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EXISTENCE OF SOLUTION FOR A THIRD-ORDER DIFFERENTIAL INCLUSION WITH THREE-POINT BOUNDARY VALUE PROBLEM INVOLVING CONVEX MULTIVALUED MAPS

Abstract. In this paper, we discuss the existence of solutions for a third-order differential inclusions with three-point boundary conditions involving convex multivalued maps. The obtained results are based on a nonlinear alternative of the Leray–Schauder type. Finally, some examples are given to illustrate our results.

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რეზიუმე. ნაშრომში შესწავლილია ამონახსნის არსებობა მესამე რიგის დიფერენციალური ჩანართებისთვის დასმული სასაზღვრო ამოცანისთვის სამწერტილიანი სასაზღვრო პირობებით, რომელიც მოიცავს ამოზნექილ მრავალსახა ასახვებს. მიღებული შედეგები ემყარება ლერეი-შაუდერის ტიპის არაწრფივ ალტერნატივას. ნაშრომის ბოლოს, შედეგების საილუსტრაციოდ, მოყვანილია რამდენიმე მაგალითი.

1 Introduction

Differential inclusions arising in the mathematical modeling of certain problems in economics, optimal control, stochastic analysis, and so forth, are widely studied by many authors (see [3–5, 14, 15, 18, 19] and the references therein). This work is concerned with the existence of solutions for boundary value problems (BVP, for short). In Section 3, we study the three-point boundary value problems of the third order differential inclusion, when the right-hand side is convex

$$-u'''(t) \in F(t, u(t)), \quad t \in (0,1), \tag{1.1}$$

with the boundary conditions

$$u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta), \quad u'(1) = \gamma u'(\eta), \tag{1.2}$$

where $\eta \in (0,1)$, $\alpha, \beta, \gamma \in \mathbb{R}$, with $(1+\alpha)\gamma \le \beta \le \frac{\gamma}{2}$, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map; and with

$$u'(0) = u''(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta), \tag{1.3}$$

where $\eta \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map.

The present paper is motivated by the recent papers [15], by S. A. Guezane-Lakoud, N. Hamidane, and [10], by R. Khaldi and D. Liu and Z. Ouyang, where problems (1.1), (1.2) and (1.1), (1.3) with single valued $F(\,\cdot\,,\,\cdot\,)$, respectively, are considered, and several existence results are obtained by using fixed point techniques.

The aim of our paper is to extend the study in [10] and [15] to the set-valued framework and to present some existence results for problems (1.1), (1.2) and (1.1), (1.3). Our results are based on the nonlinear alternative of Leray–Schauder type [9]. The method used is standard, however, its exposition in the framework of problems (1.1), (1.2) and (1.1), (1.3) are new. In Section 4, we complete our work by giving some examples to illustrate the obtained results.

2 Preliminaries

We begin this section by introducing some notation. Let $C([0,1];\mathbb{R})$ denote the Banach space of all continuous functions $u : [0,1] \to \mathbb{R}$ with the norm

$$\|u\| = \sup\big\{|u(t)| \ \text{ for all } t \in [0,1]\big\},$$

let $L^1([0,1];\mathbb{R})$ be the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ which are Lebesgue integrable, normed by

$$\|u\|_{L^1} = \int_0^1 |u(t)|\,dt,$$

and $AC^i([0,1];\mathbb{R})$ be the space of i-times differentiable functions $u : [0,1] \to \mathbb{R}$, whose ith derivative $u^{(i)}$ is absolutely continuous. Let (X,d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Denote

$$\begin{aligned}
\mathcal{P}_0(X) &= \big\{A \in \mathcal{P}(X) : \ A \neq \varnothing\big\},\\
\mathcal{P}_{cl}(X) &= \big\{A \in \mathcal{P}_0(X) : \ A \text{ is closed}\big\},\\
\mathcal{P}_b(X) &= \big\{A \in \mathcal{P}_0(X) : \ A \text{ is bounded}\big\},\\
\mathcal{P}_c(X) &= \big\{A \in \mathcal{P}_0(X) : \ A \text{ is convex}\big\},\\
\mathcal{P}_{comp}(X) &= \big\{A \in \mathcal{P}_0(X) : \ A \text{ is compact}\big\}.
\end{aligned}$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\},$$

where

$$d(a,B)=\inf_{b\in B} d(a,b) \text{ and } d(b,A)=\inf_{a\in A} d(a,b).$$

Then $(\mathcal{P}_{b,cl}(X),H_d)$ is a metric space and $(\mathcal{P}_{cl}(X),H_d)$ is a generalized metric space (see [12]).

