Memoirs on Differential Equations and Mathematical Physics

Volume 78, 2019, 1–162

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THE INITIAL PROBLEM FOR LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS, LINEAR IMPULSIVE AND ORDINARY DIFFERENTIAL SYSTEMS. NUMERICAL SOLVABILITY

Abstract. For the system of generalized linear ordinary differential equations the initial problem

$$dx = dA(t) \cdot x + df(t) \quad (t \in I),$$
$$x(t_0) = c_0$$

is considered, where $I \subset \mathbb{R}$ is an interval, $A : I \to \mathbb{R}^{n \times n}$ and $f : I \to \mathbb{R}^n$ are, respectively, matrix- and vector-functions with components of local bounded variation, $t_0 \in I$, $c_0 \in \mathbb{R}^n$.

Under a solution of the system is understood a vector-function $x : I \to \mathbb{R}^n$ with components of bounded local variation satisfying the corresponding integral equality, where the integral is understood in the Kurzweil sense.

Along with a number of questions, we investigate the problems of the well-posedness and stability in Liapunov sense. Effective sufficient conditions, as well as effective necessary and sufficient conditions, are established for each of these problems.

The obtained results are realized for the initial problem for linear impulsive system

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \quad (l = 1, 2, \dots),$$

where P and q are, respectively, the matrix- and the vector-functions with Lebesgue local integrable components, τ_l $(l = 1, 2, \dots)$ are the points of impulse actions, and $G(\tau_l)$ $(l = 1, 2, \dots)$ and $u(\tau_l)$ $(l = 1, 2, \dots)$ are the matrix– and the vector-functions of discrete variables.

Using the well-posedness results, the effective sufficient conditions, as well as the effective necessary and sufficient conditions, are established for the convergence of difference schemes to the solution of the initial problem for the linear systems of ordinary differential equations.

2010 Mathematics Subject Classification. 34A12, 34A30, 34A37, 34D20, 34K06, 34K07, 34K20.

Key words and phrases. Generalized linear ordinary differential equations in the Kurzwei sense, initial problem, well-posedness, the Liapunov stability, linear impulsive differential equations, linear ordinary differential equations, numerical solvability, convergence of difference schemes, effective necessary and sufficient conditions.

რეზიუმე. განზოგადებულ ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემისთვის განხილულია საწყისი ამოცანა

$$dx = dA(t) \cdot x + df(t) \quad (t \in I),$$
$$x(t_0) = c_0,$$

სადაც $I \subset \mathbb{R}$ არის ნებისმიერი ინტერვალი, $A : I \to \mathbb{R}^{n \times n}$ და $f : I \to \mathbb{R}^n$ არის შესაბამისად მატრიცული და ვექტორული ფუნციები, რომელთა კომპონენტები არის ლოკალურად სასრული ვარიაციის ფუნქციები, $t_0 \in I$, $c_0 \in \mathbb{R}^n$.

აღნიშნული სისტემის ამონახსნის ქვეშ გაიგება ისეთი ლოკალურად სასრული ვარიაციის ვექტორული ფუნქცია $x : I \to \mathbb{R}^n$, რომელიც აკმაყოფილებს შესაბამის ინტეგრალურ ტოლობას, სადაც ინტეგრალი გაიგება კურცვაილის აზრით.

სხვა საკითხებთან ერთად განხილულია ამ ამოცანის კორექტულობისა და ლიაპუნოვის აზრით მდგრადობის საკითხები. თითოეული აღნიშნული ამოცანისთვის დადგენილია როგორც ეფექტური საკმარისი პირობები, ასევე ეფექტური აუცილებელი და საკმარისი პირობები.

მიღებული შედეგები რეალიზებულია საწყისი ამოცანისთვის წრფივ იმპულსურ განტოლებათა

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \quad (l = 1, 2, \dots),$$

სისტემისთვის, სადაც P და q არის შესაბამისად ლებეგის აზრით ლოკალურად ინტეგრებადი მატრიცული და ვექტორული ფუნქციები, τ_l $(l = 1, 2, \dots)$ არის იმპულსური ქმედების წერტილები, ხოლო $G(\tau_l)$ $(l = 1, 2, \dots)$ და $u(\tau_l)$ $(l = 1, 2, \dots)$ კი შესაბამისად დისკრეტული არგუმენტის მატრიცული და ვექტორული ფუნქციებია.

კორექტულობის შედეგების საფუძველზე დადგენილია როგორც ეფექტური საკმარისი პირობები, ასევე ეფექტური აუცილებელი და საკმარისი პირობები, რომლებიც უზრუნველყოფს სხვაობიანი სქემების კრებადობას ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემებისთვის კოში ამოცანის ამონახსნისკენ.

Introduction

In the present paper, the initial problem for systems of the so-called linear generalized ordinary differential equations in the sense of Kurzweil is considered. We present the solvability conditions for the problem and consider the related questions such as well-posedness of the problem and stability of solutions in the Liapunov sense. The obtained results are realized for the initial problem for linear systems of impulsive differential equations. Moreover, the obtained results on the well-posedness are used for the numerical solvability of the initial problem for systems of linear ordinary differential equations.

The theory of generalized ordinary differential equations is has been introduced by the Czech mathematician J. Kurzweil in 1957. In [37], he investigated the question on the well-posedness of the initial problem for linear ordinary differential equations, i.e., the problem where small perturbations of the right-hand sides and the initial data of the given problem imply the nearness, in a uniform sense, of the solutions of the perturbed initial problems to the solutions of the given problem. He constructed an example of the problem which fails to have a solution in the classical sense, i.e., the "solution" has the points of discontinuity. The perturbation problems have a classical solution which converges to the "solution" of the given problem only in a pointwise sense. So, in this case, the convergence may not be in a uniform sense. In this connection, J. Kurzweil has introduced certain type of integral (see [37–39, 45, 53, 55, 56]) known in literature as the Kurzweil–Hanstock integral. He considered the solutions of differential equations which were defined as the functions satisfying the corresponding integral equations, where the integral was understood in the introduced sense. Such differential equations, called as generalized ordinary differential equations, may have solutions with the points of discontinuity. For such differential equations J. Kurzweil has proved the well-posed theorem. In such a case, the convergence occurs only in the pointwise sense. So, the above-constructed example was in conformity with the theorem.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enabled one to investigate ordinary differential, impulsive and difference equations from a unified point of view. In particular, all of them can be rewritten in the form of generalized ordinary differential equations

$$dx = dA(t)\cdot x + df(t),$$

where A and f are the matrix- and vector-functions of bounded variation, respectively, for the following systems: (a) the impulsive system

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \quad (l = 1, 2, \dots),$$

where P and q are, respectively, the matrix- and vector-functions with Lebesgue integrable components, τ_l $(l = 1, 2, \dots)$ are the points of impulse actions, and $G(\tau_l)$ $(l = 1, 2, \dots)$ and $u(\tau_l)$ $(l = 1, 2, \dots)$ are the matrix- and vector-functions of discrete variables; (b) the difference system

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + g_0(k) \ \ (k = 1, \dots, m_0),$$

where m_0 is a fixed natural number, and G_1, G_2 and g_0 are, respectively, the matrix- and vector-functions of the discrete variables; (c) the differential-difference systems, and so on.

Therefore, we can consider the ordinary differential, impulsive differential and difference equations as equations of the same type. In particular, if for the generalized ordinary differential equations we investigate the question of well-posedness in the uniform sense, then, as a particular case, we will obtain the results on the convergence of difference schemes to the solutions of the initial problem for the ordinary differential equations. Analogous concept we can use for the investigation of the same problem for linear boundary value problems and the initial and general boundary value problems for nonlinear cases. In the present paper, we consider the question dealing only with the initial problem in the linear case.

Note that another conception of the investigation enabling one to study the continuous and discrete problems, one can find in [22].

The initial and boundary value problems for generalized ordinary differential equations are investigated sufficiently well for the linear and nonlinear cases. The questions of the existence and well-posedness for linear problems are considered, e.g., in [2, 4, 11–13, 15, 17, 20, 21, 32, 33, 39, 45, 53–56] (see also the references therein). The same questions for the nonlinear case are studied in [5–10, 14, 31, 37–39, 45, 53, 55] (see also the references therein).

We give a short description of the results obtained in the present paper.

The work consists of three chapters. Section 1.1 is devoted to consideration of general properties of initial problems given for systems of generalized linear ordinary differential equations most of which are included, e.g., in [45, 53, 55]. Some of the results are precise and some are given as supplementary. In particular, we suggest the method of successive approximations for constructing of solutions of the problem. In addition, both questions on the nonnegativity of the Cauchy matrix and the systems of linear generalized differential and integral inequalities are investigated. Moreover, the subject on the relationship between the stability in the Liapunov sense and the well-posedness of the initial problem on the infinity intervals is studied.

Section 1.2 considers the question of the well-posedness of the initial problem for systems of generalized linear ordinary differential equations.

In the past century, the question on the well-posedness of the problems for the systems of ordinary differential equations was of the utmost interest. In particular, such a question for the initial problem for linear systems was treated very thoroughly (see, e.g., [3, 11, 34, 35, 40, 46, 48, 57] and the references therein). Note that unprovable sufficient conditions, as well as unprovable necessary and sufficient conditions both for the initial and for the linear boundary value problems were obtained in [3].

The same question for the initial and boundary value problems for the nonlinear systems are investigated, for example, in [34, 36, 57] (see also the references therein).

In the same section, we establish new effective sufficient conditions, as well as the effective criterion for the well-posedness of the problem. Moreover, the effective conditions guaranteeing the uniform convergence of the solutions of the perturbed problems on every closed subsegment are also established. Some results obtained in the paper are new for ordinary differential case, as well.

Section 1.3 proposes investigation of the stability in Liapunov sense of the solutions of systems of generalized linear ordinary differential equations. Such a subject-matter is classical. Our earlier results concerning the problem for ordinary differential equations can be found in [26, 34] (see also the references therein). As for the case of generalized ordinary differential equations, one can see, e.g., the works [2, 13, 20, 53] (see also the references therein). In the present paper, we make more precise already known results for generalized case: the effective sufficient conditions and effective necessary and sufficient conditions for stability, uniform stability, asymptotic and the so-called ξ-exponentially asymptotic stability. The obtained results are new for the case of ordinary differential equations, as well.

In Chapter 2, the results of Chapter 1 are realized for linear impulsive differential systems.

Some questions, such as solvability, well-posedness, stability in the Liapunov sense, etc., are studied in [2, 15, 16, 18, 19, 23, 24, 42, 44, 47, 51, 58] (see also the references therein).

The results obtained in the monograph is the generalization of our earlier results. In particular, we obtain effective sufficient and necessary and sufficient conditions for the well-posedness and stability in the Liapunov sense. Moreover, we also give the method of successive approximations for constructing of solution of the impulsive initial problem.

In Chapter 3, we realize the results of Chapter 1 for the initial problem for ordinary differential systems. The results obtained for this case generalize our earlier results. Moreover, we establish the effective sufficient conditions and the effective necessary and sufficient conditions for the criterion of convergence of difference schemes to the solutions of the initial problems for ordinary differential case. Note that for the convergence of difference schemes we have used the concept that it is possible to consider both continuous and difference problems as generalized ones, and therefore, the convergence is a particular case of the well-posedness for the latter problems.

Such problems, and among them the question of the solvability, stability, convergence of difference schemes and others were investigated earlier in [1, 2, 15, 27, 28, 30, 41, 43, 49, 52] for linear as well as nonlinear difference systems.

Basic notation and definitions

In the paper, the use will be made of the following notation and definitions.

$\mathbb{N} = \{1, 2, \dots\}$, $\widetilde{\mathbb{N}} = \{0, 1, \dots\}$, $\mathbb{Z}$ is the set of all integers.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{C}$ is the space of all complex numbers z; $|z|$ is the modulus of z.

I is an arbitrary finite or infinite interval from $\mathbb{R}$. We say that some properties are valid in I if they are valid on every closed interval from I.

$[t]$ is the integer part of $t \in \mathbb{R}$.

$\mathbb{R}^{n\times m}$ $(\mathbb{C}^{n\times n})$ is the space of all real (complex) $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}_+^{n\times m} = \big\{(x_{ij})_{i,j=1}^{n,m} : x_{ij} \geq 0 \ \ (i = 1, \dots, n;\ \ j = 1, \dots, m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix. We designate the zero n-vector by 0_n or 0.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then

$$|X| = (|x_{ij}|)_{i,j=1}^{n,m}.$$

X^T is the matrix transposed to X.

$\limsup\limits_{k\to+\infty} x_k$ is the upper limit of the sequence $x_k \in \mathbb{R}$ $(k, 2 \dots)$.

Sometimes, by $[X]_{ij}$ we denote the element x_{ij} in the i-th row and in the j-th column of the matrix $X = (x_{ij})_{i,j=1}^{n,m}$, i.e., $x_{ij} = [X]_{ij}$ $(i = 1, \dots, n;\ j = 1, \dots, m)$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

$x * y$ is the scalar product of the vectors $x, y \in \mathbb{R}^n$.

If $X \in \mathbb{C}^{n\times n}$, then X^{-1}, $\det(X)$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X.

$\operatorname{diag}(X_1, \dots, X_m)$, where $X_i \in \mathbb{C}^{n_i\times n_i}$ $(i = 1, \dots, m)$, $n_1 + \dots + n_m = n$, is a quasidiagonal $n \times n$-matrix. In particular, if $X = (x_{ij})_{i,j=1}^n$, then $\operatorname{diag}(X) = \operatorname{diag}(x_{11}, \dots, x_{nn})$.

$\lambda_0(X)$ and $\lambda^0(X)$ are, respectively, the minimum and maximum eigenvalue of the symmetric matrix $X \in \mathbb{R}^{n\times m}$,

I_n is the identity $n \times n$-matrix; $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$; δ_{ij} is the Kroneker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, \dots)$; $Z_n = (\delta_{i+1\,j})_{i,j=1}^n$.

The inequalities between the real matrices are understood componentwise.

We say that some property holds in the set I if it holds on every closed interval from I.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

$\bigvee\limits_a^b(X)$ is the sum of total variations of components x_{ij} $(i = 1, \dots, m;\ j = 1, \dots, m)$ of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$; $\bigvee\limits_b^a(X) = -\bigvee\limits_a^b(X)$; $\bigvee\limits_I(X) = \lim\limits_{a\to\alpha+,\, b\to\beta-} \bigvee\limits_a^b(X)$, where $\alpha = \inf I$ and $\beta = \sup I$; $\bigvee\limits_{(b,a)}(X) = -\bigvee\limits_{(b,a)}(X)$.

If $X : I \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_I(X)$ is the sum of total variations on I of its components x_{ij} $(i = 1, \dots, m;\ j = 1, \dots, m)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$ for $t \in I$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) \equiv \bigvee\limits_a^t(x_{ij})$, and $a \in I$ is some fixed point.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(\alpha-) = X(\alpha)$ if $\alpha \in I$ and $X(\beta+) = X(\beta)$ if $\gamma \in I$; if α or β do not belong to I, then $X(t)$ is defined by continuity outside of $I)$.

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\|X\|_\infty = \sup\left\{\|X(t)\| : t \in I\right\}$, $\ |X|_\infty = (|x_{ij}|_\infty)_{i,j=1}^{n,m}$.

$\mathrm{BV}(I;\mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X : I \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee_I(X) < \infty$) with the norm $\|X\|_s$.

$\mathrm{BV}(I;D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all bounded variation matrix-functions $X : I \to D$.

$\mathrm{BV}_{loc}(I;D)$ is the set of all $X : I \to D$ for which the restriction on $[a,b]$ belongs to $\mathrm{BV}([a,b];D)$ for every closed interval $[a,b]$ from I.

$\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m}_+) = \big\{X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m}) : X(t) \ge O_{n\times m} \text{ for } t \in I\big\}$.

$C([a,b];\mathbb{R}^{n\times m})$ is the space of all continuous on $[a,b]$ matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ with the standard norm

$$\|X\|_c = \max\{\|X(t)\| : t \in [a,b]\}.$$

$C(I;\mathbb{R}^{n\times m})$ is the space of all continuous and bounded matrix-functions $X : I \to \mathbb{R}^{n\times m}$ with the norm $\|X\|_{c,I} = \sup\{\|X(t)\| : t \in I\}$.

$C(I;D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all continuous and bounded matrix-functions $X : I \to D$.

$C_{loc}(I;D)$ is the set of all continuous matrix-functions $X : I \to D$.

$\mathrm{AC}([a,b];D)$ is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$.

$\mathrm{AC}_{loc}(I;D)$ is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from I belong to $\mathrm{AC}([a,b];D)$.

$\mathrm{AC}_{loc}(I \setminus T;D)$, where $T = \{\tau_1,\tau_2,\dots\}$, $\tau_l \in I$ $(l = 1,2,\dots)$, $\tau_l \ne \tau_k$ $(i \ne k)$, is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I \setminus T$ belong to $\mathrm{AC}([a,b];D)$.

$\mathrm{ACV}([a,b],T;\mathbb{R}^n) = \mathrm{AC}([a,b] \setminus T;\mathbb{R}^n) \cap \mathrm{BV}([a,b];\mathbb{R}^n)$.

$\mathrm{ACV}_{loc}([a,b],T;\mathbb{R}^n) = \mathrm{AC}_{loc}([a,b] \setminus T;\mathbb{R}^n) \cap \mathrm{BV}_{loc}([a,b];\mathbb{R}^n)$.

$T_J = T \cap J$ for every interval $J \subset I$.

$T_{s,t} = T_J$ if $J = [\min\{s,t\},\max\{s,t\}[$ for $s,t \in I$.

