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QUALITATIVE ANALYSIS OF GENERALIZED PROPORTIONAL FRACTIONAL FUNCTIONAL INTEGRO-DIFFERENTIAL LANGEVIN EQUATION WITH VARIABLE COEFFICIENT AND NONLOCAL INTEGRAL CONDITIONS

Abstract. In this paper, the existence and uniqueness of solutions for a nonlinear generalized proportional fractional functional integro-differential Langevin equation involving variable coefficient via nonlocal multi-point integral conditions are investigated by using Banach's, Schaefer's and Krasnoselskii's fixed point theorems. Different types of Ulam–Hyers stability are also established. Finally, an example is given to demonstrate applicability to the theoretical findings.

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Key words and phrases. Existence and uniqueness, fixed point theorem, fractional Langevin equation, generalized proportional fractional derivative, nonlocal integral condition, Ulam–Hyers stability.

რეზიუმე. ნაშრომში ბანახის, შეფერისა და კრასნოსელსკის უძრავი წერტილის თეორემების გამოყენებით გამოკვლეულია ამონახსნების არსებობა და ერთადერთობა არაწრფივი განზოგადებული პროპორციული წილად-ფუნქციონალური ინტეგრო-დიფერენციალური ლანჟევანის განტოლებისთვის ცვლადი კოეფიციენტის შემცველი არალოკალური მრავალწერტილიანი ინტეგრალური პირობებით, აგრეთვე დადგენილია სხვადასხვა ტიპის ულამ-ჰაიერსის მდგრადობა. ნაშრომის ბოლოს მოცემულია თეორიული დასკვნების გამოყენების მაგალითი.

1 Introduction

Fractional differential equations have used to be an excellent instrument in the mathematical modelling of dynamical systems and real world problems, such as aerodynamics, polymer science, fractals and chaotic, nonlinear control theory, signal and image processing, bioengineering and chemical engineering, etc. However, a number of various definitions of fractional derivative and integral operators of non-integer order can be found in literature. For more details, we refer the reader to the books [20, 24, 29, 32]. Recently, Jarad *et al.* [22] introduced a new type of fractional derivative operator, the so-called generalized proportional fractional (GPF) derivatives extended by local derivatives [9]. The characteristic of the new derivative is that it involves two fractional orders, preserves the semigroup property, possesses nonlocal character and upon limiting cases it converges to the original function and its derivative. The GPF derivative is well behaved and has a various helpful over the classical derivatives in the sense that it generalizes previously defined derivatives in the literature. We list some recent papers which have been refined in frame of GPF derivative and other related works [2, 7, 8, 37].

Several interesting and important areas of investigation fractional differential equations are devoted to the existence theory and stability analysis of the solutions. In recent years, many authors have discussed the questions on existence, uniqueness and different types of Ulam–Hyers (UH) stability of solutions of initial and boundary value problems for fractional differential equations. The UH stability is the essential and special type of stability analysis that researchers studied in the field of mathematical analysis. The concept of Ulam stability of functional equations was firstly initiated by Ulam [40, 41] and Hyers [21] who presented the partial answer to the Ulam question in the case of Banach space. Thereafter, this type of stability is called the UH stability. In 1950, the Hyers stability was generalized by Aoki [10]. Rassias [33,34] provided an interesting generalization of the UH stability of linear and nonlinear mappings. The UH stability was initially applied to a linear differential equation by Obloza [31]. We refer the reader to the recent works [1,5,11,12,14,17,23,28,36,42,43]. It should be noted that the above-said areas of interest (existence and stability) have been fabulously deliberated within the Riemann–Liouville, Caputo, Hilfer or Hadamard derivatives.

In 1908, Paul Langevin [26] introduced a concept of Langevin equation in a sense of ordinary derivative which is an important equation of mathematical physics. It is well known that a Langevin equation have been widely used to describe the dynamical processes of various fluctuating environments such as physics, chemistry and electrical engineering [16, 30, 44]. However, for a system in complex media, the ordinary Langevin equation does not provide the correct representation of dynamical systems. One of the possible ways of the ordinary Langevin equation is to replace the ordinary (integer-order) derivative by the fractional-order derivative. The fractional Langevin equation was studied by various researchers (for some recent works on fractional Langevin equations, see [6,13,15,18,27,38,39,45]). It is to be noted that most exiting in literature results dealt with a fractional Langevin equation, have been reported in the case of a constant coefficient $\mathcal{H}(t)$. However, the paper [4] has first discussed fractional Langevin equation containing variable coefficient and supplemented with nonlocal-terminal fractional boundary conditions. On the other hand, we claim that our approach in this paper is totally different from paper [4] in the sense that different fractional derivative is accommodated, different boundary conditions are associated, different fixed point theorems are used and UH stability is discussed which has not studied in [4].

Motivated by [4, 15, 38, 39], in this paper we study th existence, uniqueness and different types of UH stability for a nonlinear GPF functional integro-differential Langevin equation involving a variable coefficient via nonlocal multi-point integral conditions:

$$\begin{cases} {}^C_a\mathcal{D}^{\beta,\rho}\big({}^C_a\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\big)x(t) = f(t, x(t), x(\theta(t)), (\mathcal{S}x)(t)), \quad t \in (a, T], \quad a > 0, \\ x(a) = \gamma, \quad x(\eta) = \displaystyle\sum_{i=1}^{m} \delta_i {}_a\mathcal{I}^{\mu_i,\rho} x(\xi_i) + \kappa, \end{cases} \tag{1.1}$$

where ${}^C_a\mathcal{D}^{q,\rho}$ denotes the GPF derivative operator of Caputo type of order $q \in \{\alpha, \beta\}$, $0 < \alpha, \beta \le 1$, $1 < \alpha + \beta \le 2$, $\rho > 0$, ${}_a\mathcal{I}^{\mu_i,\rho}$ denotes the GPF integral opertator of order $\mu_i > 0$, $\rho > 0$, $i = 1, \dots, m$,

$\mathcal{H} \in C([a,T],\mathbb{R})$, $f \in C([a,T]\times\mathbb{R}^3,\mathbb{R})$, $\theta : [a,T] \to [a,T]$,

$$(\mathcal{S}x)(t) = \int_a^t \phi(t,s,x(s))\,ds, \quad t \in [a,T],$$

$\phi : [a,T]^2 \times \mathbb{R} \to [a,\infty)$ is a continuous function. $\gamma, \kappa, \delta_i \in \mathbb{R}$ and $\eta, \xi_i \in (a,T)$, $i = 1,2,\ldots,m$.

The manuscript is structured as follows. In Section 2, we give some definitions and lemmas. In Section 3, we establish some appropriate conditions for the existence results of solutions of problem (1.1) by applying a variety of fixed point theorems due to Banach, Schaefer and Krasnoselskii. In Section 4, we set up applicable results for different types of Ulam–Hyers stability to the solution of problem (1.1). An example illustrating our results is given in Section 5.

2 Preliminaries

This section is devoted to definitions and lemmas that will be used throughout the paper. For their justifications and proofs, we refer the reader to [22].

Definition 2.1 ([22])**.** For $0 < \rho \le 1$, $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > 0$, the GPF integral of f of order α is

$$({}_a\mathcal{I}^{\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\alpha-1} f(s)\,ds = \rho^{-\alpha} e^{\frac{\rho-1}{\rho}t}\, {}_a\mathcal{I}^{\alpha}(e^{\frac{1-\rho}{\rho}s} f)(t),$$

where ${}_a\mathcal{I}^{\alpha}$ is the Riemann–Liouville fractional integral [24].

Definition 2.2 ([22])**.** For $0 < \rho \le 1$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \ge 0$, the Caputo type GPF derivative of f of order α is

$$({}^C_a\mathcal{D}^{\alpha,\rho} f)(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{n-\alpha-1} (\mathcal{D}^{n,\rho} f)(s)\,ds,$$

where $n = [\mathrm{Re}(\alpha)] + 1$ and $[\mathrm{Re}(\alpha)]$ represents the integer part of the real number α.

