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EXISTENCE AND SOLUTION SETS FOR SYSTEMS OF IMPULSIVE DIFFERENTIAL INCLUSIONS

Abstract. In this paper, we consider the existence of solutions and some properties of the set of solutions, as well as the solution operator for a system of differential inclusions with impulse effects. For the Cauchy problem, under various assumptions on the nonlinear term, we present several existence results. We appeal to some fixed point theorems in vector metric spaces. Finally, we prove some characterizing geometric properties about the structure of the solution set such as AR, R_δ, contractibility and acyclicity, with these properties corresponding to Aronszajn–Browder–Gupta type results.

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Key words and phrases. System of differential inclusions, impulsive, fixed point, existence, vector metric space, R_δ-set, acyclic, matrix.

რეზიუმე. ნაშრომში შესწავლილია იმპულსური ეფექტის მქონე დიფერენციალური ჩანართების სისტემის ამონახსნის არსებობა და ამონახსნთა სიმრავლის ზოგიერთი თვისება, აგრეთვე განხილულია ამოხსნის ოპერატორი. კოშის ამოცანისთვის, სხვადასხვა დაშვებებით არაწრფივ წევრზე, მოყვანილია ამონახსნის არსებობის რამდენიმე შედეგი. კვლევისას გამოყენებულია ზოგიერთი თეორემა ვექტორულ მეტრიკულ სივრცეებში უძრავი წერტილის შესახებ. დაბოლოს, დამტკიცებულია ამონახსნთა სიმრავლის სტრუქტურის ზოგიერთი დამახასიათებელი გეომეტრიული თვისება, როგორიცაა AR, R_δ, კუმშვადობა და აციკლურობა. ეს თვისებები შეესაბამება აირონშაინ-ბროუდერ-გუფთას ტიპის შედეგებს.

1 Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [41] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [31]. The dynamics of many processes in physics, population dynamics, biology, medicine, and so on, may be subject to abrupt changes such as shocks or perturbations (see, e.g., [1, 39, 40] and the references therein). These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. Important contributions to the study of the mathematical aspects of such equations have been undertaken in [25, 37, 50] among others.

In this work, we consider the following problem:

$$\begin{cases} x'(t) \in F_1(t, x(t), y(t)), & \text{a.e. } t \in [0,1], \\ y'(t) \in F_2(t, x(t), y(t)), & \text{a.e. } t \in [0,1], \\ x(t_k^+) = x(t_k^-) + I_{1,k}(x(t_k), y(t_k)), & k = 1, \dots, m, \\ y(t_k^+) = y(t_k^-) + I_{2,k}(x(t_k), y(t_k)), & k = 1, \dots, m, \\ x(0) = x_0, \quad y(0) = y_0, \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_m < 1$, $F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, $i = 1,2$, is a multifunction and $I_{1,k}, I_{2,k} \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The notations $x(t_k^+) = \lim\limits_{h \to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim\limits_{h \to 0^+} x(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively.

For single valued framework, the above system was used to analyze initial value and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [42] and mathematical economics [34]; this can be set in the operator form (1.1).

Recently, Precup [48] proved the role of matrix convergence and vector metric in the study of semilinear operator systems. In recent years, many authors studied the existence of solutions for systems of differential equations and impulsive differential equations by using vector version of fixed point theorems (see [11, 12, 26, 32, 35, 44–46, 49] and in the references therein).

In general, for the ordinary Cauchy problems, the uniqueness property does not hold. Kneser [36] proved in 1923 that the solution set is a continuum, i.e., closed and connected. For differential inclusions, Aronszajn [7] proved in 1942 that the solution set is, in fact, compact and acyclic, and he even specified this continuum to be an R_δ-set.

An analogous result was obtained for differential inclusions with upper semi-continuous (*u.s.c.*) convex valued nonlinearities by several authors (we cite [2–4, 6, 24, 30, 33]).

The topological and geometric structure of solution sets for impulsive differential inclusions on compact intervals, which were investigated in [18, 27–29, 53], are a contractibility, AR, acyclicity and R_δ-sets. Also, the topological structure of solution sets for some Cauchy problems without impulses posed on non-compact intervals were studied by various techniques in [4, 10, 16, 17].

The goal of this paper is to study the existence of solutions and solution sets for systems of impulsive differential inclusions with initial conditions. The paper is organized as follows. In Section 2, we recall some definitions and facts which will be needed in our analysis. In Section 3, we prove some existence results based on a nonlinear alternative of Leray–Schauder type theorem in generalized Banach spaces in the convex case, and a multivalued version of Perov's fixed point theorem (Theorem 2.3) for the nonconvex case. Finally, we present some topological and geometric structures for solution sets of (1.1).

2 Preliminaries

In this section, we introduce notations and definitions which are used throughout this paper.

Denote by

$$\begin{aligned}
\mathcal{P}(X) &= \big\{Y \subset X : \ \ Y \neq \varnothing\big\};\\
\mathcal{P}_{cl}(X) &= \big\{Y \in \mathcal{P}(X) : \ \ Y \text{ closed}\big\};\\
\mathcal{P}_{b}(X) &= \big\{Y \in \mathcal{P}(X) : \ \ Y \text{ bounded}\big\};\\
\mathcal{P}_{cv}(X) &= \big\{Y \in \mathcal{P}(X) : \ \ Y \text{ convex}\big\};\\
\mathcal{P}_{cp}(X) &= \big\{Y \in \mathcal{P}(X) : \ \ Y \text{ compact}\big\};\\
\mathcal{M}_{n\times n}(\mathbb{R}_+) &: \text{Designate the set of real nonnegative } \ n \times n \ \text{ matrices.}
\end{aligned}$$

Definition 2.1. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u,v) \geq 0$ for all $u, v \in X$, if $d(u,v) = 0$ if and only if $u = v$;

(ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$.

We call the pair (X,d) a generalized metric space. For $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, we denote by

$$B(x_0, r) = \big\{x \in X : \ \ d(x_0, x) < r\big\}$$

the open ball of radius r centered at x_0 and by

$$\overline{B(x_0, r)} = \big\{x \in X : \ \ d(x_0, x) \leq r\big\}$$

the closed ball of radius r centered at x_0.

We mention that for a generalized metric space, the notation of an open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces. If, $x, y \in \mathbb{R}^n, x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also, $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x,y) = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$.

Definition 2.2. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc (i.e., $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$).

Theorem 2.1 ([51]). *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M is convergent towards zero;*

(ii) *$M^k \to 0$ as $k \to \infty$;*

(iii) *the matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ;$$

(iv) *the matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 2.3. We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the absolute value property if

$$A^{-1}|A| \leq I,$$

where

$$|A| = \big(|a_{ij}|\big)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}).$$

Definition 2.4. Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x), N(y)) \leq M d(x, y), \ \ \forall\, x, y \in X.$$

Theorem 2.2 ([23, 47])**.** *Let (X, d) be a complete generalized metric space and $N : X \to X$ be a contractive operator with Lipschitz matrix M. Then N has a unique fixed point x_* and for each $x_0 \in X$ we have*

$$d(N^k(x_0), x_*) \leq M^k (I - M)^{-1} d(x_0, n(x_0)), \ \ \forall\, k \in \mathbb{N}.$$

Let (X, d) be a metric space. We denote by H_{d_*} the Pompeiu–Hausdorff pseudo-metric distance on $\mathcal{P}(X)$ defined as

$$H_{d_*} : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}, \quad H_{d_*}(A, B) = \max\Big\{ \sup_{a \in A} d_*(a, B),\ \sup_{b \in B} d_*(A, b) \Big\},$$

where $d_*(A, b) = \inf_{a \in A} d_*(a, b)$ and $d_*(a, B) = \inf_{b \in B} d_*(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_{d_*})$ is a metric space and $(\mathcal{P}_{cl}(X), H_{d_*})$ is a generalized metric space. In particular, H_{d_*} satisfies the triangle inequality.

Let (X, d) be a generalized metric space with

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_n(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1, \ldots, n$, are metrics on X. Consider the generalized Hausdorff pseudo-metric distance

$$H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$$

defined by

$$H_d(A, B) := \begin{pmatrix} H_{d_1}(A, B) \\ \vdots \\ H_{d_n}(A, B) \end{pmatrix}.$$

Definition 2.5. Let (X, d) be a generalized metric space. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is said to be contractive if there exists a metrix $M \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ such that

$$M^k \to 0 \ \text{ as } \ k \to \infty$$

and

$$H_d(N(u), N(v)) \leq M d(u, v), \ \ \forall\, u, v \in X.$$

Theorem 2.3 ([23])**.** *Let (X, d) be a generalized complete metric space, and let $N : X \to \mathcal{P}_{cl}(X)$ be a multivalued map. Assume that there exist $A, B, C \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ such that*

$$H_d(N(x), N(y)) \leq A d(x, y) + B d(y, N(x)) + C d(x, N(x)), \tag{2.1}$$

where $A + C$ converges to zero. Then there exists $x \in X$ such that $x \in N(x)$.

Definition 2.6. Let E be a vector space on $K = \mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm on E we mean a map $\|\cdot\| : E \to \mathbb{R}^n$ with the following properties:

(i) $\|x\| \geq 0$ for all $x \in E$; if $\|x\| = 0$, then $x = (0, \ldots, 0)$;

(ii) $\|\lambda x\| = |\lambda|\, \|x\|$ for all $x \in E$ and $\lambda \in K$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

The pair $(E, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|$ (i.e., $d(x,y) = \|x-y\|$) is complete, then the space $(E, \|\cdot\|)$ is called a generalized Banach space.

Lemma 2.1 ([43, Theorem 19.7]). *Let Y be a separable metric space and $F : [a,b] \to \mathcal{P}(Y)$ be a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.*

Lemma 2.2 ([38]). *Let X be a Banach space. Let $F : [a,b] \times X \to \mathcal{P}_{cp,cv}(X)$ be an L^1-Carathéodory multifunction with $S_{F,y} \neq \varnothing$, and let Γ be a continuous linear operator from $L^1([a,b],X)$ to $C([a,b],X)$. Then the operator*

$$\begin{aligned} \Gamma \circ S_F : C([0,b],X) &\longrightarrow \mathcal{P}_{cp,cv}(C([a,b],X)), \\ y &\longrightarrow (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \end{aligned}$$

has a closed graph in $C([a,b],X) \times C([a,b],X)$, where

$$S_{F,y} = \big\{ v \in L^1([0,b],X) : \ \ v(t) \in F(t,y(t)); \ \ t \in [a,b] \big\}.$$

Lemma 2.3 ([23, 47]). *Let X be a generalized Banach space and $F : X \to \mathcal{P}_{cp,cv}(X)$ be an u.s.c. compact multifunction. Moreover, assume that the set*

$$\mathcal{A} = \big\{ x \in X : \ \ x \in \lambda N(x) \ \ \textit{for some} \ \ \lambda \in (0,1) \big\}$$

is bounded. Then N has at least one fixed point.

Theorem 2.4 ([23]). *Let X be a generalized Banach space and $N : X \to X$ be a continuous compact mapping. Moreover, assume that the set*

$$\mathcal{K} = \big\{ x \in X : \ \ x = \lambda N(x) \ \ \textit{for some} \ \ \lambda \in (0,1) \big\}$$

is bounded. Then N has a fixed point.

Definition 2.7. Let X be a Banach space. A is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$, where I is Lebesgue measurable in $[a,b]$ and D is Borel measurable in X.

Definition 2.8. A subset $B \subset L^1([a,b],X)$ is decomposable if for all $u, v \in A$ and for every Lebesgue measurable set $I \subset [a,b]$, we have

$$u\chi_{_I} + v\chi_{_{[a,b]\setminus I}} \in B,$$

where $\chi_{_I}$ stands for the characteristic function of the set I.

Let $F : J \times X \to \mathcal{P}_{cl}(X)$ be multi-valued. Assign to F the multi-valued operator $\mathcal{F} : C(J,X) \to \mathcal{P}(L^1([a,b],X))$ defined by $\mathcal{F}(y) = S_{F,y}$. The operator $\mathcal{F}$ is called the Nemyts'kiĭ operator associated to F.

Definition 2.9. Let $F : J \times X \to \mathcal{P}_{cp}(X)$ be multi-valued. We say that F is of lower semi-continuous type (*l.s.c.* type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 2.4 ([19]). *Let $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be an integrable bounded multi-valued map such that*

(a) *$(t,x,y) \to F(t,x,y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;*

(b) *$(x,y) \to F(t,x,y)$ is l.s.c. a.e. $t \in [a,b]$.*

Then F is lower semi-continuous.

Next, we state a classical selection theorem due to Bressan and Colombo.

Theorem 2.5 ([13, 20]) (Theorem of "Bressan–Colombo" selection). *Let X be a metric separable space, and let E be a Banach space. Then each l.s.c. operator $N : X \to \mathcal{P}_{cl}(L^1([a,b],X))$ which has a decomposable closed value, also has a continuous selection.*

2.1 σ-selectionable multi-valued maps

The following four definitions and the theorem can be found in [22,30] (see also [8, p. 86]). Let (X,d) and (Y,d') be two metric spaces.

Definition 2.10. We say that a map $F : X \to \mathcal{P}(Y)$ is σ-Ca-selectionable if there exists a decreasing sequence of compact-valued *u.s.c.* maps $F_n : X \to Y$ satisfying:

(a) F_n has a Carathédory selection for all $n \geq 0$ (F_n are called Ca-selectionable);

(b) $F(x) = \bigcap_{n\geq 0} F_n(x)$ for all $x \in X$.

Definition 2.11. A single-valued map $f : [0,a] \times X \to Y$ is said to be measurable-locally-Lipschitz (mLL) if $f(\,\cdot\,,x)$ is measurable for every $x \in X$, and for every $x \in X$ there exist a neighborhood $V_x \subset X$ of x and an integrable function $L_x : [0,a] \to [0,\infty)$ such that

$$d'(f(t,x_1), f(t,x_2)) \leq L_x(t) d(x_1,x_2) \ \text{ for every } \ t \in [0,a], \ \ x_1, x_2 \in V_x.$$

Definition 2.12. A multi-valued mapping $F : [0,a] \times X \to \mathcal{P}(Y)$ is mLL-selectionable if it has an mLL-selection.

Definition 2.13. We say that a multi-valued map $\phi : [0,a] \times E \to \mathcal{P}(E)$ with closed values is upper-Scorza–Dragoni if, given $\delta > 0$, there exists a closed subset $A_\delta \subset [0,a]$ such that the measure $\mu([0,a] \setminus A_\delta) \leq \delta$ and the restriction ϕ_δ of ϕ to $A_\delta \times E$ is *u.s.c.*

Theorem 2.6 (see [22, Theorem 19.19])**.** *Let E, E_1 be two separable Banach spaces and let $F : [a,b] \times E \to \mathcal{P}_{cp,cv}(E_1)$ be an upper-Scorza–Dragoni map. Then F is σ-Ca-selectionable, the maps $F_n : [a,b] \times E \to \mathcal{P}(E_1)$, $n \in \mathbb{N}$, are almost upper semicontinuous, and we have*

$$F_n(t,e) \subset \overline{co}\Big(\bigcup_{x\in E} F(t,x)\Big).$$

Moreover, if F is integrably bounded, then F is σ-mLL-selectionable.

