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SOLVABILITY OF THE MIXED TYPE INTERACTION PROBLEM OF ACOUSTIC WAVES AND ELECTRO-MAGNETO-ELASTIC STRUCTURES

Abstract. In the present paper, we consider a three-dimensional model of fluid-solid acoustic interaction when an electro-magneto-elastic body occupying a bounded region Ω^+ is embedded in an unbounded fluid domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. In this case, we have a five-dimensional electro-magneto-elastic field (the displacement vector with three components, electric potential and magnetic potential) in the domain Ω^+, while we have a scalar acoustic pressure field in the unbounded domain Ω^-. The physical kinematic and dynamic relations are described mathematically by appropriate boundary and transmission conditions. We consider less restrictions on matrix differential operator of electro-magneto-elasticity by introducing asymptotic classes, in particular, we allow the corresponding characteristic polynomial of the matrix operator to have multiple real zeros. Using the potential method and the theory of pseudodifferential equations based on the Wiener–Hopf factorization method, the uniqueness and existence theorems are proved in Sobolev–Slobodetskii spaces.

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Key words and phrases. Boundary-transmission problems, fluid-solid interaction, potential method, pseudodifferential equations, Helmholtz equation, steady state oscillations, Jones modes, Jones eigenfrequencies.

რეზიუმე. ნაშრომში განხილულია სითხისა და სხეულის აკუსტიკური ურთიერთქმედების სამგანზომილებიანი მოდელი, როდესაც ელექტრო-მაგნიტო-დრეკად სხეულს უკავია Ω^+ შემოსაზღვრული არე, რომელიც ჩადგმულია $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$ შემოუსაზღვრელ სითხის არეში. ამ შემთხვევაში შემოსაზღვრულ Ω^+ არეში არის ხუთგანზომილებიანი ელექტრო-მაგნიტო-დრეკადი ველი (გადაადგილების ვექტორის სამი კომპონენტი, ელექტრული პოტენციალი და მაგნიტური პოტენციალი), ხოლო შემოუსაზღვრელ Ω^- არეში – აკუსტიკური წნევის სკალარული ველი. ფიზიკური კინემატიკური და დინამიკური ურთიერთქმედებები მათემატიკურად აღწერილია შესაბამისი სასაზღვრო და ტრანსმისიის პირობებით. ნაშრომში მოთხოვნილია ნაკლები შეზღუდვები ელექტრო-მაგნიტო-დრეკადობის დიფერენციალურ ოპერატორზე და შემოღებულია შესაბამისი ასიმპტოტური კლასები. კერძოდ, მატრიცული დიფერენციალური ოპერატორის შესაბამის მახასიათებელ პოლინომს შესაძლოა გააჩნდეს ჯერადი ნამდვილი ნულები. პოტენციალთა მეთოდისა და ფსევდოდიფერენციალურ განტოლებათა თეორიის გამოყენებით, რომელიც ეფუძნება ვინერ-ჰოფის ფაქტორიზაციის მეთოდს, დამტკიცებულია არსებობისა და ერთადერთობის თეორემები სობოლევ-სლობოდეცკის სივრცეებში.

1 Formulation of the problems

1.1 Introduction

Interaction problems of different dimensional fields appear in mathematical models of electro-magneto transducers. Further examples of similar models are related to phased array microphones, ultrasound equipment, inkjet droplet actuators, sonar transducers, bioimaging, immunochemistry, and acousto-biotherapeutics (see [34–36, 41]).

The Dirichlet type, Neumann type and mixed type interaction problems of acoustic waves and piezoelectric structures are studied in [10, 12, 14].

Similar interaction problems for the classical model of elasticity have been investigated by a number of authors. An exhaustive information concerning theoretical and numerical results, for the case when both interacting media are isotropic, can be found in [2–5, 17, 20–22, 25, 26, 28]. The cases when the elastic body is homogeneous and anisotropic and the fluid is isotropic, are considered in [24, 31, 32]. In this case, one has a three-dimensional elastic field, the displacement vector with three components in the bounded domain Ω^+, and a scalar pressure field in the unbounded domain Ω^-.

In our case, in the domain Ω^+ we have additional electric and magnetic fields that essentially complicate the investigation of the transmission problems in question. In particular, except transmission conditions, electric and magnetic potentials are given on one part of the boundary of Ω^+ (the Dirichlet type condition), while on the other part of Ω^+, normal components of electric displacement and magnetic induction are given (the Neumann type condition).

In contrast to the classical elasticity, the differential operator of electro-magneto-elasticity is not self-adjoint and positive-definite.

The Dirichlet and Neumann type interaction problems of acoustic waves and piezo-electro-magnetic structures are studied in [11].

We consider less restrictions on the matrix differential operator of electro-magneto-elasticity by introducing asymptotic classes $M_{m_1,m_2,m_3}(\mathbf{P})$, where $\mathbf{P}$ is a determinant of the electro-magneto-elasticity matrix operator, in particular, we allow the corresponding characteristic polynomial of the matrix operator to have multiple real zeros.

We investigate the aforementioned problem with the use of the potential method and the theory of pseudodifferential equations on manifolds with boundary and prove the existence and uniqueness theorems in Sobolev–Slobodetskii spaces.

1.2 Electro-magnetic field

Let Ω^+ be a bounded three-dimensional domain in $\mathbb{R}^3$ with a compact, C^∞-smooth boundary $S = \partial\Omega^+$ and let $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domain Ω^+ is filled with an anisotropic homogeneous piezo-electro-magnetic material.

The basic equations of steady state oscillations of piezoelectro-magneticity for anisotropic homogeneous media are written as follows:

$$\begin{gathered}
c_{ijkl}\partial_i\partial_l u_k + \rho_1\omega^2\delta_{jk}u_k + e_{lij}\partial_l\partial_i\varphi + q_{lij}\partial_i\partial_l\psi + F_j = 0, \quad j = 1,2,3,\\
-e_{ikl}\partial_i\partial_l u_k + \varepsilon_{il}\partial_i\partial_l\varphi + a_{il}\partial_i\partial_l\psi + F_4 = 0,\\
-q_{ikl}\partial_i\partial_l u_k + a_{il}\partial_i\partial_l\varphi + \mu_{il}\partial_i\partial_l\psi + F_5 = 0,
\end{gathered}$$

or in the matrix form

$$A(\partial,\omega)U + F = 0 \ \text{ in } \Omega^+,$$

where $U = (u,\varphi,\psi)^\top$, $u = (u_1,u_2,u_3)^\top$ is the displacement vector, $\varphi = u_4$ is the electric potential, $\psi = u_5$ is the magnetic potential and $F = (F_1,F_2,F_3,F_4,F_5)^\top$ is a given vector-function. The three-dimensional vector (F_1,F_2,F_3) is the mass force density, while F_4 is the electric charge density, F_5 is the electric current density, and $A(\partial,\omega)$ is the matrix differential operator,

$$A(\partial,\omega) = [A_{jk}(\partial,\omega)]_{5\times5}, \tag{1.1}$$

$$A_{jk}(\partial,\omega) = c_{ijkl}\partial_i\partial_l + \rho_1\omega^2\delta_{jk}, \quad A_{j4}(\partial,\omega) = e_{lij}\partial_l\partial_i, \quad A_{j5}(\partial,\omega) = q_{lij}\partial_l\partial_i,$$

$$A_{4k}(\partial,\omega) = -e_{ikl}\partial_i\partial_l, \quad A_{44}(\partial,\omega) = \varepsilon_{il}\partial_i\partial_l, \quad A_{45}(\partial,\omega) = a_{il}\partial_i\partial_l,$$
$$A_{5k}(\partial,\omega) = -q_{ikl}\partial_i\partial_l, \quad A_{54}(\partial,\omega) = a_{il}\partial_i\partial_l, \quad A_{55}(\partial,\omega) = \mu_{il}\partial_i\partial_l,$$

$j,k = 1,2,3$, where $\omega \in \mathbb{R}$ is a frequency parameter, ρ_1 is the density of the piezoelectro-magnetic material, c_{ijlk}, e_{ikl}, q_{ikl}, ε_{il}, μ_{il}, a_{il} are elastic, piezoelectric, piezomagnetic, dielectric, magnetic permeability and electromagnetic coupling constants, respectively, δ_{jk} is the Kronecker symbol and summation over repeated indices is meant from 1 to 3, unless otherwise stated. These constants satisfy the standard symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klij}, \quad e_{ijk} = e_{ikj}, \quad q_{ijk} = q_{ikj}, \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad \mu_{jk} = \mu_{kj}, \quad a_{jk} = a_{kj},$$
$$i,j,k,l = 1,2,3.$$

Moreover, from physical considerations related to the positiveness of internal energy, it follows that the quadratic forms $c_{ijkl}\xi_{ij}\xi_{kl}$ and $\varepsilon_{ij}\eta_i\eta_j$ are positive definite:

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij} \quad \forall\, \xi_{ij} = \xi_{ji} \in \mathbb{R}, \tag{1.2}$$
$$\varepsilon_{ij}\eta_i\eta_j \geq c_2|\eta|^2, \quad q_{ij}\eta_i\eta_j \geq c_3|\eta|^2, \quad \mu_{ij}\eta_i\eta_j \geq c_1\,|\eta|^2 \quad \forall\, \eta = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3, \tag{1.3}$$

where c_0, c_1, c_2 and c_3 are positive constants.

More careful analysis related to the positive definiteness of the potential energy insure that the following matrix

$$\Lambda := \begin{pmatrix} [\varepsilon_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} \end{pmatrix}_{6\times 6}$$

is positive definite, i.e.,

$$\varepsilon_{kj}\zeta'_k\overline{\zeta'_j} + a_{kj}(\zeta'_k\overline{\zeta''_j} + \overline{\zeta'_k}\zeta''_j) + \mu_{kj}\zeta''_k\overline{\zeta''_j} \geq c_4\big(|\zeta'|^2 + |\zeta''|^2\big) \quad \forall\, \zeta',\zeta'' \in \mathbb{C}^3, \tag{1.4}$$

where c_4 is some positive constant.

The principal homogeneous symbol matrix of the operator $A(\partial,\omega)$ has the following form:

$$A^{(0)}(\xi) = \begin{pmatrix} [-c_{ijlk}\xi_i\xi_l]_{3\times 3} & [-e_{lij}\xi_l\xi_i]_{3\times 1} & [-q_{lij}\xi_l\xi_i]_{3\times 1} \\ [e_{ikl}\xi_i\xi_l]_{1\times 3} & -\varepsilon_{il}\xi_i\xi_l & -a_{il}\xi_i\xi_l \\ [q_{ikl}\xi_i\xi_l]_{1\times 3} & -a_{il}\xi_i\xi_l & -\mu_{il}\xi_i\xi_l \end{pmatrix}_{5\times 5}.$$

With the help of inequalities (1.2) and (1.3), it can be easily shown that

$$-\operatorname{Re} A^{(0)}(\xi)\zeta\cdot\zeta \geq c|\zeta|^2|\xi|^2 \quad \forall\, \zeta \in \mathbb{C}^4, \ \ \forall\, \xi \in \mathbb{R}^3, \ \ c = \mathit{const} > 0,$$

implying that $A(\partial,\omega)$ is a strongly elliptic formally nonselfadjoint differential operator.

Here and in the sequel, $a\cdot b$ denotes the scalar product of two vectors $a,b \in \mathbb{C}^N$, $a\cdot b := \sum\limits_{k=1}^{N} a_k\overline{b}_k$.

In the theory of electro-magneto-elasticity, the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1,n_2,n_3)$ have the form

$$\sigma_{ij}n_i := c_{ijlk}n_i\partial_l u_k + e_{lij}n_i\partial_l\varphi + q_{lij}n_i\partial_l\psi, \qquad j = 1,2,3,$$

while the normal component of the electric displacement vector $D = (D_1,D_2,D_3)^\top$ and the normal component of the magnetic induction vector $B = (B_1,B_2,B_3)^\top$ read as

$$-D_i n_i = -e_{ikl}n_i\partial_l u_k + \varepsilon_{il}n_i\partial_l\varphi + a_{il}n_i\partial_l\psi,$$
$$-B_i n_i = -q_{ikl}n_i\partial_l u_k + a_{il}n_i\partial_l\varphi + \mu_{il}n_i\partial_l\psi.$$

Let us introduce the boundary matrix differential operator

$$T(\partial,n) = [T_{jk}(\partial,n)]_{5\times 5},$$

$$
\begin{aligned}
T_{jk}(\partial,n) &= c_{ijlk}n_i\partial_l, \quad T_{j4}(\partial,n) = e_{lij}n_i\partial_l, \quad T_{j5}(\partial,n) = q_{lij}n_i\partial_l,\\
T_{4k}(\partial,n) &= -e_{ikl}n_i\partial_l, \quad T_{44}(\partial,n) = \varepsilon_{il}n_i\partial_l, \quad T_{45}(\partial,n) = a_{il}n_i\partial_l,\\
T_{5k}(\partial,n) &= -q_{ikl}n_i\partial_l, \quad T_{54}(\partial,n) = a_{il}n_i\partial_l, \quad T_{55}(\partial,n) = \mu_{il}n_i\partial_l,
\end{aligned}
$$

$j,k=1,2,3$. For a vector $U=(u,\varphi,\psi)^\top$, we have

$$
T(\partial,n)U = (\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_in_i, -B_in_i)^\top. \tag{1.5}
$$

The components of the vector TU given by (1.5) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of electro-magneto-elasticity, while the fourth one is the normal component of the electric displacement vector and the fifth one is the normal component of the magnetic induction vector.

In Green's formulae, one also has the following boundary operator associated with the adjoint differential operator

$$
\begin{gathered}
A^*(\partial,\omega) = A^\top(-\partial,\omega) = A^\top(\partial,\omega),\\
\widetilde{T}(\partial,n) = \big[\widetilde{T}_{jk}(\partial,n)\big]_{5\times5},
\end{gathered}
$$

where

$$
\begin{aligned}
\widetilde{T}_{jk}(\partial,n) &= T_{jk}(\partial,n), \quad \widetilde{T}_{j4}(\partial,n) = -T_{j4}(\partial,n), \quad \widetilde{T}_{j5}(\partial,n) = -T_{j5}(\partial,n),\\
\widetilde{T}_{4k}(\partial,n) &= -T_{4k}(\partial,n), \quad \widetilde{T}_{44}(\partial,n) = T_{44}(\partial,n), \quad \widetilde{T}_{45}(\partial,n) = T_{45}(\partial,n),\\
\widetilde{T}_{5k}(\partial,n) &= -T_{5k}(\partial,n), \quad \widetilde{T}_{54}(\partial,n) = T_{54}(\partial,n), \quad \widetilde{T}_{55}(\partial,n) = T_{55}(\partial,n),
\end{aligned}
$$

$j,k=1,2,3$.