Lemma 2.1 ([22])**.** *For $0 < \rho \le 1$ and $n = [\mathrm{Re}(\alpha)] + 1$, we have $({}^C_a\mathcal{D}^{\alpha,\rho}\ {}_a\mathcal{I}^{\alpha,\rho} f)(t) = f(t)$, and*

$$({}_a\mathcal{I}^{\alpha,\rho}\ {}^C_a\mathcal{D}^{\alpha,\rho} f)(t) = f(t) - e^{\frac{\rho-1}{\rho}(t-a)} \sum_{k=0}^{n-1} \frac{(\mathcal{D}^{k,\rho} f)(a)}{\rho^k k!} (t-a)^k.$$

Lemma 2.2 ([22])**.** *Let $\alpha, \beta \in \mathbb{C}$ be such that $\mathrm{Re}(\alpha) \ge 0$ and $\mathrm{Re}(\beta) > 0$. Then, for any $0 < \rho \le 1$ and $n = [\mathrm{Re}(\alpha)] + 1$, we have*

(i)
$$\left({}_a\mathcal{I}^{\alpha,\rho} e^{\frac{\rho-1}{\rho}s}(s-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\rho^\alpha \Gamma(\beta+\alpha)}\, e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta+\alpha-1}, \quad \mathrm{Re}(\alpha) > 0.$$

(ii)
$$\left({}^C_a\mathcal{D}^{\alpha,\rho} e^{\frac{\rho-1}{\rho}s}(s-a)^{\beta-1}\right)(t) = \frac{\rho^\alpha \Gamma(\beta)}{\Gamma(\beta-\alpha)}\, e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-\alpha-1}, \quad \mathrm{Re}(\beta) > n.$$

(iii)
$$\left({}^C_a\mathcal{D}^{\alpha,\rho} e^{\frac{\rho-1}{\rho}s}(s-a)^{k}\right)(t) = 0, \quad \mathrm{Re}(\alpha) > n, \quad k = 0,1,\ldots,n-1.$$

Lemma 2.3 (Arzelá–Ascoli theorem [3])**.** *A subset $\mathbb{M}$ in $C([a,b],\mathbb{R})$ with norm*

$$\|f\| = \sup_{t\in[a,b]} |f(t)|$$

is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.

Lemma 2.4 (Banach's fixed point theorem [19]). *Let $\mathbb{M}$ be a non-empty closed subset of a Banach space $\mathbb{E}$. Then any contraction mapping $\mathbb{T}$ from $\mathbb{M}$ into itself has a unique fixed point.*

Lemma 2.5 (Schaefer's fixed point theorem [19]). *Let $\mathbb{M}$ be a Banach space and $\mathbb{T}:\mathbb{M}\to\mathbb{M}$ be a completely continuous operator and let the set $\mathbb{G}=\{x\in\mathbb{M}:\ x=\kappa\mathbb{T}x,\ 0<\kappa\le 1\}$ be bounded. Then $\mathbb{T}$ has a fixed point in $\mathbb{M}$.*

Lemma 2.6 (Krasnoselskii's fixed point theorem [25]). *Let $\mathbb{M}$ be a closed, bounded, convex and nonempty subset of a Banach space $\mathbb{X}$. Let $\mathcal{A}$, $\mathcal{B}$ be the operators such that*

(i) *$\mathcal{A}x+\mathcal{B}y\in\mathbb{M}$ whenever $x,y\in\mathbb{M}$;*

(ii) *$\mathcal{A}$ is compact and continuous;*

(iii) *$\mathcal{B}$ is a contraction mapping.*

Then there exists $z\in\mathbb{M}$ such that $z=\mathcal{A}z+\mathcal{B}z$.

For the sake of computational convenience, we make use of the following constants:

$$\Lambda := \frac{(\eta-a)^{\alpha}e^{\frac{\rho-1}{\rho}(\eta-a)}}{\rho^{\alpha}\Gamma(\alpha+1)} - \sum_{i=1}^{m}\frac{\delta_i(\xi_i-a)^{\alpha+\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\mu_i+1)} \neq 0, \tag{2.1}$$

$$\Omega_1 := \frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\right), \tag{2.2}$$

$$\Omega_2 := \frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)} + \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)} + \frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\right), \tag{2.3}$$

$$\Omega_3 := {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T) + \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\right), \tag{2.4}$$

$$\Omega_4 := \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\right) + |\gamma|. \tag{2.5}$$

Let $\mathbb{E}=C([a,T],\mathbb{R})$ be the Banach space of all continuous functions from $[a,T]$ into $\mathbb{R}$ equipped with the norm $\|x\|_{\mathbb{E}} = \sup_{t\in[a,T]}\{|x(t)|\}$. In order to transform the main problem into a fixed point problem, problem (1.1) must be converted to an equivalent Volterra integral equation. Next, we provide the following lemma.

Lemma 2.7. *Let $h:[a,T]\to\mathbb{R}$ be a continuous function, $0<\alpha,\beta\le 1$, $1<\alpha+\beta\le 2$, and $\rho,\mu_i>0$, $i=1,2,\dots,m$. Then the function $x\in\mathbb{E}$ is the solution to the following linear GPF Langevin equation equipped with the nonlocal integral conditions*

$$\begin{cases} {}^{C}_{a}\mathcal{D}^{\beta,\rho}\big({}^{C}_{a}\mathcal{D}^{\alpha,\rho}+\mathcal{H}(t)\big)x(t)=h(t), \quad t\in(a,T], \\ x(a)=\gamma, \quad x(\eta)=\displaystyle\sum_{i=1}^{m}\delta_i{}_a\mathcal{I}^{\mu_i,\rho}x(\xi_i)+\kappa, \end{cases} \tag{2.6}$$

if and only if x satisfies the following Volterra integral equation:

$$\begin{aligned} x(t) = {}_a\mathcal{I}^{\alpha+\beta,\rho}h(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(t)x(t) \\ + \frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m \delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}h(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}h(\eta) \\ - \sum_{i=1}^m \delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(\xi_i)x(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(\eta)x(\eta) \\ + \sum_{i=1}^m \frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa\Bigg) + \gamma e^{\frac{\rho-1}{\rho}(t-a)}, \end{aligned} \tag{2.7}$$

where Λ is given by (2.1).

Proof. Let x be a solution of problem (2.6). By using Lemma 2.1 with Lemma 2.2(i), the first equation of (2.6) can be written as an equivalent integral equation

$$x(t) = {}_a\mathcal{I}^{\alpha+\beta,\rho}h(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(t)x(t) + c_1\frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^\alpha\Gamma(\alpha+1)} + c_2 e^{\frac{\rho-1}{\rho}(t-a)}, \tag{2.8}$$

where $c_1, c_2 \in \mathbb{R}$.

From the first condition, $x(a) = \gamma$, we get $c_2 = \gamma$. Taking the GPF integral operator ${}_a\mathcal{I}^{\mu_i,\rho}$ into both sides of (2.8), we have

$${}_a\mathcal{I}^{\mu_i,\rho}x(t) = {}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}h(t) - {}_a\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(t)x(t) + c_1\frac{(t-a)^{\alpha+\mu_i}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\mu_i+1)} + \frac{\gamma(t-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}.$$

From the second condition, we obtain c_1 as follows:

$$\begin{aligned} c_1 = \frac{1}{\Lambda}\Bigg(\sum_{i=1}^m \delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}h(\xi_i) - {}_aI^{\alpha+\beta,\rho}h(\eta) - \sum_{i=1}^m \delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(\xi_i)x(\xi_i) \\ + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(\eta)x(\eta) + \sum_{i=1}^m \frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa\Bigg), \end{aligned}$$

where Λ is defined by (2.1). Substituting c_1 and c_2 into (2.8), we get the Volterra integral equation (2.7).

