2\pi).\end{aligned}$$

For an explicit expression of the $\int\limits_0^{2\pi} II(r,\theta)\,d\theta$, using first (4.6) and substituting in (4.5), we have

$$\begin{aligned}II(r,\theta) = \Big(&\sum_{i=0}^{[\frac{m}{2}]} e_{2i}r^{2i+2\alpha-1}B_{2i+1,2\alpha}(\theta) + e_3r^{2\alpha+2}B_{4,2\alpha}(\theta)\\ &- \sum_{i=0}^{n_1} a_i\,r^{i+2\alpha-1}B_{i,2\alpha+1}(\theta) - \sum_{i=0}^{[\frac{n_2-1}{2}]} b_{2i+1}\,r^{2i+2\alpha+1}B_{2i+1,2\alpha+2}(\theta)\\ &- \sum_{i=0}^{n_3} c_ir^{i+2\alpha+1}B_{i,2\alpha+3}(\theta) - \Big(\frac{3}{2\alpha+1}\,e_3 - d_0^1(2\alpha+3)\Big)r^{2\alpha+2}B_{2,2\alpha+2} - d_0^1r^{2\alpha+2}B_{0,2\alpha+4}\\ &- \sum_{i=0}^{\mu} \frac{b_{2i}}{2i+1}\,r^{2i+2\alpha}((2i+1)B_{2i,2\alpha+2}(\theta) - (2\alpha+1)B_{2i+2,2\alpha}(\theta))\Big)\\ \times \Big(&\sum_{i=0}^{[\frac{m}{2}]} e_{2i}r^{2i+2\alpha}B_{2i,2\alpha+1}(\theta) + \sum_{i=0}^{\mu} e_{2i+1}r^{2i+2\alpha+1}B_{2i+1,2\alpha+1}(\theta) + e_3r^{2\alpha+3}B_{3,2\alpha+1}\\ &+ \sum_{i=0}^{n_1} a_ir^{i+2\alpha}B_{i+1,2\alpha}(\theta) + \sum_{i=0}^{\mu} b_{2i}r^{2i+2\alpha+1}B_{2i+1,2\alpha+1}(\theta) + \sum_{i=0}^{n_3} c_ir^{i+2\alpha+2}B_{i+1,2\alpha+2}(\theta)\\ &+ \sum_{i=0}^{[\frac{n_2-1}{2}]} e_{2i}r^{2i+2\alpha}B_{2i,2\alpha+1}(\theta) + b_2\,r^{2\alpha+3}B_{3,2\alpha+1}(\theta) + d_0^1\,r^{2\alpha+3}B_{1,2\alpha+3}\Big).\end{aligned}$$

From the 42 products between the different sums only 18 will not be zero after the integration with respect to θ between 0 and 2π. So, the terms of $\int\limits_0^{2\pi} II(r,\theta)\,d\theta$ which will contribute to $F^2_{20}(r)$ are

$$(b_1) = \int\limits_0^{2\pi}\Big(\sum_{i=0}^{[\frac{m}{2}]} e_{2i}r^{2i+2\alpha-1}B_{2i+1,2\alpha}(\theta)\Big)\Big(\sum_{j=0}^{n_1} a_jr^{j+2\alpha}B_{j+1,2\alpha}(\theta)\Big)\,d\theta$$

$$= \sum_{i=0}^{[\frac{m}{2}]} \sum_{j=0}^{[\frac{n_1}{2}]} e_{2i} a_{2j} r^{2i+2j+4\alpha-1} A_{2i+2j+2,4\alpha}(2\pi),$$

$$(b_2) = \int_0^{2\pi} \Big(\sum_{i=0}^{[\frac{m}{2}]} e_{2i} r^{2i+2\alpha-1} B_{2i+1,2\alpha}(\theta) \Big) \Big(\sum_{j=0}^{n_3} c_j r^{j+2\alpha+2} B_{j+1,2\alpha+2}(\theta) \Big)\, d\theta$$

$$= \sum_{i=0}^{[\frac{m}{2}]} \sum_{j=0}^{[\frac{n_3}{2}]} e_{2i} c_{2j} r^{2i+2j+4\alpha+1} A_{2i+2j+2,4\alpha+2}(2\pi),$$

