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AN APPLICATION OF THE LEGENDRE POLYNOMIALS FOR THE NUMERICAL SOLUTION OF THE NONLINEAR DYNAMICAL KIRCHHOFF STRING EQUATION

Abstract. In the present work, the classical nonlinear Kirchhoff string equation is considered. A three-layer symmetrical semi-discrete scheme with respect to the temporal variable is applied for finding an approximate solution to the initial-boundary value problem for this equation, in which the value of the gradient of a non-linear term is taken at the middle point. This approach is essential because the inversion of the linear operator is sufficient for computations of approximate solutions for each temporal step. The variation method is applied to the spatial variable. Differences of the Legendre polynomials are used as coordinate functions. This choice of Legendre polynomials is also important for numerical realization. This way makes it possible to get a system whose structure does not essentially differ from the corresponding system of difference equations allowing us to use the methods developed for solving a system of difference equations. An application of the suggested variational-difference scheme for the numerical treatment of the stated nonlinear problem gives us an opportunity to solve the system of linear equations instead of a nonlinear one. It is proved that a matrix of the system of Galerkin's linear equations is positively defined and the stability of the factorization method is established.

The program of the numerical implementation with the corresponding interface is created based on the suggested algorithm, and numerical computations are carried out for the model problems.

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Key words and phrases. Non-linear Kirchhoff string equation, Cauchy problem, three-layer semi-discrete scheme, Galerkin method, Cholesky decomposition.

რეზიუმე. ნაშრომში განხილულია კირხჰოფის არაწრფივი კლასიკური სიმის განტოლება. ამ განტოლებისათვის დასმული საწყის-სასაზღვრო ამოცანის მიახლოებითი ამოხსნისათვის გამოყენებულია სამშრიანი, სიმეტრიული, ნახევრადდისკრეტული სქემა დროითი ცვლადის მიხედვით, სადაც გრადიენტის მნიშვნელობა არაწრფივ წევრში აღებულია შუა წერტილში. ეს მიდგომა მნიშვნელოვანია, რადგან მიახლოებითი ამონახსნის ყოველ დროით ბიჯზე გამოთვლისათვის საკმარისია წრფივი ოპერატორის შებრუნება. სივრცითი ცვლადების მიხედვით გამოყენებულია ვარიაციული მეთოდი. საკოორდინატო ფუნქციებად აღებულია ლეჟანდრის პოლინომების სხვაობა. საბაზისო ფუნქციებად ლეჟანდრის პოლინომების სხვაობის აღება მნიშვნელოვანია რიცხვითი რეალიზაციის თვალსაზრისით. ამ გზით მიიღება ისეთი სისტემა, რომლის სტრუქტურა არსებითად არ განსხვავდება შესაბამის სხვაობიან განტოლებათა სისტემისგან, რაც გვაძლევს საშუალებას გამოყენებულ იქნას სხვაობიანი სისტემის ამოხსნისთვის დამუშავებული მეთოდები. დასმული არაწრფივი ამოცანის რიცხვითი ამონახსნის საპოვნელად შემოთავაზებული ვარიაციულ-სხვაობიანი სქემის გამოყენება საშუალებას გვაძლევს ამოიხსნას წრფივი განტოლებათა სისტემა ნაცვლად არაწრფივისა. დამტკიცებულია გალიორკინის წრფივ განტოლებათა სისტემის მატრიცის დადებითად განსაზღვრულობა და დადგენილია ფაქტორიზაციის მეთოდის მდგრადობა.

შემოთავაზებული ალგორითმის საფუძველზე შეიქმნა რიცხვითი რეალიზაციის პროგრამა შესაბამისი ინტერფეისით, ჩატარდა რიცხვითი გამოთვლები მოდელური ამოცანებისათვის.

1 Introduction

For the first time, G. Kirchhoff generalized D'Alembert's classical linear model with the addition of a nonlinear term (see [14]). The issues on the existence and uniqueness of local and global solutions of initial-boundary value problems for the Kirchhoff string equation were first studied by S. Bernstein in 1940 (see [4]). The issues of the solvability of the classical and generalized Kirchhoff equations were later considered by many authors: Arosio, Panizzi [1], Arosio and Spagnolo [2], Berselli, Manfrin [5], D'Ancona, Spagnolo [7,8], Manfrin [17], Medeiros [19], Liu, Rincon [15], Matos [18] and Nishihara [20]. To the approximate solutions of initial-boundary value problems for classical equations the following works are devoted: Christie, Sanz-Serna [6], Peradze [3, 21, 22] and Temimi et al. [28]. Construction of algorithms of finding approximate solutions and their investigations for initial-boundary value problems of some classes integro-differential equations are considered in the monograph of Jangveladze, Kiguradze and Neta [13]. As far as we know, issues on the approximate solution in terms of a part of numerical realization to the Kirchhoff string equation are less studied.

We consider the nonlinear dynamical Kirchhoff string equation and look for an approximate solution to a Cauchy problem for this equation using the symmetric three-layer semi-discrete scheme with respect to the temporal variable. The value of the gradient in the nonlinear term of the equation is taken at the middle point. This type of semi-discrete schemes for a generalized Kirchhoff equation have been studied by Rogava and Tsiklauri [24–26]. Inversion of the liner operator makes it possible to find an approximate solution at each temporal step. The variation method is applied to a spatial variable. The differences of the Legendre polynomials are used as coordinate functions. An application of the Legendre polynomials to boundary value problems of equations of the theory of elasticity are considered in the monograph of Vashakmadze [30]. The Gauss-Legendre quadrature (see [16, 27]) is applied for numerical integration, where $[-1, 1]$ is the domain.