$B_{loc}(T;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G : T \to \mathbb{R}^{n\times m}$ such that

$$\sum_{\tau_l \in T_{[a,b]}} \|G(\tau_l)\| < +\infty \quad \text{for every } \ [a,b] \subset I.$$

We say that a matrix-function $X : I \to \mathbb{R}^{n\times n}$ is nonsingular if $\det(X(t)) \ne 0$ for every $t \in I$.

$L([a,b];\mathbb{R}^{n\times m})$ is the set of all the Lebesgue integrable matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$.

$L_{loc}(I;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from I belong to $L([a,b];\mathbb{R}^{n\times m})$.

s_1, s_2, s_c and $J : \mathrm{BV}_{loc}(I;\mathbb{R}) \to \mathrm{BV}_{loc}(I;\mathbb{R})$ are the operators defined, respectively, as follows:

$$s_1(x)(a) = s_2(x)(a) = 0 \quad s_c(x) = x(a);$$
$$s_1(x)(t) = s_1(x)(s) + \sum_{s<\tau\le t} d_1x(\tau), \;\; s_2(x)(t) = s_2(x)(s) + \sum_{s\le\tau<t} d_2x(\tau)$$
$$s_c(x)(t) = s_c(x)(s) + x(t) - x(s) - \sum_{j=1}^{2}(s_j(x)(t) - s_j(x)(s)) \ \text{ for } \ s < t;$$
$$J(x)(a) = x(a),$$
$$J(x)(t) = J(x)(s) + s_c(x)(t) - s_c(x)(s) - \sum_{s<\tau\le t} \ln|1 - d_1x(\tau)| + \sum_{s\le\tau<t} \ln|1 + d_2x(\tau)| \ \text{ for } \ s < t,$$

where $a \in I$ is an arbitrary fixed point.

If $g \in \mathrm{BV}([a,b];\mathbb{R})$, $f : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then we assume

$$\int_s^t x(\tau)\,dg(\tau) = (L-S)\int_{]s,t[} x(\tau)\,dg(\tau) + f(t)d_1g(t) + f(s)d_2g(s),$$

where $(L-S)\int\limits_{]s,t[} f(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$. It is known (see [45, 55]) that if the integral exists, then the right-hand side of the integral equality equals to the Kurzeil–Stieltjes integral $(K-S)\int\limits_s^t f(\tau)\,dg(\tau)$ and, therefore, $\int\limits_s^t f(\tau)\,dg(\tau) = (K-S)\int\limits_s^t f(\tau)\,dg(\tau)$.

If $a=b$, then we assume

$$\int\limits_a^b x(t)\,dg(t) = 0.$$

Moreover, we put

$$\int\limits_s^{t+} x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int\limits_s^{t+\varepsilon} x(\tau)\,dg(\tau), \quad \int\limits_s^{t-} x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int\limits_s^{t-\varepsilon} x(\tau)\,dg(\tau).$$

$L([a,b];\mathbb{R};g)$ is the set of all functions $x:[a,b]\to\mathbb{R}$, measurable and integrable with respect to the measures $\mu(g_i)$ $(i=1,2)$, i.e., such that

$$\int\limits_a^b |x(t)|\,dg_i(t) < +\infty \quad (i=1,2).$$

If $G=(g_{ik})_{i,k=1}^{l,n} \in \mathrm{BV}([a,b];\mathbb{R}^{l\times n})$ and $X=(x_{kj})_{k,j=1}^{n,m} : [a,b];\to\mathbb{R}^{n\times m}$, then

$$S_c(G)(t) \equiv (c_c(g_{ik})(t))_{i,k=1}^{l,n} \quad S_j(G)(t) \equiv (S_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j=1,2)$$

and

$$\int\limits_a^b dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int\limits_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

Sometimes we use the designation $\int\limits_a^{\cdot} dG(s)\cdot X(s)$ for the integral $\int\limits_a^t dG(s)\cdot X(s)$ as the vector-function to the variable t.

Let $a\in I$ be a fixed point. We introduce the operators:

(a) if $X\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^j d_jX(t))\neq 0$ for $t\in I$ $(j=1,2)$, and $Y\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$, then

$$\begin{gathered}\mathcal{A}(X,Y)(a) = O_{n\times m},\\ \mathcal{A}(X,Y)(t)-\mathcal{A}(X,Y)(s) = Y(t)-Y(s)+\sum_{s<\tau\le t} d_1X(\tau)\,(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\ -\sum_{s\le\tau<t} d_2X(\tau)\,(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau) \ \text{ for } \ s<t;\end{gathered} \tag{0.0.1}$$

(b) if $X\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $Y\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$, then

$$\mathcal{B}(X,Y)(t) = X(t)Y(t)-X(a)Y(a)-\int\limits_a^t dX(\tau)\cdot Y(\tau) \quad \text{for} \quad t\in I; \tag{0.0.2}$$

(c) if $X\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det X(t))\neq 0$, and $Y\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, then

$$\mathcal{I}(X,Y)(t) = \int\limits_a^t d\big(X(\tau)+\mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau) \quad \text{for} \quad t\in I; \tag{0.0.3}$$

(d)

$$\mathcal{D}_{\mathcal{I}}(Y_1, X_1; Y_2, X_2)(t) = \mathcal{I}(X_1, Y_1)(t) - \mathcal{I}(X_2, Y_2)(t) \quad \text{for} \quad t \in I \tag{0.0.4}$$

and

$$\mathcal{D}_{\mathcal{B}}(Y_1, X_1; Y_2, X_2)(t) = \mathcal{B}(X_1, Y_1)(t) - \mathcal{B}(X_2, Y_2)(t) \quad \text{for} \quad t \in I. \tag{0.0.5}$$

moreover, we introduce the following operator: if $a \in I$, and $X \in \mathrm{BV}_{loc}(I, \mathbb{R}^{l\times n})$ and $Y : I \to \mathbb{R}^{n\times m}$, then we put

$$\Delta_a X(t) = \begin{cases} -d_1X(t) & \text{for} \quad t < a, \quad t \in I, \\ d_2X(t) & \text{for} \quad t > a, \quad t \in I, \\ O_{l\times n} & \text{for} \quad t = a. \end{cases} \tag{0.0.6}$$

If $l \in \mathbb{N}$, then $\mathbb{N}_l = \{1, \dots, l\}$, $\widetilde{\mathbb{N}}_l = \{0, 1, \dots, l\}$.

$E(J, \mathbb{R}^{n\times m})$, where $J \subset \mathbb{Z}$, is the space of all matrix-functions $Y = (y_{ij})_{i,j=1}^{n,m} : J \to \mathbb{R}^{n\times m}$ with the norm

$$\|Y\|_J = \max\left\{\|Y(k)\| : k \in J\right\}, \quad |Y|_J = (\|y_{ij}\|_J)_{i,j=1}^{n,m}.$$

Δ is the difference operator of the first order, i.e.,

$$\Delta Y(k-1) = Y(k) - Y(k-1) \quad \text{for} \quad Y \in E(\widetilde{\mathbb{N}}_l, \mathbb{R}^{n\times m}), \ \ k \in \mathbb{N}_l.$$

If a function Y is defined on $\mathbb{N}_l$ or $\widetilde{\mathbb{N}}_{l-1}$, then we assume $Y(0) = O_{n\times m}$, or $Y(l) = O_{n\times m}$, respectively, if necessary.

We say that the matrix-function $X \in \mathrm{BV}_{loc}(I, \mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition if there exists $t_* \in I$ such that the matrices $S_c(X)(t) - S_c(X)(t_*)$, $S_1(X)(t) - S_1(X)(t_*)$ and $S_2(X)(t) - S_2(X)(t_*)$ are pairwise permutable and

$$\int_{t_*}^{t} S_c(X)(\tau)\, dS_c(X)(\tau) = \int_{t_*}^{t} dS_c(X)(\tau) \cdot S_c(X)(\tau) \quad \text{for} \quad t \in I.$$

Here, the use will be made of the following formulas:

$$\int_a^b f(t)\, dg(t) = \int_a^b f(t)\, dg(t-) + f(b)d_1g(b), \tag{0.0.7}$$

$$\int_a^b f(t)\, dg(t) = \int_a^b f(t)\, dg(t+) + f(a)d_2g(a),$$

$$\int_a^{t-} x(\tau)\, dg(\tau) = \int_a^t x(\tau)\, dg(\tau) - x(t)\, d_1g(t), \tag{0.0.8}$$

$$\int_a^{t+} x(\tau)\, dg(\tau) = \int_a^t x(\tau)\, dg(\tau) + x(t)\, d_2g(t),$$

$$\int_a^b f(t)\, dg(t) + \int_a^b f(t)\, dg(t) = f(b)g(b) - f(a)g(a)$$
$$+ \sum_{a<t\le b} d_1f(t) \cdot d_1g(t) - \sum_{a\le t<b} d_2f(t) \cdot d_2g(t)$$
$$\text{(integration-by-parts formula)}, \tag{0.0.9}$$

$$\int_a^b h(t)\,d\,(f(t)g(t)) = \int_a^b h(t)f(t)\,dg(t) + \int_a^b h(t)g(t)\,df(t) - \sum_{a<t\leq b} h(t)d_1f(t)\cdot d_1g(t) + \sum_{a\leq t<b} h(t)d_2f(t)\cdot d_2g(t)$$

(general integration-by-parts formula), (0.0.10)

$$\int_a^b f(t)ds_1(g)(t) = \sum_{a<t\leq b} f(t)\,d_1g(t), \int_a^b f(t)ds_2(g)(t) = \sum_{a<t\leq b} f(t)d_2g(t), \tag{0.0.11}$$

$$\int_a^b f(t)\,d\bigg(\int_a^s g(s)dh(s)\bigg) = \int_a^b f(t)g(t)\,dh(t) \quad \text{for} \quad t\in I, \tag{0.0.12}$$

$$d_j\bigg(\int_a^t f(s)\,dg(s)\bigg) = f(t)\,d_jg(t) \quad \text{for} \quad t\in I \ \ (j=1,2) \tag{0.0.13}$$

and

$$\int_a^b f^k(t)\,df(t) = \frac{1}{k+1}\bigg[f^{k+1}(b) - f^{k+1}(a) + \sum_{m=0}^{k-1}\bigg(\sum_{a<t\leq b} f^m(t)\,d_1f(t)\cdot d_1f^{k-m}(t) - \sum_{a\leq t<b} f^m(t)\,d_2f(t)\cdot d_2f^{k-m}(t)\bigg)\bigg] \ \ (k=1,2,\dots) \tag{0.0.14}$$

for $f,g\in \mathrm{BV}([a,b];\mathbb{R})$.

The proof of formulas (0.0.7), (0.0.9) (0.0.11) and (0.0.12) one can find e.g., in [55, Theorems I.4.25, I.4.33, Lemma I.4.23]). As to formulas (0.0.10) and (0.0.14), they are proved in Subsection 1.1.3 (see Lemma 1.1.1).

Chapter 1

Systems of generalized linear ordinary differential equations

1.1 The initial problem. Unique solvability

1.1.1 Statement of the problem and formulation of the results

Let $I \subset \mathbb{R}$ be some interval, non-degenerate into a point. In this section, for the system of linear generalized ordinary differential equations

$$dx = dA(t) \cdot x + df(t) \tag{1.1.1}$$

we consider the initial problem

$$x(t_0) = c_0, \tag{1.1.2}$$

where $A = (a_{ik})_{i,k=1}^{n} : I \to \mathbb{R}^{n\times n}$ and $f = (f_i)_{i=1}^{n} : I \to \mathbb{R}^n$ are, respectively, the matrix- and the vector-functions with bounded variation components on every closed interval from I, i.e., $A \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$; $t_0 \in I$, and $c_0 \in \mathbb{R}^n$.

A vector-function $x \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ is said to be a solution of system (1.1.1) if

$$x(t) - x(s) = \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \ \text{ for } \ s < t; \ \ s,t \in I.$$

Note that if the vector-function $x \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$, then the above integral exists for every $s,t \in I$ (see [55]). If we define a solution of system (1.1.1) as an arbitrary vector-function $x : I \to \mathbb{R}^n$ such that the integral $\int_s^t dA(\tau) \cdot x(\tau)$ exists for $s < t$ $(s,t \in I)$, then by Theorem III.1.3 from [55], x will have bounded variation on every closed interval from I and so x will be from the set $\mathrm{BV}_{loc}(I;\mathbb{R}^n)$.

Under a solution of the system of generalized ordinary differential inequalities

$$dx \le dA(t) \cdot x + df(t) \ \ (\text{resp. } \ \ge)$$

we mean a vector-function $x \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ such that

$$x(t) \le x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \ \ (\text{resp. } \ \ge) \ \text{ for } \ s < t; \ \ s,t \in I.$$

Under a solution of problem (1.1.1), (1.1.2) we understand a solution $x \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ of system (1.1.1), satisfying condition (1.1.2).

We give here some known as well as new results concerning the solvability and properties of solutions of the initial problem (1.1.1), (1.1.2).

Theorem 1.1.1. *Let $t_0 \in I$. Then:*

(a) *the initial value problem* (1.1.1), (1.1.2) *possesses a unique solution x defined on $\{t \in I:\ t > t_0\}$ for any $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$ if and only if $\det(I_n - d_1 A(t)) \neq 0$ for any $t \in I$, $t > t_0$;*

(b) *the initial value problem* (1.1.1), (1.1.2) *possesses a unique solution x defined on $\{t \in I:\ t < t_0\}$ for any $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$ if and only if $\det(I_n + d_2 A(t)) \neq 0$ for any $t \in I$, $t < t_0$;*

(c) *the initial value problem* (1.1.1), (1.1.2) *possesses a unique solution x defined on I for any $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$ if and only if*

$$\det\big(I_n + (-1)^j d_j A(t)\big) \neq\ 0 \ \text{ for } \ t \in I,\ \ (-1)^j (t - t_0) < 0 \ \ (j = 1,2). \tag{1.1.3}$$

Proposition 1.1.1. *Let $s \in I$, and $\alpha \in \mathrm{BV}_{loc}(I;\mathbb{R})$ be such that*

$$1 + (-1)^j d_j \alpha(t) \neq 0 \ \text{ for } \ t \in I,\ \ (-1)^j (t - s) < 0 \ \ (j = 1,2).$$

Then the initial problem

$$d\gamma = \gamma d\alpha(t),\ \ \gamma(s) = 1 \tag{1.1.4}$$

has the unique solution $\gamma_\alpha(\,\cdot\,, s)$ defined by

$$\gamma_\alpha(t,s) = \begin{cases} \exp(s_0(\alpha)(t) - s_0(\alpha)(s)) \prod\limits_{s<\tau\le t} (1 - d_1\alpha(\tau))^{-1} \prod\limits_{s\le\tau<t} (1 + d_2\alpha(\tau)) & \text{for } \ t > s, \\ \exp(s_0(\alpha)(t) - s_0(\alpha)(s)) \prod\limits_{t<\tau\le s} (1 - d_1\alpha(\tau)) \prod\limits_{t\le\tau<s} (1 + d_2\alpha(\tau))^{-1} & \text{for } \ t < s, \\ 1 & \text{for } \ t = s. \end{cases} \tag{1.1.5}$$

Proposition 1.1.2. *Let a vector-function x be a solution of system* (1.1.1). *Then*

$$d_j x(t) = d_j A(t) \cdot x(t) + d_j f(t) \ \text{ for } \ t \in I \ \ (j = 1,2). \tag{1.1.6}$$

Theorem 1.1.2. *Let $A \in \mathrm{BV}_{loc}([a,b];\mathbb{R}^{n\times n})$ and $t_0 \in [a,b]$ be such that condition* (1.1.3) *holds for $I =]a,b[$. Then there exists a constant $r \in \mathbb{R}_+$ such that*

$$\|x(t)\| \le r\Big(\|x(t_0)\| + \bigvee_a^{t_0}(f)\Big) \exp\Big(r \bigvee_t^{t_0}(A)\Big) \ \text{ for } \ a < t \le t_0 \tag{1.1.7}$$

and

$$\|x(t)\| \le r\Big(\|x(t_0)\| + \bigvee_{t_0}^{b}(f)\Big) \exp\Big(r \bigvee_{t_0}^{t}(A)\Big) \ \text{ for } \ t_0 \le t < b, \tag{1.1.8}$$

where x is an arbitrary solution of system (1.1.1) *with $f \in \mathrm{BV}([a,b];\mathbb{R}^n)$.*

Alongside with system (1.1.1), we consider the corresponding homogeneous system

$$dx = dA(t) \cdot x. \tag{1.1.1$_0$}$$

The assumption (1.1.3) on the regularity of the matrices is essential. We present a simple example of a generalized ordinary differential system from [55] concerning the role of the condition.

Example 1.1.1. Let us set

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ \text{ for } \ t \in \Big[0, \frac{1}{2}\Big),\quad A(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ \text{ for } \ t \in \Big[\frac{1}{2}, 1\Big].$$

It is evident that $A \in \mathrm{BV}([0,1];\mathbb{R}^{2\times 2})$, $d_2 A(t) \equiv O_{2\times 2}$, $d_1 A(t) \equiv O_{2\times 2}$ for $t \neq 1/2$, and

$$d_1 A\Big(\frac{1}{2}\Big) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ \text{ and } \ I_n - d_1 A\Big(\frac{1}{2}\Big) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the matrix $I_n - d_1A(1/2)$ is not regular. Consider the initial value problem

$$dx(t) = dA(t) \cdot x(t), \ \ x(0) = c_0, \tag{1.1.9}$$

where $c_0 = (c_{0i})_{i=1}^2$, $c_{01}, c_{02} \in \mathbb{R}$. Let $x = (x_i)_{i=1}^2$ be a solution of the problem. Then by the definition of the matrix-function A, we have $x(t) = c_0$ for $t \in [0, 1/2)$. Moreover, by (1.1.5) we have $x(1/2-) = (I_n - d_1A(1/2))x(1/2)$, i.e., $c_{01} = x_1(1/2)$, and $c_{02} = 0$. Hence problem (1.1.8) cannot have a solution on $[0, 1/2]$ when $c_{02} \neq 0$.