Lemma 2.5 ([9])**.** *For an u.s.c. multifunction $F : X \to \mathcal{P}_{cp}(Y)$, we have*

$$\forall\, x_0 \in X, \ \ \lim_{x\to x_0} \sup F(x) \subseteq F(x_0).$$

Lemma 2.6 ([9])**.** *Let $(K_n)_n \subset K$ such that K is a compact subset of X, and X is a separable Banach space. Then*

$$\overline{co}\Big(\lim_{n\to\infty} \sup K_n\Big) = \bigcap_{N>0} \overline{co}\Big(\bigcup_{n\geq N} K_n\Big),$$

where co is the convex envelope.

Lemma 2.7 ([21])**.** *Let X be a metric compact space. If X is R_δ-set, then X is an acyclic space.*

Theorem 2.7 ([22])**.** *Let E be a normed space, X be a metric space, and let $f : X \to E$ be a continuous map. Then $\forall\, \varepsilon > 0$ there is a locally Lipschitz function $f_\varepsilon : X \to E$ such that*

$$\|f(x) - f_\varepsilon(x)\| \leq \varepsilon, \ \ \forall\, x \in X. \tag{2.2}$$

Theorem 2.8 (Theorem of Browder and Gupta, [14])**.** *Let $(E, \|\cdot\|)$ be a Banach space, $f : X \to E$ be a proper map, and suppose that for every $\varepsilon > 0$, we have a proper map $f_\varepsilon : X \to E$ satisfying:*

(i) *$\|f_\varepsilon(x) - f(x)\| < \varepsilon$ for all $x \in X$;*

(ii) *for all $u \in E$ such that $\|u\| \leq \varepsilon$, the equation $f_\varepsilon(x) = u$ has a unique solution.*

Then the set $S = f^{-1}(0)$ is R_δ.

3 Existence results

Let $J := [0,1]$. In order to define a solution for problem (1.1), consider the space $PC(J,\mathbb{R})\times PC(J,\mathbb{R})$, where

$$PC(J,\mathbb{R}) := \Big\{ y:\ \ J\to\mathbb{R},\ \ y\in C(J\setminus\{t_k\},\mathbb{R}):\ \ k=1,\dots,m,\\ y(t_k^-)\ \text{ and }\ y(t_k^+)\ \text{ exist and satisfy }\ y(t_k^-)=y(t_k)\Big\}.$$

Endowed with the norm

$$\|y\|_{PC} = \sup\big\{\|y(t)\|:\ \ t\in J\big\},$$

PC is a Banach space.

3.1 Convex case

Theorem 3.1. *Assume there exist a continuous nondecreasing map $\psi : [0,+\infty)\to(0,+\infty)$ and $p\in L^1(J,\mathbb{R}_+)$ such that*

$$\|F_i(t,u,v)\| \le p(t)\psi\big(\|u\|+\|v\|\big)\ \ a.e.\ \ t\in J,\ \ i\in\{1,2\},\ \ (u,v)\in\mathbb{R}^2.$$

Assume also that $F_1,F_2 : J\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathéodory. Then problem (1.1) *has at least one solution.*

Proof. Consider the operator $N : PC\times PC\to\mathcal{P}(PC\times PC)$ defined by

$$N(x,y)=\left\{(h_1,h_2)\in PC\times PC:\ \ \begin{pmatrix}h_1(t)\\ h_2(t)\end{pmatrix}=\begin{pmatrix} x_0+\int_0^t f_1(s)\,ds+\sum_{0<t_k<t} I_1(x(t_k),y(t_k)),\ \ t\in J\\ y_0+\int_0^t f_2(s)\,ds+\sum_{0<t_k<t} I_2(x(t_k),y(t_k)),\ \ t\in J\end{pmatrix}\right\},$$

where $f_i\in S_{F_i}=\{f\in L^1(J,\mathbb{R}) : f(t)\in F_i(t,x(t),y(t)),\ \text{a.e. } t\in J\}$. Fixed points of the operator N are the solutions of problem (1.1).

We are going to prove that N is *u.s.c.* compact and that N has convex compact values. The proof is given by the following steps.

Step 1. $N(x,y)$ is convex for all $(x,y)\in PC\times PC$.

Let $(h_1,h_2),(h_3,h_4)\in N(x,y)$. So, there exist $f_1,f_3\in S_{F_1(\,\cdot\,,x(\,\cdot\,),y(\,\cdot\,))}$ and $f_2,f_4\in S_{F_2(\,\cdot\,,x(\,\cdot\,),y(\,\cdot\,))}$ such that for all $t\in J$, we have

$$h_1(t)=x_0+\int_0^t f_1(s)\,ds+\sum_{0<t_k<t} I_1(x(t_k),y(t_k)),$$

$$h_2(t)=y_0+\int_0^t f_2(s)\,ds+\sum_{0<t_k<t} I_2(x(t_k),y(t_k))$$

and

$$h_3(t)=x_0+\int_0^t f_3(s)\,ds+\sum_{0<t_k<t} I_1(x(t_k),y(t_k)),$$

$$h_4(t)=y_0+\int_0^t f_4(s)\,ds+\sum_{0<t_k<t} I_2(x(t_k),y(t_k)).$$

Let $l \in [0,1]$. For each $t \in J$, we have

$$\left(l \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (1-l) \begin{pmatrix} h_3 \\ h_4 \end{pmatrix} \right)(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \int\limits_0^t (lf_1 + (1-l)f_3)(s)\, ds \\ \int\limits_0^t (lf_2 + (1-l)f_4)(s)\, ds \end{pmatrix} + \begin{pmatrix} \sum\limits_{0<t_k<t} I_1(x(t_k), y(t_k)) \\ \sum\limits_{0<t_k<t} I_2(x(t_k), y(t_k)) \end{pmatrix}.$$

As S_{F_1} and S_{F_2} are convex (since F_1 and F_2 have convex values),

$$l \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (1-l) \begin{pmatrix} h_3 \\ h_4 \end{pmatrix} \in N(x,y).$$

Step 2. N transforms every bounded set to a bounded set in $PC \times PC$.
It suffices to show that

$$\exists\, \ell := \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} > 0 \ \text{ such that}$$
$$\forall\, (x,y) \in \mathcal{B}_q := \left\{ (x,y) \in PC \times PC : \ \|(x,y)\|_{PC\times PC} \le q, \ \ q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} > 0 \right\},$$
$$\text{if } \ (h,g) \in N(x,y), \ \text{ then we have } \ \|(h,g)\|_{PC\times PC} \le \ell.$$

Let $(h,g) \in N(x,y)$, then there exist $f_1 \in S_{F_1(\,\cdot\,,x(\,\cdot\,),y(\,\cdot\,))}$ and $f_2 \in S_{F_2(\,\cdot\,,x(\,\cdot\,),y(\,\cdot\,))}$ such that for all $t \in J$,

$$h(t) = x_0 + \int\limits_0^t f_1(s)\, ds + \sum_{0<t_k<t} I_1(x(t_k), y(t_k)),$$
$$g(t) = y_0 + \int\limits_0^t f_2(s)\, ds + \sum_{0<t_k<t} I_2(x(t_k), y(t_k)),$$
$$\|(h,g)\|_{PC\times PC} = \begin{pmatrix} \|h\|_{PC} \\ \|g\|_{PC} \end{pmatrix}.$$

For all $t \in J$, we have

$$\begin{aligned} \|h(t)\| &\le \|x_0\| + \int\limits_0^t \|f_1(s)\|\, ds + \sum_{0<t_k<t} \big\|I_1(x(t_k), y(t_k))\big\| \\ &\le \|x_0\| + \int\limits_0^1 \big\|F_1(s, x(s), y(s))\big\|\, ds + \sum_{k=1}^m \sup_{(x,y)\in\mathcal{B}_q} \|I_1(x,y)\| \\ &\le \|x_0\| + \psi(q_1+q_2)\|p\|_{L^1} + \sum_{k=1}^m \sup_{(x,y)\in\mathcal{B}_q} \|I_1(x,y)\| := \widetilde{\ell} \end{aligned}$$

and

$$\begin{aligned} \|g(t)\| &\le \|y_0\| + \int\limits_0^t \|f_2(s)\|\, ds + \sum_{0<t_k<t} \big\|I_2(x(t_k), y(t_k))\big\| \\ &\le \|y_0\| + \int\limits_0^b \|F_2(s, x(s), y(s))\|\, ds + \sum_{k=1}^m \sup_{(x,y)\in\mathcal{B}_q} \|I_2(x,y)\| \end{aligned}$$

$$\leq \|y_0\| + \psi(q_1+q_2)\|p\|_{L^1} + \sum_{k=1}^{m} \sup_{(x,y)\in\mathcal{B}_q} \|I_2(x,y)\| := \widetilde{\widetilde{\ell}}.$$

Then

$$\begin{pmatrix} \|h\|_{PC} \\ \|g\|_{PC} \end{pmatrix} \leq \begin{pmatrix} \widetilde{\ell} \\ \widetilde{\widetilde{\ell}} \end{pmatrix} := \ell.$$

Step 3. N transforms every bounded set to an equicontinuous set in $PC \times PC$.

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and let $\mathcal{B}_q$ be as above in Step 2. For each $(x,y) \in \mathcal{B}_q$ and $(h,g) \in N(x,y)$, there exist $f_1 \in S_{F_1(\cdot, x(\cdot), y(\cdot))}$ and $f_2 \in S_{F_2(\cdot, x(\cdot), y(\cdot))}$ such that for all $t \in J$, we have

$$h(t) = x_0 + \int_0^t f_1(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)),$$

$$g(t) = y_0 + \int_0^t f_2(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)).$$

Then

$$\begin{aligned} \|h(\tau_2) - h(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2} \|f_1(s)\|\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \big\|I_{1,k}(x(t_k), y(t_k))\big\| \\ &\leq \psi(q_1+q_2) \int_{\tau_1}^{\tau_2} p(s)\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in\mathcal{B}_q} \|I_{1,k}(x,y)\| \longrightarrow 0 \ \text{ as } \ \tau_2 \to \tau_1 \end{aligned}$$

and

$$\begin{aligned} \|g(\tau_2) - g(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2} \|f_2(s)\|\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \big\|I_{2,k}(x(t_k), y(t_k))\big\| \\ &\leq \psi(q_1+q_2) \int_{\tau_1}^{\tau_2} p(s)\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in\mathcal{B}_q} \|I_{2,k}(x,y)\| \longrightarrow 0 \ \text{ as } \ \tau_2 \to \tau_1. \end{aligned}$$

So, by Step 2 and Step 3, N is compact.

Step 4. The graph of N is closed.

Let $(x_n, y_n) \to (x_*, y_*)$, $(h_n, g_n) \in N(x_n, y_n)$, and $h_n \to h_*$ and $g_n \to g_*$. It suffices to show that there exist $f_1 \in S_{F_1(\cdot, x_*(\cdot), y_*(\cdot))}$ and $f_2 \in S_{F_2(\cdot, x_*(\cdot), y_*(\cdot))}$ such that for all $t \in J$, we have

$$h_*(t) = x_0 + \int_0^t f_1(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x_*(t_k), y_*(t_k)),$$

$$g_*(t) = y_0 + \int_0^t f_2(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x_*(t_k), y_*(t_k)).$$

With $(h_n, g_n) \in N(x_n, y_n)$, there exist $f_{1,n} \in S_{F_1(\cdot, x_n(\cdot), y_n(\cdot))}$ and $f_{2,n} \in S_{F_2(\cdot, x_n(\cdot), y_n(\cdot))}$ such that for all $t \in J$,

$$h_n(t) = x_0 + \int_0^t f_{1,n}(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)),$$

$$g_n(t) = y_0 + \int_0^t f_{2,n}(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k)).$$

Since $I_{i,k}$, $k = 1, \dots, m$, $i = 1, 2$, are continuous,

$$\left\| \Big(h_n(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k))\Big) - \Big(h_*(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_*(t_k), y_*(t_k))\Big) \right\|_{PC} \longrightarrow 0$$

and

$$\left\| \Big(g_n(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k))\Big) - \Big(g_*(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_*(t_k), y_*(t_k))\Big) \right\|_{PC} \longrightarrow 0$$

as $n \to \infty$.

Let Γ be a continuous linear operator defined as

$$\Gamma : L^1(J, \mathbb{R}) \longrightarrow PC(J, \mathbb{R}),$$
$$r \longrightarrow \Gamma(r)$$

such that

$$\Gamma(r)(t) = \int_0^t r(s)\,ds, \quad \forall\, t \in J.$$

By Lemma 2.2, the operator $\Gamma \circ S_F$ has a closed graph and, moreover, we have

$$\Big(h_n(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k))\Big) \in \Gamma(S_{F_1(\,\cdot\,, x_n(\,\cdot\,), y_n(\,\cdot\,))})$$

and

$$\Big(g_n(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k))\Big) \in \Gamma(S_{F_2(\,\cdot\,, x_n(\,\cdot\,), y_n(\,\cdot\,))}).$$

So,

$$\Big(h_*(t) - x_0 - \sum_{0<t_k<t} I_{1,k}(x_*(t_k), y_*(t_k))\Big) = \int_0^t f_1(s)\,ds,$$
$$\Big(g_*(t) - y_0 - \sum_{0<t_k<t} I_{2,k}(x_*(t_k), y_*(t_k))\Big) = \int_0^t f_2(s)\,ds,$$

and then $f_1 \in S_{F_1(\,\cdot\,, x_*(\,\cdot\,), y_*(\,\cdot\,))}$ and $f_2 \in S_{F_2(\,\cdot\,, x_*(\,\cdot\,), y_*(\,\cdot\,))}$.

Step 5. A priori estimation.

