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INFINITELY MANY SOLUTIONS FOR A SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATION WITH p-LAPLACIAN OPERATOR

Abstract. In this paper, by using the critical point theory, specially the fountain theorem given in [18], we prove the existence of infinitely many solutions for a second order impulsive differential equation governed by the one-dimensional p-Laplacian operator. Finally, an example is presented to illustrate our main result.

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Key words and phrases. p-Laplacian operator, Impulsive differential equations, Critical point theory, Fountain theorem.

რეზიუმე. ნაშრომში, კრიტიკული წერტილის თეორიის გამოყენებით, განსაკუთრებით [18]-ში მოცემული შადრევნების თეორემით, ვამტკიცებთ უსასრულოდ ბევრი ამონახსნის არსებობას მეორე რიგის იმპულსური დიფერენციალური განტოლებისთვის, რომელიც მართავს ერთგანზომილებიან p-ლაპლასიურ ოპერატორს. ნაშრომის ბოლოში მთავარი შედეგის საილუსტრაციოდ მოყვანილია მაგალითი.

1 Introduction

We consider the problem

$$\begin{gathered} -(\rho(t)\Phi_p(u'(t)))' + s(t)\Phi_p(u(t)) = f(t,u(t)), \ \ t \neq t_i, \ \text{ a.e. } \ t \in [0,T], \\ -\Delta_p(\rho(t_i)\Phi_p(u'(t_i))) = I_i(u(t_i)), \ \ i = 1,2,\dots,l, \\ u(0) = u(T) = 0, \end{gathered} \tag{1.1}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\Phi_p(x) = |x|^{p-2}x$, $p > 1$ and $\rho, s \in L^\infty([0,T])$ with

$$\operatorname*{ess\,inf}_{t\in[0,T]} \rho(t) > 0, \quad \operatorname*{ess\,inf}_{t\in[0,T]} s(t) > 0, \ \ < \rho(0), \ \ \rho(T) < +\infty, \ \ t_0 = 0 < t_1 < t_2 < \dots < t_l < t_{l+1} = T,$$

are given points and the functions $I_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,\dots,l$, are continuous. The operator Δ_p is defined as

$$\Delta_p\big(\rho(t_i)\Phi_p(u'(t_i))\big) = \rho(t_i^+)\Phi_p(u'(t_i^+)) - \rho(t_i^-)\Phi_p(u'(t_i^-)),$$

where $u'(t_i^+)$ and $u'(t_i^-)$ denote the right and left limits of $u'(t)$ at $t = t_i$, respectively.

Differential equations with impulsive effects arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur. We refer to some recent works on the theory of impulsive differential equations that have been developed by a large number of mathematicians [4, 7, 9, 10, 12, 16, 17]. There are many approaches to study the existence of solutions of impulsive differential equations such as fixed point theory [8], topological degree theory [13], comparison method [15], and so on. On the other hand, many researchers have used variational methods to study the existence of solutions for boundary value problems [1–3, 5, 6, 11, 14]. However, to the best of our knowledge, there are few papers dealing with the existence of infinitely many solutions for impulsive boundary value problems by using fountain theorems. Recently, in [14], the authors considered the following problem:

$$\begin{gathered} -u''(t) + g(t)u(t) = f(t,u(t)), \ \ t \neq t_j, \ \text{ a.e. } \ t \in [0,T], \\ \Delta(u'(t_j)) = I_j(u(t_j)), \ \ j = 1,2,\dots,p, \\ u(0) = u(T) = 0. \end{gathered} \tag{1.2}$$

They obtained the existence of infinitely many solutions for (1.2) in both cases, superlinear and asymptotically linear, by using the fountain theorems without using the Ambrosetti–Rabinowitz condition in the superlinear case which is given as follows, that is, there exist $\eta > 2$ and $K > 0$ such that

$$0 < \eta F(t,u) \le f(t,u)u, \quad |u| \ge K \ \text{ for all } \ t \in [0,T], \tag{1.3}$$

where F is a primitive of f with respect to the second variable, that is, $F(t,u) = \int\limits_0^x f(t,x)\,\mathrm{d}x$.

However, there are the functions which are superlinear, but do not satisfy condition (1.3). For example,

$$f(t,u) = |\sin(t)|\Big(2u\ln(1+|u|) + \frac{|u|u}{1+|u|}\Big) \ \text{ for } \ t \in [0,T] \ \text{ and } \ u \in \mathbb{R}\setminus\{0\}. \tag{1.4}$$

Inspired by the above-mentioned works, in the present paper we study the existence of infinitely many solutions for problem (1.1), when the nonlinearity $f(t,u)$ and I_i $(i = 1,2,\dots,l)$ satisfy some sub-critical conditions.

The remainder of this paper is organized as follows. In Section 2, we present preliminaries and main results. In Section 3, we give an example that satisfies the assumptions of our main result.

2 Variational setting and main results

Here and in what follows, X denotes the Sobolev space $W_0^{1,p}([0,T])$ endowed with the norm

$$\|u\| = \bigg(\int\limits_0^T \big(\rho(t)|u'(t)|^p + s(t)|u(t)|^p\big)\,\mathrm{d}t\bigg)^{\frac{1}{p}}, \tag{2.1}$$

which is equivalent to the usual one. As usual, for $1 < p < +\infty$, we define the norms in $L^p([0,T])$ and $C([0,T])$, respectively, by

$$\|u\|_p = \Big(\int_0^T |u(t)|^p \,\mathrm{d}t\Big)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

Lemma 2.1 ([1]). *For $u \in W_0^{1,p}([0,T])$, we have $\|u\|_\infty \le M\|u\|$, where*

$$M = 2^{\frac{1}{q}} \max\left\{\frac{1}{T^{\frac{1}{p}}(\operatorname*{ess\,inf}_{t\in[0,T]} s(t))^{\frac{1}{p}}}, \frac{T^{\frac{1}{q}}}{(\operatorname*{ess\,inf}_{t\in[0,T]} \rho(t))^{\frac{1}{p}}}\right\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. For $u \in W_0^{1,p}([0,T])$, it follows from the mean value theorem that