Conversely, it is easily shown by direct calculation that the solution $x(t)$ is given by (2.7) and satisfies problem (2.6) under the given boundary conditions. □

3 Main results

In this section, we establish the existence results of solutions for problem (1.1), which is studied by applying Banach's, Schaefer's and Krasnolselskii's fixed point theorems. Throughout this paper, the expression ${}_a\mathcal{I}^{b,\rho}f(s, x(s), x(\theta(s)), (\mathcal{S}x)(s))(c)$ means that

$${}_a\mathcal{I}^{b,\rho}F_x(s)(c) := \frac{1}{\rho^b\Gamma(b)}\int_a^c e^{\frac{\rho-1}{\rho}(c-s)}(c-s)^{b-1}F_x(s)\,ds, \quad c \in [a, T],$$

where $b \in \{\alpha, \alpha+\mu_i, \alpha+\beta, \alpha+\beta+\mu_i\}$ and $c \in \{t, T, \eta, \xi_i\}$, $i = 1, 2, \ldots, m$. For simplicity, we set

$$F_x(t) = f\big(s, x(s), x(\theta(s)), (\mathcal{S}x)(s)\big)(t).$$

In view of Lemma 2.7, an operator $\mathcal{A}:\mathbb{E}\to\mathbb{E}$ is defined by

$$\begin{aligned}(\mathcal{A}x)(t) &= {}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(t)\\ &\quad+\frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_x(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(\eta)\\ &\quad-\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta)\\ &\quad+\sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)}+\kappa\Bigg) + \gamma e^{\frac{\rho-1}{\rho}(t-a)}, \end{aligned}\tag{3.1}$$

where Λ is defined by (2.1).

To proceed further, we introduce the following assumptions:

(H_1) The functions $f:[a,T]\times\mathbb{R}^3\to\mathbb{R}$ and $\mathcal{H}:[a,T]\to\mathbb{R}$ are continuous.

(H_2) There exist the positive constants L_1, L_2 such that

$$\big|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)\big| \le L_1\big(|u_1-v_1|+|u_2-v_2|\big)+L_2|u_3-v_3|,$$

for each $t\in[a,T]$ and u_i, $v_i\in\mathbb{R}$, $i=1,2,3$.

(H_3) The function $\phi:[a,T]^2\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a constant $\phi_0>0$ such that

$$|\phi(t,s,u)-\phi(t,s,v)|\le\phi_0|u-v|,$$

for each t, $s\in[a,T]$ and u, $v\in\mathbb{R}$.

(H_4) There exist the functions σ, τ, φ, $\omega\in C([a,T],\mathbb{R}^+)$ such that

$$|f(t,u,v,w)|\le\sigma(t)+\tau(t)|u|+\varphi(t)|v|+\omega(t)|w|,\quad u,v,w\in\mathbb{R},\quad t\in[a,T],$$

with

$$\sigma^*=\sup_{t\in[a,T]}\sigma(t),\quad \tau^*=\sup_{t\in[a,T]}\tau(t),\quad \varphi^*=\sup_{t\in[a,T]}\varphi(t),\quad \omega^*=\sup_{t\in[a,T]}\omega(t).$$

(H_5) $|f(t,u,v,w)|\le g(t)$, $\forall\,(t,u,v,w)\in[a,T]\times\mathbb{R}^3$ and $g\in C([a,T],\mathbb{R}^+)$.

3.1 Existence and uniqueness result via Banach's fixed point theorem

The existence and uniqueness result of a solution for problem (1.1) will be proved by using Banach's fixed point theorem (Banach contraction mapping principle).

Theorem 3.1. *Assume that* (H_1)–(H_3) *hold. If* $\mathcal{L}<1$, *where*

$$\mathcal{L}:=2L_1\Omega_1+L_2\phi_0\Omega_2+\Omega_3,\tag{3.2}$$

and Ω_i, $i=1,2,3$, *are given by* (2.2)–(2.4), *respectively, then* (1.1) *has a unique solution in* $\mathbb{E}$.

Proof. Firstly, we transform problem (1.1) into a fixed point problem, $x=\mathcal{A}x$, where $\mathcal{A}$ is defined as in (3.1). Observe that the fixed points of the operator $\mathcal{A}$ are solutions of problem (1.1). Applying Banach's fixed point theorem, we show that $\mathcal{A}$ has a fixed point which is a unique solution of problem (1.1).

Let $\sup_{t\in[a,T]}|f(t,0,0,0)|:=M_1<\infty$. Next, we define a set $B_{r_1}:=\{x\in\mathbb{E}:\ \|x\|_{\mathbb{E}}\le r_1\}$ with

$$r_1\ge\frac{\Omega_1M_1+\Omega_4}{1-[2L_1\Omega_1+L_2\phi_0\Omega_2+\Omega_3]}.$$

Notice that B_{r_1} is a bounded, closed and convex subset of $\mathbb{E}$. The proof is divided into two steps.

Step 1. We show that $\mathcal{A}B_{r_1} \subset B_{r_1}$.

For any $x \in B_{r_1}$, we have

$$
\begin{aligned}
|(\mathcal{A}x)(t)| &\leq {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(T) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(T) \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m |\delta_i| {}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(\eta) \\
&\quad + \sum_{i=1}^m |\delta_i| {}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)|(\xi_i) + {}_aI^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(\eta) \\
&\quad + \sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma| \\
&\leq {}_a\mathcal{I}^{\alpha+\beta,\rho}\big(|F_x(s)-f(s,0,0,0)| + |f(s,0,0,0)|\big)(T) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(T) \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m |\delta_i| {}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}\big(|F_x(s)-f(s,0,0,0)| + |f(s,0,0,0)|\big)(\xi_i) \\
&\quad + {}_a\mathcal{I}^{\alpha+\beta,\rho}\big(|F_x(s)-f(s,0,0,0)| + |f(s,0,0,0)|\big)(\eta) + \sum_{i=1}^m |\delta_i| {}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)|(\xi_i) \\
&\quad + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(\eta) + \sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma|.
\end{aligned}
$$

By using the property $0 < e^{\frac{\rho-1}{\rho}(u-s)} \leq 1$ for $a \leq s < u < t \leq T$ and (H_2)–(H_3), we obtain

$$
\begin{aligned}
|(\mathcal{A}x)(t)| &\leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-1}\big((2L_1 + L_2\phi_0(s-a))r_1 + M_1\big)\,ds + r_1 {}_aI^{\alpha,\rho}|\mathcal{H}(s)|(T) \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m \frac{|\delta_i|}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i)} \\
&\quad \times \int_a^{\xi_i} (\xi_i-s)^{\alpha+\beta+\mu_i-1}\big((2L_1 + L_2\phi_0(s-a))r_1 + M_1\big)\,ds \\
&\quad + \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^\eta (\eta-s)^{\alpha+\beta-1}\big((2L_1 + L_2\phi_0(s-a))r_1 + M_1\big)\,ds + r_1 {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta) \\
&\quad + r_1\sum_{i=1}^m |\delta_i| {}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i) + \sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma| \\
&= (2L_1r_1 + M_1)\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)} \\
&\quad \times \Bigg(\sum_{i=1}^m \frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg] \\
&\quad + L_2\phi_0 r_1\Bigg[\frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)} \\
&\quad \times \Bigg(\sum_{i=1}^m \frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)} + \frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\Bigg)\Bigg] \\
&\quad + r_1\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T) + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m |\delta_i| {}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg]
\end{aligned}
$$

$$+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|+|\kappa|\right)+|\gamma|$$
$$\leq(2L_1r_1+M_1)\Omega_1+L_2\phi_0\Omega_2r_1+\Omega_3r_1+\Omega_4\leq r_1,$$

then $\|\mathcal{A}x\|_{\mathbb{E}}\leq r_1$, which implies that $\mathcal{A}B_{r_1}\subset B_{r_1}$.

Step 2. We show that the operator $\mathcal{A}:\mathbb{E}\to\mathbb{E}$ is a contraction mapping.

Let $x,y\in\mathbb{E}$. Then for $t\in[a,T]$, we have

$$\begin{aligned}|(\mathcal{A}x)(t)-(\mathcal{A}y)(t)|&\leq {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)-F_y(s)|(T)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)-y(s)|(T)\\
&+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)-F_y(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha+\beta,\rho}\big(|F_x(s)-F_y(s)|\big)(\eta)\\
&+\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)-y(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)-y(s)|(\eta)\Bigg)\\
&\leq\Bigg\{2L_1\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}+\frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\right)\Bigg]\\
&+L_2\phi_0\Bigg[\frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)}+\frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\right)\Bigg]\\
&+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T)+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\right)\Bigg\}\|x-y\|_{\mathbb{E}}\\
&=\big[2L_1\Omega_1+L_2\phi_0\Omega_2+\Omega_3\big]\|x-y\|_{\mathbb{E}}=\mathcal{L}\|x-y\|_{\mathbb{E}},\end{aligned}$$

which implies that $\|\mathcal{A}x-\mathcal{A}y\|_{\mathbb{E}}\leq\mathcal{L}\|x-y\|_{\mathbb{E}}$. As $\mathcal{L}<1$, hence, by Banach's fixed point theorem (Lemma 2.4), the operator $\mathcal{A}$ is a contraction mapping. Therefore, $\mathcal{A}$ has only one fixed point, which implies that problem (1.1) has a unique solution in $\mathbb{E}$. □

3.2 Existence result via Schaefer's fixed point theorem

Next, the second existence result is based on Schaefer's fixed point theorem.