Let $(x, y) \in PC(J, \mathbb{R})$ such that $(x, y) \in \lambda N(x, y)$, and $0 < \lambda < 1$. So, $\exists\, f_1 \in S_{F_1(\,\cdot\,, x(\,\cdot\,), y(\,\cdot\,))}$ and $\exists\, f_2 \in S_{F_2(\,\cdot\,, x(\,\cdot\,), y(\,\cdot\,))}$ such that for all $t \in [0, t_1]$,

$$x(t) = \lambda x_0 + \lambda \int_0^t f_1(s, x(s), y(s))\,ds,$$
$$y(t) = \lambda y_0 + \lambda \int_0^t f_2(s, x(s), y(s))\,ds.$$

Then

$$\|x(t)\| \le \|x_0\| + \int_0^t p(s)\psi\big(\|x(s)\| + \|y(s))\|\big)\,ds, \quad t \in [0, t_1],$$

$$\|y(t)\| \le \|y_0\| + \int_0^t p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\, ds, \quad t \in [0, t_1].$$

Consider the functions ϑ_1, $\mathcal{W}_1$ defined by

$$\vartheta_1(t) = \|x_0\| + \int_0^t p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\, ds, \quad t \in [0, t_1],$$

$$\mathcal{W}_1(t) = \|y_0\| + \int_0^t p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\, ds, \quad t \in [0, t_1].$$

So,

$$(\vartheta_1(0), \mathcal{W}_1(0)) = \big(\|x_0\|, \|y_0\|\big), \quad \|x(t)\| \le \vartheta_1(t), \quad \|y(t)\| \le \mathcal{W}_1(t), \quad t \in [0, t_1],$$

and

$$\dot{\mathcal{W}}_1(t) = \dot{\vartheta}_1(t) = p(t)\psi\big(\|(x(t)\| + \|y(t))\|\big), \quad t \in [0, t_1].$$

As ψ is a nondecreasing map, we have

$$\dot{\vartheta}_1(t) \le p(t)\psi(\vartheta_1(t)), \quad \dot{\mathcal{W}}_1(t) \le p(t)\psi(\mathcal{W}_1(t)), \quad t \in [0, t_1].$$

This implies that for every $t \in [0, t_1]$,

$$\int_{\vartheta_1(0)}^{\vartheta_1(t)} \frac{du}{\psi(u)} \le \int_0^{t_1} p(s)\, ds, \quad \int_{\mathcal{W}_1(0)}^{\mathcal{W}_1(t)} \frac{du}{\psi(u)} \le \int_0^{t_1} p(s)\, ds.$$

The maps $\Gamma_0^1(z) = \int_{\vartheta_1(0)}^{z} \frac{du}{\psi(u)}$ and $\Gamma_0^2(z) = \int_{\mathcal{W}_1(0)}^{z} \frac{du}{\psi(u)}$ are continuous and increasing. Then $(\Gamma_0^1)^{-1}$ and $(\Gamma_0^2)^{-1}$ exist and are increasing, and we get

$$\vartheta_1(t) \le (\Gamma_0^1)^{-1}\left(\int_0^{t_1} p(s)\, ds\right) := M_0, \quad \mathcal{W}_1(t) \le (\Gamma_0^2)^{-1}\left(\int_0^{t_1} p(s)\, ds\right) := \ell_0.$$

As for every $t \in [0, t_1]$, $\|x(t)\| \le \vartheta_1(t)$ and $\|y(t)\| \le \mathcal{W}_1(t)$, so,

$$\sup_{t\in[0,t_1]} \|y(t)\| \le \ell_0, \quad \sup_{t\in[0,t_1]} \|x(t)\| \le M_0.$$

Now, for $t \in (t_1, t_2]$, we have

$$\|x(t_1^+)\| \le \big\|I_{1,1}(x(t_1), y(t_1))\big\| + \|x(t_1)\| \le \sup_{(\alpha,\beta)\in\overline{B}(0,M_0)\times\overline{B}(0,\ell_0)} \|I_{1,1}(\alpha, \beta)\| + M_0 := N_1,$$

$$\|y(t_1^+)\| \le \big\|I_{2,1}(x(t_1), y(t_1))\big\| + \|y(t_1)\| \le \sup_{(\alpha,\beta)\in\overline{B}(0,M_0)\times\overline{B}(0,\ell_0)} \|I_{2,1}(\alpha, \beta)\| + \ell_0 := D_1.$$

Also,

$$x(t) = \lambda\big(x(t_1) + I_{1,1}(x(t_1), y(t_1))\big) + \lambda \int_{t_1}^t f_1(s, x(s), y(s))\, ds,$$

$$y(t) = \lambda\big(y(t_1) + I_{2,1}(x(t_1), y(t_1))\big) + \lambda \int_{t_1}^t f_2(s, x(s), y(s))\, ds,$$

and so,

$$\|x(t)\| \leq N_1 + \int_{t_1}^{t} p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\,ds, \ \ t \in [t_1, t_2],$$

$$\|y(t)\| \leq D_1 + \int_{t_1}^{t} p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\,ds, \ \ t \in [t_1, t_2].$$

Let us consider the maps ϑ_2 and $\mathcal{W}_2$ defined by

$$\vartheta_2(t) = N_1 + \int_{t_1}^{t} p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\,ds, \quad \mathcal{W}_2(t) = D_1 + \int_{t_1}^{t} p(s)\psi\big(\|(x(s)\| + \|y(s))\|\big)\,ds, \ \ t \in [t_1, t_2].$$

Then

$$\vartheta_2(t_1^+) = N_1, \quad \|x(t)\| \leq \vartheta_2(t), \ \ t \in [t_1, t_2],$$
$$\mathcal{W}_2(t_1^+) = D_1, \quad \|y(t)\| \leq \mathcal{W}_2(t), \ \ t \in [t_1, t_2],$$

and

$$\dot{\vartheta}_2(t) = p(t)\psi\big(\|(x(t)\| + \|y(t))\|\big), \quad \dot{\mathcal{W}}_2(t) = p(t)\psi\big(\|(x(t)\| + \|y(t))\|\big), \ \ t \in [t_1, t_2].$$

As ψ is nondecreasing,

$$\dot{\vartheta}_2(t) \leq p(t)\psi(\vartheta_2(t)), \quad \dot{\mathcal{W}}_2(t) \leq p(t)\psi(\mathcal{W}_2(t)), \ \ t \in [t_1, t_2].$$

This implies that for every $t \in [t_1, t_2]$,

$$\int_{\vartheta_2(t_1^+)}^{\vartheta_2(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)\,ds, \qquad \int_{\mathcal{W}_2(t_1^+)}^{\mathcal{W}_2(t)} \frac{du}{\psi(u)} \leq \int_{t_1}^{t_2} p(s)\,ds.$$

If we consider the maps $\Gamma_1^1(z) = \int_{\vartheta_2(t_1^+)}^{z} \frac{du}{\psi(u)}$ and $\Gamma_1^2(z) = \int_{\mathcal{W}_2(t_1^+)}^{z} \frac{du}{\psi(u)}$, we get

$$\vartheta_2(t) \leq (\Gamma_1^1)^{-1}\left(\int_{t_1}^{t_2} p(s)\,ds\right) := M_1,$$

$$\mathcal{W}_2(t) \leq (\Gamma_1^2)^{-1}\left(\int_{t_1}^{t_2} p(s)\,ds\right) := \ell_1.$$

For all $t \in [t_1, t_2]$, $\|x(t)\| \leq \vartheta_2(t)$ and $\|y(t)\| \leq \mathcal{W}_2(t)$, and then

$$\sup_{t\in[t_1,t_2]} \|x(t)\| \leq M_1, \qquad \sup_{t\in[t_1,t_2]} \|y(t)\| \leq \ell_1.$$

We continue the process to the interval $(t_m, 1]$. We get the existence of M_m and ℓ_m such that

$$\sup_{t\in[t_m,1]} \|x(t)\| \leq (\Gamma_m^1)^{-1}\left(\int_{t_m}^{1} p(s)\,ds\right) := M_m, \qquad \sup_{t\in[t_m,1]} \|y(t)\| \leq (\Gamma_m^2)^{-1}\left(\int_{t_m}^{1} p(s)\,ds\right) := \ell_m.$$

As we chose y arbitrarily, then for all solutions of problem (1.1), we get

$$\|(x,y)\|_{PC\times PC} \leq \max\left\{ \begin{pmatrix} M_k \\ \ell_k \end{pmatrix} : \ \ k = 0, 1, \ldots, m \right\} := b^*.$$

Then the set

$$\mathcal{A}=\Big\{(x,y)\in PC\times PC:\quad (x,y)\in \lambda N(x,y),\quad \lambda\in(0,1)\Big\}$$

is bounded. So, $N : PC\times PC \to \mathcal{P}_{cv}(PC\times PC)$ is compact and *u.s.c.* Then, by Lemma 2.3, we obtain that problem (1.1) has at least one solution. □

3.2 Nonconvex case

Assume that the following conditions hold:

$(\mathcal{H}_1)$ $F_i : J\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$, $t\to F_i(t,u,v)$ are measurable for each $u,v\in\mathbb{R}$, $i=1,2$.

$(\mathcal{H}_2)$ There exist the functions $l_i\in L^1(J,\mathbb{R}^+)$, $i=1,\dots,4$, such that

$$H_d\big(F_1(t,u,v),F_1(t,\overline{u},\overline{v})\big)\le l_1(t)\|u-\overline{u}\|+l_2(t)\|v-\overline{v}\|,\quad t\in J,\quad \forall\, u,\overline{u},v,\overline{v}\in\mathbb{R},$$
$$H_d\big(F_2(t,u,v),F_2(t,\overline{u},\overline{v})\big)\le l_3(t)\|u-\overline{u}\|+l_4(t)\|v-\overline{v}\|,\quad t\in J,\quad \forall\, u,\overline{u},v,\overline{v}\in\mathbb{R}$$

and

$$H_d(0,F_1(t,0,0))\le l_1(t)\ \text{ for a.e. } t\in J,\qquad H_d(0,F_2(t,0,0))\le l_3(t)\ \text{ for a.e. } t\in J.$$

$(\mathcal{H}_3)$ There exist the constants $a_i,b_i\ge 0$, $i=1,2$, such that

$$\big\|I_1(u,v)-I_1(\overline{u}-\overline{v}\big\|\le a_1\|u-\overline{u}\|+a_2\|v-\overline{v}\|,\quad \forall\, u,\overline{u},v,\overline{v}\in\mathbb{R}$$

and

$$\big\|I_2(u,v)-I_2(\overline{u}-\overline{v}\big\|\le b_1\|u-\overline{u}\|+b_2\|v-\overline{v}\|,\quad \forall\, u,\overline{u},v,\overline{v}\in\mathbb{R}.$$

Theorem 3.2. *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *are satisfied and the matrix*

$$M=\begin{pmatrix}\|l_1\|_{L^1}+a_1 & \|l_2\|_{L^1}+a_2\\ \|l_3\|_{L^1}+b_1 & \|l_4\|_{L^1}+b_2\end{pmatrix}$$

converges to zero. Then problem (1.1) *has at least one solution.*

Proof. Consider the operator $N : PC\times PC\to\mathcal{P}(PC\times PC)$ defined by

$$N(x,y)=\left\{(h_1,h_2)\in PC\times PC:\quad \begin{pmatrix}h_1(t)\\ h_2(t)\end{pmatrix}=\begin{pmatrix} x_0+\displaystyle\int_0^t f_1(s)\,ds+\sum_{0<t_k<t} I_1(x(t_k),y(t_k)),\quad t\in J\\ y_0+\displaystyle\int_0^t f_2(s)\,ds+\sum_{0<t_k<t} I_2(x(t_k),y(t_k)),\quad t\in J\end{pmatrix}\right\},$$

where

$$f_i\in S_{F_i}=\Big\{f\in L^1(J,\mathbb{R}):\quad f(t)\in F_i(t,x(t),y(t)),\ \text{ a.e. } t\in J\Big\}.$$

Fixed points of the operator N are the solutions of problem (1.1).

Let, for $i=1,2$,

$$N_i(x,y)=\Big\{h\in PC:\quad h(t)=x_i(t)+\int_0^t f_i(s)\,ds+\sum_{0<t_k<t} I_i(x(t_k),y(t_k)),\quad t\in J\Big\},$$

where $x_1=x_0$ and $x_2=y_0$. We show that N satisfies the assumptions of Theorem 2.3.

Let $(x,y),(\overline{x},\overline{y})\in PC\times PC$ and $(h_1,h_2)\in N(x,y)$. Then there exist $f_i\in S_{F_i}, i=1,2,,$ such that

$$\begin{pmatrix} h_1(t)\\ h_2(t)\end{pmatrix}=\begin{pmatrix} x_0+\int\limits_0^t f_1(s)\,ds+\sum\limits_{0<t_k<t} I_1(x(t_k),y(t_k)), & t\in J\\ y_0+\int\limits_0^t f_2(s)\,ds+\sum\limits_{0<t_k<t} I_2(x(t_k),y(t_k)), & t\in J\end{pmatrix}.$$

$(\mathcal{H}_2)$ implies that

$$H_{d_1}\big(F_1(t,x(t),y(t)),F_1(t,\overline{x}(t),\overline{y}(t))\big)\le l_1(t)|x(t)-\overline{x}(t)|+l_2(t)|y(t)-\overline{y}(t)|,\ \ t\in J,$$

and

$$H_{d_2}\big(F_2(t,x(t),y(t)),F_2(t,\overline{x}(t),\overline{y}(t))\big)\le l_3(t)|x(t)-\overline{x}(t)|+l_4(t)|y(t)-\overline{y}(t)|,\ \ t\in J.$$

Hence, there is some $(\omega,\overline{\omega})\in F_1(t,\overline{x}(t),\overline{y}(t))\times F_2(t,\overline{x}(t),\overline{y}(t))$ such that

$$|f_1(t)-\omega|\le l_1(t)|x(t)-\overline{x}(t)|+l_2(t)|y(t)-\overline{y}(t)|,\ \ t\in J,$$

and

$$|f_2(t)-\overline{\omega}|\le l_3(t)|x(t)-\overline{x}(t)|+l_4(t)|y(t)-\overline{y}(t)|,\ \ t\in J.$$

Consider the multi-valued maps $U_i: J\to\mathcal{P}(\mathbb{R})$, $i=1,2$, defined by

$$U_1(t)=\Big\{\omega\in F_1(t,\overline{x}(t),\overline{y}(t)):\ \ |f_1(t)-\omega|\le l_1(t)|x(t)-\overline{x}(t)|+l_2(t)|y(t)-\overline{y}(t)|,\ \text{ a.e. } t\in J\Big\}$$

and

$$U_2(t)=\Big\{\omega\in F_2(t,\overline{x}(t),\overline{y}(t)):\ \ |f_1(t)-\omega|\le l_1(t)|x(t)-\overline{x}(t)|+l_2(t)|y(t)-\overline{y}(t)|,\ \text{ a.e. } t\in J\Big\}.$$

Then each $U_i(t)$ is a nonempty set and Theorem III.4.1 in [15] implies that U_i is measurable. Moreover, the multi-valued intersection operator $V_i(\cdot):=U_i(\cdot)\cap F_i(\cdot,\overline{x}(\cdot),\overline{y}(\cdot))$ is measurable. Therefore, for each $i=1,2$, by Lemma 2.1, there exists a function $t\to\overline{f}_i(t)$, which is a measurable selection for V_i, that is, $\overline{f}_i(t)\in F_i(t,\overline{x}(t),\overline{y}(t))$ and

$$|f_1(t)-\overline{f}_1(t)|\le l_1(t)|x(t)-\overline{x}(t)|+l_2(t)|y(t)-\overline{y}(t)|,\ \text{ a.e. } t\in J,$$

and

$$|f_2(t)-\overline{f}_2(t)|\le l_3(t)|x(t)-\overline{x}(t)|+l_4(t)|y(t)-\overline{y}(t)|,\ \text{ a.e. } t\in J.$$

Define $\overline{h}_1$ and $\overline{h}_2$ by

$$\overline{h}_1(t)=x_0+\int\limits_0^t\overline{f}_1(s)\,ds+\sum_{0<t_k<t}I_1(\overline{x}(t_k),\overline{y}(t_k)),\ \ t\in J,$$

and

$$\overline{h}_2(t)=y_0+\int\limits_0^t\overline{f}_2(s)\,ds+\sum_{0<t_k<t}I_2(\overline{x}(t_k),\overline{y}(t_k)),\ \ t\in J.$$

Then for $t\in J$,

$$|h_1(t)-\overline{h}_1(t)|\le\big(\|l_1\|_{L^1}+a_1\big)|x-\overline{x}|_{PC}+\big(\|l_2\|_{L^1}+a_2\big)\|y-\overline{y}\|_{PC}.$$

Thus

$$\|h_1-\overline{h}_1\|_{PC}\le\big(\|l_1\|_{L^1}+a_1\big)|x-\overline{x}|_{PC}+\big(\|l_2\|_{L^1}+a_2\big)\|y-\overline{y}\|_{PC}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at the estimate

$$H_{d_1}\big(N_1(x,y),N_1(\overline{x},\overline{y})\big) \le \big(\|l_1\|_{L^1}+a_1\big)\|x-\overline{x}\|_{PC}+\big(\|l_2\|_{L^1}+a_2\big)\|y-\overline{y}\|_{PC}.$$

Similarly, we get

$$H_{d_2}\big(N_2(x,y),N_2(\overline{x},\overline{y})\big) \le \big(\|l_3\|_{L^1}+b_1\big)\|x-\overline{x}\|_{PC}+\big(\|l_4\|_{L^1}+b_2\big)\|y-\overline{y}\|_{PC}.$$