$$u(\zeta) = \frac{1}{T}\int_0^T u(\tau)\,\mathrm{d}\tau$$

for some $\zeta \in [0,T]$. Hence, for $t \in [0,T]$, using Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned}
|u(t)| &= \Big|u(\zeta) + \int_\zeta^t u'(\tau)\,\mathrm{d}\tau\Big| \\
&\le \int_0^T |u(\tau)|\,\mathrm{d}\tau + \int_0^T |u'(\tau)|\,\mathrm{d}\tau \le T^{-\frac{1}{p}}\Big(\int_0^T |u(\tau)|^p\,\mathrm{d}\tau\Big)^{\frac{1}{p}} + T^{\frac{1}{q}}\Big(\int_0^T |u'(\tau)|^p\,\mathrm{d}\tau\Big)^{\frac{1}{p}} \\
&\le \frac{1}{T^{\frac{1}{p}}(\operatorname*{ess\,inf}_{t\in[0,T]} s(t))^{\frac{1}{p}}}\Big(\int_0^T s(\tau)|u(\tau)|^p\,\mathrm{d}\tau\Big)^{\frac{1}{p}} + \frac{T^{\frac{1}{q}}}{(\operatorname*{ess\,inf}_{t\in[0,T]} \rho(t))^{\frac{1}{p}}}\Big(\int_0^T \rho(\tau)|u'(\tau)|^p\,\mathrm{d}\tau\Big)^{\frac{1}{p}} \\
&\le 2^{\frac{1}{q}} \max\left\{\frac{1}{T^{\frac{1}{p}}(\operatorname*{ess\,inf}_{t\in[0,T]} s(t))^{\frac{1}{p}}}, \frac{T^{\frac{1}{q}}}{(\operatorname*{ess\,inf}_{t\in[0,T]} \rho(t))^{\frac{1}{p}}}\right\}\|u\|,
\end{aligned}$$

which completes the proof. □

Now, we introduce the following concept for the solution of problem (1.1).

Definition 2.1. We say that a function $u \in W_0^{1,p}([0,T])$ is a weak solution of problem (1.1) if the identity

$$\int_0^T \rho(t)|u'(t)|^{p-2}u'(t)v'(t)\,\mathrm{d}t + \int_0^T s(t)|u(t)|^{p-2}u(t)v(t)\,\mathrm{d}t + \sum_{i=1}^{l} I_i(u(t_i))v(t_i) = \int_0^T f(t,u(t))v(t)\,\mathrm{d}t$$

holds for any $v \in W_0^{1,p}([0,T])$.

Definition 2.2. A function $u \in \{u \in W_0^{1,p}([0,T]) :\ \rho|u'|^{p-2}u' \in W^{1,\infty}([0,T] \setminus \{t_1,t_2,\dots,t_l\})\}$ is a classical solution of problem (1.1) if u satisfies the equation a.e. on $[0,T] \setminus \{t_1,t_2,\dots,t_l\}$, the limits $u'(t_i^+)$, $u'(t_i^-)$, $i = 1,2,\dots,l$, exist and satisfy the impulsive condition

$$-\Delta_p\big(\rho(t_i)\Phi_p(u'(t_i))\big) = \rho(t_i^-)\Phi_p(u'(t_i^-)) - \rho(t_i^+)\Phi_p(u'(t_i^+)) = I_i(u(t_i)),$$

and the boundary conditions $u(0) = u(T) = 0$ holds.

Next, we begin describing the variational formulation of our problem. Consider the energy functional $J : W_0^{1,p}([0,T]) \to \mathbb{R}$ associated to (1.1) as follows

$$J(u) = \frac{1}{p}\int_0^T \big(\rho(t)|u'(t)|^p + s(t)|u(t)|^p\big)\,\mathrm{d}t + \sum_{i=1}^{l}\int_0^{u(t_i)} I_i(x)\,\mathrm{d}x - \int_0^T F(t,u(t))\,\mathrm{d}t$$
$$= \frac{1}{p}\,\|u\|^p + \sum_{i=1}^{l}\int_0^{u(t_i)} I_i(x)\,\mathrm{d}x - \int_0^T F(t,u(t))\,\mathrm{d}t. \tag{2.2}$$

Since f and I_i $(i = 1,2,\ldots,l)$ are continuous, we deduce that J is of the class $C^1(W_0^{1,p}([0,T]),\mathbb{R})$ and its derivative is given by

$$\langle J'(u), v\rangle = \int_0^T \rho(t)|u'(t)|^{p-2}u'(t)v'(t)\,\mathrm{d}t$$
$$+ \int_0^T s(t)|u(t)|^{p-2}u(t)v(t)\,\mathrm{d}t + \sum_{i=1}^{l} I_i(u(t_i))v(t_i) - \int_0^T f(t,u(t))v(t)\,\mathrm{d}t \tag{2.3}$$

for all $u, v \in W_0^{1,p}([0,T])$.

Then it is clear that the critical points of J are weak solutions of problem (1.1).

Lemma 2.2. *If $u \in W_0^{1,p}([0,T])$ is a weak solution of problem* (1.1)*, then u is a classical solution of problem* (1.1)*.*

Proof. The proof is similar to that of Lemma 1 in [2]. □

To prove our main results, we need the following variant fountain theorem introduced in [18] to handle our problem. Let X be a Banach space with the norm $\|\cdot\|$ and $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < +\infty$ for any $j \in \mathbb{N}$. Set

$$Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j} \ \text{ and } \ B_k = \big\{u \in Y_k :\ \|u\| \le \rho_k\big\}.$$

Consider the C^1-functional $J_\lambda : X \to \mathbb{R}$ defined by

$$J_\lambda(u) = A(u) - \lambda B(u), \ \ \lambda \in [1,2],$$

where

$$A(u) = \frac{1}{p}\,\|u\|^p + \sum_{i=1}^{l}\int_0^{u(t_i)} I_i(x)\,\mathrm{d}x \ \text{ and } \ B(u) = \int_0^T F(t,u(t))\,\mathrm{d}t.$$

For convenience, we list the following assumptions:

(H_1) $I_i(u)$ $(i = 1,2,\ldots,l)$ are odd about u and satisfy $I_i(u)u \ge 0$ for all $u \in \mathbb{R}$.

(H_2) For any $i \in \{1,2,\ldots,l\}$, there exist the positive constants a_i, b_i and $\gamma_i \in [0, p-1[$ such that

$$|I_i(u)| \le a_i + b_i|u|^{\gamma_i} \ \text{ for } \ u \in \mathbb{R}.$$

(H_3) There exist the constants $\theta_1 > 0$, $\theta_2 > 0$, and $\nu > p$ such that

$$|f(t,u)| \le \theta_1|u|^{p-1} + \theta_2|u|^{\nu-1} \ \text{ for all } \ (t,u) \in [0,T] \times \mathbb{R}.$$

(H_4) $F(t,0)=0$ and $F(t,u)\geq 0$, $\forall (t,u)\in[0,T]\times\mathbb{R}$ and

$$\lim_{|u|\to+\infty}\frac{F(t,u)}{|u|^p}+\infty \text{ uniformly for } t\in[0,T].$$

(H_5) $F(t,-u)=F(t,u)$, $\forall\,(t,u)\in[0,T]\times\mathbb{R}$.