Theorem 3.2. *Assume that* (H_1) *and* (H_4) *hold. Then problem* (1.1) *has at least one solution on* $[a,T]$.

Proof. To show that $\mathcal{A}$ has at least a fixed point in $\mathbb{E}$, the proof is divided into four steps.

Step 1. We show that the operator $\mathcal{A}$ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n\to x$ in $\mathbb{E}$. Then, for each $t\in[a,T]$, we get

$$\begin{aligned}|(\mathcal{A}x_n)(t)-(\mathcal{A}x)(t)|&\leq {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_{x_n}(s)-F_x(s)|(T)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x_n(s)-x(s)|(T)\\
&+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_{x_n}(s)-F_x(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha+\beta,\rho}|F_{x_n}(s)-F_x(s)|(\eta)\\
&+\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x_n(s)-x(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)||x_n(s)-x(s)|(\eta)\Bigg)\\
&\leq\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}\\
&+\frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\|F_{x_n}-F_x\|_{\mathbb{E}}+\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T)\\
&+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\right)\Bigg]\|x_n-x\|_{\mathbb{E}}\\
&=\Omega_1\|F_{x_n}-F_x\|_{\mathbb{E}}+\Omega_3\|x_n-x\|_{\mathbb{E}}.\end{aligned}$$

Since f and $\mathcal{H}$ are continuous, by the Lebesgue dominated convergent theorem, we have

$$|(\mathcal{A}x_n)(t) - (\mathcal{A}x)(t)| \longrightarrow 0 \text{ as } n \to \infty.$$

Hence,

$$\|\mathcal{A}x_n - \mathcal{A}x\|_{\mathbb{E}} \longrightarrow 0 \text{ as } n \to \infty.$$

Therefore, the operator $\mathcal{A}$ is continuous.

Step 2. We show that the operator $\mathcal{A}$ maps a bounded set into the bounded set in $\mathbb{E}$.

Indeed, we show that for any $r_2 > 0$, there exists a constant $M_2 > 0$ such that for each $x \in \overline{B}_{r_2} = \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} \leq r_2\}$, we have $\|\mathcal{A}x\|_{\mathbb{E}} \leq M_2$.

Then, for any $t \in [a, T]$ and $x \in \overline{B}_{r_2}$, we have

$$\begin{aligned}
|(\mathcal{A}x)(t)| &\leq {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(T) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(T) \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(\eta) \\
&\quad + \sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(\eta) \\
&\quad + \sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma| \\
&\leq {}_a\mathcal{I}^{\alpha+\beta,\rho}\Big(\sigma(s) + \tau(s)|x(s)| + \varphi(s)|x(\theta(s))| + \omega(s)|(\mathcal{S}x)(s)|\Big)(T) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(T) \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}\Big(\sigma(s) + \tau(s)|x(s)| + \varphi(s)|x(\theta(s))| + \omega(s)|(\mathcal{S}x)(s)|\Big)(\xi_i) \\
&\quad + {}_a\mathcal{I}^{\alpha+\beta,\rho}\Big(\sigma(s) + \tau(s)|x(s)| + \varphi(s)|x(\theta(s))| + \omega(s)|(\mathcal{S}x)(s)|\Big)(\eta) \\
&\quad + \sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|x(s)|(\eta) + \sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma| \\
&\leq (\sigma^* + \tau^* r_2 + \varphi^* r_2)\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)} \\
&\quad \times \Bigg(\sum_{i=1}^m \frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg] \\
&\quad + \omega^* r_2\Bigg[\frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)} \\
&\quad \times \Bigg(\sum_{i=1}^m \frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)} + \frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\Bigg)\Bigg] \\
&\quad + r_2\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T) + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m |\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg] \\
&\quad + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^m \frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa|\Bigg) + |\gamma| \\
&= (\sigma^* + \tau^* r_2 + \varphi^* r_2)\Omega_1 + (\omega^*\Omega_2 + \Omega_3)r_2 + \Omega_4,
\end{aligned}$$

and we get the estimate

$$\|\mathcal{A}x\|_{\mathbb{E}} \leq \big[(\tau^* + \varphi^*)\Omega_1 + \omega^*\Omega_2 + \Omega_3\big]r_2 + \sigma^*\Omega_1 + \Omega_4 := M_2,$$

where Ω_i, $i = 1, 2, 3, 4$, are given by (2.2)–(2.5), respectively.

Step 3. We show that the operator $\mathcal{A}$ is equicontinuous.

Let $\overline{B}_{r_2}$ be a bounded set of $\mathbb{E}$ as defined in Step 2, then, for $x \in \overline{B}_{r_2}$ and $t_1, t_2 \in [a,T]$ with $t_1 < t_2$, we have

$$
\begin{aligned}
&|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)| \\
&\leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha+\beta-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha+\beta-1} \right| |F_x(s)|\, ds \\
&\quad + \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha+\beta-1} |F_x(s)|\, ds \\
&\quad + \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha-1} \right| |\mathcal{H}(s)|\, |x(s)|\, ds \\
&\quad + \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} |\mathcal{H}(s)|\, |x(s)|\, ds + |\gamma| \left| e^{\frac{\rho-1}{\rho}(t_2-a)} - e^{\frac{\rho-1}{\rho}(t_1-a)} \right| \\
&\quad + \frac{\left|(t_2-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_2-a)} - (t_1-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} \frac{|\delta_i|}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i)} \right. \\
&\quad \times \int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\xi_i-s)}(\xi_i-s)^{\alpha+\beta+\mu_i-1} \left| f\big(s, x(s), x(\theta(s)), (\mathcal{S}x)(s)\big) \right| ds \\
&\quad + \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{\eta} e^{\frac{\rho-1}{\rho}(\eta-s)}(\eta-s)^{\alpha+\beta-1} \left| f\big(s, x(s), x(\theta(s)), (\mathcal{S}x)(s)\big) \right| ds \\
&\quad + \sum_{i=1}^{m} \frac{|\delta_i|}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\mu_i)} \int_a^{\xi_i} e^{\frac{\rho-1}{\rho}(\xi_i-s)}(\xi_i-s)^{\alpha+\mu_i-1} |\mathcal{H}(s)|\, |x(s)|\, ds \\
&\quad \left. + \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^{\eta} e^{\frac{\rho-1}{\rho}(\eta-s)}(\eta-s)^{\alpha-1} |\mathcal{H}(s)|\, |x(s)|\, ds + \sum_{i=1}^{m} \frac{|\gamma|\, |\delta_i| (\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)} + |\gamma| + |\kappa| \right) \\
&\leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha+\beta-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha+\beta-1} \right| \\
&\quad \times \big(\sigma^* + \tau^* r_2 + \varphi^* r_2 + \omega^* r_2 (s-a)\big)\, ds + \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha+\beta-1} \\
&\quad \times \big(\sigma^* + \tau^* r_2 + \varphi^* r_2 + \omega^* r_2 (s-a)\big)\, ds \\
&\quad + \frac{r_2}{\rho^{\alpha}\Gamma(\alpha)} \int_a^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha-1} \right| |\mathcal{H}(s)|\, ds \\
&\quad + \frac{r_2}{\rho^{\alpha}\Gamma(\alpha)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} |\mathcal{H}(s)|\, ds \\
&\quad + |\gamma| \left| e^{\frac{\rho-1}{\rho}(t_2-a)} - e^{\frac{\rho-1}{\rho}(t_1-a)} \right| + \frac{\left|(t_2-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_2-a)} - (t_1-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}
\end{aligned}
$$

$$\times\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i)}\int_a^{\xi_i}(\xi_i-s)^{\alpha+\beta+\mu_i-1}\big(\sigma^*+\tau^*r_2+\varphi^*r_2+\omega^*r_2(s-a)\big)\,ds$$

$$+\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{\eta}(\eta-s)^{\alpha+\beta-1}\big(\sigma^*+\tau^*r_2+\varphi^*r_2+\omega^*r_2(s-a)\big)\,ds$$

$$+r_2\sum_{i=1}^{m}\frac{|\delta_i|}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\mu_i)}\int_a^{\xi_i}e^{\frac{\rho-1}{\rho}(\xi_i-s)}(\xi_i-s)^{\alpha+\mu_i-1}|\mathcal{H}(s)|\,ds$$

$$+\frac{r_2}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\eta}e^{\frac{\rho-1}{\rho}(\eta-s)}(\eta-s)^{\alpha-1}|\mathcal{H}(s)|\,ds+\sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|+|\kappa|\Bigg),$$

which implies that

$$|(\mathcal{A}x)(t_2)-(\mathcal{A}x)(t_1)|\longrightarrow 0,\ \text{ as } t_1\to t_2,$$