Let us now consider the case when $c_{02} = 0$. Then the vector $(x_i(1/2))_{i=1}^2 = (c_i)_{i=1}^2$, where $c_1 = c_{01}$ and $c_2 \in \mathbb{R}$ is arbitrary, satisfying the last equality. Therefore, according to the equality $x(t) = x(1/2)$ for $t \in [1/2, 1]$, the vector-function $x = (x_i)_{i=1}^2 \in \mathrm{BV}([0, 1]; \mathbb{R}^2)$ defined by $x_1(t) = c_{01}$ for $t \in [0, 1]$, and $x_2(t) = 0$ for $t \in [0, 1/2)$, $x_2(t) = c_2$ for $t \in [1/2, 1]$, will be a solution of problem (1.1.9) for every $c_2 \in \mathbb{R}$.

Summarizing the above-said, we have: if $c_0 = (c_{0i})_{i=1}^2$, where $c_{02} \neq 0$, then problem (1.1.9) is unsolvable on the whole interval $[0, 1]$; if $c_{02} = 0$, then problem (1.1.9) has solutions on the whole interval $[0, 1]$, but the uniqueness is violated.

Note that by equalities (1.1.5), the singularities of matrices $(I_n + (-1)^j d_jA(t_0))$ $(j = 1, 2)$ at the initial point t_0, is irrelevant for the existence and uniqueness of solutions to the initial value problem.

Therefore, problem (1.1.1), (1.1.2) is uniquely solvable for every $t_0 \in I$ and $c_0 \in \mathbb{R}^n$ if and only if

$$\det\big(I_n + (-1)^j d_jA(t)\big) \neq 0 \ \text{ for } \ t \in I \ \ (j = 1, 2). \tag{1.1.10}$$

Theorem 1.1.3. *Let $A \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ and $t_0 \in I$ be such that condition* (1.1.3) *holds. Then the set of all solutions x of the homogeneous system* $(1.1.1_0)$ *is an n-dimensional subset of* $\mathrm{BV}_{loc}(I; \mathbb{R}^n)$.

Theorem 1.1.4. *Let $A \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ and $t_0 \in I$ be such that the condition* (1.1.3) *holds. Then there exists a unique $n \times n$ matrix-function $U(t, s)$ defined for $a \le t \le s \le t_0$ and $t_0 \le s \le t \le b$ such that the matrix function $X(t) = U(t, s)$ satisfies the matrix initial value problem*

$$dX = dA(t) \cdot X, \ \ X(s) = I_n, \tag{1.1.11}$$

and

$$U(t, s) = U(t, r)\, U(r, s) \ \text{ for } \ t, s \in I, \ \ t \le r \le s \le t_0 \ \text{ or } \ t_0 \le s \le r \le t. \tag{1.1.12}$$

In addition, if $t_1 \in I$, then every solution of the homogeneous system $(1.1.1_0)$ *defined on $\{t \in I, t \le t_1\}$ if $t_1 \le t_0$ and on $\{t \in I, t \ge t_1\}$ if $t_0 \le t_1$ is given by the relation*

$$x(t) = U(t, t_1)\, x(t_1) \tag{1.1.13}$$

on the intervals of definition.

Theorem 1.1.5 (Variation-of-constants formula)**.** *Let $A \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ and $t_0 \in I$ be such that condition* (1.1.3) *holds. Then every solution of system* (1.1.1) *admits the representation*

$$x(t) = U(t, t_0)x(t_0) + f(t) - f(t_0) - \int_{t_0}^{t} d_s\, U(t, s) \cdot (f(s) - f(t_0)) \ \text{ for } \ t \in I \tag{1.1.14}$$

for every $f \in \mathrm{BV}_{loc}(I; \mathbb{R}^n)$, where $U(t, s)$ is the matrix-function appearing in Theorem 1.1.4.

Proposition 1.1.3. *Let the matrix-function $A \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ satisfy the Lappo–Danilevskiĭ condition and condition* (1.1.10) *hold. Then the fundamental matrix X, $X(a) = I_n$, where $a \in I$, of system* $(1.1.1_0)$ *is defined by*

$$X(t) = \begin{cases} \exp\big(S_0(A)(t) - S_0(A)(a)\big) \prod\limits_{a \le \tau < t} (I_n + d_2A(\tau)) \prod\limits_{a < \tau \le t} (I_n - d_1A(\tau))^{-1} & \text{for } t > a, \\ \exp\big(S_0(A)(t) - S_0(A)(a)\big) \prod\limits_{t < \tau \le a} (I_n - d_1A(\tau)) \prod\limits_{t \le \tau < a} (I_n + d_2A(\tau))^{-1} & \text{for } t < a. \end{cases} \tag{1.1.15}$$

Remark 1.1.1. In the general case, the expression of the fundamental matrix can be found, for example, in [31, 33, 53].

Theorem 1.1.6. *Let the matrix-function $A \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ be such that condition* (1.1.10) *holds. Then there exists a unique $n\times n$ matrix-function $U : I\times I \to \mathbb{R}^{n\times n}$ such that the matrix function $X(t) = U(t,s)$ satisfies the matrix initial value problem* (1.1.11) *for every $s \in I$. In addition, the matrix-function $U(t,s)$ has the following properties:*

(a) $U(t,t) = I_n$ *for* $t \in I$*;*

(b) *relation* (1.1.12) *holds for* $r, s, t \in I$*;*

(c)

$$U(t-,s) = (I_n - d_1A(t))U(t,s), \quad U(t+,s) = (I_n + d_2A(t))U(t,s),$$
$$U(t,s) - U(t-,s) = d_1A(t)U(t,s), \quad U(t+,s) - U(t,s) = d_2A(t)C(t,s) \ \text{ for } \ s,t \in I;$$

(d)

$$U(t,s-) = (I_n - d_1A(t))^{-1}U(t,s), \quad U(t,s+) = (I_n + d_2A(t))^{-1}U(t,s),$$
$$U(t,s) - U(t,s-) = -(I_n - d_1A(t))^{-1}d_1A(t)U(t,s),$$
$$U(t,s+) - U(t,s) = -(I_n + d_2A(t))^{-1}d_2A(t)U(t,s) \ \text{ for } \ s,t \in I;$$

(e) $\det(U(t,s)) \neq 0$ *for* $s,t \in I$*;*

(f) *the matrices $U(t,s)$ and $U(s,t)$ are mutually reciprocal, i.e.,* $U^{-1}(t,s) = U(s,t)$ *for* $s,t \in I$*;*

(g) $U(t,s) = X(t)X^{-1}(s)$*, where* $X(t) = U(t,a)$ *for* $s,t \in I$*.*

The matrix-function defined in the theorem is called the Cauchy matrix of the homogeneous generalized differential system $(1.1.1_0)$, and the matrix-function $X(t) = U(t,a)$ is called the fundamental matrix of the system.

Theorem 1.1.7 (Variation-of-constants formula)**.** *Let the matrix-function $A \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ be such that condition* (1.1.10) *holds. Then every solution of system* (1.1.1) *admits the representation* (1.1.14) *for every $t_0 \in I$.*

Corollary 1.1.1. *Let the matrix-function $A \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ be such that condition* (1.1.10) *holds. Then representation* (1.1.14) *can be written in the form*

$$x(t) = f(t) - f(t_0) + X(t)\bigg\{X^{-1}(t_0)x(t_0) - \int\limits_{t_0}^{t} dX^{-1}(s)\cdot (f(s) - f(t_0))\bigg\} \ \text{ for } \ t, t_0 \in I, \tag{1.1.16}$$

where X is a fundamental matrix of the homogeneous system $(1.1.1_0)$*.*

Proposition 1.1.4. *Let the matrix-function $A \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ be such that condition* (1.1.10) *holds, and let X be a fundamental matrix of the homogeneous system* $(1.1.1_0)$*. Then*

$$\begin{aligned} X^{-1}(t) &= X^{-1}(s) - X^{-1}(t)A(t) + X^{-1}(s)A(s) + \int\limits_{s}^{t} dX^{-1}(\tau)\cdot A(\tau) \\ &= X^{-1}(s) - \mathcal{B}(X^{-1},A)(t) + \mathcal{B}(X^{-1},A)(s) \ \text{ for } \ s,t \in I, \ \ s < t, \end{aligned} \tag{1.1.17}$$

and

$$d_jX^{-1}(t) = -X^{-1}(t)d_jA(t)\cdot \big(I_n + (-1)^j d_jA(t)\big)^{-1} \ \text{ for } \ t \in [a,b] \ \ (j = 1,2). \tag{1.1.18}$$

In addition,

$$dX^{-1}(t) = -X^{-1}(t)d\mathcal{A}(A,A)(t) \ \text{ for } \ t \in I, \tag{1.1.19}$$

where $\mathcal{A}$ is the operator defined by (0.0.1)*.*

We give also a method of successive approximations for constructing the solution of the initial problem (1.1.1), (1.1.2).

Here and in the sequel, we use the following designations:

$$J_j = \big\{t \in I:\ (-1)^j (t - t_0) < 0\big\} \ \ (j = 1, 2).$$

Theorem 1.1.8. *Let $A \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ and $t_0 \in I$ be such that condition* (1.1.3) *holds, and let x be a unique solution of the initial problem* (1.1.1), (1.1.2)*. Then*

$$\lim_{k\to+\infty} x_k(t) = x(t) \ \ \textit{uniformly on} \ \ [a, b] \tag{1.1.20}$$

for every $[a, b] \subset I$, where

$$\begin{aligned}
x_k(t_0) &= c_0 \quad (k = 0, 1, \dots),\\
x_0(t) &= \big(I_n + (-1)^j d_j A(t)\big)^{-1} c_0 \ \ \textit{for} \ \ t \in J_j \ \ (j = 1, 2),\\
x_k(t) &= \big(I_n + (-1)^j d_j A(t)\big)^{-1} \bigg\{ c_0 + \int_{t_0}^{t} dA(\tau) \cdot x_{k-1}(\tau) + (-1)^j d_j A(t) \cdot x_{k-1}(t)\\
&\qquad + f(t) - f(t_0) \bigg\} \ \ \textit{for} \ \ t \in J_j \ \ (j = 1, 2;\ k = 1, 2, \dots).
\end{aligned} \tag{1.1.21}$$

1.1.2 Nonnegativity of the Cauchy matrix. The systems of linear generalized differential and integral inequalities

In this subsection, we establish the sufficient conditions guaranteeing the nonnegativity of the Cauchy matrix of system ($1.1.1_0$). Moreover, we investigate the question of the estimates of solutions of linear systems of differential and integral inequalities.

Theorem 1.1.9. *Let $t_0 \in I$, $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ and $Q = \mathrm{diag}(\alpha_1, \dots, \alpha_n) \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ be such that conditions* (1.1.3),

$$1 + (-1)^j d_j \alpha_i(t) > 0 \ \ \textit{for} \ \ t \in J_j \cup \{t_0\} \ \ (j = 1, 2;\ i = 1, \dots, n), \tag{1.1.22}$$

$$\det\big(I_n + (-1)^j d_j(\widetilde{A}(t) + Q(t))\big) \neq 0 \ \ \textit{for} \ \ t \in J_j \ \ (j = 1, 2) \tag{1.1.23}$$

and

$$\big(I_n + (-1)^j d_j(\widetilde{A}(t) + Q(t))\big)^{-1} \geq O_{n\times n} \ \ \textit{for} \ \ t \in J_j \ \ (j = 1, 2) \tag{1.1.24}$$

hold, where $\widetilde{A}(t) = A(t) - \mathrm{diag}(A(t))$. Let, moreover, for every $j \in \{1, 2\}$, the functions $(-1)^{j+1} a_{ik}$ ($i \neq k$; $i, k = 1, \dots, n$) be non-decreasing on the set J_j. Then

$$U(t, s) \geq O_{n\times n} \ \ \textit{for} \ \ t \leq s \leq t_0 \ \ \textit{or} \ \ t_0 \leq s \leq t, \tag{1.1.25}$$

where U is the Cauchy matrix of system ($1.1.1_0$).

If the matrix-function Q appearing in Theorem 1.1.9 is continuous, then the theorem has the following form.

Corollary 1.1.2. *Let $t_0 \in I$ and $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ be such that conditions* (1.1.3),

$$\det\big(I_n + (-1)^j d_j(\widetilde{A}(t))\big) \neq 0 \ \ \textit{for} \ \ t \in J_j \ \ (j = 1, 2) \tag{1.1.26}$$

and

$$\big(I_n + (-1)^j d_j(\widetilde{A}(t))\big)^{-1} \geq O_{n\times n} \ \ \textit{for} \ \ t \in J_j \ \ (j = 1, 2) \tag{1.1.27}$$

hold, where $\widetilde{A}(t) = A(t) - \mathrm{diag}(A(t))$. Let, moreover, for every $j \in \{1, 2\}$, the functions $(-1)^{j+1} a_{ik}$ ($i \neq k$; $i, k = 1, \dots, n$) be non-decreasing on the set J_j. Then the conclusion of Theorem 1.1.9 *is true.*

Remark 1.1.2. We will prove an estimate (see estimate (1.1.64) in the proof of the theorem) which is more strong than (1.1.25). Note also that the condition

$$\|d_j A(t)\| < 1 \ \text{ for } \ t \in J_j \ \ (j = 1, 2) \tag{1.1.28}$$

guarantees the validity of condition (1.1.3). If

$$\big\|(I_n + (-1)^j d_j A(t))^{-1} \, d_j(Q(t) - \operatorname{diag}(A(t)))\big\| < 1 \ \text{ for } \ t \in J_j \ \ (j = 1, 2),$$

then condition (1.1.26) follows from (1.1.3). If the condition

$$(-1)^j d_j(\widetilde{A}(t) + Q(t)) \le O_{n\times n} \ \text{ for } \ t \in J_j \ \ (j = 1, 2)$$

holds together with (1.1.26), then condition (1.1.27) holds, as well. If $Q(t) \equiv \operatorname{diag}(A(t))$, then condition (1.1.26) coincides with (1.1.3).

Theorem 1.1.10. *Let $t_0 \in I$, $f \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ and let $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ be such that a_{ik} $(i \ne k;\ i,k = 1,\dots,n)$ are non-decreasing functions on the sets J_1 and J_2 and the conditions*

$$\det(I_n - d_j A(t)) \ne 0 \ \textit{ for } \ t \in J_j \ \ (j = 1, 2), \tag{1.1.29}$$

$$1 - d_j a_{ii}(t) > 0 \ \textit{ for } \ t \in J_j \cup \{t_0\} \ \ (j = 1, 2;\ i = 1, \dots, n) \tag{1.1.30}$$

and

$$\big(I_n - d_j A(t)\big)^{-1} \ge O_{n\times n} \ \textit{ for } \ t \in J_j \ \ (j = 1, 2) \tag{1.1.31}$$

hold. Let, moreover, a vector-function $x \in \mathrm{BV}_{loc}(I \setminus \{t_0\}, \mathbb{R}^n)$ satisfy be the system of linear differential inequalities

$$\operatorname{sgn}(t - t_0) dx(t) \le dA(t) \cdot x(t) + df(t) \tag{1.1.32}$$

on the intervals J_1 and J_2, satisfying the condition

$$x(t_0) + (-1)^j d_j x(t_0) \le c_0 + (-1)^j d_j A(t_0) \cdot c_0 + (-1)^j d_j f(t_0) \ \ (j = 1, 2), \tag{1.1.33}$$

where $c_0 \in \mathbb{R}^n$. Then the estimate

$$x(t) \le y(t) \ \textit{ for } \ t \in I \setminus \{t_0\} \tag{1.1.34}$$

holds, where $y \in \mathrm{BV}_{loc}(I \setminus \{t_0\};\mathbb{R}^n)$ is a solution of the system

$$\operatorname{sgn}(t - t_0)\, dy = dA(t) \cdot y + df(t) \tag{1.1.35}$$

on the intervals J_1 and J_2, satisfying the conditions

$$(-1)^j d_j y(t_0) = d_j A(t_0) \cdot y(t_0) + d_j f(t_0) \ \ (j = 1, 2) \tag{1.1.36}$$

and

$$y(t_0) = c_0. \tag{1.1.37}$$

Remark 1.1.3. It is evident that if we assume

$$x(t_0) \le c_0$$

in Theorem 1.1.10, then inequality (1.1.34) is fulfilled on the whole I. Moreover, note that in this case, inequalities (1.1.33) follow from the inequalities

$$(-1)^j \, d_j x(t_0) \le (-1)^j \, d_j A(t) \cdot c_0 + (-1)^j \, d_j f(t) \ \ (j = 1, 2).$$

Remark 1.1.4. If for some $j \in \{1, 2\}$ estimate (1.1.28) holds and

$$d_j A(t) \ge O_{n\times n} \ \text{ for } \ t \in J_j,$$

then condition (1.1.31) holds, as well.

It is clear that condition (1.1.31) automatically holds if the matrix-function A is continuous, in particular, for the case of ordinary differential equations.