(H_6) There exist $\theta_1\geq 1$, $\theta_2\geq 1$ such that

$$\theta_1\mathcal{G}(u(t_i))\geq\mathcal{G}(\tau u(t_i)),\ \ \forall\, i\in\{1,2,\ldots,l\},\ \ \tau\in[0,1] \text{ and } \forall\, u\in\mathbb{R},$$

and

$$\theta_2\mathcal{F}(t,u)\geq\mathcal{F}(t,\tau u),\ \ \forall\,(t,u)\in[0,T]\times\mathbb{R} \text{ and } \tau\in[0,1],$$

where

$$\mathcal{G}(u(t_i))=pG(u(t_i))-I_i(u(t_i))u(t_i),\quad G(u(t_i))=\int\limits_0^{u(t_i)}I_i(x)\,\mathrm{d}x$$

and

$$\mathcal{F}(t,u)=f(t,u)u-pF(t,u).$$

Theorem 2.1. *Assume that* (H_1)*–*(H_6) *are satisfied. Then problem* (1.1) *possesses infinitely many high energy solutions* $\{u_k\}\subset W_0^{1,p}([0,T])\setminus\{0\}$ *satisfying*

$$\frac{1}{p}\int\limits_0^T\big(\rho(t)|u_k'(t)|^p+s(t)|u_k(t)|^p\big)\,\mathrm{d}t+\sum_{i=1}^{l}\int\limits_0^{u_k(t_i)}I_i(x)\,\mathrm{d}x-\int\limits_0^T F(t,u_k(t))\,\mathrm{d}t\longrightarrow+\infty \ \text{ as } \ k\to+\infty.$$

To prove our main result, we will show that J_λ satisfies the assumptions of the following variant fountain theorem.

Theorem 2.2 ([18])**.** *Assume that the functional* J_λ *defined above satisfies*

(A_1) *J_λ maps bounded sets into bounded sets uniformly for $\lambda\in[1,2]$, and $J_\lambda(-u)=J_\lambda(u)$ for all $(\lambda,u)\in[1,2]\times X$;*

(A_2) *$B(u)\geq 0$ for all $u\in X$, $A(u)\to+\infty$ or $B(u)\to+\infty$ as $\|u\|\to+\infty$;*

or

(A_3) *$B(u)\leq 0$ for all $u\in X$, $B(u)\to-\infty$ as $\|u\|\to+\infty$;*

(A_4) *there exist $\rho_k>r_k>0$ such that*

$$b_k(\lambda)=\inf_{u\in Z_k,\ \|u\|=r_k}J_\lambda(u)>a_k(\lambda)=\max_{u\in Y_k,\ \|u\|=\rho_k}J_\lambda(u),\ \ \forall\,\lambda\in[1,2].$$

Then

$$b_k(\lambda)\leq c_k(\lambda)=\inf_{\gamma\in\Gamma_k}\max_{u\in B_k}J_\lambda(\gamma(u)),\ \ \forall\,\lambda\in[1,2],$$

where

$$\Gamma_k=\big\{\gamma\in C(B_k,X):\ \gamma \text{ is odd},\ \gamma_{|\partial B_k}=id\equiv \text{ identity}\big\}.$$

Moreover, for almost every $\lambda\in[1,2]$, there exists a sequence $\{u_{n,k}(\lambda)\}_{n\in\mathbb{N}}$ such that

$$\sup_n\|u_{n,k}(\lambda)\|<+\infty,\quad J_\lambda'(u_{n,k}(\lambda))\to 0 \text{ and } J_\lambda(u_{n,k}(\lambda))\to c_k(\lambda) \text{ as } n\to+\infty.$$

Proof of Theorem 2.1. By (H_3), there exist positive numbers θ_3 and θ_4 such that

$$|F(t,u)| \leq \theta_3 |u|^p + \theta_4 |u|^\nu. \tag{2.4}$$

Combining (2.4), (H_2) and Lemma 2.1, it is easily seen that J_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. By (H_1) and (H_5), $J_\lambda(-u) = J_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$. Thus condition (A_1) holds. Assumption (H_4) means that $B(u) \geq 0$. Condition (A_2) holds for the fact that $A(u) \geq \frac{1}{p}\|u\|^p \to +\infty$ as $\|u\| \to +\infty$ and $B(u) \geq 0$. Next, to show assumption (A_4), we first show the following useful lemmas.

Lemma 2.3. *Let*

$$\alpha_r(k) = \sup_{u \in Z_k,\ \|u\|=1} \|u\|_r$$

with $r \geq p$. Then

$$\alpha_r(k) \to 0 \ \text{ as } \ k \to +\infty.$$

Proof. We aim to prove that $\alpha_r(k) \to 0$ as $k \to +\infty$. The function $\alpha_r(k)$ is decreasing with respect to k, then there exists $\alpha_r \geq 0$ for all $r \geq p$ such that $\alpha_p(k) \to \alpha_p$ and $\alpha_r(k) \to \alpha_r$ as $k \to +\infty$. For any $k \geq 0$, there exits $u_k \in Z_k$ such that

$$\|u_k\| = 1 \ \text{ and } \ \|u_k\|_p \geq \frac{\alpha_p(k)}{2}\,.$$

By the fact that X is a reflexive space, we can assume that $u_k \rightharpoonup u$ in X. Let $\{e_j^*\}_{j\in\mathbb{N}}$ be the family of the dual space of X and for any $e_n^* \in \{e_j^*\}_{j\in\mathbb{N}}$, we have

$$\langle e_n^*, u_k\rangle = 0 \ \text{ for } \ k > n.$$

Therefore,

$$0 = \langle e_n^*, u_k\rangle \to \langle e_n^*, u\rangle \ \text{ as } \ k \to +\infty$$

for any $e_n^* \in \{e_j^*\}_{j\in\mathbb{N}}$, which implies that $u = 0$, then $u_k \rightharpoonup 0$ in X, $u_k \to 0$ in $L^p([0,T])$ and therefore $u_k \to 0$ in $C([0,T])$ which implies that $\alpha_p = 0$. Similarly, we prove that $\alpha_r = 0$ for all $r \geq p$. □

Lemma 2.4. *There exists $r_k > 0$ such that*

$$b_k(\lambda) = \inf_{u \in Z_k,\ \|u\|=r_k} J_\lambda(u) > 0, \ \ \forall\, \lambda \in [1,2].$$

Proof. For any $u \in Z_k$ and $\lambda \in [1,2]$, by (2.4) and (H_1) and the above definition of $\alpha_r(k)$, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\,\|u\|^p + \sum_{i=1}^{l} \int_0^{u(t_i)} I_i(x)\,\mathrm{d}x - \lambda \int_0^T F(t,u(t))\,\mathrm{d}t \\ &\geq \frac{1}{p}\,\|u\|^p - 2\theta_3\|u\|_p^p - 2\theta_4\|u\|_\nu^\nu \geq \frac{1}{p}\,\|u\|^p - 2\theta_3\alpha_p^p(k)\|u\|^p - 2\theta_4\alpha_\nu^\nu(k)\|u\|^\nu. \end{aligned}$$

