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George Chkadua

ASYMPTOTIC ANALYSIS AND REGULARITY RESULTS FOR A MIXED TYPE INTERACTION PROBLEM OF ACOUSTIC WAVES AND ELECTRO-MAGNETO-ELASTIC STRUCTURES

Abstract. In the paper, we consider a three-dimensional model of fluid-solid acoustic interaction when an electro-magneto-elastic body occupying a bounded region Ω^+ is embedded in an unbounded fluid domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. In this case, we have a five-dimensional electro-magneto-elastic field (the displacement vector with three components, electric potential and magnetic potential) in the domain Ω^+, while we have a scalar acoustic pressure field in the unbounded domain Ω^-. The physical kinematic and dynamic relations are mathematically described by the appropriate boundary and transmission conditions. We consider less restrictions on a matrix differential operator of electro-magneto-elasticity by introducing asymptotic classes, in particular, we allow the corresponding characteristic polynomial of the matrix operator to have multiple real zeros.

In the paper, we consider mixed type interaction problem. In particular, except transmission conditions, electric and magnetic potentials are given on one part of the boundary of Ω^+ (the Dirichlet type condition), while on the other part, normal components of electric displacement and magnetic induction are given (the Neumann type condition).

We derive asymptotic expansion of solutions near the line where different boundary conditions change, and on the basis of asymptotic analysis, we establish optimal Hölder's smoothness results for solutions of the problem.

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რეზიუმე. ნაშრომში განვიხილავთ სითხისა და სხეულის აკუსტიკური ურთიერთქმედების სამგანზომილებიან მოდელს, როდესაც ელექტრო-მაგნეტო-დრეკად სხეულს უკავია Ω^+ შემოსაზღვრული არე, რომელიც ჩადგმულია $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$ შემოუსაზღვრელ სითხის არეში. ამ შემთხვევაში შემოსაზღვრულ Ω^+ არეში არის ხუთგანზომილებიანი ელექტრო-მაგნეტო-დრეკადი ველი (გადაადგილების ვექტორის სამი კომპონენტი, ელექტრული პოტენციალი და მაგნიტური პოტენციალი), ხოლო შემოუსაზღვრელ Ω^- არეში - აკუსტიკური წნევის სკალარული ველი. ფიზიკური კინემატიკური და დინამიკური ურთიერთქმედებები მათემატიკურად აღწერილია შესაბამისი სასაზღვრო და ტრანსმისიის პირობებით. ნაშრომში მოთხოვნილია ნაკლები შეზღუდვები ელექტრო-მაგნეტო-დრეკადობის დიფერენციალურ ოპერატორზე და შემოღებულია შესაბამისი ასიმპტოტური კლასები. კერძოდ, მატრიცული დიფერენციალური ოპერატორის შესაბამის მახასიათებელ პოლინომს შესაძლოა გააჩნდეს ჯერადი ნამდვილი ნულები.

ნაშრომში განვიხილავთ შერეული ტიპის ურთიერთქმედების ამოცანას, კერძოდ, ტრანსმისიის პირობების გარდა, Ω^+ არის საზღვრის ერთ ნაწილზე მოცემულია ელექტრული და მაგნიტური ველის პოტენციალები (დირიხლეს პირობა), ხოლო მეორე ნაწილზე - ელექტრული გადაადგილების და მაგნიტური ინდუქციის ნორმალური კომპონენტები (ნეიმანის პირობა).

მიღებულია ამონახსნის ასიმპტოტური გაშლა იმ წირის მახლობლობაში, სადაც იცვლება სასაზღვრო პირობები, და ასიმპტოტური ანალიზის გამოყენებით დადგენილია ამოცანის ამონახსნის ოპტიმალური ჰელდერის სიგლუვე.

1 Formulation of the problems

1.1 Introduction

Solvability of the mixed type interaction problem of acoustic waves and electro-magneto-elastic structures is investigated in the paper [8] with the use of the potential method and the theory of pseudodifferential equations on manifolds with boundary and is proved existence and uniqueness theorems in Sobolev–Slobodetskii spaces.

The Dirichlet type, Neumann type and mixed type interaction problems of acoustic waves and piezoelectric structures are studied in [4, 6, 9].

In [7], the Dirichlet and Neumann type interaction problems of acoustic waves and piezo-electromagnetic structures are studied.

We consider less restrictions on matrix differential operator of electro-magneto-elasticity by introducing asymptotic classes $M_{m_1,m_2,m_3}(\mathbf{P})$, where $\mathbf{P}$ is determinant of the electro-magneto-elasticity matrix operator, in particular, we allow the corresponding characteristic polynomial of the matrix operator to have multiple real zeros.

In this paper, we derive asymptotic expansions of solutions of the problems (M_τ) and (M_Ω) near the line where different boundary conditions meet, and on the basis of asymptotic analysis, we obtain optimal Hölder smoothness results for solutions. In particular, it turns out that the acoustic pressure has Hölder smoothness higher by one than themechanical displacement vector and the electric potential. It means that the acoustic pressure has $C^{\delta'+1}$ smoothness, while the displacement vector and the electric potential have $C^{\delta'}$ smoothness, with some $\delta' \in (0,1)$. In the case when the domain Ω^+ is occupied by a special class of solids, which belong to the 422 (Tetragonal) or 622 (Hexagonal) class of crystals, one has $\delta' = 1/2$. Note that in the general anisotropic case, the smoothness of solutions depends on the material constants and also on the geometry of the line where the different boundary conditions meet.

1.2 Electro-magnetic field

Let Ω^+ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a compact, C^∞-smooth boundary $S = \partial\Omega^+$ and let $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domain Ω^+ is filled with an anisotropic homogeneous piezoelectro-magnetic material.

The basic equations of steady state oscillations of piezoelectro-magneticity for anisotropic homogeneous media are written as follows:

$$\begin{gathered}
c_{ijkl}\partial_i\partial_l u_k + \rho_1\omega^2\delta_{jk}u_k + e_{lij}\partial_l\partial_i\varphi + q_{lij}\partial_i\partial_l\psi + F_j = 0, \quad j = 1,2,3, \\
-e_{ikl}\partial_i\partial_l u_k + \varepsilon_{il}\partial_i\partial_l\varphi + a_{il}\partial_i\partial_l\psi + F_4 = 0, \\
-q_{ikl}\partial_i\partial_l u_k + a_{il}\partial_i\partial_l\varphi + \mu_{il}\partial_i\partial_l\psi + F_5 = 0,
\end{gathered}$$

or in the matrix form

$$A(\partial,\omega)U + F = 0 \ \text{ in } \Omega^+,$$

where $U = (u,\varphi,\psi)^\top$, $u = (u_1,u_2,u_3)^\top$ is the displacement vector, $\varphi = u_4$ is the electric potential, $\psi = u_5$ is the magnetic potential and $F = (F_1,F_2,F_3,F_4,F_5)^\top$ is a given vector-function. The three-dimensional vector (F_1,F_2,F_3) is the mass force density, while F_4 is the electric charge density, F_5 is the electric current density, and $A(\partial,\omega)$ is the matrix differential operator,

$$\begin{gathered}
A(\partial,\omega) = [A_{jk}(\partial,\omega)]_{5\times 5}, \\
A_{jk}(\partial,\omega) = c_{ijkl}\partial_i\partial_l + \rho_1\omega^2\delta_{jk}, \quad A_{j4}(\partial,\omega) = e_{lij}\partial_l\partial_i, \quad A_{j5}(\partial,\omega) = q_{lij}\partial_l\partial_i, \\
A_{4k}(\partial,\omega) = -e_{ikl}\partial_i\partial_l, \quad A_{44}(\partial,\omega) = \varepsilon_{il}\partial_i\partial_l, \quad A_{45}(\partial,\omega) = a_{il}\partial_i\partial_l, \\
A_{5k}(\partial,\omega) = -q_{ikl}\partial_i\partial_l, \quad A_{54}(\partial,\omega) = a_{il}\partial_i\partial_l, \quad A_{55}(\partial,\omega) = \mu_{il}\partial_i\partial_l,
\end{gathered}$$

$j,k = 1,2,3$, where $\omega \in \mathbb{R}$ is a frequency parameter, ρ_1 is the density of the piezoelectro-magnetic material, c_{ijlk}, e_{ikl}, q_{ikl}, ε_{il}, μ_{il}, a_{il} are elastic, piezoelectric, piezomagnetic, dielectric, magnetic permeability and electromagnetic coupling constants respectively, δ_{jk} is the Kronecker symbol and summation over repeated indices is meant from 1 to 3, unless otherwise stated. These constants satisfy the standard symmetry conditions:

$$c_{ijkl}=c_{jikl}=c_{klij}, \quad e_{ijk}=e_{ikj}, \quad q_{ijk}=q_{ikj}, \quad \varepsilon_{ij}=\varepsilon_{ji}, \quad \mu_{jk}=\mu_{kj}, \quad a_{jk}=a_{kj}, \quad i,j,k,l=1,2,3.$$

Moreover, from physical considerations related to the positiveness of the internal energy, it follows that the quadratic forms $c_{ijkl}\xi_{ij}\xi_{kl}$ and $\varepsilon_{ij}\eta_i\eta_j$ are positive definite:

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\,\xi_{ij}\xi_{ij} \quad \forall\,\xi_{ij}=\xi_{ji}\in\mathbb{R}, \tag{1.1}$$

$$\varepsilon_{ij}\eta_i\eta_j \geq c_2|\eta|^2, \quad q_{ij}\eta_i\eta_j \geq c_3|\eta|^2, \quad \mu_{ij}\eta_i\eta_j \geq c_1|\eta|^2 \quad \forall\,\eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3, \tag{1.2}$$

where c_0, c_1, c_2 and c_3 are positive constants.

More careful analysis related to the positive definiteness of the potential energy insure that the matrix

$$\Lambda := \begin{pmatrix} [\varepsilon_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} \end{pmatrix}_{6\times 6}$$

is positive definite, i.e.,

$$\varepsilon_{kj}\zeta'_k\overline{\zeta'_j} + a_{kj}\big(\zeta'_k\overline{\zeta''_j} + \overline{\zeta'_k}\zeta''_j\big) + \mu_{kj}\zeta''_k\overline{\zeta''_j} \geq c_4\big(|\zeta'|^2+|\zeta''|^2\big) \quad \forall\,\zeta',\zeta''\in\mathbb{C}^3,$$

where c_4 is some positive constant.

The principal homogeneous symbol matrix of the operator $A(\partial,\omega)$ has the following form:

$$A^{(0)}(\xi) = \begin{pmatrix} [-c_{ijlk}\xi_i\xi_l]_{3\times 3} & [-e_{lij}\xi_l\xi_i]_{3\times 1} & [-q_{lij}\xi_l\xi_i]_{3\times 1} \\ [e_{ikl}\xi_i\xi_l]_{1\times 3} & -\varepsilon_{il}\xi_i\xi_l & -a_{il}\xi_i\xi_l \\ [q_{ikl}\xi_i\xi_l]_{1\times 3} & -a_{il}\xi_i\xi_l & -\mu_{il}\xi_i\xi_l \end{pmatrix}_{5\times 5}.$$

With the help of inequalities (1.1) and (1.2) it can be easily shown that

$$-\operatorname{Re}A^{(0)}(\xi)\zeta\cdot\zeta \geq c|\zeta|^2|\xi|^2 \quad \forall\,\zeta\in\mathbb{C}^4, \ \ \forall\,\xi\in\mathbb{R}^3, \ \ c=const>0,$$

implying that $A(\partial,\omega)$ is a strongly elliptic formally nonself-adjoint differential operator.

Here and in the sequel, $a\cdot b$ denotes the scalar product of two vectors $a,b\in\mathbb{C}^N$, $a\cdot b := \sum\limits_{k=1}^{N} a_k\bar{b}_k$.