Theorem 1.1.11. *Let $t_0 \in [a,b]$, $c_0 \in \mathbb{R}^n$, $f \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ and let $A = (a_{ik})_{i,k=1}^n : I \to \mathbb{R}^{n\times n}$ be a non-decreasing matrix-function satisfying conditions* (1.1.29) *and* (1.1.31)*. Let, moreover, $x \in \mathrm{BV}_{loc}(I \setminus \{t_0\};\mathbb{R}^n)$ be a solution of the system of linear integral inequalities*

$$x(t) \le c_0 + \Big(\int_{t_0}^{t} dA(\tau)\cdot x(\tau) + f(t) - f(t_0)\Big)\cdot \mathrm{sgn}(t-t_0) \tag{1.1.38}$$

on the sets J_1 and J_2, satisfying (1.1.33)*. Then the conclusion of Theorem 1.1.10 is true.*

1.1.3 Auxiliary propositions. The lemmas on the general differential and integral inequalities

Lemma 1.1.1. *Let $f,g,h \in \mathrm{BV}([a,b];\mathbb{R})$. Then equalities* (0.0.10) *and* (0.0.14) *hold.*

Proof. First we show (0.0.10). Using (0.0.9), (0.0.11) and (0.0.12), we have

$$\begin{aligned}
&\int_a^b h(t)\,d(f(t)g(t))\\
&\quad= \int_a^b h(t)\,d\Big(\int_a^t f(s)\,dg(s) + \int_a^t f(s)\,dg(s) + \sum_{a<s\le t} d_1 f(s)\cdot d_1 g(s) - \sum_{a\le s<t} d_2 f(s)\cdot d_2 g(s)\Big)\\
&\quad= \int_a^b h(t)f(t)\,dg(t) + \int_a^b h(t)g(t)\,df(t) + \sum_{a<t\le b} h(t)d_1 f(t)\cdot d_1 g(t) - \sum_{a\le t<b} h(t)d_2 f(t)\cdot d_2 g(t).
\end{aligned}$$

Let us show (0.0.14). By (0.0.9), we conclude

$$\begin{aligned}
&\int_a^b f^m(t)\,df^{k-m+1}(t) = \int_a^b f^{m+1}(t)\,df^{k-m}(t) + \int_a^b f^k(t)\,d\beta(t) -\\
&\qquad - \sum_{a<t\le b} f^m(t)d_1 f(t)\cdot d_1 f^{k-m}(t) + \sum_{a\le t<b} f^m(t)d_2 f(t)\cdot d_2 f^{k-m}(t)\ \ (m=0,\dots,k-1).
\end{aligned}$$

Summing over m these equalities, we obtain (0.0.14). □

Lemma 1.1.2. *Let $g \in \mathrm{BV}([a,b];R)$. Then*

$$\begin{aligned}
&\int_a^b \mathrm{sgn}\,g(t)\,dg(t) = |g(b)| - |g(a)|\\
&\qquad + \sum_{a<t\le b}\big(|g(t-)| - g(t-)\,\mathrm{sgn}\,g(t)\big) - \sum_{a\le t<b}\big(|g(t+)| - g(t+)\,\mathrm{sgn}\,g(t)\big).
\end{aligned} \tag{1.1.39}$$

Proof. It is evident that $\mathrm{sgn}\,g(t)$ is the break function. So, using the integration-by-parts formula and equality (0.0.11), we find that

$$\int\limits_a^b \operatorname{sgn} g(t)\,dg(t) = g(b)\operatorname{sgn} g(b) - g(a)\operatorname{sgn} g(a)$$

$$- \int\limits_a^b g(t)\,d\operatorname{sgn} g(t) + \sum_{a<t\le b} d_1 g(t) d_1 \operatorname{sgn} g(t) - \sum_{a\le t<b} d_2 g(t) d_2 \operatorname{sgn} g(t)$$

$$= |g(b)| - |g(a)| - \sum_{a<t\le b} g(t)\,d_1 \operatorname{sgn} g(t) - \sum_{a\le t<b} g(t)\,d_2 \operatorname{sgn} g(t)$$

$$+ \sum_{a<t\le b} d_1 g(t) d_1 \operatorname{sgn} g(t) - \sum_{a\le t<b} d_2 g(t)\,d_2 \operatorname{sgn} g(t)$$

$$= |g(b)| - |g(a)| - \sum_{a<t\le b} g(t-)\,d_1 \operatorname{sgn} g(t) - \sum_{a\le t<b} g(t+)\,d_2 \operatorname{sgn} g(t).$$

From this immediately follows equality (1.1.39). □

We give here the following lemma dealing with the differential inequalities.

Lemma 1.1.3. *Let* $t_1,\dots,t_n \in [a,b]$; $q=(q_i)_{i=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^n)$ *and* $(b_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be such that the functions* b_{il} $(i\ne l;\ i,l=1,\dots,n)$ *are non-decreasing. Let, moreover,* $C=(c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b];R^{n\times n})$ *be a matrix-function satisfying the conditions*

$$s_c(b_{ii})(t)-s_c(b_{ii})(s) \le \big(s_c(c_{ii})(t)-s_c(c_{ii})(s)\big)\operatorname{sgn}(t-s) \ \text{ for } \ (t-s)(s-t_i)>0 \ \ (i=1,\dots,n), \tag{1.1.40}$$

$$(-1)^{j+m}\big(|1+(-1)^m d_m b_{ii}(t)|-1\big) \le d_m c_{ii}(t) \ \text{ for } \ (-1)^j(t-t_i)\ge 0 \ \ (j,m=1,2;\ i=1,\dots,n), \tag{1.1.41}$$

$$\big|s_c(b_{il})(t)-s_c(b_{il})(s)\big| \le s_c(c_{il})(t)-s_c(c_{il})(s) \ \text{ for } \ a\le s<t\le b \ \ (i\ne l;\ i,l=1,\dots,n) \tag{1.1.42}$$

and

$$|d_j b_{il}(t)| \le d_j c_{il}(t) \ \text{ for } \ t\in[a,b] \ \ (i\ne l;\ i,l=1,\dots,n). \tag{1.1.43}$$

Then every solution $x=(x_i)_{i=1}^n$ *of the system*

$$dx = dB(t)\cdot x + dq(t) \tag{1.1.44}$$

will be a solution of the system

$$\begin{gathered}\bigg(d|x_i(t)| - \operatorname{sgn}(t-t_i)\sum_{l=1}^n |x_l(t)|dc_{il}(t) - \operatorname{sgn} x_i(t) dq_i(t)\bigg)\operatorname{sgn}(t-t_i) \le 0 \ \ (i=1,\dots,n),\\ (-1)^j d_j|x_i(t_i)| \le \sum_{l=1}^n |x_l(t_i)|d_j c_{il}(t_i) + (-1)^j \operatorname{sgn} x_i(t_i) d_j q_i(t_i) \ \ (j=1,2;\ i=1,\dots,n).\end{gathered} \tag{1.1.45}$$

Proof. First, we note that from (1.1.41) follows

$$d_j c_{ii}(t_i)\ge 0,\ \ d_j c_{ii}(t)\ge -1 \ \text{ for } \ (-1)^j(t-t_i)>0 \ \ (j=1,2;\ i=1,\dots,n). \tag{1.1.46}$$

Taking into account (1.1.39) and the definition of the solution of system (1.1.44), it can be easily shown that

$$|x_i(t)| - |x_i(s)| = \int\limits_s^t |x_i(\tau)|\,ds_c(b_{ii})(\tau)$$

$$+ \sum_{l\ne i,\,l=1}^n \int\limits_s^t x_l(\tau)\operatorname{sgn} x_i(\tau)\,ds_c(b_{il})(\tau) + \sum_{s<\tau\le t} \big(|x_i(\tau)| - |x_i(\tau-)|\big)$$

$$+\sum_{s\le\tau<t}\big(|x_i(\tau+)|-|x_i(\tau)|\big)+\int_s^t \operatorname{sgn} x_i(\tau)\,dq_i(\tau)\ \text{ for }\ a\le s\le t\le b\ \ (i=1,\dots,n).$$

By (1.1.40)–(1.1.43) and (1.1.46), from the above equality, we have

$$|x_i(t)|-|x_i(s)|\le\sum_{l=1}^n\int_s^t|x_l(\tau)|\,ds_c(c_{il})(\tau)+\sum_{s<\tau\le t}\big[\,|x_i(\tau)|-|x_i(\tau-)|\big]$$

$$+\sum_{s\le\tau<t}\big(|x_i(\tau+)|-|x_i(\tau)|\big)+\int_s^t\operatorname{sgn}x_i(\tau)\,dq_i(\tau)\le\sum_{l=1}^n\int_s^t|x_l(\tau)|\,dc_{il}(\tau)$$

$$+\sum_{s<\tau\le t}\bigg\{|x_i(\tau)|(1-d_1c_{ii}(\tau))+\Big|\sum_{l\ne i,\,l=1}^n x_l(\tau)d_1b_{il}(\tau)\Big|-|x_i(\tau)|\,|1-d_1b_{ii}(\tau)|\bigg\}$$

$$-\sum_{s\le\tau<t}\bigg\{|x_i(\tau)|(1+d_2c_{ii}(\tau))-\Big|\sum_{l\ne i,\,l=1}^n x_l(\tau)d_2b_{il}(\tau)\Big|-|x_i(\tau)|\,|1+d_2b_{ii}(\tau)|\bigg\}$$

$$-\sum_{s<\tau\le t}\sum_{l\ne i,\,l=1}^n|x_l(\tau)|d_1c_{il}(\tau)-\sum_{s\le\tau<t}\sum_{l\ne i,\,l=1}^n|x_l(\tau)|d_2c_{il}(\tau)+\int_s^t\operatorname{sgn}x_i(\tau)dq_i(\tau)$$

$$\le\sum_{l=1}^n\int_s^t|x_l(\tau)|\,dc_{il}(\tau)+\int_s^t\operatorname{sgn}x_i(\tau)\,dq_i(\tau)$$

$$+\sum_{s<\tau\le t}\bigg\{|x_i(\tau)|\big(1-d_1c_{ii}(\tau)-|1-d_1b_{ii}(\tau)|\big)+\sum_{l\ne i,\,l=1}^n|x_l(\tau)|\big(|d_1b_{il}(\tau)|-d_1c_{il}(\tau)\big)\bigg\}$$

$$-\sum_{s\le\tau<t}\bigg\{|x_i(\tau)|\big(1+d_2c_{ii}(\tau)-|1+d_2b_{ii}(\tau)|\big)-\sum_{l\ne i,\,l=1}^n|x_l(\tau)|\big(|d_2b_{il}(\tau)|-d_2c_{il}(\tau)\big)\bigg\}$$

$$\le\sum_{l=1}^n\int_s^t|x_l(\tau)|\,dc_{il}(\tau)+\int_s^t\operatorname{sgn}x_i(\tau)\,dq_i(\tau)\ \text{ for }\ t_i<s\le t\le b\ \ (i=1,\dots,n).$$

Therefore inequalities (1.1.45) are fulfilled for $t>t_i$ and $j=2$.

Analogously, we can show that

$$|x_i(t)|-|x_i(s)|\ge-\sum_{l=1}^n\int_s^t|x_l(\tau)|\,dc_{il}(\tau)+\int_s^t\operatorname{sgn}x_i(\tau)\,dq_i(\tau)\ \text{ for }\ a\le s\le t<t_i\ \ (i=1,\dots,n).$$

The above inequality implies (1.1.45) for $t<t_i$ and $j=1$. □

Lemma 1.1.3 has the following form for $n=1$.

Lemma 1.1.4. *Let $t_0\in[a,b]$, α and $q\in\mathrm{BV}_{loc}([a,t_0[,\mathbb{R})\cap\mathrm{BV}_{loc}(]t_0,b];\mathbb{R})$ be such that*

$$1+(-1)^j\operatorname{sgn}(t-t_0)d_j\alpha(t)>0\ \text{ for }\ t\in[a,b]\ \ (j=1,2).$$

Let, moreover, $x\in\mathrm{BV}_{loc}([a,t_0[,\mathbb{R})\cap\mathrm{BV}_{loc}(]t_0,b];\mathbb{R})$ satisfy the linear generalized differential inequality

$$\operatorname{sgn}(t-t_0)\,dx(t)\le x(t)d\alpha(t)+dq(t)$$

on the intervals $[a,t_0[$ and $]t_0,b]$, and

$$x(t_0+)\le y(t_0+)\ \text{ and }\ x(t_0-)\le y(t_0-),$$

where $y \in \mathrm{BV}_{loc}([a,t_0[,\mathbb{R}) \cap \mathrm{BV}_{loc}(]t_0,b];\mathbb{R})$ *is a solution of the general differential equality*

$$\operatorname{sgn}(t-t_0)\,dy = y\,d\alpha(t) + dq(t). \tag{1.1.47}$$

Then

$$x(t) \le y(t) \ \ \textit{for} \ \ t \in [a,t_0[\cup]t_0,b].$$

Lemma 1.1.5 (Gronwall)**.** *Let* $t_0 \in [a,b]$, $c_0 \in \mathbb{R}_+$, $x \in \mathrm{BV}([a,b];\mathbb{R}_+)$ *and the non-decreasing functions* $\alpha, q : [a,b] \to \mathbb{R}_+$ *be such that*

$$x(t) \le c_0 + \bigg|\int\limits_{t_0}^{t} x(\tau)\,d\alpha(\tau) + q(t) - q(t_0)\bigg| \ \ \textit{for} \ \ t \in [a,b]. \tag{1.1.48}$$

Then

$$x(t) \le q(t) - q(t_0) + \gamma_{\widetilde{\alpha}}(t,t_0)\bigg\{c_0 + \int\limits_{t_0}^{t} \gamma_{\widetilde{\alpha}}(s,t_0)\cdot(q(s)-q(t_0))\,d\mathcal{A}(\widetilde{\alpha},\widetilde{\alpha})(s)\bigg\} \ \ \textit{for} \ \ t \in [a,b], \tag{1.1.49}$$

where $\widetilde{\alpha}(t) \equiv \alpha(t)\operatorname{sgn}(t-t_0)$, *and the function* $\gamma_{\widetilde{\alpha}}(t,t_0)$ *is defined according to* (1.1.5)*.*

Proof. Let

$$z(t) = c_0 + \bigg|\int\limits_{t_0}^{t} x(\tau)\,d\alpha(\tau) + q(t) - q(t_0)\bigg| \ \ \text{for} \ \ t \in [a,b].$$

By (1.1.48), it is evident that

$$x(t) \le z(t) \ \ \text{for} \ \ t \in [a,b]. \tag{1.1.50}$$

First, consider the case $t \in [t_0,b]$. Assume $t_0 < b$. In this case, we have

$$z(t) = c_0 + \int\limits_{t_0}^{t} x(\tau)\,d\alpha(\tau) + q(t) - q(t_0) \ \ \text{for} \ \ t \in [t_0,b].$$

Then it is evident that $x(t) \le z(t)$ for $t \in [t_0,b]$. Using this estimate and the conditions of the lemma, we have

$$\begin{aligned}\operatorname{sgn}(t-t_0)\cdot(z(t)-z(s)) &= z(t)-z(s)\\ &= \int\limits_{s}^{t} x(\tau)\,dg(\tau) + q(t) - q(s) \le \int\limits_{s}^{t} z(\tau)\,dg(\tau) + q(t) - q(s) \ \ \text{for} \ \ t_0 \le s < t < b.\end{aligned}$$

Analogous estimate we obtain for $a \le s < t < b$. Therefore, the function z satisfies the general differential inequality

$$\operatorname{sgn}(t-t_0)dz(t) \le z(t)d\alpha(\tau) + dq(t) \ \ \text{for} \ \ t \in [a,t_0) \quad \textit{and} \quad t \in (t_0,b].$$

Let now y be a solution of equation (1.1.47) under the condition $y(t_0) = c_0$. It is easy to see that

$$z(t_0+) \le y(t_0+) \ \ \text{and} \ \ z(t_0-) \le y(t_0-0).$$

Therefore, by Lemma 1.1.4, we obtain

$$z(t) \le y(t) \ \ \text{for} \ \ t \in [a,b].$$

According to (1.1.50), Corollary 1.1.1 (see (1.1.16)) and (1.1.50), estimate (1.1.49) holds. □

Remark 1.1.5. In conditions of Lemma 1.1.5, every function $x \in \mathrm{BV}([a,b];\mathbb{R}_+)$ satisfying the integral inequality

$$x(t) \le c_0 + \left| \int_{t_0}^{t} x(\tau)\, d\alpha(\tau) \right| \text{ for } t \in [a,b]$$

admits the estimate

$$x(t) \le \begin{cases} c_0 \exp(s_c(\alpha)(t) - s_c(\alpha)(t_0)) \prod\limits_{t_0<\tau\le t} (1-d_1\alpha(\tau))^{-1} \prod\limits_{t_0\le\tau<t} (1+d_2\alpha(\tau)) & \text{for } t > t_0, \\ c_0 \exp(s_c(\alpha)(t_0) - s_c(\alpha)(t)) \prod\limits_{t<\tau\le t_0} (1+d_1\alpha(\tau)) \prod\limits_{t\le\tau<t_0} (1-d_2\alpha(\tau))^{-1} & \text{for } t < t_0. \end{cases}$$

From the above remark, according to estimate (1.1.49), the following lemma given in [55] (see Theorem I.4.30) immediately follows.