As a result of Steps 1–3 together with the Arzelá–Ascoli theorem (Lemma 2.3), we conclude that the operator $\mathcal{A}:\mathbb{E}\to\mathbb{E}$ is completely continuous.

Step 4. We show that the set $\mathbb{D}=\{x\in\mathbb{E}:\ x=\varepsilon\mathcal{A}x,\ 0<\varepsilon<1\}$ is bounded (A priori bounds).

Let $x\in\mathbb{D}$, then $x=\varepsilon\mathcal{A}x$. For any $t\in[a,T]$, one can get the estimate

$$(\mathcal{A}x)(t)=\varepsilon\Bigg[{}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(t)-{}_aI^{\alpha,\rho}\mathcal{H}(s)x(s)(t)$$

$$+\frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\delta_i{}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_x(s)(\xi_i)-{}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(\eta)$$

$$-\sum_{i=1}^{m}\delta_i{}_a\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i)+{}_aI^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta)$$

$$+\sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}-\gamma e^{\frac{\rho-1}{\rho}(\eta-a)}+\kappa\Bigg)+\gamma e^{\frac{\rho-1}{\rho}(t-a)}\Bigg].$$

It follows from (H_3)-(H_4) and $0<\varepsilon<1$ that for any $t\in[a,T]$,

$$|x(t)|=|\varepsilon(\mathcal{A}x)(t)|\le(\sigma^*+\tau^*r_2+\varphi^*r_2)\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}$$

$$+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}+\frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]$$

$$+\omega^*r_2\Bigg[\frac{(T-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}$$

$$\times\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i+1}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+2)}+\frac{(\eta-a)^{\alpha+\beta+1}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+2)}\Bigg)\Bigg]$$

$$+r_2\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T)+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|{}_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg]$$

$$+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|+|\kappa|\Bigg)+|\gamma|$$

$$=(\sigma^*+\tau^*r_2+\varphi^*r_2)\Omega_1+(\omega^*\Omega_2+\Omega_3)r_2+\Omega_4.$$

Thus,

$$\|x\|_{\mathbb{E}}\le\big[(\tau^*+\varphi^*)\Omega_1+\omega^*\Omega_2+\Omega_3\big]r_2+\sigma^*\Omega_1+\Omega_4:=N<\infty.$$

This implies that $\mathbb{D}$ is bounded.

Hence, as a consequence of Schaefer's fiexd point theorem (Lemma 2.5), the operator $\mathcal{A}$ has at least one fixed point which is the solution of problem (1.1). $\square$

3.3 Existence result via Krasnoselskii's fixed point theorem

By using Krasnoselskii's fixed point theorem, we obtain the last existence theorem.

Theorem 3.3. *Assume that* (H_1), (H_3), (H_5) *hold. Then problem* (1.1) *has at least one solution on* $[a,T]$ *if* $\Omega_3<1$, *where* Ω_3 *is defined by* (2.4).

Proof. Let $\sup\limits_{t\in[a,T]}|g(t)|=\|g\|_{\mathbb{E}}$. By choosing a suitable $B_{\overline{r}_3}=\{x\in\mathbb{E}:\ \|x\|_{\mathbb{E}}\le\overline{r}_3\}$, where

$$\overline{r}_3\ge\frac{\Omega_1\|g\|_{\mathbb{E}}+\Omega_4}{1-\Omega_3}$$

with $\|g\|_{\mathbb{E}}=\sup\limits_{t\in[a,T]}|g(t)|$, we define the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on $B_{\overline{r}_3}$ by

$$\begin{aligned}(\mathcal{A}_1x)(t)&={}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(t)+\frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Big(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_x(s)(\xi_i)-{}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(\eta)\Big),\\(\mathcal{A}_2x)(t)&=\frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Bigg({}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta)-\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i)\\&\qquad+\sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}-\gamma e^{\frac{\rho-1}{\rho}(\eta-a)}+\kappa\Bigg)+\gamma e^{\frac{\rho-1}{\rho}(t-a)}-{}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(t).\end{aligned}$$

To show that $\mathcal{A}_1x+\mathcal{A}_2y\in B_{\overline{r}_3}$, let $x,y\in B_{\overline{r}_3}$. Then we have

$$\begin{aligned}\|\mathcal{A}_1x+\mathcal{A}_2y\|_{\mathbb{E}}&\le\sup_{t\in[a,T]}\Bigg\{{}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(t)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|y(s)|(t)\\&\quad+\frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(\eta)\\&\qquad+\sum_{i=1}^{m}|\delta_i|_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|\,|y(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|\,|y(s)|(\eta)\\&\quad+\sum_{i=1}^{m}\frac{|\gamma|\,|\delta_i|(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|e^{\frac{\rho-1}{\rho}(\eta-a)}+|\kappa|\Bigg)+|\gamma|e^{\frac{\rho-1}{\rho}(t-a)}\Bigg\}\\&\le\|g\|_{\mathbb{E}}\Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}+\frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\\&\quad+\|x\|_{\mathbb{E}}\Bigg[{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(T)+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}|\delta_i|_a\mathcal{I}^{\alpha+\mu_i,\rho}|\mathcal{H}(s)|(\xi_i)+{}_a\mathcal{I}^{\alpha,\rho}|\mathcal{H}(s)|(\eta)\Bigg)\Bigg]\\&\quad+\frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\frac{|\gamma||\delta_i|(\xi_i-a)^{\mu_i}}{\rho^{\mu_i}\Gamma(\mu_i+1)}+|\gamma|+|\kappa|\Bigg)+|\gamma|\le\Omega_1\|g\|_{\mathbb{E}}+\Omega_3\overline{r}_3+\Omega_4\le\overline{r}_3.\end{aligned}$$

This implies that $\mathcal{A}_1x+\mathcal{A}_2y\in B_{\overline{r}_3}$, which satisfies assumption (i) of Lemma 2.6.

Show that assumption (ii) of Lemma 2.6 is satisfied, the continuity of f and $\mathcal{H}$ implies that the operator $\mathcal{A}_1$ is continuous. For $x\in B_{\overline{r}_3}$, we obtain $\|\mathcal{A}_1x\|_{\mathbb{E}}\le\Omega_1\|g\|_{\mathbb{E}}$. This means that the operator $\mathcal{A}_1$ is uniformly bounded on $B_{\overline{r}_3}$. Next, we show that the operator $\mathcal{A}_1$ is equicontinuous. Setting

$$\sup_{(t,z_1,z_2,z_3)\in[a,T]\times B^3_{\overline{r}_3}}|f(t,z_1,z_2,z_3)|=f^*<\infty,$$

