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A FAMILY OF PLANAR DIFFERENTIAL SYSTEMS WITH EXPLICIT EXPRESSION FOR ALGEBRAIC AND NON-ALGEBRAIC LIMIT CYCLES

Abstract. This paper is devoted to the study of a family of planar polynomial differential systems. First, we prove that the considered family has invariant algebraic curves which are given explicitly. Then, we introduce an explicit expression for their first integral. Moreover, we provide sufficient conditions for the systems to possess two limit cycles explicitly given: one is an algebraic and the other is shown to be non-algebraic. The applicability of our result was illustrated by concrete examples.

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1 Introduction

One of the main problems in the qualitative theory of differential equations is the study of limit cycles of planar differential systems and especially of the planar polynomial differential systems of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = P(x,y), \\ y' = \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{1.1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y. The degree of the system is the maximum of the degrees of the polynomials P and Q.

Recall that:

- A limit cycle of system (1.1) is an isolated periodic orbit in the set of its periodic orbits and is said to be algebraic if it is contained in the zero set of an invariant algebraic curve of the system.
- An algebraic curve defined by $U(x,y) = 0$ is an invariant curve for (1.1) if there exists a polynomial $K(x,y)$ (called the cofactor) such that

$$P(x,y)\frac{\partial U(x,y)}{\partial x} + Q(x,y)\frac{\partial U(x,y)}{\partial y} = K(x,y)U(x,y).$$

- System (1.1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non-constant analytic function $H : \Omega \to \mathbb{R}$, called a first integral, such that

$$\frac{dH(x,y)}{dt} = P(x,y)\,\frac{\partial H(x,y)}{\partial x} + Q(x,y)\,\frac{\partial H(x,y)}{\partial y} \equiv 0.$$

Among the important and attractive problems in the qualitative theory of differential equations [8, 14] is the study of limit cycles of system (1.1) related to the Hilbert's 16th problem [11]; several works and papers in this field investigate their number, stability and location in the phase plane [1,12].

The notion of integrability of (1.1) is based on the existence of a first integral [5, 16]. There is a strong relationship between the integrability of polynomial systems and the number of invariant algebraic curves they have [7], and questions about the existence of a first integral, determining its expression explicitly, when it exists, are always presents.

The results and examples [2–4, 9, 10] about algebraic and non-algebraic limit cycle are given, but it is not easy work to decide whether a limit cycle is algebraic or not. Thus, the well-known limit cycle of the van der Pol differential system exhibited in 1926 (see [15]), was not proved until 1995 by Odani [13] that it was non-algebraic. An invariant algebraic curve is a principal topic for several authors and researchers because of its importance in understanding the dynamics of a system (we refer to [6] for an exhaustive survey on this topic).

In this paper, we give an explicit expression of invariant algebraic curves, then we prove that these systems are integrable, and we introduce an explicit expression of a first integral of a multi-parameter planar polynomial differential system of thirteenth degree of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = x + (x^2+y^2)^2\big(P_3(x,y) - x(x^2+y^2)^3R_2(x,y)\big), \\ y' = \dfrac{dy}{dt} = y + (x^2+y^2)^2\big(Q_3(x,y) - y(x^2+y^2)^3R_2(x,y)\big), \end{cases} \tag{1.2}$$

where

$$\begin{aligned} P_3(x,y) &= ax^3 + bx^2y + cxy^2 - dy^3, \\ Q_3(x,y) &= ax^2y + dx^3 + (b+2\,d)xy^2 + cy^3, \\ R_2(x,y) &= (a+1)x^2 + (b+d)xy + (c+1)y^2, \end{aligned}$$

in which a, b, c, d are the real constants.

Moreover, we provide sufficient conditions for a polynomial differential system to possess two limit cycles explicitly given: one is algebraic and the other is shown to be non-algebraic. Concrete examples exhibiting the applicability of our result are introduced.

We define the trigonometric functions

$$G(\theta) = \frac{a+c}{2} + \frac{a-c}{2}\cos 2\theta + \frac{b+d}{2}\sin 2\theta,$$

$$A(\theta) = \int_0^\theta \frac{6+6G(t)}{d}\exp\left(\int_0^t \frac{-12-6G(\omega)}{d}\,d\omega\right)dt,$$

$$B(\theta) = \exp\left(\int_0^\theta \frac{-12-6G(\omega)}{d}\,d\omega\right).$$

Our main result is contained in the following theorem.