The results of the numerical computations of test problems are presented at the end of the paragraph. According to the numerical experiments, the order of convergence of the scheme is practically stated and it is shown that the constructed scheme describes well the behavior of an oscillating solution.

2 Statement of the problem and discretization for a temporal variable

Let us consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \left(\alpha + \beta \int\limits_{-1}^{1} \left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx\right) \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in]-1,1[\times]0,T], \tag{2.1}$$

where $\alpha > 0$ and $\beta > 0$; $f(x,t)$ is a continuous function; $u(x,t)$ is an unknown function.

For equation (2.1), the following initial-boundary conditions

$$u(x,0) = \psi_0(x), \quad u_t'(x,0) = \psi_1(x), \tag{2.2}$$

$$u(-1,t) = 0, \quad u(1,t) = 0 \tag{2.3}$$

hold, where $\psi_0(x)$ and $\psi_1(x)$ are continuous functions, and, in addition, the compatibility condition $\psi_0(-1) = 0$, $\psi_0(1) = 0$ is fulfilled.

The segment $[0, 1]$ is divided into equal parts with uniform meshes τ, i.e.,

$$0 = t_0 < t_1 < \cdots < t_M = T,$$

where

$$t_k = k\tau \;\; (k = 0, 1, \ldots, M), \;\; \tau = \frac{T}{M}\,.$$

We would like to find an approximate solution of problem (2.1)–(2.3) by using the following semi-discrete scheme:

$$\frac{u_{k+1}(x)-2u_k(x)+u_{k-1}(x)}{\tau^2}-\frac{1}{2}\,q_k\Big(\frac{d^2u_{k+1}(x)}{dx^2}+\frac{d^2u_{k-1}(x)}{dx^2}\Big)=f_k(x),\quad k=1,2,\dots,M-1,\tag{2.4}$$

where $f_k(x)=f(x,t_k)$,

$$q_k=\alpha+\beta\int\limits_{-1}^{1}\Big(\frac{du_k(x)}{dx}\Big)^2\,dx.$$

As an approximate solution of $u(x,t)$ of problem (2.1)–(2.3) at the point $t_k=k\tau$, we declare $u_k(x)$, $u(x,t_k)\approx u_k(x)$.

From equation (2.4) we obtain

$$\Big(2I-\tau^2q_k\,\frac{d^2}{dx^2}\Big)u_{k+1}(x)=g_k(x),\tag{2.5}$$

where

$$g_k(x)=2\tau^2f_k(x)+4u_k(x)+\tau^2q_k\,\frac{d^2u_{k-1}(x)}{dx^2}-2u_{k-1}(x).$$

The values of the unknown functions on the zeroth and first layers are described by the initial conditions (2.2) and equation (2.1),

$$u_0(x)=\psi_0(x),\tag{2.6}$$

$$u_1(x)=\psi_0(x)+\tau\psi_1(x)+\frac{1}{2}\,\tau^2\Big(q_0\,\frac{d^2\psi_0(x)}{dx^2}+f_0(x)\Big).\tag{2.7}$$

Let us rewrite the boundary conditions (2.3) in the following form:

$$u_k(-1)=0,\quad u_k(1)=0.\tag{2.8}$$

3 A solution of the system of equations with the Galerkin method using the Legendre polynomials as coordinate functions

To find approximate solutions of problem (2.1)–(2.3) per temporal step we apply the following linear combination:

$$\widetilde{u}_k(x)=\sum_{m=1}^{N}c_m^k\varphi_m(x),\tag{3.1}$$

where the coordinate functions $\varphi_m(x)$ represent differences of the Legendre polynomials, i.e.,

$$\varphi_m(x)=\sqrt{\frac{2m+1}{2}}\int\limits_{-1}^{x}P_m(s)\,ds=A_m\big(P_{m+1}(x)-P_{m-1}(x)\big),\quad A_m=\frac{1}{\sqrt{2(2m+1)}}\,.\tag{3.2}$$

For any $(k+1)$-th layers, the coefficients c_m^{k+1} $(k=1,2,\dots,M-1)$ can be found from the following equation:

$$\bigg(\Big(2I-\tau^2q_k\,\frac{d^2}{dx^2}\Big)u_{k+1}(x)-g_k(x),\varphi_m(x)\bigg)=0.\tag{3.3}$$

Putting (3.1) into equation (3.3), we finally get

$$\bigg(\sum_{i=1}^{N}c_i^{k+1}\Big(2I-\tau^2q_k\,\frac{d^2}{dx^2}\Big)\varphi_i(x),\varphi_m(x)\bigg)=\big(g_k(x),\varphi_m(x)\big).\tag{3.4}$$

The key property of the Legendre polynomials is given (see [9, 12]) in the form

$$\int_{-1}^{1} P_i(x)P_n(x)\,dx = \frac{2}{\sqrt{(2i+1)(2n+1)}}\,\delta_{in}, \tag{3.5}$$

where δ_{in} is the Kronecker symbol.