In the theory of electro-magneto-elasticity, the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form

$$\sigma_{ij}n_i := c_{ijlk}n_i\partial_l u_k + e_{lij}n_i\partial_l\varphi + q_{lij}n_i\partial_l\psi, \ \ j=1,2,3,$$

while the normal component of the electric displacement vector $D=(D_1,D_2,D_3)^\top$ and the normal component of the magnetic induction vector $B=(B_1,B_2,B_3)^\top$ read as

$$\begin{aligned} -D_in_i &= -e_{ikl}n_i\partial_l u_k + \varepsilon_{il}n_i\partial_l\varphi + a_{il}n_i\partial_l\psi, \\ -B_in_i &= -q_{ikl}n_i\partial_l u_k + a_{il}n_i\partial_l\varphi + \mu_{il}n_i\partial_l\psi. \end{aligned}$$

Let us introduce the boundary matrix differential operator

$$\begin{gathered} T(\partial,n) = [T_{jk}(\partial,n)]_{5\times 5}, \\ T_{jk}(\partial,n) = c_{ijlk}n_i\partial_l, \quad T_{j4}(\partial,n) = e_{lij}n_i\partial_l, \quad T_{j5}(\partial,n) = q_{lij}n_i\partial_l, \\ T_{4k}(\partial,n) = -e_{ikl}n_i\partial_l, \quad T_{44}(\partial,n) = \varepsilon_{il}n_i\partial_l, \quad T_{45}(\partial,n) = a_{il}n_i\partial_l, \\ T_{5k}(\partial,n) = -q_{ikl}n_i\partial_l, \quad T_{54}(\partial,n) = a_{il}n_i\partial_l, \quad T_{55}(\partial,n) = \mu_{il}n_i\partial_l, \end{gathered}$$

$j,k=1,2,3$. For a vector $U=(u,\varphi,\psi)^\top$, we have

$$T(\partial,n)U = (\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_in_i, -B_in_i)^\top. \tag{1.3}$$

The components of the vector TU given by (1.3) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of electro-magneto-elasticity, while the fourth one is the normal component of the electric displacement vector and the fifth one is the normal component of the magnetic induction vector.

In Green's formulae, one also has the following boundary operator associated with the adjoint differential operator

$$A^*(\partial,\omega) = A^\top(-\partial,\omega) = A^\top(\partial,\omega),$$
$$\widetilde{T}(\partial,n) = \big[\widetilde{T}_{jk}(\partial,n)\big]_{5\times 5},$$

where

$$\begin{aligned}
&\widetilde{T}_{jk}(\partial,n) = T_{jk}(\partial,n), \quad \widetilde{T}_{j4}(\partial,n) = -T_{j4}(\partial,n), \quad \widetilde{T}_{j5}(\partial,n) = -T_{j5}(\partial,n),\\
&\widetilde{T}_{4k}(\partial,n) = -T_{4k}(\partial,n), \quad \widetilde{T}_{44}(\partial,n) = T_{44}(\partial,n), \quad \widetilde{T}_{45}(\partial,n) = T_{45}(\partial,n),\\
&\widetilde{T}_{5k}(\partial,n) = -T_{5k}(\partial,n), \quad \widetilde{T}_{54}(\partial,n) = T_{54}(\partial,n), \quad \widetilde{T}_{55}(\partial,n) = T_{55}(\partial,n),
\end{aligned}$$

$j,k = 1,2,3$. Let us consider the equation

$$\Phi_A(\xi,\omega) := \det A(i\xi,\omega) = \det\begin{pmatrix} [c_{ijlk}\xi_i\xi_l - \rho_1\omega^2\delta_{jk}]_{3\times 3} & [e_{lij}\xi_l\xi_i]_{3\times 1} & [q_{lij}\xi_l\xi_i]_{3\times 1}\\ [-e_{ikl}\xi_i\xi_l]_{1\times 3} & \varepsilon_{il}\xi_i\xi_l & a_{il}\xi_i\xi_l\\ [-q_{ikl}\xi_i\xi_l]_{1\times 3} & a_{il}\xi_i\xi_l & \mu_{il}\xi_i\xi_l \end{pmatrix}_{5\times 5} = 0, \tag{1.4}$$
$$\xi\in\mathbb{R}^3\setminus\{0\},\ \ \omega\in\mathbb{R},\ \ i,j,k,l = 1,2,3,$$

where $\Phi_A(\xi,\omega)$ is the characteristic polynomial of the operator $A(\partial,\omega)$. The origin is an isolated zero of (1.4).

We are interested in the real zeros of the function $\Phi_A(\xi,\omega)$, $\xi\in\mathbb{R}^3\setminus\{0\}$.

Denote

$$\lambda := \frac{\rho_1\omega^2}{|\xi|^2}, \quad \widehat{\xi} := \frac{\xi}{|\xi|} \ \text{ for } \ |\xi|\neq 0,$$
$$B(\lambda,\widehat{\xi}) := \begin{pmatrix} [c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l - \lambda\delta_{jk}]_{3\times 3} & [A_{j4}(\widehat{\xi})]_{3\times 1} & [A_{j5}(\widehat{\xi})]_{3\times 1}\\ [-A_{j4}(\widehat{\xi})]_{1\times 3} & \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l & a_{il}\widehat{\xi}_i\widehat{\xi}_l\\ [-A_{j5}(\widehat{\xi})]_{1\times 3} & a_{il}\widehat{\xi}_i\widehat{\xi}_l & \mu_{il}\widehat{\xi}_i\widehat{\xi}_l \end{pmatrix}_{5\times 5}.$$

Then (1.4) can be rewritten as

$$\Psi(\lambda,\widehat{\xi}) := \det B(\lambda,\widehat{\xi}) = 0. \tag{1.5}$$

This is a cubic equation in λ with real coefficients.

The following theorem holds (see [7]).

Theorem 1.1. *Equation* (1.5) *possesses three real positive roots* $\lambda_1(\widehat{\xi})$, $\lambda_2(\widehat{\xi})$, $\lambda_3(\widehat{\xi})$.

Denote the roots of equation (1.5) by λ_1, λ_2, λ_3. Clearly, the equation of the surface $S_{\omega,j}$, $j = 1,2,3$, in the spherical coordinates reads as

$$r = r_j(\theta,\varphi) = \frac{\sqrt{\rho_1}\,\omega}{\sqrt{\lambda_j(\widehat{\xi})}}\,,$$

where

$$\xi_1 = r\cos\varphi\sin\theta, \quad \xi_2 = r\sin\varphi\sin\theta, \quad \xi_3 = r\cos\theta$$

with $0\le\varphi\le 2\pi$, $0\le\theta\le\pi$, $r = |\xi|$.

We have also the following identity:

$$\Phi_A(\xi,\omega)=\det A(i\xi,\omega)=\Phi_A(\widehat{\xi},0)r^4\prod_{j=1}^{3}\big(r^2-r_j^2(\widehat{\xi})\big)=\Phi_A(\widehat{\xi},0)r^4\prod_{j=1}^{3}P_j(\xi).$$

It can be easily shown that the vector

$$n(\xi)=(-1)^j\big|\nabla\Phi_A(\xi,\omega)\big|^{-1}\nabla\Phi_A(\xi,\omega),\ \ \xi\in S_{\omega,j},$$

is an external unit normal vector to $S_{\omega,j}$ at the point ξ.

Further, we assume that the following conditions are fulfilled (cf. [5,17,21,22]):

(i) If $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)r^4P_1(\xi)P_2(\xi)P_3(\xi)$, then $\nabla_\xi(P_1(\xi)P_2(\xi)P_3(\xi))\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (1.4), or

If $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)r^4P_1^2(\xi)P_2(\xi)$, then $\nabla_\xi(P_1(\xi)P_2(\xi))\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (1.4), or

If $\Phi_A(\xi,\omega)=\Phi_A(\widehat{\xi},0)r^4P_1^3(\xi)$, then $\nabla_\xi P_1(\xi)\neq 0$ at real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial (1.4).

(ii) The Gaussian curvature of the surface defined by the real zeros of the polynomial $\Phi_A(\xi,\omega)$, $\xi\in\mathbb{R}^3\setminus\{0\}$, does not vanish anywhere.

It follows from the above conditions (i) and (ii) that the real zeros $\xi\in\mathbb{R}^3\setminus\{0\}$ of the polynomial $\Phi_A(\xi,\omega)$ form non-self-intersecting, closed, convex two-dimensional surfaces $S_{\omega,1}$, $S_{\omega,2}$, $S_{\omega,3}$, enclosing the origin. For an arbitrary unit vector $\eta=x/|x|$ with $x\in\mathbb{R}^3\setminus\{0\}$, there exists only one point on each $S_{\omega,j}$, namely, $\xi^j=(\xi_1^j,\xi_2^j,\xi_3^j)\in S_{\omega,j}$ such that the outward unit normal vector $n(\xi^j)$ to $S_{\omega,j}$ at the point ξ^j has the same direction as η, i.e., $n(\xi^j)=\eta$. In this case, we say that the points ξ^j, $j=1,2,3$, correspond to the vector η.

From (i) we see that the surfaces $S_{\omega,j}$, $j=1,2,3$, may have multiplicites.

We say that a vector-function $U=(u_1,u_2,u_3,u_4,u_5)^\top$ belongs to the $M_{m_1,m_2,m_3}(\mathbf{P})$ class if $U\in[C^\infty(\Omega^-)]^5$ and the relation

$$U(x)=\sum_{p=1}^{5}u^p(x)$$

holds, where u^p has the following uniform asymptotic expansion as $r=|x|\to\infty$:

$$u^p\sim\sum_{j=1}^{3}e^{-i\,r\xi^j}\Big\{d^p_{0,m_j}(\eta)r^{m_j-2}+\sum_{q=1}^{\infty}d^p_{q,m_j}(\eta)r^{m_j-2-q}\Big\},\ \ p=1,2,3,$$

$$u^4(x)=O(r^{-1}),\quad \partial_ku^4(x)=O(r^{-2}),\quad u^5(x)=O(r^{-1}),\quad \partial_ku^5(x)=O(r^{-2}),\ \ k=1,2,3,$$

here $\mathbf{P}=\det A(i\partial_x,\omega)$ and $d^p_{q,m_j}\in C^\infty$, $j=1,2,3$ (see [5]).

These conditions are the generalized Sommerfeld–Kupradze type radiation conditions in the anisotropic elasticity (cf. [16,17]).

From condition (i) it follows that our $M_{m_1,m_2,m_3}(\mathbf{P})$ class is $M_{1,1,1}(\mathbf{P})$ or $M_{2,1}(\mathbf{P})$ or $M_3(\mathbf{P})$.

The class $M_{1,1,1}(\mathbf{P})$ is a subset of generalized Sommerfeld–Kupradze class.

We introduce the single and double layer potentials associated with the differential operator $A(\partial,\omega)$,

$$\mathbf{V}_\omega(g)(x)=\int_S\Gamma(x-y,\omega)g(y)\,d_yS,\ \ x\in\Omega^\pm,$$

$$\mathbf{W}_\omega(f)(x)=\int_S\big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(x-y,\omega)\big]^\top f(y)\,d_yS,\ \ x\in\Omega^\pm,$$

where $g=(g_1,\dots,g_4)^\top$ and $f=(f_1,\dots,f_4)^\top$ are density vector-functions and $\Gamma(x-y,\omega)$ is the fundamental solution of equation (1.8).

The following theorem holds (see [1]).

Theorem 1.2. *Let $g \in [H^{-1+s}(S)]^4$, $s > 0$. Then*

$$\{\mathbf{V}_\omega(g)(z)\}^\pm = \boldsymbol{H}_\omega(g)(z), \quad z \in S,$$

where $\boldsymbol{H}_\omega$ is a weakly singular integral operator,

$$\boldsymbol{H}_\omega(g)(z) := \int\limits_S \Gamma(z-y,\omega)g(y)\,d_yS, \quad z \in S.$$

The mapping properties of these potentials and the boundary integral operators are described in Appendix of [8].