Theorem 1.1. *For system* (1.2)*, the following statements hold.*

(1) *If $d \neq 0$, then the origin of coordinates $O(0,0)$ is the unique critical point of system* (1.2) *at a finite distance.*

(2) *The curve $U(x,y) = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 1$ is an invariant algebraic curve of system* (1.2) *with a cofactor*

$$K(x,y) = -6(x^2+y^2)^3\Big(1 + (x^2+y^2)^2\big((a+1)x^2 + (b+d)xy + (c+1)y^2\big)\Big).$$

(3) *System* (1.2) *has the first integral*

$$H(x,y) = \frac{(1-(x^2+y^2)^3)A(\arctan\frac{y}{x}) + B(\arctan\frac{y}{x})}{(x^2+y^2)^3 - 1}.$$

(4) *System* (1.2) *has an explicit limit cycle, given in Cartesian coordinates by*

$$(\Gamma_1): x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 1 = 0.$$

(5) *If $d < 0$, $-2-(a+c) > |b+d| + |c-a|$ and $4+a+c > |b+d| + |c-a|$, then system* (1.2) *has non-algebraic limit cycle (Γ_2), explicitly given in the polar coordinates (r,θ) by*

$$r(\theta, r_*) = \left(\frac{(B(\theta)+A(\theta))(B(2\pi)-1)+A(2\pi)}{A(\theta)(B(2\pi)-1)+A(2\pi)}\right)^{\frac{1}{6}}.$$

Moreover, the algebraic limit cycle (Γ_1) lies inside the non-algebraic limit cycle (Γ_2).

2 Proof of Theorem 1.1

Proof of Statement (1). By definition, $A(x_0, y_0) \in \mathbb{R}^2$ is a critical point of system (1.2) if

$$\begin{cases} x_0 + (x_0^2+y_0^2)\big(P_3(x_0,y_0) - x_0(x_0^2+y_0^2)^3 R_2(x_0,y_0)\big) = 0, \\ y_0 + (x_0^2+y_0^2)\big(Q_3(x_0,y_0) - y_0(x_0^2+y_0^2)^3 R_2(x_0,y_0)\big) = 0, \end{cases}$$

and we have

$$(x_0^2+y_0^2)^2(y_0P_3(x_0,y_0) - x_0Q_3(x_0,y_0)) = -d({x_0}^2+{y_0}^2)^4.$$

Since $d \neq 0$, we have that $(x_0, y_0) = (0,0)$ is the unique solution of this equation. Thus the origin is the unique critical point at a finite distance.

This completes the proof of Statement (1) of Theorem 1.1.

Proof of Statement (2). A computation shows that

$$U(x,y) = x^6 + 3\,x^4y^2 + 3\,x^2y^4 + y^6 - 1$$

satisfies the linear partial differential equation

$$\frac{\partial U(x,y)}{\partial x}\,P(x,y) + \frac{\partial U(x,y)}{\partial y}\,Q(x,y) = U(x,y)K(x,y),$$

the associated cofactor being

$$K(x,y) = -6(x^2+y^2)^3\Big(1 + (x^2+y^2)^2\big((a+1)x^2 + (b+d)xy + (c+1)y^2\big)\Big).$$

This completes the proof of Statement (2) of Theorem 1.1.

Proof of Statement (3). To prove Statement (3), we need to convert system (1.2) in polar coordinates (r,θ) given by $x = r\cos\theta$ and $y = r\sin\theta$, then system (1.2) takes the form

$$\begin{cases} r' = \dfrac{dr}{dt} = r + G(\theta)r^7 + (-G(\theta)-1)r^{13}, \\ \theta' = \dfrac{d\theta}{dt} = dr^6. \end{cases} \tag{2.1}$$

Taking θ as an independent variable, we obtain the equation

$$\frac{dr}{d\theta} = \frac{1}{d}\,r^{-5} + \frac{G(\theta)}{d}\,r + \frac{-G(\theta)-1}{d}\,r^7. \tag{2.2}$$

Using the change of variables $\rho = r^6$, equation (2.2) is transformed into the Riccati equation

$$\frac{d\rho}{d\theta} = \frac{6}{d} + \frac{6G(\theta)}{d}\,\rho + \frac{-6G(\theta)-6}{d}\,\rho^2. \tag{2.3}$$

This equation is integrable, since it possesses the particular solution $\rho = 1$.