Choose

$$r_k = \frac{1}{\alpha_p(k) + \alpha_\nu(k)}\,.$$

Then $r_k \to +\infty$ as $k \to +\infty$. Hence for $u \in Z_k$ with $\|u\| = r_k$, we obtain

$$J_\lambda(u) \geq \frac{1}{p}\,\|u\|^p - 2\theta_3\,\frac{\alpha_p^p(k)}{(\alpha_p(k)+\alpha_\nu(k))^p} - 2\theta_4\,\frac{\alpha_\nu^\nu(k)}{(\alpha_p(k)+\alpha_\nu(k))^\nu} \geq \frac{1}{p}\,r_k^p - 2\theta_3 - 2\theta_4 > 0.$$

Therefore,

$$b_k(\lambda) = \inf_{u \in Z_k,\ \|u\|=r_k} J_\lambda(u) > 0, \ \ \forall\, \lambda \in [1,2].$$

□

Lemma 2.5. *There exists ρ_k large enough and $\rho_k > r_k$ such that*

$$a_k(\lambda) = \max_{u\in Y_k,\ \|u\|=\rho_k} J_\lambda(u) < 0, \ \ \forall\,\lambda \in [1,2].$$

Proof. First, we claim that for any $u \in Y_k$, there exists $\epsilon_1 > 0$ such that

$$\operatorname{meas}\big\{t \in [0,T] : |u(t)| \geq \epsilon_1 \|u\|\big\} \geq \epsilon_1, \ \ \forall u \in Y_k \setminus \{0\}. \tag{2.5}$$

Proof of claim. We argue by the contradiction and suppose that for any positive integer n there exists $u_n \in Y_k \setminus \{0\}$ such that

$$\operatorname{meas}\Big\{t \in [0,T] :\ |u_n(t)| \geq \frac{1}{n}\,\|u_n\|\Big\} < \frac{1}{n}\,, \ \ \forall\, n \in \mathbb{N}.$$

Set $v_n(t) = \frac{u_n(t)}{\|u_n\|} \in Y_k \setminus \{0\}$. Then $\|v_n\| = 1$ and

$$\operatorname{meas}\Big\{t \in [0,T] :\ |v_n(t)| \geq \frac{1}{n}\Big\} < \frac{1}{n}\,, \ \ \forall n \in \mathbb{N}. \tag{2.6}$$

Since $\dim Y_k < +\infty$, it follows from the unit sphere of Y_k that there exists a subsequence denoted by $\{v_n\}$ such that v_n converges to some v in Y_k. Therefore, we have $\|v\| = 1$. By the fact that all norms are equivalent on Y_k, we deduce that $v_n \to v$ in $L^p([0,T])$, i.e.,

$$\int_0^T |v_n(t) - v(t)|^p \,\mathrm{d}t \to 0 \ \text{ as } \ n \to +\infty. \tag{2.7}$$

Thus there exist $\xi_1, \xi_2 > 0$ such that

$$\operatorname{meas}\big\{t \in [0,T] :\ |v(t)| \geq \xi_1\big\} \geq \xi_2. \tag{2.8}$$

In fact, if not, for all positive integers n, we have

$$\operatorname{meas}\Big\{t \in [0,T] :\ |v(t)| \geq \frac{1}{n}\Big\} = 0, \ \text{ i.e., } \ \operatorname{meas}\Big\{t \in [0,T] :\ |v(t)| < \frac{1}{n}\Big\} = T.$$

It implies that

$$0 < \int_0^T |v(t)|^p \,\mathrm{d}t < \frac{1}{n^p}\,T \to 0 \ \text{ as } \ n \to +\infty.$$

Hence $v = 0$, which contradicts that $\|v\| = 1$. Therefore, (2.8) holds.

Now let

$$\Omega_0 = \operatorname{meas}\big\{t \in [0,T] :\ |v(t)| \geq \xi_1\big\}, \quad \Omega_n = \Big\{t \in [0,T] :\ |v(t)| < \frac{1}{n}\Big\} \ \text{ and } \ \Omega_n^c = [0,T] \setminus \Omega_n.$$

By (2.6) and (2.8), we have

$$\operatorname{meas}(\Omega_0 \cap \Omega_n) = \operatorname{meas}\big(\Omega_0 \setminus (\Omega_n^c \cap \Omega_0)\big) \geq \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_n^c \cap \Omega_0) \geq \xi_2 - \frac{1}{n}$$

for all positive integers n. Let n be large enough such that $\xi_2 - \frac{1}{n} \geq \frac{1}{2}\,\xi_2$ and $\xi_1 - \frac{1}{n} \geq \frac{1}{2}\,\xi_1$.

Then we have

$$|v_n(t) - v(t)|^p \geq \Big(\xi_1 - \frac{1}{n}\Big)^p \geq \frac{1}{2^p}\,\xi_1^p, \ \ \forall\, t \in \Omega_0 \cap \Omega_n.$$

Also,

$$\int_0^T |v_n(t) - v(t)|^p \,\mathrm{d}t \geq \int_{\Omega_0\cap\Omega_n} |v_n - v|^p \,\mathrm{d}t \geq \frac{1}{2^p}\,\xi_1^p \operatorname{meas}(\Omega_0 \cap \Omega_n) \geq \frac{1}{2^p}\,\xi_1^p\Big(\xi_2 - \frac{1}{n}\Big) \geq \frac{1}{2^{p+1}}\,\xi_1^p \xi_2$$

for all large n, which is a contradiction to (2.7). Therefore, (2.5) holds.

Now, using the fact that Y_k is finite-dimensional and the claim, we can find $\epsilon_k > 0$ such that

$$\operatorname{meas}\left\{t \in [0,T] : \ |u(t)| \geq \epsilon_k \|u\|\right\} \geq \epsilon_k, \quad \forall\, u \in Y_k \setminus \{0\}. \tag{2.9}$$

By (H_4), for any $k \in \mathbb{N}$, there exists $R_k > 0$ such that

$$F(t,u) \geq \frac{|u|^p}{\epsilon_k^{p+1}} \ \text{ uniformly for } \ t \in [0,T] \ \text{ and } \ |u| \geq R_k.$$

Set

$$\Omega_u^k = \left\{t \in [0,T] : \ |u(t)| \geq \epsilon_k \|u\|\right\}$$

and let us observe that, by (2.9), $\operatorname{meas}(\Omega_u^k) \geq \epsilon_k$ for any $u \in Y_k \setminus \{0\}$. Then for any $u \in Y_k$ with $\|u\| \geq \frac{R_k}{\epsilon_k}$, it follows from (H_2), (H_4), (2.5) and Lemma 2.1 that