Lemma 1.1.4′. *Let $g : [a,b] \to \mathbb{R}_+$ be a non-decreasing function, and $\varphi : [a,b] \to \mathbb{R}_+$ be a bounded function, i.e., $\varphi(t) \le r$.*

(a) *If g is continuous from the right on $[a,b)$ and if there exist nonnegative constants r_1 and r_2 such that*

$$\varphi(t) \le r_1 + r_2 \int_{t}^{b} \varphi(\tau)\, dg(\tau) \ \ \textit{for} \ \ t \in [a,b],$$

then

$$\varphi(t) \le r_1 \exp(r_2(g(b) - g(t))) \ \text{ for } \ t \in [a,b].$$

(b) *If g is continuous from the left on $(a,b]$ and if there exist nonnegative constants r_1 and r_2 such that*

$$\varphi(t) \le r_1 + r_2 \int_{a}^{t} \varphi(\tau)\, dg(\tau) \ \ \textit{for} \ \ t \in [a,b],$$

then

$$\varphi(t) \le r_1 \exp(r_2(g(t) - g(a))) \ \text{ for } \ t \in [a,b].$$

Lemma 1.1.6. *Let $t_0 \in [a,b]$; $\alpha, \beta \in \mathrm{BV}([a,b];\mathbb{R})$ and*

$$1 + (-1)^j d_j\alpha(t) \ne 0 \ \ \textit{for} \ \ t \in [a,b] \ \ (j=1,2). \tag{1.1.51}$$

Let, moreover, $\xi \in \mathrm{BV}([a,b];\mathbb{R})$ be a solution of the equation

$$d\xi = \xi\, d\alpha(t) + d\beta(t).$$

Then

$$\gamma^{-1}(t)\xi(t) - \gamma^{-1}(s)\xi(s) = \int_{s}^{t} \gamma^{-1}(\tau)\, d\beta(\tau) - \sum_{s<\tau\le t} d_1\gamma^{-1}(\tau)\cdot d_1\beta(\tau) + \sum_{s\le\tau<t} d_2\gamma^{-1}(\tau)\cdot d_2\beta(\tau) \ \ \textit{for} \ \ a \le s < t \le b, \tag{1.1.52}$$

where $\gamma \in \mathrm{BV}([a,b];\mathbb{R})$ is a solution of the problem

$$d\gamma = \gamma\, d\alpha(t), \quad \gamma(t_0) = 1. \tag{1.1.53}$$

Proof. By (1.1.51), problem (1.1.53) has the unique solution γ and $\gamma(t)\neq 0$ for $t\in[a,b]$.

Let $a\le s<t\le b$. Due to the integration-by-parts formula (0.0.9) and (1.1.17), we have

$$\begin{aligned}
&\gamma^{-1}(t)\xi(t)-\gamma^{-1}(s)\xi(s)\\
&\quad=\int_s^t\gamma^{-1}(\tau)\,d\xi(\tau)+\int_s^t\xi(\tau)\,d\gamma^{-1}(\tau)-\sum_{s<\tau\le t}d_1\gamma^{-1}(\tau)\cdot d_1\beta(\tau)+\sum_{s\le\tau<t}d_2\gamma^{-1}(\tau)\cdot d_2\beta(\tau)\\
&\quad=\int_s^t\gamma^{-1}(\tau)\xi(\tau)\,d\alpha(\tau)+\int_s^t\gamma^{-1}(\tau)\,d\beta(\tau)+\int_s^t\xi(\tau)\,d\gamma^{-1}(\tau)\\
&\qquad-\sum_{s<\tau\le t}d_1\gamma^{-1}(\tau)\cdot\big(\xi(\tau)\,d_1\alpha(\tau)+d_1\beta(\tau)\big)+\sum_{s\le\tau<t}d_2\gamma^{-1}(\tau)\cdot\big(\xi(\tau)\,d_2\alpha(\tau)+d_2\beta(\tau)\big)
\end{aligned}$$

and

$$\begin{aligned}
\gamma^{-1}(\tau)=\gamma^{-1}(s)-\int_s^\tau\gamma^{-1}(\sigma)\,d\alpha(\sigma)\\
+\sum_{s<\sigma\le\tau}d_1\gamma^{-1}(\sigma)\cdot d_1\alpha(\sigma)-\sum_{s\le\sigma<\tau}d_2\gamma^{-1}(\sigma)\cdot d_2\alpha(\sigma)\ \text{ for }\ s<\tau\le t.
\end{aligned}$$

Therefore, (1.1.52) holds, since by the latter equality,

$$\begin{aligned}
\int_s^t\xi(\tau)\,d\gamma^{-1}(\tau)=-\int_s^t\xi(\tau)\gamma^{-1}(\tau)\,d\alpha(\tau)\\
+\sum_{s<\tau\le t}\xi(\tau)d_1\gamma^{-1}(\tau)\cdot d_1\alpha(\tau)-\sum_{s\le\tau<t}\xi(\tau)d_2\gamma^{-1}(\tau)\cdot d_2\alpha(\tau)\ \text{ for }\ s<t.
\end{aligned}$$

□

1.1.4 Proof of the results

Theorems 1.1.1, 1.1.2 and Propositions 1.1.1, 1.1.2 (except equality (1.1.19)) follow immediately from the corresponding results of [53] (see Part III).

Let us show equality (1.1.19). By (1.1.17), using the integration-by-parts formula, equalities (1.1.18) and the definition of the operator $\mathcal{A}$, we obtain

$$\begin{aligned}
X^{-1}(t)-X^{-1}(s)&=-\int_s^t X^{-1}(\tau)\,dA(\tau)+\sum_{s<\tau\le t}d_1X^{-1}(\tau)\cdot d_1A(\tau)-\sum_{s\le\tau<t}d_2X^{-1}(\tau)\cdot d_2A(\tau)\\
&=-\int_s^t X^{-1}(\tau)\,dA(\tau)-\sum_{s<\tau\le t}X^{-1}(\tau)d_1(\tau)\cdot(I_n-d_1A(t))^{-1}d_1A(\tau)\\
&\quad+\sum_{s\le\tau<t}X^{-1}(\tau)d_2A(\tau)\cdot(I_n+d_2A(t))^{-1}d_2A(\tau)=-\int_s^t X^{-1}(\tau)\,d\mathcal{A}(\tau)
\end{aligned}$$

for $a\le s<t\le b$. Thus equality (1.1.19) holds on $[a,b]$.

Proof of Theorem 1.1.8. By (1.1.3), according to Theorem 1.1.1, problem (1.1.1), (1.1.2) has the unique solution x.

Let us show (1.1.20). Let

$$A_j(t)=\begin{cases}A(t)+(-1)^jd_jA(t) & \text{for }\ t\in J_j,\\ A(t_0) & \text{for }\ t\notin J_j\ \ (j=1,2).\end{cases}$$

Then, by (0.0.7) and (1.1.21),

$$x_k(t) = \big(I_n + (-1)^j d_j A(t)\big)^{-1} \times \left\{ c_0 + \int_{t_0}^{t} dA_j(\tau)\cdot x_{k-1}(\tau) + f(t) - f(t_0) \right\} \text{ for } t \in J_j \ \ (j=1,2;\ k=1,2,\dots). \quad (1.1.54)$$

Besides, in view of (1.1.3), there exists a positive number r such that

$$\big\|(J_n + (-1)^j d_j A(t))^{-1}\big\| \le \frac{r}{n} \text{ for } t \in J_j \ \ (j=1,2). \quad (1.1.55)$$

Put

$$l_{0j}(t) = \left\| \int_{t_0}^{t} dV(A_j)(\tau)\cdot |c_0| \right\| + \big\|V(f)(t) - V(f)(t_0)\big\| \text{ for } t \in [a,b] \ \ (j=1,2)$$

and

$$l_{1j}(t) = \big\|V(A_j)(t) - V(A_j)(t_0)\big\| \text{ for } t \in [a,b] \ \ (j=1,2).$$

It is evident that the functions $(-1)^{j+1} l_{mj}$ $(m=0,1)$ are non-decreasing on J_j for every $j \in \{1,2\}$. In addition, l_{11} is continuous from the left on J_1, and l_{12} is continuous from the right on J_2. Taking into account this, (0.0.14), (1.1.54) and (1.1.55), and using the inductive method, it is not difficult to verify that

$$\|x_k(t) - x_{k-1}(t)\| \le \frac{r^k}{(k-1)!}\, l_{0j}(t) l_{1j}^{k-1}(t) \text{ for } t \in J_j \ \ (j=1,2;\ k=1,2,\dots).$$

It follows from this that the functional series

$$x_0(t) + \sum_{k=1}^{\infty} \big(x_k(t) - x_{k-1}(t)\big)$$

converges uniformly on $[a,b]$, because the convergent series

$$\|c_0\| + r_{0j} \sum_{k=1}^{\infty} \frac{r^k r_{1j}^{k-1}}{(k-1)!}\,,$$

where

$$r_{mj} = \sup\big\{ l_{mj}(t) :\ t \in J_j \big\} \ \ (m=0,1),$$

majorizes it on every $J_j \cup \{t_0\}$ $(j=1,2)$.

Let a vector-function $x_* : [a,b] \to \mathbb{R}^n$ be such that

$$\lim_{k\to+\infty} x_k(t) = x_*(t) \text{ uniformly on } [a,b]. \quad (1.1.56)$$

Then, by Theorem I.4.17 from [55], the integral $\int_{t_0}^{t} dA_j(\tau)\cdot x_*(\tau)$ exists and

$$\lim_{k\to+\infty} \int_{t_0}^{t} dA_j(\tau)\cdot x_{k-1}(\tau) = \int_{t_0}^{t} dA_j(\tau)\cdot x_*(\tau) \text{ for } t \in [a,b] \ \ (j=1,2).$$

The latter equality, (1.1.54), (1.1.56) and the definition of A_j $(j=1,2)$ imply

$$x_*(t) = c_0 + \int_{t_0}^{t} dA(\tau)\cdot x_*(\tau) + f(t) - f(t_0) \text{ for } t \in [a,b].$$

Hence $x_* \in \mathrm{BV}([a,b];\mathbb{R}^n)$ and it is a solution of problem (1.1.1), (1.1.2). But the latter problem has the unique solution x. Therefore, (1.1.20) follows from (1.1.56). □

Proof of Theorem 1.1.9. Let $s \in I$ $(s \neq t_0)$ and $j \in \{1,2\}$ be such that $s \in J_j$. Let $k \in \{\,1,\dots,n\}$ be fixed, and let $u_k(t,s) = (u_{ik}(t,s))_{i=1}^n$ be the k-th column of the matrix $U(t,s)$.

Assume

$$y(t) = (y_i(t))_{i=1}^n \ \text{ for } \ t \in J_j,$$
$$y_i(t) = \gamma_s^{-1}(\alpha_i)(t) \cdot u_{ik}(t,s) \ \ (i=1,\dots,n),$$

where $\gamma_s(\alpha_i)(t) = \gamma^{-1}(\alpha_i)(s) \cdot \gamma(\alpha_i)(t)$, and $\gamma(\alpha_i)(t)$ is a solution of problem (1.1.4) for $\alpha(t) = \alpha_i(t)$. Here, in view of (1.1.5) and (1.1.22), $\gamma(a_{ii})(t)$ is positive for $t \in J_j$.

According to Lemma 1.1.6 and the integration-by-parts formula, we find

$$\begin{aligned} y_i(t) - y_i(r) = \sum_{l\neq i,\, l=1}^{n} \bigg(\int_r^t \gamma_s^{-1}(\alpha_i)(\tau) \cdot u_{lk}(\tau,s)\, du_{il}(\tau) \\ - \sum_{r<\tau\leq t} d_1\gamma_s^{-1}(\alpha_i)(\tau) \cdot u_{lk}(\tau,s)\, d_1 u_{il}(\tau) + \sum_{r\leq\tau<t} d_2\gamma_s^{-1}(\alpha_i)(\tau) \cdot u_{lk}(\tau,s)\, d_2 u_{il}(\tau) \bigg) \\ = \sum_{l\neq i,\, l=1}^{n} \bigg(\int_r^t \gamma_s^{-1}(\alpha_i)(\tau) \cdot u_{lk}(\tau,s)\, ds_c(u_{il})(\tau) \\ + \sum_{r<\tau\leq t} \gamma_s^{-1}(\alpha_i)(\tau-) \cdot u_{lk}(\tau,s)\, d_1 u_{il}(\tau) + \sum_{r\leq\tau<t} \gamma_s^{-1}(\alpha_i)(\tau+) \cdot u_{lk}(\tau,s)\, d_2 u_{il}(\tau) \bigg) \\ = \sum_{l\neq i,\, l=1}^{n} \bigg(\int_r^t \gamma_s^{-1}(\alpha_i)(\tau) \cdot \gamma_s(\alpha_l)(\tau) y_l(\tau)\, ds_c(u_{il})(\tau) \\ + \sum_{r<\tau\leq t} \gamma_s^{-1}(\alpha_i)(\tau-) \cdot \gamma_s(\alpha_i)(\tau)\, d_1 u_{il}(\tau) + \sum_{r\leq\tau<t} \gamma_s^{-1}(\alpha_i)(\tau+) \cdot \gamma_s(\alpha_l)(\tau)\, d_2 u_{il}(\tau) \bigg) \\ \text{for } \ a \leq \tau \leq t \leq b \ \ (i=1,\dots,n). \end{aligned}$$

Hence $y = (y_i)_{i=1}^n$ is a solution of the initial problem

$$dy = dA^*(t) \cdot y, \ \ y(s) = e_k,$$

where $e_k = (\delta_{ik})_{i=1}^n$, $A^*(t) = (a^*_{il}(t))_{i,l=1}^n$, $a^*_{ii}(t) \equiv 0$ and

$$a^*_{il}(t) \equiv \int_s^t \gamma_s^{-1}(\alpha_i)(\tau) \cdot \gamma_s(\alpha_l)(\tau)\, da_{il}(\tau) \ \ (i \neq l;\ i,l=1,\dots,n).$$

In view of the conditions of the lemma, the functions $(-1)^{j+1} a^*_{il}$ $(i \neq l;\ i,l=1,\dots,n)$ are nondecreasing on J_j.

Let

$$\Lambda_s(t) = \operatorname{diag}(\gamma_s(\alpha_1)(t),\dots,\gamma_s(\alpha_n)(t)) \ \text{ for } \ t \in J_j.$$

Using (1.1.18), for the matrix-function $Q(t)$, we have

$$\begin{aligned} I_n + (-1)^j d_j A^*(t) = I_n + (-1)^j \big(\Lambda_s^{-1}(t) + (-1)^j d_j \Lambda_s^{-1}(t)\big)\, d_j \widetilde{A}(t)\, \Lambda_s(t) \\ = I_n - \big(\Lambda_s^{-1}(t) + (-1)^j d_j \Lambda_s^{-1}(t)\big)\big(I_n + (-1)^j d_j Q(t)\big)\Lambda_s(t) \\ + \big(\Lambda_s^{-1}(t) + (-1)^j\, d_j \Lambda_s^{-1}(t)\big)\big(I_n + (-1)^j d_j(\widetilde{A}(t) + Q(t))\big)\Lambda_s(t) \ \text{ for } \ t \in J_j \end{aligned}$$

and

$$I_n + (-1)^j d_j A^*(t) = \big(\Lambda_s^{-1}(t) + (-1)^j\, d_j \Lambda_s^{-1}(t)\big)\big(I_n + (-1)^j d_j(\widetilde{A}(t) + Q(t))\big)\Lambda_s(t) \ \text{ for } \ t \in J_j. \quad (1.1.57)$$

Hence, due to (1.1.23), we obtain

$$\det(I_n + (-1)^j d_j A^*(t)) \neq 0 \text{ for } t \in J_j.$$

Therefore, according to Theorem 1.1.8,

$$\lim_{m\to+\infty} z_m(t) = y(t) \text{ uniformly into } J_j, \tag{1.1.58}$$

where

$$\begin{gathered} z_m(s) = e_k \ \ (m = 0, 1, \dots), \\ z_0(t) = \big(I_n + (-1)^j d_j A^*(t)\big)^{-1} e_k \text{ for } (-1)^j (t-s) < 0, \ \ ; s, t \in J_j, \\ z_m(t) = \big(I_n + (-1)^j d_j A^*(t)\big)^{-1} \left(e_k + \int\limits_s^t dA^*(\tau) \cdot z_{m-1}(\tau) + (-1)^j \, d_j A^*(t) \cdot z_{m-1}(t) \right) \\ \text{for } (-1)^j (t-s) < 0, \ \ s, t \in J_j \ \ (m = 1, 2, \dots). \end{gathered} \tag{1.1.59}$$

Taking into account the equalities

$$d_j \Lambda_s(t) = d_j Q(t) \cdot \Lambda_s(t) \text{ for } t \in J_j,$$

from (1.1.57) we get

$$\begin{aligned} & I_n + (-1)^j d_j A^*(t) \\ & \quad = \big(\Lambda_s^{-1}(t) + (-1)^j \, d_j \Lambda_s^{-1}(t)\big)\big(I_n + (-1)^j B_j(t)\big)\big(\Lambda_s(t) + (-1)^j \, d_j \Lambda_s(t)\big) \text{ for } t \in J_j, \end{aligned} \tag{1.1.60}$$

where

$$B_j(t) \equiv d_j \widetilde{A}(t) \left(I_n + (-1)^j d_j Q(t)\right)^{-1}.$$

Based on this, it is not difficult to verify that

$$\big(I_n + (-1)^j B_j(t)\big)^{-1} = \big(I_n + (-1)^j \, d_j Q(t)\big) \left(I_n + (-1)^j \, d_j(\widetilde{A}(t) + Q(t))\right)^{-1} \text{ for } t \in J_j.$$

Taking into account the above equality, by (1.1.22) and (1.1.24) we have

$$\big(I_n + (-1)^j B_j(t)\big)^{-1} \geq 0 \text{ for } t \in J_j.$$

Therefore, due to (1.1.60),

$$\big(I_n + (-1)^j d_j A^*(t)\big)^{-1} \geq O_{n\times n} \text{ for } t \in J_j, \tag{1.1.61}$$

since by (1.1.22) we have

$$\Lambda_s(t) \geq O_{n\times n} \text{ for } t \in J_j. \tag{1.1.62}$$

(1.1.59) and (1.1.61) imply the estimates

$$z_m(t) \geq \big(I_n + (-1)^j d_j A^*(t)\big)^{-1} e_k \text{ for } (-1)^j (t-s) < 0, \ \ t, s \in J_j \ \ (m = 0, 1, \dots),$$

where $e_k = (\delta_{ik})_{i,k=1}^n$ (δ_{ik} is the Kronecker symbol).