We introduce the notation

$$\widetilde{P}_i(x) = \sqrt{\frac{2i+1}{2}}\,P_i(x).$$

It is easy to see that

$$\varphi'_m(x) = \widetilde{P}_m(x). \tag{3.6}$$

If we apply the integration by parts with the boundary conditions (2.8), we get

$$\int_{-1}^{1} \Big(\frac{du_k(x)}{dx}\Big)^2\,dx = -\int_{-1}^{1} \frac{d^2u_k(x)}{dx^2}\,u_k(x)\,dx. \tag{3.7}$$

The usage of the integration by parts, due to (3.5) and (3.6), yields

$$\int_{-1}^{1} \frac{d^2\varphi_i(x)}{dx^2}\,\varphi_m(x)\,dx = -\delta_{im}. \tag{3.8}$$

Now, let us rewrite equality (3.5) in terms of A_i and A_m:

$$\int_{-1}^{1} P_i(x)P_m(x)\,dx = 4A_iA_m\delta_{im}. \tag{3.9}$$

According to (3.9), we get

$$\begin{aligned}\int_{-1}^{1} \varphi_i(x)\varphi_m(x)\,dx = 4A_iA_m\big(&A_{i+1}A_{m+1}\delta_{i+1,m+1}\\ &- A_{i+1}A_{m-1}\delta_{i+1,m-1} - A_{i-1}A_{m+1}\delta_{i-1,m+1} + A_{i-1}A_{m-1}\delta_{i-1,m-1}\big).\end{aligned} \tag{3.10}$$

If we take equalities (3.7) and (3.8) into account, we obtain

$$q_k = \alpha + \beta\sum_{m=1}^{N} \big(c_m^k\big)^2. \tag{3.11}$$

From (3.10) we get

$$\begin{aligned}\big(u_{k+1}(x), \varphi_m(x)\big) &= \sum_{i=1}^{N} c_i^{k+1}\int_{-1}^{1} \varphi_i(x)\varphi_m(x)\,dx\\ &= 4\big(-A_{m-2}A_{m-1}^2A_mc_{m-2}^{k+1} + A_m^2(A_{m-1}^2 + A_{m+1}^2)c_m^{k+1} - A_mA_{m+1}^2A_{m+2}c_{m+2}^{k+1}\big),\end{aligned}$$

Let us introduce the following notation:

$$B_m = 4A_{m-1}A_m^2A_{m+1}, \qquad B_m = \frac{1}{(2m+1)\sqrt{(2m-1)(2m+3)}}, \tag{3.12}$$

$$C_m = 4A_m^2(A_{m-1}^2 + A_{m+1}^2) = 8A_{m-1}^2A_{m+1}^2, \qquad C_m = \frac{2}{(2m-1)(2m+3)}. \tag{3.13}$$

According to (3.12) and (3.13), the inner product of $(u_{k+1}(x),\varphi_m(x))$ can be rewritten in the following form:

$$\big(u_{k+1}(x),\varphi_m(x)\big) = -B_{m-1}c_{m-2}^{k+1} + C_m c_m^{k+1} - B_{m+1}c_{m+2}^{k+1}. \tag{3.14}$$

From (3.8) we conclude that

$$\Big(\frac{d^2u_{k+1}(x)}{dx^2},\varphi_m(x)\Big) = -c_m^{k+1}. \tag{3.15}$$

Finally, if we use (3.14) and (3.15), for the calculation of inner product of the left-hand side of equation (3.4), we get the equality

$$\Big(\sum_{i=1}^{N} c_i^{k+1}(2I-\tau^2 q_k\frac{d^2}{dx^2})\varphi_i(x),\varphi_m(x)\Big) = -2B_{m-1}c_{m-2}^{k+1} + (2C_m+\tau^2 q_k)c_m^{k+1} - 2B_{m+1}c_{m+2}^{k+1}. \tag{3.16}$$

For the right-hand side of equation (3.4), we have

$$\begin{aligned}\big(g_k(x),\varphi_m(x)\big) &= -2B_{m-1}(2c_{m-2}^k - c_{m-2}^{k-1})\\ &\quad + 2C_m(2c_m^k - c_m^{k-1}) - \tau^2(q_k c_m^{k-1} - 2I_m^k) - 2B_{m+1}(2c_{m+2}^k - c_{m+2}^{k-1}).\end{aligned} \tag{3.17}$$

For every $k=1,2,\dots,M-1$, we obtain the following system of linear equations:

$$\begin{aligned}&-2B_{m-1}c_{m-2}^{k+1} + (2C_m+\tau^2 q_k)c_m^{k+1} - 2B_{m+1}c_{m+2}^{k+1}\\ &\qquad = -2B_{m-1}(2c_{m-2}^k - c_{m-2}^{k-1}) + 2C_m(2c_m^k - c_m^{k-1})\\ &\qquad\qquad - \tau^2(q_k c_m^{k-1} - 2I_m^k) - 2B_{m+1}(2c_{m+2}^k - c_{m+2}^{k-1}).\end{aligned} \tag{3.18}$$

To find coefficients c_m^{k+1} ($k=1,2,\dots,M-1$), we have first to find c_m^0 and c_m^1. To this end, we calculate the inner products $(u_0(x),\varphi_m(x))$ and $(u_1(x),\varphi_m(x))$:

$$-B_{m-1}c_{m-2}^0 + C_m c_m^0 - B_{m+1}c_{m+2}^0 = \widetilde{I}_m^0, \tag{3.19}$$

$$-B_{m-1}c_{m-2}^1 + C_m c_m^1 - B_{m+1}c_{m+2}^1 = \widetilde{I}_m^0 + \tau\widetilde{I}_m^1 - \frac{1}{2}\,\tau^2(q_0c_m^0 - I_m^0). \tag{3.20}$$

The values of summands with negative indices in (3.18), (3.19) and (3.20) we set equal to zeros.