$$(b_3) = \int_0^{2\pi} \Big(e_3 r^{2\alpha+2} \Big(B_{4,2\alpha}(\theta) - \frac{3}{2\alpha+1}\, B_{2,2\alpha+2}(\theta) \Big) \Big) \Big(\sum_{j=0}^{n_1} a_j r^{j+2\alpha} B_{j+1,2\alpha}(\theta) \Big)\, d\theta$$

$$= \sum_{j=0}^{[\frac{n_1-1}{2}]} e_3 a_{2j+1} r^{2j+4\alpha+3} \Big(A_{2j+6,4\alpha}(2\pi) - \frac{3}{2\alpha+1}\, A_{2j+4,4\alpha+2}(2\pi) \Big),$$

$$(b_4) = \int_0^{2\pi} \Big(e_3 r^{2\alpha+2} \Big(B_{4,2\alpha}(\theta) - \frac{3}{2\alpha+1}\, B_{2,2\alpha+2}(\theta) \Big) \Big) \Big(\sum_{j=0}^{n_3} c_j r^{j+2\alpha+2} B_{j+1,2\alpha+2}(\theta) \Big)\, d\theta$$

$$= \sum_{j=0}^{[\frac{n_3-1}{2}]} e_3 c_{2j+1} r^{2j+4\alpha+5} \Big(A_{2j+6,4\alpha+2}(2\pi) - \frac{3}{2\alpha+1} A_{2j+4,4\alpha+4}(2\pi) \Big),$$

$$(b_5) = \int_0^{2\pi} \Big(- \sum_{i=0}^{n_1} a_i r^{i+2\alpha-1} B_{i,2\alpha+1}(\theta) \Big) \Big(\sum_{j=0}^{[\frac{m}{2}]} e_{2j} r^{2j+2\alpha} B_{2j,2\alpha+1}(\theta) \Big)\, d\theta$$

$$= - \sum_{i=0}^{[\frac{n_1}{2}]} \sum_{j=0}^{[\frac{m}{2}]} a_{2i} e_{2j} r^{2i+2j+4\alpha-1} A_{2i+2j,4\alpha+2}(2\pi),$$

$$(b_6) = \int_0^{2\pi} \Big(- \sum_{i=0}^{n_1} a_i r^{i+2\alpha-1} B_{i,2\alpha+1}(\theta) \Big) \Big(\sum_{j=0}^{[\frac{n_2-1}{2}]} b_{2j+1} r^{2j+2\alpha+2} B_{2j+2,2\alpha+1}(\theta) \Big)\, d\theta$$

$$= - \sum_{i=0}^{[\frac{n_1}{2}]} \sum_{j=0}^{[\frac{n_2-1}{2}]} a_{2i} b_{2j+1} r^{2i+2j+4\alpha+1} A_{2i+2j+2,4\alpha+2}(2\pi),$$

$$(b_7) = \int_0^{2\pi} \Big(- \sum_{i=0}^{[\frac{n_1}{2}]} a_i r^{i+2\alpha-1} B_{i,2\alpha+1}(\theta) \Big) \Big(\sum_{j=0}^{\mu} \frac{2j+2\alpha+2}{2j+1} b_{2j} r^{2j+2\alpha+1} B_{2j+1,2\alpha+1}(\theta) \Big)\, d\theta$$

$$= - \sum_{i=0}^{[\frac{n_1-1}{2}]} \sum_{j=0}^{\mu} \frac{2j+2\alpha+2}{2j+1} a_{2i+1} b_{2j} r^{2i+2j+4\alpha+1} A_{2i+2j+2,4\alpha+2}(2\pi),$$

$$(b_8) = \Big(- \sum_{i=0}^{n_1} a_i r^{i+2\alpha-1} B_{i,2\alpha+1}(\theta) \Big)$$

$$\times \Big(\Big(\frac{2\alpha+4}{2\alpha+1} \Big) e_3 B_{3,2\alpha+1}(\theta) + d_0^1 \big(B_{1,2\alpha+3}(\theta) - (2\alpha+3) B_{3,2\alpha+1}(\theta) \big) r^{2\alpha+3} \Big)$$