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\,\|u\|^p + \sum_{i=1}^{l} \int_0^{u(t_i)} I_i(x)\,\mathrm{d}x - \lambda \int_0^T F(t,u(t))\,\mathrm{d}t \\
&\leq \frac{1}{p}\,\|u\|^p + \sum_{i=1}^{l} \left(a_i M\|u\| + b_i M^{\gamma_i+1}\|u\|^{\gamma_i+1}\right) - \int_{\Omega_u^k} F(t,u(t))\,\mathrm{d}t \\
&\leq \frac{1}{p}\,\|u\|^p + \sum_{i=1}^{l} \left(a_i M\|u\| + b_i M^{\gamma_i+1}\|u\|^{\gamma_i+1}\right) - \frac{\|u\|^p}{\epsilon_k^{p+1}}\,\epsilon_k^p \operatorname{meas}(\Omega_u^k) \\
&\leq \frac{1}{p}\,\|u\|^p + \sum_{i=1}^{l} \left(a_i M\|u\| + b_i M^{\gamma_i+1}\|u\|^{\gamma_i+1}\right) - \|u\|^p \\
&= -\frac{(p-1)}{p}\,\|u\|^p + \sum_{i=1}^{l} \left(a_i M\|u\| + b_i M^{\gamma_i+1}\|u\|^{\gamma_i+1}\right)
\end{aligned}$$

for all $u \in Y_k$. Since $\gamma_i < p-1$, choosing ρ_k large enough such that

$$\rho_k > \max\left\{r_k, \frac{R_k}{\epsilon_k}\right\} \ \text{ for all } \ k > k_1,$$

it follows that

$$a_k(\lambda) = \max_{u \in Y_k,\ \|u\|=\rho_k} J_\lambda(u) < 0, \quad \forall\, k > k_1.$$

□

Since all assumptions of Theorem 2.2 hold, for $\lambda \in [1,2]$, there exists a sequence $\{u_{n,k}(\lambda)\}_{n=1}^\infty$ such that

$$\sup_n \|u_{n,k}(\lambda)\| < +\infty, \quad J'_\lambda(u_{n,k}(\lambda)) \to 0 \ \text{ and } \ J_\lambda(u_{n,k}(\lambda)) \to c_k(\lambda) \ \text{ as } \ n \to +\infty,$$

where

$$c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} J_\lambda(\gamma(u)).$$

From the proof of Lemma 2.4, we deduce that for any $k > k_1$ and $\lambda \in [1,2]$,

$$c_k(\lambda) \geq b_k(\lambda) \geq \frac{1}{p}\,r_k^p - 2\theta_3 - 2\theta_4 = \bar{b}_k \to +\infty \ \text{ as } \ k \to +\infty,$$

and

$$c_k(\lambda) \leq \max_{u \in B_k} J_1(u) = \bar{c}_k.$$

Thus

$$\bar{b}_k \leq c_k(\lambda) \leq \bar{c}_k \ \text{ for all } \ \lambda \in [1,2].$$

As a consequence, for any $k \geq k_1$, we can choose $\lambda_m \to 1$, $m \to +\infty$, and get the corresponding sequences satisfying

$$\sup_n \|u_{n,k}(\lambda_m)\| < +\infty, \quad J'_{\lambda_m}(u_{n,k}(\lambda_m)) \to 0 \text{ and } J_{\lambda_m}(u_{n,k}(\lambda_m)) \to c_k(\lambda_m) \text{ as } n \to +\infty.$$

Now, we prove that for any $k \geq k_1$, $\{u_{n,k}(\lambda_m)\}_{n\in\mathbb{N}}$ admits a strongly convergent subsequence and that such subsequence is bounded.

Lemma 2.6. *For each λ_m given above, the sequence $\{u_{n,k}(\lambda_m)\}_{n\in\mathbb{N}}$ has a strong convergent subsequence.*

Proof. The fact that $\sup_n \|u_{n,k}(\lambda_m)\| < +\infty$ implies that $\{u_{n,k}(\lambda_m)\}_{n\in\mathbb{N}}$ is bounded in X. Since X is a reflexive Banach space, passing to a subsequence, if necessary, we may assume that there is a $u^k(\lambda_m) \in X$ such that

$$\begin{aligned} &u_{n,k}(\lambda_m) \rightharpoonup u_k(\lambda_m) \text{ in } X \text{ as } n \to +\infty, \\ &u_{n,k}(\lambda_m) \to u_k(\lambda_m) \text{ in } L^p([0,T]) \text{ as } n \to +\infty \end{aligned}$$

and

$$\{u_{n,k}(\lambda_m)\}_{n\in\mathbb{N}} \text{ converges uniformly to } u_k(\lambda_m) \text{ on } [0,T].$$

Thus we have

$$\sum_{i=1}^{l} \big(I_i(u_{n,k}(\lambda_m)(t_i)) - I_i(u_k(\lambda_m)(t_i))\big)\big(u_{n,k}(\lambda_m)(t_i) - u_k(\lambda_m)(t_i)\big) \longrightarrow 0 \text{ as } n \to +\infty, \tag{2.10}$$

$$\int_0^T \big(f(t, u_{n,k}(\lambda_m)) - f(t, u_k(\lambda_m))\big)\big(u_{n,k}(\lambda_m) - u_k(\lambda_m)\big)\,\mathrm{d}t \longrightarrow 0 \text{ as } n \to +\infty. \tag{2.11}$$

Notice that

$$\begin{aligned} &\Big\langle J'_{\lambda_m}(u_{n,k}(\lambda_m)) - J'_{\lambda_m}(u_k(\lambda_m)), u_{n,k}(\lambda_m) - u_k(\lambda_m)\Big\rangle \\ &\quad = \int_0^T \rho(t)\big(\Phi_p(u'_{n,k}(\lambda_m)) - \Phi_p(u'_k(\lambda_m))\big)\big(u'_{n,k}(\lambda_m) - u'_k(\lambda_m)\big)\,\mathrm{d}t \\ &\qquad + \int_0^T s(t)\big(\Phi_p(u_{n,k}(\lambda_m)) - \Phi_p(u_k(\lambda_m))\big)\big(u_{n,k}(\lambda_m) - u_k(\lambda_m)\big)\,\mathrm{d}t \\ &\qquad + \sum_{i=1}^{l} \big(I_i(u_{n,k}(\lambda_m)(t_i)) - I_i(u_k(\lambda_m)(t_i))\big)\big(u_{n,k}(\lambda_m)(t_i) - u_k(\lambda_m)(t_i)\big) \\ &\qquad - \lambda_m \int_0^T \big(f(t, u_{n,k}(\lambda_m)) - f(t, u_k(\lambda_m))\big)\big(u_{n,k}(\lambda_m) - u_k(\lambda_m)\big)\,\mathrm{d}t. \end{aligned} \tag{2.12}$$