The notation of I_m^k, $\widetilde{I}_m^0$ and $\widetilde{I}_m^1$ denote the inner products $(f_k(x),\varphi_m(x))$, $(u_0(x),\varphi_m(x))$ and $(u_1(x),\varphi_m(x))$, respectively. We calculate approximately the already-mentioned inner products using the Gauss–Legendre quadrature rule (see [16,27]), which is exact for polynomials of degree $2N-1$ or less.

We rewrite the system of linear equations (3.18) in a matrix form. Let us introduce the following notation:

$$\begin{aligned}D_m^k &= 2C_m + \tau^2 q_k,\\ F_m^k &= -2B_{m-1}(2c_{m-2}^k - c_{m-2}^{k-1}) + 2C_m(2c_m^k - c_m^{k-1})\\ &\qquad - \tau^2(q_k c_m^{k-1} - 2I_m^k) - 2B_{m+1}(2c_{m+2}^k - c_{m+2}^{k-1}).\end{aligned}$$

According to the above-mentioned notation, the system of linear equations has the form

$$\begin{pmatrix} D_1^k & 0 & -2B_2 & 0 & \cdots & 0\\ 0 & D_2^k & 0 & -2B_3 & \ddots & \vdots\\ -2B_2 & 0 & D_3^k & 0 & \ddots & 0\\ 0 & -2B_3 & 0 & \ddots & \ddots & -2B_{m-1}\\ \vdots & \ddots & \ddots & \ddots & D_{m-1}^k & 0\\ 0 & \cdots & 0 & -2B_{m-1} & 0 & D_m^k \end{pmatrix} \begin{pmatrix} c_1^{k+1}\\ c_2^{k+1}\\ c_3^{k+1}\\ c_4^{k+1}\\ \vdots\\ c_m^{k+1}\end{pmatrix} = \begin{pmatrix} F_1^k\\ F_2^k\\ F_3^k\\ F_4^k\\ \vdots\\ F_m^k\end{pmatrix}. \tag{3.21}$$

The following statement takes place.

Theorem 3.1. *The matrix of the system of Galerkin's linear equations* (3.21) *is positively defined.*

This theorem is a result of the following

Lemma 3.1. *Let us consider a general operator equation in a Hilbert space H,*

$$Au = f, \quad f \in H,$$

where the operator A is symmetric and satisfies the condition

$$(Au, u) \geq \alpha(Bu, u) + \nu\|u\|^2, \quad \forall\, u \in D(A) \subset D(B), \tag{3.22}$$

B is also a symmetric operator, besides $D(A) \subset D(B)$; α and ν are the positive constants.

The matrix of the system of linear equations (3.21) *is positively defined when the basis functions $\{\varphi_k\}_{k=1}^{\infty}$ are B-orthogonal, which means that*

$$(B\varphi_k, \varphi_i) = \delta_{ki}. \tag{3.23}$$

Proof. We denote the Galerkin system of equations by S_N. Let us introduce the vector

$$v_N = (c_1, c_2, \dots, c_N)^\top.$$

We can straightforwardly show that

$$S_N v_N = \big((Au_N, \varphi_1), (Au_N, \varphi_2), \dots, (Au_N, \varphi_N)\big)^T,$$

where

$$u_N = \sum_{k=1}^{N} c_k \varphi_k. \tag{3.24}$$

Indeed,

$$(Au_N, \varphi_i) = \Big(\sum_{k=1}^{N} c_k A\varphi_k, \varphi_i\Big) = \sum_{k=1}^{N} (A\varphi_k, \varphi_i) c_k \quad (i = 1, 2, \dots, N). \tag{3.25}$$

Due to (3.25), we have

$$\begin{aligned}(S_N v_N, v_N) &= c_1(Au_N, \varphi_1) + c_2(Au_N, \varphi_2) + \dots + c_N(Au_N, \varphi_N) \\ &= (Au_N, c_1\varphi_1) + (Au_N, c_2\varphi_2) + \dots + (Au_N, c_N\varphi_N) = \Big(Au_N, \sum_{k=1}^{N} c_k\varphi_k\Big) = (Au_N, u_N),\end{aligned}$$

and obtain

$$(S_N v_N, v_N) = (Au_N, u_N). \tag{3.26}$$

From (3.22) and (3.26) it follows that

$$(S_N v_N, v_N) \geq \alpha(Bu_N, u_N) + \nu\|u_N\|^2. \tag{3.27}$$

Inserting (3.24) into inequality (3.27) and also taking into account the B-orthogonality (3.23), we get

$$\begin{aligned}(S_N v_N, v_N) &\geq \alpha\Big(\sum_{k=1}^{N} c_k B\varphi_k, \sum_{i=1}^{N} c_i B\varphi_i\Big) + \nu\|u_N\|^2 \\ &\geq \alpha \sum_{k=1}^{N}\sum_{i=1}^{N} c_k c_i (B\varphi_k, \varphi_i) = \alpha \sum_{k=1}^{N} c_k^2 = \alpha\|v_N\|^2.\end{aligned}$$

□

Remark 3.1. Obviously, for equation (2.5) we have

$$(Au, u) = 2\|u\|^2 + \tau^2 q_k (Bu, u),$$

where $A = 2I + \tau^2 q_k B$ and $B = -\frac{d^2}{dx^2}$, $D(A) = D(B) = \{u(x) \in C^2([-1, 1]) \mid u(-1) = u(1) = 0\}$. It is well-known that the operator B is positive (see [23]).