$$= - \sum_{i=0}^{[\frac{n_1-1}{2}]} a_{2i+1} r^{2i+4\alpha+3} \Big[\Big(\Big(\frac{2\alpha+4}{2\alpha+1} \Big) e_3 - d_0^1 (2\alpha+3) \Big) A_{2i+4,4\alpha+2}(2\pi) + d_0^1 A_{2i+2,4\alpha+4}(2\pi) \Big],$$

$$(b_9) = \int_0^{2\pi} \Big(- \sum_{i=0}^{n_3} c_i r^{i+2\alpha+1} B_{i,2\alpha+3}(\theta) \Big) \Big(\sum_{j=0}^{[\frac{m}{2}]} e_{2j} r^{2j+2\alpha} B_{2j,2\alpha+1}(\theta) \Big)\, d\theta$$

$$= -\sum_{i=0}^{[\frac{n_3}{2}]}\sum_{j=0}^{[\frac{m}{2}]} c_{2i}e_{2j}r^{2i+2j+4\alpha+1}A_{2i+2j,4\alpha+4}(2\pi),$$

$$(b_{10}) = \int_0^{2\pi}\Big(-\sum_{i=0}^{n_3} c_i r^{i+2\alpha+1}B_{i,2\alpha+3}(\theta)\Big)\Big(\sum_{j=0}^{[\frac{n_2-1}{2}]} b_{2j+1}r^{2j+2\alpha+2}B_{2j+2,2\alpha+1}(\theta)\Big)\,d\theta$$

$$= -\sum_{i=0}^{[\frac{n_3}{2}]}\sum_{j=0}^{[\frac{n_2-1}{2}]} c_{2i}b_{2j+1}r^{2i+2j+4\alpha+3}A_{2i+2j+2,4\alpha+4}(2\pi),$$

$$(b_{11}) = \int_0^{2\pi}\Big(-\sum_{i=0}^{n_3} c_i r^{i+2\alpha+1}B_{i,2\alpha+3}(\theta)\Big)\Big(\sum_{j=0}^{\mu}\frac{2j+2\alpha+2}{2j+1}b_{2j}r^{2j+2\alpha+1}B_{2j+1,2\alpha+1}(\theta)\Big)\,d\theta$$

$$= -\sum_{i=0}^{[\frac{n_3-1}{2}]}\sum_{j=0}^{\mu}\frac{2j+2\alpha+2}{2j+1}c_{2i+1}b_{2j}r^{2i+2j+4\alpha+1}A_{2i+2j+2,4\alpha+4}(2\pi),$$

$$(b_{12}) = \int_0^{2\pi}\Big(-\sum_{i=0}^{n_3} c_i r^{i+2\alpha+1}B_{i,2\alpha+3}(\theta)\Big)$$

$$\times\Big(\Big(\frac{2\alpha+4}{2\alpha+1}\Big)e_3B_{3,2\alpha+1}(\theta) + d_0^1\big(B_{1,2\alpha+3}(\theta) - (2\alpha+3)B_{3,2\alpha+1}(\theta)\big)r^{2\alpha+3}\Big)\,d\theta$$

$$= -\sum_{i=0}^{[\frac{n_3-1}{2}]} c_{2i+1}r^{2i+4\alpha+5}\Big[\Big(\Big(\frac{2\alpha+4}{2\alpha+1}\Big)e_3 - d_0^1(2\alpha+3)\Big)A_{2i+4,4\alpha+4}(2\pi) + d_0^1A_{2i+2,4\alpha+6}(2\pi)\Big],$$

$$(b_{13}) = \int_0^{2\pi}\Big(\sum_{i=0}^{[\frac{n_2-1}{2}]} b_{2i+1}r^{2i+2\alpha+1}B_{2i+1,2\alpha+2}(\theta)\Big)\Big(\sum_{j=0}^{n_1} a_j r^{j+2\alpha}B_{j+1,2\alpha}(\theta)\Big)\,d\theta$$