By introducing the standard change of variables $z = \rho - 1$, we obtain the Bernoulli equation

$$\frac{dz}{d\theta} = \frac{-6-6G(\theta)}{d}\,z^2 + \frac{-12-6G(\theta)}{d}\,z. \tag{2.4}$$

We note that $z = 0$ is the solution for (2.4), and by introducing the standard change of variables $y = \frac{1}{z}$, we obtain the linear equation

$$\frac{dy}{d\theta} = -\frac{6+6G(\theta)}{d} - \frac{12+6G(\theta)}{d}\,y. \tag{2.5}$$

The general solution of linear equation (2.5) is

$$y(\theta) = \frac{\alpha + A(\theta)}{B(\theta)}\,,$$

where $\alpha \in \mathbb{R}$. Then the general solution of equation (2.4) is

$$z(\theta) = 0, \quad z(\theta) = \frac{B(\theta)}{\alpha + A(\theta)}, \text{ where } \alpha \in \mathbb{R}.$$

The general solution of equation (2.3) is

$$\rho(\theta) = 1, \quad \rho(\theta) = \frac{\alpha + A(\theta) + B(\theta)}{\alpha + A(\theta)}, \text{ where } \alpha \in \mathbb{R}.$$

Consequently, the general solution of (2.2) is

$$r(\theta)=1,\quad r(\theta)=\Big(\frac{\alpha+A(\theta)+B(\theta)}{\alpha+A(\theta)}\Big)^{\frac{1}{6}},\ \text{ where }\ \alpha\in\mathbb{R}.$$

From this solution we obtain a first integral in the variables (x,y) of the form

$$H(x,y)=\frac{(1-(x^2+y^2)^3)A(\arctan\frac{y}{x})+B(\arctan\frac{y}{x})}{(x^2+y^2)^3-1}\,.$$

Hence, Statement (3) of Theorem 1.1 is proved.

Proof of Statement (4). The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (1.2), in Cartesian coordinates are written as

$$x^2+y^2=1,\quad (x^2+y^2)^3=\frac{\alpha+A(\theta)+B(\theta)}{\alpha+A(\theta)}\,,$$

where $\alpha\in\mathbb{R}$.

Notice that system (1.2) has a periodic orbit if and only if equation (2.2) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of (2.2) that fulfils $r(0,r_*)=r(2\pi,r_*)$ and $r(\theta,r_*)>0$ for any θ in $[0,2\pi]$.

The solution $r(\theta,r_0)$ of the differential equation (2.2) such that $r(0,r_0)=r_0$ is

$$r(\theta,r_0)=\left(\frac{\frac{1}{r_0^6-1}+A(\theta)+B(\theta)}{\frac{1}{r_0^6-1}+A(\theta)}\right)^{\frac{1}{6}},$$

where $r_0=r(0)$.

We have the particular solution $\rho(\theta)=1$ of the differential equation (2.3); from this solution we obtain $r^6(\theta)=1>0$ for all θ in $[0,2\pi]$, which is a particular solution of the differential equation (2.2).

This is an algebraic limit cycle for the differential systems (1.2), corresponding, of course, to an invariant algebraic curve $U(x,y)=0$.

More precisely, in Cartesian coordinates $r^2=x^2+y^2$ and $\theta=\arctan(\frac{y}{x})$ the curve (Γ_1) defined by this limit cycle is $(\Gamma_1): x^6+3x^4y^2+3x^2y^4+y^6-1=0$.

Hence, Statement (4) of Theorem 1.1 is proved.