Using now (1.1.58) and (1.1.59), we obtain

$$y(s) \geq e_k, \ \ y(t) \geq \big(I_n + (-1)^j d_j A^*(t)\big)^{-1} e_k \text{ for } (-1)^j (t-s) < 0, \ \ t, s \in J_j. \tag{1.1.63}$$

On the other hand, by the equalities

$$y(t) = \Lambda_s^{-1}(t) \, u_k(t, s) \text{ for } t \in J_j,$$

inequalities (1.1.63) imply

$$u_k(t,s) \geq \Lambda_s(t)\big(I_n + (-1)^j d_j A^*(t)\big)^{-1} e_k \ \text{ for } \ (-1)^j(t-s) < 0, \ \ t,s \in J_j.$$

Since the latter inequalities are fulfilled for every $k \in \{1,\dots,n\}$, we have

$$U(t,s) \geq \Lambda_s(t)\big(I_n + (-1)^j d_j A^*(t)\big)^{-1} \ \text{ for } \ (-1)^j(t-s) < 0 \ \ (j=1,2). \tag{1.1.64}$$

So, by virtue of (1.1.61) and (1.1.62), condition (1.1.64) implies estimate (1.1.25). □

Proof of Theorem 1.1.10. Assume $t_0 < \sup I$ and consider the interval $\{t \in I : t \geq t_0\}$. Then problem (1.1.35)–(1.1.37) has the form

$$dy = dA(t)\cdot y + df(t) \ \text{ for } \ t \geq t_0, \ \ y(t_0) = c_0.$$

Let Z $(Z(t_0) = I_n)$ be a fundamental matrix of the system

$$dz = dA(t)\cdot z \ \text{ for } \ t \geq t_0. \tag{1.1.65}$$

Then by the variation-of-constants formula (see (1.1.14)),

$$y(t) = f(t) - f(s) + Z(t)\Big\{Z^{-1}(s)y(s) - \int_s^t dZ^{-1}(\tau)\cdot(f(\tau)-f(s))\Big\} \ \text{ for } \ s,t \geq t_0. \tag{1.1.66}$$

Put

$$g(t) = -x(t) + x(t_0) + \int_{t_0}^t dA(\tau)\cdot x(\tau) + f(t) - f(t_0) \ \text{ for } \ t \geq t_0.$$

Evidently,

$$dx(t) = dA(t)\cdot x(t) + d(f(t) - g(t)) \ \text{ for } \ t \geq t_0.$$

Let ε be an arbitrary small positive number. Then

$$\begin{aligned} x(t) = f(t) - f(t_0+\varepsilon) - g(t) + g(t_0+\varepsilon) + Z(t)\Big\{&Z^{-1}(t_0+\varepsilon)x(t_0+\varepsilon) \\ &- \int_{t_0+\varepsilon}^t dZ^{-1}(\tau)\cdot(f(\tau) - f(t_0+\varepsilon) - g(\tau) + g(t_0+\varepsilon))\Big\} \ \text{ for } \ t \geq t_0+\varepsilon. \end{aligned}$$

Hence, by (1.1.66), we get

$$x(t) = y(t) + Z(t)Z^{-1}(t_0+\varepsilon)\big(x(t_0+\varepsilon) - y(t_0+\varepsilon)\big) + g_\varepsilon(t) \ \text{ for } \ t \geq t_0+\varepsilon, \tag{1.1.67}$$

where

$$g_\varepsilon(t) = -g(t) + g(t_0+\varepsilon) + Z(t)\int_{t_0+\varepsilon}^t dZ^{-1}(\tau)\cdot(g(\tau) - g(t_0+\varepsilon)).$$

Using the integration-by-parts formula, we have

$$\begin{aligned} g_\varepsilon(t) = -\int_{t_0+\varepsilon}^t U(t,\tau)\,ds_c(g)(\tau) &- \sum_{t_0+\varepsilon<\tau\leq t} U(t,\tau-)\,d_1 g(\tau) \\ &- \sum_{t_0+\varepsilon\leq\tau<t} U(t,\tau+)\,d_2 g(\tau) \ \text{ for } \ t \geq t_0+\varepsilon, \end{aligned} \tag{1.1.68}$$

where $U(t,\tau) = Z(t)Z^{-1}(\tau)$ is the Cauchy matrix of system (1.1.65).

On the other hand, conditions (1.1.29)–(1.1.31) guarantee conditions (1.1.22)–(1.1.24). Hence, according to Theorem 1.1.9, where $Q(t) \equiv \operatorname{diag}(A(t))$, estimate (1.1.25) holds, and by (1.1.68),

$$g_\varepsilon(t) \le 0 \ \text{ for } \ t \ge t_0 + \varepsilon,$$

since by (1.1.32), the function g is non-decreasing on $]t_0, b]$. Hence, this and (1.1.67) result in

$$x(t) \le y(t) + U(t, t_0 + \varepsilon)(x(t_0 + \varepsilon) - y(t_0 + \varepsilon)) \ \text{ for } \ t \ge t_0 + \varepsilon.$$

Passing to the limit as $\varepsilon \to 0$ in the latter inequality and taking into account (1.1.25) and (1.1.33), we get

$$x(t) \le y(t) \ \text{ for } \ t > t_0$$

since by (1.1.36) and (1.1.37),

$$y(t_0+) = c_0 + d_2 A(t_0) \cdot c_0 + d_2 f(t_0).$$

Analogously, we can show the validity of inequality (1.1.34) for $t < t_0$. □

In particular, Theorem 1.1.10 yields Theorem 1.1.11.

Proof of Theorem 1.1.11. Let us introduce the vector-function

$$z(t) = c_0 + \bigg(\int\limits_{t_0}^{t} dA(\tau) \cdot x(\tau) + f(t) - f(t_0) \bigg) \cdot \operatorname{sgn}(t - t_0) \ \text{ for } \ t \in I.$$

It is clear that $z \in \mathrm{BV}_{loc}(I \setminus \{t_0\}; \mathbb{R}^n)$. Moreover, due to (1.1.38), the function z satisfies (1.1.33) and

$$x(t) \le z(t) \ \text{ for } \ t \in I. \tag{1.1.69}$$

Since A is a non-decreasing matrix-function, from the latter inequality we find that x satisfies (1.1.32) on the intervals J_1 and J_2. Therefore, according to Theorem 1.1.10 and (1.1.69), the theorem is proved. □

1.2 The well-posedness of the initial problem

1.2.1 Statement of the problem and formulation of the results

Let $A_0 \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}_{loc}(I; \mathbb{R}^n)$ and $t_0 \in I$, where $I \subset \mathbb{R}$ is an arbitrary interval non-degenerated at the point. Consider the system

$$dx = dA_0(t) \cdot x + df_0(t) \ \text{ for } \ t \in I \tag{1.2.1}$$

under the initial condition

$$x(t_0) = c_0, \tag{1.2.2}$$

where $c_0 \in \mathbb{R}^n$ is an arbitrary constant vector.

Let x_0 be a unique solution of problem (1.2.1), (1.2.2).

Along with the initial problem (1.2.1), (1.2.2), consider the sequence of the initial problems

$$dx = dA_k(t) \cdot x + df_k(t), \tag{1.2.1$_k$}$$
$$x(t_k) = c_k \tag{1.2.2$_k$}$$

$(k = 1, 2, \dots)$, where $A_k \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$, $f_k \in \mathrm{BV}_{loc}(I; \mathbb{R}^n)$ $(k = 1, 2, \dots)$, $t_k \in I$ $(k = 1, 2, \dots)$ and $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$.

If $t \in I$, then we denote $I_t = I \setminus \{t\}$. Moreover, we use the designations

$$\|x\|_{kl} = \sup\big\{ \|x(t)\| : \ t \in I; \ (-1)^l (t - t_k) > 0 \big\} \ \text{ for } \ x \in \mathrm{BV}(I; \mathbb{R}^n) \ \ (l = 1, 2; k = 0, 1, \dots).$$

We assume that $A_k=(a_{kil})_{i,l=1}^n$ and $f_k=(f_{kl})_{l=1}^n$ $(k=0,1,\dots)$ and, without loss of generality, either $t_k<t_0$ $(k=1,2,\dots)$, or $t_k=t_0$ $(k=1,2,\dots)$, or $t_k>t_0$ $(k=1,2,\dots)$.

In this section, we establish the necessary and sufficient and the effective sufficient conditions for the initial problem $(1.2.1_k),(1.2.2_k)$ to have a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty} x_k(t)=x_0(t) \tag{1.2.3}$$

or

$$\lim_{k\to+\infty}\big(x_k(t)+\Delta_{t_k}x_k(t)\big)=x_0(t)+\Delta_{t_0}x_0(t) \tag{1.2.4}$$

uniformly on I, where $\Delta_{t_k}x_k(t)$ $(k=0,1,\dots)$ are the functions defined by (0.0.6).

Note that by (0.0.6) we have

$$x_k(t)+\Delta_{t_k}x_k(t)=\begin{cases} x_k(t-) & \text{for } t<t_k,\\ x_k(t_k) & \text{for } t=t_k,\\ x_k(t+) & \text{for } t>t_k\end{cases}$$
$$(k=0,1,\dots).$$

Along with systems (1.2.1) and $(1.2.1_k)$, we consider the corresponding homogeneous systems

$$dx=dA_0(t)\cdot x \tag{$1.2.1_0$}$$

and

$$dx=dA_k(t)\cdot x \tag{$1.2.1_{k0}$}$$

for any natural k.

Definition 1.2.1. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(A_0,f_0;t_0)$ if for every $c_0\in\mathbb{R}^n$ and a sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k=c_0, \tag{1.2.5}$$

problem $(1.2.1_k),(1.2.2_k)$ has the unique solution x_k for any sufficiently large k, and condition (1.2.3) holds uniformly on I.

We also consider the case where the condition

$$\lim_{k\to+\infty} c_{kj}=c_{0j}, \tag{$1.2.5_j$}$$

if $j\in\{1,2\}$ is such that $(-1)^j(t_k-t_0)\ge 0$ $(k=0,1,\dots)$, holds instead or along with (1.2.5), where

$$c_{kj}=c_k+(-1)^j\big(d_jA_k(t_k)c_k+d_jf_k(t_k)\big)\ \ (j=1,2;\ k=0,1,\dots). \tag{1.2.6}$$

Note that if

$$\lim_{k\to+\infty} d_jA_k(t_k)=d_jA_0(t_0)\ \text{ and }\ \lim_{k\to+\infty} d_jf_k(t_k)=d_jf_0(t_0) \tag{1.2.7}$$

for some $j\in\{1,2\}$, then condition $(1.2.5_j)$ follows from (1.2.5).

Definition 1.2.2. We say that the sequence $(A_k,f_k;t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}_\Delta(A_0,f_0;t_0)$ if for every $c_0\in\mathbb{R}^n$ and a sequence $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ satisfying condition $(1.2.5_j)$, problem $(1.2.1_k),(1.2.2_k)$ has the unique solution x_k for any sufficiently large k, and condition (1.2.4) holds uniformly on I.

It is evident that $S(A_0,f_0;t_0)\subset\mathcal{S}_\Delta(A_0,f_0;t_0)$, but the inverse inclusion is not true, in general. The corresponding example can be easily constructed based on the Example 1.2.1 given below.

From equalities

$$x_k(t-)\equiv(I_n-d_1A(t))x_k(t)\ \text{ and }\ x_k(t+)\equiv(I_n+d_2A(t))x_k(t)\ \ (k=0,1,\dots)$$

follow some conditions guaranteeing the inverse inclusion $S_\Delta(A_0,f_0;t_0)\subset\mathcal{S}(A_0,f_0;t_0)$.

We consider separately the cases of the sets $\mathcal{S}(A_0,f_0;t_0)$ and $\mathcal{S}_\Delta(A_0,f_0;t_0)$.

First, we give the results concerning the set $\mathcal{S}(A_0,f_0;t_0)$.

Theorem 1.2.1. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k=1,2,\dots)$ be such that*

$$\det\big(I_n + (-1)^j d_j A_0(t)\big) \neq 0 \ \text{ for } \ t \in I, \ \ (-1)^j(t-t_0)<0$$
$$\text{and for } \ t=t_0 \ \text{ if } \ j\in\{1,2\} \ \text{ is such that } \ (-1)^j(t_k-t_0)>0 \ \text{ for every } \ k\in\{1,2,\dots\} \tag{1.2.8}$$

and

$$\lim_{k\to+\infty} t_k = t_0. \tag{1.2.9}$$

Then

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in \mathcal{S}(A_0, f_0; t_0) \tag{1.2.10}$$

if and only if there exists a sequence of matrix-functions $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that

$$\inf\big\{\,|\det(H_0(t))| : \ t\in I\big\} > 0, \tag{1.2.11}$$

and the conditions

$$\lim_{k\to+\infty} H_k(t) = H_0(t), \tag{1.2.12}$$

$$\lim_{k\to+\infty}\bigg\{\Big\|\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)(\tau)\Big|_{t_k}^{t}\Big\|\Big(1+\Big|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)\big)\Big|\Big)\bigg\} = 0, \tag{1.2.13}$$

$$\lim_{k\to+\infty}\bigg\{\Big\|\mathcal{D}_{\mathcal{B}}(f_k,H_k;f_0,H_0)(\tau)\Big|_{t_k}^{t}\Big\|\Big(1+\Big|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)\big)\Big|\Big)\bigg\} = 0 \tag{1.2.14}$$

hold uniformly on I, where the operators $\mathcal{D}_{\mathcal{I}}$ and $\mathcal{D}_{\mathcal{B}}$ are defined, respectively, by (0.0.4) *and* (0.0.5).

The following theorem together with Remark 1.2.1 is analogous of the Opial type theorem (see [46]) concerning the case of ordinary differential equations.

Theorem 1.2.2. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$, $c_0\in\mathbb{R}^n$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.8) *and* (1.2.9) *hold. Let, moreover, the sequences of matrix- and vector-functions $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $f_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ and bounded sequences of constant vectors $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ be such that conditions* $(1.2.5_j)$,

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\bigg\{\|A_{kj}(t)-A_{0j}(t)\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k-A_0)\Big|\Big)\bigg\} = 0 \tag{1.2.15}$$

and

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\bigg\{\|f_{kj}(t)-f_{0j}\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k-A_0)\Big|\Big)\bigg\} = 0 \tag{1.2.16}$$

hold if $j\in\{1,2\}$ is such that $(-1)^j(t_k-t_0)\geq 0$ for every $k\in\{1,2,\dots\}$, where c_{kj} $(k=0,1,\dots)$ are defined by (1.2.6),

$$A_{kj}(t) \equiv (-1)^j(A_k(t)-A_k(t_k)) - d_jA_k(t_k) \ \ (j=1,2; k=0,1,\dots)$$

and

$$f_{kj}(t) \equiv (-1)^j(f_k(t)-f_k(t_k)) - d_jf_k(t_k) \ \ (j=1,2; k=0,1,\dots).$$

Then the initial problem $(1.2.1_k), (1.2.2_k)$ has the unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\|x_k(t)-x_0(t)\| = 0. \tag{1.2.17}$$

Remark 1.2.1. In Theorem 1.2.2, it is evident that the sequence x_k $(k = 1, 2, \dots)$ converges to x_0 uniformly on the set $\{t \in I, t \le t_0\}$ if $t_k > t_0$ $(k = 1, 2, \dots)$, and on the set $\{t \in I, t \ge t_0\}$ if $t_k < t_0$ $(k = 1, 2, \dots)$. Moreover, in Theorem 1.2.2, if conditions (1.2.15) and (1.2.16) hold uniformly on the set I, then these conditions are equivalent, respectively, to the conditions

$$\lim_{k\to+\infty}\bigg\{\big\|(A_k(t)-A_k(t_k))-(A_0(t)-A_0(t_0))\big\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k-A_0)\Big|\Big)\bigg\}=0 \tag{1.2.18}$$

and

$$\lim_{k\to+\infty}\bigg\{\big\|(f_k(t)-f_k(t_k))-(f_0(t)-f_0(t_0))\big\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k-A_0)\Big|\Big)\bigg\}=0 \tag{1.2.19}$$

uniformly on I, since (1.2.15) and (1.2.16) imply that

$$\lim_{k\to+\infty} d_jA_k(t)=d_jA_0(t) \ \text{ and } \ \lim_{k\to+\infty} d_jf_k(t)=d_jf_0(t)$$

uniformly on I for every $j \in \{1, 2\}$. In addition, equalities (1.2.7) hold and therefore, in view of (1.2.4) and (1.2.6), conditions $(1.2.5_j)$ $(j = 1, 2)$ hold, too. Thus, in this case, condition (1.2.3) holds uniformly on I.