Let E be a separable Banach space, Y be a nonempty closed subset of E and $G: Y\to\mathcal{P}_{cl}(E)$ be a multivalued operator. G has a fixed point if there is $x\in Y$ such that $x\in G(x)$. G is said to be completely continuous if $G(\Omega)$ is relatively compact for every $\Omega\in\mathcal{P}_b(Y)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is upper semicontinuous ($u.s.c$) if and only if G has a closed graph, that is, $x_n\to x_*, y_n\to y_*, y_n\in G(x_n)$ imply that $y_*\in G(x_*)$. For more details on the multi-valued maps, see the books by Aubin and Cellina [1], by Aubin and Frankowska [2], by Deimling [7], by Gorniewicz [8] and by Hu and Papageorgiou [11].

Definition 2.1. A multivalued map $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(1) $t\to F(t,u)$ is measurable for each $u\in\mathbb{R}$,

(2) $u\to F(t,u)$ is upper semicontinuous for almost all $t\in(0,1)$,

and, further, a Carathéodory function F is called L^1-Carathéodory if

(3) for each $r>0$, there exists $\Phi_r\in L^1((0,1);\mathbb{R}^+)$ such that

$$\|F(t,u)\|=\sup\big\{|v|:\ v\in F(t,u)\big\}\leqslant\Phi_r(t)$$

for all $\|u\|\le r$ and for a.e. $t\in(0,1)$.

For each $u\in C((0,1);\mathbb{R})$, define the set of selections of F by

$$S_{F,u}=\Big\{v\in L^1((0,1);\mathbb{R}):\ v(t)\in F(t,u(t)) \text{ for a.e. } t\in(0,1)\Big\}.$$

Lemma 2.1 ([13])**.** *Let E be a Banach space, let $F:[0,T]\times\to\mathcal{P}_{comp,c}(E)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],E)$ to $C([0,1],E)$. Then the operator*

$$\Theta\circ S_F: C([0,1],E)\to\mathcal{P}_{comp,c}(C([0,1],E)),\quad u\to(\Theta\circ S_F)(u)=\Theta(S_{F,u})$$

is a closed graph operator in $C([0,1],E)\times C([0,1],E)$.

Lemma 2.2. *Assume*

$$\xi=2\big(\eta(\alpha(\gamma+1)-\beta)+(\beta-\alpha)\big)-\gamma-1\neq0,$$

then for $y\in C([0,1];\mathbb{R})$, the problem

$$\begin{gathered}u'''(t)+y(t)=0,\quad t\in(0,1),\\ u(0)=\alpha u'(0),\quad u(1)=\beta u'(\eta),\quad u'(1)=\gamma u'(\eta)\end{gathered}\tag{2.1}$$

has a unique solution

$$\begin{aligned}u(t)=&-\frac12\int\limits_0^t(t-s)^2y(s)\,ds-\frac1\xi\left[t^2(\beta-\gamma-\alpha\gamma)+(t+\alpha)(\gamma-2\beta)\right]\int\limits_0^\eta(\eta-s)y(s)\,ds\\ &+\frac1\xi\int\limits_0^1(1-s)\Big[\frac{t^2}{2}(s-2\alpha+2\beta-\gamma-s\gamma-1)+(t+\alpha)(\gamma\eta-2\beta\eta-s+s\gamma\eta)\Big]y(s)\,ds.\end{aligned}$$

Lemma 2.3. *Assume*

$$\xi=1-\beta\Big(\frac{\eta^2}{2}+\eta-\frac32\Big)-\alpha\neq0,$$

then for $y \in C([0,1];\mathbb{R})$*, the problem*

$$u'''(t) + y(t) = 0, \quad t \in (0,1),$$
$$u'(0) = u''(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta) \tag{2.2}$$

has a unique solution

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{1}{2\xi}\Big(-\frac{\beta}{2}t^2 - \beta t + \Big(\frac{3}{2}\beta - \alpha\Big)\Big)\int_0^\eta (\eta-s)^2 y(s)\,ds$$
$$+ \frac{1}{2\xi}\Big(\frac{\beta}{2}t^2 + \beta t + 1 - \frac{1}{2}\beta\eta^2 - \beta\eta\Big)\int_0^1 (1-s)^2 y(s)\,ds.$$