Therefore,

$$H_d\big(N(x,y),N(\overline{x},\overline{y})\big) \le M\big(\|x-\overline{x}\|_{PC},\|y-\overline{y}\|_{PC}\big),\ \ \forall\,(x,y),(\overline{x},\overline{y})\in PC\times PC.$$

Hence, by Theorem 2.3, the operator N has at least one fixed point which is a solution of (1.1). $\square$

Theorem 3.3. *Assume, for each $i=1,2$, that there exist a continuous nondecreasing map $\psi_i : [0,+\infty[\to(0,+\infty)$ and $p_i\in L^1(J,\mathbb{R}_+)$ such that*

$$\|F_i(t,u,v)\| \le p_i(t)\psi_i\big(\|u\|+\|v\|\big)\ \ a.e.\ \ t\in J,\ \ (u,v)\in\mathbb{R}^2.$$

Assume also that $F_1,F_2 : J\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathéodory, and

(a) $(t,x,y)\to F_i(t,x,y)$ *is $\mathcal{L}\otimes\mathcal{B}$ measurable for $i=1,2$.*

(b) $(x,y)\to F_i(t,x,y)$ *is l.s.c. a.e. $t\in J$.*

Then problem (1.1) *has at least one solution.*

Proof. For each $i=1,2$, since F_i is *l.s.c.*, by Theorem 2.5, there exists a continuous function $f_i : PC\to L^1(J,\mathbb{R})$ such that $f_i(x,y)\in S_{F_i(\,\cdot\,,x,y)}$ for all $(x,y)\in PC(J,\mathbb{R})\times PC(J,\mathbb{R})$. Consider the impulsive system

$$\begin{cases} x'(t)=f_1(t,x,y), & \text{a.e. } t\in J,\\ y'(t)=f_2(t,x,y), & \text{a.e. } t\in J,\\ x(t_k^+)-x(t_k^-)=I_1(x(t_k),y(t_k)), & k=1,2,\dots,m,\\ y(t_k^+)-y(t_k^-)=I_2(x(t_k),y(t_k)), & k=1,2,\dots,m,\\ x(0)=x_0,\quad y(0)=y_0. \end{cases} \tag{3.1}$$

It is clear that if (x,y) is a solution of problem (3.1), then (x,y) is also a solution of problem (1.1). When the proof of Theorem 3.1 is applied to the operator $N_* : PC\times PC\to\mathcal{P}(PC\times PC)$ defined by

$$N_*(x,y)=\left\{(h_1,h_2)\in PC\times PC:\ \ \begin{pmatrix}h_1(t)\\ h_2(t)\end{pmatrix}=\begin{pmatrix} x_0+\int_0^t f_1(s)\,ds+\sum_{0<t_k<t} I_1(x(t_k),y(t_k)),\ \ t\in J\\ y_0+\int_0^t f_2(s)\,ds+\sum_{0<t_k<t} I_2(x(t_k),y(t_k)),\ \ t\in J\end{pmatrix}\right\},$$

there is a solution of problem (1.1). $\square$

4 Structure of solutions sets

Consider the first-order impulsive single-valued problem

$$\begin{cases} x'(t)=f_1(t,x(t),y(t)), & \text{a.e. } t\in[0,1],\\ y'(t)=f_2(t,x(t),y(t)), & \text{a.e. } t\in[0,1],\\ x(t_k^+)-x(t_k^-)=I_1(x(t_k),y(t_k)), & k=1,\dots,m,\\ y(t_k^+)-y(t_k^-)=I_2(x(t_k),y(t_k)), & k=1,\dots,m,\\ x(0)=x_0,\quad y(0)=y_0, \end{cases} \tag{4.1}$$

where $f_1, f_2 \in L^1(J \times \mathbb{R}^2, \mathbb{R})$ are te given functions and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. Then (x, y) is a solution of (4.1) if and only if (x, y) is a solution of the impulsive integral system

$$\begin{cases} x(t) = x_0 + \int\limits_0^t f_1(s, x(s), y(s))\,ds + \sum\limits_{0<t_k<t} I_1(x(t_k), y(t_k)), \ \text{ a.e. } t \in J, \\ y(t) = y_0 + \int\limits_0^t f_2(s, x(s), y(s))\,ds + \sum\limits_{0<t_k<t} I_2(x(t_k), y(t_k)), \ \text{ a.e. } t \in J. \end{cases} \tag{4.2}$$

Denote by $S(f_{1,2}, (x_0, y_0))$ the set of all solutions of problem (4.1).

Theorem 4.1. *Suppose that there are the functions $\ell_i \in L^1(J, \mathbb{R}_+)$, $i = 1, 2$, such that*

$$\big|f_i(t, x_1, y_1) - f_i(t, x_2, y_2)\big| < \ell_i(t)\big(|x_1 - x_2| + |y_1 - y_2|\big), \ \ \forall\,(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2.$$

Then problem (4.1) *has a unique solution.*

Proof.
1. The existence:

- We consider problem (4.1) on $[0, t_1]$,

$$\begin{gathered} x'(t) = f_1(t, x(t), y(t)), \quad y'(t) = f_2(t, x(t), y(t)), \ \text{ a.e. } t \in [0, t_1], \\ x(0) = x_0, \quad y(0) = y_0. \end{gathered} \tag{4.3}$$

We consider the operator N_1 defined by

$$\begin{gathered} N_1 : C([0, t_1], \mathbb{R}) \times C([0, t_1], \mathbb{R}) \longrightarrow C([0, t_1], \mathbb{R}) \times C([0, t_1], \mathbb{R}), \\ (x, y) \longrightarrow N_1(x, y), \end{gathered}$$

$$N_1(x, y)(t) = \bigg(x_0 + \int\limits_0^t f_1(s, x(s), y(s))\,ds; y_0 + \int\limits_0^t f_2(s, x(s), y(s))\,ds\bigg), \ \ t \in [0, t_1].$$

Let $(x_1, y_1), (x_2, y_2) \in C([0, t_1], \mathbb{R}) \times C([0, t_1], \mathbb{R})$, $t \in [0, t_1]$, and

$$\big\|N_1(x_1, y_1)(t) - N_1(x_2, y_2)(t)\big\| = \big\|(\alpha, \beta)\big\| = \begin{pmatrix} \|\alpha\| \\ \|\beta\| \end{pmatrix},$$

where

$$\alpha = \int\limits_0^t \big(f_1(s, x_1(s), y_1(s)) - f_1(s, x_2(s), y_2(s))\big)\,ds$$

and

$$\beta = \int\limits_0^t \big(f_2(s, x_1(s), y_1(s)) - f_2(s, x_2(s), y_2(s))\big)\,ds.$$

Then

$$\begin{aligned} \|\alpha\| &\le \int\limits_0^t \ell_1(s)\big\|(x_1(s), y_1(s)) - (x_2(s), y_2(s))\big\|\,ds \\ &\le \frac{1}{\tau}\int\limits_0^t \tau\ell(s)e^{\tau L(s)}\,ds\, \bigg\|\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\bigg\| |_{BC} \le \frac{1}{\tau}\,e^{\tau L(t)}\,\bigg\|\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\bigg\|_{BC} = \frac{1}{\tau}\,e^{\tau L(t)}\,\bigg\|\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\bigg\|_{BC} \\ &= \frac{1}{\tau}\,e^{\tau L(t)}\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big) = e^{\tau L(t)}\Big(\frac{1}{\tau}\,\|x_1 - x_2\| + \frac{1}{\tau}\,\|y_1 - y_2\|\Big), \end{aligned}$$

where

$$L(t) = \int_0^t \ell(s)\,ds, \quad \text{and} \quad \tau > 2.$$

Similarly,

$$\|\beta\| \le e^{\tau L(t)}\Big(\frac{1}{\tau}\,\|x_1 - x_2\| + \frac{1}{\tau}\,\|y_1 - y_2\|\Big).$$

Thus

$$e^{-\tau L(t)}\big\|N_1(x_1,y_1)(t) - N_1(x_2,y_2)(t)\big\| \le \begin{pmatrix} \frac{1}{\tau} & \frac{1}{\tau} \\ \frac{1}{\tau} & \frac{1}{\tau} \end{pmatrix} \begin{pmatrix} \|x_1 - x_2\| \\ \|y_1 - y_2\| \end{pmatrix}, \quad t \in [0, t_1].$$

Then

$$\big\|N_1(x_1,y_1) - N_1(x_2,y_2)\big\|_{BC} \le \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x_1 - x_2\| \\ \|y_1 - y_2\| \end{pmatrix},$$

where

$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{BC} = \sup_{t\in[0,t_1]} e^{-\tau L(t)}\left\|\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right\|.$$

Let

$$B = \frac{1}{\tau}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then we have

$$\det(B - \lambda I) = \Big(\frac{1}{\tau} - \lambda\Big)^2 - \frac{1}{\tau^2},$$

hence $\rho(B) = \frac{2}{\tau}$. For $\tau \in (2, +\infty)$, N_1 is contractive, so there exists a unique

$$(x^0, y^0) \in C([0,t_1],\mathbb{R}) \times C([0,t_1],\mathbb{R}) \ \text{ such that } \ N_1(x^0,y^0) = (x^0,y^0).$$

Then (x^0, y^0) is the solution of (4.3).

- We consider problem (4.1) on $(t_1, t_2]$,

$$\begin{aligned} &x'(t) = f_1(t,x(t),y(t)), \quad y'(t) = f_2(t,x(t),y(t)), \ \text{ a.e. } t \in J_1 = (t_1,t_2], \\ &x(t_1^+) = x^0(t_1) + I_1(x^0(t_1),y^0(t_1)), \quad y(t_1^+) = y^0(t_1) + I_1(x^0(t_1),y^0(t_1)). \end{aligned} \tag{4.4}$$

  Consider the space $C_* = \{(x,y) \in C(J_1,\mathbb{R}) \times C(J_1,\mathbb{R}) :\ (x(t_1^+), y(t_1^+)) \text{ exist}\}$, $(C_*, \|\cdot\|_{J_1})$ is a Banach space.

Let

$$\begin{aligned} N_2 : C_* &\longrightarrow C_*, \\ (x,y) &\longrightarrow N_2(x,y), \end{aligned}$$

$$N_2(x,y)(t) = \Big(x^0(t_1) + I_1(x^0(t_1),y^0(t_1)) + \int_{t_1}^t f_1(s,x(s),y(s))\,ds,$$
$$y^0(t_1) + I_2(x^0(t_1),y^0(t_1)) + \int_{t_1}^t f_2(s,x(s),y(s))\,ds\Big), \quad t \in (t_1,t_2].$$

Let $(x_1,y_1),(x_2,y_2) \in C_* \times C_*$, and $t \in (t_1,t_2]$,

$$\big\|N_2(x_1,y_1)(t) - N_2(x_2,y_2)(t)\big\| = \|(\alpha,\beta)\| = \begin{pmatrix} \|\alpha\| \\ \|\beta\| \end{pmatrix},$$

where

$$\|\alpha\| \le \int_{t_1}^{t} \ell(s)\big\|(x_1(s),y_1(s))-(x_2(s),y_2(s))\big\|\,ds \le \frac{1}{\tau}\int_{t_1}^{t} \tau\ell(s)e^{\tau L(s)}\,ds \left\|\begin{pmatrix} x_1-x_2\\ y_1-y_2\end{pmatrix}\right\|_{BC}$$
$$\le \frac{1}{\tau}\,e^{\tau L(t)}\left\|\begin{pmatrix} x_1-x_2\\ y_1-y_2\end{pmatrix}\right\|_{BC} = e^{\tau L(t)}\Big(\frac{1}{\tau}\,\|x_1-x_2\|+\frac{1}{\tau}\,\|y_1-y_2\|\Big)$$

and

$$L(t)=\int_{t_1}^{t}\ell(s)\,ds.$$

Similarly,

$$\|\beta\| \le e^{\tau L(t)}\Big(\frac{1}{\tau}\,\|x_1-x_2\|+\frac{1}{\tau}\,\|y_1-y_2\|\Big).$$

So,

$$e^{-\tau L(t)}\big\|N_2(x_1,y_1)(t)-N_2(x_2,y_2)(t)\big\| \le \frac{1}{\tau}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\|x_1-x_2\|\\ \|y_1-y_2\|\end{pmatrix}, \quad t\in(t_1,t_2].$$

Then

$$\big\|N_2(x_1,y_1)-N_2(x_2,y_2)\big\|_{BC} \le \frac{1}{\tau}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\|x_1-x_2\|\\ \|y_1-y_2\|\end{pmatrix}.$$

Then for $\tau\in(2,+\infty)$, N_2 is a contraction and, so, there exists a unique $(x^1,y^1)\in C((t_1,t_2],\mathbb{R})$ such that

$$N_2(x^1,y^1)=(x^1,y^1).$$

We have

$$(x^1,y^1)(t_1^+)=N_2(x^1,y^1)(t_1^+)=\bigg(x^0(t_1)+I_1(x^0(t_1),y^0(t_1))+\lim_{t\to t_1}\int_{t_1}^{t} f_1(s,x(s),y(s))\,ds,$$
$$y^0(t_1)+I_1(x^0(t_1),y^0(t_1))+\lim_{t\to t_1}\int_{t_1}^{t} f_2(s,x(s),y(s))\,ds\bigg).$$

Then (x^1,y^1) is the solution of problem (4.4). As a consequence, arguing inductively, the solution of problem (4.1) is given by

$$(x^*,y^*)(t):=\begin{cases}(x^0,y^0)(t), & t\in[0,t_1],\\ (x^1,y^1)(t), & t\in(t_1,t_2],\\ \quad\vdots & \\ (x^m,y^m)(t), & t\in(t_m,1].\end{cases}$$

2. The uniqueness:
   Let $(x^*,y^*),(x^{**},y^{**})$ be two solutions of problem (4.1). We are going to show that

$$(x^*,y^*)(t)=(x^{**},y^{**})(t), \ \ \forall\, t\in J=[0,1].$$

Again, the process is inductive.

If $t\in J_0=[0,t_1]$, then $(x^*,y^*)(t)=(x^{**},y^{**})(t)$, $\forall\, t\in[0,t_1]$.

Now, suppose that if $t\in J_i=(t_i,t_{i+1}]$, then $(x^*,y^*)(t)=(x^{**},y^{**})(t)$, $\forall\, t\in(t_i,t_{i+1}]$. It is enough to show that $(x^*,y^*)(t_k^+)=(x^{**},y^{**})(t_k^+)$, $k\in\{1,2,\dots,m\}$. To that end, we have

$$(x^*,y^*)(t_i^+)-(x^*,y^*)(t_i^-)=\big(I_{1i}(x^*(t_i),y^*(t_i)),I_{2i}(x^*(t_i),y^*(t_i))\big),$$

which implies that

$$(x^*, y^*)(t_i^+) = (x^*, y^*)(t_i^-) + I_{1i}(x^*(t_i), y^*(t_i))$$

and

$$I_{2i}(x^*(t_i), y^*(t_i)) = (x^{**}, y^{**})(t_i) + \big(I_{1i}(x^{**}(t_i), y^{**}(t_i)), I_{2i}(x^{**}(t_i), y^{**}(t_i))\big) = (x^{**}, y^{**})(t_i^+). \quad \square$$

Theorem 4.2. *Suppose there exist a continuous function $\psi : [0, \infty) \to (0, \infty)$ which is nondecreasing, and a function $p \in L^1(J, \mathbb{R}_+)$ such that*

$$\|f^i(t, x, y)\| \le p(t)\psi\big(\|x\| + \|y\|\big), \ \ \forall\, t \in J, \ \ \forall\, x, y \in \mathbb{R},$$

with

$$\int_0^1 p(s)\, ds < \int_{\|x_0\|}^{\infty} \frac{du}{\psi(u)}\, .$$

Then problem (4.1) *has at least one solution.*

Proof. For the proof we use "the nonlinear alternative of Leray–Schauder". Consider the operator

$$N : PC(J, \mathbb{R}) \times PC(J, \mathbb{R}) \longrightarrow PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$$

defined by

$$N(x, y)(t) = \Bigg(x_0 + \int_0^t f_1(s, x(s), y(s))\, ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)), \\ y_0 + \int_0^t f_2(s, x(s), y(s))\, ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k))\Bigg).$$

The fixed points of N are the solutions of problem (4.1). It is enough to prove that N is completely continuous. This is established in the following steps.