1.3 Green's formulae for electro-magneto-elastic vector fields

For arbitrary vector-functions $U=(u_1,u_2,u_3,u_4,u_5)^\top \in [C^2(\overline{\Omega^+})]^5$ and $V=(v_1,v_2,v_3,v_4,v_5)^\top \in [C^2(\overline{\Omega^+})]^5$, we have the following Green's formulae (see [6]):

$$
\int\limits_{\Omega^+} \big[A(\partial,\omega)U\cdot V + E(U,\overline{V})\big]\,dx = \int\limits_S \{TU\}^+\cdot\{V\}^+\,dS,
$$

$$
\int\limits_{\Omega^+} \big[A(\partial,\omega)U\cdot V - U\cdot A^*(\partial,\omega)V\big]\,dx = \int\limits_S \big[\{TU\}^+\cdot\{V\}^+ - \{U\}^+\cdot\{\widetilde{T}V\}^+\big]\,dS,
$$

where

$$
\begin{aligned}
E(U,\overline{V}) = {} & c_{ijlk}\partial_iu_j\partial_l\overline{v}_k - \rho_1\omega^2u\cdot v + e_{lij}(\partial_lu_4\partial_i\overline{v}_j - \partial_iu_j\partial_l\overline{v}_4)\\
&+ q_{lij}(\partial_lu_5\partial_i\overline{v}_j - \partial_iu_j\partial_l\overline{v}_5) + \varepsilon_{jl}\partial_ju_4\partial_l\overline{v}_4 + a_{jl}(\partial_lu_4\partial_j\overline{v}_5 - \partial_ju_5\partial_l\overline{v}_4) + \mu_{jl}\partial_ju_5\partial_l\overline{v}_5
\end{aligned}
$$

with $u=(u_1,u_2,u_3)^\top$ and $v=(v_1,v_2,v_3)^\top$. The symbol $\{\,\cdot\,\}^+$ denotes the one-sided limits (the trace operator) on S from Ω^+. Note that by the standard limiting procedure, the above Green's formulae can be generalized to vector-functions $U\in[H^1(\Omega^+)]^5$ and $V\in[H^1(\Omega^+)]^5$ with $A(\partial,\omega)U\in[L_2(\Omega^+)]^5$ and $A^*(\partial,\omega)V\in[L_2(\Omega^+)]^5$.

Using these Green's formulae, we can define a generalized trace vector $\{T(\partial,n)U\}^+\in[H^{-1/2}(S)]^5$ for a function $U\in[H^1(\Omega^+)]^5$ with $A(\partial,\omega)U\in[L_2(\Omega^+)]^5$:

$$
\big\langle\{T(\partial,n)U\}^+,\{V\}^+\big\rangle_S := \int\limits_{\Omega^+} \big[A(\partial,\omega)U\cdot V + E(U,\overline{V})\big]\,dx,
$$

where $V\in[H^1(\Omega^+)]^5$ is an arbitrary vector-function.

Here and in what follows, the symbol $\langle\,\cdot\,,\cdot\,\rangle_S$ denotes the duality between the mutually adjoint function spaces $[H^{-1/2}(S)]^N$ and $[H^{1/2}(S)]^N$, which extends the usual L_2 scalar product

$$
\langle f,g\rangle_S = \int\limits_S \sum_{j=1}^N f_j\,\overline{g}_j\,dS \ \text{ for } \ f,g\in[L_2(S)]^N.
$$

1.4 Scalar acoustic pressure field and Green's formulae

We assume that the exterior domain Ω^- is filled with a homogeneous isotropic inviscid fluid medium with constant density ρ_2. Further, let the propagation of acoustic wave in Ω^- be described by a complex-valued scalar function (scalar field) w being a solution of the homogeneous Helmholtz equation

$$\Delta \mathrm{w} + \rho_2 \omega^2 \mathrm{w} = 0 \ \text{ in } \Omega^-, \tag{1.6}$$

where

$$\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$$

is the Laplace operator and $\omega > 0$. The function $\mathrm{w}(x) = P^{sc}(x)$ is the pressure of a scattered acoustic wave.

We say that a solution w to the Helmholtz equation (1.6) belongs to the class $\mathrm{Som}_p(\Omega^-)$, $p = 1, 2$, if w satisfies the classical Sommerfeld radiation condition

$$\frac{\partial \mathrm{w}(x)}{\partial |x|} + i(-1)^p \sqrt{\rho_2}\, \omega \mathrm{w}(x) = O(|x|^{-2}) \ \text{ as } \ |x| \to \infty. \tag{1.7}$$

Note that if a solution w of the Helmholtz equation (1.6) in Ω^- satisfies the Sommerfeld radiation condition (1.7), then (see [42])

$$\mathrm{w}(x) = O(|x|^{-1}) \ \text{ as } \ |x| \to \infty.$$

Let Ω be a domain in $\mathbb{R}^3$ with a compact simply connected boundary $\partial\Omega \in C^\infty$.

We denote by $H^s(\Omega)$ $(H^s_{loc}(\Omega))$ and $H^s(\partial\Omega)$, $s \in \mathbb{R}$, the L_2 based Sobolev–Slobodetskii (Bessel potential) spaces in Ω and on the closed manifold $\partial\Omega$. Respectively, we denote by $H^s_{comp}(\Omega)$ the subspace of $H^s(\Omega)$ $(H^s_{loc}(\Omega))$ consisting of functions with compact supports.

If M is a smooth proper submanifold of a manifold $\partial\Omega$, then we denote by $\widetilde{H}^s(M)$ the following subspace of $H^s(\partial\Omega)$:

$$\widetilde{H}^s(M) := \big\{ g :\ g \in H^s(\partial\Omega),\ \operatorname{supp} g \subset \overline{M} \big\},$$

while $H^s(M)$ denotes the space of restrictions to M of functions from $H^s(\partial\Omega)$,

$$H^s(M) := \big\{ r_M f :\ f \in H^s(\partial\Omega) \big\},$$

where r_M is the restriction operator to M.

Let

$$\mathrm{w}_1 \in H^1_{loc}(\Omega^-) \cap \mathrm{Som}_p(\Omega^-), \ \ p = 1, 2, \quad \Delta \mathrm{w}_1 \in L_{2,loc}(\Omega^-), \quad \mathrm{w}_2 \in H^1_{comp}(\overline{\Omega^-}),$$

then the following Green's first formula holds:

$$\int_{\Omega^-} (\Delta + k^2) \mathrm{w}_1 \overline{\mathrm{w}}_2 \, dx + \int_{\Omega^-} \nabla \mathrm{w}_1 \nabla \overline{\mathrm{w}}_2 \, dx - k^2 \int_{\Omega^-} \mathrm{w}_1 \overline{\mathrm{w}}_2 \, dx = -\big\langle \{\partial_n \mathrm{w}_1\}^-, \{\mathrm{w}_2\}^- \big\rangle_S, \tag{1.8}$$

where $n = (n_1, n_2, n_3)$ is the exterior unit normal vector to S directed outward with respect to the domain Ω^+, and $\partial_n = \frac{\partial}{\partial n}$ denotes the normal derivative.

1.5 Formulation of mixed type interaction problem for steady state oscillation equation

Now we formulate the fluid-solid interaction problems. Let the boundary $S = \partial\Omega^+ = \partial\Omega^- \in C^\infty$ be divided into two disjoint parts S_D and S_N, i.e.

$$S = \overline{S_D} \cup \overline{S_N}, \quad S_D \cap S_N = \varnothing \ \text{ and } \ l_m := \partial S_D = \partial S_N \in C^\infty.$$

Mixed type problem (M_ω)**:** Find a vector-function $U = (u, u_4, u_5)^\top = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$ satisfying the following differential equations

$$A(\partial, \omega)U = 0 \ \text{ in } \Omega^+, \tag{1.9}$$

$$\Delta \mathrm{w} + \rho_2 \omega^2 \mathrm{w} = 0 \ \text{ in } \Omega^-, \tag{1.10}$$

the transmission conditions

$$\{u \cdot n\}^+ = b_1 \{\partial_n \mathrm{w}\}^- + f_0 \ \text{ on } S, \tag{1.11}$$

$$\big\{[T(\partial, n)U]_j\big\}^+ = b_2 \{\mathrm{w}\}^- n_j + f_j \ \text{ on } S, \ \ j = 1, 2, 3, \tag{1.12}$$

and the mixed boundary conditions

$$\{\varphi\}^+ = f_1^{(D)} \ \text{ on } S_D, \tag{1.13}$$

$$\{\psi\}^+ = f_2^{(D)} \ \text{ on } S_D, \tag{1.14}$$

$$\big\{[T(\partial, n)U]_4\big\}^+ = f_1^{(N)} \ \text{ on } S_N, \tag{1.15}$$

$$\big\{[T(\partial, n)U]_5\big\}^+ = f_2^{(N)} \ \text{ on } S_N, \tag{1.16}$$

where b_1 and b_2 are given complex constants satisfying the conditions

$$b_1 b_2 \neq 0, \quad \mathrm{Im}[\overline{b}_1 b_2] = 0, \tag{1.17}$$

and

$$f_0 \in H^{-1/2}(S), \quad f_j \in H^{-1/2}(S), \ \ j = 1, 2, 3,$$
$$f_1^{(D)} \in H^{1/2}(S_D), \quad f_2^{(D)} \in H^{1/2}(S_D), \quad f_1^{(N)} \in H^{-1/2}(S_N), \quad f_2^{(N)} \in H^{-1/2}(S_N).$$

The transmission conditions (1.11)–(1.12) are called the *kinematic and dynamic* conditions. For an interaction problem of fluid and electro-magneto-elastic body

$$\begin{gathered} b_1 = [\rho_2 \omega^2]^{-1}, \quad b_2 = -1, \quad f_0(x) \equiv f_0^{inc}(x) = [\rho_2 \omega^2]^{-1} \partial_n P^{inc}(x), \\ f_j = -P^{inc}(x)\, n_j(x), \quad j = 1, 2, 3, \end{gathered} \tag{1.18}$$

where P^{inc} is an incident plane wave,

$$P^{inc}(x) = e^{i d \cdot x}, \quad d = \omega \sqrt{\rho_2}\, \eta, \quad \eta \in \mathbb{R}^3, \quad |\eta| = 1.$$

2 Uniqueness of solutions of the problem (M_ω)

We denote by $J_M(\Omega^+)$ the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A(\partial, \omega)U = 0 \ \text{ in } \Omega^+, \tag{2.1}$$

$$\{u \cdot n\}^+ = 0 \ \text{ on } S, \tag{2.2}$$

$$\big\{[T(\partial, n)U]_j\big\}^+ = 0 \ \text{ on } S, \ \ j = 1, 2, 3, \tag{2.3}$$

$$\{\varphi\}^+ = 0 \ \text{ on } S_D, \tag{2.4}$$

$$\{\psi\}^+ = 0 \ \text{ on } S_D, \tag{2.5}$$

$$\{u \cdot n\}^+ = 0 \ \text{ on } S_N, \tag{2.6}$$

$$\big\{[T(\partial, n)U]\big\}^+ = 0 \ \text{ on } S_N, \tag{2.7}$$

has a nontrivial solution $U = (u, \varphi, \psi)^\top \in [H^1(\Omega^+)]^5$ (cf., [24]).

Nontrivial solutions of problem (2.1)–(2.7) will be referred to as *Jones modes*, while the corresponding values of ω are called *Jones eigenfrequencies*, as they were first discussed by D. S. Jones [24] in a related context (a thin layer of ideal fluid between an elastic body and surrounding elastic exterior). For example, Jones eigenfrequencies exist for any axisymmetric body, such bodies can sustain torsional oscillations in which only the azimuthal component of displacement is nonzero. However, we do not expect Jones eigenfrequencies to exist for an arbitrary body. The spaces of Jones modes corresponding to ω we denote by $X_{M,\omega}(\Omega^+)$.

Let $J_M^*(\Omega^+)$ be the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$A^*(\partial,\omega)V = 0 \ \text{ in } \Omega^+, \tag{2.8}$$
$$\{v\cdot n\}^+ = 0 \ \text{ on } S, \tag{2.9}$$
$$\big\{[\widetilde{T}(\partial,n)V]_j\big\}^+ = 0 \ \text{ on } S, \ \ j=1,2,3, \tag{2.10}$$
$$\{v_4\}^+ = 0 \ \text{ on } S_D, \tag{2.11}$$
$$\{v_5\}^+ = 0 \ \text{ on } S_D, \tag{2.12}$$
$$\{v\cdot n\}^+ = 0 \ \text{ on } S_N, \tag{2.13}$$
$$\big\{[\widetilde{T}(\partial,n)V]\big\}^+ = 0 \ \text{ on } S_N \tag{2.14}$$

has a nontrivial solution $V = (v, v_4, v_5)^\top \in [H^1(\Omega^+)]^5$.

The spaces of Jones modes corresponding to ω for the differential operator $A^*(\partial,\omega)$ we denote by $X^*_{M,\omega}(\Omega^+)$.

It can be shown that $J_M(\Omega^+)$ and $J_M^*(\Omega^+)$ are at most countable. Note that for each ω the corresponding spaces of Jones modes $X_{M,\omega}(\Omega^+)$ and $X^*_{M,\omega}(\Omega^+)$ are of finite dimension.

Using Green's formulas and the Rellich–Vekua lemma, we obtain the following uniqueness theorem for the problem (M_ω) (cf. [11]).

Theorem 2.1. *Let a pair (U,w) be a solution of the homogeneous problem (M_ω) and $\omega > 0$. Then $\mathrm{w}=0$ in Ω^- and either $U=0$ in Ω^+ if $\omega \notin J_M(\Omega^+)$ or $U \in X_{M,\omega}(\Omega^+)$ if $\omega \in J_M(\Omega^+)$.*

Remark 1. Let a pair $(V,\mathrm{w}) \in [H^1(\Omega^+)]^5 \times [H^1_{loc}(\Omega^-) \cap \mathrm{Som}_2(\Omega^-)]$ be a solution of the homogeneous problem

$$\begin{aligned}
A^*(\partial,\omega)V &= 0 \ \text{ in } \Omega^+,\\
(\Delta+\rho_2\omega^2)\mathrm{w} &= 0 \ \text{ in } \Omega^-,\\
\{v\cdot n\}^+ + \overline{b}_2^{\,-1}\{\partial_n \mathrm{w}\}^- &= 0 \ \text{ on } S,\\
\big\{[\widetilde{T}(\partial,n)V]_j\big\}^+ + \overline{b}_1^{\,-1}\{\mathrm{w}\}^- n_j &= 0 \ \text{ on } S, \ \ j=1,2,3,\\
\{v_4\}^+ &= 0 \ \text{ on } S_D,\\
\{v_5\}^+ &= 0 \ \text{ on } S_D,\\
\big\{[\widetilde{T}(\partial,n)V]_4\big\}^+ &= 0 \ \text{ on } S_N,\\
\big\{[\widetilde{T}(\partial,n)V]_5\big\}^+ &= 0 \ \text{ on } S_N,
\end{aligned}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.17). Then $\mathrm{w}=0$ in Ω^- and either $V=0$ in Ω^+ if $\omega \notin J_M^*(\Omega^+)$ or $V \in X^*_{M,\omega}(\Omega^+)$ if $\omega \in J_M^*(\Omega^+)$.

3 Layer potentials

3.1 Potentials associated with the Helmholtz equation

Let us introduce the single and double layer potentials

$$V_\omega(g)(x) := \int\limits_S \gamma(x-y,\omega)g(y)\,d_yS, \ \ x\notin S,$$

$$W_\omega(f)(x) := \int\limits_S \partial_{n(y)}\gamma(x-y,\omega)f(y)\,d_yS, \quad x \notin S,$$

where

$$\gamma(x,\omega) := -\frac{\exp(i\sqrt{\rho_2}\,\omega|x|)}{4\pi|x|}$$

is the fundamental solution of the Helmholtz equation (1.6). These potentials satisfy the Sommerfeld radiation condition, i.e., belong to the class $\mathrm{Som}_1(\Omega^-)$.

For these potentials the following theorems are valid (see [15, 33]).