The proofs of Lemmas 2.2 and 2.3 are given by integrating three times $u'''(t) + y(t) = 0$ over the interval $[0,t]$. We obtain

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + A_1 t^2 + A_2 t + A_3, \quad \text{where } A_1, A_2, A_3 \in \mathbb{R}.$$

The constants A_1, A_2 and A_3 in Lemmas 2.2 and 2.3 are given by the three-point boundary conditions (2.1) and (2.2), respectively.

3 Main results

Before presenting the existence result for problem (1.1), (1.2), let us introduce the following hypotheses which are assumed hereafter:

(H_1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_c(\mathbb{R})$ is Carathéodory;

(H_2) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1];\mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\big\{|w| : w \in F(t,u)\big\} \le p(t)\psi(\|u\|) \text{ for each } (t,u) \in [0,1] \times \mathbb{R}.$$

Definition 3.1. A function $u \in AC^2((0,1);\mathbb{R})$ is called a solution to the BVP (1.1), (1.2) if u satisfies the differential inclusion (1.1) a.e. on $(0,1)$ and conditions (1.2).

Theorem 3.1. *Assume that* (H_1), (H_2) *hold and let the function* ψ *be bounded satisfying the condition: there exists a number* $M > 0$ *such that*

$$\bigg(\frac{1}{2} + \frac{\frac{1}{2} + \gamma + |\alpha - \beta| - \eta(\beta + \alpha\gamma)}{|\xi_1|}\bigg)\psi(\|u\|)\|p\|_{L^1} < M.$$

Then the BVP (1.1), (1.2) *has at least one solution on* $[0,1]$.

Proof. Define the operator $T : C([0,1];\mathbb{R}) \to \mathcal{P}(C[0,1];\mathbb{R})$ by

$$T(u) = \bigg\{h \in C([0,1];\mathbb{R}) : \quad h(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(u)\,ds$$
$$+ \frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1 - s + 2(\alpha-\beta) + \gamma(1+s)\big) + (t+\alpha)\big(\eta(\gamma - 2\beta) - s(1+\gamma\eta)\big)\Big) f(u)\,ds$$
$$- \frac{1}{\xi_1}\big(t^2(\beta - (1+\alpha)\gamma) + (t+\alpha)(\gamma - 2\beta)\big)\int_0^\eta (\eta - s) f(u)\,ds\bigg\}$$

for $f \in \mathcal{S}_{F,u}$. It is not difficult to show that T has a fixed point which is a solution of problem (1.1), (1.2). We show that T satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. The proof consists of several steps.

Step 1. First, we show that T is convex for each $u \in C([0,1];\mathbb{R})$.

Let $h_1, h_2 \in Tu$. Then there exist $w_1, w_2 \in \mathcal{S}_{F,u}$ such that for each $t \in [0,1]$, we have

$$h_i(t) = -\frac{1}{2}\int_0^t (t-s)^2 w_i(s)\,ds + \frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big) + (t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)w_i(s)\,ds - \frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w_i(s)\,ds, \quad i=1,2.$$

Let $0 \le \mu \le 1$. So, for each $t \in [0,1]$, we have

$$\mu h_1(t) + (1-\mu)h_2(t) = \frac{1}{2}\int_0^t (t-s)^2\big(\mu w_1(s)+(1-\mu)w_2(s)\big)\,ds + \frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big) + (t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big) \times \big(\mu w_1(s)+(1-\mu)w_2(s)\big)\,ds - \frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)(\mu w_1(s+(1-\mu)w_2(s))\,ds.$$

Since $\mathcal{S}_{F,u}$ is convex, it follows that $\mu h_1 + (1-\mu)h_2 \in Tu$.

Step 2. Here we show that T maps bounded sets into bounded sets in $C([0,1];\mathbb{R})$.