Proof of Statement (5). A periodic solution of system (1.2) must satisfy the condition $r(0,r_*)=r(2\pi,r_*)$, which leads to a unique value $r_0=r_*$ given by

$$r_*=\Big(\frac{A(2\pi)+B(2\pi)-1}{A(2\pi)}\Big)^{\frac{1}{6}}.$$

The value r_* is the intersection of the periodic orbit with the OX_+ axis. After the substitution of this value r_* into $r(\theta,r_0)$, we obtain

$$r(\theta,r_*)=\left(\frac{(B(\theta)+A(\theta))(B(2\pi)-1)+A(2\pi)}{A(\theta)(B(2\pi)-1)+A(2\pi)}\right)^{\frac{1}{6}}.$$

In what follows, it is proved that $r(\theta,r_*)>0$. Indeed,

$$A(2\pi)-A(\theta)=\int_{\theta}^{2\pi}\frac{6+6G(t)}{d}\exp\left(\int_0^t\frac{-12-6G(\omega)}{d}\,\mathrm{d}\omega\right)\mathrm{d}t.$$

According to $d<0$, $-2-(a+c)>|b+d|+|c-a|$ and $4+a+c>|b+d|+|c-a|$, hence $\frac{-2-G(\theta)}{d}$ and $\frac{1+G(\theta)}{d}>0$ for all θ in $[0,2\pi]$, then we have $A(2\pi)-A(\theta)>0$ and $B(2\pi)>1$; therefore, we have

$r_* > 0$ and $r(\theta, r_*) > 0$ for all θ in $[0, 2\pi]$. This is the second limit cycle for the differential system (1.2), we denote it by (Γ_2). This limit cycle is not algebraic, due to the expression

$$B(\theta) = \exp\Big(\int_0^\theta \frac{-12 - 6G(\omega)}{d}\,\mathrm{d}\omega\Big).$$

More precisely, in the Cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan(\frac{y}{x})$, the curve defined by this limit cycle (Γ_2) is $F(x,y) = 0$, where

$$F(x,y) = (x^2+y^2)^3 - \frac{\big(B(\arctan\frac{y}{x}) + A(\arctan\frac{y}{x})\big)(B(2\pi)-1) + A(2\pi)}{A(\arctan\frac{y}{x})(B(2\pi)-1) + A(2\pi)}.$$

If the limit cycle is algebraic, this curve should be given by a polynomial, but a polynomial $F(x,y)$ in the variables x and y satisfies that there is a positive integer n such that $\frac{\partial^n F(x,y)}{\partial x^n} = 0$, but this is not the case, therefore, the curve $(\Gamma_2) : F(x,y) = 0$ is non-algebraic and the limit cycle will also be non-algebraic.

According to $d < 0$, $-2 - (a+c) > |b+d| + |c-a|$ and $4 + a + c > |b+d| + |c-a|$, we get

$$r_* = (1 + \frac{B(2\pi)-1}{A(2\pi)})^{\frac{1}{6}} > 1,$$

and

$$r(\theta, r_*) = \left(1 + \frac{B(\theta)}{\frac{1}{r_*^6 - 1} + A(\theta)}\right)^{\frac{1}{6}} > 1.$$

We conclude that system (1.2) has two limit cycles, the algebraic (Γ_1) lies inside the non-algebraic one (Γ_2).

This completes the proof of Statement (5) of Theorem 1.1.

3 Examples

Example 3.1. We take $a = c = -\frac{6}{5}$, $d = -5$ and $b = \frac{51}{10}$, then system (1.2) reads as

$$\begin{cases} x' = x + (x^2+y^2)^2\Big(-\frac{6}{5}x^3 + \frac{51}{10}x^2y - \frac{6}{5}xy^2 + 5y^3\Big) - x(x^2+y^2)^5\Big(-\frac{1}{5}x^2 + \frac{1}{10}xy - \frac{1}{5}y^2\Big), \\ y' = y + (x^2+y^2)^2\Big(-\frac{6}{5}x^2y - 5x^3 - \frac{49}{10}xy^2 - \frac{6}{5}y^3\Big) - y(x^2+y^2)^5\Big(-\frac{1}{5}x^2 + \frac{1}{10}xy - \frac{1}{5}y^2\Big). \end{cases} \tag{3.1}$$

In this case, we get

$$A(\theta) = -\frac{3}{50}\int_0^\theta (\sin(2t) - 4)\exp\Big(\frac{3}{100} + \frac{24}{25}t - \frac{3}{100}\cos(2\theta)\Big)\,dt,$$
$$B(\theta) = \exp\Big(-\frac{3}{100}\cos(2\theta) + \frac{24}{25}\theta + \frac{3}{100}\Big).$$