1.3 Scalar acoustic pressure field and Green's formulae

We assume that the exterior domain Ω^- is filled by a homogeneous isotropic inviscid fluid medium with the constant density ρ_2. Further, let the propagation of acoustic wave in Ω^- be described by a complex-valued scalar function (scalar field) w being a solution of the homogeneous Helmholtz equation

$$\Delta \mathrm{w} + \rho_2\omega^2 \mathrm{w} = 0 \ \text{ in } \Omega^-, \tag{1.6}$$

where $\Delta = \sum\limits_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator and $\omega > 0$. The function $\mathrm{w}(x) = P^{sc}(x)$ is the pressure of a scattered acoustic wave.

We say that a solution w to the Helmholtz equation (1.6) belongs to the class $Som_p(\Omega^-)$, $p = 1, 2$, if w satisfies the classical Sommerfeld radiation condition

$$\frac{\partial \mathrm{w}(x)}{\partial |x|} + i(-1)^p\sqrt{\rho_2}\,\omega \mathrm{w}(x) = O(|x|^{-2}) \ \text{ as } \ |x| \to \infty. \tag{1.7}$$

Note that if a solution w of the Helmholtz equation (1.6) in Ω^- satisfies the Sommerfeld radiation condition (1.7), then (see [23])

$$\mathrm{w}(x) = O(|x|^{-1}) \ \text{ as } \ |x| \to \infty.$$

Let us introduce the single and double layer potentials

$$V_\omega(g)(x) := \int\limits_S \gamma(x-y,\omega)g(y)\,d_yS, \quad x \notin S,$$

$$W_\omega(f)(x) := \int\limits_S \partial_{n(y)}\gamma(x-y,\omega)f(y)\,d_yS, \quad x \notin S,$$

where

$$\gamma(x,\omega) := -\frac{\exp(i\sqrt{\rho_2}\,\omega|x|)}{4\pi|x|}$$

is the fundamental solution of the Helmholtz equation (1.6). These potentials satisfy the Sommerfeld radiation condition, i.e., belong to the class $\mathrm{Som}_1(\Omega^-)$.

For these potentials the following theorems are valid (see [12,18]).

Theorem 1.3. *Let $g \in H^{-1/2}(S)$, $f \in H^{1/2}(S)$. Then on the manifold S the following jump relations hold:*

$$\{V_\omega(g)\}^\pm = \mathcal{H}_\omega(g), \quad \{W_\omega(f)\}^\pm = \pm 2^{-1}f + \mathcal{K}^*_\omega(f),$$
$$\big\{\partial_n V_\omega(g)\big\}^\pm = \mp 2^{-1}g + \mathcal{K}_\omega(g), \quad \big\{\partial_n W_\omega(f)\big\}^+ = \big\{\partial_n W_\omega(f)\big\}^- =: \mathcal{L}_\omega(f),$$

where $\mathcal{H}_\omega$, $\mathcal{K}^*_\omega$ and $\mathcal{K}_\omega$ are integral operators with the weakly singular kernels,

$$\mathcal{H}_\omega(g)(z) := \int\limits_S \gamma(z-y,\omega)g(y)\,d_yS, \quad z \in S,$$

$$\mathcal{K}^*_\omega(f)(z) := \int\limits_S \partial_{n(y)}\gamma(z-y,\omega)f(y)\,d_yS, \quad z\in S,$$

$$\mathcal{K}_\omega(g)(z) := \int\limits_S \partial_{n(z)}\gamma(z-y,\omega)g(y)\,d_yS, \quad z\in S,$$

while $\mathcal{L}_\omega$ *is a singular integro-differential operator (pseudodifferential operator) of order* 1*.*

Theorem 1.4. *The operators*

$$\mathcal{N} := -2^{-1}I_1 + \mathcal{K}^*_\omega + \mu\mathcal{H}_\omega : H^{1/2}(S) \to H^{1/2}(S),$$

$$\mathcal{M} := \mathcal{L}_\omega + \mu\big(2^{-1}I_1 + \mathcal{K}_\omega\big) : H^{1/2}(S) \to H^{-1/2}(S)$$

are invertible provided $\operatorname{Im}\mu \neq 0$*. Here,* I_1 *is the scalar identity operator.*

The mapping properties of the above potentials and the boundary integral operators are described in Appendix of [8].

1.4 Formulation of Mixed type interaction problem for steady state oscillation equation

Now we formulate the fluid-solid interaction problems. Let the boundary $S=\partial\Omega^+=\partial\Omega^-\in C^\infty$ be divided into two disjoint parts S_D and S_N, i.e., $S=\overline{S_D}\cup\overline{S_N}$, $S_D\cap S_N=\varnothing$ and $l_m:=\partial S_D=\partial S_N\in C^\infty$.

Mixed type problem (M_ω)**:** Find a vector-function $U=(u,u_4,u_5)^\top=(u,\varphi,\psi)^\top\in[H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w}\in H^1_{loc}(\Omega^-)\cap\mathrm{Som}_1(\Omega^-)$ satisfying the following differential equations:

$$A(\partial,\omega)U=0 \ \text{ in } \Omega^+, \tag{1.8}$$

$$\Delta\mathrm{w}+\rho_2\omega^2\mathrm{w}=0 \ \text{ in } \Omega^-, \tag{1.9}$$

the transmission conditions

$$\{u\cdot n\}^+ = b_1\{\partial_n\mathrm{w}\}^- + f_0 \ \text{ on } S, \tag{1.10}$$

$$\{[T(\partial,n)U]_j\}^+ = b_2\{\mathrm{w}\}^-n_j + f_j \ \text{ on } S, \quad j=1,2,3, \tag{1.11}$$

and the mixed boundary conditions

$$\{\varphi\}^+ = f_1^{(D)} \ \text{ on } S_D, \tag{1.12}$$

$$\{\psi\}^+ = f_2^{(D)} \ \text{ on } S_D, \tag{1.13}$$

$$\big\{[T(\partial,n)U]_4\big\}^+ = f_1^{(N)} \ \text{ on } S_N, \tag{1.14}$$

$$\big\{[T(\partial,n)U]_5\big\}^+ = f_2^{(N)} \ \text{ on } S_N, \tag{1.15}$$

where b_1 and b_2 are the given complex constants satisfying the conditions

$$b_1b_2\neq 0 \ \text{ and } \ \operatorname{Im}[\overline{b}_1b_2]=0, \tag{1.16}$$

and

$$f_0\in H^{-1/2}(S), \quad f_j\in H^{-1/2}(S), \ \ j=1,2,3,$$

$$f_1^{(D)}\in H^{1/2}(S_D), \quad f_2^{(D)}\in H^{1/2}(S_D), \quad f_1^{(N)}\in H^{-1/2}(S_N), \quad f_2^{(N)}\in H^{-1/2}(S_N).$$

Theorem 1.5. *Let a pair* (U,w) *be a solution of the homogeneous problem* (M_ω) *and* $\omega>0$*. Then* $\mathrm{w}=0$ *in* Ω^- *and either* $U=0$ *in* Ω^+ *if* $\omega\notin J_M(\Omega^+)$*, or* $U\in X_{M,\omega}(\Omega^+)$ *if* $\omega\in J_M(\Omega^+)$*.*

We denote by $J_M(\Omega^+)$ Jones eigenfrequencies and by $X_{M,\omega}(\Omega^+)$ Jones modes corresponding to ω (see [8, 15]).

1.5 Formulation of Mixed type interaction problem for pseudo-oscillation equations

In this subsection, we consider the mixed type interaction problem for the so-called pseudo-oscillation equations. These problems are intermediate auxiliary problems for investigation of interaction problems for the steady state oscillation equations.

The matrix differential operator corresponding to the basic pseudo-oscillation equations of the electro-magneto-elasticity for anisotropic homogeneous media is written as follows:

$$\begin{gathered} A(\partial,\tau)=[A_{jk}(\partial,\tau)]_{5\times 5},\\ A_{jk}(\partial,\tau)=c_{ijkl}\partial_i\partial_l+\rho_1\tau^2\delta_{jk},\quad A_{j4}(\partial,\tau)=e_{lij}\partial_l\partial_i,\quad A_{j5}(\partial,\tau)=q_{lij}\partial_l\partial_i,\\ A_{4k}(\partial,\tau)=-e_{ikl}\partial_i\partial_l,\quad A_{44}(\partial,\tau)=\varepsilon_{il}\partial_i\partial_l,\quad A_{45}(\partial,\tau)=a_{il}\partial_i\partial_l,\\ A_{5k}(\partial,\tau)=-q_{ikl}\partial_i\partial_l,\quad A_{54}(\partial,\tau)=a_{il}\partial_i\partial_l,\quad A_{55}(\partial,\tau)=\mu_{il}\partial_i\partial_l, \end{gathered}$$

$j,k=1,2,3$, where τ is a purely imaginary complex parameter: $\tau=i\sigma$, $\sigma\neq 0$, $\sigma\in\mathbb{R}$.

Mixed type problem (M_τ)**:** Find a vector-function $U=(u,u_4,u_5)^\top\in[H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w}\in H^1_{loc}(\Omega^-)\cap\mathrm{Som}_1(\Omega^-)$ satisfying the differential equations

$$A(\partial,\tau)U=0 \ \text{ in } \Omega^+, \tag{1.17}$$

$$\Delta\mathrm{w}+\rho_2\omega^2\mathrm{w}=0 \ \text{ in } \Omega^-, \tag{1.18}$$

the transmission conditions

$$\{u\cdot n\}^+=b_1\big\{\partial_n\mathrm{w}\big\}^-+f_0 \ \text{ on } S, \tag{1.19}$$

$$\big\{[TU]_j\big\}^+=b_2\{\mathrm{w}\}^-n_j+f_j \ \text{ on } S,\ \ j=1,2,3, \tag{1.20}$$

and the mixed boundary conditions

$$\{u_4\}^+=f_1^{(D)} \ \text{ on } S_D, \tag{1.21}$$

$$\{u_5\}^+=f_2^{(D)} \ \text{ on } S_D, \tag{1.22}$$

$$\big\{[TU]_4\big\}^+=f_1^{(N)} \ \text{ on } S_N, \tag{1.23}$$

$$\big\{[TU]_5\big\}^+=f_2^{(N)} \ \text{ on } S_N, \tag{1.24}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.16), $f_0\in H^{-1/2}(S)$, $f_j\in H^{-1/2}(S)$, $j=1,2,3$, $f_1^{(D)}\in H^{1/2}(S_D)$, $f_2^{(D)}\in H^{1/2}(S_D)$, $f_1^{(N)}\in H^{-1/2}(S_N)$, $f_2^{(N)}\in H^{-1/2}(S_N)$.

The following uniqueness theorem holds for the problem (M_τ) (see [8]).