For a positive number r, let $B_r = \{u \in C([0,1];\mathbb{R}) : \|u\| \le r\}$ be a bounded ball in $C([0,1];\mathbb{R})$. So, for each $h \in Tu$, $u \in B_r$, there exists $w \in \mathcal{S}_{F,u}$ such that

$$h(t) = -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds + \frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big) + (t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)w(s)\,ds - \frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w(s)\,ds.$$

If $(1+\alpha)\gamma \le \beta \le \frac{\gamma}{2}$, we obtain

$$|h(t)| \le \frac{\psi(\|u\|)}{2}\int_0^1 p(s)\,ds + \frac{\psi(\|u\|)}{|\xi_1|}\Big[\frac{1}{2}+|\alpha-\beta|+|\gamma|-\beta\eta\Big]\int_0^1 p(s)\,ds - \frac{\psi(\|u\|)}{|\xi_1|}\gamma\eta\Big(\frac{1}{2}+\alpha\Big)\int_0^\eta p(s)\,ds,$$

Thus

$$\|h\| \leq \frac{\psi(\|u\|)}{2} \int_0^1 p(s)\,ds + \frac{\psi(\|u\|)}{|\xi_1|}\Big[1+|\alpha-\beta|+2\gamma-\beta\eta\Big]\int_0^1 p(s)\,ds$$
$$-\frac{\psi(\|u\|)}{|\xi_1|}\gamma\eta\Big(\frac{1}{2}+\alpha\Big)\int_0^\eta p(s)\,ds.$$

Step 3. Now we show that T maps the bounded sets into equicontinuous sets of $C([0,1];\mathbb{R})$.

Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and let B_r be a bounded set of $C([0,1];\mathbb{R})$. Then, for each $h\in Tu$, we obtain that the bounded sets of $C([0,1];\mathbb{R})$ are mapped into the equicontinuous sets,

$$|h(t_2)-h(t_1)| \leq \frac{1}{2}\int_{t_1}^{t_2}(t_2-s)^2|w(s)|\,ds + \frac{1}{2}\int_0^{t_1}\big((t_1-s)^2-(t_2-s)^2\big)|w(s)|\,ds$$
$$+\frac{1}{|\xi_1|}\int_0^1(1-s)\Big(\frac{-(t_2^2-t_1^2)}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t_2-t_1)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)|w(s)|\,ds$$
$$+\frac{1}{|\xi_1|}\big((t_2^2-t_1^2)(\beta-(1+\alpha)\gamma)+(t_2-t_1)(\gamma-2\beta)\big)\int_0^\eta(\eta-s)|w(s)|\,ds,$$
$$\leq \frac{\psi(\|u\|)}{2}\int_{t_1}^{t_2}(t_2-s)^2p(s)\,ds + \frac{\psi(\|u\|)}{2}\int_0^{t_1}\big((t_1-s)^2-(t_2-s^2\big)p(s)\,ds$$
$$+\frac{\psi(\|u\|)}{|\xi_1|}\int_0^1(1-s)\Big(\frac{-(t_2^2-t_1^2)}{2}(1-s+2(\alpha-\beta)+\gamma(1+s))+(t_2-t_1)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)p(s)\,ds$$
$$+\frac{\eta\psi(\|u\|)}{|\xi_1|}\big((t_2^2-t_1^2)(\beta-(1+\alpha)\gamma)+(t_2-t_1)(\gamma-2\beta)\big)\int_0^\eta p(s)\,ds.$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u\in B_r$ as $t_2-t_1\to 0$. Since T satisfies the above three assumptions, it follows by the Ascoli–Arzelèa theorem that $T: C([0,1];\mathbb{R})\to P(C[0,1];\mathbb{R})$ is completely continuous.

Step 4. We show that T has a closed graph.

Let $u_n\to u_*$, $h_n\in T(u_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in Tu_*$.