Step 1. N is continuous.

Let $(x_n, y_n)_n$ be a sequence in $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ such that $(x_n, y_n) \to (x, y)$. It is enough to prove that $N(x_n, y_n) \to N(x, y)$. For all $t \in J$, we have

$$N(x_n, y_n)(t) = \Bigg(x_0 + \int_0^t f_1(s, x_n(s), y_n(s))\, ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k), y_n(t_k)), \\ y_0 + \int_0^t f_2(s, x_n(s), y_n(s))\, ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k), y_n(t_k))\Bigg).$$

Then

$$\big\|N(x_n, y_n)(t) - N(x, y)(t)\big\| = \big\|(\alpha, \beta)\big\| = \begin{pmatrix} \|\alpha\| \\ \|\beta\| \end{pmatrix},$$

where

$$\|\alpha\| = \Bigg\| \int_0^t \big(f_1(s, x_n(s), y_n(s)) - f_1(s, x(s), y(s))\big)\, ds + \sum_{0<t_k<t} \big(I_{1,k}(x_n(t_k), y_n(t_k)) - I_{1,k}(x(t_k), y(t_k))\big) \Bigg\|$$

$$\le \int_0^t \big\|f_1(s, x_n(s), y_n(s)) - f_1(s, x(s), y(s))\big\|\, ds + \sum_{0<t_k<t} \big\|I_{1,k}(x_n(t_k), y_n(t_k)) - I_{1,k}(x(t_k), y(t_k))\big\|.$$

As I_k, $k = 1, \dots, m$, are continuous functions, and f^1 and f^2 are L^1-Carathéodory functions, by the Lebesgue dominated convergence theorem, we have

$$\|\alpha\| \le \int_0^b \big\|f_1(s, x_n(s), y_n(s)) - f_1(s, x(s), y(s))\big\|\, ds + \sum_{k=1}^m \big\|I_{1,k}(x_n(t_k), y_n(t_k)) - I_{1,k}(x(t_k), y(t_k))\big\| \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

Similarly,

$$\|\beta\| \le \int_0^b \big\|f_2(s, x_n(s), y_n(s)) - f_2(s, x(s), y(s))\big\|\, ds + \sum_{k=1}^m \big\|I_{2,k}(x_n(t_k), y_n(t_k)) - I_{2,k}(x(t_k), y(t_k))\big\| \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

So,

$$\big\|N(x_n, y_n) - N(x, y)\big\| \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \text{ as } \ n \to \infty.$$

Then N is continuous.

Step 2. N transforms every bounded set into a bounded set in $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$.

It suffices to show that

$$\forall\, q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} > 0, \ \ \exists\, \ell = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} > 0 \ \text{ such that } \ \forall\, (x, y) \in \mathcal{B}_q = \big\{(x, y) \in PC \times PC : \ \ \|(x, y)\| \le q\big\}, \ \text{ we have } \ \|N(x, y)\| \le \ell.$$

Let $(x, y) \in \mathcal{B}_q$. We have

$$\|N(x, y)\| \le \bigg(\|x_0\| + \int_0^b \|f_1(s, x(s), y(s))\|\, ds + \sum_{k=1}^m \|I_{1,k}(x(t_k), y(t_k))\|, \ \|y_0\| + \int_0^b \|f_2(s, x(s), y(s))\|\, ds + \sum_{k=1}^m \|I_{2,k}(x(t_k), y(t_k))\|\bigg) = (\alpha, \beta),$$

where

$$\begin{aligned}\|\alpha\| &\le \|x_0\| + \int_0^b p(t)\psi\big(\|x\|_{PC} + \|y\|_{PC}\big)\, dt + \sum_{k=1}^m \|I_{1,k}(x(t_k), y(t_k))\| \\ &\le \|x_0\| + \int_0^b p(t)\psi\big(\|x\|_{PC} + \|y\|_{PC}\big)\, dt + \sum_{k=1}^m \sup_{(x,y)\in\overline{B_q}} \|I_{1,k}(x, y)\| := \ell_1.\end{aligned}$$

Similarly,

$$\|\beta\| \le \|y_0\| + \int_0^b p(t)\psi\big(\|x\|_{PC} + \|y\|_{PC}\big)\, dt + \sum_{k=1}^m \sup_{(x,y)\in\overline{B_q}} \|I_{2,k}(x, y)\| := \ell_2.$$

Step 3. N transforms every bounded set into an equicontinuous set to $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$.

Let τ_1, $\tau_2 \in J$, $\tau_1 < \tau_2$ and let $\mathcal{B}_q$ be as in Step 2.

Let $(x, y) \in \mathcal{B}_q$. Then:

1. If $\tau_1 \neq t_k$ (or $\tau_2 \neq t_k$), $\forall\, k \in \{1, 2, \ldots, m\}$, we have

$$\big\|N(x,y)(\tau_2) - N(x,y)(\tau_1)\big\| \leq \Bigg(\int\limits_{\tau_1}^{\tau_2} p(s)\psi(q_1+q_2)\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in \overline{B_q}} \|I_{1,k}(x,y)\|,$$
$$\int\limits_{\tau_1}^{\tau_2} p(s)\psi(q_1+q_2)\,ds + \sum_{\tau_1 \leq t_k < \tau_2} \sup_{(x,y)\in \overline{B_q}} \|I_{2,k}(x,y)\| \Bigg) \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\text{ as }\; \tau_1 \to \tau_2.$$

2. If $\tau_1 = t_i^-$, we consider $\delta_1 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \varnothing$, so, for $0 < h < \delta_1$, we have

$$\big\|N(x,y)(t_i) - N(x,y)(t_i - h)\big\|$$
$$\leq \Bigg(\int\limits_{t_i - h}^{t_i} p(s)\psi(q_1+q_2)\,ds,\; \int\limits_{t_i - h}^{t_i} p(s)\psi(q_1+q_2)\,ds \Bigg) \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\text{ as }\; h \to 0.$$

3. If $\tau_2 = t_i^+$, we consider $\delta_2 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \varnothing$, so, for $0 < h < \delta_2$, we have

$$\big\|N(x,y)(t_i + h) - N(x,y)(t_i)\big\|$$
$$\leq \Bigg(\int\limits_{t_i}^{t_i + h} p(s)\psi(q_1+q_2)\,ds,\; \int\limits_{t_i}^{t_i + h} p(s)\psi(q_1+q_2)\,ds \Bigg) \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\text{ as }\; h \to 0.$$

So by Steps 1, 2 and 3, and by Arzelà-Ascoli's theorem, N is completely continuous.

Step 4. A Priori Estimates.

Let $(x,y) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ such that $(x,y) = \lambda N(x,y)$, and $0 < \lambda < 1$. Then for all $t \in [0, t_1]$, we have

$$x(t) = \lambda x_0 + \lambda \int\limits_0^t f_1(s, x(s), y(s))\,ds,$$
$$y(t) = \lambda y_0 + \lambda \int\limits_0^t f_2(s, x(s), y(s))\,ds,$$

and so,

$$\|(x,y)(t)\| \leq \Bigg(\|x_0\| + \int\limits_0^t p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds,\, \|y_0\| + \int\limits_0^t p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds \Bigg),\;\; t \in [0, t_1].$$

Consider the map $\vartheta = (\vartheta_1, \vartheta_2)$ such that

$$\vartheta_1(t) = \|x_0\| + \int\limits_0^t p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds,\;\; t \in [0, t_1],$$
$$\vartheta_2(t) = \|y_0\| + \int\limits_0^t p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\,ds,\;\; t \in [0, t_1].$$

Then we have

$$\vartheta(0) = \big(\|x_0\|, \|y_0\|\big), \quad \|(x,y)(t)\| \leq \vartheta(t), \ \ t \in [0, t_1],$$

and

$$\dot{\vartheta}_i(t) = p(t)\psi\big(\|x(s)\| + \|y(t)\|\big), \ \ \forall\, i = 1, 2, \ \ t \in [0, t_1].$$

As ψ is a nondecreasing map, we have

$$\dot{\vartheta}_i(t) \leq p(t)\psi(\vartheta_i(t)), \ \ \forall\, i = 1, 2, \ \ t \in [0, t_1],$$

which implies that for every $t \in [0, t_1]$,

$$\int\limits_{\vartheta_i(0)}^{\vartheta_i(t)} \frac{du}{\psi(u)} \leq \int\limits_0^{t_1} p(s)\, ds, \ \ \forall\, i = 1, 2.$$

The map $\Gamma_{i,0}(z) = \int\limits_{\vartheta_i(0)}^{z} \frac{du}{\psi(u)}$, $i = 1, 2$, is continuous and increasing. Then $\Gamma_{i,0}^{-1}$ exists and is increasing, and we get

$$\vartheta_i(t) \leq \Gamma_{i,0}^{-1}\Bigg(\int\limits_0^{t_1} p(s)\, ds\Bigg) := M_{i,0}, \ \ i = 1, 2.$$

As for all $t \in [0, t_1]$, $\|(x,y)(t)\| \leq \vartheta(t)$, and so,

$$\sup_{t \in [0, t_1]} \|(x,y)(t)\| \leq \begin{pmatrix} M_{1,0} \\ M_{2,0} \end{pmatrix}.$$

Now, for $t \in (t_1, t_2]$, we have

$$\|x(t_1^+)\| \leq \|I_{1,1}(x(t_1), y(t_1))\| + \|x(t_1)\| \leq \sup_{(x,y) \in \overline{B}_q} \|I_{1,1}(x,y)\| + M_{1,0} := N_1,$$
$$\|y(t_1^+)\| \leq \|I_{2,1}(x(t_1), y(t_1))\| + \|y(t_1)\| \leq \sup_{(x,y) \in \overline{B}_q} \|I_{2,1}(x,y)\| + M_{2,0} := N_2,$$

where

$$q = \begin{pmatrix} M_{1,0} \\ M_{2,0} \end{pmatrix},$$

$$y(t) = \lambda\big(x(t_1) + I_{1,1}(x(t_1), y(t_1))\big) + \lambda \int\limits_{t_1}^{t} f_1(s, x(s), y(s))\, ds,$$

$$y(t) = \lambda\big(y(t_1) + I_{2,1}(x(t_1), y(t_1))\big) + \lambda \int\limits_{t_1}^{t} f_2(s, x(s), y(s))\, ds.$$

Then

$$\|x(t)\| \leq N_1 + \int\limits_{t_1}^{t} p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\, ds, \ \ t \in [t_1, t_2],$$

$$\|y(t)\| \leq N_2 + \int\limits_{t_1}^{t} p(s)\psi\big(\|x(s)\| + \|y(s)\|\big)\, ds, \ \ t \in [t_1, t_2].$$

Consider the map $W=(W_1,W_2)$ such that

$$W_1(t)=N_1+\int_{t_1}^{t} p(s)\psi\big(\|x(s)\|+\|y(s)\|\big)\,ds,\ \ t\in[t_1,t_2],$$

$$W_2(t)=N_2+\int_{t_1}^{t} p(s)\psi\big(\|x(s)\|+\|y(s)\|\big)\,ds,\ \ t\in[t_1,t_2].$$

So,

$$W(t_1^+)=(N_1,N_2),\quad \|(x,y)(t)\|\le W(t),\ \ t\in[t_1,t_2],$$

and

$$\dot W_i(t)=p(t)\psi\big(\|x(s)\|+\|y(t)\|\big),\ \ \forall\, i=1,2,\ \ t\in[t_1,t_2].$$

Since ψ is nondecreasing, we get

$$\dot W_i(t)\le p(t)\psi(W_i(t)),\ \ \forall\, i=1,2,\ \ t\in[t_1,t_2],$$

what implies that for every $t\in[t_1,t_2]$, we have

$$\int_{W_i(t_1^+)}^{W_i(t)}\frac{du}{\psi(u)}\le\int_{t_1}^{t_2}p(s)\,ds,\ \ i=1,2.$$

If we consider the map $\Gamma_{i,1}(z)=\int\limits_{W_i(t_1^+)}^{z}\frac{du}{\psi(u)}$, $i=1,2$, we get

$$W_i(t)\le\Gamma_{i,1}^{-1}\Bigg(\int_{t_1}^{t_2}p(s)\,ds\Bigg):=M_{i,1},\ \ i=1,2.$$

For all $t\in[t_1,t_2]$,

$$\|(x,y)(t)\|=\begin{pmatrix}\|x(t)\|\\ \|y(t)\|\end{pmatrix}\le\begin{pmatrix}W_1(t)\\ W_2(t)\end{pmatrix},$$

so,

$$\sup_{t\in[t_1,t_2]}\|(x,y)(t)\|\le\begin{pmatrix}M_{1,1}\\ M_{2,1}\end{pmatrix}).$$

We continue this process to the interval $(t_m,1]$, and $(x,y)|_{(t_m,1]}$ is the solution of the problem $(x,y)=\lambda N(x,y)$ for $0<\lambda<1$. There exists $M_{i,m}$, $i=1,2$, such that

$$\sup_{t\in[t_m,b]}\|(x,y)(t)\|\le\Gamma_{i,m}^{-1}\Bigg(\int_{t_m}^{b}p(s)\,ds\Bigg):=M_{i,m}.$$

As we choose (x,y) arbitrarily, for all solution of problem (4.1) we have

$$\|(x,y)\|\le\begin{pmatrix}\max\limits_{k=0,1,\dots,m}(M_{1,k})\\ \max\limits_{k=0,1,\dots,m}(M_{2,k})\end{pmatrix}:=\begin{pmatrix}b_1^*\\ b_2^*\end{pmatrix}.$$

Thus, the set

$$\mathcal{K}=\Big\{(x,y)\in PC\times PC:\ \ (x,y)=\lambda N(x,y),\ \ \lambda\in(0,1)\Big\}.$$

Since $N:PC\times PC\to PC\times PC$ is completely continuous and the set $\mathcal{K}$ is bounded, from Theorem 2.4, N has a fixed point $(x,y)\in PC\times PC$ which is the solution of problem (4.1). $\square$

Theorem 4.3. *Suppose that the conditions of Theorem 4.2 hold. Then the set of all solutions of problem* (4.1) *is nonempty, compact,* R_δ*, and acyclic. Moreover, the solution operator* S *is u.s.c., where*

$$S:\mathbb{R}\times\mathbb{R}\longrightarrow \mathcal{P}_{cp}(PC\times PC),$$
$$(x_0,y_0)\longrightarrow S(x_0,y_0),$$
$$S(x_0,y_0)=\Big\{(x,y)\in PC\times PC:\ (x,y) \textit{ is a solution of problem } (4.1) \textit{ with } (x(0),y(0))=(x_0,y_0)\Big\}.$$

Proof.

- The solution set is compact.