$$= \sum_{i=0}^{[\frac{n_2-1}{2}]}\sum_{j=0}^{[\frac{n_1}{2}]} b_{2i+1}a_{2j}r^{2i+2j+4\alpha+1}A_{2i+2j+2,4\alpha+2}(2\pi),$$

$$(b_{14}) = \int_0^{2\pi}\Big(\sum_{i=0}^{[\frac{n_2-1}{2}]} b_{2i+1}r^{2i+2\alpha+1}B_{2i+1,2\alpha+2}(\theta)\Big)\Big(\sum_{j=0}^{n_3} c_j r^{j+2\alpha+2}B_{j+1,2\alpha+2}(\theta)\Big)\,d\theta$$

$$= \sum_{i=0}^{[\frac{n_2-1}{2}]}\sum_{j=0}^{[\frac{n_3}{2}]} b_{2i+1}c_{2j}r^{2i+2j+4\alpha+3}A_{2i+2j+2,4\alpha+4}(2\pi),$$

$$(b_{15}) = \int_0^{2\pi}\big(d_{0^1}r^{2\alpha+2}\big(-B_{0,2\alpha+4}(\theta) + (2\alpha+3)B_{0,2\alpha+4}(\theta)\big)\big)\Big(\sum_{j=0}^{n_1} a_j r^{j+2\alpha}B_{j+1,2\alpha}(\theta)\Big)\,d\theta$$

$$= \sum_{j=0}^{[\frac{n_1-1}{2}]} d_0^1a_{2j+1}r^{2j+4\alpha+3}\big(-A_{2j+2,4\alpha+4}(2\pi) + (2\alpha+3)A_{2j+4,4\alpha+2}(2\pi)\big),$$

$$(b_{16}) = \int_0^{2\pi}\big(d_{0^1}r^{2\alpha+2}\big(-B_{0,2\alpha+4}(\theta) + (2\alpha+3)B_{0,2\alpha+4}(\theta)\big)\big)\Big(\sum_{j=0}^{n_3} c_j r^{j+2\alpha+2}B_{j+1,2\alpha+2}(\theta)\Big)\,d\theta$$

$$= \sum_{j=0}^{[\frac{n_3-1}{2}]} d_0^1c_{2j+1}r^{2j+4\alpha+5}\big(-A_{2j+2,4\alpha+6}(2\pi) + (2\alpha+3)A_{2j+4,4\alpha+4}(\theta)\big),$$

$$
\begin{aligned}
(b_{17}) &= \int_0^{2\pi} \Big(-\sum_{i=0}^{\mu} \frac{b_{2i}}{2i+1} r^{2i+2\alpha+1}\big((2i+1)B_{2i,2\alpha+2}(\theta)-(2\alpha+1)B_{2i+2,2\alpha}(\theta)\big)\Big) \\
&\qquad\qquad \times \Big(\sum_{j=0}^{n_1} a_j r^{j+2\alpha} B_{j+1,2\alpha}(\theta)\Big)\,d\theta \\
&= -\sum_{j=0}^{\mu}\sum_{j=0}^{[\frac{n_1-1}{2}]} \frac{b_{2i}}{2i+1} a_{2j+1} r^{2j+4\alpha+1}\big((2i+1)A_{2i+2j+2,4\alpha+2}(2\pi)-(2\alpha+1)A_{2i+2j+4,4\alpha}(\theta)\big),\\
(b_{18}) &= \int_0^{2\pi} \Big(-\sum_{i=0}^{\mu} \frac{b_{2i}}{2i+1} r^{2i+2\alpha+1}\big((2i+1)B_{2i,2\alpha+2}(\theta)-(2\alpha+1)B_{2i+2,2\alpha}(\theta)\big)\Big) \\
&\qquad\qquad \times \Big(\sum_{j=0}^{n_3} c_j r^{j+2\alpha+2} B_{j+1,2\alpha+2}(\theta)\Big)\,d\theta \\
&= -\sum_{j=0}^{\mu}\sum_{j=0}^{[\frac{n_1-1}{2}]} \frac{b_{2i}}{2i+1} c_{2j+1} r^{2j+4\alpha+3}\big((2i+1)A_{2i+2j+2,4\alpha+4}(2\pi)-(2\alpha+1)A_{2i+2j+4,4\alpha+2}(\theta)\big).
\end{aligned}
$$