Associated with $h_n\in T(u_n)$, there exists $w_n\in\mathcal{S}_{F,u_n}$ such that for each $t\in[0,1]$, we have

$$h(t) = -\frac{1}{2}\int_0^t(t-s)^2w_n(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1(1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)(\eta(\gamma-2\beta)-s(1+\gamma\eta))\Big)w_n(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta(\eta-s)w_n(s)\,ds.$$

Thus we have to show that there exists $w_*\in\mathcal{S}_{F,u_*}$ such that for each $t\in[0,1]$,

$$h_*(t) = -\frac{1}{2}\int_0^t (t-s)^2 w_*(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)(\eta(\gamma-2\beta)-s(1+\gamma\eta))\Big)w_*(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w_*(s)\,ds.$$

Let us consider the continuous linear operator $\Theta : L^1([0,1];\mathbb{R}) \to C([0,1];\mathbb{R})$ given by

$$w \longrightarrow \Theta w(t) = -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)w(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w(s)\,ds.$$

Observe that

$$\|h_n(t)-h_*(t)\| = \Bigg\| -\frac{1}{2}\int_0^t (t-s)^2 (w_n(s)-w_*(s))\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)(w_n(s)-w_*(s))\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)(w_n(s)-w_*(s))\,ds\Bigg\|,$$

then $\|h_n(t)-h_*(t)\| \to 0$ as $n \to \infty$.

Thus, it follows by Lemma 2.1 that $\Theta \circ \mathcal{F}$ is a closed graph operator.

Further, we have $h_n(t) \in \Theta(S_{F,u_n})$. Since $u_n \to u_*$, we get

$$h_*(t) = -\frac{1}{2}\int_0^t (t-s)^2 w_*(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big[\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big]w_*(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w_*(s)\,ds$$

for some $w_* \in S_{F,u_*}$.

Step 5. We discuss a priori bounds on solutions.

Let u be a solution of (1.1), (1.2). So, there exists $w \in L^1([0,1];\mathbb{R})$ with $w \in S_{F,u}$ such that for $t \in [0,1]$, we have

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 w(s)\,ds$$
$$+\frac{1}{\xi_1}\int_0^1 (1-s)\Big(\frac{-t^2}{2}\big(1-s+2(\alpha-\beta)+\gamma(1+s)\big)+(t+\alpha)\big(\eta(\gamma-2\beta)-s(1+\gamma\eta)\big)\Big)w(s)\,ds$$
$$-\frac{1}{\xi_1}\big(t^2(\beta-(1+\alpha)\gamma)+(t+\alpha)(\gamma-2\beta)\big)\int_0^\eta (\eta-s)w(s)\,ds.$$

In view of (H_2), for each $t\in[0,1]$, and $(1+\alpha)\gamma \le \beta \le \frac{\gamma}{2}$, we obtain

$$|u(t)| \le \frac{\psi(\|u\|)}{2}\int_0^1 p(s)\,ds + \frac{\psi x(\|u\|)}{|\xi_1|}\int_0^1 \Big(\frac{1}{2}+|\alpha-\beta|+\gamma+\frac{1}{2}\eta(\gamma-2\beta)\Big)\int_0^1 p(s)\,ds$$
$$-\frac{1}{2}\gamma\eta\,\frac{\psi(\|u\|)}{|\xi_1|}((2\alpha+1))\int_0^1 p(s)\,ds.$$

Consequently,
$$\frac{\|u\|}{\big(\frac{1}{2}+\frac{\frac{1}{2}+\gamma+|\alpha-\beta|-\eta(\beta+\alpha\gamma)}{|\xi_1|}\big)\psi(\|u\|)\|p\|_{L^1}} \le 1.$$

So, there exists M such that $\|u\|\neq M$. Let us set $U=\{u\in C([0,1];\mathbb{R}):\ \|u\|<M+1\}$. Note that the operator $T:\overline{U}\to\mathcal{P}C([0,1];\mathbb{R})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $u\in\partial U$ such that $u\in\lambda Tx$ for some $\lambda\in(0,1)$.

Consequently, by the nonlinear alternative of Leray–Schauder type [19], we deduce that T has a fixed point $u\in\overline{U}$ which is a solution of problem $(1.1),(1.2)$. This completes the proof. □

The next result concerns the four-point BVP $(1.1),(1.3)$. Before stating and proving this result, we give the definition of a solution of the four-point BVP $(1.1),(1.3)$.

Definition 3.2. A function $u\in AC^2((0,1);\mathbb{R})$ is called a solution to the BVP $(1.1),(1.3)$ if u satisfies the differential inclusion (1.1) a.e. on $(0,1)$ and conditions (1.3).