Theorem 3.1. *Let $g \in H^{-1/2}(S)$, $f \in H^{1/2}(S)$. Then the following jump relations hold on the manifold S:*

$$\{V_\omega(g)\}^\pm = \mathcal{H}_\omega(g), \quad \{W_\omega(f)\}^\pm = \pm 2^{-1}f + \mathcal{K}^*_\omega(f),$$
$$\{\partial_n V_\omega(g)\}^\pm = \mp 2^{-1}g + \mathcal{K}_\omega(g), \quad \{\partial_n W_\omega(f)\}^+ = \{\partial_n W_\omega(f)\}^- =: \mathcal{L}_\omega(f),$$

where $\mathcal{H}_\omega$, $\mathcal{K}^*_\omega$ and $\mathcal{K}_\omega$ are integral operators with weakly singular kernels,

$$\mathcal{H}_\omega(g)(z) := \int\limits_S \gamma(z-y,\omega)g(y)\,d_yS, \quad z \in S,$$
$$\mathcal{K}^*_\omega(f)(z) := \int\limits_S \partial_{n(y)}\gamma(z-y,\omega)f(y)\,d_yS, \quad z \in S,$$
$$\mathcal{K}_\omega(g)(z) := \int\limits_S \partial_{n(z)}\gamma(z-y,\omega)g(y)\,d_yS, \quad z \in S,$$

while $\mathcal{L}_\omega$ is a singular integro-differential operator (pseudodifferential operator) of order 1.

Theorem 3.2. *The operators*

$$\mathcal{N} := -2^{-1}I_1 + \mathcal{K}^*_\omega + \mu\mathcal{H}_\omega : H^{1/2}(S) \to H^{1/2}(S), \tag{3.1}$$
$$\mathcal{M} := \mathcal{L}_\omega + \mu\big(2^{-1}I_1 + \mathcal{K}_\omega\big) : H^{1/2}(S) \to H^{-1/2}(S) \tag{3.2}$$

are invertible, provided $\operatorname{Im}\mu \neq 0$. *Here, I_1 is the scalar identity operator.*

the mapping properties of the above potentials and the boundary integral operators are described in Appendix.

3.2 Fundamental solution and potentials of the steady state oscillation equations of electro-magneto-elasticity

Let us consider the equation

$$\Phi_A(\xi,\omega) := \det A(i\xi,\omega) = \det\begin{pmatrix} [c_{ijlk}\xi_i\xi_l - \rho_1\omega^2\delta_{jk}]_{3\times3} & [e_{lij}\xi_l\xi_i]_{3\times1} & [q_{lij}\xi_l\xi_i]_{3\times1} \\ [-e_{ikl}\xi_i\xi_l]_{1\times3} & \varepsilon_{il}\xi_i\xi_l & a_{il}\xi_i\xi_l \\ [-q_{ikl}\xi_i\xi_l]_{1\times3} & a_{il}\xi_i\xi_l & \mu_{il}\xi_i\xi_l \end{pmatrix}_{5\times5} = 0, \tag{3.3}$$
$$\xi \in \mathbb{R}^3 \setminus \{0\}, \quad \omega \in \mathbb{R}, \quad i,j,k,l = 1,2,3,$$

where $\Phi_A(\xi,\omega)$ is the characteristic polynomial of the operator $A(\partial,\omega)$. The origin is an isolated zero of (3.3).

We are interested in the real zeros of the function $\Phi_A(\xi,\omega)$, $\xi \in \mathbb{R}^3 \setminus \{0\}$.

Denote

$$\lambda := \frac{\rho_1\omega^2}{|\xi|^2}\,, \quad \widehat{\xi} := \frac{\xi}{|\xi|} \ \text{ for } \ |\xi| \neq 0,$$

$$B(\lambda,\widehat{\xi}) := \begin{pmatrix} [c_{ijkl}\widehat{\xi}_i\widehat{\xi}_l - \lambda\delta_{jk}]_{3\times 3} & [A_{j4}(\widehat{\xi})]_{3\times 1} & [A_{j5}(\widehat{\xi})]_{3\times 1} \\ [-A_{j4}(\widehat{\xi})]_{1\times 3} & \varepsilon_{il}\widehat{\xi}_i\widehat{\xi}_l & a_{il}\widehat{\xi}_i\widehat{\xi}_l \\ [-A_{j5}(\widehat{\xi})]_{1\times 3} & a_{il}\widehat{\xi}_i\widehat{\xi}_l & \mu_{il}\widehat{\xi}_i\widehat{\xi}_l \end{pmatrix}_{5\times 5}.$$

Then (3.3) can be rewritten as

$$\Psi(\lambda,\widehat{\xi}) := \det B(\lambda,\widehat{\xi}) = 0. \tag{3.4}$$

This is a cubic equation in λ with real coefficients.

The following theorem holds (see [11]).

Theorem 3.3. *Equation* (3.4) *possesses three real positive roots* $\lambda_1(\widehat{\xi})$, $\lambda_2(\widehat{\xi})$, $\lambda_3(\widehat{\xi})$.

Denote the roots of equation (3.4) by λ_1, λ_2, λ_3. Clearly, the equation of the surface $S_{\omega,j}$, $j = 1,2,3$, in the spherical coordinates reads as

$$r = r_j(\theta,\varphi) = \frac{\sqrt{\rho_1}\,\omega}{\sqrt{\lambda_j(\widehat{\xi})}}\,,$$

where

$$\xi_1 = r\cos\varphi\sin\theta, \quad \xi_2 = r\sin\varphi\sin\theta, \quad \xi_3 = r\cos\theta$$

with $0 \le \varphi \le 2\pi$, $0 \le \theta \le \pi$, $r = |\xi|$.

We have also the following identity

$$\Phi_A(\xi,\omega) = \det A(i\xi,\omega) = \Phi_A(\widehat{\xi},0)\,r^4\prod_{j=1}^{3}\big(r^2 - r_j^2(\widehat{\xi})\big) = \Phi_A(\widehat{\xi},0)\,r^4\prod_{j=1}^{3}P_j(\xi).$$

It can easily be shown that the vector

$$n(\xi) = (-1)^j|\nabla\Phi_A(\xi,\omega)|^{-1}\nabla\Phi_A(\xi,\omega), \ \ \xi \in S_{\omega,j},$$

is an external unit normal vector to $S_{\omega,j}$ at the point ξ.

Further, we assume that the following conditions are fulfilled (cf. [13,30,39,40]):

(i) If $\Phi_A(\xi,\omega) = \Phi_A(\widehat{\xi},0)\,r^4P_1(\xi)P_2(\xi)P_3(\xi)$, then $\nabla_\xi\big(P_1(\xi)P_2(\xi)P_3(\xi)\big) \neq 0$ at real zeros $\xi \in \mathbb{R}^3\setminus\{0\}$ of polynomial (3.3), or

If $\Phi_A(\xi,\omega) = \Phi_A(\widehat{\xi},0)\,r^4P_1^2(\xi)P_2(\xi)$, then $\nabla_\xi\big(P_1(\xi)P_2(\xi)\big) \neq 0$ at real zeros $\xi \in \mathbb{R}^3\setminus\{0\}$ of polynomial (3.3), or

If $\Phi_A(\xi,\omega) = \Phi_A(\widehat{\xi},0)\,r^4P_1^3(\xi)$, then $\nabla_\xi P_1(\xi) \neq 0$ at real zeros $\xi \in \mathbb{R}^3\setminus\{0\}$ of polynomial (3.3).

(ii) The Gaussian curvature of the surface, defined by the real zeros of the polynomial $\Phi_A(\xi,\omega)$, $\xi \in \mathbb{R}^3\setminus\{0\}$, does not vanish anywhere.

It follows from the above conditions (i) and (ii) that the real zeros $\xi \in \mathbb{R}^3\setminus\{0\}$ of the polynomial $\Phi_A(\xi,\omega)$ form non-self-intersecting, closed, convex two-dimensional surfaces $S_{\omega,1}, S_{\omega,2}, S_{\omega,3}$, enclosing the origin. For an arbitrary unit vector $\eta = x/|x|$ with $x \in \mathbb{R}^3\setminus\{0\}$, there exists only one point on each $S_{\omega,j}$, namely, $\xi^j = (\xi_1^j,\xi_2^j,\xi_3^j) \in S_{\omega,j}$ such that the outward unit normal vector $n(\xi^j)$ to $S_{\omega,j}$ at the point ξ^j has the same direction as η, i.e., $n(\xi^j) = \eta$. In this case, we say that the points ξ^j, $j = 1,2,3$, correspond to the vector η. From (i), we see that the surfaces $S_{\omega,j}$, $j = 1,2,3$, might have multiplicites.

We say that a vector-function $U = (u_1, u_2, u_3, u_4, u_5)^\top$ belongs to the $M_{m_1,m_2,m_3}(\mathbf{P})$ class if $U \in [C^\infty(\Omega^-)]^5$ and the relation

$$U(x) = \sum_{p=1}^{5} u^p(x)$$

holds, where u^p has the following uniform asymptotic expansion as $r = |x| \to \infty$:

$$u^p \sim \sum_{j=1}^{3} e^{-i\,r\xi^j} \Big\{ d^p_{0,m_j}(\eta) r^{m_j-2} + \sum_{q=1}^{\infty} d^p_{q,m_j}(\eta) r^{m_j-2-q} \Big\}, \ \ p = 1,2,3, \tag{3.5}$$
$$u^4(x) = O(r^{-1}), \quad \partial_k u^4(x) = O(r^{-2}), \quad u^5(x) = O(r^{-1}), \quad \partial_k u^5(x) = O(r^{-2}), \ \ k = 1,2,3,$$

here $\mathbf{P} = \det A(i\partial_x, \omega)$ and $d^p_{q,m_j} \in C^\infty$, $j = 1,2,3$ (see [13]).

These conditions are the generalized Sommerfeld–Kupradze type radiation conditions in the anisotropic elasticity (cf. [27,30]).

From condition (i) follows that our $M_{m_1,m_2,m_3}(\mathbf{P})$ class is $M_{1,1,1}(\mathbf{P})$ or $M_{2,1}(\mathbf{P})$ or $M_3(\mathbf{P})$.

The class $M_{1,1,1}(\mathbf{P})$ is a subset of the generalized Sommerfeld–Kupradze class.

We can show the following uniqueness theorems.

Theorem 3.4. *The homogeneous exterior Dirichlet boundary value problem*

$$A(\partial, \omega) U = 0 \ \ \text{in } \Omega^-, \quad \{U\}^- = 0 \ \ \text{on } S,$$

has only the trivial solution in the class $[H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\boldsymbol{P})$.

Theorem 3.5. *The homogeneous exterior Dirichlet boundary value problem*

$$A^*(\partial, \omega) V = 0 \ \ \text{in } \Omega^-, \quad \{V\}^- = 0 \ \ \text{on } S,$$

has only the trivial solution in the class $[H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\boldsymbol{P}^*)$, *where* $\boldsymbol{P}^* = \det A^*(\partial, \omega)$.

If the surfaces $S_{\omega,j}$, $j = 1,2,3$, have no multipicity, Theorems 3.4 and 3.5 are valid in the generalized Sommerfeld–Kupradze class (cf. [27]).

Denote by $\Gamma(x,\omega)$ the fundamental matrix of the operator $A(\partial,\omega)$. By means of the Fourier transform method and the limiting absorption principle we can construct this matrix explicitly (see [40, Chapter 1, Section 1])

$$\Gamma(x,\omega) = \lim_{\varepsilon \to 0+} F^{-1}_{\xi \to x} \big[A^{-1}(i\xi, \omega + i\varepsilon) \big], \tag{3.6}$$

where F^{-1} is the inverse Fourier transform. Columns of the matrix $\Gamma(x,\omega)$ are infinitely differentiable in $\mathbb{R}^3 \setminus \{0\}$ and belong to the class $SK_1(\mathbb{R}^3 \setminus \{0\})$.

Further, we introduce the single and double layer potentials associated with the differential operator $A(\partial,\omega)$,

$$\mathbf{V}_\omega(g)(x) = \int_S \Gamma(x-y,\omega) g(y)\, d_yS, \ \ x \in \Omega^\pm,$$
$$\mathbf{W}_\omega(f)(x) = \int_S \big[\widetilde{T}(\partial_y, n(y)) \Gamma^\top(x-y,\omega) \big]^\top f(y)\, d_yS, \ \ x \in \Omega^\pm,$$

where $g = (g_1, \dots, g_5)^\top$ and $f = (f_1, \dots, f_5)^\top$ are the density vector-functions.

For a solution $U \in [H^1(\Omega^+)]^5$ to the homogeneous equation (1.9) in Ω^+ we have the following integral representation

$$U = \mathbf{W}_\omega\big(\{U\}^+\big) - \mathbf{V}_\omega\big(\{TU\}^+\big) \ \ \text{in } \Omega^+.$$

For these potentials the following theorem holds (see [6]).

Theorem 3.6. *Let* $g \in [H^{-1+s}(S)]^5$ *and* $f \in [H^s(S)]^5$, $s > 0$. *Then*

$$\begin{gathered}
\{\mathbf{V}_\omega(g)(z)\}^\pm = \boldsymbol{H}_\omega(g)(z), \ \ z \in S, \\
\{\mathbf{W}_\omega(f)(z)\}^\pm = \pm 2^{-1} f(z) + \widetilde{\boldsymbol{K}}_\omega(f)(z), \ \ z \in S, \\
\big\{T(\partial_y, n(y))\mathbf{V}_\omega(g)(z)\big\}^\pm = \mp 2^{-1} g(z) + \boldsymbol{K}_\omega(g)(z), \ \ z \in S, \\
\big\{T(\partial_z, n(z))\mathbf{W}_\omega(f)(z)\big\}^+ = \big\{T(\partial_z, n(z))\mathbf{W}_\omega(f)(z)\big\}^- := \boldsymbol{L}_\omega(f)(z), \ \ z \in S,
\end{gathered}$$

where $\boldsymbol{H}_\omega$ *is a weakly singular integral operator,* $\widetilde{\boldsymbol{K}}_\omega$ *and* $\boldsymbol{K}_\omega$ *are singular integral operators, while* $\boldsymbol{L}_\omega$ *is a pseudodifferential operator of order* 1,

$$\begin{aligned}
\boldsymbol{H}_\omega(g)(z) &:= \int\limits_S \Gamma(z-y,\omega) g(y)\, d_y S, \ \ z \in S, \\
\widetilde{\boldsymbol{K}}_\omega(f)(z) &:= \int\limits_S \big[\widetilde{T}(\partial_y, n(y))\Gamma^\top(z-y,\omega)\big]^\top f(y)\, d_y S, \ \ z \in S, \\
\boldsymbol{K}_\omega(g)(z) &:= \int\limits_S T(\partial_z, n(z))\Gamma(z-y,\omega) g(y) d_y S, \ \ z \in S.
\end{aligned}$$

The mapping properties of these potentials and the boundary integral operators are described in Appendix.

4 Mixed type interaction problems for pseudo-oscillation equations

In this section, we consider the mixed type interaction problems for the so-called pseudo-oscillation equations. These problems are intermediate auxiliary ones used for investigation of interaction problems for the steady state oscillation equations.