Let $(a,b)\in\mathbb{R}\times\mathbb{R}$,

$$S(a,b)=\Big\{(x,y)\in PC\times PC:\ (x,y) \text{ is a solution of problem (4.1) with } (x(0),y(0))=(a,b)\Big\}.$$

1. $S(a,b)$ is a closed set.
Let $(x_q,y_q)_q$ be a sequence in $S(a,b)$ such that

$$\lim_{q\to\infty}(x_q,y_q)=(x,y).$$

Let

$$Z_1(t)=a+\int_0^t f_1(s,x(s),y(s))\,ds+\sum_{0<t_k<t} I_{1,k}(x(t_k),y(t_k)),\quad t\in[0,1],$$
$$Z_2(t)=b+\int_0^t f_2(s,x(s),y(s))\,ds+\sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k)),\quad t\in[0,1].$$

For $t\in[0,1]$, we have

$$\begin{aligned}
&\|x_q(t)-Z_1(t)\|\\
&\leq \int_0^t \big\|f_1(s,x_q(s),y_q(s))-f_1(s,x(s),y(s))\big\|\,ds+\sum_{0<t_k<t}\big\|I_{1,k}(x_q(t_k),y_q(t))-I_{1,k}(x(t_k),y(t_k))\big\|\\
&\quad\leq \int_0^1 \big\|f_1(s,x_q(s),y_q(s))-f_1(s,x(s),y(s))\big\|\,ds+\sum_{k=1}^m\big\|I_{1,k}(x_q(t_k),y_q(t))-I_{1,k}(x(t_k),y(t_k))\big\|.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\|x_q(t)-Z_1(t)\|\longrightarrow 0 \ \text{ as } \ q\to\infty.$$

Similarly,

$$\|y_q(t)-Z_2(t)\|\longrightarrow 0 \ \text{ as } \ q\to\infty.$$

So, $\lim_{q\to\infty}(x_q,y_q)=(x,y)=(Z_1,Z_2)\in S(a,b)$.

2. $S(a,b)$ is bounded uniformly.
Let $(x,y)\in S(a,b)$; then (x,y) is a solution of problem (4.1) and hence, $\exists\, b^*>0$ such that

$$\|(x,y)\|\leq (b^*,b^*).$$

3. $S(a,b)$ is equicontinuous.

Let $r_1, r_2 \in [0,1]$, $r_1 < r_2$ and $(x,y) \in S(a,b)$. Then

$$\|(x,y)(r_1)-(x,y)(r_2)\| \le \Big(\int_{r_1}^{r_2} \|f_1(s,x(s),y(s))\|\,ds + \sum_{r_1<t_k<r_2} \|I_{1,k}(x(t),y(t))\|,$$

$$\int_{r_1}^{r_2} \|f_2(s,x(s),y(s))\|\,ds + \sum_{r_1<t_k<r_2} \|I_{2,k}(x(t),y(t))\|\Big)$$

and

$$\int_{r_1}^{r_2} \|f_1(s,x(s),y(s))\|\,ds + \sum_{r_1<t_k<r_2} \|I_{1,k}(x(t),y(t))\|$$

$$\le \int_{r_1}^{r_2} p(s)\psi\big(\|x(s)\|+\|y(s)\|\big)\,ds + \sum_{r_1<t_k<r_2} \sup_{(x,y)\in \overline{B}_{b^*}} \|I_{1,k}(x,y)\|$$

$$\le \int_{r_1}^{r_2} p(s)\psi(b_1^*+b_2^*)\,ds + \sum_{r_1<t_k<r_2} \sup_{(x,y)\in \overline{B}_{b^*}} \|I_{1,k}(x,y)\| \longrightarrow 0 \text{ as } r_1 \to r_2.$$

Then $S(a,b)$ is compact.

- The solution set $S(a,b)$ is R_δ.

Let $N : PC\times PC \longrightarrow PC \times PC$ be defined by

$$N(x,y)(t) = \Big(a + \int_0^t f_1(s,x(s),y(s))\,ds + \sum_{0<t_k<t} I_{1,k}(x(t_k),y(t_k)),$$

$$b + \int_0^t f_2(s,x(s),y(s))\,ds + \sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k))\Big), \quad t \in J.$$

Then $\operatorname{Fix} N = S(a,b)$, and by Step 4 of the proof of Theorem 4.2, $\exists\, b^* > 0$ such that

$$\|(x,y)\| \le (b^*,b^*), \quad \forall\,(x,y) \in S(a,b).$$

For $i = 1,2$, we define

$$\widetilde{f}_i(t,y(t)) = \begin{cases} f_i(t,x(t),y(t)), & \text{if } \|(x,y)(t)\| \le (b^*,b^*),\\ f_i\Big(t, \dfrac{b^*x(t)}{\|x(t)\|}, \dfrac{b^*y(t)}{\|y(t)\|}\Big), & \text{if } \|(x,y)(t)\|_{PC\times PC} \ge (b^*,b^*) \end{cases}$$

and

$$\widetilde{I}_{i,k}(x(t),y(t)) = \begin{cases} I_{i,k}(x(t),y(t)) & \text{if } \|(x,y)(t)\| \le (b^*,b^*),\\ I_{i,k}\Big(\dfrac{b_1^*x(t)}{\|x(t)\|}, \dfrac{b_2^*y(t)}{\|y(t)\|}\Big) & \text{if } \|(x,y)(t)\| \ge (b^*,b^*). \end{cases}$$

Since the functions $f_i,\ i=1,2$, are L^1-Carathéodory, $\widetilde{f}^i$ are also L^1-Carathéodory, and $\exists\, h \in L^1(J,\mathbb{R}_+)$ such that

$$\|\widetilde{f}_i(t,x,y)\| \le h(t), \quad \forall i=1,2, \quad \text{a.e. } t \in J, \quad \text{and} \quad (x,y) \in \mathbb{R}\times\mathbb{R}. \tag{4.5}$$

Consider the problem

$$\begin{cases} \dot{x}(t) = \widetilde{f}_1(t, x(t), y(t)), & t \in [0,1], \\ \dot{y}(t) = \widetilde{f}_2(t, x(t), y(t)), & t \in [0,1], \\ x(t_k^+) - x(t_k^-) = \widetilde{I}_{1,k}(x(t_k), y(t_k^-)), & k = 1, 2, \dots, m, \\ y(t_k^+) - y(t_k^-) = \widetilde{I}_{2,k}(x(t_k), y(t_k^-)), & k = 1, 2, \dots, m, \\ x(0) = a, \quad y(0) = b. \end{cases}$$

We can easily prove that $\operatorname{Fix} N = \operatorname{Fix} \widetilde{N}$, where $\widetilde{N} : PC \times PC \to PC \times PC$ is defined by

$$\widetilde{N}(x,y)(t) = \Big(a + \int_0^t \widetilde{f}_i(s, x(s), y(s))\, ds + \sum_{0<t_k<t} \widetilde{I}_{1,k}(x(t_k), y(t_k)),$$
$$b + \int_0^t \widetilde{f}_2(s, x(s), y(s))\, ds + \sum_{0<t_k<t} \widetilde{I}_{2,k}(x(t_k), y(t_k))\Big), \ \ t \in J.$$

By inequalities (4.5) and the continuity of $I_{i,k}$, $i = 1, 2$, we get

$$\|\widetilde{N}(x,y)\| \le \Big(\|a\| + \|h\|_{L^1} + \sum_{k=1}^m \sup_{(x,y)\in \overline{B}_b^*} \|I_{1,k}(x,y)\|,$$
$$\|b\| + \|h\|_{L^1} + \sum_{k=1}^m \sup_{(x,y)\in \overline{B}_b^*} \|I_{2,k}(x,y)\| \Big) := (r_1, r_2) = r.$$

Then $\widetilde{N}$ is bounded uniformly.

We can easily prove that the function $\mathcal{M}$ defined by $\mathcal{M}(x,y) = (x,y) - \widetilde{N}(x,y)$ is well defined, and since $\widetilde{N}$ is compact, by the Lasota–Yorke theorem (Theorem 2.7), it is easy to prove that the conditions of Theorem 2.8 are satisfied. Then the set $\mathcal{M}^{-1}(0) = \operatorname{Fix} \widetilde{N} = S(a,b)$ is the R_δ-set and, by Lemma 2.7, it is also acyclic.

- The solution operator is *u.s.c.*

1. S has a closed graph.
To see this, first we note that the graph of S is the set

$$G_S = \Big\{ ((a,b), (x,y)) \in (\mathbb{R} \times \mathbb{R}) \times (PC \times PC) : \ \ (x,y) \in S(a,b) \Big\}.$$

Let $((a_q, b_q), (x_q, y_q))_q$ be a sequence in G_S, and let $((a_q, b_q), (x_q, y_q))_q \to ((a,b), (x,y))$ as $q \to \infty$. Since $(x_q, y_q) \in S(a_q, b_q)$, we have

$$x_q(t) = a_q + \int_0^t f_1(s, x_q(s), y_q(s))\, ds + \sum_{0<t_k<t} I_{1,k}(x_q(s), y_q(t_k)), \ \ t \in J,$$

$$y_q(t) = b_q + \int_0^t f_2(s, x_q(s), y_q(s))\, ds + \sum_{0<t_k<t} I_{2,k}(x_q(s), y_q(t_k)), \ \ t \in J.$$

Let

$$Z(t) = (Z_1(t), Z_2(t)) = \Big(a + \int_0^t f_1(s, x(s), y(s))\, ds + \sum_{0<t_k<t} I_{1,k}(x(s), y(t_k)),$$
$$b + \int_0^t f_2(s, x(s), y(s))\, ds + \sum_{0<t_k<t} I_{2,k}(x(s), y(t_k)) \Big), \ \ t \in J.$$

Let $t \in J$, then

$$\|(x_q, y_q)(t) - Z(t)\|$$
$$\leq \Bigg(\|a_q - a\| + \int_0^b \big\|f_1(s, x_q(s), y_q(s)) - f_1(s, x(s), y(s))\big\|\, ds + \sum_{k=1}^{m} \big\|I_{1,k}(x_q(t), y_q(t)) - I_{1,k}(x(t), y(t))\big\|,$$
$$\|b_q - b\| + \int_0^b \big\|f_2(s, x_q(s), y_q(s)) - f_2(s, x(s), y(s))\big\|\, ds + \sum_{k=1}^{m} \big\|I_{2,k}(x_q(t), y_q(t)) - I_{2,k}(x(t), y(t))\big\| \Bigg)$$

and, by the Lebesgue dominated convergence theorem, we have

$$\|(x_q, y_q)(t) - Z(t)\| \longrightarrow 0 \text{ as } q \to \infty.$$

Then

$$(x, y)(t) = Z(t),$$

which implies that $(x, y) \in S(a, b)$.

2. S transforms every bounded set into a relatively compact set.
Let $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} > 0$ and $\overline{B}_r := \{(x, y) \in PC \times PC : \ \|(x, y)\| \leq r\}$.

(a) $S(\overline{B}_r)$ is bounded uniformly.

Let $(x, y) \in S(\overline{B}_r)$, then there exists $(a, b) \in \overline{B}_r$ such that

$$x(t) = a + \int_0^t f_1(s, x(s), y(s))\, ds + \sum_{0<t_k<t} I_{1,k}(x(t_k), y(t_k)), \quad t \in J,$$

$$y(t) = b + \int_0^t f_2(s, x(s), y(s))\, ds + \sum_{0<t_k<t} I_{2,k}(x(t_k), y(t_k)), \quad t \in J.$$

By the same method detailed in Step 4 of the proof of Theorem 4.2, we find that there exists $b^* > 0$ such that

$$\|(x, y)\|_{PC\times PC} \leq (b^*, b^*).$$

(b) $S(\overline{B}_r)$is an equicontinuous set.

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and $(x, y) \in S(\overline{B}_r)$. Then

$$\|(x, y)(\tau_2) - (x, y)(\tau_1)\|$$
$$\leq \Bigg(\int_{\tau_1}^{\tau_2} \|f_1(s, x(s), y(s))\|\, ds + \sum_{\tau_1<t_k<\tau_2} \|I_{1,k}(x(t_k), y(t_k))\|,$$
$$\int_{\tau_1}^{\tau_2} \|f_2(s, x(s), y(s))\|\, ds + \sum_{\tau_1<t_k<\tau_2} \|I_{2,k}(x(t_k), y(t_k))\| \Bigg)$$
$$\leq \Bigg(\int_{\tau_1}^{\tau_2} p(s)\psi(\|x(s)\| + \|y(s)\|)\, ds + \sum_{\tau_1<t_k<\tau_2} \|I_{1,k}(x(t_k), y(t_k))\|,$$
$$\int_{\tau_1}^{\tau_2} p(s)\psi(\|x(s)\| + \|y(s)\|)\, ds + \sum_{\tau_1<t_k<\tau_2} \|I_{2,k}(x(t_k), y(t_k))\| \Bigg)$$

$$\leq \Bigg(\int_{\tau_1}^{\tau_2} p(s)\psi(b_1^* + b_2^*)\,ds + \sum_{\tau_1<t_k<\tau_2} \sup_{(x,y)\in\overline{B_{b^*}}} \|I_{1,k}(x,y)\|,$$
$$\int_{\tau_1}^{\tau_2} p(s)\psi(b_1^* + b_2^*)\,ds + \sum_{\tau_1<t_k<\tau_2} \sup_{(x,y)\in\overline{B_{b^*}}} \|I_{2,k}(x,y)\| \Bigg) \longrightarrow 0 \ \text{ as } \ \tau_1 \to \tau_2.$$

Thus the set $\overline{S(\overline{B}_r)}$ is compact.

The operator S is locally compact and has a closed graph, so, S is *u.s.c.* □

Theorem 4.4. *Assume that the conditions of Theorem 3.1 hold, where $F_1, F_2 : J\times\mathbb{R}\times\mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ are Carathédory, u.s.c. and* mLL*-sectionnable. Then the set of all solutions of problem* (1.1) *is contractible.*

Proof. Let $f^i \in S_{F_i}$ be a locally Lipschitzian measurable selection of F_i, $i = 1,2$. Let us consider the problem

$$\begin{cases} x'(t) = f_1(t,x(t),y(t)), & \text{a.e. } t\in J,\\ y'(t) = f_2(t,x(t),y(t)), & \text{a.e. } t\in J,\\ x(t_k^+) - x(t_k^-) = I_{1k}(x(t_k),y(t_k)), & k=1,\dots,m,\\ y(t_k^+) - y(t_k^-) = I_{2k}(x(t_k),y(t_k)), & k=1,\dots,m,\\ x(0)=x_0, \quad y(0)=y_0. \end{cases} \tag{4.6}$$

By Theorem 4.1, problem (4.6) has a unique solution.