Theorem 3.2. *Assume that (H_1), (H_2) hold and let the function ψ be bounded satisfying the condition: there exists a number $M>0$ such that*

$$\Big(\frac{1}{2}+\frac{1}{2|\xi|}\Big(\eta^2|\alpha|+\Big(\frac{7}{2}\eta^2+\eta+\frac{3}{2}\Big)|\beta|+1\Big)\Big)\psi(\|u\|)\|p\|_{L^1}<M.$$

Then the BVP $(1.1),(1.3)$ has at least one solution on $[0,1]$.

Proof. Define the operator $T:C([0,1];\mathbb{R})\to\mathcal{P}(C[0,1];\mathbb{R})$ by

$$T(u)=\Big\{h\in C([0,1];\mathbb{R}):\ h(t)=-\frac{1}{2}\int_0^t (t-s)^2 f(u)\,ds$$
$$+\frac{1}{2\xi}\Big(-\frac{\beta}{2}t^2-\beta t+\Big(\frac{3}{2}\beta-\alpha\Big)\Big)\int_0^\eta (\eta-s)^2 f(u)\,ds$$
$$+\frac{1}{2\xi}\Big(\frac{\beta}{2}t^2+\beta t+1-\frac{1}{2}\beta\eta^2-\beta\eta\Big)\int_0^1 (1-s)^2 f(u)\,ds\Big\}$$

for $f\in\mathcal{S}_{F,u}$. We can easily show that T has a fixed point which is a solution of problem $(1.1),(1.3)$, following the steps of Theorem 3.1. We omit the details. □

4 Examples

Example 4.1. Consider the boundary value problem

$$\begin{aligned} &-u'''(t) \in F(t,u(t)), \quad t \in (0,1), \\ &u(0) = -u'(0), \quad u(1) = \frac{1}{3}\, u'\Big(\frac{1}{3}\Big), \quad u'(1) = u'\Big(\frac{1}{3}\Big), \end{aligned} \tag{4.1}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t,u) = \left[\frac{\exp(u)}{3+\exp(u)}, -2\log(t+1) + t^3 + t + 1\right].$$

For $f \in F$, we have

$$|f| \leqslant \max\left(\frac{\exp(u)}{3+\exp(u)}\,, -2\log(t+1) + t^3 + t + 1\right) \leqslant 2, \quad u \in \mathbb{R}.$$

Thus

$$\|F(t,u)\|_{\mathcal{P}} = \sup\big\{|w| : \ w \in F(t,u)\big\} \le 2 = p(t)\psi(\|u\|), \quad u \in \mathbb{R},$$

with $p(t) = \frac{1}{2}$, $\psi(\|u\|) = 4$. Further, using the condition

$$\left(\frac{1}{2} + \frac{\frac{1}{2} + \gamma + |\alpha - \beta| - \eta(\beta + \alpha\gamma)}{|\xi|}\right)\psi(\|u\|)\|p\|_{L^1} < M.$$

we find that $M > \frac{63}{8}$. By Theorem 3.1, the boundary value problem (4.1), has at least one solution on $[0,1]$.

Example 4.2. Consider the boundary value problem

$$\begin{aligned} &-u'''(t) \in F(t,u(t)), \quad t \in (0,1), \\ &u'(0) = u''(0) = -u\Big(\frac{1}{7}\Big), \quad u(1) = -2u\Big(\frac{1}{7}\Big), \end{aligned} \tag{4.2}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t,u) = \Big[\sin(u), \frac{u}{\exp u} + t\Big].$$

For $f \in F$, we have

$$|f| \leqslant \max\Big(\sin(u), \frac{u}{\exp u} + t\Big) \leqslant 1 + t, \quad u \in \mathbb{R}.$$

Thus

$$\|F(t,u)\|_{\mathcal{P}} = \sup\big\{|w| : \ w \in F(t,u)\big\} \le 1 + t = p(t)\psi(\|u\|), \quad u \in \mathbb{R},$$

with $p(t) = 1 + t$, $\psi(\|u\|) = 1$. Further, we use the condition

$$\left(\frac{1}{2} + \frac{1}{2|\xi|}\,(\eta^2|\alpha| + \Big(\frac{7}{2}\,\eta^2 + \eta + \frac{3}{2}\Big)|\beta| + 1)\right)\psi(\|u\|)\|p\|_{L^1} < M$$

with $M > 2$. By Theorem 3.2, the boundary value problem (4.2) has at least one solution on $[0,1]$.

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