Recalling the following inequalities, for any $x, y \in \mathbb{R}$, there exist $c_p, d_p > 0$ such that

$$\big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\big\rangle \geq c_p|x-y|^p \text{ if } p \geq 2 \tag{2.13}$$

and

$$\big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\big\rangle \geq d_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}} \text{ if } 1 < p < 2. \tag{2.14}$$

Then if $p \geq 2$, there exists $c_p > 0$ such that

$$\begin{aligned}\int_0^T \rho(t)\big(\Phi_p(u'_{n,k}(\lambda_m)) - \Phi_p(u'_k(\lambda_m))\big)\big(u'_{n,k}(\lambda_m) - u'_k(\lambda_m)\big)\,\mathrm{d}t \\ + \int_0^T s(t)\big(\Phi_p(u_{n,k}(\lambda_m)) - \Phi_p(u_k(\lambda_m))\big)\big(u_{n,k}(\lambda_m) - u_k(\lambda_m)\big)\,\mathrm{d}t \\ \geq c_p \int_0^T \Big(\rho(t)\big|u'_{n,k}(\lambda_m) - u'_k(\lambda_m)\big|^p + s(t)\big|u_{n,k}(\lambda_m) - u_k(\lambda_m)\big|^p\Big)\,\mathrm{d}t \\ = c_p\|u_{n,k}(\lambda_m) - u_k(\lambda_m)\|^p.\end{aligned} \tag{2.15}$$

Since

$$\lim_{n\to+\infty} J'_{\lambda_m}(u_{n,k}(\lambda_m)) = 0$$

and $u_{n,k}(\lambda_m)$ converges weakly to $u_k(\lambda_m)$, one has

$$\lim_{n\to+\infty} \Big\langle J'_{\lambda_m}(u_{n,k}(\lambda_m)) - J'_{\lambda_m}(u_k(\lambda_m)), u_{n,k}(\lambda_m) - u_k(\lambda_m)\Big\rangle = 0. \tag{2.16}$$

By (2.10)–(2.12), (2.15) and (2.16), we have

$$\begin{aligned}c_p\|u_{n,k}(\lambda_m) - u_k(\lambda_m)\|^p \leq \Big\langle J'_{\lambda_m}(u_{n,k}(\lambda_m)) - J'_{\lambda_m}(u_k(\lambda_m)), u_{n,k}(\lambda_m) - u_k(\lambda_m)\Big\rangle \\ - \sum_{i=1}^{l}\big(I_i(u_{n,k}(\lambda_m)(t_i)) - I_i(u_k(\lambda_m)(t_i))\big)\big(u_{n,k}(\lambda_m)(t_i) - u_k(\lambda_m)(t_i)\big) \\ + \lambda_m \int_0^T \big(f(t, u_{n,k}(\lambda_m)) - f(t, u_k(\lambda_m))\big)\big(u_{n,k}(\lambda_m) - u_k(\lambda_m)\big)\,\mathrm{d}t \longrightarrow 0 \ \text{ as } \ n \to +\infty.\end{aligned}$$

Then

$$\|u_{n,k}(\lambda_m) - u_k(\lambda_m)\| \to 0 \ \text{ as } \ n \to +\infty.$$

If $1 < p < 2$, by (2.14), there exists $d_p > 0$ such that

$$\begin{aligned}\int_0^T \rho(t)\big(\Phi_p(u'_{n,k}(\lambda_m)) - \Phi_p(u'_k(\lambda_m))\big)\big(u'_{n,k}(\lambda_m) - u'_k(\lambda_m)\big)\,\mathrm{d}t \\ + \int_0^T s(t)\big(\Phi_p(u_{n,k}(\lambda_m)) - \Phi_p(u_k(\lambda_m))\big)\big(u_{n,k}(\lambda_m) - u_k(\lambda_m)\big)\,\mathrm{d}t \\ \geq d_p \int_0^T \bigg(\frac{\rho(t)|u'_{n,k}(\lambda_m) - u'_k(\lambda_m)|^2}{(|u'_{n,k}(\lambda_m)| + |u'_k(\lambda_m)|)^{2-p}} + \frac{s(t)|u_{n,k}(\lambda_m) - u_k(\lambda_m)|^2}{(|u_{n,k}(\lambda_m)| + |u_k(\lambda_m)|)^{2-p}}\bigg)\,\mathrm{d}t.\end{aligned} \tag{2.17}$$

Furthermore, by the Hölder inequality, one has

$$\begin{aligned}\int_0^T \rho(t)|u'_{n,k}(\lambda_m) - u'_k(\lambda_m)|^p\,\mathrm{d}t \\ \leq \int_0^T \bigg(\frac{\rho(t)|u'_{n,k}(\lambda_m) - u'_k(\lambda_m)|^2}{(|u'_{n,k}(\lambda_m)| + |u'_k(\lambda_m)|)^{2-p}}\,\mathrm{d}t\bigg)^{\frac{p}{2}}\bigg(\int_0^T \rho(t)\big(|u'_{n,k}(\lambda_m)| + |u'_k(\lambda_m)|\big)^p\,\mathrm{d}t\bigg)^{\frac{2-p}{p}}\end{aligned}$$

$$\leq 2^{\frac{(p-1)(2-p)}{2}} \int_0^T \left(\frac{\rho(t)|u'_{n,k}(\lambda_m) - u'_k(\lambda_m)|^2}{(|u'_{n,k}(\lambda_m)| + |u'_k(\lambda_m)|)^{2-p}} \,\mathrm{d}t \right)^{\frac{p}{2}} \left(\int_0^T \rho(t)(|u'_{n,k}(\lambda_m)|^p + |u'_k(\lambda_m)|^p) \,\mathrm{d}t \right)^{\frac{2-p}{p}}$$

$$\leq 2^{\frac{(p-1)(2-p)}{2}} \int_0^T \left(\frac{\rho(t)|u'_{n,k}(\lambda_m) - u'_k(\lambda_m)|^2}{(|u'_{n,k}(\lambda_m)| + |u'_k(\lambda_m)|)^{2-p}} \,\mathrm{d}t \right)^{\frac{p}{2}} \big(\|u_{n,k}(\lambda_m)\| + \|u_k(\lambda_m)\| \big)^{\frac{(2-p)p}{2}}. \tag{2.18}$$

Similarly,

$$\int_0^T s(t)\big|u_{n,k}(\lambda_m) - u_k(\lambda_m)\big|^p \,\mathrm{d}t$$

$$\leq 2^{\frac{(p-1)(2-p)}{2}} \int_0^T \left(\frac{\rho(t)|u_{n,k}(\lambda_m) - u_k(\lambda_m)|^2}{(|u_{n,k}(\lambda_m)| + |u_k(\lambda_m)|)^{2-p}} \,\mathrm{d}t \right)^{\frac{p}{2}} \big(\|u_{n,k}(\lambda_m)\| + \|u_k(\lambda_m)\| \big)^{\frac{(2-p)p}{2}}. \tag{2.19}$$