Theorem 1.6. *Let $\tau=i\sigma$, $\sigma\neq 0$, $\sigma\in\mathbb{R}$. The homogeneous problem (M_τ) has only trivial solutions.*

Investigation of the problem (M_τ) is reduced to the following scalar pseudodifferential equations on the manifold S_N with the boundary with respect to the unknown functions $g_0^{(1)}$, $g_0^{(2)}\in\widetilde{H}^{1/2}(S_N)$ (see [8]),

$$\begin{gathered} r_{S_N}\boldsymbol{A}_\tau^{(1)}g_0^{(1)}=F^{(1)} \ \text{ on } S_N,\\ r_{S_N}\boldsymbol{A}_\tau^{(2)}g_0^{(2)}=F^{(2)} \ \text{ on } S_N, \end{gathered}$$

where

$$\begin{gathered} \boldsymbol{A}_\tau^{(1)}g_0^{(1)}:=\big[\mathcal{A}_\tau\,\mathcal{B}_\tau^{-1}(0,0,0,g_0^{(1)},0)^\top\big]_4,\ \ F^{(1)}\in H^{-1/2}(S_N),\\ \boldsymbol{A}_\tau^{(2)}g_0^{(2)}:=\big[\mathcal{A}_\tau\,\mathcal{B}_\tau^{-1}(0,0,0,0,g_0^{(2)})^\top\big]_5,\ \ F^{(2)}\in H^{-1/2}(S_N), \end{gathered}$$

and

$$\mathcal{A}_\tau:=(-2^{-1}I_5+\mathbf{K}_\tau)\mathbf{H}_\tau^{-1}=[\mathcal{A}_\tau^{jk}]_{5\times 5},\ \ j,k=\overline{1,5},$$

is the Steklov–Poincaré type operator on S. This operator is a strongly elliptic pseudodifferential operator of order 1 (see [2] and [3] for details),

$$\mathcal{B}_\tau = \begin{pmatrix} [C_\tau]_{3\times 3} & [\mathcal{A}_\tau^{j4}]_{3\times 1} & [\mathcal{A}_\tau^{j5}]_{3\times 1} \\ [0]_{1\times 3} & I_1 & 0 \\ [0]_{1\times 3} & 0 & I_1 \end{pmatrix}_{5\times 5},$$

$$[C_\tau]_{3\times 3} = [\mathcal{A}_\tau^{jk}]_{3\times 3} - b_2 b_1^{-1}[n_j\mathcal{N}]_{3\times 1}[\mathcal{M}^{-1}n_k]_{1\times 3}, \quad j,k = 1,2,3.$$

Let us introduce the single and double layer pseudo-oscillation potentials

$$\mathbf{V}_\tau(h) = \int\limits_S \Gamma(x-y,\tau)h(y)\,d_yS,$$

$$\mathbf{W}_\tau(h) = \int\limits_S \big[\widetilde{T}(\partial_y, n(y))\Gamma^\top(x-y,\tau)\big]^\top h(y)\,d_yS,$$

where $h = (h_1,h_2,h_3,h_4,h_5)^\top$ is a density vector-function and $\Gamma(x-y,\tau)$ is the fundamental solution of equation (1.17).

Theorem 1.7. *Let $h \in [H^{-1+s}(S)]^5$, $s > 0$. Then*

$$\big\{\mathbf{V}_\tau(h^{(1)})(z)\big\}^\pm = \int\limits_S \Gamma(z-y,\tau)h(y)\,d_yS.$$

Further, we introduce the following boundary operator:

$$\mathbf{H}_\tau(h)(z) = \int\limits_S \Gamma(z-y,\tau)h(y)\,d_yS,$$

Note that $\mathbf{H}_\tau$ is a weakly singular integral operator (pseudodifferential operator of order -1).

The mapping properties of these potentials are described in Appendix of [8].

1.6 Formulation of the existence and uniqueness theorems of the mixed type problems (M_τ) and (M_ω)

We introduce the notation

$$\delta' := \inf_{x'\in l_m,\, j=1,2} \operatorname{Re}\varkappa_j(x'), \quad \delta'' := \sup_{x'\in l_m,\, j=1,2} \operatorname{Re}\varkappa_j(x'), \ \text{ where } \ 0 < \delta' \le \delta'' < 1,$$

where $\varkappa_j(x)$, $j=1,2$, are the factorization indices of the symbols $\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(x,\xi) = \mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(\xi)$, $j=1,2$, at the "frozen" point $x \in \partial S_N$, whose real part is calculated by the formula [14]:

$$\operatorname{Re}\varkappa_j(x) = \frac{1}{2} + \frac{1}{2\pi}\arg\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(x,0,-1) - \frac{1}{2\pi}\arg\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(x,0,+1),$$

$$-\frac{\pi}{2} < \arg\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(x,0,\pm 1) < \frac{\pi}{2}, \quad j=1,2, \quad x \in \partial S_N.$$

It is evident that $0 < \operatorname{Re}\varkappa_j(x) < 1$ $\ j=1,2$, for $x \in \partial S_N$.

The following theorem holds (see [8]).

Theorem 1.8. *The operators $r_{S_N}\boldsymbol{A}_\tau^{(1)}, r_{S_N}\boldsymbol{A}_\tau^{(2)} : \widetilde{H}^s(S_N) \to H^{s-1}(S_N)$ are invertible for all s satisfying*

$$-\frac{1}{2} + \sup_{x\in\partial S_N}\operatorname{Re}\kappa_j(x) < s < \frac{1}{2} + \inf_{x\in\partial S_N}\operatorname{Re}\kappa_j(x).$$

The following existence theorem holds for the problem (M_τ) (see [8,19]).

Theorem 1.9. *Let $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$, and let $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1,2,3$, $f_1^{(D)}, f_2^{(D)} \in H^{1/2}(S_D)$ and $f_1^{(N)}, f_2^{(N)} \in H^{-1/2}(S_N)$. Then the problem (M_τ) has a unique solution (U, w), $U \in [H^1(\Omega^+)]^5$, $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$ which is represented by the potentials*

$$U = \mathbf{V}_\tau \boldsymbol{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega) h \ \text{ in } \Omega^-,$$

where the densities $g \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are defined from the uniquely solvable system in [8]*. If the conditions $f_0 \in H^{s-1}(S)$, $f_j \in H^{s-1}(S)$, $j = 1,2,3$, $f_1^{(D)}, f_2^{(D)} \in H^s(S_D)$, $f_1^{(N)}, f_2^{(N)} \in H^{s-1}(S_N)$ hold for the data in* (1.19)–(1.24) *and*

$$\frac{1}{2} < s < \frac{1}{2} + \inf_{x \in \partial S_N,\, j=1,2} \operatorname{Re} \varkappa_j(x), \tag{1.25}$$

then the solution (U, w) of the mixed type problem (M_τ) exists, is unique and $U \in [H^{s+1/2}(\Omega^+)]^5$, $\mathrm{w} \in H^{s+1/2}_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$.

Moreover, if the conditions $f_0 \in H^s(S)$, $f_j \in H^{s-1}(S)$, $j = 1,2,3$, $f_1^{(D)}, f_2^{(D)} \in H^s(S_D)$, $f_1^{(N)}, f_2^{(N)} \in H^{s-1}(S_N)$ hold for the data in (1.19)–(1.24) *and* (1.25) *is satisfied, then the solution (U, w) of the mixed type problem (M_τ) exists, is unique and $U \in [H^{s+1/2}(\Omega^+)]^5$, $\mathrm{w} \in H^{s+3/2}_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$.*

Theorem 1.10. *If $\omega \notin J_M(\Omega^+)$, then the problem (M_ω) is uniquely solvable, and if $\omega \in J_M(\Omega^+)$, then the mixed type problem (M_ω) is solvable if and only if the following orthogonality condition*

$$\sum_{j=1}^{3} \big\langle f_j, \{\widetilde{U}_j\}^+ \big\rangle_S - \Big\langle \big\{ [\overline{\widetilde{T}\widetilde{U}}]_4 \big\}^+, \overline{f}_1^{(D)} \Big\rangle_S - \Big\langle \big\{ [\overline{\widetilde{T}\widetilde{U}}]_5 \big\}^+, \overline{f}_2^{(D)} \Big\rangle_S$$
$$+ \Big\langle \big\{ [\overline{\widetilde{U}}]_4 \big\}^+, \overline{f}_1^{(N)} \Big\rangle_S + \Big\langle \big\{ [\overline{\widetilde{U}}]_5 \big\}^+, \overline{f}_2^{(N)} \Big\rangle_S = 0 \quad \forall\, \widetilde{U} \in X^*_{D,\omega}(\Omega^+) \tag{1.26}$$

holds, and a solution is defined modulo Jones modes $X_{M,\omega}(\Omega^+)$.

The following theorem holds.

Theorem 1.11. *Let*

$$\frac{1}{2} < s < \frac{1}{2} + \inf_{x \in \partial S_N} \operatorname{Re} \varkappa_j(x), \tag{1.27}$$

where $\varkappa_j(x)$, $j = 1,2$, are the factorization indices of the principal homogeneous symbol of the operators $\boldsymbol{A}_\tau^{(j)}$, $j = 1,2$ (see Subsection 1.5*), and let $U \in [H^1(\Omega^+)]^5$, $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$ be the solution of the mixed type problem (M_ω). Then the following regularity result holds:*

> *if $f_0 \in H^{s-1}(S)$, $f_j \in H^{s-1}(S)$, $j = 1,2,3$, $f_1^{(D)}, f_2^{(D)} \in H^s(S_D)$, $f_1^{(N)}, f_2^{(N)} \in H^{s-1}(S_N)$, then $U \in [H^{s+1/2}(\Omega^+)]^5$, $\mathrm{w} \in H^{s+1/2}_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$.*

Moreover, if

$$f_0 \in H^s(S), \quad f_j \in H^{s-1}(S), \ \ j = 1,2,3, \quad f_1^{(D)}, f_2^{(D)} \in H^s(S_D), \quad f_1^{(N)}, f_2^{(N)} \in H^{s-1}(S_N),$$

and (1.27) *is satisfied, then $U \in [H^{s+1/2}(\Omega^+)]^5$, $\mathrm{w} \in H^{s+3/2}_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$.*

Remark 1.12. In the last statement of Theorem 1.11, the smoothness of w follows from the representation of h (see [8])

$$h = b_1^{-1} \mathcal{M}^{-1} [\mathbf{H}_\omega g]_l n_l - b_1^{-1} \mathcal{M}^{-1}(f_0) \in H^{s+1}(S) \ \text{ on } S$$

and the mapping properties of potentials W_ω and V_ω (see [8, Appendix, Theorem 6.1]), where $f_0 \in H^s(S)$, $g \in [H^{s-1}(S)]^5$ and s satisfies (1.27).

2 Asymptotics of solutions and regularity results for the mixed type problems (M_τ) and (M_ω)

2.1 Asymptotic analysis of the mixed type problem (M_τ) and regularity result

Here, we investigate the asymptotic behavior of a solution of the mixed type problem (M_τ) near the line $l_m = \partial S_N$.

Let $x' \in l_m$ and $\Pi^{(m)}_{x'}$ be the plane passing through the point x' and orthogonal to the curve l_m at x'. We introduce the polar coordinates (r, α), $r \geq 0$, $-\pi \leq \alpha \leq \pi$, in the plane $\Pi^{(m)}_{x'}$ with origin at the point x'. Denote by $S^{\pm}_N$ two different faces of the surface S_N. It is clear that $(r, \pm\pi) \in S^{\pm}_N$.

The intersection of the plane $\Pi^{(m)}_{x'}$ and Ω^- is identified with the half-plane $r \geq 0$ and $-\pi \leq \alpha \leq 0$, while the intersection of the plane $\Pi^{(m)}_{x'}$ and Ω^+ is identified with the half-plane $r \geq 0$ and $0 \leq \alpha \leq \pi$.

For simplicity of the description of the method applied below, we assume that the boundary data of the mixed type problem (M_τ) are infinitely smooth, $f_0, f_j \in C^\infty(S)$, $j = 1, 2, 3$, $f^{(D)}_1, f^{(D)}_2 \in C^\infty(\overline{S_D})$, $f^{(N)}_1, f^{(N)}_2 \in C^\infty(\overline{S_N})$.