Remark 3.2. The matrix of system (3.21) is diagonally dominant of order $\mathcal{O}(\frac{1}{m^3})$ and the following inequality holds:

$$C_m + \frac{m+4}{(2m-1)(2m+3)(m-1)(m+1)} > B_{m-1} + B_{m+1} \ \ (m = 3, 4, \dots, N-2).$$

Proof. We note that for the coefficient B_m $(m = 2, 3, \dots, N-1)$ in (3.12) the following double inequality holds:

$$(2m)^2 < (2m-1)(2m+3) < (2m+1)^2 \tag{3.28}$$

Due to (3.28), for B_{m-1} and B_{m+1}, the inequalities

$$4(m-1)^2 < (2m-3)(2m+1) < (2m-1)^2 \tag{3.29}$$

and

$$4(m+1)^2 < (2m+1)(2m+5) < (2m+3)^2 \tag{3.30}$$

are fulfilled, respectively.

Let us evaluate the expression $B_{m-1} + B_{m+1} - C_m$ $(m = 3, 4, \dots, N-2)$. Taking into account (3.29) and (3.30) we get

$$\frac{16}{(2m-1)^2(2m+3)^2} < B_{m-1} + B_{m+1} - C_m < \frac{m+4}{(2m-1)(2m+3)(m-1)(m+1)}.$$

For the first two and the last two rows of the matrix of system (3.21), we have the following estimations:

$$\begin{gathered}
\frac{7}{20} < C_1 - B_2 < \frac{9}{25}\,, \\
\frac{1}{14} < C_2 - B_3 < \frac{11}{147}\,, \\
\frac{2N-9}{2(2N-3)(2N+1)(N-2)} < C_{N-1} - B_{N-2} < \frac{2N-7}{(2N-3)^2(2N+1)}\,, \\
\frac{2N-7}{2(2N-1)(2N+3)(N-1)} < C_N - B_{N-1} < \frac{2N-5}{(2N-1)^2(2N+3)}\,.
\end{gathered}$$

□

For the solution of system (3.21) we consider the so-called Cholesky decomposition (see [10, 11, 27, 29])

$$A = LDL^\top \tag{3.31}$$

of a symmetric, positively defined matrix $A = (a_{i,j})_{N\times N}$, where L is a lower triangular matrix having identities of the main diagonal, $L^\top$ is the transposed matrix of L and D is a diagonal matrix. Applying the decomposition similar to (3.31), the system of linear equations

$$Ax = b$$

can be split into the following sub-systems:

$$\begin{cases} Lz = b, \\ Dy = z, \\ L^\top x = y. \end{cases}$$

For the system of equations on the layers $k = 0$ and $k = 1$, we get

$$Ac^{(n)} = b^{(n)}, \ \ n = 0, 1, \tag{3.32}$$

a solution of system (3.32) has the following form ($n = 0, 1$):

$$\begin{cases} z_m^{(n)} = b_m^{(n)}, & m \in \{1, 2\}; \\ z_m^{(n)} = b_m^{(n)} + \dfrac{B_{m-1}}{d_{m-2}}\, z_{m-2}^{(n)}, & m \in \{3, 4, \ldots, N\}; \\ y_m^{(n)} = \dfrac{z_m^{(n)}}{d_m}, & m \in \{1, 2, \ldots, N\}; \\ c_m^{(n)} = y_m^{(n)}, & m \in \{N, N-1\}; \\ c_m^{(n)} = y_m^{(n)} + \dfrac{B_{m+1}}{d_m}\, c_{m+2}^{(n)}, & m \in \{N-2, N-3, \ldots, 1\}, \end{cases}$$

where

$$\begin{cases} d_m = C_m, & m \in \{1, 2\}; \\ d_m = C_m - \dfrac{B_{m-1}^2}{d_{m-2}}, & m \in \{3, 4, \ldots, N\}. \end{cases}$$

Any $(k+1)$-th layers, a solution of linear algebraic system of equations $A^{(k)} c^{(k+1)} = F^{(k)}$, where $k = 1, 2, \ldots, M-1$, has the following form:

$$\begin{cases} z_m^{(k+1)} = F_m^{(k)}, & m \in \{1, 2\}; \\ z_m^{(k+1)} = F_m^{(k)} + \dfrac{2B_{m-1}}{d_{m-2}^{(k)}}\, z_{m-2}^{(k+1)}, & m \in \{3, 4, \ldots, N\}; \\ y_m^{(k+1)} = \dfrac{z_m^{(k+1)}}{d_m^{(k)}}, & m \in \{1, 2, \ldots, N\}; \\ c_m^{(k+1)} = y_m^{(k+1)}, & m \in \{N, N-1\}; \\ c_m^{(k+1)} = y_m^{(k+1)} + \dfrac{2B_{m+1}}{d_m^{(k)}}\, c_{m+2}^{(k+1)}, & m \in \{N-2, N-3, \ldots, 1\}, \end{cases}$$

where

$$\begin{cases} d_m^{(k)} = 2C_m + \tau^2 q_k, & m \in \{1, 2\}; \\ d_m^{(k)} = (2C_m + \tau^2 q_k) - \dfrac{4B_{m-1}^2}{d_{m-2}^{(k)}}, & m \in \{3, 4, \ldots, N\}. \end{cases}$$

4 Analysis of the numerical results

Let us consider the initial-boundary value problem (2.1)–(2.3) with the constants $\alpha = \beta = 1$ and $t \in [0, 1]$. For this problem we take two cases of tests, which are also considered in [25].