4.1 Formulation of the problems

The matrix differential operator corresponding to the basic pseudo-oscillation equations of the electro-magneto-elasticity for anisotropic homogeneous media is written as follows:

$$\begin{gathered}
A(\partial,\tau) = [A_{jk}(\partial,\tau)]_{5\times 5}, \\
A_{jk}(\partial,\tau) = c_{ijkl}\partial_i\partial_l + \rho_1\tau^2\delta_{jk}, \quad A_{j4}(\partial,\tau) = e_{lij}\partial_l\partial_i, \quad A_{j5}(\partial,\tau) = q_{lij}\partial_l\partial_i, \\
A_{4k}(\partial,\tau) = -e_{ikl}\partial_i\partial_l, \quad A_{44}(\partial,\tau) = \varepsilon_{il}\partial_i\partial_l, \quad A_{45}(\partial,\tau) = a_{il}\partial_i\partial_l, \\
A_{5k}(\partial,\tau) = -q_{ikl}\partial_i\partial_l, \quad A_{54}(\partial,\tau) = a_{il}\partial_i\partial_l, \quad A_{55}(\partial,\tau) = \mu_{il}\partial_i\partial_l,
\end{gathered}$$

$j,k = 1,2,3$, where τ is a purely imaginary complex parameter: $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$.

Mixed type problem (M_τ)**:** Find a vector-function $U = (u, u_4, u_5)^\top \in [H^1(\Omega^+)]^5$ and a scalar function $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$ satisfying the differential equations

$$A(\partial,\tau)U = 0 \ \text{ in } \Omega^+, \tag{4.1}$$

$$\Delta \mathrm{w} + \rho_2\omega^2 \mathrm{w} = 0 \ \text{ in } \Omega^-, \tag{4.2}$$

the transmission conditions

$$\{u\cdot n\}^+ = b_1\{\partial_n \mathrm{w}\}^- + f_0 \ \text{ on } S, \tag{4.3}$$

$$\big\{[TU]_j\big\}^+ = b_2\{\mathrm{w}\}^- n_j + f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{4.4}$$

and the mixed boundary conditions

$$\{u_4\}^+ = f_1^{(D)} \quad \text{on } S_D, \tag{4.5}$$

$$\{u_5\}^+ = f_2^{(D)} \quad \text{on } S_D, \tag{4.6}$$

$$\big\{[TU]_4\big\}^+ = f_1^{(N)} \quad \text{on } S_N, \tag{4.7}$$

$$\big\{[TU]_5\big\}^+ = f_2^{(N)} \quad \text{on } S_N, \tag{4.8}$$

where b_1 and b_2 are the given complex constants satisfying conditions (1.17),

$$f_0 \in H^{-1/2}(S), \quad f_j \in H^{-1/2}(S), \;\; j = 1,2,3,$$
$$f_1^{(D)} \in H^{1/2}(S_D), \quad f_2^{(D)} \in H^{1/2}(S_D), \quad f_1^{(N)} \in H^{-1/2}(S_N), \quad f_2^{(N)} \in H^{-1/2}(S_N).$$

Using Green's formulas and the Rellich–Vekua lemma, we obtain the following uniqueness theorem for the problem (M_τ).

Theorem 4.1. *Let $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$. The homogeneous problem (M_τ) has only the trivial solutions.*

4.2 Fundamental solution and potentials for the pseudo-oscillation equations of piezoelectro-magneto-elasticity

The full symbol of the pseudo-oscillation operator $A(\partial,\tau)$ is elliptic provided $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$, i.e.,

$$\det A(-i\xi,\tau) \neq 0 \quad \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Moreover, the entries of the inverse matrix $A^{-1}(-i\xi,\tau)$ are locally integrable functions decaying at infinity as $O(|\xi|^{-2})$. Therefore, we can construct the fundamental matrix $\Gamma(x,\tau) = [\Gamma_{kj}(x,\tau)]_{5\times 5}$ of the operator $A(\partial,\tau)$ by the Fourier transform technique,

$$\Gamma(x,\tau) = F^{-1}_{\xi\to x}\big[A^{-1}(-i\xi,\tau)\big]. \tag{4.9}$$

Note that in a neighbourhood of the origin the following estimates hold $(0 < |x| < 1)$:

$$\big|\Gamma_{jk}(x,\tau) - \Gamma_{jk}(x,\omega)\big| \le c(\tau,\omega), \tag{4.10}$$

$$\Big|\partial_l\big[\Gamma_{jk}(x,\tau) - \Gamma_{jk}(x,\omega)\big]\Big| \le c(\tau,\omega)\ln|x|^{-1}, \tag{4.11}$$

$$\Big|\partial^\alpha\big[\Gamma_{jk}(x,\tau) - \Gamma_{jk}(x,\omega)\big]\Big| \le c(\tau,\omega)|x|^{1-|\alpha|}, \;\; j,k = \overline{1,5}, \tag{4.12}$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ is a multi-index with $|\alpha| = \alpha_1+\alpha_2+\alpha_3 \ge 2$, while $c(\tau,\omega)$ is a positive constant depending on $\tau = i\sigma$ and ω with $\sigma,\omega \in \mathbb{R}\setminus\{0\}$ (cf. [30]).

Let us introduce the single and double layer pseudo-oscillation potentials

$$\mathbf{V}_\tau(h) = \int_S \Gamma(x-y,\tau)h(y)\,d_yS,$$
$$\mathbf{W}_\tau(h) = \int_S \big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(x-y,\tau)\big]^\top h(y)\,d_yS,$$

where $h = (h_1,h_2,h_3,h_4,h_5)^\top$ is a density vector-function.

These pseudo-oscillation potentials have the following jump properties (see [6]).

Theorem 4.2. *Let $h^{(1)} \in [H^{-1+s}(S)]^5$, $h^{(2)} \in [H^s(S)]^5$, $s > 0$. Then the following jump relations hold on S:*

$$\begin{aligned}
\big\{\mathbf{V}_\tau(h^{(1)})(z)\big\}^{\pm} &= \int\limits_S \Gamma(z-y,\tau)h^{(1)}(y)\,d_yS,\\
\big\{\mathbf{W}_\tau(h^{(2)})(z)\big\}^{\pm} &= \pm 2^{-1}h^{(2)}(z) + \int\limits_S \big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(z-y,\tau)\big]^\top h^{(2)}(y)\,d_yS,\\
\big\{T\mathbf{V}_\tau(h^{(1)})(z)\big\}^{\pm} &= \mp 2^{-1}h^{(1)}(z) + \int\limits_S T(\partial_z,n(z))\Gamma(z-y,\tau)h^{(1)}(y)\,d_yS,\\
\big\{T\mathbf{W}_\tau(h^{(2)})(z)\big\}^{+} &= \big\{T\mathbf{W}_\tau(h^{(2)})(z)\big\}^{-}.
\end{aligned}$$

Further, we introduce the boundary operators

$$\begin{aligned}
\mathbf{H}_\tau(h)(z) &= \int\limits_S \Gamma(z-y,\tau)h(y)\,d_yS,\\
\mathbf{K}_\tau(h)(z) &= \int\limits_S T(\partial_z,n(z))\Gamma(z-y,\tau)h(y)\,d_yS,\\
\widetilde{\mathbf{K}}_\tau(h)(z) &= \int\limits_S \big[\widetilde{T}(\partial_y,n(y))\Gamma^\top(z-y,\tau)\big]^\top h(y)\,d_yS,\\
\mathbf{L}_\tau(h)(z) &= \big\{T\mathbf{W}_\tau(h)(z)\big\}^{+} = \big\{T\mathbf{W}_\tau(h)(z)\big\}^{-}.
\end{aligned}$$

Note that $\mathbf{H}_\tau$ is a weakly singular integral operator (pseudodifferential operator of order -1), $\mathbf{K}_\tau$ and $\widetilde{\mathbf{K}}_\tau$ are singular integral operators (pseudodifferential operator of order 0), and $\mathbf{L}_\tau$ is a pseudodifferential operator of order 1.

The mapping properties of these potentials are described in Appendix.

4.3 Some results for pseudodifferential equations on a manifold with boundary

The Fredholm properties of strongly elliptic pseudodifferential operators on a compact manifold with boundary are studied in [1, 16, 18, 19, 37].

The spaces H^s_p and $B^s_{p,p}$ (with $s \in \mathbb{R}$, $1 < p < \infty$) denote the well-known Bessel potential and Besov function spaces, respectively (see [38]).

Let $\Sigma \in C^\infty$ be a compact n-dimensional, nonselfintersecting manifold with boundary $\partial\Sigma \in C^\infty$ and let $\boldsymbol{A}$ be a strongly elliptic scalar pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\Sigma}$, that is, there is a positive constant c_0 such that

$$\operatorname{Re}\mathfrak{S}_{\boldsymbol{A}}(x,\xi) \geq c_0$$

for $x \in \overline{\Sigma}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. Here, we denote by $\mathfrak{S}_{\boldsymbol{A}}(x,\xi)$ the principal homogeneous symbol of the operator $\boldsymbol{A}$ in some local coordinate system ($x \in \overline{\mathbb{R}^n_+}$, $\xi \in \mathbb{R}^n \setminus \{0\}$).

The Fredholm properties of strongly elliptic scalar pseudodifferential operators on a compact manifold with boundary are investigated by the Wiener–Hopf factorization method. The factorization index of the principal homogeneous symbol of the operator $\boldsymbol{A}$ is calculated by the following formula (see [16]):

$$\begin{aligned}
\kappa(x) &= \frac{\nu}{2} + \frac{1}{2\pi i}\ln\frac{\mathfrak{S}_{\boldsymbol{A}}(x,0,\dots,0,-1)}{\mathfrak{S}_{\boldsymbol{A}}(x,0,\dots,0,+1)}\\
&= \frac{\nu}{2} + \frac{1}{2\pi}\Big(\arg\mathfrak{S}_{\boldsymbol{A}}(x,0,\dots,0,-1) - \arg\mathfrak{S}_{\boldsymbol{A}}(x,0,\dots,0,+1)\Big) - \frac{i}{2\pi}\ln\left|\frac{\mathfrak{S}_{\boldsymbol{A}}(x,0,\dots,0,-1)}{\mathfrak{S}_{\boldsymbol{A}}(x,0,\dots,0,+1)}\right|,
\end{aligned}$$

where

$$-\frac{\pi}{2} < \arg\mathfrak{S}_{\boldsymbol{A}}(x,0,\pm 1) < \frac{\pi}{2}\,,\ \ x \in \partial\Sigma.$$

Theorem 4.3. *Let $s \in \mathbb{R}$, $1 < p < \infty$, and $\boldsymbol{A}$ be a strongly elliptic pseudodifferential scalar operator of order $\nu \in \mathbb{R}$. Then the operator*

$$\boldsymbol{A} : \widetilde{H}^s_p(\Sigma) \to H^{s-\nu}_p(\Sigma) \quad \Big(\widetilde{B}^s_{p,p}(\Sigma) \to B^{s-\nu}_{p,p}(\Sigma)\Big) \tag{4.13}$$

is Fredholm with index zero if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\Sigma} \operatorname{Re}\kappa(x) < s < \frac{1}{p} + \inf_{x\in\partial\Sigma} \operatorname{Re}\kappa(x). \tag{4.14}$$

Moreover, the null space and index of operator (4.13) *are the same for all s and p satisfying inequality* (4.14).

4.4 Existence of solutions of the problem (M_τ)

By Theorem 6.4 (see Appendix), the operator $\mathbf{H}_\tau : [H^s(S)]^5 \to [H^{s+1}(S)]^5$ is invertible for all $s \in \mathbb{R}$ and we can look for a solution of the problem (D_τ) in the form

$$U = \mathbf{V}_\tau \mathbf{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-, \ \ \mu \in \mathbb{C}, \ \ \operatorname{Im}\mu \neq 0,$$

where $g = (\widetilde{g}, g_4, g_5)^\top \in [H^{1/2}(S)]^5$, $\widetilde{g} = (g_1, g_2, g_3)^\top$, $h \in H^{1/2}(S)$ are unknown densities. From Theorems 6.1, 6.3 and 6.4 (see Appendix) it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

The transmission conditions (4.3), (4.4), and the Dirichlet type conditions (4.5), (4.6) lead to the following system of pseudodifferential equations with respect to the unknowns $\widetilde{g}$, g_4, g_5 and h:

$$\widetilde{g} \cdot n - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.15}$$
$$\big[\big(-2^{-1}I_5 + \mathbf{K}_\tau\big)\mathbf{H}_\tau^{-1} g\big]_j - b_2 n_j \mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{4.16}$$
$$r_{S_D} g_4 = f_1^{(D)} \ \text{ on } S_D, \tag{4.17}$$
$$r_{S_D} g_5 = f_2^{(D)} \ \text{ on } S_D, \tag{4.18}$$
$$r_{S_N} [\mathcal{A}_\tau g]_4 = f_1^{(N)} \ \text{ on } S_N, \tag{4.19}$$
$$r_{S_N} [\mathcal{A}_\tau g]_5 = f_2^{(N)} \ \text{ on } S_N, \tag{4.20}$$

where

$$\mathcal{N} = -2^{-1}I_1 + \mathcal{K}^*_\omega + \mu\mathcal{H}_\omega, \quad \mathcal{M} = \mathcal{L}_\omega + \mu\big(2^{-1}I_1 + \mathcal{K}_\omega\big).$$

Here and in what follows, I_m stands for the $m \times m$ unit matrix.

The matrix operator generated by the left-hand side expressions in system (4.15)–(4.20) reads as

$$\mathcal{P}_{\tau,M} := \begin{pmatrix} [n]_{1\times3} & 0 & 0 & -b_1\mathcal{M} \\ [\mathcal{A}^{jk}_\tau]_{3\times3} & [\mathcal{A}^{j4}_\tau]_{3\times1} & [\mathcal{A}^{j5}_\tau]_{3\times1} & [-b_2 n_j \mathcal{N}]_{3\times1} \\ [0]_{1\times3} & r_{S_D} I_1 & 0 & 0 \\ [0]_{1\times3} & 0 & r_{S_D} I_1 & 0 \\ r_{S_N}[\mathcal{A}^{4j}_\tau]_{1\times3} & r_{S_N}[\mathcal{A}^{44}_\tau] & r_{S_N}[\mathcal{A}^{45}_\tau] & 0 \\ r_{S_N}[\mathcal{A}^{5j}_\tau]_{1\times3} & r_{S_N}[\mathcal{A}^{54}_\tau] & r_{S_N}[\mathcal{A}^{55}_\tau] & 0 \end{pmatrix}_{8\times6}, \quad j,k = 1,2,3,$$

where

$$\mathcal{A}_\tau := \big(-2^{-1}I_5 + \mathbf{K}_\tau\big)\mathbf{H}_\tau^{-1} = [\mathcal{A}^{jk}_\tau]_{5\times5}, \ \ j,k = \overline{1,5}, \tag{4.21}$$

is the Steklov–Poincaré type operator on S. This operator is a strongly elliptic pseudodifferential operator of order 1 (see [6,9] for details).

By Theorems 6.2 and 6.4 (see Appendix), the operator $\mathcal{P}_{\tau,M}$ possesses the following mapping property:

$$\mathcal{P}_{\tau,M} : [H^s(S)]^6 \to [H^{s-1}(S)]^4 \times [H^s(S_D)]^2 \times [H^{s-1}(S_N)]^2, \ \ s \in \mathbb{R}. \tag{4.22}$$

Next, we show that system (4.15)–(4.20) is uniquely solvable in the space $[H^{1/2}(S)]^6$.