Theorem 1.2.3. *Let $A_0^* \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0^* \in \mathrm{BV}(I;\mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, $t_0 \in I$, and a sequence of points $t_k \in I$ $(k = 1, 2, \dots)$ be such that condition* (1.2.9) *holds,*

$$\begin{gathered}\det\big(I_n+(-1)^j d_jA_0^*(t)\big)\neq 0 \ \text{ for } \ t\in I, \ \ (-1)^j(t-t_0)<0\\ \text{and for } \ t=t_0 \ \text{ if } \ j\in\{1,2\} \ \text{ is such that } \ (-1)^j(t_k-t_0)>0 \ \text{ for every } \ k\in\{1,2,\dots\},\end{gathered} \tag{1.2.20}$$

and the initial problem

$$dx = dA_0^*(t)\cdot x + df_0^*(t), \tag{1.2.21}$$

$$x(t_0) = c_0^* \tag{1.2.22}$$

has a unique solution x_0^*. Let, moreover, the sequences of matrix- and vector-functions $A_k, H_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ and $f_k, h_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k = 1, 2, \dots)$ and the sequence of constant vectors c_k $(k = 1, 2, \dots)$ be such that the sequence $c_k^* \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ is bounded and the conditions

$$\inf\big\{\,|\det(H_k(t)|:\ t\in I_{t_k}\big\}>0 \ \text{ for every sufficiently large } k, \tag{1.2.23}$$

$$\lim_{k\to+\infty} c_{kj}^* = c_{0j}^*, \tag{1.2.24}$$

$$\lim_{k\to+\infty}\ \sup_{t\in I,\, t\neq t_k}\bigg\{\|A_{kj}^*(t)-A_{0j}^*(t)\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k^*-A_0^*)\Big|\Big)\bigg\}=0 \tag{1.2.25}$$

$$\lim_{k\to+\infty}\ \sup_{t\in I,\, t\neq t_k}\bigg\{\|f_{kj}^*(t)-f_{0j}^*\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k^*-A_0^*)\Big|\Big)\bigg\}=0 \tag{1.2.26}$$

hold for $j \in \{1, 2\}$ being such that $(-1)^j(t_k - t_0) \ge 0$ for every $k \in \{1, 2, \dots\}$, where

$$\begin{gathered}A_{kj}^*(t)=(-1)^j(A_k^*(t)-A_k^*(t_k))-d_jA_k^*(t_k) \ \text{ and}\\ f_{kj}^*(t)=(-1)^j(f_k^*(t)-f_k^*(t_k))-d_jf_k^*(t_k) \ \text{ for } \ t\in I \ \ (j=1,2;\ k=0,1,\dots);\\ A_k^*(t)=\mathcal{I}(H_k,A_k)(t) \ \text{ and } \ f_k^*(t)=h_k(t)-h_k(t_k)+\mathcal{B}(H_k,f_k)(t)\\ -\mathcal{B}(H_k,f_k)(t_k)-\int_{t_k}^{t} dA_k^*(s)\cdot h_k(s) \ \text{ for } \ t\in I \ \ (k=1,2,\dots);\end{gathered}$$

$$c_k^* = H_k(t_k)c_k + h_k(t_k) \ \ (k = 1, 2, \dots),$$
$$c_{kj}^* = c_k^* + (-1)^j \big(d_j A_k^*(t_k)c_k^* + d_j f_k^*(t_k)\big) \ \ (j = 1, 2;\ k = 0, 1, \dots).$$

Then problem $(1.2.1_k), (1.2.2_k)$ *has the unique solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \big\| H_k(t)x_k(t) + h_k(t) - x_0^*(t) \big\| = 0. \tag{1.2.27}$$

Remark 1.2.2. In Theorem 1.2.3, the vector-function $x_k^*(t) \equiv H_k(t)\, x_k(t) + h_k(t)$ is a solution of the problem

$$dx = dA_k^*(t) \cdot x + df_k^*(t), \tag{1.2.21$_k$}$$
$$x(t_k) = c_k^* \tag{1.2.22$_k$}$$

for every sufficiently large k.

Remark 1.2.3. It is evident that if condition (1.2.3) holds uniformly on I, then condition (1.2.17) holds, as well. But the inverse proposition is not true, in general.

We give the corresponding example, which is simple modification of the one given in [32, 56].

Example 1.2.1. Let $I = [-1, 1]$, $n = 1$, and α_k $(k = 1, 2, \dots)$ and β_k $(k = 1, 2, \dots)$ be, respectively, an arbitrary increasing in $[-1, 0)$ and decreasing in $(0, 1]$ sequences such that

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \gamma_k = \gamma_0 \in [0, 1),$$

where $\gamma_k = \alpha_k(\alpha_k - \beta_k)^{-1}$.

Let $t_k = t_0 = 0$ $(k = 1, 2, \dots)$, $c_k = \exp(\gamma_k - \gamma_0)c_0$ $(k = 1, 2, \dots)$, where c_0 is arbitrary, $f_k(t) = f_0(t) \equiv 0_n$ $(k = 1, 2, \dots)$,

$$A_k(t) = \begin{cases} 0 & \text{for } \ t \in [-1, \alpha_k[, \\ (t - \alpha_k)/(\beta_k - \alpha_k) & \text{for } \ t \in [\alpha_k, \beta_k], \\ 1 & \text{for } \ t \in]\beta_k, 1] \ \ (k = 1, 2, \dots). \end{cases}$$

It is not difficult to verify that the unique solution of the corresponding homogeneous initial problem has the form

$$x_k(t) = \begin{cases} c_k & \text{for } \ t \in [-1, \alpha_k[, \\ c_k \ \exp\big(t(\beta_k - \alpha_k)^{-1}\big) & \text{for } \ t \in [\alpha_k, \beta_k], \\ c_k \ \exp(1) & \text{for } \ t \in]\beta_k, 1] \ \ (k = 1, 2, \dots). \end{cases}$$

So, condition (1.2.17) holds, where

$$x_0(t) = \begin{cases} c_0 & \text{for } \ t \in [-1, 0[\,, \\ c_0 \ \exp(\gamma_0) & \text{for } \ t = 0, \\ c_0 \ \exp(1) & \text{for } \ t \in]0, 1], \end{cases}$$

but (1.2.3) does not hold uniformly on $[0, 1]$, since the function $x_0(t)$ is discontinuous at the point $t = 0$.

On the other hand, in the "limit" equation

$$dx = dA_0^*(t) \cdot x,$$

where the function A_0^* is defined as

$$A_0^*(t) = \begin{cases} 0 & \text{for } \ t \in [-1, 0[, \\ \gamma_0 & \text{for } \ t = 0, \\ 1 & \text{for } \ t \in]0, 1], \end{cases}$$

and, therefore, the unique solution of the equation under the condition $x(0)=c_0(1-\gamma_0)^{-1}$ has the form

$$x_0^*(t)=\begin{cases} c_0 & \text{for } t\in[-1,0[\,,\\ c_0(1-\gamma_0)^{-1} & \text{for } t=0,\\ c_0(2-\gamma_0)\,(1-\gamma_0)^{-1} & \text{for } t\in]0,1]. \end{cases}$$

It is evident that $x_0^*\neq x_0$.

On the other hand, x_0 is the solution of the initial problem

$$dx=dA_0(t)\cdot x,\quad x(0)=c_0\exp(\gamma_0),$$

where

$$A_0(t)=\begin{cases} 0 & \text{for } t\in[-1,0[\,,\\ 1-\exp(-\gamma_0) & \text{for } t=0,\\ \exp(1-\gamma_0)-\exp(-\gamma_0) & \text{for } t\in]0,1]. \end{cases}$$

The obtain "anomaly" corresponds to the statement of Theorem 1.2.3, in particular, to condition (1.2.27), where $H_k(t)\equiv I_n$ $(k=1,2,\dots)$, and

$$h_k(t)=\begin{cases} c_0-c_k & \text{for } t\in[-1,\alpha_k[\,,\\ c_0(1-\gamma_k)^{-1}-c_k\,\exp\big(t(\beta_k-\alpha_k)^{-1}\big) & \text{for } t\in[\alpha_k,\beta_k],\\ c_0(2-\gamma_k)\,(1-\gamma_k)^{-1}-c_k\,\exp(1) & \text{for } t\in]\beta_k,1]\ \ (k=1,2,\dots). \end{cases}$$

Indeed, in view of Remark 1.2.2, the function $x_k^*(t)=x_k(t)$ will be a solution of the problem

$$dx=dA_k^*(t)\cdot x,\quad x(0)=c_0(1-\gamma_k)^{-1}$$

for every natural k, where

$$A_k^*(t)=\begin{cases} 0 & \text{for } t\in[-1,\alpha_k[\,,\\ \gamma_k & \text{for } t\in[\alpha_k,\beta_k],\\ 1 & \text{for } t\in]\beta_k,1] \end{cases}\qquad (k=1,2,\dots).$$

So, due to the conditions $\lim\limits_{k\to+\infty}\gamma_k=\gamma_0$, we have

$$\lim_{k\to+\infty}\ \sup_{t\in I,\,t\neq t_k}\|A_k^*(t)-A_0^*(t)\|=0.$$

Below, we consider, mainly, the well-posedness question only on the whole interval I. For the last case, instead of (1.2.24) we consider the condition

$$\lim_{k\to+\infty}c_k^*=c_0^*, \tag{1.2.28}$$

and instead of conditions (1.2.25) and (1.2.26), we consider, respectively, the conditions

$$\lim_{k\to+\infty}\bigg\{\big\|(A_k^*(t)-A_k^*(t_k))-(A_0^*(t)-A_0^*(t_0))\big\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k^*-A_0^*)\Big|\Big)\bigg\}=0 \tag{1.2.29}$$

and

$$\lim_{k\to+\infty}\bigg\{\big\|(f_k^*(t)-f_k^*(t_k))-(f_0^*(t)-f_0^*(t_0))\big\|\Big(1+\Big|\bigvee_{t_k}^{t}(A_k^*-A_0^*)\Big|\Big)\bigg\}=0 \tag{1.2.30}$$

uniformly on I.

Corollary 1.2.1. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$, $t_0 \in I$ and the sequences $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $c_k \in \mathbb{R}^n$ $(k=1,2,\dots)$ and $t_k \in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.8), (1.2.9), (1.2.11) *and*

$$\lim_{k\to+\infty}(c_k-\varphi_k(t_k))=c_0 \tag{1.2.31}$$

hold, and conditions (1.2.12), (1.2.13) *and*

$$\lim_{k\to+\infty}\left\{\left\|\mathcal{D}_{\mathcal{B}}(f_k-\varphi_k,H_k;f_0,H_0)(\tau)\Big|_{t_k}^{t}+\int_{t_k}^{t}d\mathcal{I}(H_k,A_k)(\tau)\cdot\varphi_k(\tau)\right\| \times\left(1+\Big|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)\big)\Big|\right)\right\}=0 \tag{1.2.32}$$

hold uniformly on I, where $H_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$, $\varphi_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, and the operators $\mathcal{D}_{\mathcal{B}}$ and $\mathcal{D}_{\mathcal{I}}$ are defined, respectively, by (0.0.4) *and* (0.0.5). *Then problem* $(1.2.1_k),(1.2.2_k)$ *has the unique solution x_k for any sufficiently large k and*

$$\lim_{k\to+\infty}\|x_k(t)-\varphi_k(t)-x_0(t)\|=0 \tag{1.2.33}$$

uniformly on I.

Below, we will give some sufficient conditions guaranteeing inclusion (1.2.10). Towards this end, we establish a theorem, other than Theorem 1.2.1, concerning the necessary and sufficient conditions for the inclusion, as well as the corresponding propositions.

Theorem 1.2.1′. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$, $t_0 \in I$ and the sequence of points $t_k \in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.8) *and* (1.2.9) *hold. Then inclusion* (1.2.10) *holds if and only if there exists a sequence of the matrix-functions $H_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that conditions* (1.2.11) *and*

$$\limsup_{k\to+\infty}\bigvee_I(H_k+\mathcal{B}(H_k,A_k))<+\infty \tag{1.2.34}$$

hold, and conditions (1.2.12),

$$\lim_{k\to+\infty}\big(\mathcal{B}(H_k,A_k)(t)-\mathcal{B}(H_k,A_k)(t_k)\big)=\mathcal{B}(H_0,A_0)(t)-\mathcal{B}(H_0,A_0)(t_0) \tag{1.2.35}$$

and

$$\lim_{k\to+\infty}\big(\mathcal{B}(H_k,f_k)(t)-\mathcal{B}(H_k,f_k)(t_k)\big)=\mathcal{B}(H_0,f_0)(t)-\mathcal{B}(H_0,f_0)(t_0) \tag{1.2.36}$$

hold uniformly on I.

Remark 1.2.4. Due to (1.2.11), (1.2.12), there exists a positive number r such that

$$\sup\left\{\Big|\bigvee_{t_0}^{t}(\mathcal{I}(H_k,A_k))\Big|:\ t\in I\right\}\le r\bigvee_I(H_k+\mathcal{B}(H_k,A_k))\ \ (k=0,1,\dots).$$

In addition, in view of Lemma 1.2.4 (see below), by conditions (1.2.35) and (1.2.36), we get

$$\lim_{k\to+\infty}\left\|\mathcal{D}_{\mathcal{I}}(A_k,H_k;A_0,H_0)(\tau)\Big|_{t_k}^{t}\right\|=0$$

uniformly on I. Therefore, owing to (1.2.34) and (1.2.36), conditions (1.2.13) and (1.2.14) hold uniformly on I.

Theorem 1.2.1″. *Let conditions* (1.2.9) *and*

$$\det\big(I_n+(-1)^j d_j A_k(t)\big)\neq 0 \ \ for \ \ t\in[a,b] \ \ (j=1,2; \ \ k=0,1,\dots)$$

hold. Then inclusion (1.2.10) *holds if and only if the conditions*

$$\lim_{k\to+\infty} X_k^{-1}(t)=X_0^{-1}(t) \tag{1.2.37}$$

and

$$\lim_{k\to+\infty}\big(\mathcal{B}(X_k^{-1},f_k)(t)-\mathcal{B}(X_k^{-1},f_k)(t_k)\big)=\mathcal{B}(X_0^{-1},f_0)(t)-\mathcal{B}(X_0^{-1},f_0)(t_0)$$

hold uniformly on $[a,b]$, *where* X_k *is the fundamental matrix of the homogeneous system* $(1.2.1_{k0})$ *for every* $k\in\{0,1,\dots\}$.

Theorem 1.2.2′. *Let* $A_0\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in \mathrm{BV}(I;\mathbb{R}^{n})$, $c_0\in\mathbb{R}^n$, $t_0\in I$ *and the sequence of points* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.8) *and* (1.2.9) *hold. Let, moreover, the sequences* $A_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n})$ $(k=1,2,\dots)$ *and* $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ *be such that conditions* (1.2.5) *and*

$$\limsup_{k\to+\infty}\bigvee_I(A_k)<+\infty \tag{1.2.38}$$

hold, and the conditions

$$\lim_{k\to+\infty}\big(A_k(t)-A_k(t_k)\big)=A_0(t)-A_0(t_0) \tag{1.2.39}$$

and

$$\lim_{k\to+\infty}\big(f_k(t)-f_k(t_k)\big)=f_0(t)-f_0(t_0) \tag{1.2.40}$$

hold uniformly on I. *Then the initial problem* $(1.2.1_k),(1.2.2_k)$ *has the unique solution* x_k *for any sufficiently large* k *and condition* (1.2.3) *holds uniformly on* I.

Theorem 1.2.3′. *Let* $A_0^*\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0^*\in \mathrm{BV}(I;\mathbb{R}^{n})$, $c_0^*\in\mathbb{R}^n$, $t_0\in I$ *and the sequence of points* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.9) *and* (1.2.20) *hold, and the initial problem* (1.2.21), (1.2.22) *has a unique solution* x_0^*. *Let, moreover, the sequences* $A_k,H_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k,h_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n})$ $(k=1,2,\dots)$ *and* $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ *be such that conditions* (1.2.23),

$$\lim_{k\to+\infty}(H_k(t_k)c_k+h_k(t_k))=c_0^* \tag{1.2.41}$$

and

$$\limsup_{k\to+\infty}\bigvee_I(A_k^*)<+\infty \tag{1.2.42}$$

hold, and the conditions

$$\lim_{k\to+\infty}\big(A_k^*(t)-A_k^*(t_k)\big)=A_0^*(t)-A_0^*(t_0) \tag{1.2.43}$$

and

$$\lim_{k\to+\infty}\big(f_k^*(t)-f_k^*(t_k)\big)=f_0^*(t)-f_0^*(t_0) \tag{1.2.44}$$

hold uniformly on I, *where the matrix-functions* $A_k^*(t)$ $(k=1,2,\dots)$ *and vector-functions* $f_k^*(t)$ $(k=1,2,\dots)$ *are defined as in Theorem* 1.2.3. *Then problem* $(1.2.1_k),(1.2.2_k)$ *has the unique solution* x_k *for any sufficiently large* k *and condition* (1.2.27) *holds uniformly on* I.