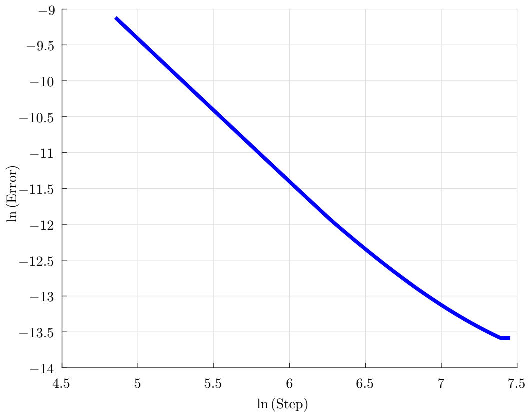
Test 1:

$$\psi_0(x) = 0, \quad \psi_1(x) = m\pi \sin(\pi x),$$
$$f(x, t) = \pi^2\big(-m^2 + (\alpha + \beta\pi^2 \sin^2(m\pi t))\big) \sin(m\pi t) \sin(\pi x).$$

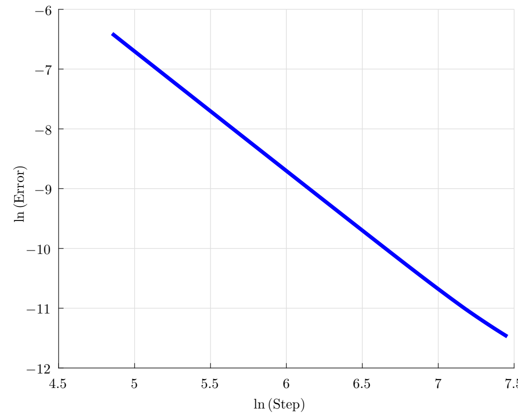
Test 2:

$$\psi_0(x) = \sin(m\pi x), \quad \psi_1(x) = \pi \sin(m\pi x),$$
$$f(x, t) = \pi^2\big(1 + m^2(\alpha + \beta m^2 \pi^2 \mathrm{e}^{2\pi t})\big) \mathrm{e}^{\pi t} \sin(m\pi x).$$

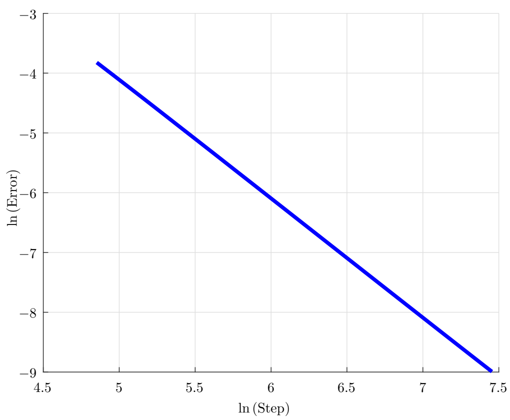
The solutions of Test 1 and Test 2 are $u(x, t) = \sin(m\pi t) \sin(\pi x)$ and $u(x, t) = \mathrm{e}^{\pi t} \sin(m\pi x)$, respectively.



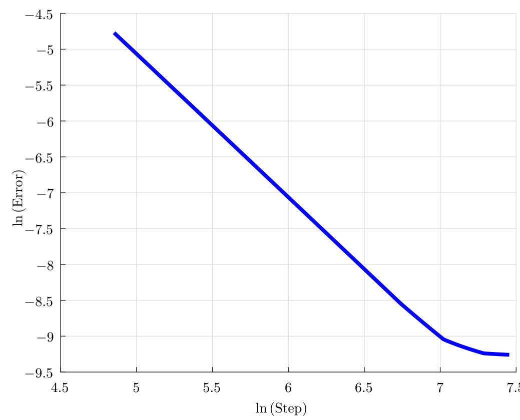
(a) $m = 1$.



(b) $m = 3$.



(c) $m = 5$.



(d) $m = 7$.