So, by (2.17)–(2.19), it follows that

$$\begin{aligned}
&\int_0^T \rho(t)\big(\Phi_p(u'_{n,k}(\lambda_m)) - \Phi_p(u'_k(\lambda_m))\big)(u'_{n,k}(\lambda_m) - u'_k(\lambda_m)) \,\mathrm{d}t \\
&\qquad + \int_0^T s(t)\big(\Phi_p(u_{n,k}(\lambda_m)) - \Phi_p(u_k(\lambda_m))\big)\big(u_{n,k}(\lambda_m) - u_k(\lambda_m)\big) \,\mathrm{d}t \\
&\geq \frac{2^{\frac{(p-1)(p-2)}{2}} d_p}{(\|u_{n,k}(\lambda_m)\| + \|u_k(\lambda_m)\|)^{2-p}} \left[\left(\int_0^T \rho(t)\big|u'_{n,k}(\lambda_m) - u'_k(\lambda_m)\big|^p \,\mathrm{d}t \right)^{\frac{2}{p}} \right. \\
&\qquad \left. + \left(\int_0^T s(t)|u_{n,k}(\lambda_m) - u_k(\lambda_m)|^p \,\mathrm{d}t \right)^{\frac{2}{p}} \right] \\
&\geq \frac{2^{p-2} d_p}{(\|u_{n,k}(\lambda_m)\| + \|u_k(\lambda_m)\|)^{2-p}} \|u_{n,k}(\lambda_m) - u_k(\lambda_m)\|^2,
\end{aligned} \tag{2.20}$$

which implies by (2.10), (2.11), (2.12) and (2.16) that

$$\begin{aligned}
&\frac{2^{p-2} d_p}{(\|u_{n,k}(\lambda_m)\| + \|u_k(\lambda_m)\|)^{2-p}} \|u_{n,k}(\lambda_m) - u_k(\lambda_m)\|^2 \\
&\quad \leq \Big\langle J'_{\lambda_m}(u_{n,k}(\lambda_m)) - J'_{\lambda_m}(u_k(\lambda_m)), u_{n,k}(\lambda_m) - u_k(\lambda_m) \Big\rangle \\
&\qquad - \sum_{i=1}^{l} \big(I_i(u_{n,k}(\lambda_m)(t_i)) - I_i(u_k(\lambda_m)(t_i))\big)\big(u_{n,k}(\lambda_m)(t_i) - u_k(\lambda_m)(t_i)\big) \\
&\qquad + \lambda_m \int_0^T \big(f(t, u_{n,k}(\lambda_m)) - f(t, u_k(\lambda_m))\big)\big(u_{n,k}(\lambda_m) - u_k(\lambda_m)\big) \,\mathrm{d}t \longrightarrow 0 \text{ as } n \to +\infty.
\end{aligned}$$

Then

$$\|u_{n,k}(\lambda_m) - u_k(\lambda_m)\| \to 0 \text{ as } n \to +\infty.$$

Therefore, in all cases, $\{u_{n,k}(\lambda_m)\}_{n\in\mathbb{N}}$ converges strongly to $u_k(\lambda_m)$ in X for all $m \in \mathbb{N}$ and $k \geq k_1$. As a consequence, we obtain

$$J'_{\lambda_m}(u_k(\lambda_m)) = 0, \quad J_{\lambda_m}(u_k(\lambda_m)) \in [\bar{b}_k, \bar{c}_k], \ \forall\, m \in \mathbb{N} \text{ and } k \geq k_1. \tag{2.21}$$

The lemma is proved. □

Lemma 2.7. *For any $k \geq k_1$, the sequence $\{u_k(\lambda_m)\}_{m\in\mathbb{N}}$ is bounded.*

Proof. For simplicity, we set $u_k(\lambda_m) = u_m$. We suppose by contradiction that

$$\|u_m\| \to +\infty \ \text{ as } \ m \to +\infty. \tag{2.22}$$

Let $z_m = \frac{u_m}{\|u_m\|}$ for any $m \in \mathbb{N}$, $\{z_m\}_{m\in\mathbb{N}}$ be bounded and $\|z_m\| = 1$. Then there exists a subsequence of z_m denoted again by z_m such that

$$z_m \rightharpoonup z \ \text{ in } X \ \text{ as } \ m \to +\infty, \tag{2.23}$$

$$z_m \to z \ \text{ in } L^p([0,T]) \ \text{ as } \ m \to +\infty, \tag{2.24}$$

$$\{z_m\}_{m\in\mathbb{N}} \ \text{ converges uniformly to } z \text{ on } [0,T]. \tag{2.25}$$

Now we distinguish two cases.

Case $z = 0$**.** We can say that for any $m \in \mathbb{N}$, there exists $t_m \in [0,1]$ such that

$$J_{\lambda_m}(t_m u_m) = \max_{t\in[0,1]} J_{\lambda_m}(tu_m). \tag{2.26}$$

By (2.22), we can choose $r_j = (2jp)^{\frac{1}{p}} z_m$ such that

$$0 < \frac{r_j}{\|u_m\|} < 1, \tag{2.27}$$

with m large enough. By (2.25), $F(\,\cdot\,,0) = 0$ and the continuity of F, we have

$$F(t, r_j z_m) \to F(t, r_j z) = 0 \ \text{ as } \ \ m \to +\infty \ \text{ for any } \ j \in \mathbb{N} \ \text{ and uniformly for } \ t \in [0,T]. \tag{2.28}$$

By (H_3), (H_4), Lemma 2.1, (2.25), (2.28) and by applying the dominated convergence theorem, we deduce that

$$F(t, r_j z_m) \to 0 \ \text{ in } L^1([0,T]) \ \text{ as } \ m \to +\infty \ \text{ for any } \ j \in \mathbb{N}. \tag{2.29}$$

Then by (2.26), (2.27) and (2.29), we have

$$J_{\lambda_m}(t_m u_m) \geq J_{\lambda_m}(r_j z_m) = \frac{1}{p}\,\|r_j z_m\|^p + \sum_{i=1}^{l} G(r_j z_m(t_i) - \lambda_m \int_0^T F(t, r_j z_m(t))\,\mathrm{d}t \geq 2j - j = j,$$

provided n is large enough, for any $j \in \mathbb{N}$. Therefore,

$$J_{\lambda_m}(t_m u_m) \to +\infty \ \text{ as } \ \ m \to +\infty. \tag{2.30}$$

Since $J_{\lambda_m}(0) = 0$ and $J_{\lambda_m}(t_m u_m) \in [\bar{b}_k, \bar{c}_k]$, we deduce that $t_m \in]0,1[$ for m large enough.