In [8], we have shown that the mixed type problem (M_τ) is uniquely solvable and a solution (U, w) can be represented in the form

$$U = \mathbf{V}_\tau \mathbf{H}^{-1}_\tau (\widetilde{g}, g_4, g_5)^\top \ \text{ in } \Omega^+, \tag{2.1}$$

$$\mathrm{w} = (W_\omega + \mu V_\omega) h \ \text{ in } \Omega^-, \tag{2.2}$$

where $(\widetilde{g}, g_4, g_5, h)^\top$ is the unique solution of the system (see [8])

$$\mathcal{P}_{\tau,M} (\widetilde{g}, g_4, g_5, h)^\top = \Phi,$$

with

$$\Phi = \big(f_0, f_1, f_2, f_3, f^{(D)}_1, f^{(D)}_2, f^{(N)}_1, f^{(N)}_2\big)^\top,$$

and

$$\mathcal{P}_{\tau,M} := \begin{pmatrix} [n]_{1\times 3} & 0 & 0 & -b_1\mathcal{M} \\ [\mathcal{A}^{jk}_\tau]_{3\times 3} & [\mathcal{A}^{j4}_\tau]_{3\times 1} & [\mathcal{A}^{j5}_\tau]_{3\times 1} & [-b_2 n_j \mathcal{N}]_{3\times 1} \\ [0]_{1\times 3} & r_{S_D} I_1 & 0 & 0 \\ [0]_{1\times 3} & 0 & r_{S_D} I_1 & 0 \\ r_{S_N}[\mathcal{A}^{4j}_\tau]_{1\times 3} & r_{S_N}[\mathcal{A}^{44}_\tau] & r_{S_N}[\mathcal{A}^{45}_\tau] & 0 \\ r_{S_N}[\mathcal{A}^{5j}_\tau]_{1\times 3} & r_{S_N}[\mathcal{A}^{54}_\tau] & r_{S_N}[\mathcal{A}^{55}_\tau] & 0 \end{pmatrix}_{8\times 6}, \quad j, k = 1, 2, 3. \tag{2.3}$$

To establish the asymptotic behaviour of the vector U near the curve l_m, we rewrite (2.1) as

$$U = \mathbf{V}_\tau \mathbf{H}^{-1}_\tau (\widetilde{g}, g^{(1)}_0, g^{(2)}_0)^\top + R, \tag{2.4}$$

where $R := \mathbf{V}_\tau \mathbf{H}^{-1}_\tau (0, 0, 0, G^{(1)}_0, G^{(2)}_0)^\top \in C^\infty(\overline{\Omega^+})$, $G^{(1)}_0, G^{(2)}_0 \in C^\infty(S)$ are some fixed extensions of $f^{(D)}_1, f^{(D)}_2 \in C^\infty(\overline{S}_D)$, respectively, and $g^{(1)}_0$, $g^{(2)}_0$ are the unique solutions of the scalar strongly elliptic pseudodifferential equation on the manifold S_N with the boundary:

$$r_{S_N} \boldsymbol{A}^{(j)}_\tau g^{(j)}_0 = F^{(j)}, \ \ j = 1, 2, \ \text{ on } S_N, \tag{2.5}$$

where

$$\begin{aligned} \boldsymbol{A}^{(1)}_\tau g^{(1)}_0 &:= \big[\mathcal{A}_\tau \mathcal{B}^{-1}_\tau (0, 0, 0, g^{(1)}_0, 0)^\top\big]_4, \\ F^{(1)} &:= f^{(N)}_1 - r_{S_N} \big[\mathcal{A}_\tau \mathcal{B}^{-1}_\tau (\Psi_1, \Psi_2, \Psi_3, G^{(1)}_0, 0)^\top\big]_4 \in H^{-1/2}(S_N), \\ \boldsymbol{A}^{(2)}_\tau g^{(2)}_0 &:= \big[\mathcal{A}_\tau \mathcal{B}^{-1}_\tau (0, 0, 0, 0, g^{(2)}_0)^\top\big]_5, \end{aligned}$$

$$F^{(2)} := f_2^{(N)} - r_{S_N}\big[\mathcal{A}_\tau \mathcal{B}_\tau^{-1}(\Psi_1,\Psi_2,\Psi_3,0,G_0^{(2)})^\top\big]_5 \in H^{-1/2}(S_N),$$
$$\Psi_j = f_j - b_2 b_1^{-1} n_j \mathcal{N}\mathcal{M}^{-1} f_0, \ \ j=1,2,3.$$

Applying the results from [11, 14], we can derive the following asymptotic expansion of the solution $g_0^{(j)}$, $j=1,2$, of the strongly elliptic pseudodifferential equation (2.5) near the line l_m

$$g_0^{(j)}(x',r) = a_0^{(j)}(x')r^{\varkappa_j(x')} + \sum_{k=1}^{N}\sum_{i=0}^{k} a_{ki}^{(j)}(x')r^{\varkappa_j(x')+k}\ln^i r + R_{N+1}^{(j)}(x',r), \ \ j=1,2 \tag{2.6}$$

where N is an arbitrary positive integer, $a_0^{(j)}, a_{ik}^{(j)} \in C^\infty(l_m)$, and the remainder term $R_{N+1}^{(j)} \in C^{\delta'_j+N+1-\varepsilon}(l^+_{m,\varepsilon'})$, $l^+_{m,\varepsilon'} := l_m \times [0,\varepsilon']$ with $\forall\,\varepsilon>0$, $\forall\,\varepsilon'>0$, $x'\in l_m$, $j=1,2$.

The vector-function $(\widetilde{g}, g_4, g_5)^\top$ satisfies the uniquely solvable equation (see [8])

$$\mathcal{B}_\tau(\widetilde{g}, g_4, g_5)^\top = \Psi \ \text{ on } S,$$

where

$$\Psi = (\Psi', \Psi_4, \Psi_5)^\top, \quad \Psi' = (\Psi_1,\Psi_2,\Psi_3),$$
$$\Psi_j = f_j - b_2 b_1^{-1} n_j \mathcal{N}\mathcal{M}^{-1} f_0 \in C^\infty(S), \ \ j=1,2,3,$$
$$\Psi_4 = G_0^{(1)} + g_0^{(1)}, \quad \Psi_5 = G_0^{(2)} + g_0^{(2)}.$$

Then we get

$$(\widetilde{g}, g_4, g_5)^\top = \mathcal{B}_\tau^{-1}(\Psi) = \mathcal{B}_\tau^{-1}\big(\Psi', G_0^{(1)}+g_0^{(1)}, G_0^{(2)}+g_0^{(2)}\big)^\top,$$

whence

$$(\widetilde{g},0,0)^\top = \mathcal{B}_\tau^{-1}\big(\Psi', G_0^{(1)}+g_0^{(1)}, G_0^{(2)}+g_0^{(2)}\big)^\top - (0,0,0,g_4,g_5)^\top. \tag{2.7}$$

Since

$$\mathcal{B}_\tau^{-1} = \begin{pmatrix} [C_\tau]^{-1}_{3\times3} & -[C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j4}]_{3\times1} & -[C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j5}]_{3\times1} \\ [0]_{1\times3} & I_1 & 0 \\ [0]_{1\times3} & 0 & I_1 \end{pmatrix}_{5\times5},$$

taking into account $g_4 = G_0^{(1)} + g_0^{(1)}$, $g_5 = G_0^{(2)} + g_0^{(2)}$, from (2.7) we get

$$(\widetilde{g},0,0)^\top = \begin{pmatrix} [C_\tau]^{-1}_{3\times3}\Psi' - [C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j4}]_{3\times1}(G_0^{(1)}+g_0^{(1)}) - [C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j5}]_{3\times1}(G_0^{(2)}+g_0^{(2)}) \\ G_0^{(1)}+g_0^{(1)} \\ G_0^{(2)}+g_0^{(2)} \end{pmatrix}$$
$$- (0,0,0,0,g_5)^\top = \begin{pmatrix} [C_\tau]^{-1}_{3\times3}\Psi' - [C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j4}]_{3\times1}(G_0^{(1)}+g_0^{(1)}) \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$\widetilde{g} = -[C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j4}]_{3\times1}\, g_0^{(1)} - [C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j5}]_{3\times1}\, g_0^{(2)} + R_1, \tag{2.8}$$

where

$$R_1 = [C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j4}]_{3\times1}\, G_0^{(1)} + [C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j5}]_{3\times1}\, G_0^{(2)} \in [C^\infty(S)]^3.$$

From system (2.3) we obtain

$$h = b_1^{-1}\mathcal{M}^{-1}(\widetilde{g}\cdot n) - b_1^{-1}\mathcal{M}^{-1}(f_0), \tag{2.9}$$

where $b_1^{-1}\mathcal{M}^{-1}(f_0) \in C^\infty(S)$.

Denote

$$[\widetilde{\mathcal{C}}_\tau^{(1)}]_{3\times1} := -[C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j4}]_{3\times1}, \quad [\widetilde{\mathcal{D}}_\tau^{(1)}]_{5\times1} := \begin{pmatrix} [\widetilde{\mathcal{C}}_\tau^{(1)}]_{3\times1} \\ I_1 \\ I_1 \end{pmatrix},$$

$$[\widetilde{\mathcal{C}}_\tau^{(2)}]_{3\times1} := -[C_\tau]^{-1}_{3\times3}[\mathcal{A}_\tau^{j5}]_{3\times1}, \quad [\widetilde{\mathcal{D}}_\tau^{(2)}]_{5\times1} := \begin{pmatrix} [\widetilde{\mathcal{C}}_\tau^{(2)}]_{3\times1} \\ I_1 \\ I_1 \end{pmatrix},$$

which are the operators of order 0.

Substituting (2.8) and (2.9) in (2.4) and (2.2), respectively, the solutions of the problem (M_ω) can be represented in the form of potential type functions

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1}\Big([\widetilde{\mathcal{D}}_\tau^{(1)}]_{5\times1} g_0^{(1)} + [\widetilde{\mathcal{D}}_\tau^{(2)}]_{5\times1} g_0^{(2)}\Big) + \widetilde{R}_1 \ \text{ in } \Omega^+, \tag{2.10}$$

$$\mathrm{w} = (W_\omega + \mu V_\omega) b_1^{-1} \mathcal{M}^{-1} n_j \Big([\widetilde{\mathcal{C}}_\tau^{(1)}]_j g_0^{(1)} + [\widetilde{\mathcal{C}}_\tau^{(2)}]_j g_0^{(2)}\Big) + \widetilde{R}_2 \ \text{ in } \Omega^-, \tag{2.11}$$

where $\widetilde{R}_1 \in [C^\infty(\overline{\Omega^+})]^5$, $\widetilde{R}_2 \in C^\infty(\overline{\Omega^-})$.

By using the asymptotic expansion (2.6) and by means of the asymptotic expansion of potential type functions (see [10, Theorem 2.2, Remark 2.11]), from (2.10) and (2.11) we obtain the following asymptotic expansions of solutions U and w of the mixed type problem (M_τ) near the line l_m:

$$U(x', r, \alpha) = \sum_{j=1}^{2} p_0^{(j)}(x', \alpha) r^{\varkappa_j(x')} + \sum_{j=1}^{2}\sum_{i=0}^{N}\sum_{k=1}^{N} p_{ki}^{(j)}(x', \alpha) r^{\varkappa_j(x')+k} \ln^i r + U_{N+1}(x', r, \alpha), \tag{2.12}$$

$$\mathrm{w}(x', r, \alpha) = \sum_{j=1}^{2} q_0^{(j)}(x', \alpha) r^{\varkappa_j(x')+1} + \sum_{j=1}^{2}\sum_{i=0}^{N}\sum_{k=1}^{N} q_{ki}^{(j)}(x', \alpha) r^{\varkappa_j(x')+k+1} \ln^i r + \mathrm{w}_{N+1}(x', r, \alpha), \tag{2.13}$$

where $p_0, p_{jk} \in [C^\infty(l_m \times [0,\pi])]^5$, $q_0, q_{jk} \in C^\infty(l_m \times [-\pi, 0])$, and the remainder terms $U_{N+1} \in [C^{\delta'+N+1-\varepsilon}(\overline{\Omega^+})]^5$, $\mathrm{w}_{N+1} \in C^{\delta'+N+2-\varepsilon}(\overline{\Omega^-})$ for $\forall\,\varepsilon > 0$, $x' \in l_m$.