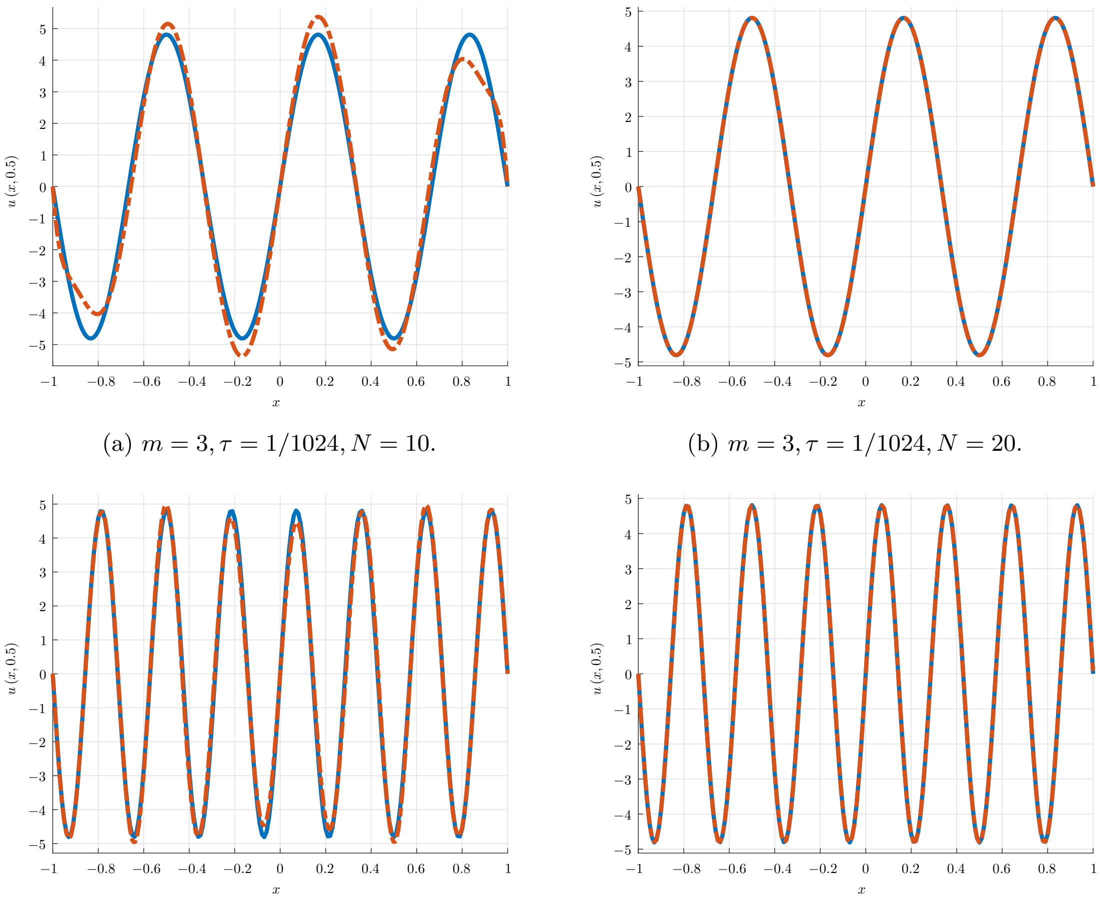
Figure 1: Dependence of logarithm of relative error on logarithm of the temporal step.

In Figure 1, there is a dependence of the logarithm of relative error of the approximated solution of Test 1 on the logarithm of the temporal step. On the horizontal axis there is the logarithm of temporal step, and on the vertical axis there is the logarithm of a relative error of the approximated solution. In all the four pictures, starting from the certain time step, the curve approaches the line, whose angular coefficient is -2, which confirms that the approximate solution obtained by the considered scheme is of the second order accuracy. For this case, eleven ($N = 11$) coordinate functions are taken and the errors of each temporal step are calculated with a maximum norm.

In Figure 2, there are approximate and exact solutions of Test 2 at the point $t = 0.5$. The approximate and exact solutions are shown as dashed and continuous curves, respectively. The errors between the exact and approximate solutions are calculated by a maximum norm and in each cases they represent the following values:

$$\begin{aligned}
\|u(x,0.5) - \widetilde{u}(x,0.5)\|_\infty &\approx 1.00 \times 10^0, \\
\|u(x,0.5) - \widetilde{u}(x,0.5)\|_\infty &\approx 4.44 \times 10^{-5}, \\
\|u(x,0.5) - \widetilde{u}(x,0.5)\|_\infty &\approx 3.43 \times 10^{-1}, \\
\|u(x,0.5) - \widetilde{u}(x,0.5)\|_\infty &\approx 3.31 \times 10^{-5}
\end{aligned}$$

with respect to the cases (a), (b), (c) and (d). In Figure 2, (a) and (b) represent the case $m = 3$, and (c) and (d) represent the case $m = 7$. In figures (a) and (b), the value of τ is the same, but the amount of the coordinate functions is different. Analogously, figures (c) and (d) have the same



(a) $m = 3, \tau = 1/1024, N = 10$.

(b) $m = 3, \tau = 1/1024, N = 20$.

(c) $m = 7, \tau = 1/4096, N = 30$.

(d) $m = 7, \tau = 1/4096, N = 35$.

Figure 2: Exact and approximate solutions at the point of 0.5 with respect to the temporal variable, which are represented by solid and dashed lines, respectively.

mesh length, however, the number of the coordinate functions is not equal to each others. As the tests show, increasing of only temporal layers is not enough to reach high order accuracy, we need to rise the amount of the coordinate functions. Nevertheless, there exists some relationship between numbers of layers and the coordinate functions.

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References

[1] A. Arosio and S. Panizzi, On the well-posedness of the Kirchhoff string. *Trans. Amer. Math. Soc.* **348** (1996), no. 1, 305–330.

[2] A. Arosio and S. Spagnolo, Global existence for abstract evolution equations of weakly hyperbolic type. *J. Math. Pures Appl. (9)* **65** (1986), no. 3, 263–305.