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axis is the point

$$r_* = \Big(\frac{116.8 + \exp(\frac{48\pi}{25}) - 1}{116.8}\Big)^{\frac{1}{6}} \simeq 1.2876.$$

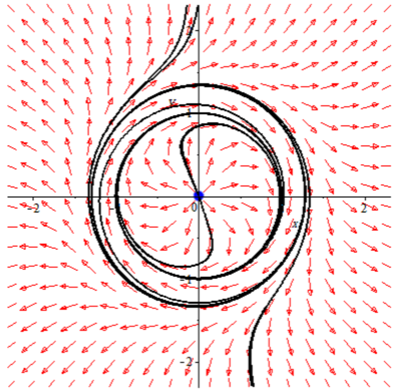

Figure 3.1. Limit cycles of system (3.1).

Example 3.2. We take $a = \frac{-11}{10}$, $c = \frac{-115}{100}$, $d = -7$ and $b = \frac{141}{20}$, then system (1.2) reads as

$$\begin{cases} x' = x + (x^2+y^2)^2\Big(\dfrac{-11}{10}\,x^3 + \dfrac{141}{20}\,x^2y - \dfrac{23}{20}\,xy^2 + 7y^3\Big) \\ \qquad -x(x^2+y^2)^5\Big(-\dfrac{1}{10}\,x^2 + \dfrac{1}{20}\,xy - \dfrac{3}{20}y^2\Big), \\ y' = y + (x^2+y^2)^2\Big(-\dfrac{11}{10}\,x^2y - 7\,x^3 - \dfrac{139}{20}\,xy^2 - \dfrac{23}{20}\,y^3\Big) \\ \qquad -y(x^2+y^2)^5\Big(-\dfrac{1}{10}\,x^2 + \dfrac{1}{20}\,xy - \dfrac{3}{20}\,y^2\Big). \end{cases} \tag{3.2}$$

In this case, we get

$$A(\theta) = -\frac{3}{140}\int_0^{\theta}(\cos(2t)+\sin(2t)-5)\exp\Big(\frac{3}{280}+\frac{3}{280}\,\sin(2t)-\frac{3}{280}\,\cos(2t)+\frac{3}{4}\Big)\,dt,$$

$$B(\theta) = \exp\Big(-\frac{3}{280}\,\sin(2\theta)-\frac{3}{280}\,\cos(2\theta)+\frac{3}{4}\,\theta+\frac{3}{280}\Big).$$

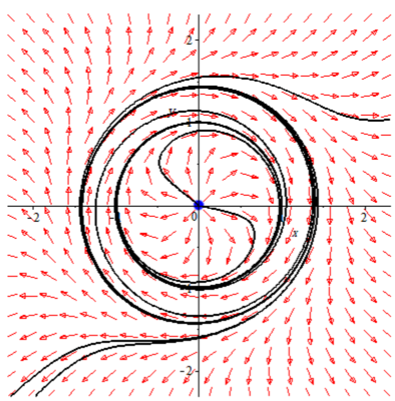

Figure 3.2. Limit cycles of system (3.2).

The intersection of the non-algebraic (Γ_2) limit cycle with the OX_+ axis is the point

$$r_* = \Big(\frac{16.509+\exp(\frac{2\pi}{3})-1}{16.509}\Big)^{\frac{1}{6}} \simeq 1.4047.$$

Example 3.3. We take $a = \frac{-101}{100}$, $c = \frac{-105}{100}$, $d = -1$ and $b = \frac{151}{150}$, then system (1.2) reads as

$$\begin{cases} x' = x + (x^2+y^2)^2\Big(-\dfrac{101}{100}x^3 + \dfrac{151}{50}x^2y - \dfrac{21}{20}xy^2 + y^3\Big) \\ \qquad -x(x^2+y^2)^5\Big(-\dfrac{1}{100}x^2 + \dfrac{1}{50}xy - \dfrac{1}{20}y^2\Big), \\ y' = y + (x^2+y^2)^2\Big(-\dfrac{101}{100}x^2y - x^3 + \dfrac{149}{150}xy^2 - \dfrac{21}{20}y^3\Big) \\ \qquad -y(x^2+y^2)^5\Big(-\dfrac{1}{100}x^2 + \dfrac{1}{150}xy - \dfrac{1}{20}y^2\Big). \end{cases} \tag{3.3}$$