Now we can obtain a regularity result. From the asymptotic expansions (2.12), (2.13), we obtain the optimal Hölder smoothness of solutions of the problem (M_τ),

$$U \in [C^{\delta'}(\overline{\Omega^+})]^5, \quad \mathrm{w} \in C^{\delta'+1}(\overline{\Omega^-}).$$

where

$$\delta' := \inf_{x' \in l_m,\, j=1,2} \operatorname{Re} \varkappa_j(x').$$

2.2 Regularity result and asymptotic analysis of the mixed type problem (M_ω)

Here, we establish the asymptotic behavior and optimal regularity results for the solution of the mixed type problem (M_ω) near the line l_m. To this end, we will need theorems in Bessel potential and Besov spaces.

The following assertions hold.

Theorem 2.1. *Let* $\frac{2}{2-\delta''} < p < \frac{2}{1-\delta'}$, $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$ *and let the boundary data of the problem* (M_τ) *belong to the following Besov spaces:*

$$f_0 \in B^{-1/p}_{p,p}(S)\big(f_0 \in B^{1/p'}_{p,p}(S)\big), \quad f_j \in B^{-1/p}_{p,p}(S), \ \ j = 1, 2, 3,$$
$$f_1^{(D)}, f_2^{(D)} \in B^{1/p'}_{p,p}(S_D), \quad f_1^{(N)}, f_2^{(N)} \in B^{-1/p}_{p,p}(S_N), \ \ \frac{1}{p'} = 1 - \frac{1}{p}\,.$$

Then the unique solution pair (U, w) *of the mixed type problem* (M_τ) *belongs to the space* $[H^1_p(\Omega^+)]^5 \times [H^1_{p,loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)]$ $\big([H^1_p(\Omega^+)]^5 \times [H^2_{p,loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)]\big)$.

Theorem 2.2. *Let f_0, f_j, $j=1,2,3$ and (U,w) be as in Theorem 2.1, and the conditions*

$$\frac{1}{t}-1+\delta''<s<\frac{1}{t}+\delta',\ \ 1<t<\infty$$

be fulfilled. If

$$f_0\in B^s_{t,t}(S),\quad f_j\in B^s_{t,t}(S),\ \ j=1,2,3,$$
$$f_1^{(D)},f_2^{(D)}\in B^s_{t,t}(S_D),\quad f_1^{(N)},f_2^{(N)}\in B^{s-1}_{t,t}(S_N),$$

then

$$(U,\mathrm{w})\in[H_t^{s+1/t}(\Omega^+)]^5\times[H^{s+1+1/t}_{t,loc}(\Omega^-)\cap\mathrm{Som}_1(\Omega^-)].$$

Theorem 2.3. *Let the right-hand side of transmission conditions* (1.10), (1.11) *and boundary conditions* (1.12)–(1.15) *of the mixed type problem (M_ω) satisfy* (1.26) *if $\omega\in J_M(\Omega^+)$, and let*

$$f_0\in B^{-1/p}_{p,p}(S)\big(f_0\in B^{1/p'}_{p,p}(S)\big),\quad f_j\in B^{-1/p}_{p,p}(S),\ \ j=1,2,3,$$
$$f_1^{(D)},f_2^{(D)}\in B^{1/p'}_{p,p}(S_D),\quad f_1^{(N)},f_2^{(N)}\in B^{-1/p}_{p,p}(S_N),\ \ \frac{1}{p'}=1-\frac{1}{p}$$

with

$$\frac{2}{2-\delta''}<p<\frac{2}{1-\delta'}\,.$$

Then the solution pair (U,w) of the mixed type problem (M_ω) belongs to the space $[H^1_p(\Omega^+)]^5\times[H^1_{p,loc}(\Omega^-)\cap\mathrm{Som}_1(\Omega^-)]$ $\big([H^1_p(\Omega^+)]^5\times[H^2_{p,loc}(\Omega^-)\cap\mathrm{Som}_1(\Omega^-)]\big)$.

Theorem 2.4. *Let f_0, f_j, $j=1,2,3$ and (U,w) be as in Theorem 2.3, and the conditions*

$$\frac{1}{t}-1+\delta''<s<\frac{1}{t}+\delta',\ \ 1<t<\infty$$

be fulfilled. If

$$f_0\in B^s_{t,t}(S),\quad f_j\in B^s_{t,t}(S),\ \ j=1,2,3,$$
$$f_1^{(D)},f_2^{(D)}\in B^s_{t,t}(S_D),\quad f_1^{(N)},f_2^{(N)}\in B^{s-1}_{t,t}(S_N),$$

then

$$(U,\mathrm{w})\in[H_t^{s+1/t}(\Omega^+)]^5\times[H^{s+1+1/t}_{t,loc}(\Omega^-)\cap\mathrm{Som}_1(\Omega^-)].$$

Proofs of Theorems 2.1–2.4 are similar to those of Theorems 7.1–7.4 from [6].

Now we investigate the regularity and asymptotics of solutions of the mixed type problem (M_ω). Let the boundary data of the mixed problem (M_ω) belong to the following Besov spaces:

$$f_0\in B^{s+1}_{t,t}(S),\quad f_j\in B^{s+1}_{t,t}(S),\ \ j=1,2,3,$$
$$f_1^{(D)},f_2^{(D)}\in B^{s+1}_{t,t}(S_D),\quad f_1^{(N)},f_2^{(N)}\in B^{s}_{t,t}(S_N),$$

where the numbers t and s satisfy the conditions of Theorem 2.4.

Then the solution of the problem (M_ω) can be represented in the form

$$U=\mathbf{V}_\omega g\ \text{ in }\Omega^+,\tag{2.14}$$
$$\mathrm{w}=(W_\omega+\mu V_\omega)h\ \text{ in }\Omega^-,\tag{2.15}$$

where g and h are the solutions of the system

$$Q_{\omega,M}(g,h)^\top=\Phi,\tag{2.16}$$

$$Q_{\omega,M} := \begin{pmatrix} [n_l \mathbf{H}_\omega^{lk}]_{1\times5} & -b_1\mathcal{M} \\ [(-2^{-1}I_5+\mathbf{K}_\omega)^{jk}]_{3\times5} & [-b_2 n_j \mathcal{N}]_{3\times1} \\ [\mathbf{H}_\omega^{4k}]_{1\times5} & 0 \\ [\mathbf{H}_\omega^{5k}]_{1\times5} & 0 \\ [(-2^{-1}I_5+\mathbf{K}_\omega)^{4k}]_{1\times5} & 0 \\ [(-2^{-1}I_5+\mathbf{K}_\omega)^{5k}]_{1\times5} & 0 \end{pmatrix}_{8\times6}, \quad j=1,2,3, \quad k=\overline{1,5},$$

with

$$\Phi = \big(f_0, f_1, f_2, f_3, f_1^{(D)}, f_2^{(D)}, f_1^{(N)}, f_2^{(N)}\big)^\top.$$

Rewrite (2.16) in the form

$$Q_{\tau,M}(g,h)^\top = \widetilde{\Phi},$$

where

$$\widetilde{\Phi} = \Phi + \big(Q_{\tau,M} - Q_{\omega,M}\big)(g,h)^\top$$

with

$$\widetilde{\Phi} = \big(\widetilde{f}_0, \widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3, \widetilde{f}_1^{(D)}, \widetilde{f}_2^{(D)}, \widetilde{f}_1^{(N)}, \widetilde{f}_2^{(N)}\big)^\top \in [B_{t,t}^{s+1}(S)]^4 \times [B_{t,t}^{s+1}(S_D)]^2 \times [B_{t,t}^{s}(S_N)]^2.$$

To establish the asymptotic behaviour of the vector U near the line l_m, we rewrite (2.14) as

$$U = \mathbf{V}_\omega \mathbf{H}_\tau^{-1}(\widetilde{g}, g_0^{(1)}, g_0^{(2)})^\top + R \ \text{ in } \Omega^+, \tag{2.17}$$

where

$$\big(\widetilde{g}, g_0^{(1)} + G_0^{(1)}, g_0^{(2)} + G_0^{(2)}\big)^\top = \mathbf{H}_\tau g, \quad R := \mathbf{V}_\omega \mathbf{H}_\tau^{-1}(0,0,0,G_0^{(1)}, G_0^{(2)})^\top \in \big[H_t^{s+1+1/t}(\Omega^+)\big]^5,$$

$G_0^{(1)}, G_0^{(2)} \in B_{t,t}^{s+1}(S)$ is some fixed extension of $\widetilde{f}_1^{(D)}, \widetilde{f}_2^{(D)} \in B_{t,t}^{s+1}(S_D)$ and $g_0^{(1)}$, $g_0^{(2)}$ are the unique solutions of the scalar strongly elliptic pseudodifferential equations on the manifold S_N with boundary:

$$r_{S_N} \boldsymbol{A}_\tau^{(j)} g_0^{(j)} = \widetilde{F} \ \text{ on } S_N, \quad j=1,2, \tag{2.18}$$

with

$$\begin{gathered} \widetilde{F}^{(1)} = \widetilde{f}_1^{(N)} - r_{S_N}\big[\mathcal{A}_\tau \mathcal{B}_\tau^{-1}(\widetilde{\Psi}_1, \widetilde{\Psi}_2, \widetilde{\Psi}_3, G_0^{(1)}, 0)^\top\big]_4 \in B_{t,t}^s(S_N), \\ \widetilde{F}^{(2)} = \widetilde{f}_2^{(N)} - r_{S_N}\big[\mathcal{A}_\tau \mathcal{B}_\tau^{-1}(\widetilde{\Psi}_1, \widetilde{\Psi}_2, \widetilde{\Psi}_3, 0, G_0^{(2)})^\top\big]_4 \in B_{t,t}^s(S_N), \\ \widetilde{\Psi}_j = \widetilde{f}_j - b_2 b_1^{-1} n_j \mathcal{N}\mathcal{M}^{-1}\widetilde{f}_0, \quad \widetilde{\Psi}_j \in B_{t,t}^{s+1}(S), \quad j=1,2,3. \end{gathered}$$

For any $\gamma < \delta'$, one can find s satisfying $s < 1/t + \delta'$ and $\varepsilon > 0$ such that $s = 1/t + \varepsilon + \gamma$. It follows from the embedding theorem (see [20, Theorem 4.6.2(b)]) that $B_{t,t}^{1/t+\varepsilon+\gamma}(S_N) \subset H_t^{1/t+\gamma}(S_N)$. Therefore, $\widetilde{F} \in H_t^{1/t+\gamma}(S_N)$, where $\gamma < \delta'$.

Applying the results on asymptotic expansions of solutions to strongly elliptic pseudodifferential equations on a manifold with the boundary (see [11, 14]), we can derive the following asymptotics of the solution g_0 of the strongly elliptic pseudodifferential equation (2.18) near the line l_m:

$$g_0^{(j)}(x', r) = a_0^{(j)}(x') r^{\varkappa_j(x')} + R_1^{(j)}(x', r), \quad j=1,2, \tag{2.19}$$

where $a_0^{(j)} \in H_t^{\gamma+1-\delta''}(l_m)$ and the remainder term $R_1^{(j)} \in \widetilde{H}_t^{\gamma+1+1/t}(S_N)$ for any $1 < t < \infty$, $\gamma < \delta'$, $j = 1, 2$.