[3] G. Berikelashvili, A. Papukashvili, G. Papukashvili and J. Peradze, Iterative solution of a nonlinear static beam equation. *arXiv preprint* arXiv:1709.08687, [math.NA], 2017; `https://arxiv.org/abs/1709.08687`.

[4] S. Bernstein, Sur une classe d'équations fonctionnelles aux dérivées partielles. (Russian) *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **4** (1940), 17–26.

[5] L. C. Berselli and R. Manfrin, Linear perturbations of the Kirchhoff equation. *Comput. Appl. Math.* **19** (2000), no. 2, 157–178.

[6] I. Christie and J. M. Sanz-Serna, A Galerkin method for a nonlinear integro-differential wave system. *Comput. Methods Appl. Mech. Engrg.* **44** (1984), no. 2, 229–237.

[7] P. D'Ancona and S. Spagnolo, On an abstract weakly hyperbolic equation modelling the nonlinear vibrating string. *Developments in partial differential equations and applications to mathematical physics (Ferrara, 1991)*, 27–32, *Plenum, New York*, 1992.

[8] P. D'Ancona and S. Spagnolo, A class of nonlinear hyperbolic problems with global solutions. *Arch. Rational Mech. Anal.* **124** (1993), no. 3, 201–219.

[9] O. J. Farrell and B. Ross, *Solved Problems in Analysis. As Applied to Gamma, Beta, Legendre and Bessel functions.* Reprint edition, Dover Publications, Inc. VI, New York, 2013.

[10] G. H. Golub and Ch. F. Van Loan, *Matrix Computations.* Fourth edition. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2013.

[11] R. A. Horn and Ch. R. Johnson, *Matrix Analysis.* Second edition. Cambridge University Press, Cambridge, 2013.

[12] D. Jackson, *Fourier Series and Orthogonal Polynomials.* Carus Monograph Series, no. 6. Mathematical Association of America, Oberlin, Ohio, 1941.

[13] T. Jangveladze, Z. Kiguradze and B. *Neta, Numerical Solutions of Three Classes of Nonlinear Parabolic Integro-Differential Equations.* Elsevier/Academic Press, Amsterdam, 2015.

[14] G. Kirchhoff, *Vorlesungen über mathematische Physik: Mechanik*, volume 1. BG Teubner, Leipzig, 1876.

[15] I-Sh. Liu and M. A. Rincon, Effect of moving boundaries on the vibrating elastic string. 2nd International Workshop on Numerical Linear Algebra, Numerical Methods for Partial Differential Equations and Optimization (Curitiba, 2001). *Appl. Numer. Math.* **47** (2003), no. 2, 159–172.

[16] A. N. Lowan and N. Davids and A. Levenson, Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula. *Bull. Amer. Math. Soc.* **48** (1942), 739–743.

[17] R. Manfrin, Global solvability to the Kirchhoff equation for a new class of initial data. *Port. Math. (N.S.)* **59** (2002), no. 1, 91–109.

[18] M. P. Matos, Mathematical analysis of the nonlinear model for the vibrations of a string. *Nonlinear Anal.* **17** (1991), no. 12, 1125–1137.

[19] L. A. Medeiros, On a new class of nonlinear wave equations. *J. Math. Anal. Appl.* **69** (1979), no. 1, 252–262.

[20] K. Nishihara, On a global solution of some quasilinear hyperbolic equation. *Tokyo J. Math.* **7** (1984), no. 2, 437–459.

[21] J. Peradze, A numerical algorithm for the nonlinear Kirchhoff string equation. *Numer. Math.* **102** (2005), no. 2, 311–342.

[22] J. Peradze, A numerical algorithm for a Kirchhoff-type nonlinear static beam. *J. Appl. Math.* **2009**, Art. ID 818269, 12 pp.

[23] K. Rektorys, *Variational Methods in Mathematics, Science and Engineering.* Springer Science & Business Media, Springer, Dordrecht, 2012.

[24] J. Rogava and M. Tsiklauri, Three-layer semidiscrete scheme for generalized Kirchhoff equation. In *Proceedings of the 2nd WSEAS International Conference on Finite Differences, Finite Elements, Finite Volumes, Boundary Elements*, pp. 193–199, 2009.

[25] J. Rogava and M. Tsiklauri, On local convergence of a symmetric semi-discrete scheme for an abstract analogue of the Kirchhoff equation. *J. Comput. Appl. Math.* **236** (2012), no. 15, 3654–3664.

[26] J. Rogava and M.Tsiklauri, Convergence of a semi-discrete scheme for an abstract nonlinear second order evolution equation. *Appl. Numer. Math.* **75** (2014), 22–36.

[27] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis.* Translated from the German by R. Bartels, W. Gautschi and C. Witzgall. Third edition. Texts in Applied Mathematics, 12. Springer-Verlag, New York, 2002.

[28] H. Temimi, A. R. Ansari and A. M. Siddiqui, An approximate solution for the static beam problem and nonlinear integro-differential equations. *Comput. Math. Appl.* **62** (2011), no. 8, 3132–3139.

[29] L. N. Trefethen and D. Bau, III. *Numerical Linear Algebra.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[30] T. S. Vashakmadze, *The Theory of Anisotropic Elastic Plates.* Mathematics and its Applications, 476. Kluwer Academic Publishers, Dordrecht, 1999.

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