In this case, we get

$$A(\theta) = -\frac{1}{50}\int_0^\theta (6\cos(2t) + \sin(2t) - 9)\exp\Big(\frac{1}{100} + \frac{3}{50}\sin(2t) - \frac{1}{100}\cos(2t) + \frac{291}{50}t\Big)\,dt,$$

$$B(\theta) = \exp\Big(\frac{3}{50}\sin(2\theta) - \frac{1}{100}\cos(2t) + \frac{291}{50}\theta + \frac{1}{100}\Big).$$

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axis is the point

$$r_* = \Big(\frac{1.019\times 10^{14} + \exp(\frac{291\pi}{25}) - 1}{1.019\times 10^{14}}\Big)^{\frac{1}{6}} \simeq 2.0566.$$

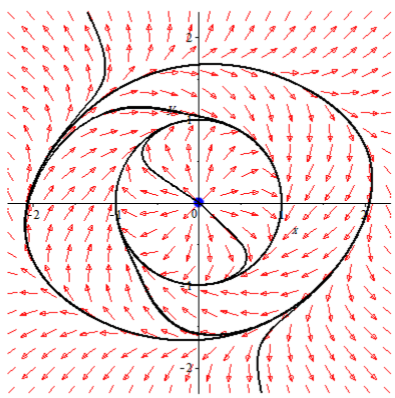

Figure 3.3. Limit cycles of system (3.3).

Example 3.4. We take $a = \frac{-107}{100}$, $c = \frac{-109}{100}$, $d = -5$ and $b = \frac{507}{100}$, then system (1.2) reads as

$$\begin{cases} x' = x + (x^2+y^2)^2\Big(-\dfrac{107}{100}x^3 + \dfrac{507}{100}x^2y - \dfrac{109}{100}xy^2 + 5y^3\Big) \\ \qquad -x(x^2+y^2)^5\Big(-\dfrac{7}{100}x^2 + \dfrac{7}{100}xy - \dfrac{9}{100}y^2\Big), \\ y' = y + (x^2+y^2)^2\Big(-\dfrac{107}{100}x^2y - 5x^3 - \dfrac{493}{100}xy^2 - \dfrac{109}{100}y^3\Big) \\ \qquad -y(x^2+y^2)^5\Big(-\dfrac{7}{100}x^2 + \dfrac{7}{100}xy - \dfrac{9}{100}y^2\Big). \end{cases} \tag{3.4}$$

In this case, we get

$$A(\theta) = -\frac{3}{500}\int_0^\theta (2\cos(2t) + 7\sin(2t) - 16)\exp\Big(\frac{21}{1000} + \frac{3}{500}\sin(2t) - \frac{21}{1000}\cos(2t) + \frac{138}{125}t\Big)\,dt,$$

$$B(\theta) = \exp\Big(\frac{3}{500}\sin(2\theta) - \frac{21}{1000}\cos(2t) + \frac{138}{25}\theta + \frac{21}{1000}\Big).$$

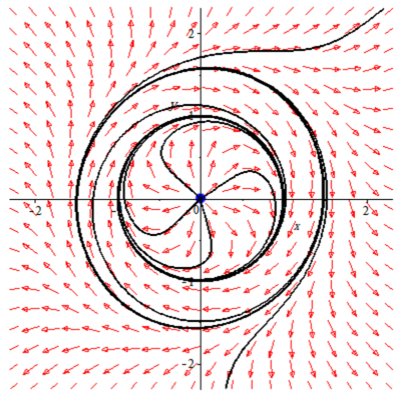

Figure 3.4. Limit cycles of system (3.4).

The intersection of the non-algebraic limit cycle (Γ_2) with the OX_+ axis is the point

$$r_* = \left(\frac{104.804 + \exp(\frac{276\pi}{125}) - 1}{104.804}\right)^{\frac{1}{6}} \simeq 1.4870.$$

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