Any extensions of the Dirichlet datum $f_1^{(D)} \in H^{1/2}(S_D)$ and $f_2^{(D)} \in H^{1/2}(S_D)$ onto the whole boundary S have the form

$$G_0^{(1)} + g_0^{(1)} \in H^{1/2}(S), \quad G_0^{(2)} + g_0^{(2)} \in H^{1/2}(S),$$

where $G_0^{(1)} \in H^{1/2}(S)$ and $G_0^{(2)} \in H^{1/2}(S)$ are some fixed extensions of $f_1^{(D)}$ and $f_2^{(D)}$, respectively, while $g_0^{(1)} \in \widetilde{H}^{1/2}(S_N)$, $g_0^{(2)} \in \widetilde{H}^{1/2}(S_N)$.

Rewrite system (4.15)–(4.20) in the equivalent form with respect to $\widetilde{g}$, g_4, g_5, h, $g_0^{(1)}$, $g_0^{(2)}$:

$$\widetilde{g}\cdot n - b_1\mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.23}$$
$$\big[\mathcal{A}_\tau(\widetilde{g}, g_4)^\top\big]_j - b_2 n_j\mathcal{N}(h) = f_j \ \text{ on } S, \ \ j=1,2,3, \tag{4.24}$$
$$g_4 - g_0^{(1)} = G_0^{(1)} \ \text{ on } S, \tag{4.25}$$
$$g_5 - g_0^{(2)} = G_0^{(2)} \ \text{ on } S, \tag{4.26}$$
$$r_{S_N}\big[\mathcal{A}_\tau(\widetilde{g}, g_4, g_5)^\top\big]_4 = f_1^{(N)} \ \text{ on } S_N, \tag{4.27}$$
$$r_{S_N}\big[\mathcal{A}_\tau(\widetilde{g}, g_4, g_5)^\top\big]_5 = f_2^{(N)} \ \text{ on } S_N. \tag{4.28}$$

Remark 2. Systems (4.15)–(4.20) and (4.23)–(4.28) are equivalent in the following sense:

(i) if $(\widetilde{g}, g_4, g_5, h)$ solves system (4.15)–(4.20), then $(\widetilde{g}, g_4, g_5, h, g_0)$ solves system (4.23)–(4.28), where $g_0^{(1)} = g_4 - G_0^{(1)}$, $g_0^{(2)} = g_5 - G_0^{(2)}$ and $G_0^{(1)}$, $G_0^{(2)}$ are some fixed extensions of the functions $f_1^{(D)}$ and $f_2^{(D)}$ from S_D onto the whole boundary S;

(ii) if $(\widetilde{g}, g_4, g_5, h, g_0^{(1)}, g_0^{(2)})$ solves system (4.23)–(4.28), then $(\widetilde{g}, g_4, g_5, h)$ solves system (4.15)–(4.20).

From the results obtained in [11], system (4.23)–(4.26) is uniquely solvable with respect to $\widetilde{g}$, g_4, g_5, h for any extensions $G_0^{(1)} + g_0^{(1)}$ and $G_0^{(2)} + g_0^{(2)}$ of the Dirichlet boundary data $f_1^{(D)}$ and $f_2^{(D)}$. We have to find such extensions of $f_1^{(D)}$ and $f_2^{(D)}$ (i.e., such functions $g_0^{(1)}$ and $g_0^{(2)}$) that equations (4.27) and (4.28) are satisfied.

The operator corresponding to system (4.23)–(4.28) has the following form:

$$\mathcal{P}_{\tau,M}^{(1)} := \begin{pmatrix} [n]_{1\times 3} & 0 & 0 & -b_1\mathcal{M} & 0 & 0 \\ [\mathcal{A}_\tau^{jk}]_{3\times 3} & [\mathcal{A}_\tau^{j4}]_{3\times 1} & [\mathcal{A}_\tau^{j5}]_{3\times 1} & [-b_2 n_j\mathcal{N}]_{3\times 1} & [0]_{3\times 1} & [0]_{3\times 1} \\ [0]_{1\times 3} & I_1 & 0 & 0 & -I_1 & 0 \\ [0]_{1\times 3} & 0 & I_1 & 0 & 0 & -I_1 \\ r_{S_N}[\mathcal{A}_\tau^{4j}]_{1\times 3} & r_{S_N}[\mathcal{A}_\tau^{44}] & r_{S_N}[\mathcal{A}_\tau^{45}] & 0 & 0 & 0 \\ r_{S_N}[\mathcal{A}_\tau^{5j}]_{1\times 3} & r_{S_N}[\mathcal{A}_\tau^{54}] & r_{S_N}[\mathcal{A}_\tau^{55}] & 0 & 0 & 0 \end{pmatrix}_{8\times 8}, \quad j,k=1,2,3.$$

The operator $\mathcal{P}_{\tau,M}^{(1)}$ is bounded in the spaces

$$\mathcal{P}_{\tau,M}^{(1)} : [H^s(S)]^6 \times [\widetilde{H}^s(S_N)]^2 \to [H^{s-1}(S)]^4 \times [H^s(S)]^2 \times [H^{s-1}(S_N)]^2, \ \ s\in\mathbb{R}.$$

Let us consider the system generated by equations (4.23)–(4.26) with respect to the unknowns $\widetilde{g}$, g_4, g_5, and h:

$$\widetilde{g}\cdot n - b_1\mathcal{M}(h) = f_0 \ \text{ on } S, \tag{4.29}$$
$$\big[\mathcal{A}_\tau(\widetilde{g}, g_4, g_5)^\top\big]_j - b_2 n_j\mathcal{N}(h) = f_j \ \text{ on } S, \ \ j=1,2,3, \tag{4.30}$$
$$g_4 = G_0^{(1)} + g_0^{(1)} \ \text{ on } S, \tag{4.31}$$
$$g_5 = G_0^{(2)} + g_0^{(2)} \ \text{ on } S. \tag{4.32}$$

System (4.29)–(4.32) is uniquely solvable in the class $[H^{1/2}(S)]^6$ and the corresponding operator of this system

$$\mathcal{P}_{\tau,D} : [H^s(S)]^6 \to [H^{s-1}(S)]^4 \times [H^s(S)]^2$$

is invertible for all $s \in \mathbb{R}$ (see Appendix, Theorem 6.6).

Define h from equation (4.29) (see Theorem 3.2) and substitute in equation (4.30). Then we obtain a uniquely solvable system with respect to $\widetilde{g}$, g_4, g_5:

$$\mathcal{B}_\tau(\widetilde{g}, g_4, g_5)^\top = \Psi \text{ on } S, \tag{4.33}$$

where

$$\mathcal{B}_\tau = \begin{pmatrix} [C_\tau]_{3\times 3} & [\mathcal{A}_\tau^{j4}]_{3\times 1} & [\mathcal{A}_\tau^{j5}]_{3\times 1} \\ [0]_{1\times 3} & I_1 & 0 \\ [0]_{1\times 3} & 0 & I_1 \end{pmatrix})_{5\times 5},$$
$$[C_\tau]_{3\times 3} = [\mathcal{A}_\tau^{jk}]_{3\times 3} - b_2\, b_1^{-1}[n_j\mathcal{N}]_{3\times 1}[\mathcal{M}^{-1}n_k]_{1\times 3}, \quad j,k = 1,2,3,$$
$$\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5)^\top \in [H^{-1/2}(S)]^3 \times [H^{1/2}(S)]^2,$$
$$\Psi_j = f_j - b_2 b_1^{-1} n_j\, \mathcal{N}\mathcal{M}^{-1} f_0, \quad j = 1,2,3,$$
$$\Psi_4 = G_0^{(1)} + g_0^{(1)}, \quad \Psi_5 = G_0^{(2)} + g_0^{(2)}.$$

Since the operator

$$\mathcal{B}_\tau : [H^{1/2}(S)]^5 \to [H^{-1/2}(S)]^3 \times [H^{1/2}(S)]^2$$

is invertible, from equation (4.33) we deduce

$$(\widetilde{g}, g_4, g_5)^\top = \mathcal{B}_\tau^{-1}\Psi \text{ on } S. \tag{4.34}$$

After substituting (4.34) into equations (4.27) and (4.28), on the manifold S_N with boundary we obtain the following scalar pseudodifferential equations with respect to the unknown functions $g_0^{(1)}$ and $g_0^{(2)}$:

$$r_{S_N} \boldsymbol{A}_\tau^{(1)} g_0^{(1)} = F^{(1)} \text{ on } S_N,$$
$$r_{S_N} \boldsymbol{A}_\tau^{(2)} g_0^{(2)} = F^{(2)} \text{ on } S_N,$$

where

$$\boldsymbol{A}_\tau^{(1)} g_0^{(1)} := \big[\mathcal{A}_\tau\, \mathcal{B}_\tau^{-1}(0,0,0,g_0^{(1)},0)^\top\big]_4,$$
$$F^{(1)} := f_1^{(N)} - r_{S_N}\big[\mathcal{A}_\tau\, \mathcal{B}_\tau^{-1}(\Psi_1, \Psi_2, \Psi_3, G_0^{(1)}, 0)^\top\big]_4 \in H^{-1/2}(S_N),$$
$$\boldsymbol{A}_\tau^{(2)} g_0^{(2)} := \big[\mathcal{A}_\tau\, \mathcal{B}_\tau^{-1}(0,0,0,0,g_0^{(2)})^\top\big]_5,$$
$$F^{(2)} := f_2^{(N)} - r_{S_N}\big[\mathcal{A}_\tau\, \mathcal{B}_\tau^{-1}(\Psi_1, \Psi_2, \Psi_3, 0, G_0^{(2)})^\top\big]_5 \in H^{-1/2}(S_N).$$

Thus system (4.23)–(4.28) can be reduced to the following equivalent system with respect to the unknowns $\widetilde{g}$, g_4, g_5, h, $g_0^{(1)}$, $g_0^{(2)}$:

$$\mathcal{P}_{\tau,M}^{(2)}(\widetilde{g}, g_4, g_5, h, g_0^{(1)}, g_0^{(2)})^\top = \Phi,$$

where

$$\mathcal{P}^{(2)}_{\tau,M} = \begin{pmatrix} [n]_{1\times 3} & 0 & 0 & -b_1\mathcal{M} & 0 & 0 \\ [\mathcal{A}^{jk}_\tau]_{3\times 3} & [\mathcal{A}^{j4}_\tau]_{3\times 1} & [\mathcal{A}^{j5}_\tau]_{3\times 1} & [-b_2 n_j \mathcal{N}]_{3\times 1} & [0]_{3\times 1} & [0]_{3\times 1} \\ [0]_{1\times 3} & I_1 & 0 & 0 & -I_1 & 0 \\ [0]_{1\times 3} & 0 & I_1 & 0 & 0 & -I_1 \\ [0]_{1\times 3} & 0 & 0 & 0 & r_{S_N}\boldsymbol{A}^{(1)}_\tau & 0 \\ [0]_{1\times 3} & 0 & 0 & 0 & 0 & r_{S_N}\boldsymbol{A}^{(2)}_\tau \end{pmatrix}_{8\times 8}$$

$$= \begin{pmatrix} \mathcal{P}_{\tau,D} & \widetilde{I}^*_{6\times 1} & I^*_{6\times 1} \\ [0]_{1\times 6} & r_{S_N}\boldsymbol{A}^{(1)}_\tau & 0 \\ [0]_{1\times 6} & 0 & r_{S_N}\boldsymbol{A}^{(2)}_\tau \end{pmatrix}_{8\times 8}, \quad j,k=1,2,3,$$

with

$$I^*_{6\times 1} = (0,0,0,0,0,-I_1)^\top, \quad \widetilde{I}^*_{6\times 1} = (0,0,0,0,-I_1,0)^\top$$

and

$$\Phi = \big(f_0, f_1, f_2, f_3, G^{(1)}_0, G^{(2)}_0, F^{(1)}, F^{(2)}\big)^\top \in [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^2 \times [H^{-1/2}(S_N)]^2.$$

Therefore. the operator

$$\mathcal{P}^{(2)}_{\tau,M} : [H^{1/2}(S)]^6 \times [\widetilde{H}^{1/2}(S_N)]^2 \to [H^{-1/2}(S)]^4 \times [H^{1/2}(S)]^2 \times [H^{-1/2}(S_N)]^2$$

is Fredholm with index zero if the operators

$$r_{S_N}\boldsymbol{A}^{(1)}_\tau, r_{S_N}\boldsymbol{A}^{(2)}_\tau : \widetilde{H}^{1/2}(S_N) \to H^{-1/2}(S_N)$$

are Fredholm with index zero.

Let us show that the operators $r_{S_N}\boldsymbol{A}^{(1)}_\tau$, $r_{S_N}\boldsymbol{A}^{(2)}_\tau$ are Fredholm with index zero. First, we show that the pseudodifferential operators of order 1

$$r_{S_N}\boldsymbol{A}^{(1)}_\tau, r_{S_N}\boldsymbol{A}^{(2)}_\tau : \widetilde{H}^{s}(S_N) \to H^{s-1}(S_N), \quad s\in\mathbb{R},$$

are strongly elliptic.

By $\mathfrak{S}_{\boldsymbol{A}^{(1)}_\tau}(x,\xi) = \mathfrak{S}_{\boldsymbol{A}^{(1)}_\tau}(\xi)$ and $\mathfrak{S}_{\boldsymbol{A}^{(2)}_\tau}(x,\xi) = \mathfrak{S}_{\boldsymbol{A}^{(2)}_\tau}(\xi)$ we denote the principal homogeneous symbol of the operators $\boldsymbol{A}^{(1)}_\tau$ and $\boldsymbol{A}^{(2)}_\tau$ at the "frozen" point $x \in \overline{S_N}$, where $\xi \in \mathbb{R}^2$. Below, we suppose that the principal homogeneous symbols of the operators are written at the "frozen" point $x \in \overline{S_N}$.