Consider a homotopy function $h : S(x_0,y_0)\times[0,1] \to S(x_0,y_0)$ defined by

$$h((x,y),\alpha)(t) = \begin{cases} (x,y)(t) & \text{if } 0\le t\le\alpha,\\ (x^*,y^*)(t) & \text{if } \alpha<t\le 1, \end{cases}$$

where (x^*,y^*) is the solution of problem (4.6), and $S(x_0,y_0)$ is the set of all solutions of problem (1.1). In particular

$$h((x,y),\alpha) = \begin{cases} (x,y), & \text{if } \alpha=1,\\ (x^*,y^*), & \text{if } \alpha=0. \end{cases}$$

Thus to prove that $S(x_0,y_0)$ is contractible, it is enough to show that the homotopy h is continuous. Let $((x_n,y_n),\alpha_n) \in S(x_0,y_0)\times[0,1]$ be such that $((x_n,y_n),\alpha_n) \to ((x,y),\alpha)$ as $n\to\infty$. We have

$$h((x_n,y_n),\alpha_n)(t) = \begin{cases} (x_n,y_n)(t) & \text{if } 0\le t\le\alpha_n,\\ (x^*,y^*)(t) & \text{if } \alpha_n<t\le 1. \end{cases}$$

(a) If $\lim\limits_{n\to\infty} \alpha_n = 0$, then

$$h((x,y),0)(t) = (x^*,y^*)(t) \ \text{ for all } \ t\in J.$$

Thus

$$\big\|h((x_n,y_n),\alpha_n) - h((x,y),\alpha)\big\|_\infty \le \|(x_n,y_n)-(x^*,y^*)\|_{[0,\alpha_n]} \longrightarrow 0 \ \text{ as } \ n\to\infty.$$

(b) If $\lim\limits_{n\to\infty} \alpha_n = 1$, then

$$h((x,y),1)(t) = (x,y)(t) \ \text{ for all } \ t\in J.$$

Thus

$$\big\|h((x_n,y_n),\alpha_n) - h((x,y),\alpha)\big\|_\infty \le \|(x_n,y_n)-(x,y)\|_{[0,\alpha_n]} \longrightarrow 0 \ \text{ as } \ n\to\infty.$$

(c) If $0 < \lim_{n\to\infty}\alpha_n = \alpha < 1$, then we distinguish the following two cases.

(1) If $t \in [0,\alpha]$, we have $(x_n,y_n) \in S(x_0,y_0)$, thus there exists $(v_{1n},v_{2n}) \in S_{F_1} \times S_{F_2}$ such that for all $t \in [0,\alpha_n]$,

$$x_n(t) = x_0 + \int_0^t v_{1n}(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x_n(t_k),y_n(t_k)),$$

$$y_n(t) = y_0 + \int_0^t v_{2n}(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x_n(t_k),y_n(t_k)).$$

By Step 5 of the proof of Theorem 3.1, we have

$$\|(x_n,y_n)\|_{PC\times PC} \le b^* = \begin{pmatrix} b_1^* \\ b_2^* \end{pmatrix},$$

and, by hypothesis, we get

$$\|(v_{1n},v_{2n})(t)\| \le p(t)\psi(b_1^*+b_2^*)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for all } n\in\mathbb{N} \Longrightarrow (v_{1n},v_{2n})(t) \in p(t)\psi(b_1^*+b_2^*)\overline{B}(0,1).$$

The sequences $\{v_{1n}(\,\cdot\,),v_{2n}(\,\cdot\,)\}_{n\in\mathbb{N}}$ are integrably bounded. By the Dunford–Pettis theorem [52], there are subsequences, still denoted by $(v_{1n})_{n\in\mathbb{N}}$, $(v_{2n})_{n\in\mathbb{N}}$ which converge weakly to elements $v_1(\,\cdot\,) \in L^1$ and $v_2(\,\cdot\,) \in L^1$, respectively. Mazur's Lemma implies the existence of $\alpha_i^n \ge 0$, $i = n,\dots,k(n)$, such that $\sum\limits_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n^i(\,\cdot\,) = \sum\limits_{j=1}^{k(n)} \alpha_j^n v_{ij}(\,\cdot\,)$, $i=1,2$, converges strongly to v_i in L^1. Since F_1 and F_2 take convex values, using Lemma 2.6, we obtain

$$\begin{aligned} v_i(t) &\in \bigcap_{n\ge 1} \overline{\{g_n^i(t)\}}, \quad \text{a.e. } t\in J, \\ &\subset \bigcap_{n\ge1} \overline{co}\big\{v_{ik}(t),\ \ k\ge n\big\} \subset \bigcap_{n\ge1}\overline{co}\Big\{\bigcup_{k\ge n} F_i(t,x_k(t),y_k(t))\Big\} \\ &= \overline{co}\Big(\limsup_{k\to\infty} F_i(t,x_k(t),y_k(t))\Big). \end{aligned} \tag{4.7}$$

Since F is $u.s.c.$ with compact values, by Lemma 2.5, we have

$$\limsup_{n\to\infty} F_i(t,x_n(t),y_n(t)) \subseteq F_i(t,x(t),y(t)) \text{ for a.e. } t\in[0,\alpha].$$

This, together with (4.7), imply that

$$v_i(t) \in \overline{co}\,F_i(t,x(t),y(t)), \quad i=1,2.$$

Hence, for every $t\in[0,\alpha]$,

$$x(t) = x_0 + \int_0^t v_1(s)\,ds + \sum_{0<t_k<t} I_{1,k}(x(t_k),y(t_k))$$

and

$$y(t) = y_0 + \int_0^t v_2(s)\,ds + \sum_{0<t_k<t} I_{2,k}(x(t_k),y(t_k)).$$

(2) If $t \in\,]\alpha_n,1]$, then

$$h((x_n,y_n),\alpha_n)(t) = h((x,y),\alpha)(t) = (x^*,y^*)(t).$$

Thus

$$\big\|h((x_n,y_n),\alpha_n) - h((x,y),\alpha)\big\| \to 0 \text{ as } n\to\infty.$$

Hence, h is continuous, so, the set $S(x_0,y_0)$ is contractible. □

Theorem 4.5. *Suppose the conditions of Theorem 3.1 hold, and $F_1, F_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R} \times \mathbb{R})$ are Carathéodory, u.s.c. and σ-*Ca*-selectionnable. Then the set of all solutions of problem* (1.1) *is R_δ-contractible and acyclic.*

Proof. Let $f^i \in S_{F_i}$ be a Carathéodory selection of F_i, $i = 1, 2$. Consider the homotopy multifunction $\Pi : S(x_0, y_0) \times [0, 1] \to \mathcal{P}(S(x_0, y_0))$ defined by

$$\Pi((x, y), \alpha) = \begin{cases} S(x_0, y_0)(t) & \text{if } 0 \le t \le \alpha, \\ S(f, \alpha, (x, y)) & \text{if } \alpha < t \le 1, \end{cases}$$

where

- $S(x_0, y_0)$ is the set of all solutions of problem (1.1);
- $S(f, \alpha, (x, y))$ is the set of all solutions of the problem

$$\begin{cases} z_1'(t) = f_1(t, z_1(t), z_2(t)), & \text{a.e. } t \in [\alpha, 1], \\ z_2'(t) = f_2(t, z_1(t), z_2(t)), & \text{a.e. } t \in [\alpha, 1], \\ z_1(t_k^+) - z_1(t_k^-) = I_{1,k}(z_1(t_k), z_2(t_k)), & k = 1, \dots, m, \\ z_2(t_k^+) - z_2(t_k^-) = I_{2,k}(z_1(t_k), z_2(t_k)), & k = 1, \dots, m, \\ z_1(\alpha) = x(\alpha), \quad z_2(\alpha) = y(\alpha). \end{cases} \tag{4.8}$$

By the definition of Π, for all $(x, y) \in S(x_0, y_0)$, $(x, y) \in \Pi((x, y), 1)$ and $\Pi((x, y), 0) = S(f, 0, (x, y))$, which is an R_δ-set by Theorem 4.3.

It remains to show that Π is *u.s.c.* and $\Pi((x, y), \alpha)$ is an R_δ-set for all $((x, y), \alpha) \in S(x_0, y_0) \times [0, 1]$. The proof is given by the following steps.

Step 1. Π is locally compact.

(a) The multifunction $\widetilde{S} : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(PC(J, \mathbb{R}) \times PC(J, \mathbb{R}))$ defined by

$$\widetilde{S}\big(\widetilde{t}, (\widetilde{x}, \widetilde{y})\big) = S\big(f, \widetilde{t}, (\widetilde{x}, \widetilde{y})\big)$$

is *u.s.c.* where $S(f, \widetilde{t}, (\widetilde{x}, \widetilde{y}))$ is the set of all solutions of the problem

$$\begin{cases} z_1'(t) = f_1(t, z_1(t), z_2(t)), & \text{a.e. } t \in [\widetilde{t}, 1], \\ z_2'(t) = f_2(t, z_1(t), z_2(t)), & \text{a.e. } t \in [\widetilde{t}, 1], \\ z_1(t_k^+) - z_1(t_k^-) = I_{1,k}(z_1(t_k), z_2(t_k)), & k = 1, \dots, m, \\ z_2(t_k^+) - z_2(t_k^-) = I_{2,k}(z_1(t_k), z_2(t_k)), & k = 1, \dots, m, \\ z_1(\widetilde{t}) = \widetilde{x}, \quad z_2(\widetilde{t}) = \widetilde{y}. \end{cases} \tag{4.9}$$

Assume the opposite, i.e., $\widetilde{S}$ is not *u.s.c.* Then for some point $(\widetilde{t}, (\widetilde{x}, \widetilde{y}))$, there is an open neighborhood U of $\widetilde{S}(\widetilde{t}, (\widetilde{x}, \widetilde{y}))$ in $PC([0, 1], \mathbb{R}) \times PC([0, 1], \mathbb{R})$ such that for any open neighborhood V of $(\widetilde{t}, (\widetilde{x}, \widetilde{y}))$ in $[0, 1] \times \mathbb{R} \times \mathbb{R}$, there exists $(\widetilde{t_1}, (\widetilde{x_1}, \widetilde{y_1})) \in V$ such that $\widetilde{S}(\widetilde{t_1}, (\widetilde{x_1}, \widetilde{y_1})) \not\subset U$.

Let

$$V_n = \left\{ (t, (x, y)) \in [0, 1] \times \mathbb{R} \times \mathbb{R} : \;\; d\big((t, (x, y)), (\widetilde{t}, (\widetilde{x}, \widetilde{y}))\big) < \begin{pmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \frac{1}{n} \end{pmatrix} \right\}, \quad n \in \mathbb{N},$$

where d is the generalized metric of the space $[0, 1] \times (\mathbb{R} \times \mathbb{R})$. Then for each $n \in \mathbb{N}$ we take $(t_n, (x_n, y_n)) \in V_n$ and $(x_n, y_n) \in \widetilde{S}(t_n, (x_n, y_n))$ such that $(x_n, y_n) \notin U$. We define the functions

$$G_{\widetilde{t}, (\widetilde{x}, \widetilde{y})}, F_{\widetilde{t}, (\widetilde{x}, \widetilde{y})} : PC([0, 1], \mathbb{R}) \times PC([0, 1], \mathbb{R}) \longrightarrow PC([0, 1], \mathbb{R}) \times PC([0, 1], \mathbb{R})$$

by

$$F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=\Bigg(\widetilde{x}+\int\limits_{\widetilde{t}}^{t} f_1(s,(x(s),y(s)))\,ds+\sum_{\widetilde{t}<t_k<t} I_{1k}(x(t_k),y(t_k)),$$

$$\widetilde{y}+\int\limits_{\widetilde{t}}^{t} f_2(s,(x(s),y(s)))\,ds+\sum_{\widetilde{t}<t_k<t} I_{2k}(x(t_k),y(t_k))\Bigg),\quad t\in[\widetilde{t},1],$$

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)=(x,y)-F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)\ \text{ for }\ t\in[0,1],\ \ (x,y)\in PC(J,\mathbb{R})\times PC(J,\mathbb{R}).$$

Then for $(x,y)\in PC(J,\mathbb{R})\times PC(J,\mathbb{R}),t,\widetilde{t}\in[0,1]$, and $(\widetilde{x},\widetilde{y})\in\mathbb{R}\times\mathbb{R}$, we have

$$F_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=(\widetilde{x},\widetilde{y})-F_{0,(\widetilde{x},\widetilde{y})}(x,y)(\widetilde{t})+F_{0,(\widetilde{x},\widetilde{y})}(x,y)(t).$$

Consequently,

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x,y)(t)=-(\widetilde{x},\widetilde{y})+F_{0,(\widetilde{x},\widetilde{y})}(x,y)(t)+G_{0,(\widetilde{x},\widetilde{y})}(x,y)(t).$$

Then, we obtain

$$\widetilde{S}(\widetilde{t},(\widetilde{x},\widetilde{y}))=G^{-1}_{\widetilde{t},(\widetilde{x},\widetilde{y})}(0)\ \text{ for all }\ (\widetilde{t},(\widetilde{x},\widetilde{y}))\in[0,1]\times\mathbb{R}\times\mathbb{R}.$$

Since $F_{\widetilde{t},(\widetilde{x},\widetilde{y})}$ is compact (see the proof of Theorem 4.3), $G_{\widetilde{t},(\widetilde{x},\widetilde{y})}$ is proper. And as $(x_n,y_n)\in\widetilde{S}(t_n,(x_n,y_n))$, we have

$$x_n(t)=x_n(t_n)+\int\limits_{t_n}^{t} f_1(s,x_n(s),y_n(s))\,ds+\sum_{t_n<t_k<t} I_{1,k}(x_n(t_k),y_n(t_k)),\quad t\in[t_n,1],$$

$$y_n(t)=y_n(t_n)+\int\limits_{t_n}^{t} f_2(s,x_n(s),y_n(s))\,ds+\sum_{t_n<t_k<t} I_{2,k}(x_n(t_k),y_n(t_k)),\quad t\in[t_n,1],$$

which in turn gives

$$0=G_{t_n,(x_n,y_n)}(x_n,y_n)(t)=-(x_n,y_n)(t_n)+F_{0,(x_n,y_n)}(x_n,y_n)(t_n)+G_{0,(x_n,y_n)}(x_n,y_n)(t)$$

and

$$G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t)=-(\widetilde{x},\widetilde{y})+F_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(\widetilde{t})+G_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(t).$$

Then

$$\begin{aligned}&\big\|G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t)-G_{t_n,(x_n,y_n)}(x_n,y_n)(t)\big\|=\big\|G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)(t)\big\|\\&\quad=\|-(\widetilde{x},\widetilde{y})+(x_n,y_n)(t_n)+F_{0,(\widetilde{x},\widetilde{y})}(x_n,y_n)(\widetilde{t})-F_{0,(x_n,y_n)}(x_n,y_n)(t_n)\|=\left\|\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\|=\begin{pmatrix}\|\alpha\|\\ \|\beta\|\end{pmatrix},\end{aligned}$$

where

$$\alpha=-\widetilde{x}+x_n(t_n)+\Bigg(\widetilde{x}+\int\limits_{0}^{\widetilde{t}} f_1(s,x_n(s),y_n(s))\,ds+\sum_{0<t_k<\widetilde{t}} I_{1,k}(x_n(t_k),y_n(t_k))\Bigg)$$

$$-\Bigg(x_n(t_n)+\int\limits_{0}^{t_n} f_1(s,x_n(s),y_n(s))\,ds+\sum_{0<t_k<t_n} I_{1,k}(x_n(t_k),y_n(t_k))\Bigg).$$

Therefore,

$$\begin{aligned}\|\alpha\| &\leq \int_{t_n}^{\widetilde{t}} \|f_1(s,x_n(s),y_n(s))\|\,ds + \sum_{t_n<t_k<\widetilde{t}} \|I_{1,k}(x_n(t_k),y_n(t_k))\| \\ &\leq \int_{t_n}^{\widetilde{t}} p(s)\psi(b_1^*+b_2^*)\,ds + \sum_{t_n<t_k<\widetilde{t}} \|I_{1,k}(x_n(t_k),y_n(t_k))\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\beta = -\widetilde{y} + y_n(t_n) + \Bigg(\widetilde{y} + \int_0^{\widetilde{t}} f_2(s,x_n(s),y_n(s))\,ds + \sum_{0<t_k<\widetilde{t}} I_{2,k}(x_n(t_k),y_n(t_k))\Bigg) \\ - \Bigg(y_n(t_n) + \int_0^{t_n} f_2(s,x_n(s),y_n(s))\,ds + \sum_{0<t_k<t_n} I_{2,k}(x_n(t_k),y_n(t_k))\Bigg),\end{aligned}$$

$$\|\beta\| \leq \int_{t_n}^{\widetilde{t}} p(s)\psi(b_1^*+b_2^*)\,ds + \sum_{t_n<t_k<\widetilde{t}} \|I_{2,k}(x_n(t_k),y_n(t_k))\|.$$

Now,

$$\lim_{n\to\infty}(x_n,y_n) = (\widetilde{x},\widetilde{y}) \ \text{ and } \ \lim_{n\to\infty} t_n = \widetilde{t}$$

imply that

$$\lim_{n\to\infty} G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n) = 0.$$

Then the set $A = \overline{\{G_{\widetilde{t},(\widetilde{x},\widetilde{y})}(x_n,y_n)\}}$ is compact, thus $G^{-1}_{\widetilde{t},(\widetilde{x},\widetilde{y})}(A)$ is also compact. It is clear that $\{(x_n,y_n)\} \subset A$. As $\lim\limits_{n\to\infty}(x_n,y_n) = (\widetilde{x},\widetilde{y})$, it follows $(\widetilde{x},\widetilde{y}) \in \widetilde{S}(\widetilde{t},(\widetilde{x},\widetilde{y})) \subset U$, so we have a contradiction to the hypothesis $(x_n,y_n) \notin U$ for every n.