From the embedding theorem (see [20, Theorem 4.6.1(e)]), it follows that

$$H_t^{\gamma+1-\delta''}(l_m) \subset C^{\gamma+1-\delta''-1/t}(l_m), \quad \widetilde{H}_t^{\gamma+1+1/t}(S_N) \subset C^{\gamma+1-1/t}(\overline{S_N}), \tag{2.20}$$

where

$$\frac{1}{\gamma+1-\delta''} < t < \infty, \ \ 0 < \gamma < \delta'.$$

We assume that $\gamma = \delta' - \varepsilon$ with an arbitrarily small $\varepsilon > 0$ and $\max\{\frac{1}{1-\delta''-\varepsilon}, \frac{1}{1-\varepsilon}\} < t < \infty$. Then from the asymptotic expansion (2.19) and embeddings (2.20) we obtain that $g_0^{(j)} \in C^{\delta'}(S)$, where $\operatorname{supp} g_0^{(j)} \subset \overline{S_N}$, $j = 1, 2$.

The vector-function $(\widetilde{g}, G_0^{(1)} + g_0^{(1)}, G_0^{(2)} + g_0^{(2)})^\top$ satisfies the uniquely solvable equation (see [8])

$$\mathcal{B}_\tau\big(\widetilde{g}, G_0^{(1)} + g_0^{(1)}, G_0^{(2)} + g_0^{(2)}\big)^\top = \widetilde{\Psi} \ \text{ on } S,$$

where

$$\widetilde{\Psi} = (\widetilde{\Psi}', \widetilde{\Psi}_4, \widetilde{\Psi}_5)^\top, \quad \widetilde{\Psi}' = (\widetilde{\Psi}_1, \widetilde{\Psi}_2, \widetilde{\Psi}_3) \in [H_t^{\gamma+1/t}(S)]^3$$

and

$$\widetilde{\Psi}_4 = G_0^{(1)} + g_0^{(1)}, \quad \widetilde{\Psi}_5 = G_0^{(2)} + g_0^{(2)}.$$

Then we get

$$\big(\widetilde{g}, G_0^{(1)} + g_0^{(1)}, G_0^{(2)} + g_0^{(2)}\big)^\top = \mathcal{B}_\tau^{-1}(\widetilde{\Psi}) = \mathcal{B}_\tau^{-1}\big(\widetilde{\Psi}', G_0^{(1)} + g_0^{(1)}, G_0^{(2)} + g_0^{(2)}\big)^\top,$$

whence

$$(\widetilde{g}, 0, 0)^\top = \mathcal{B}_\tau^{-1}\big(\widetilde{\Psi}', G_0^{(1)} + g_0^{(1)}, G_0^{(2)} + g_0^{(2)}\big)^\top - \big(0, 0, 0, G_0^{(1)} + g_0^{(1)}, G_0^{(2)} + g_0^{(2)}\big)^\top. \tag{2.21}$$

Therefore, from (2.21) we get

$$\widetilde{g} = -[C_\tau]_{3\times3}^{-1}[\mathcal{A}_\tau^{j4}]_{3\times1}\, g_0^{(1)} - [C_\tau]_{3\times3}^{-1}[\mathcal{A}_\tau^{j5}]_{3\times1}\, g_0^{(2)} + R_2, \tag{2.22}$$

where

$$R_2 = [C_\tau]_{3\times3}^{-1}[\mathcal{A}_\tau^{j4}]_{3\times1}\, G_0^{(1)} + [C_\tau]_{3\times3}^{-1}[\mathcal{A}_\tau^{j5}]_{3\times1}\, G_0^{(2)} \in [H_t^{\gamma+1+1/t}(S)]^3.$$

From the first equation of system (2.16) with the right-hand side function $\widetilde{f}_0$, we obtain

$$h = b_1^{-1}\mathcal{M}^{-1}(\widetilde{g}\cdot n) - b_1^{-1}\mathcal{M}^{-1}(\widetilde{f}_0), \tag{2.23}$$

where

$$b_1^{-1}\mathcal{M}^{-1}(\widetilde{f}_0) \in H_t^{\gamma+2+1/t}(S).$$

Substituting (2.22) and (2.23) in (2.17) and (2.15), respectively, the solutions of the problem (M_ω) can be represented in the form of potential type functions

$$U = \mathbf{V}_\tau\mathbf{H}_\tau^{-1}\big([\widetilde{\mathcal{D}}_\tau^{(1)}]_{5\times1}g_0^{(1)} + [\widetilde{\mathcal{D}}_\tau^{(2)}]_{5\times1}g_0^{(2)}\big) + \widetilde{R}_1 \ \text{ in } \Omega^+, \tag{2.24}$$

$$\mathrm{w} = (W_\omega + \mu V_\omega)b_1^{-1}\mathcal{M}^{-1}n_j\big([\widetilde{\mathcal{C}}_\tau^{(1)}]_j g_0^{(1)} + [\widetilde{\mathcal{C}}_\tau^{(2)}]_j g_0^{(2)}\big) + \widetilde{R}_2 \ \text{ in } \Omega^-, \tag{2.25}$$

where

$$\widetilde{R}_1 \in [H_t^{\gamma+1+2/t}(\Omega^+)]^5 \subset [C^{\gamma+1-1/t}(\overline{\Omega^+})]^5, \quad \widetilde{R}_2 \in H_{t,loc}^{\gamma+2+2/t}(\Omega^-) \subset C^{\gamma+2-1/t}(\overline{\Omega^-}).$$

Now, we can obtain the regularity result. Since the potential type operators

$$\mathbf{V}_\omega\mathbf{H}_\tau^{-1}[\widetilde{\mathcal{D}}_\tau^{(1)}]_{5\times1}, \mathbf{V}_\omega\mathbf{H}_\tau^{-1}[\widetilde{\mathcal{D}}_\tau^{(2)}]_{5\times1} : [C^{\delta'}(S)]^5 \to [C^{\delta'}(\overline{\Omega^+})]^5,$$
$$(W_\omega + \mu V_\omega)b_1^{-1}\mathcal{M}^{-1}n_j[\widetilde{\mathcal{C}}_\tau^{(1)}]_j, (W_\omega + \mu V_\omega)b_1^{-1}\mathcal{M}^{-1}n_j[\widetilde{\mathcal{C}}_\tau^{(2)}]_j : C^{\delta'}(S) \to C^{\delta'+1}(\overline{\Omega^-})$$

are continuous (cf. [16, Chapter 5], [12, Chapter 2]), taking into account that $g_0^{(1)}, g_0^{(2)} \in C^{\delta'}(S)$, from (2.24), (2.25) we obtain optimal Hölder smoothness of solutions of the mixed type problem (M_ω)

$$U \in \big[C^{\delta'}(\overline{\Omega^+})\big]^5, \quad \mathrm{w} \in C^{\delta'+1}(\overline{\Omega^-}),$$

where

$$\delta' := \inf_{x'\in l_m,\, j=1,2} \operatorname{Re} \varkappa_j(x').$$

By using the asymptotic expansion (2.19) and by means of the asymptotic expansion of potential type functions (see [10, Theorem 2.2, Theorem 2.3, Remark 2.11]), from (2.24) and (2.25) we obtain the following asymptotic expansions of the solution (U, w) of the mixed type problem (M_ω) near the line l_m:

$$U(x', r, \alpha) = \sum_{j=1}^{2} p_0^{(j)}(x', \alpha) r^{\varkappa_j(x')} + U_1(x', r, \alpha), \tag{2.26}$$

$$\mathrm{w}(x', r, \alpha) = \sum_{j=1}^{2} q_0^{(j)}(x', \alpha) r^{\varkappa_j(x')+1} + \mathrm{w}_1(x', r, \alpha), \tag{2.27}$$

where $p_0 \in [C^\beta(l_m \times [0,\pi])]^5$, $q_0 \in C^\beta(l_m \times [-\pi, 0])$, and the remainder terms $U_1 \in [C^\beta(\overline{\Omega^+})]^5$, $\mathrm{w}_1 \in C^\beta(\overline{\Omega^-})$ with $\beta = \gamma + 1 - \delta'' - 1/t$ for any $\max\{\frac{1}{1-\delta''-\varepsilon}, \frac{1}{1-\varepsilon}\} < t < \infty$, where $\varepsilon = \delta' - \gamma > 0$ is an arbitrarily small number.

Remark 2.5. Note that the first coefficients $p_0^{(j)}$ and $q_0^{(j)}$, $j = 1, 2$, of the asymptotic expansions (2.26) and (2.27) have the same smoothness as the first coefficient $a_0^{(j)}$, $j = 1, 2$, of the asymptotic expansion (2.19), since the coefficients $p_0^{(j)}$ and $q_0^{(j)}$, $j = 1, 2$, are defined by the coefficient a_0 (see [10, Theorem 2.3]).

Let us consider the above investigated mixed type interaction problem for particular components. We assume that the medium occupying the domain Ω^+ belongs to the 422 (Tetragonal) or 622 (Hexagonal) class of crystals. The corresponding system of differential equations reads as follows (see, e.g., [13]):

$$\begin{aligned}
&(c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2)u_1 + (c_{12} + c_{66})\partial_1\partial_2 u_2 + (c_{13} + c_{44})\partial_1\partial_3 u_3\\
&\qquad\qquad -e_{14}\partial_2\partial_3\varphi - q_{15}\partial_2\partial_3\psi + \rho_1\omega^2 u_1 = F_1,\\
&(c_{12} + c_{66})\partial_2\partial_1 u_1 + (c_{66}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2)u_2 + (c_{13} + c_{44})\partial_2\partial_3 u_3\\
&\qquad\qquad +e_{14}\partial_1\partial_3\varphi + q_{15}\partial_1\partial_3\psi + \rho_1\omega^2 u_2 = F_2,\\
&(c_{13} + c_{44})\partial_3\partial_1 u_1 + (c_{13} + c_{44})\partial_3\partial_2 u_2 + (c_{44}\partial_1^2 + c_{44}\partial_2^2 + c_{33}\partial_3^2)u_3 + \rho_1\omega^2 u_3 = F_3,\\
&e_{14}\partial_2\partial_3 u_1 - e_{14}\partial_1\partial_3 u_2 + (\varepsilon_{11}\partial_1^2 + \varepsilon_{11}\partial_2^2 + \varepsilon_{33}\partial_3^2)\varphi = F_4,\\
&q_{15}\partial_2\partial_3 u_1 - q_{15}\partial_1\partial_3 u_2 + (\mu_{11}\partial_1^2 + \mu_{11}\partial_2^2 + \mu_{33}\partial_3^2)\psi = F_5,
\end{aligned}$$

where c_{11}, c_{12}, c_{13}, c_{33}, c_{44} and $c_{66} = \frac{c_{11}-c_{12}}{2}$ are the elastic constants, e_{14} is the piezoelastic constant, q_{15} is the piezomagnetic constant, ε_{11} and ε_{33} are the dielectric constants, μ_{11} and μ_{33} are the magnetic permeability constants, satisfying the inequalities which follow from the positive definiteness of the internal energy form (see (1.1), (1.2)):

$$\begin{gathered}
c_{11} > |c_{12}|, \quad c_{44} > 0, \quad c_{66} > 0, \quad c_{33}(c_{11} + c_{12}) > 2c_{13}^2,\\
\varepsilon_{11} > 0, \quad \varepsilon_{33} > 0, \quad \mu_{11} > 0, \quad \mu_{33} > 0.
\end{gathered} \tag{2.28}$$

The following proposition holds.