First, we find the form of the principal homogeneous symbol of the pseudodifferential operator $\mathcal{A}_\tau\mathcal{B}^{-1}_\tau$

$$\mathfrak{S}_{\mathcal{A}_\tau\mathcal{B}^{-1}_\tau}(\xi) = \mathfrak{S}_{\mathcal{A}_\tau}(\xi)\mathfrak{S}^{-1}_{\mathcal{B}_\tau}(\xi),$$

where

$$\mathfrak{S}_{\mathcal{A}_\tau}(\xi) = \begin{pmatrix} [\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]_{3\times 3} & [\mathfrak{S}^{j4}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} & [\mathfrak{S}^{j5}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} \\ [\mathfrak{S}^{4k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3} & \mathfrak{S}^{44}_{\mathcal{A}_\tau}(\xi) & \mathfrak{S}^{45}_{\mathcal{A}_\tau}(\xi) \\ [\mathfrak{S}^{5k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3} & \mathfrak{S}^{54}_{\mathcal{A}_\tau}(\xi) & \mathfrak{S}^{55}_{\mathcal{A}_\tau}(\xi) \end{pmatrix}_{5\times 5}, \quad j,k=1,2,3,$$

is the principal homogeneous symbol of the Steklov–Poincaré type operator $\mathcal{A}_\tau$. This operator is strongly elliptic, since the pseudodifferential operator $\mathcal{N}\mathcal{M}^{-1}$ is of order -1. Therefore, the principal homogeneous symbol of the operator $\mathcal{B}_\tau$ has the form

$$\mathfrak{S}_{\mathcal{B}_\tau}(\xi) = \begin{pmatrix} [\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]_{3\times 3} & [\mathfrak{S}^{j4}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} & [\mathfrak{S}^{j5}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} \\ [0]_{1\times 3} & I_1 & 0 \\ [0]_{1\times 3} & 0 & I_1 \end{pmatrix}_{5\times 5}, \quad j,k=1,2,3,$$

and the inverse symbol matrix of $\mathfrak{S}_{\mathcal{B}_\tau}(\xi)$ reads as

$$\mathfrak{S}^{-1}_{\mathcal{B}_\tau}(\xi) = \begin{pmatrix} [\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3} & -[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j4}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} & -[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j5}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} \\ [0]_{1\times 3} & I_1 & 0 \\ [0]_{1\times 3} & 0 & I_1 \end{pmatrix}_{5\times 5}, \quad j,k=1,2,3.$$

Then we obtain

$$\mathfrak{S}_{\mathcal{A}_\tau}(\xi)\mathfrak{S}^{-1}_{\mathcal{B}_\tau}(\xi) = \begin{pmatrix} I_3 & [0]_{3\times 1} & [0]_{3\times 1} \\ [\mathcal{D}_1]_{1\times 3} & \mathcal{D}_2 & \mathcal{D}_3 \\ [\mathcal{D}_4]_{1\times 3} & \mathcal{D}_5 & \mathcal{D}_6 \end{pmatrix}_{5\times 5},$$

where

$$\begin{aligned}
\mathcal{D}_1 &= [\mathfrak{S}^{4k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\,[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3},\\
\mathcal{D}_2 &= -[\mathfrak{S}^{4k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\,[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j4}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} + \mathfrak{S}^{44}_{\mathcal{A}_\tau}(\xi),\\
\mathcal{D}_3 &= -[\mathfrak{S}^{4k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\,[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j5}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} + \mathfrak{S}^{45}_{\mathcal{A}_\tau}(\xi),\\
\mathcal{D}_4 &= [\mathfrak{S}^{5k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\,[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3},\\
\mathcal{D}_5 &= -[\mathfrak{S}^{5k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\,[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j4}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} + \mathfrak{S}^{54}_{\mathcal{A}_\tau}(\xi),\\
\mathcal{D}_6 &= -[\mathfrak{S}^{5k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\,[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j5}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} + \mathfrak{S}^{55}_{\mathcal{A}_\tau}(\xi),\\
&\qquad j,k=1,2,3.
\end{aligned}$$

Therefore, the principal homogeneous symbols of the pseudodifferential operators $\boldsymbol{A}^{(1)}_\tau$ and $\boldsymbol{A}^{(2)}_\tau$ have the form

$$\begin{aligned}
\mathfrak{S}_{\boldsymbol{A}^{(1)}_\tau}(\xi) &= -[\mathfrak{S}^{4k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\,[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j4}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} + \mathfrak{S}^{44}_{\mathcal{A}_\tau}(\xi), \quad j,k=1,2,3,\\
\mathfrak{S}_{\boldsymbol{A}^{(2)}_\tau}(\xi) &= -[\mathfrak{S}^{5k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\,[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j5}_{\mathcal{A}_\tau}(\xi)]_{3\times 1} + \mathfrak{S}^{55}_{\mathcal{A}_\tau}(\xi), \quad j,k=1,2,3.
\end{aligned}$$

Since the symbol $\mathfrak{S}_{\mathcal{A}_\tau}(\xi)$ is strongly elliptic, it is easy to check that the symbols $[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}$, $j,k=1,2,3$, $\mathfrak{S}^{44}_{\mathcal{A}_\tau}(\xi)$ and $\mathfrak{S}^{55}_{\mathcal{A}_\tau}(\xi)$ are also strongly elliptic.

From the strong ellipticity of the operator $\mathcal{A}_\tau$, the following inequality follows:

$$\operatorname{Re}\mathfrak{S}_{\mathcal{A}_\tau}(\xi)\zeta\cdot\zeta \geq c_0|\xi|\,|\zeta|^2,$$

where c_0 is a positive constant, $\xi\in\mathbb{R}^2$, $\zeta=(\zeta^{'},\zeta_4,\zeta_5)^\top\in\mathbb{C}^4$.

Suppose now that

$$\zeta^{'} = -[\mathfrak{S}^{jk}_{\mathcal{A}_\tau}(\xi)]^{-1}_{3\times 3}\,[\mathfrak{S}^{j4}_{\mathcal{A}_\tau}(\xi)]_{3\times 1}\zeta_4, \quad j,k=1,2,3.$$

Then

$$\begin{aligned}
&\operatorname{Re}\mathfrak{S}_{\mathcal{A}_\tau}(\xi)(\zeta^{'},\zeta_4,0)^\top\cdot(\zeta^{'},\zeta_4,0)^\top\\
&= \operatorname{Re}\begin{pmatrix} [0]_{3\times 1} \\ \mathfrak{S}_{\boldsymbol{A}^{(1)}_\tau}(\xi)\zeta_4 \\ [\mathfrak{S}^{5k}_{\mathcal{A}_\tau}(\xi)]_{1\times 3}\zeta' + \mathfrak{S}^{54}_{\mathcal{A}_\tau}(\xi)\zeta_4 \end{pmatrix}\cdot\begin{pmatrix}\zeta' \\ \zeta_4 \\ 0\end{pmatrix} = \operatorname{Re}\mathfrak{S}_{\boldsymbol{A}^{(1)}_\tau}(\xi)|\zeta_4|^2 \geq c_0|\xi|\big(|\zeta^{'}|^2+|\zeta_4|^2\big) \geq c_0|\xi|\,|\zeta_4|^2,
\end{aligned}$$

i.e.,

$$\operatorname{Re}\mathfrak{S}_{\boldsymbol{A}^{(1)}_\tau}(\xi) \geq c_0|\xi|.$$

Thus we find that the pseudodifferential operator $r_{S_N}\boldsymbol{A}^{(1)}_\tau$ is strongly elliptic. Analogously, we can show that the pseudodifferential operator $r_{S_N}\boldsymbol{A}^{(2)}_\tau$ is strongly elliptic. Therefore, it follows from Theorem 4.3 for $p=2$ that the operators

$$r_{S_N}\boldsymbol{A}^{(j)}_\tau : \widetilde{H}^s(S_N) \to H^{s-1}(S_N)$$

are Fredholm with index zero if

$$-\frac{1}{2}+\sup_{x\in\partial S_N}\operatorname{Re}\kappa_j(x)<s<\frac{1}{2}+\inf_{x\in\partial S_N}\operatorname{Re}\kappa_j(x), \tag{4.35}$$

where $\kappa_j(x)$, $j=1,2$, are the factorization indices of the symbols $\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(x,\xi)=\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(\xi)$, $j=1,2$, at the "frozen" point $x\in\partial S_N$, whose real part is calculated by the formula (see Subsection 4.3):

$$\begin{gathered}\operatorname{Re}\kappa_j(x)=\frac{1}{2}+\frac{1}{2\pi}\arg\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(x,0,-1)-\frac{1}{2\pi}\arg\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(x,0,+1),\\ -\frac{\pi}{2}<\arg\mathfrak{S}_{\boldsymbol{A}_\tau^{(j)}}(x,0,\pm1)<\frac{\pi}{2}\,,\quad j=1,2,\quad x\in\partial S_N.\end{gathered} \tag{4.36}$$

It is evident that $0<\operatorname{Re}\kappa_j(x)<1$, $j=1,2$, for $x\in\partial S_N$ and $s=\frac{1}{2}$ satisfies condition (4.35).

Thus we find that the operators

$$\mathcal{P}^{(1)}_{\tau,M},\mathcal{P}^{(2)}_{\tau,M}:[H^s(S)]^6\times[\widetilde{H}^s(S_N)]^2\to[H^{s-1}(S)]^4\times[H^s(S)]^2\times[H^{s-1}(S_N)]^2$$

are Fredholm with index zero for all s satisfying

$$-\frac{1}{2}+\sup_{x\in\partial S_N,\,j=1,2}\operatorname{Re}\kappa_j(x)<s<\frac{1}{2}+\inf_{x\in\partial S_N,\,j=1,2}\operatorname{Re}\kappa_j(x). \tag{4.37}$$

Now we show that the operator $\mathcal{P}^{(1)}_{\tau,M}$ is injective.

Let

$$\big(\widetilde{g},g_4,g_5,h,g_0^{(1)},g_0^{(2)}\big)^\top\in[H^{1/2}(S)]^6\times[\widetilde{H}^{1/2}(S_N)]^2$$

be some solutions of the homogeneous system

$$\mathcal{P}^{(1)}_{\tau,M}\big(\widetilde{g},g_4,g_5,h,g_0^{(1)},g_0^{(2)}\big)^\top=0.$$

It is clear that $g_4=g_0^{(1)}$, $g_5=g_0^{(2)}$ and the potentials

$$U=\mathbf{V}_\tau\mathbf{H}_\tau^{-1}g\ \text{ in }\Omega^+,\quad \mathrm{w}=(W_\omega+\mu V_\omega)h\ \text{ in }\Omega^-,$$

solve the homogeneous problem (M_τ).

It follows from the uniqueness result for the problem (M_τ) (see Theorem 4.1) that $U=0$ in Ω^+ and $\mathrm{w}=0$ in Ω^-. Then

$$\{U\}^+=\big(\widetilde{g},g_0^{(1)},g_0^{(2)}\big)^\top=0\ \text{ on } S\ \text{ and }\ \{\partial_n\mathrm{w}\}^-=\mathcal{M}h=0\ \text{ on } S.$$

From the invertibility of the operator $\mathcal{M}$ (see Theorem 3.2), we obtain that $h=0$ on S.

Hence we obtain that system (4.23)–(4.28) is uniquely solvable. It means that the corresponding operator of system (4.23)–(4.28)

$$\mathcal{P}^{(1)}_{\tau,M}:[H^{1/2}(S)]^6\times[\widetilde{H}^{1/2}(S_N)]^2\to[H^{-1/2}(S)]^4\times[H^{1/2}(S)]^2\times[H^{-1/2}(S_N)]^2$$

is invertible.

The null space and index of the operator $\mathcal{P}^{(1)}_{\tau,M}$ are the same for all s satisfying (4.37).

Hence it follows that the operator

$$\mathcal{P}^{(1)}_{\tau,M}:[H^s(S)]^6\times[\widetilde{H}^s(S_N)]^2\to[H^{s-1}(S)]^4\times[H^s(S)]^2\times[H^{s-1}(S_N)]^2$$

is invertible for all s satisfying (4.37).

Note that the solutions $\widetilde{g}$, g_4, g_5, h of system (4.23)–(4.28) do not depend on the extensions $G_0^{(1)}+g_0^{(1)}$ and $G_0^{(2)}+g_0^{(2)}$ of the functions $f_1^{(D)}$ and $f_2^{(D)}$, respectively, while $g_0^{(1)}$ and $g_0^{(2)}$ do. However, the sums $G_0^{(1)}+g_0^{(1)}$ and $G_0^{(2)}+g_0^{(2)}$ are defined uniquely.

Due to Remark 2, we conclude that the operator

$$\mathcal{P}_{\tau,M}:[H^s(S)]^6\to[H^{s-1}(S)]^4\times[H^s(S_D)]^2\times[H^{s-1}(S_N)]^2$$

is invertible for all s satisfying (4.37).

Thus for the problem (M_τ) the following existence theorem holds.

Theorem 4.4. *Let $\tau = i\sigma$, $\sigma \neq 0$, $\sigma \in \mathbb{R}$, and let $f_0 \in H^{-1/2}(S)$, $f_j \in H^{-1/2}(S)$, $j = 1,2,3$, $f_1^{(D)}, f_2^{(D)} \in H^{1/2}(S_D)$ and $f_1^{(N)}, f_2^{(N)} \in H^{-1/2}(S_N)$. Then the problem (M_τ) has a unique solution (U, w), $U \in [H^1(\Omega^+)]^5$, $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$, which is represented by the potentials*

$$U = \mathbf{V}_\tau \boldsymbol{H}_\tau^{-1} g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-,$$

where the densities $g \in [H^{1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are defined from the uniquely solvable system (4.15)–(4.20). *If the conditions*

$$f_0 \in H^{s-1}(S), \quad f_j \in H^{s-1}(S), \ \ j = 1,2,3, \quad f_1^{(D)}, f_2^{(D)} \in H^s(S_D), \quad f_1^{(N)}, f_2^{(N)} \in H^{s-1}(S_N)$$

hold for the data in (4.3)–(4.8) *and*

$$\frac{1}{2} < s < \frac{1}{2} + \inf_{x \in \partial S_N,\, j=1,2} \operatorname{Re} \kappa_j(x), \tag{4.38}$$

then the solution (U, w) of the mixed type problem (M_τ) exists, is unique and $U \in [H^{s+1/2}(\Omega^+)]^5$, $\mathrm{w} \in H^{s+1/2}_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$.

Moreover, if the conditions

$$f_0 \in H^s(S), \quad f_j \in H^{s-1}(S), \ \ j = 1,2,3, \quad f_1^{(D)}, f_2^{(D)} \in H^s(S_D), \quad f_1^{(N)}, f_2^{(N)} \in H^{s-1}(S_N)$$

hold for the data in (4.3)–(4.8) *and* (4.38) *is satisfied, then the solution (U, w) of the mixed type problem (M_τ) exists, is unique and $U \in [H^{s+1/2}(\Omega^+)]^5$, $\mathrm{w} \in H^{s+3/2}_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$.*

Remark 3. In the last statement of Theorem 4.4, the smoothness of w follows from the representation of h (see (4.15))

$$h = b_1^{-1} \mathcal{M}^{-1}(\widetilde{g} \cdot n) - b_1^{-1} \mathcal{M}^{-1}(f_0) \in H^{s+1}(S) \ \text{ on } S$$

and mapping properties of potentials W_ω and V_ω (see Appendix, Theorem 6.1), where $f_0 \in H^s(S)$, $\widetilde{g} \in [H^s(S)]^3$ and s satisfies (4.38).

Note that from the invertibility of the operator $\mathcal{P}^{(1)}_{\tau,M}$ there follows the invertibility of the operator $\mathcal{P}^{(2)}_{\tau,M}$. Therefore, the following Remark holds.

Remark 4. The operators

$$r_{S_N} \boldsymbol{A}^{(1)}_\tau, r_{S_N} \boldsymbol{A}^{(2)}_\tau : \widetilde{H}^s(S_N) \to H^{s-1}(S_N)$$

are invertible for all s satisfying (4.35).

5 Existence of a solution of the mixed type problem (M_ω)

We look for a solution of the problem (M_ω) in the form

$$U = \mathbf{V}_\omega g \ \text{ in } \Omega^+, \quad \mathrm{w} = (W_\omega + \mu V_\omega)h \ \text{ in } \Omega^-, \quad \mu \in \mathbb{C}, \ \operatorname{Im} \mu \neq 0,$$

where $g \in [H^{-1/2}(S)]^5$ and $h \in H^{1/2}(S)$ are unknown densities, and $\omega \in \mathbb{R} \setminus \{0\}$. From Theorems 6.1 and 6.3 (see Appendix), it follows that $U \in [H^1(\Omega^+)]^5$ and $\mathrm{w} \in H^1_{loc}(\Omega^-)$.