for $a \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&|(\mathcal{A}_1 x)(t_2) - (\mathcal{A}_1 x)(t_1)| \leq \left|{}_a\mathcal{I}^{\alpha+\beta,\rho} F_x(s)(t_2) - {}_a\mathcal{I}^{\alpha+\beta,\rho} F_x(s)(t_1)\right| \\
&+ \frac{\left|(t_2-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_2-a)} - (t_1-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \left(\sum_{i=1}^{m} |\delta_i| {}_a\mathcal{I}^{\alpha+\beta+\mu_i,\rho}|F_x(s)|(\xi_i) + {}_a\mathcal{I}^{\alpha+\beta,\rho}|F_x(s)|(\eta)\right) \\
&\leq f^* \Bigg[\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \left(\left|(t_2-a)^{\alpha+\beta} - (t_1-a)^{\alpha+\beta} - (t_2-t_1)^{\alpha+\beta}\right| + (t_2-t_1)^{\alpha+\beta}\right) \\
&+ \frac{\left|(t_2-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_2-a)} - (t_1-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Lambda|\rho^{\alpha+\beta}\Gamma(\alpha+1)} \left(\frac{(\eta-s)^{\alpha+\beta}}{\rho^{\alpha}\Gamma(\alpha+\beta+1)} + \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-s)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)}\right)\Bigg],
\end{aligned}$$

which is independent of x and $|(\mathcal{A}_1 x)(t_2) - (\mathcal{A}_1 x)(t_1)| \to 0$ as $t_1 \to t_2$. Therefore, the operator $\mathcal{A}_1$ is equicontinuous. So, the operator $\mathcal{A}_1$ is relatively compact on $B_{\overline{r}_3}$. Then, by the Arzelá–Ascoli theorem, the operator $\mathcal{A}_1$ is compact on $B_{\overline{r}_3}$, and assumption (ii) of Lemma 2.6 is satisfied. It is easy to see that, using $\Omega_3 < 1$, we come to the conclusion that the operator $\mathcal{A}_2$ is a contraction mapping, and also assumption (iii) of Lemma 2.6 holds. Hence, the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy all assumptions of Krasnoselskii's fixed point theorem (Lemma 2.6). Therefore, problem (1.1) has at least one solution on $[a,T]$. □

4 Ulam–Hyers stability results

In this section, we investigate some necessary and sufficient conditions for Ulam–Hyers (UH) stability, generalized Ulam–Hyers (GUH) stability, Ulam–Hyers–Rassias (UHR) stability, and generalized Ulam–Hyers–Rassias (GUHR) stability of problem (1.1).

Definition 4.1 ([35])**.** Problem (1.1) is UH stable if there exists a real number $\Phi > 0$ such that for $\epsilon > 0$ and solution $z \in \mathbb{E}^1 = C^1([a,T],\mathbb{R})$ of the inequality

$$\left|{}^C_a\mathcal{D}^{\beta,\rho}\big({}^C_a\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\big)z(t) - f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big)\right| \leq \epsilon, \quad t \in [a,T], \tag{4.1}$$

there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) with

$$|z(t) - x(t)| \leq \Phi\epsilon, \quad t \in [a,T].$$

Definition 4.2 ([35])**.** Problem (1.1) is GUH stable if there exists $\Phi_f \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\Phi_f(0) = 0$ such that for each solution $z \in \mathbb{E}^1$ of inequality (4.1) there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) such that

$$|z(t) - x(t)| \leq \Phi_f\epsilon, \quad t \in [a,T].$$

Definition 4.3 ([35])**.** Problem (1.1) is UHR stable with respect to $\Phi_f \in C([a,T],\mathbb{R}^+)$ if there exists a real number $C_{f,\Phi} > 0$ such that for $\epsilon > 0$ and for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}^C_a\mathcal{D}^{\beta,\rho}\big({}^C_a\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\big)z(t) - f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big)\right| \leq \epsilon\Phi_f(t), \quad t \in [a,T], \tag{4.2}$$

there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) with

$$|z(t) - x(t)| \leq C_{f,\Phi}\epsilon\Phi_f(t), \quad t \in [a,T].$$

Definition 4.4 ([35])**.** Problem (1.1) is GUHR stable with respect to $\Phi_f \in C([a,T],\mathbb{R}^+)$ if there exists a real number $C_{f,\Phi} > 0$ such that for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}^C_a\mathcal{D}^{\beta,\rho}\big({}^C_a\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t)\big)z(t) - f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big)\right| \leq \Phi_f(t), \quad t \in [a,T],$$

there exists a solution $x \in \mathbb{E}^1$ of problem (1.1) such that

$$|z(t) - x(t)| \leq C_{f,\Phi}\Phi_f(t), \quad t \in [a,T].$$

Remark 4.1. It is clear that

(i) Definition 4.1 $\Longrightarrow$ Definition 4.2;

(ii) Definition 4.3 $\Longrightarrow$ Definition 4.4;

(iii) Definition 4.3 for $\Phi_f(\,\cdot\,) = 1 \Longrightarrow$ Definition 4.1.

Remark 4.2. A function $z \in \mathbb{E}^1$ is a solution of inequality (4.1) if and only if there exists a function $v \in C([a,T],\mathbb{R})$ (dependent on z) such that

(i) $|v(t)| \le \epsilon$, $\forall\, t \in [a,T]$.

(ii) ${}^C_a\mathcal{D}^{\beta,\rho}({}^C_a\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t))z(t) = f(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)) + v(t)$, $t \in [a,T]$.

By Remark 4.2, the solution of the problem

$$ {}^C_a\mathcal{D}^{\beta,\rho}({}^C_a\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t))z(t) = f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big) + v(t), \quad t \in [a,T], $$

can be written by

$$ \begin{aligned} z(t) = {}&{}_a\mathcal{I}^{\alpha+\beta,\rho}F_z(s)(t) - {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)z(s)(t) \\ &+ \frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_z(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}F_z(s)(\eta) \\ &- \sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)z(s)(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)z(s)(\eta) \\ &+ \sum_{i=1}^{m}\frac{\gamma\delta_i(\xi_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa\Bigg) + \gamma e^{\frac{\rho-1}{\rho}(t-a)} + {}_a\mathcal{I}^{\alpha+\beta,\rho}v(s)(t) \\ &+ \frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)}\Big(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}v(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}v(s)(\eta)\Big). \end{aligned} $$

Firstly, we present an important lemma that will be used in the proofs of the first stability theorem.

Lemma 4.1. *If $z \in \mathbb{E}^1$ satisfies* (4.1), *then the function z is a solution of the inequality*

$$ |z(t) - (\mathcal{A}z)(t)| \le \Omega_1\epsilon, \quad 0 < \epsilon \le 1, \tag{4.3} $$

where Ω_1 is given by (2.2).

Proof. From Remark 4.2, we obtain the inequality

$$ \begin{aligned} |z(t) - (\mathcal{A}z)(t)| &\le \Bigg|{}_a\mathcal{I}^{\alpha+\beta,\rho}v(s)(t) + \frac{(t-a)^{\alpha}e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^{\alpha}\Gamma(\alpha+1)} \\ &\qquad\times\Big(\sum_{i=1}^{m}\delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}v(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}v(s)(\eta)\Big)\Bigg| \\ &\le \Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^{\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(\alpha+1)} \\ &\qquad\times\Bigg(\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Bigg)\Bigg]\epsilon = \Omega_1\epsilon, \end{aligned} $$

where Ω_1 is given by (2.2), from which inequality (4.3) follows. $\square$

Now, we present the UH and GUH results.

Theorem 4.1. *Assume that* (H_1), (H_2), (H_3) *are satisfied with* $\mathcal{L} < 1$, *where* $\mathcal{L}$ *is defined by* (3.2). *Then problem* (1.1) *is both UH stable and GUH stable on* $[a, T]$.