From (2.26), we have

$$\left\langle J'_{\lambda_m}(t_m u_m), t_m u_m \right\rangle = t_m \left.\frac{d}{dt}\right|_{t=t_m} J_{\lambda_m}(tu_m) = 0. \tag{2.31}$$

Let $\theta = \max\{\theta_1, \theta_2\}$ and taking into account (H_6) and (2.31), we have

$$\begin{aligned}\frac{p}{\theta}\,J_{\lambda_m}(t_m u_m) &= \frac{1}{\theta}\left(pJ_{\lambda_m}(t_m u_m) - \left\langle J'_{\lambda_m}(t_m u_m), t_m u_m \right\rangle\right)\\ &= \frac{1}{\theta}\sum_{i=1}^{l}\left(pG(t_m u_m(t_i)) - I_i(t_m u_m(t_i)) t_m u_m(t_i)\right)\\ &\qquad + \frac{\lambda_m}{\theta}\int_0^T \left(f(t, t_m u_m(t)) t_m u_m(t) - pF(t, t_m u_m(t))\right)\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\theta}\sum_{i=1}^{l}\mathcal{G}(t_m u_m(t_i)) + \frac{\lambda_m}{\theta}\int_0^T \mathcal{F}(t, t_m u_m(t))\,\mathrm{d}t \\
&\le \frac{1}{\theta}\sum_{i=1}^{l}\theta_1\mathcal{G}(u_m(t_i)) + \frac{\lambda_m}{\theta}\int_0^T \theta_2\mathcal{F}(t, u_m(t))\,\mathrm{d}t \\
&\le \sum_{i=1}^{l}\mathcal{G}(u_m(t_i)) + \lambda_m\int_0^T \mathcal{F}(t, u_m(t))\,\mathrm{d}t \\
&= pJ_{\lambda_m}(u_m) - \langle J'_{\lambda_m}(u_m), u_m\rangle = pJ_{\lambda_m}(u_m)
\end{aligned}
$$

which contradicts (2.21) and (2.30).

Case $z \neq 0$**.** Let $\Omega = \{t \in [0,T] : \ z(t) \neq 0\}$, then $\operatorname{meas}(\Omega) > 0$. By using (2.22) and $z \neq 0$, we obtain

$$
|z_m(t)| \to +\infty \ \text{ uniformly on } \ t \in \Omega \ \text{ as } \ m \to +\infty. \tag{2.32}
$$

Notice that

$$
\begin{aligned}
\frac{1}{p} - \frac{J_{\lambda_m}(u_m)}{\|u_m\|^p} &= \lambda_m \int_0^T \frac{F(t, u_m(t))}{\|u_m\|^p}\,\mathrm{d}t - \sum_{i=1}^{l}\frac{G(u_m(t_i))}{\|u_m\|^p} \\
&\ge \lambda_m \int_\Omega |z_m(t)|^p \frac{F(t, u_m(t))}{|u_m(t)|^p}\,\mathrm{d}t - \sum_{i=1}^{l}\frac{G(u_m(t_i))}{\|u_m\|^p}\,.
\end{aligned}
$$

Putting together (H_4), (H_2) and applying Fatou's Lemma, we deduce that

$$
\int_\Omega |z_m(t)|^p \frac{F(t, u_m(t))}{|u_m(t)|^p}\,\mathrm{d}t \longrightarrow +\infty \ \text{ as } \ m \to +\infty \tag{2.34}
$$

and

$$
\begin{aligned}
\Big|\sum_{i=1}^{l}\frac{G(u_m(t_i))}{\|u_m\|^p}\Big| &\le \sum_{i=1}^{l}\frac{a_i|u_m(t_i)|}{\|u_m\|^p} + \sum_{i=1}^{l}\frac{b_i|u_m(t_i)|^{\gamma_i+1}}{(\gamma_i+1)\|u_m\|^p} \\
&\le \sum_{i=1}^{l}\frac{a_i M\|u_m\|}{\|u_m\|^p} + \sum_{i=1}^{l}\frac{b_i M^{\gamma_i+1}\|u_m\|^{\gamma_i+1}}{(\gamma_i+1)\|u_m\|^p} \\
&= \sum_{i=1}^{l}\frac{a_i M}{\|u_m\|^{p-1}} + \sum_{i=1}^{l}\frac{b_i M^{\gamma_i+1}}{(\gamma_i+1)\|u_m\|^{p-\gamma_i-1}}\,.
\end{aligned}
$$

Since $p > 1$ and $p > \gamma_i + 1$ for all $i \in \{1, 2, \ldots, l\}$, we have

$$
\sum_{i=1}^{l}\frac{a_i M}{\|u_m\|^{p-1}} + \sum_{i=1}^{l}\frac{b_i M^{\gamma_i+1}}{(\gamma_i+1)\|u_m\|^{p-\gamma_i-1}} \longrightarrow 0 \ \text{ as } \ m \to +\infty
$$

which implies that

$$
\sum_{i=1}^{l}\frac{G(u_m(t_i))}{\|u_m\|^p} \longrightarrow 0 \ \text{ as } \ m \to +\infty. \tag{2.35}
$$

Then, by (2.21), (2.33), (2.34) and (2.35), we obtain $\frac{1}{p} \ge +\infty$, which is a contradiction.

Thus we have proved that the sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded in X. □

Therefore, $\{u_k(\lambda_m)\}_{m\in\mathbb{N}}$ is bounded in X for all $k \ge k_1$. Also, as a similar argument of the proof of Lemma 2.6, we can show that $u_k(\lambda_m) \to u_k$ in X as $m \to +\infty$ for all $k \ge k_1$. Then u_k is a critical point of $J = J_1$ with $J(u_k) \in [\bar{b}_k, \bar{c}_k]$ for all $k \ge k_1$. According to $b_k \to +\infty$ as $k \to +\infty$, we know that problem (1.1) has infinitely many nontrivial high energy solutions. □

3 Example

In this section, an example is given to illustrate our result.

Consider the following problem:

$$\begin{gathered} -\big((t+3)|u'(t)|^5u'(t)\big)'+(t^2+5t+1)|u(t)|^5u(t)=(t^9+6)|u|^5u\ln(1+|u|),\ t\neq t_1,\ \text{a.e. } t\in[0,T],\\ -\Delta_7((t_1+3)|u'(t_1)|^5u'(t_1))=u^5(t_1),\\ u(0)=u(T)=0, \end{gathered} \tag{3.1}$$

we have chosen $p=7$, $I_1(u)=u^5(t_1)$ and

$$f(t,u)=(t^9+6)|u|^5u\ln(1+|u|)\ \text{ for all }\ (t,u)\in[0,T]\times\mathbb{R}.$$

We remark that all assumptions (H_1)–(H_6) are satisfied. Therefore, by Theorem 2.1, problem (3.1) has infinitely many nontrivial high energy solutions.

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