The transmission conditions (1.11), (1.12), and the mixed boundary conditions (1.13), (1.14), (1.15), (1.16) lead then to the following system of pseudodifferential equations with respect to the unknowns g and h:

$$[\mathbf{H}_\omega g]_l n_l - b_1 \mathcal{M}(h) = f_0 \ \text{ on } S, \tag{5.1}$$

$$\big[(-2^{-1} I_4 + \mathbf{K}_\omega) g\big]_j - b_2 n_j \mathcal{N}(h) = f_j \ \text{ on } S, \ \ j = 1,2,3, \tag{5.2}$$

$$[\mathbf{H}_\omega g]_4 = f_1^{(D)} \ \text{ on } S_D, \tag{5.3}$$

$$[\mathbf{H}_\omega g]_5 = f_2^{(D)} \ \text{ on } S_D, \tag{5.4}$$

$$\big[(-2^{-1} I_5 + \mathbf{K}_\omega) g\big]_4 = f_1^{(N)} \ \text{ on } S_N, \tag{5.5}$$

$$\big[(-2^{-1} I_5 + \mathbf{K}_\omega) g\big]_5 = f_2^{(N)} \ \text{ on } S_N. \tag{5.6}$$

The operator generated by the left-hand side of system (5.1)–(5.4) reads as

$$Q_{\omega,M}=\begin{pmatrix} [n_l\mathbf{H}_\omega^{lk}]_{1\times 5} & -b_1\mathcal{M} \\ [(-2^{-1}I_5+\mathbf{K}_\omega)^{jk}]_{3\times 5} & [-b_2n_j\mathcal{N}]_{3\times 1} \\ [\mathbf{H}_\omega^{4k}]_{1\times 5} & 0 \\ [\mathbf{H}_\omega^{5k}]_{1\times 5} & 0 \\ [(-2^{-1}I_5+\mathbf{K}_\omega)^{4k}]_{1\times 5} & 0 \\ [(-2^{-1}I_5+\mathbf{K}_\omega)^{5k}]_{1\times 5} & 0 \end{pmatrix}_{8\times 6}, \quad j=1,2,3, \quad k=\overline{1,5}.$$

By Theorem 6.5, the operator

$$Q_{\omega,M}:[H^{s-1}(S)]^5\times H^s(S)\to[H^{s-1}(S)]^4\times[H^s(S_D)]^2\times[H^{s-1}(S_N)]^2, \quad s\in\mathbb{R},$$

is bounded.

In view of estimates (4.10)–(4.12), it follows that the main parts of the operators $\mathbf{H}_\omega$ and $\mathbf{H}_\tau$ (as well as the main parts of the operators $\mathbf{K}_\omega$ and $\mathbf{K}_\tau$) are the same, implying that the operators

$$\mathbf{H}_\omega-\mathbf{H}_\tau:[H^{-1/2}(S)]^5\to[H^{1/2}(S)]^5, \tag{5.7}$$

$$\mathbf{K}_\omega-\mathbf{K}_\tau:[H^{-1/2}(S)]^5\to[H^{-1/2}(S)]^5 \tag{5.8}$$

are compact. Hence the operator

$$Q_{\omega,M}-Q_{\tau,M}:[H^{s-1}(S)]^5\times H^s(S)\to[H^{s-1}(S)]^4\times[H^s(S_D)]^2\times[H^{s-1}(S_N)]^2, \quad s\in\mathbb{R},$$

is compact, where

$$Q_{\tau,M}:=\mathcal{P}_{\tau,M}\mathcal{T}_\tau$$

with

$$\mathcal{T}_\tau:=\begin{pmatrix} \mathbf{H}_\tau & [0]_{5\times 1} \\ [0]_{1\times 5} & I_1 \end{pmatrix}_{6\times 6}. \tag{5.9}$$

Therefore, from the invertibility of the operators

$$\mathcal{P}_{\tau,M}:[H^s(S)]^6\to[H^{s-1}(S)]^4\times[H^s(S_D)]^2\times[H^{s-1}(S_N)]^2$$

for all s satisfying (4.35) and

$$\mathcal{T}_\tau:[H^{s-1}(S)]^6\times H^{1/2}(S)\to[H^s(S)]^6$$

for all s (see Section 4), the invertibility of the operator

$$Q_{\tau,M}:[H^{s-1}(S)]^5\times H^s(S)\to[H^{s-1}(S)]^4\times[H^s(S_D)]^2\times[H^{s-1}(S_N)]^2$$

follows for all s satisfying (4.35). In turn, this implies that the operator

$$Q_{\omega,M}:[H^{s-1}(S)]^5\times H^s(S)\to[H^{s-1}(S)]^4\times[H^s(S_D)]^2\times[H^{s-1}(S_N)]^2 \tag{5.10}$$

is Fredholm with index zero for all s satisfying (4.35).

Let us show that for $\omega\notin J_M(\Omega^+)$ the operator $Q_{\omega,M}$ is injective. Indeed, let $g\in[H^{-1/2}(S)]^5$ and $h\in H^{1/2}(S)$ be solutions of the homogeneous system

$$Q_{\omega,M}(g,h)^\top=0 \ \text{ on } S.$$

Construct a vector-function $U=\mathbf{V}_\omega g$ and a scalar function $\mathrm{w}=(W_\omega+\mu V_\omega)h$ with $\mu\in\mathbb{C}$, $\operatorname{Im}\mu\neq 0$. Clearly, the pair (U,w) solves the homogeneous problem (M_ω). Since $\omega\notin J_M(\Omega^+)$, it follows from Theorem 2.1 that $U=\mathbf{V}_\omega g=0$ in Ω^+, $\mathrm{w}=(W_\omega+\mu V_\omega)h=0$ in Ω^-.

In view of the equation $\{\mathrm{w}\}^- = \mathcal{N}(h) = 0$ on S and invertibility of the operator $\mathcal{N}$, we deduce that $h = 0$ on S. From continuity of single layer potential, we have $\{U\}^+ = \{U\}^- = 0$ on S.

Thus $U = \mathbf{V}_\omega g$ solves the exterior homogeneous Dirichlet problem

$$A(\partial,\omega)U = 0 \ \text{ on } \Omega^-, \quad \{U\}^- = 0 \ \text{ on } S. \tag{5.11}$$

$U = \mathbf{V}_\omega g \in M_{m_1,m_2,m_3}(\mathbf{P})$ and by Theorem 3.4 $U = \mathbf{V}_\omega g \equiv 0$ in Ω^-. Using the jump formula $\{TU\}^- - \{TU\}^+ = g$ on S, we get that $g = 0$ on S. Thus the null space of the Fredholm operator (5.10) is trivial and since the index equals zero, we conclude that (5.10) is invertible.

These results imply the following assertion.

Theorem 5.1. *If $\omega \notin J_M(\Omega^+)$, then the problem (M_ω) is uniquely solvable.*

Now let us consider the case when ω is a Jones's frequency, $\omega \in J_M(\Omega^+)$.

The operator adjoint to $Q_{\omega,M}$ has the form

$$Q^*_{\omega,D} = \begin{pmatrix} [\mathbf{H}^{*\,kl}_{\ \omega} n_l]_{5\times 1} & \big[(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{kj}\big]_{5\times 3} & [\mathbf{H}^{*\,k4}_{\ \omega}]_{5\times 1} & [\mathbf{H}^{*\,k5}_{\ \omega}]_{5\times 1} & \big[(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{k4}\big]_{5\times 1} & \big[(-2^{-1}I_5 + \mathbf{K}^*_\omega)^{k5}\big]_{5\times 1} \\ -\overline{b}_1\mathcal{M}^* & [-\overline{b}_2\mathcal{N}^* n_j]_{1\times 3} & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 8},$$

$$j = 1,2,3, \quad k = \overline{1,5},$$

where

$$\begin{aligned}
\mathbf{H}^*_\omega(g)(z) &= \int\limits_S \big[\,\overline{\Gamma(y-z,\omega)}\,\big]^\top g(y)\,d_yS, \quad z \in S,\\
\mathbf{K}^*_\omega(g)(z) &= \int\limits_S \big[T(\partial_y, n(y)\,\overline{\Gamma(y-z,\omega)})\big]^\top g(y)\,d_yS, \quad z \in S,\\
\mathcal{N}^*(h)(z) &= \big(-2^{-1}I_1 + \overline{\mathcal{K}}_\omega\big)(h)(z) + \overline{\mu}\,\mathcal{H}^*_\omega(h)(z), \quad z \in S,\\
\mathcal{M}^*(h)(z) &= \mathcal{L}^*_\omega(h)(z) + \overline{\mu}\big(2^{-1}I_1 + \overline{\mathcal{K}}^*_\omega\big)(h)(z), \quad z \in S,
\end{aligned}$$

while

$$\begin{aligned}
\overline{\mathcal{K}}_\omega(h)(z) &= \int\limits_S \partial_{n(z)}\overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z \in S,\\
\overline{\mathcal{K}}^*_\omega(h)(z) &= \int\limits_S \partial_{n(y)}\overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z \in S,\\
\mathcal{H}^*_\omega(h)(z) &= \int\limits_S \overline{\gamma(z-y,\omega)}h(y)\,d_yS, \quad z \in S,\\
\mathcal{L}^*_\omega(h)(z) &= \big\{\partial_{n(z)}\widetilde{W}_\omega(h)(z)\big\}^\pm, \quad z \in S,\\
\widetilde{W}_\omega(h)(x) &= \int\limits_S \partial_{n(y)}\overline{\gamma(x-y,\omega)}h(y)\,d_yS, \quad x \notin S,\\
\widetilde{V}_\omega(h)(x) &= \int\limits_S \overline{\gamma(x-y,\omega)}h(y)\,d_yS, \quad x \notin S.
\end{aligned}$$

The adjoint operator possesses the following mapping property:

$$Q^*_{\omega,M} : [H^{1/2}(S)]^4 \times [\widetilde{H}^{-1/2}(S_D)]^2 \times [\widetilde{H}^{1/2}(S_N)]^2 \to [H^{1/2}(S)]^5 \times H^{-1/2}(S).$$

Let

$$\Psi := \big(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8\big)^\top \in [H^{1/2}(S)]^4 \times [\widetilde{H}^{-1/2}(S_D)]^2 \times [\widetilde{H}^{1/2}(S_N)]^2$$

be a solution of the homogeneous adjoint system

$$Q^*_{\omega,M}\Psi = 0. \tag{5.12}$$

Construct the potentials

$$\widetilde{U} = \widetilde{\mathbf{V}}_\omega\Psi^{(1)} + \widetilde{\mathbf{W}}_\omega\Psi^{(2)} + \widetilde{\mathbf{V}}_\omega\Psi^{(3)} + \widetilde{\mathbf{W}}_\omega\Psi^{(4)} \ \text{ in } \Omega^-, \tag{5.13}$$

$$\widetilde{\mathrm{w}} = -\overline{b}_1\widetilde{W}_\omega\psi_1 - \overline{b}_2\widetilde{V}_\omega[\Psi'\cdot n] \ \text{ in } \Omega^+, \tag{5.14}$$

where

$$\begin{gathered}
\Psi^{(1)} := (n\psi_1, 0, 0)^\top, \quad \Psi^{(2)} := (\Psi', 0, 0)^\top, \quad \Psi^{(3)} := (0, 0, 0, \psi_5, \psi_6)^\top, \\
\Psi^{(4)} := (0, 0, 0, \psi_7, \psi_8)^\top, \quad \Psi' = (\psi_2, \psi_3, \psi_4)^\top, \\
\widetilde{\mathbf{V}}_\omega(g)(x) := \int\limits_S \overline{[\Gamma(y-x,\omega)]}^\top g(y)\, d_yS, \ \ x\in\Omega^+, \\
\widetilde{\mathbf{W}}_\omega(g)(x) := \int\limits_S [T(\partial_y, n(y))\overline{\Gamma(y-x,\omega)}]^\top g(y)\, d_yS, \ \ x\in\Omega^+.
\end{gathered}$$

The vectors $\widetilde{\mathbf{V}}_\omega(g)$ and $\widetilde{\mathbf{W}}_\omega(g)$ are the single and double layer potentials associated with the operator $A^*(\partial,\omega)$.

It follows from (5.12) that

$$\{\widetilde{U}\}^- = 0 \ \text{ and } \ \{\partial_n\widetilde{\mathrm{w}} + \overline{\mu}\widetilde{\mathrm{w}}\}^+ = 0 \ \text{ on } S,$$

where $\mu = \mu_1 + i\mu_2$, $\mu_2 \neq 0$.

Since the vector $\widetilde{U} \in [H^1_{loc}(\Omega^-)]^5 \cap M_{m_1,m_2,m_3}(\mathbf{P}^*)$ and it solves the homogeneous Dirichlet problem

$$A^*(\partial,\omega)\widetilde{U} = 0 \ \text{ in } \Omega^-, \quad \{\widetilde{U}\}^- = 0 \ \text{ on } S,$$

it follows from the uniqueness Theorem 3.5 that $\widetilde{U} = 0$ in Ω^-.

On the other hand, the function $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$ solves the homogeneous Robin type problem

$$(\Delta + \rho_2\omega^2)\widetilde{\mathrm{w}} = 0 \ \text{ in } \Omega^+, \tag{5.15}$$

$$\{\partial_n\widetilde{\mathrm{w}} + \overline{\mu}\widetilde{\mathrm{w}}\}^+ = 0 \ \text{ on } S. \tag{5.16}$$

This problem possesses only the trivial solution. Indeed, the following Green's first formula holds:

$$\int\limits_{\Omega^+} (\Delta + \rho_2\omega^2)\widetilde{\mathrm{w}}\overline{\widetilde{\mathrm{w}}}\, dx + \int\limits_{\Omega^+} |\nabla\widetilde{\mathrm{w}}|\, dx - \rho_2\omega^2 \int\limits_{\Omega^+} |\widetilde{\mathrm{w}}|\, dx = \big\langle\{\partial_n\widetilde{\mathrm{w}}\}^+, \{\widetilde{\mathrm{w}}\}^+\big\rangle_S. \tag{5.17}$$

Taking into account equation (5.15) and the boundary condition (5.16), from (5.17) we get

$$\int\limits_{\Omega^+} |\nabla\widetilde{\mathrm{w}}|\, dx - \rho_2\omega^2 \int\limits_{\Omega^+} |\widetilde{\mathrm{w}}|\, dx = -\mu_1 \int\limits_S |\{\widetilde{\mathrm{w}}\}^+|^2\, dS + i\mu_2 \int\limits_S |\{\widetilde{\mathrm{w}}\}^+|^2\, dS.$$

Therefore, $\{\widetilde{\mathrm{w}}\}^+ = 0$. For a solution $\widetilde{\mathrm{w}} \in H^1(\Omega^+)$ to the homogeneous equation (5.15) we have the following integral representation:

$$\widetilde{\mathrm{w}} = W_\omega\big(\{\widetilde{\mathrm{w}}\}^+\big) - V_\omega\big(\{\partial_n\widetilde{\mathrm{w}}\}^+\big) \ \text{ in } \Omega^+. \tag{5.18}$$

Since $\{\widetilde{\mathrm{w}}\}^+ = 0$ and $\{\partial_n\widetilde{\mathrm{w}}\}^+ = 0$, from the representation formula (5.18) we get that $\widetilde{\mathrm{w}} = 0$ in Ω^+.