Proof. Let $z \in \mathbb{E}^1$ be a solution of (4.1) and let x be the unique solution of problem (1.1),

$$\begin{cases} {}^{C}_{a}\mathcal{D}^{\beta,\rho}({}^{C}_{a}\mathcal{D}^{\alpha,\rho} + \mathcal{H}(t))x(t) = f\big(t, x(t), x(\theta(t)), (\mathcal{S}x)(t)\big), \quad t \in (a, T] \\ x(a) = \gamma, \quad x(\eta) = \displaystyle\sum_{i=1}^{m} \delta_{ia} I^{\mu_i,\rho} x(\xi_i) + \kappa. \end{cases}$$

By applying the triangle inequality $|u - v| \le |u| + |v|$ and Lemma 4.1, we have

$$\begin{aligned} |z(t) - x(t)| = \Bigg| & z(t) - {}_{a}\mathcal{I}^{\alpha+\beta,\rho} F_x(s)(t) + {}_{a}\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(t) \\ & - \frac{(t-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda \rho^{\alpha}\Gamma(\alpha+1)} \Bigg(\sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\beta+\mu_i,\rho} F_x(s)(\xi_i) - {}_{a}\mathcal{I}^{\alpha+\beta,\rho} F_x(s)(\eta) \\ & - \sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i) + {}_{a}\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta) \\ & + \sum_{i=1}^{m} \frac{\gamma\delta_i(\xi_i - a)^{\mu_i} e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa \Bigg) - \gamma e^{\frac{\rho-1}{\rho}(t-a)} \Bigg| \\ = \big| z(t) - (\mathcal{A}z)(t) + (\mathcal{A}z)(t) - (\mathcal{A}x)(t) \big| & \le |z(t) - (\mathcal{A}z)(t)| + |(\mathcal{A}z)(t) - (\mathcal{A}x)(t)| \le \Omega_1\epsilon + \mathcal{L}|z(t) - x(t)|. \end{aligned}$$

This yields

$$|z(t) - x(t)| \le \frac{\Omega_1 \epsilon}{1 - \mathcal{L}}.$$

By setting $\Phi = \frac{\Omega_1}{1-\mathcal{L}}$ and $\mathcal{L} < 1$, we end up with

$$|z(t) - x(t)| \le \Phi\epsilon.$$

Hence, problem (1.1) is UH stable. Moreover, if we set $\Phi_f(\epsilon) = \Phi\epsilon$, with $\Phi_f(0) = 0$, then problem (1.1) is GUH stable. □

Remark 4.3. A function $z \in \mathbb{E}^1$ is a solution of inequality (4.2) if and only if there exists a function $w \in C([a, T], \mathbb{R})$ (dependent on z) such that

(i) $|\Theta(t)| \le \epsilon\Psi_{\Theta}(t)$, $\forall\, t \in [a, T]$.

(ii) ${}^{C}_{a}D^{\beta,\rho}({}^{C}_{a}D^{\alpha,\rho} + \lambda(t))z(t) = f(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)) + \Theta(t)$, $t \in [a, T]$.

By Remark 4.3, the solution of the problem

$${}^{C}_{a}\mathcal{D}^{\beta,\rho}({}^{C}_{a}\mathcal{H}^{\alpha,\rho} + \mathcal{H}(t))z(t) = f\big(t, z(t), z(\theta(t)), (\mathcal{S}z)(t)\big) + \Theta(t), \quad t \in [a, T],$$

can be written by

$$\begin{aligned} z(t) = {} & {}_{a}\mathcal{I}^{\alpha+\beta,\rho} F_z(s)(t) - {}_{a}\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)z(s)(t) \\ & + \frac{(t-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda \rho^{\alpha}\Gamma(\alpha+1)} \Bigg(\sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\beta+\mu_i,\rho} F_z(s)(\xi_i) - {}_{a}\mathcal{I}^{\alpha+\beta,\rho} F_z(s)(\eta) \\ & - \sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)z(s)(\xi_i) + {}_{a}\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)z(s)(\eta) \\ & + \sum_{i=1}^{m} \frac{\gamma\delta_i(\xi_i - a)^{\mu_i} e^{\frac{\rho-1}{\rho}(\xi_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa \Bigg) + \gamma e^{\frac{\rho-1}{\rho}(t-a)} + {}_{a}\mathcal{I}^{\alpha+\beta,\rho} w(s)(t) \\ & + \frac{(t-a)^{\alpha} e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda \rho^{\alpha}\Gamma(\alpha+1)} \Big(\sum_{i=1}^{m} \delta_{ia}\mathcal{I}^{\alpha+\beta+\mu_i,\rho} w(s)(\xi_i) - {}_{a}\mathcal{I}^{\alpha+\beta,\rho} w(s)(\eta) \Big). \end{aligned}$$

Next, we construct lemma that will be used in the proofs of the second stability theorem.

Lemma 4.2. *Let $z \in \mathbb{E}^1$ be a solution of inequality* (4.2). *Then the function z satisfies the inequality*

$$|z(t) - (\mathcal{A}z)(t)| \leq \Omega_1 \Psi_\Theta(t)\epsilon, \quad 0 < \epsilon \leq 1, \tag{4.4}$$

where Ω_1 is given by (2.2).

Proof. From Remark 4.3, we obtain the inequality

$$\begin{aligned}
|z(t) - (\mathcal{A}z)(t)| &\leq \Bigg| {}_a\mathcal{I}^{\alpha+\beta,\rho}\Theta(s)(t) + \frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda \rho^\alpha \Gamma(\alpha+1)} \\
&\qquad \times \left(\sum_{i=1}^m \delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}\Theta(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}\Theta(s)(\eta) \right) \Bigg| \\
&\leq \Bigg[\frac{(T-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(T-a)^\alpha}{|\Lambda|\rho^\alpha\Gamma(\alpha+1)} \\
&\qquad \times \left(\sum_{i=1}^m \frac{|\delta_i|(\xi_i - a)^{\alpha+\beta+\mu_i}}{\rho^{\alpha+\beta+\mu_i}\Gamma(\alpha+\beta+\mu_i+1)} + \frac{(\eta-a)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right) \Bigg] \Psi_\Theta(t)\epsilon \\
&= \Omega_1 \Psi_\Theta(t)\epsilon,
\end{aligned}$$

where Ω_1 is given by (2.2), which leads to inequality (4.4). □

Next, we are ready to prove UHR and GUHR stability results.

Theorem 4.2. *If assumptions* (H_1), (H_2), (H_3) *are satisfied, $\mathcal{L} < 1$, where $\mathcal{L}$ is defined by* (3.2), *then problem* (1.1) *is both UHR stable and GUHR stable on $[a, T]$.*

Proof. Let $z \in \mathbb{E}^1$ be a solution of inequality (4.2) and let x be the unique solution of problem (1.1). By applying the triangle inequality and Lemma 4.1, we get

$$\begin{aligned}
|z(t) - x(t)| &= \Bigg| z(t) - {}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(t) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(t) \\
&\qquad - \frac{(t-a)^\alpha e^{\frac{\rho-1}{\rho}(t-a)}}{\Lambda\rho^\alpha\Gamma(\alpha+1)} \Bigg(\sum_{i=1}^m \delta_{i\,a}\mathcal{I}^{\alpha+\beta+\mu_i,\rho}F_x(s)(\xi_i) - {}_a\mathcal{I}^{\alpha+\beta,\rho}F_x(s)(\eta) \\
&\qquad - \sum_{i=1}^m \delta_{i\,a}\mathcal{I}^{\alpha+\mu_i,\rho}\mathcal{H}(s)x(s)(\xi_i) + {}_a\mathcal{I}^{\alpha,\rho}\mathcal{H}(s)x(s)(\eta) \\
&\qquad + \sum_{i=1}^m \frac{\gamma\delta_i(\xi_i - a)^{\mu_i} e^{\frac{\rho-1}{\rho}(\xi_i - a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)} - \gamma e^{\frac{\rho-1}{\rho}(\eta-a)} + \kappa \Bigg) - \gamma e^{\frac{\rho-1}{\rho}(t-a)} \Bigg| \\
&= \Big| z(t) - (\mathcal{A}z)(t) + (\mathcal{A}z)(t) - (\mathcal{A}x)(t) \Big| \\
&\leq |z(t) - (\mathcal{A}z)(t)| + |(\mathcal{A}z)(t) - (\mathcal{A}x)(t)| \\
&\leq \Omega_1\Psi_\Theta(t)\epsilon + \mathcal{L}|z(t) - x(t)|,
\end{aligned}$$

where $\mathcal{L}$ is defined by (3.2), which implies that

$$|z(t) - x(t)| \leq \frac{\Omega_1 \Psi_\Theta(t)\epsilon}{1 - \mathcal{L}}.$$

By setting $C_{f,\Phi} = \frac{\Omega_1}{1-\mathcal{L}}$ with $\mathcal{L} < 1$, we get the inequality

$$|z(t) - x(t)| \leq C_{f,\Phi}\epsilon\Psi_\Theta(t).$$

Hence, problem (1.1) is UHR stable. Moreover, if we set $\Phi_f(t) = \epsilon\Psi_\Theta(t)$, with $\Phi_f(0) = 0$, then problem (1.1) is GUHR stable. □

5 An example

In this section, we present an example which illustrates the validity and applicability of the main results.