Proposition 2.6. *In the case when the domain Ω^+ is occupied by solids of a special class, which belongs to the 422 (Tetragonal) or 622 (Hexagonal) class of crystals, the factorization index of the principal homogeneous symbol of the pseudodifferential operator $\boldsymbol{A}_\tau^{(j)}$, $j = 1, 2$, is equal to $1/2$, i.e., $\varkappa_j = 1/2$, $j = 1, 2$. In this case, solutions of the problems (M_τ) and (M_ω) have optimal smoothness*

$$U \in [C^{1/2}(\overline{\Omega^+})]^5, \quad \mathrm{w} \in C^{3/2}(\overline{\Omega^-}).$$

Proof. The validity of this proposition follows from

$$\mathfrak{S}^{\pm}_{\boldsymbol{A}^{(j)}_\tau} = \mathfrak{S}_{\boldsymbol{A}^{(j)}_\tau}(0,+1) = \mathfrak{S}_{\boldsymbol{A}^{(j)}_\tau}(0,-1) > 0, \ \ j=1,2,$$

since the factorization indices of the symbols

$$\mathfrak{S}_{\boldsymbol{A}^{(j)}_\tau}(x,\xi) = \mathfrak{S}_{\boldsymbol{A}^{(j)}_\tau}(\xi), \ \ j=1,2,$$

are calculated by formula (see [14]):

$$\varkappa_j(x) = \frac{1}{2} + \frac{1}{2\pi}\arg \mathfrak{S}_{\boldsymbol{A}^{(j)}_\tau}(x,0,-1) - \frac{1}{2\pi}\arg \mathfrak{S}_{\boldsymbol{A}^{(j)}_\tau}(x,0,+1) - \frac{i}{2\pi}\ln\Big|\frac{\mathfrak{S}_{\boldsymbol{A}^{(j)}_\tau}(x,0,-1)}{\mathfrak{S}_{\boldsymbol{A}^{(j)}_\tau}(x,0,+1)}\Big|. \tag{2.29}$$

Here, it is assumed that the line l_m is parallel to the plane of isotropy, i.e., to the plane $x_3=0$.

Indeed, since

$$\mathfrak{S}^{\pm}_{\boldsymbol{A}^{(1)}_\tau}(0,\pm1) = -\big[\mathfrak{S}^{4k}_{\mathcal{A}_\tau}(0,\pm1)\big]_{1\times3}\big[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(0,\pm1)\big]^{-1}_{3\times3}\big[\mathfrak{S}^{j4}_{\mathcal{A}_\tau}(0,\pm1)\big]_{3\times1} + \mathfrak{S}^{44}_{\mathcal{A}_\tau}(0,\pm1), \ \ j,k=1,2,3,$$
$$\mathfrak{S}^{\pm}_{\boldsymbol{A}^{(2)}_\tau}(0,\pm1) = -[\mathfrak{S}^{5k}_{\mathcal{A}_\tau}(0,\pm1)]_{1\times3}\big[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(0,\pm1)\big]^{-1}_{3\times3}\big[\mathfrak{S}^{j5}_{\mathcal{A}_\tau}(0,\pm1)\big]_{3\times1} + \mathfrak{S}^{55}_{\mathcal{A}_\tau}(0,\pm1), \ \ j,k=1,2,3,$$

where

$$\mathfrak{S}_{\mathcal{A}_\tau}(0,\pm1) = \mathfrak{S}_{-2^{-1}I_5\pm\mathbf{K}_\tau}(0,1)\mathfrak{S}^{-1}_{\mathbf{H}_\tau}(0,1),$$

in this case,

$$\mathfrak{S}^{\pm}_{\boldsymbol{A}^{(1)}_\tau} = \frac{2A_{41}A_{14}\mathbf{C}_{55}}{d} - \frac{2A_{41}A_{14}\mathbf{C}_{45}}{d} - \frac{\mathbf{C}_{55}}{2d},$$
$$\mathfrak{S}^{\pm}_{\boldsymbol{A}^{(2)}_\tau} = \frac{2A_{51}A_{15}\mathbf{C}_{44}}{d} - \frac{2A_{51}A_{14}\mathbf{C}_{45}}{d} - \frac{\mathbf{C}_{44}}{2d},$$

where nonzero elements of the symbol matrix $\mathfrak{S}_{\mathbf{K}_\tau}(0,1)$ are

$$A_{14} = -i\,\frac{e_{14}c_{66}(b_2-b_1)}{2b_1b_2\sqrt{B}} - i\,\frac{e_{14}q_{15}^2}{\alpha\varepsilon_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varepsilon_{11}}{\varepsilon_{33}}} - \frac{c_{44}(b_2-b_1)(\varepsilon_{33}b_1b_2+\varepsilon_{11})}{\sqrt{B}}\right],$$
$$A_{15} = -i\,\frac{q_{15}c_{66}(b_2-b_1)}{2\alpha b_1b_2\sqrt{B}} - i\,\frac{q_{15}e_{14}^2}{\alpha\varepsilon_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varepsilon_{11}}{\varepsilon_{33}}} - \frac{c_{44}(b_2-b_1)(\varepsilon_{33}b_1b_2+\varepsilon_{11})}{\sqrt{B}}\right],$$
$$A_{41} = -i\,\frac{e_{14}\varepsilon_{33}(b_2-b_1)}{2\sqrt{B}}, \quad A_{51} = -i\,\frac{q_{15}\varepsilon_{33}(b_2-b_1)}{2\sqrt{B}},$$
$$b_1 = \sqrt{\frac{A-\sqrt{B}}{2c_{44}\varepsilon_{33}}}, \quad b_2 = \sqrt{\frac{A+\sqrt{B}}{2c_{44}\varepsilon_{33}}}, \quad \widetilde{e}_{14} = \big(e_{14}^2 + \alpha^{-1}q_{15}^2\big)^{1/2}, \quad \alpha = \frac{\mu_{11}}{\varepsilon_{11}} = \frac{\mu_{33}}{\varepsilon_{33}} > 0,$$
$$A = \widetilde{e}_{14}^2 + c_{44}\varepsilon_{11} + c_{66}\varepsilon_{33} > 0, \quad B = A^2 - 4c_{44}c_{66}\varepsilon_{11}\varepsilon_{33} > 0, \quad A > \sqrt{B}.$$

Note that

$$b_1b_2 = \sqrt{\frac{c_{66}\varepsilon_{11}}{c_{44}\varepsilon_{33}}}.$$

It can be proved that $A_{14}A_{41} < 0$, $A_{15}A_{51} < 0$ (see [3]).

Let us calculate the entries A_{23} and A_{32}. Introduce the notation

$$C := c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}, \quad D := C^2 - 4c_{44}^2c_{33}c_{11}.$$

Consider two cases.

Case 1. Let $D > 0$. Then

$$A_{23} = i\,\frac{c_{44}(d_2-d_1)(c_{11}-c_{13}d_1d_2)}{2d_1d_2\sqrt{D}}, \quad A_{32} = -i\,\frac{c_{44}(d_2-d_1)(c_{33}d_1d_2-c_{13})}{2d_1d_2\sqrt{D}}, \tag{2.30}$$

where

$$d_1 = \sqrt{\frac{C - \sqrt{D}}{2\, c_{44}\, c_{33}}}\,, \quad d_2 = \sqrt{\frac{C + \sqrt{D}}{2\, c_{44}\, c_{33}}}\,.$$

Inequalities (2.28) imply that $C > \sqrt{D}$ and

$$d_1 d_2 = \frac{\sqrt{c_{11}}}{\sqrt{c_{33}}}\,, \quad (d_2 - d_1)^2 = \frac{C - 2c_{44}\sqrt{c_{33}}\sqrt{c_{11}}}{c_{44}c_{33}} > 0.$$

Then from (2.30), we obtain $A_{23}A_{32} > 0$.

Case 2. Let $D < 0$. In this case,

$$A_{23} = i\,\frac{ac_{44}(\sqrt{c_{11}c_{33}} - c_{13})}{\sqrt{-D}}\,, \quad A_{32} = -i\,\frac{ac_{44}(\sqrt{c_{11}c_{33}} - c_{13})}{\sqrt{-D}}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}}\,,$$

where

$$a = \frac{1}{2}\sqrt{\frac{-C + 2c_{44}\sqrt{c_{11}c_{33}}}{c_{44}c_{33}}} > 0$$

and we get again

$$A_{23}A_{32} = \frac{c_{44}^2\, a^2(\sqrt{c_{11}\, c_{33}} - c_{13})^2}{-D}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}} > 0.$$

Nonzero elements of the symbol matrix $\mathfrak{S}_{\mathbf{H}_\tau}(0,+1) = \mathfrak{S}_{\mathbf{H}_\tau}(0,-1)$ are:

$$\mathbf{C}_{11} = -\frac{b_2 - b_1}{2\sqrt{B}}\Big(\varepsilon_{33} + \frac{\varepsilon_{11}}{b_1 b_2}\Big),$$

$$\mathbf{C}_{22} = \begin{cases} -\dfrac{d_2 - d_1}{2\sqrt{D}}\Big(c_{33} + c_{44}\sqrt{\dfrac{c_{33}}{c_{11}}}\Big) & \text{if } D > 0,\\[2mm] -\dfrac{a}{\sqrt{D}}\Big(c_{33} + c_{44}\sqrt{\dfrac{c_{33}}{c_{11}}}\Big) & \text{if } D < 0, \end{cases}$$

$$\mathbf{C}_{33} = \begin{cases} -\dfrac{d_2 - d_1}{2\sqrt{D}}\Big(c_{44} + \sqrt{c_{11}c_{33}}\Big) & \text{if } D > 0,\\[2mm] -\dfrac{a}{\sqrt{D}}\Big(c_{44} + \sqrt{c_{11}c_{33}}\Big) & \text{if } D < 0, \end{cases}$$

$$\mathbf{C}_{44} = -\bigg\{\frac{b_2 - b_1}{2\sqrt{B}}\Big(c_{44} + \frac{c_{66}}{b_1 b_2}\Big) + \frac{q_{15}^2}{2\alpha\varepsilon_{11}\widetilde{e}_{14}^2}\bigg[\sqrt{\frac{\varepsilon_{11}}{\varepsilon_{33}}} - \frac{c_{44}(b_2 - b_1)(\varepsilon_{33}b_1b_2 + \varepsilon_{11})}{\sqrt{B}}\bigg]\bigg\},$$

$$\mathbf{C}_{55} = -\bigg\{\frac{b_2 - b_1}{2\sqrt{B}}\Big(c_{44} + \frac{c_{66}}{b_1 b_2}\Big) + \frac{e_{14}^2}{2\alpha\varepsilon_{11}\widetilde{e}_{14}^2}\bigg[\sqrt{\frac{\varepsilon_{11}}{\varepsilon_{33}}} - \frac{c_{44}(b_2 - b_1)(\varepsilon_{33}b_1b_2 + \varepsilon_{11})}{\sqrt{B}}\bigg]\bigg\},$$

$$\mathbf{C}_{45} = \mathbf{C}_{54} = \frac{e_{14}q_{15}}{2\alpha\varepsilon_{11}\widetilde{e}_{14}^2}\bigg[\sqrt{\frac{\varepsilon_{11}}{\varepsilon_{33}}} - \frac{c_{44}(b_2 - b_1)(\varepsilon_{33}b_1b_2 + \varepsilon_{11})}{\sqrt{B}}\bigg], \quad \mathbf{C}_{66} = -\frac{1}{2\sqrt{\eta_{11}\eta_{33}}}\,.$$

Note that $\mathbf{C}_{jj} < 0$, $j = \overline{1,6}$ (see [3]).

Therefore, we obtain

$$\mathfrak{S}^{\pm}_{\boldsymbol{A}_\tau^{(j)}} = \mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(0,+1) = \mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(0,-1) > 0, \quad j = 1,2,$$

and from (2.29), we get $\varkappa_j = 1/2$.

From (2.26) and (2.27), we obtain that $U \in [C^{1/2}(\overline{\Omega^+})]^5$, $\mathrm{w} \in C^{3/2}(\overline{\Omega^-})$. □

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Author's address:

Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University, 6 Tamarashvili Str., 0186 Tbilisi, Georgia.

E-mail: g.chkadua@gmail.com