Using the jump formulae for potentials (5.13) and (5.14), we derive that on the surface S the following relations hold:

$$\begin{aligned}\{\widetilde{\mathrm{w}}\}^{-} &= \overline{b}_1\psi_1,\\ \{\partial_n\widetilde{\mathrm{w}}\}^{-} &= -\overline{b}_2\,\Psi'\cdot n,\\ \big\{[\widetilde{T}\widetilde{U}]\big\}^{+} &= -(n_j\psi_1,\psi_5,\psi_6)^\top,\\ \{\widetilde{U}\}^{+} &= (\Psi',\psi_7,\psi_8)^\top.\end{aligned}$$

Hence we deduce that $\widetilde{U}=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3,\widetilde{U}_4,\widetilde{U}_5)^\top=(\widetilde{U}',\widetilde{U}_4,\widetilde{U}_5)^\top$ with $\widetilde{U}'=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3,)^\top$ and $\widetilde{\mathrm{w}}$ solves the following homogeneous transmission problem:

$$\begin{aligned}A^*(\partial,\omega)\widetilde{U}&=0\ \text{ in } \Omega^+,\\ (\Delta+\rho_2\omega^2)\widetilde{\mathrm{w}}&=0\ \text{ in } \Omega^-,\\ \{\widetilde{U}'\cdot n\}^{+}+\overline{b}_2^{\,-1}\{\partial_n\widetilde{\mathrm{w}}\}^{-}&=0\ \text{ on } S,\\ \big\{[\widetilde{T}(\partial,n)\widetilde{U}]_j\big\}^{+}+\overline{b}_1^{\,-1}\{\widetilde{\mathrm{w}}\}^{-}n_j&=0\ \text{ on } S,\quad j=1,2,3,\\ \{\widetilde{U}_4\}^{+}&=0\ \text{ on } S_D,\\ \{\widetilde{U}_5\}^{+}&=0\ \text{ on } S_D,\\ \big\{[\widetilde{T}(\partial,n)\widetilde{U}]_4\big\}^{+}&=0\ \text{ on } S_N,\\ \big\{[\widetilde{T}(\partial,n)\widetilde{U}]_5\big\}^{+}&=0\ \text{ on } S_N.\end{aligned}$$

From the uniqueness result (see Remark 1), it follows that $\widetilde{\mathrm{w}}=0$ in Ω^- and $\widetilde{U}\in X^*_{M,\omega}(\Omega^+)$, i.e., $\widetilde{U}$ belongs to the space of Jones modes $X^*_{M,\omega}(\Omega^+)$. Then we obtain

$$\begin{gathered}\psi_1=0,\quad \psi_{j+1}=\{\widetilde{U}_j\}^{+}\ \ j=1,2,3,\quad \psi_5=-\{[\widetilde{T}\widetilde{U}]_4\}^{+},\\ \psi_6=-\{[\widetilde{T}\widetilde{U}]_5\}^{+},\quad \psi_7=\{\widetilde{U}_4\}^{+},\quad \psi_8=\{\widetilde{U}_5\}^{+}.\end{gathered}$$

Vice versa, if $\widetilde{U}\in X^*_{M,\omega}(\Omega^+)$, then from the representation formula

$$\widetilde{U}=\widetilde{\mathbf{W}}_\omega\{\widetilde{U}\}^{+}-\widetilde{\mathbf{V}}_\omega\{\widetilde{T}\widetilde{U}\}^{+}\ \text{ in } \Omega^+ \tag{5.19}$$

it is easy to show that the vector-function

$$\widetilde{\Psi}:=\Big(0,\{\widetilde{U}_1\}^{+},\{\widetilde{U}_2\}^{+},\{\widetilde{U}_3\}^{+},-\{[\widetilde{T}\widetilde{U}]_4\}^{+},-\{[\widetilde{T}\widetilde{U}]_5\}^{+},\{\widetilde{U}_4\}^{+},\{\widetilde{U}_5\}^{+}\Big)^\top$$

is a solution of the adjoint homogeneous system (5.12). Indeed, let us substitute $\widetilde{\Psi}$ in system (5.12). Therefore, we obtain the equalities

$$\begin{aligned}&\big[(-2^{-1}I_4+\mathbf{K}^*_\omega)^{kj}\big]_{5\times3}\{\widetilde{U}'\}^{+}-\big[\mathbf{H}^{*k4}_{\ \omega}\big]_{5\times1}\{[\widetilde{T}\widetilde{U}]_4\}^{+}-\big[\mathbf{H}^{*k5}_{\ \omega}\big]_{5\times1}\big\{[\widetilde{T}\widetilde{U}]_5\big\}^{+}\\ &+\big[(-2^{-1}I_5+\mathbf{K}^*_\omega)^{k4}\big]_{5\times1}\{\widetilde{U}_4\}^{+}+\big[(-2^{-1}I_5+\mathbf{K}^*_\omega)^{k5}\big]_{5\times1}\{\widetilde{U}_5\}^{+}=0,\ \ j=1,2,3,\ \ k=\overline{1,5},\end{aligned} \tag{5.20}$$

$$-\overline{b}_2\mathcal{N}^*\big(\{\widetilde{U}'\}^{+}\cdot n\big)=0, \tag{5.21}$$

where $\widetilde{U}'=(\widetilde{U}_1,\widetilde{U}_2,\widetilde{U}_3)^\top$.

By taking a trace of the representation formula (5.19), we get

$$\{\widetilde{U}\}^{+}=2^{-1}\{\widetilde{U}\}^{+}+\mathbf{K}^*_\omega\{\widetilde{U}\}^{+}-\mathbf{H}^*_\omega\{\widetilde{T}\widetilde{U}\}^{+}\ \text{ on } S,$$

i.e.,

$$(-2^{-1}I+\mathbf{K}^*_\omega)\{\widetilde{U}\}^{+}-\mathbf{H}^*_\omega\{\widetilde{T}\widetilde{U}\}^{+}=0\ \text{ on } S. \tag{5.22}$$

Since $\widetilde{U} \in X^*_{M,\omega}(\Omega^+)$, we have

$$\{\widetilde{U}_4\}^+ = 0, \quad \{\widetilde{U}_5\}^+ = 0, \quad \{[\widetilde{T}\widetilde{U}]_j\}^+ = 0, \;\; j = \overline{1,5}, \tag{5.23}$$

$$\{\widetilde{U}'\}^+ \cdot n = 0. \tag{5.24}$$

Therefore, taking into account (5.23) in equality (5.22), we find that (5.20) is true, and from (5.24) it follows that (5.21) is true, as well.

Therefore,

$$\dim\ker Q_{\omega,M} = \dim\ker Q^*_{\omega,M} = \dim X^*_{M,\omega}(\Omega^+).$$

Thus the following orthogonality condition

$$\sum_{j=1}^{3}\langle f_j, \{\widetilde{U}_j\}^+\rangle_S - \Big\langle \big\{[\overline{\widetilde{T}\widetilde{U}}]_4\big\}^+, \overline{f}_1^{(D)}\Big\rangle_S - \Big\langle \big\{[\overline{\widetilde{T}\widetilde{U}}]_5\big\}^+, \overline{f}_2^{(D)}\Big\rangle_S + \Big\langle \big\{[\overline{\widetilde{U}}]_4\big\}^+, \overline{f}_1^{(N)}\Big\rangle_S + \Big\langle \big\{[\overline{\widetilde{U}}]_5\big\}^+, \overline{f}_2^{(N)}\Big\rangle_S = 0 \quad \forall\,\widetilde{U} \in X^*_{D,\omega}(\Omega^+), \tag{5.25}$$

is necessary and sufficient for the system of pseudodifferential equations (5.1)–(5.6) to be solvable.

We can now formulate the following existence theorem.

Theorem 5.2. *If $\omega \in J_M(\Omega^+)$, then the mixed type problem (M_ω) is solvable if and only if the orthogonality condition* (5.25) *holds, and a solution is defined modulo Jones modes $X_{M,\omega}(\Omega^+)$.*

Remark 5. Let $(f_1, f_2, f_3) = n\psi$, where ψ is a scalar function and n is the unit normal vector to S (see (1.18)). Then the necessary and sufficient condition (5.25) reads as

$$\Big\langle \big\{[\widetilde{T}\widetilde{U}]_4\big\}^+, f_1^{(D)}\Big\rangle_S + \Big\langle \big\{[\widetilde{T}\widetilde{U}]_5\big\}^+, f_2^{(D)}\Big\rangle_S - \Big\langle \big\{[\overline{\widetilde{U}}]_4\big\}^+, \overline{f}_1^{(N)}\Big\rangle_S - \Big\langle \big\{[\overline{\widetilde{U}}]_5\big\}^+, \overline{f}_2^{(N)}\Big\rangle_S = 0 \quad \forall\,\widetilde{U} \in X^*_{M,\omega}(\Omega^+).$$

Clearly, if the mixed datum are constant, or $\omega \notin J^*_M(\Omega^+)$, then the problem (M_ω) is always solvable.

The following theorem holds.

Theorem 5.3. *Let*

$$\frac{1}{2} < s < \frac{1}{2} + \inf_{x\in\partial S_N} \operatorname{Re}\kappa_j(x), \tag{5.26}$$

where $\kappa_j(x)$, $j = 1,2$, are the factorization indices of the principal homogeneous symbol of the operators $\boldsymbol{A}^{(j)}_\tau$, $j = 1,2$ (see Subsection 4.4), and let $U \in [H^1(\Omega^+)]^5$, $\mathrm{w} \in H^1_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$ be the solution of the mixed type problem (M_ω). Then the following regularity result holds:

if $f_0 \in H^{s-1}(S)$, $f_j \in H^{s-1}(S)$, $j = 1,2,3$, $f_1^{(D)}, f_2^{(D)} \in H^s(S_D)$, $f_1^{(N)}, f_2^{(N)} \in H^{s-1}(S_N)$, then $U \in [H^{s+1/2}(\Omega^+)]^5$, $\mathrm{w} \in H^{s+1/2}_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$.

Moreover, if

$$f_0 \in H^s(S), \quad f_j \in H^{s-1}(S), \;\; j = 1,2,3, \quad f_1^{(D)}, f_2^{(D)} \in H^s(S_D), \quad f_1^{(N)}, f_2^{(N)} \in H^{s-1}(S_N),$$

and (5.26) *is satisfied, then $U \in [H^{s+1/2}(\Omega^+)]^5$, $\mathrm{w} \in H^{s+3/2}_{loc}(\Omega^-) \cap \mathrm{Som}_1(\Omega^-)$.*

Remark 6. In the last statement of the Theorem 5.3, the smoothness of w follows from the representation of h (see (5.1))

$$h = b_1^{-1}\mathcal{M}^{-1}[\mathbf{H}_\omega g]_l\, n_l - b_1^{-1}\mathcal{M}^{-1}(f_0) \in H^{s+1}(S) \;\text{ on } S$$

and mapping properties of potentials W_ω and V_ω (see Appendix, Theorem 6.1), where $f_0 \in H^s(S)$, $g \in [H^{s-1}(S)]^5$ and s satisfies (5.26).

6 Appendix

For the readers convenience, we collect here some results describing properties of the layer potentials. Here, we preserve the notation from the main text of the paper. For the potentials associated with the Helmholtz equation, the following theorems hold (see [15, 23, 29, 33]).

Theorem 6.1. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the single and double layer scalar potentials can be extended to the following continuous operators:*

$$V_\omega : H^s(S) \to H^{s+3/2}(\Omega^+), \quad V_\omega : Hs(S) \to H^{s+3/2}_{loc}(\Omega^-),$$
$$W_\omega : H^s(S) \to H^{s+1/2}(\Omega^+), \quad W_\omega \,:\, H^s(S) \to H^{s+1/2}_{loc}(\Omega^-).$$

Theorem 6.2. *Let $s \in \mathbb{R}$, $1 < p < \infty$,$S \in C^\infty$. Then the operators*

$$\mathcal{H}_\omega : H^s(S) \to H^{s+1}(S),$$
$$\mathcal{K}_\omega, \mathcal{K}^*_\omega : H^s(S) \to H^{s+1}(S),$$
$$\mathcal{L}_\omega : H^s(S) \to H^{s-1}(S)$$

are continuous.

For the potentials of steady state oscillation and pseudo-oscillation equations, the following theorems hold (see [6–9, 12]).

Theorem 6.3. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the vector potentials $\mathbf{V}_\omega$, $\mathbf{W}_\omega$, $\mathbf{V}_\tau$ and $\mathbf{W}_\tau$ are continuous in the following spaces:*

$$\mathbf{V}_\omega, \mathbf{V}_\tau : [H^s(S)]^5 \to [H^{s+3/2}(\Omega^+)]^5 \quad \Big([H^s(S)]^5 \to [H^{s+3/2}_{loc}(\Omega^-)]^5 \Big),$$
$$\mathbf{W}_\omega, \mathbf{W}_\tau : [H^s(S)]^5 \to [H^{s+1/2}_p(\Omega^+)]^5 \qquad \Big([H^s(S)]^5 \to [H^{s+1/2}_{loc}(\Omega^-)]^5 \Big).$$

Theorem 6.4. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the operators*

$$\boldsymbol{H}_\tau : [H^s(S)]^5 \to [H^{s+1}(S)]^5,$$
$$\boldsymbol{K}_\tau, \widetilde{\boldsymbol{K}}_\tau : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\boldsymbol{L}_\tau : [H^s(S)]^5 \to [H^{s-1}(S)]^5$$

are bounded.

The operators $\boldsymbol{H}_\tau$ and $\boldsymbol{L}_\tau$ are strongly elliptic pseudodifferential operators of order -1 and 1, respectively, while the operators $\pm 2^{-1} I_5 + \boldsymbol{K}_\tau$ and $\pm 2^{-1} I_5 + \widetilde{\boldsymbol{K}}_\tau$ are elliptic pseudodifferential operators of order 0.

Moreover, the operators $\boldsymbol{H}_\tau$, $2^{-1} I_5 + \widetilde{\boldsymbol{K}}_\tau$ and $2^{-1} I_5 + \boldsymbol{K}_\tau$ are invertible, whereas the operators $\boldsymbol{L}_\tau$, $-2^{-1} I_5 + \widetilde{\boldsymbol{K}}_\tau$ and $-2^{-1} I_5 + \boldsymbol{K}_\tau$ are Fredholm operators with index zero.

Theorem 6.5. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $S \in C^\infty$. Then the operators*

$$\boldsymbol{H}_\omega : [H^s(S)]^5 \to [H^{s+1}(S)]^5,$$
$$\pm 2^{-1} I_5 + \boldsymbol{K}_\omega : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\pm 2^{-1} I_5 + \widetilde{\boldsymbol{K}}_\omega : [H^s(S)]^5 \to [H^s(S)]^5,$$
$$\boldsymbol{L}_\omega : [H^s(S)]^5 \to [H^{s-1}(S)]^5$$

are bounded Fredholm operators with index zero.

The following theorem holds [11].

Theorem 6.6. *The operator*

$$\mathcal{P}_{\tau,D} : [H^s(S)]^6 \to [H^{s-1}(S)]^4 \times [H^s(S)]^2 \quad \Big([B^s_{p,p}(S)]^6 \to [B^{s-1}_{p,p}(S)]^4 \times [B^s_{p,p}(S)]^2 \Big)$$

is invertible for all $s \in \mathbb{R}$ $1 < p < \infty$*, where*

$$\mathcal{P}_{\tau,D} = \begin{pmatrix} [n]_{1\times3} & 0 & 0 & -b_1\mathcal{M} \\ [\mathcal{A}^{jk}_\tau]_{3\times3} & [\mathcal{A}^{j4}_\tau]_{3\times1} & [\mathcal{A}^{j5}_\tau]_{3\times1} & [-b_2 n_j \mathcal{N}]_{3\times1} \\ [0]_{1\times3} & I_1 & 0 & 0 \\ [0]_{1\times3} & 0 & I_1 & 0 \end{pmatrix}_{6\times6}, \quad j,k = 1,2,3.$$

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