(b) Π is locally compact.

For $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} > 0$, consider the set

$$B \times I = \Big\{((x,y),\alpha) \in S(x_0,y_0) \times [0,1] : \ \ \|(x,y)\| \leq r\Big\},$$

and let $\{u_n\} \in \Pi(B \times I)$. Then there exists $((x_n,y_n),\alpha_n) \in B \times I$ such that

$$u_n(t) = \begin{cases} (x_n,y_n) & \text{if } 0 \leq t \leq \alpha_n, \\ v_n(t) & \text{if } \alpha_n < t \leq 1, \ \ v_n \in S(f,\alpha_n,(x_n,y_n)). \end{cases}$$

Since $S(x_0,y_0)$ is compact, there exists a subsequence of $(x_n,\alpha_n)_n$ which converges to $((x,y),\alpha)$. $\widetilde{S}$ is *u.s.c.* implies that for all $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that $v_n(t) \in \widetilde{S}(t,(x,y)) = S(f,\alpha,(x,y))$ for all $n \geq n_0(\varepsilon)$, and by the compactness of $S(f,\alpha,(x,y))$, it is concluded that there is a subsequence of $\{v_n\}$ which converges towards $v \in S(f,\alpha,(x,y))$. Hence Π is locally compact.

Step 2. Π has a closed graph.

Let $((x_n,y_n),\alpha_n) \to ((x_*,y_*),\alpha), h_n \in \Pi(x_n,y_n,\alpha_n)$ and $h_n \to h_*$ as $n \to +\infty$. We are going to prove that $h_* \in \Pi((x_*,y_*),\alpha)$. Now, $h_n \in \Pi((x_n,y_n),\alpha_n)$ implies that there exists $z_n \in S(f^i,\alpha_n,(x_n,y_n))$ such that for all $t \in J$,

$$h_n(t) = \begin{cases} (x_n,y_n) & \text{if } 0 \leq t \leq \alpha_n, \\ z_n(t) & \text{if } \alpha_n < t \leq 1. \end{cases}$$

Therefore, it is enough to prove that there exists $z_* \in S(f^i, \alpha, (x_*, y_*))$ such that for all $t \in J$,

$$h_*(t) = \begin{cases} (x_*, y_*) & \text{if } 0 \leq t \leq \alpha, \\ z_*(t) & \text{if } \alpha < t \leq 1. \end{cases}$$

It is clear that $(\alpha_n, (x_n, y_n)) \to (\alpha, (x_*, y_*))$ as $n \to \infty$, and it can easily be proved that there exists a subsequence of $\{z_n\}$ which converges to z_*. So, we can handle the cases $\alpha = 0$ and $\alpha = 1$ as we did in the proof of Theorem 4.4, and we obtain finally that $z_* \in S(f, \alpha, (x_*, y_*))$.

Step 3. $\Pi((x, y), \alpha)$ is an R_δ-set for all $((x, y), \alpha) \in S(x_0, y_0) \times [0, 1]$.

Since F is σ-Ca-selectionnable, there is a decreasing sequence of multifunctions $F_k : [0, b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R} \times \mathbb{R})$, $k \in \mathbb{N}$, which admit Carathéodory selections and

$$F_{k+1}(t, u) \subset F_k(t, u) \ \text{ for all } \ t \in [0, 1], u \in \mathbb{R} \times \mathbb{R},$$

and

$$F(t, u) = \bigcap_{k=0}^{\infty} F_k(t, u), \quad u \in \mathbb{R} \times \mathbb{R}.$$

Then

$$\Pi((x, y), \alpha) = \bigcap_{k=0}^{\infty} S(F_k, (x, y)).$$

By Theorem 4.3, the sets $\Pi((x, y), \alpha)$ and $S(F_k, (x, y))$ are compact. Furthermore, by Theorem 4.4, the set $S(F_k, (x, y))$ is contractible. Thus, $\Pi((x, y), \alpha)$ is an R_δ-set. □

Lemma 4.1. *Suppose that the multifunction $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is Carathéodory and u.s.c. of the type of Scorza–Dragoni. Then the set of all solutions of problem* (1.1) *is R_δ-contractible.*

Proof. By Theorem 2.6, we have that F is σ-Ca-selectionnable. Thus we have the same conditions of the last theorem. □

5 Summary/Conclusion

In this paper, we investigate the existence of a solution for the system of differential inclusions under various assumptions on the multi-valued right-hand side nonlinearity. Also, we have studied some properties of solution sets of those results, such as topological properties (compactness), acyclicity properties, geometric topological properties, R_δ, etc. Theorem 4.3 is a major result entailing some of the topological properties, while Section 4 is devoted to geometric topological properties.

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References

[1] Z. Agur, L. Cojocaru, G. Mazor, R. M. Anderson and Y. L. Danon, Pulse mass measles vaccination across age cohorts. *PNAS* **90** (1993), no. 24, 11698–11702.

[2] J. Andres, G. Gabor and L. Górniewicz, Boundary value problems on infinite intervals. *Trans. Amer. Math. Soc.* **351** (1999), no. 12, 4861–4903.

[3] J. Andres, G. Gabor and L. Górniewicz, Topological structure of solution sets to multi-valued asymptotic problems. *Z. Anal. Anwendungen* **19** (2000), no. 1, 35–60.

[4] J. Andres, G. Gabor and L. Górniewicz, Acyclicity of solution sets to functional inclusions. *Nonlinear Anal.* **49** (2002), no. 5, Ser. A: Theory Methods, 671–688.

[5] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems.* Topological Fixed Point Theory and Its Applications, 1. Kluwer Academic Publishers, Dordrecht, 2003.

[6] J. Andres and M. Pavlačková, Topological structure of solution sets to asymptotic boundary value problems. *J. Differential Equations* **248** (2010), no. 1, 127–150.

[7] N. Aronszajn, Le correspondant topologique de l'unicité dans la théorie des équations différentielles. (French) *Ann. of Math. (2)* **43** (1942), 730–738.

[8] J.-P. Aubin and A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 264. Springer-Verlag, Berlin, 1984.

[9] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis.* Systems & Control: Foundations & Applications, 2. Birkhäuser Boston, Inc., Boston, MA, 1990.

[10] A. Bakowska and G. Gabor, Topological structure of solution sets to differential problems in Fréchet spaces. *Ann. Polon. Math.* **95** (2009), no. 1, 17–36.

[11] H. Berrezoug, J. Henderson and A. Ouahab, Existence and uniqueness of solutions for a system of impulsive differential equations on the half-line. *J. Nonlinear Funct. Anal.* **2017** (2017), Article ID 38, 16 pp.

[12] O. Bolojan-Nica, G. Infante and R. Precup, Existence results for systems with coupled nonlocal initial conditions. *Nonlinear Anal.* **94** (2014), 231–242.

[13] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values. *Studia Mathematica* **90** (1988), no. 1, 69–86.

[14] F. E. Browder and C. P. Gupta, Topological degree and nonlinear mappings of analytic type in Banach spaces. *J. Math. Anal. Appl.* **26** (1969), 390–402.

[15] C. Castaingand M. Valadier, *Convex Analysis and Measurable Multifunctions.* Lecture Notes in Mathematics, Vol. 580. Springer-Verlag, Berlin–New York, 1977.

[16] K. Czarnowski and T. Pruszko, On the structure of fixed point sets of compact maps in B_0 spaces with applications to integral and differential equations in unbounded domain. *J. Math. Anal. Appl.* **154** (1991), no. 1, 151–163.

[17] F. S. De Blasi and J. Myjak, On the solutions sets for differential inclusions. *Bull. Polish Acad. Sci. Math.* **33** (1985), no. 1-2, 17–23.

[18] S. Djebali, L. Górniewicz and A. Ouahab, *Solution Sets for Differential Equations and Inclusions.* De Gruyter Series in Nonlinear Analysis and Applications, 18. Walter de Gruyter & Co., Berlin, 2013.

[19] M. Frigon and A. Granas, Théorèmes d'existence pour des inclusions différentielles sans convexité. (French) [Existence theorems for differential inclusions without convexity] *C. R. Acad. Sci. Paris Sér. I Math.* **310** (1990), no. 12, 819–822.

[20] A. Fryszkowski, Continuous selections for a class of nonconvex multivalued maps. *Studia Math.* **76** (1983), no. 2, 163–174.

[21] L. Górniewicz, Homological methods in fixed-point theory of multi-valued maps. *Dissertationes Math. (Rozprawy Mat.)* **129** (1976), 71 pp.

[22] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings.* Mathematics and its Applications (Dordrecht), 495. Kluwer Academic Publishers, Dordrecht, 1999.

[23] J. R. Graef, J. Henderson and A. Ouahab, *Topological Methods for Differential Equations and Inclusions.* Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2019.

[24] J. R. Graef, J. Henderson and A. Ouahab, Differential inclusions with nonlocal conditions: existence results and topological properties of solution sets. *Topol. Methods Nonlinear Anal.* **37** (2011), no. 1, 117–145.

[25] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive Differential Inclusions. A Fixed Point Approach.* De Gruyter Series in Nonlinear Analysis and Applications, 20. De Gruyter, Berlin, 2013.

[26] J. R. Graef, H. Kadari, A. Ouahab and A. Oumansour, Existence results for systems of second-order impulsive differential equations. *Acta Math. Univ. Comenian. (N.S.)* **88** (2019), no. 1, 51–66.

[27] J. R. Graef and A. Ouahab, First order impulsive differential inclusions with periodic conditions. *Electron. J. Qual. Theory Differ. Equ.* **2008**, No. 31, 40 pp.

[28] J. R. Graef and A. Ouahab, Structure of solutions sets and a continuous version of Filippov's theorem for first order impulsive differential inclusions with periodic conditions. *Electron. J. Qual. Theory Differ. Equ.* **2009**, No. 24, 23 pp.

[29] J. R. Graef and A. Ouahab, Global existence and uniqueness results for impulsive functional differential equations with variable times and multiple delays. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **16** (2009), no. 1, 27–40.

[30] G. Haddad and J.-M. Lasry, Periodic solutions of functional-differential inclusions and fixed points of σ-selectionable correspondences. *J. Math. Anal. Appl.* **96** (1983), no. 2, 295–312.

[31] A. Halanay and D. Wexler, *Teoria Calitativă a Sistemelor cu Impulsuri.* (Romanian) [*The qualitative theory of systems with impulse*] Editura Academiei Republicii Socialiste România, Bucharest, 1968.

[32] J. Henderson, A. Ouahab and M. Slimani, Existence results for a semilinear system of discrete equations. *Int. J. Difference Equ.* **12** (2017), no. 2, 235–253.

[33] S. C. Hu, V. Lakshmikantham and N. S. Papageorgiou, On the properties of the solution set of semilinear evolution inclusions. *Nonlinear Anal.* **24** (1995), no. 12, 1683–1712.

[34] B. S. Jensen, *The Dynamic Systems of Basic Economic Growth Models.* Mathematics and its Applications, 302. Kluwer Academic Publishers Group, Dordrecht, 1994.

[35] H. Kadari, J. J. Nieto, A. Ouahab and A. Oumansour, Existence of solutions for implicit impulsive differential systems with coupled nonlocal conditions. *Int. J. Difference Equ.* **15** (2020), no. 2, 429–451.

[36] H. Kneser, Über die Lösungen eines Systems gewöhnlicher Differentialgleichungen, das der Lipschitzschen Bedingung nicht genügt. (German) *Berl. Ber.* **1923** (1923), 171–174.

[37] V. Lakshmikantham, D. D. Baĭnov and P. S. Simeonov, *Theory of Impulsive Differential Equations.* Series in Modern Applied Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[38] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 781–786.

[39] E. Krüger-Thiemer, Formal theory of drug dosage regimens. I. *Journal of Theoretical Biology* **13** (1966), 212–235.

[40] E. Krüger-Thiemer, Formal theory of drug dosage regimens: II. The exact plateau effect. *Journal of Theoretical Biology* **23** (1969), no. 2, 169–190.

[41] V. D. Mil'man and A. D. Myshkis, On the stability of motion in the presence of impulses. (Russian) *Sibirsk. Mat. Ž.* **1** (1960), 233–237.

[42] J. D. Murray, *Mathematical Biology.* Biomathematics, 19. Springer-Verlag, Berlin, 1989.

[43] J. Musielak, *Introduction to Functional Analysis.* (Polish) PWN, Warsaw, 1976.

[44] O. Nica, Existence results for second order three-point boundary value problems. *Differ. Equ. Appl.* **4** (2012), no. 4, 547–570.

[45] O. Nica, Initial-value problems for first-order differential systems with general nonlocal conditions. *Electron. J. Differential Equations* **2012**, No. 74, 15 pp.

[46] O. Nica and R. Precup, On the nonlocal initial value problem for first order differential systems. *Stud. Univ. Babeş-Bolyai Math.* **56** (2011), no. 3, 113–125.

[47] A. I. Perov, On the Cauchy problem for a system of ordinary differential equations. (Russian) *Približ. Metod. Rešen. Differencial'. Uravnen. Vyp.* **2** (1964), 115–134.

[48] R. Precup, The role of matrices that are convergent to zero in the study of semilinear operator systems. *Math. Comput. Modelling* **49** (2009), no. 3-4, 703–708.

[49] R. Precup and A. Viorel, Existence results for systems of nonlinear evolution equations. *Int. J. Pure Appl. Math.* **47** (2008), no. 2, 199–206.

[50] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive Differential Equations*. World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 14. World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[51] R. S. Varga, *Matrix Iterative Analysis*. Second revised and expanded edition. Springer Series in Computational Mathematics, 27. Springer-Verlag, Berlin, 2000.

[52] K. Yosida, *Functional Analysis*. Sixth edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 123. Springer-Verlag, Berlin-New York, 1980.

[53] Y. Zhou, R.-N. Wang and L. Peng, *Topological Structure of the Solution Set for Evolution Inclusions*. Developments in Mathematics, 51. Springer, Singapore, 2017.

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