Example. Consider the following boundary value problem for the nonlinear GPF integro-differential Langevin equation

$$\begin{cases} {}_0^C\mathcal{D}^{\frac{\sqrt{\pi}}{2},\frac{\sqrt{2}}{2}}\Big({}_0^C\mathcal{D}^{\frac{\sqrt{3}}{2},\frac{\sqrt{2}}{2}}+\dfrac{1}{16}\,(t-a)^2e^{\frac{\rho-1}{\rho}(t-a)}\Big)x(t)=f\big(t,x(t),x(\theta(t)),(\mathcal{S}x)(t)\big),\quad t\in[0,2],\\ x(0)=0,\quad x(1)=\sqrt{2}\,{}_0\mathcal{I}^{\frac{1}{2},\frac{\sqrt{2}}{2}}x\Big(\dfrac{1}{2}\Big)-\dfrac{1}{2}\,{}_0\mathcal{I}^{\frac{3}{2},\frac{\sqrt{2}}{2}}x\Big(\dfrac{4}{3}\Big)-{}_0\mathcal{I}^{\frac{5}{2},\frac{\sqrt{2}}{2}}x\Big(\dfrac{3}{2}\Big)+\dfrac{1}{10}\,. \end{cases} \tag{5.1}$$

Here,

$$\begin{gathered} \alpha=\frac{\sqrt{3}}{2}\,,\ \ \beta=\frac{\sqrt{\pi}}{2}\,,\ \ \rho=\frac{\sqrt{2}}{2}\,,\\ a=0,\ \ T=2,\ \ m=3,\ \ \gamma=0,\ \ \eta=1,\\ \kappa=\frac{1}{10}\,,\ \ \mu_1=\frac{1}{2}\,,\ \ \mu_2=\frac{3}{2}\,,\ \ \mu_3=\frac{5}{2}\,,\\ \xi_1=\frac{1}{2}\,,\ \ \xi_2=\frac{4}{3}\,,\ \ \xi_3=\frac{3}{2}\,,\\ \delta_1=\sqrt{2}\,,\ \ \delta_2=-\frac{1}{2}\,,\ \ \delta_3=-1,\ \ \theta(t)=\frac{t}{2} \end{gathered}$$

and

$$\mathcal{H}(t)=\frac{1}{16}\,(t-a)^2e^{\frac{\rho-1}{\rho}\,(t-a)}.$$

Obviously, the function $\mathcal{H}$ satisfies the assumption (H_1) for all $t\in[a,T]$. From the all given all data, we obtain that $\Lambda\approx 1.49603\neq 0$, $\Omega_1\approx 8.26497$, $\Omega_2\approx 4.17132$, $\Omega_3\approx 0.17389$ and $\Omega_4\approx 0.17303$.

(i) Let $f:[a,T]\times\mathbb{R}^3\to\mathbb{R}$ be a function defined by

$$f(t,x(t),x(\theta(t)),(\mathcal{S}x)(t))=\frac{1}{4}+\frac{1}{9}\,t^3+\frac{2\sin^2(\pi t)}{(t+5)^2}\,\frac{|x|}{1+|x|}-\frac{x(1.5t)}{(t+5)^2}+\frac{(t+1)^3}{e^t+2}\int_a^t\frac{\cos^2(\pi t)}{(e^{s^2}+3)^2}\,x(s)\,ds.$$

For $x_1,x_2,y_1,y_2,z_1,z_2\in\mathbb{R}$ and $t\in[a,T]$, we have

$$\begin{gathered} \big|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)\big|\le\frac{1}{25}\,\big(|x_1-y_1|+|x_2-y_2|\big)+\frac{1}{3}\,|z_1-z_2|,\\ |\phi(t,s,x_1)-\phi(t,s,y_1)|\le\frac{1}{16}\,|x_1-y_1|. \end{gathered}$$

The assumptions (H_1)–(H_3) are satisfied with $L_1=\frac{1}{25}$, $L_2=\frac{1}{3}$, and $\phi_0=\frac{1}{16}$. Hence

$$\mathcal{L}:=2L_1\Omega_1+L_2\phi_0\Omega_2+\Omega_3\approx 0.92199<1.$$

This ensures the existence of the unique solution for (5.1) according to Theorem 3.1. Further, we compute

$$\Phi:=\frac{\Omega_1}{1-\mathcal{L}}\approx 105.95156>0.$$

Thus, by Theorem (4.1), problem (5.1) is UH stable and, consequently, GUH stable.

(ii) Let $f:[a,T]\times\mathbb{R}^3\to\mathbb{R}$ be a function defined by

$$\begin{aligned} f\big(t,x(t),x(\theta(t)),(\mathcal{S}x)(t)\big)&=\frac{e^{-t}}{(t+8)^2}+\frac{6e^{-2t}}{(t+8)^2}\,\frac{|x|}{2+|x|}\\ &\quad+\frac{5}{4(2+t)^2}\,\frac{|x(0.25t)|}{|x(0.25t)|+9}+\frac{(t+3)^3\cos^2(\pi t)}{(e^t+2)^2}\int_a^t\frac{\sin^2(t-s)}{(e^{t-s}+2)^2}\,x(s)\,ds. \end{aligned}$$

It is easy to see that for all $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [a, T]$, we get

$$\begin{aligned} \big|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\big| &\le \frac{1}{32}\big(|x_1-y_1| + |x_2-y_2|\big) + \frac{1}{3}\,|z_1-z_2|, \\ |\phi(t,s,x_1) - \phi(t,s,y_1)| &\le \frac{1}{9}|x_1-y_1|. \end{aligned}$$

The assumptions (H_1)–(H_3) are satisfied with $L_1 = \frac{1}{32}$, $L_2 = \frac{1}{3}$, and $\phi_0 = \frac{1}{9}$. Hence

$$\mathcal{L} := 2L_1\Omega_1 + L_2\phi_0\Omega_2 + \Omega_3 \approx 0.84495 < 1.$$

Furthermore, for $x, y, z \in \mathbb{R}$ and $t \in [a, T]$, it follows that

$$|f(t,x,y,z)| \le \frac{e^{-t}}{(t+8)^2} + \frac{2e^{-2t}}{(t+8)^2}\,|x| + \frac{1}{8(2+t)^2}\,|y| + \frac{27}{(e^t+2)^4}\,|z|.$$

The hypothesis (H_4) is also valid with

$$\sigma(t) = \frac{e^{-t}}{(t+8)^2}\,, \quad \tau(t) = \frac{2e^{-2t}}{(t+8)^2}\,, \quad \varphi(t) = \frac{1}{8(2+t)^2}\,, \quad \omega(t) = \frac{27}{(e^t+2)^4}$$

and

$$\sigma^* = \frac{1}{64}\,, \quad \tau^* = \frac{1}{32}\,, \quad \varphi^* = \frac{1}{32}\,, \quad \omega^* = \frac{1}{3}\,.$$

Therefore, all the assumptions of Theorem (3.2) are fulfilled, which allow to conclude that system (5.1) has at least one solution on $[a, T]$. Moreover, we obtain

$$C_{f,\Phi} := \frac{\Omega_1}{1-\mathcal{L}} \approx 53.30408555 > 0.$$

Thus, by Theorem 4.2, system (5.1) is UHR stable and, consequently, GUHR stable.

6 Conclusion

In this paper, we construct the equivalence between problem (1.1) and the Volterra integral equation. We prove the existence results of solutions for the GPF integro-differential Langevin equation via a variable coefficient with nonlocal integral conditions (1.1) using a variety of fixed point theorems due to Banach, Schaefer and Krasnoselskii. Moreover, we discuss the stability analysis of UH, GUH, UHR and GUHR for the proposed problem (1.1). In addition, an example was given to illustrate our main results. We believe that the all results of this paper will provide considerable potential to interested researchers to develop relevant results concerning qualitative properties of nonlinear GPF differential equations. In a forthcoming work, we shall focus on studying the different types of existence results and stability analysis to an impulsive GPF differential equation with nonlocal integral multi-point conditions.

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