We have that the sum of the integral $\int_0^{2\pi} I(r,\theta)\,d\theta$ and the integrals from (b_1) to (b_{18}) is the polynomial (4.11). Hence Lemma 4.3 is proved. □

Finally, we obtain that $F_{20}(r)$ is a polynomial in the variable r^2 of the form

$$2\pi F_{20}(r) = r^{2\alpha+1}\big[P_1(r^2)+r^2P_2(r^2)+r^4P_3(r^2)\big],$$

where $P_1(r^2)$, $P_2(r^2)$ and $P_3(r^2)$ are polynomials in the variable r^2 of degree, respectively,

$$
\begin{aligned}
\Lambda_1 &= \max\Big\{\Big[\frac{n_1}{2}\Big]+\Big[\frac{n_2-1}{2}\Big]+\alpha, \Big[\frac{n_1}{2}\Big]+\Big[\frac{m}{2}\Big]-1+\alpha, \Big[\frac{n_3}{2}\Big]+\Big[\frac{m}{2}\Big]+\alpha, \Big[\frac{n_2}{2}\Big],\\
&\qquad \Big[\frac{m-1}{2}\Big], \Big[\frac{n_1-1}{2}\Big]+\mu+\alpha, 1\Big\},\\
\Lambda_2 &= \max\Big\{\Big[\frac{n_1-1}{2}\Big]+\alpha, \Big[\frac{n_2-1}{2}\Big]+\Big[\frac{n_3}{2}\Big]+\alpha, \Big[\frac{n_3-1}{2}\Big]+\mu+\alpha\Big\},\\
\Lambda_3 &= \Big[\frac{n_3-1}{2}\Big]+\alpha.
\end{aligned}
$$

Then to find the real positive roots of $F_{20}(r)$ we have to find the zeros of a polynomial in r^2 of degree

$$\lambda = \max\big\{\Lambda_1, \Lambda_2+1, \Lambda_3+2\big\}.$$

We conclude that $F_{20}(r)$ has at most λ positive roots. Hence statement (b) of Theorem 1.1 follows.

Example

The computations of the following example have been obtained by using Maple.

We have $m=2$, $n_2=2$, $n_1=1$ and $n_3=1$.

If $\alpha=1$, we consider system (1.3), where

$$
\begin{gathered}
l_1(x)=1+3x+x^3,\quad f_1(x)=1+\frac{373}{189}\,x,\quad g_1(x)=1+x-5x^2,\quad h_1(x)=\frac{3}{189}-\frac{64}{63}\,x,\quad d_0^1=1,\\
l_2(x)=x+x^2,\quad f_2(x)=x,\quad g_2(x)=-3+\frac{2090}{189}\,x^2,\quad h_2(x)=x,\quad d_0^2=1.
\end{gathered}
$$

We have that F_{10} is identically zero, so, to look for the limit cycles, we have to solve the equation $F_{20}(r)=0$ which is equivalent to

$$F_{20}(r)=\frac{r^3}{36}\,(r^6-14r^4+49r^2-36).$$

This equation has exactly three positive zeros.

If $\alpha=2$, we consider system (1.3), where

$$l_1(x)=\frac{5}{9}+5x+\frac{9005}{594}\,x^2,\quad f_1(x)=2-\frac{1312}{2097}\,x,\quad g_1(x)=1+x-7x^2,\quad h_1(x)=\frac{32}{297}\,x,\quad d_0^1=1,$$
$$l_2(x)=11x+x^2,\quad f_2(x)=x,\quad g_2(x)=-1+24x^2,\quad h_2(x)=x,\quad d_0^2=\frac{8}{9}\,.$$

An easy computation shows that $F_{10}\equiv 0$ and

$$F_{20}(r)=\frac{r^5}{576}\,(r^8-30r^6+273r^4-820r^2+576).$$

This equation has exactly four positive zeros.

In [15], this system for $\alpha=0$ has at most two positive zeros using the averaging theory of the second order.

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