Corollary 1.2.1′. *Let* $A_0\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in \mathrm{BV}(I;\mathbb{R}^{n})$, $c_0\in\mathbb{R}^n$, $t_0\in I$, *and the sequences* $A_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n})$ $(k=1,2,\dots)$, $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ *and*

$t_k \in I$ $(k = 1, 2, \dots)$ *be such that conditions* (1.2.8), (1.2.9), (1.2.11), (1.2.31) *and* (1.2.34) *hold, and conditions* (1.2.12), (1.2.35) *and*

$$\lim_{k\to+\infty}\bigg(\mathcal{B}(H_k, f_k - \varphi_k)(t) - \mathcal{B}(H_k, f_k - \varphi_k)(t_k) + \int_{t_k}^{t} d\mathcal{B}(H_k, A_k)(\tau)\cdot\varphi_k(\tau)\bigg) = \mathcal{B}(H_0, f_0)(t) - \mathcal{B}(H_0, f_0)(t_0) \tag{1.2.45}$$

hold uniformly on I*, where* $H_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$*, and* $\varphi_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k = 1, 2, \dots)$*. Then problem* $(1.2.1_k), (1.2.2_k)$ *has the unique solution* x_k *for any sufficiently large* k *and condition* (1.2.33) *holds uniformly on* I*.*

Corollary 1.2.2. *Let* $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$*,* $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$*,* $t_0 \in I$*, and the sequences* $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$*,* $f_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k = 1, 2, \dots)$ *and* $t_k \in I$ $(k = 1, 2, \dots)$ *be such that conditions* (1.2.9), (1.2.11) *and* (1.2.34) *hold, and conditions* (1.2.12),

$$\lim_{k\to+\infty}\int_{t_k}^{t} H_k(s)\,dA_k(s) = \int_{t_0}^{t} H_0(s)\,dA_0(s), \tag{1.2.46}$$

$$\lim_{k\to+\infty}\int_{t_k}^{t} H_k(s)\,df_k(s) = \int_{t_0}^{t} H_0(s)\,df_0(s), \tag{1.2.47}$$

$$\lim_{k\to+\infty} d_j A_k(t) = d_j A_0(t) \ \ (j = 1, 2), \tag{1.2.48}$$

$$\lim_{k\to+\infty} d_j f_k(t) = d_j f_0(t) \ \ (j = 1, 2) \tag{1.2.49}$$

hold uniformly on I*, where* $H_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$*. Let, moreover, either*

$$\limsup_{k\to+\infty}\sum_{t\in I}\big(\|d_j A_k(t)\| + \|d_j f_k(t)\|\big) < +\infty \ \ (j = 1, 2) \tag{1.2.50}$$

or

$$\limsup_{k\to+\infty}\sum_{t\in I}\|d_j H_k(t)\| < +\infty \ \ (j = 1, 2). \tag{1.2.51}$$

Then inclusion (1.2.10) *holds.*

Corollary 1.2.3. *Let* $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$*,* $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$*,* $t_0 \in I$*, and the sequences* $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$*,* $f_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k = 1, 2, \dots)$ *and* $t_k \in I$ $(k = 1, 2, \dots)$ *be such that conditions* (1.2.9), (1.2.11) *and* (1.2.34) *hold, and conditions* (1.2.12), (1.2.39), (1.2.40),

$$\lim_{k\to+\infty}\int_{t_k}^{t} dH_k(s)\cdot A_k(s) = A^*(t) - A^*(t_0) \tag{1.2.52}$$

and

$$\lim_{k\to+\infty}\int_{t_k}^{t} dH_k(s)\cdot f_k(s) = f^*(t) - f^*(t_0) \tag{1.2.53}$$

hold uniformly on I*, where* $H_0(t) \equiv I_n$*,* $H_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$*,* $A^* \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ *and* $f^* \in \mathrm{BV}(I;\mathbb{R}^n)$*. Let, moreover, system* (1.2.21)*, where* $A_0^*(t) \equiv A_0(t) - A^*(t)$*,* $f_0^*(t) \equiv f_0(t) - f^*(t)$*, have a unique solution satisfying condition* (1.2.2)*. Then*

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in \mathcal{S}(A_0^*, f_0^*; t_0).$$

Corollary 1.2.4. *Let* $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$*,* $f_0\in\mathrm{BV}(I;\mathbb{R}^n)$*,* $t_0\in I$*, and the sequences* $A_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$*,* $f_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.8) *and* (1.2.9) *hold. Let, moreover, there exist a natural number* m *and matrix-functions* $B_j\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(j=1,\dots,m-1)$ *such that*

$$\limsup_{k\to+\infty}\bigvee_I(A_{km})<+\infty \tag{1.2.54}$$

and the conditions

$$\lim_{k\to+\infty}(A_{kj}(t)-A_{kj}(t_k))=B_j(t)-B_j(t_0)\ \ (j=1,\dots,m-1), \tag{1.2.55}$$

$$\lim_{k\to+\infty}(A_{km}(t)-A_{km}(t_k))=A_0(t)-A_0(t_0), \tag{1.2.56}$$

$$\lim_{k\to+\infty}(f_{km}(t)-f_{km}(t_k))=f_0(t)-f_0(t_0) \tag{1.2.57}$$

hold uniformly on I*, where*

$$\begin{gathered}A_{kj}(t)=H_{k\,j-1}(t)+\mathcal{B}(H_{k\,j-1},A_k)(t),\ \ f_{kj}(t)=\mathcal{B}(H_{k\,j-1},f_k)(t)\\ \text{for}\ \ t\in I\ \ (j=1,\dots,m;\ k=1,2,\dots);\\ H_{k0}(t)=I_n,\ \ H_{kj}(t)=\big(I_n-A_{kj}(t)+A_{kj}(t_k)+B_j(t)-B_j(t_0)\big)H_{k\,j-1}(t)\\ \text{for}\ \ t\in I\ \ (j=1,\dots,m-1;\ k=1,2,\dots).\end{gathered}$$

Then inclusion (1.2.10) *holds.*

If $m=1$, then Corollary 1.2.4 coincides with Theorem 1.2.2′.
If $m=2$, then Corollary 1.2.4 has the following form.

Corollary 1.2.4′. *Let* $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$*,* $f_0\in\mathrm{BV}(I;\mathbb{R}^n)$*,* $t_0\in I$*, and the sequences* $A_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$*,* $f_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.8)*,* (1.2.9) *and* (1.2.34) *hold, and the conditions*

$$\begin{gathered}\lim_{k\to+\infty}\big(A_k(t)-A_k(t_k)\big)=B(t)-B(t_0),\\ \lim_{k\to+\infty}\big(\mathcal{B}(H_k,A_k)(t)-\mathcal{B}(H_k,A_k)(t_k)\big)=A_0(t)-A_0(t_0),\\ \lim_{k\to+\infty}(\mathcal{B}(H_k,f_k)(t)-\mathcal{B}(H_k,f_k)(t_k))=f_0(t)-f_0(t_0)\end{gathered}$$

hold uniformly on I*, where* $B\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ *and*

$$H_k(t)=I_n-A_k(t)+A_k(t_k)+B(t)-B(t_k)\ \ \text{for}\ \ t\in I\ \ (k=1,2,\dots).$$

Then inclusion (1.2.10) *holds.*

Corollary 1.2.5. *Let* $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$*,* $f_0\in\mathrm{BV}(I;\mathbb{R}^n)$*,* $t_0\in I$*, and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.8) *and* (1.2.9) *hold. Then inclusion* (1.2.10) *holds if and only if there exist matrix-functions* $B_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *such that*

$$\limsup_{k\to+\infty}\bigvee_I(A_k-B_k)<+\infty \tag{1.2.58}$$

and

$$\det\big(I_n+(-1)^jd_jB_k(t)\big)\neq0\ \ \text{for}\ \ t\in I\ \ (j=1,2;k=0,1,\dots), \tag{1.2.59}$$

and the conditions

$$\lim_{k\to+\infty}Z_k^{-1}(t)=Z_0^{-1}(t), \tag{1.2.60}$$

$$\lim_{k\to+\infty}\big(\mathcal{B}(Z_k^{-1},A_k)(t)-\mathcal{B}(Z_k^{-1},A_k)(t_k)\big)=\mathcal{B}(Z_0^{-1},A_0)(t)-\mathcal{B}(Z_0^{-1},A_0)(t_0) \tag{1.2.61}$$

and

$$\lim_{k\to+\infty}\big(\mathcal{B}(Z_k^{-1},f_k)(t)-\mathcal{B}(Z_k^{-1},f_k)(t_k)\big)=\mathcal{B}(Z_0^{-1},f_0)(t)-\mathcal{B}(Z_0^{-1},f_0)(t_0) \tag{1.2.62}$$

hold uniformly on I, where Z_k $(Z_k(t_k)=I_n)$ is a fundamental matrix of the homogeneous system

$$dx=dB_k(t)\cdot x \tag{1.2.63}$$

for every $k\in\{0,1\dots\}$.

Corollary 1.2.6. *Let $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in\mathrm{BV}(I;\mathbb{R}^n)$, $t_0\in I$, $A_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, and $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.8) *and* (1.2.9) *hold. Let, moreover, there exist matrix-functions $B_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$, satisfying the Lappo–Danilevskiĭ condition, such that conditions* (1.2.58) *and*

$$\det\big(I_n+(-1)^j d_jB_0(t)\big)\neq 0 \ \ \textit{for}\ \ t\in I \ \ (j=1,2) \tag{1.2.64}$$

hold, and the conditions

$$\lim_{k\to+\infty}(B_k(t)-B_k(t_k))=B_0(t)-B_0(t_0), \tag{1.2.65}$$

$$\lim_{k\to+\infty}\int_{t_k}^{t} Z_k^{-1}(\tau)\,d\mathcal{A}(B_k,A_k)(\tau)=\int_{t_0}^{t} Z_0^{-1}(\tau)\,d\mathcal{A}(B_0,A_0)(\tau) \tag{1.2.66}$$

and

$$\lim_{k\to+\infty}\int_{t_k}^{t} Z_k^{-1}(\tau)\,d\mathcal{A}(B_k,f_k)(\tau)=\int_{t_0}^{t} Z_0^{-1}(\tau)\,d\mathcal{A}(B_0,f_0)(\tau) \tag{1.2.67}$$

hold uniformly on I, where $\mathcal{A}$ is the operator defined by (0.0.1), *and Z_k $(Z_k(t_k)=I_n)$ is a fundamental matrix of the homogeneous system* (1.2.63) *for every $k\in\{0,1\dots\}$. Then inclusion* (1.2.10) *holds.*

Remark 1.2.5. In Corollary 1.2.6, due to (1.2.65), it follows from (1.2.64) that condition (1.2.59) holds for every sufficiently large k and, therefore, the fundamental matrices Z_k $(k=0,1,\dots)$ exist. Hence conditions (1.2.66) and (1.2.67) of the corollary are correct.

Remark 1.2.6. In Corollaries 1.2.5 and 1.2.6, if we assume that the matrix functions B_k $(k=0,1,\dots)$ are continuous, then conditions (1.2.59) and (1.2.64) are, obviously, valid. Moreover, due to the integration-by-parts formula and definitions of operators $\mathcal{A}$ and $\mathcal{B}$, each of conditions (1.2.61) and (1.2.66) has the form

$$\lim_{k\to+\infty}\int_{t_k}^{t} Z_k^{-1}(\tau)\,dA_k(\tau)=\int_{t_0}^{t} Z_0^{-1}(\tau)\,dA_0(\tau),$$

and each of conditions (1.2.62) and (1.2.67) has the form

$$\lim_{k\to+\infty}\int_{t_k}^{t} Z_k^{-1}(\tau)\,df_k(\tau)=\int_{t_0}^{t} Z_0^{-1}(\tau)\,df_0(\tau).$$

Remark 1.2.7. If the matrix-function $B\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, satisfying the Lappo–Danilevskiĭ condition, and $s\in I$ are such that $\det(I_n+(-1)^j d_jB(t))\neq 0$ for $t\in I$, $(-1)^j(t-s)<0$ $(j=1,2)$, then owing to (1.1.15), the fundamental matrix Z $(Z(s)=I_n)$ of the homogeneous system

$$dx=dB(t)\cdot x$$

has the form

$$Z(t)=\begin{cases}\exp\big(S_c(B)(t)-S_c(B)(s)\big)\prod\limits_{s<\tau\le t}(I_n-d_1B(\tau))^{-1}\prod\limits_{s\le\tau<t}(I_n+d_2B(\tau)) & \text{for } t>s,\\ \exp\big(S_c(B)(s)-S_c(B)(t)\big)\prod\limits_{t<\tau\le s}(I_n-d_1B(\tau))\prod\limits_{t\le\tau<s}(I_n+d_2B(\tau))^{-1} & \text{for } t<s,\\ I_n & \text{for } t=s.\end{cases} \tag{1.2.68}$$

Corollary 1.2.7. *Let $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in\mathrm{BV}(I;\mathbb{R}^n)$, $t_0\in I$, and the sequences $A_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ and $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.8), (1.2.9) *and*

$$\limsup_{k\to+\infty}\sum_{t\in I}\|d_jA_k(t)\|<+\infty\ \ (j=1,2) \tag{1.2.69}$$

hold. Let, moreover, the matrix-functions $S_c(A_k)$ $(k=0,1,\dots)$ satisfy the Lappo–Danilevskiĭ condition and the conditions

$$\lim_{k\to+\infty}\big(S_c(A_k)(t)-S_c(A_k)(t_k)\big)=S_c(A_0)(t)-S_c(A_0)(t_0), \tag{1.2.70}$$

$$\lim_{k\to+\infty}S_j(A_k)(t)=S_j(A_0)(t)\ \ (j=1,2), \tag{1.2.71}$$

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}\exp\big(-S_c(A_k)(\tau)+S_c(A_k)(t_k)\big)\,dA_k(\tau)=\int\limits_{t_0}^{t}\exp(-S_c(A_0)(\tau)+S_c(A_0)(t_0))\,dA_0(\tau) \tag{1.2.72}$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}\exp\big(-S_c(A_k)(\tau)+S_c(A_k)(t_k)\big)\,df_k(\tau)=\int\limits_{t_0}^{t}\exp(-S_c(A_0)(\tau)+S_c(A_0)(t_0))\,df_0(\tau) \tag{1.2.73}$$

hold uniformly on I. Then inclusion (1.2.10) *holds.*

Corollary 1.2.8. *Let $A_0\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in\mathrm{BV}(I;\mathbb{R}^n)$, $t_0\in I$, and the sequences $A_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k\in\mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ and $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.8), (1.2.9),

$$\limsup_{k\to+\infty}\sum_{i,l=1;\,i\ne l}^{n}\bigvee_I(a_{kil})<+\infty$$

and

$$1+(-1)^jd_ja_{0ii}(t))\neq0\ \ \textit{for}\ \ t\in I\ \ (j=1,2;\ i=1,\dots,n)$$

hold, and the conditions

$$\lim_{k\to+\infty}\big(a_{kii}(t)-a_{kii}(t_k)\big)=a_{0ii}(t)-a_{0ii}(t_0)\ \ (i=1,\dots,n),$$

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}z_{kii}^{-1}(\tau)\,d\mathcal{A}(a_{kii},a_{kil})(\tau)=\int\limits_{t_0}^{t}z_{0ii}^{-1}(\tau)\,d\mathcal{A}(a_{0ii},a_{0il})(\tau)\ \ (i\neq l;\ i,l=1,\dots,n)$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} z_{kii}^{-1}(\tau)\,d\mathcal{A}(a_{kii},f_{ki})(\tau)=\int\limits_{t_0}^{t} z_{0ii}^{-1}(\tau)\,d\mathcal{A}(a_{0ii},f_{0i})(\tau)\ \ (i=1,\dots,n)$$

hold uniformly on I, where $\mathcal{A}$ is the operator defined by (0.0.1)*, and z_{kii}, defined according to* (1.2.68)*, is a solution of the initial problem*

$$dz(t)=z(t)da_{kii}(t),\ \ z(t_k)=1\ \ (i=1,\dots,n)$$

for every sufficiently large k. Then inclusion (1.2.10) *holds.*

Remark 1.2.8. For Corollary 1.2.8, the remark analogous to Remark 1.2.6 is true, i.e.,

$$1+(-1)^j d_j a_{0ii}(t))\neq 0\ \ \text{for}\ \ t\in I\ \ (j=1,2;\ i=1,\dots,n)$$

for every sufficiently large k and, therefore, all conditions of the corollary are correct.

Remark 1.2.9. In Theorems 1.2.1′–1.2.3′ and Corollaries 1.2.1′, 1.2.2–1.2.8, as well as in the statement below, we may, without loss of generality, assume that $H_0(t)=I_n$. In this case, it is evident that

$$\mathcal{I}(H_0,Y)(t)-\mathcal{I}(H_0,Y)(s)=Y(t)-Y(s)\ \ \text{for}\ \ Y\in \mathrm{BV}(I;\mathbb{R}^{n\times n})\ \ \text{and}\ \ t,s\in I.$$

Remark 1.2.10. If for some k the matrix-function A_k is such that $A_k(t)=const$ for $t\in I_0$, where $I_0\subset I$ is an interval, then, due to the proof of the necessity in Theorem 1.2.1, we conclude that $H_k(t)=const$ for $t\in I_0$, as well, since $H_k(t)=X_k^{-1}(t)$, where X_k is the fundamental matrix of homogeneous system $(1.2.1_{k0})$. Therefore, $X_k(t)=const$ for $t\in I_0$. So, everywhere in the results given above we can assume that the matrix-function H_k has the described property.

Remark 1.2.11. The following example shows that if condition (1.2.69) is violated, then the statement of Corollary 1.2.7 is not true, in general.

Example 1.2.2. Let $I=[0,1]$, $A_0(t)=0$, $f_0(t)=f_k(t)=0$, $t_k=t_0=0$, $c_k=c_0=1$,

$$A_k(t)=\begin{cases} k^{-1} & \text{for}\ \ t\in\bigcup\limits_{i=1}^{2k^2}]t_{2i-1\,k},t_{2i\,k}],\\[2ex] 0 & \text{for}\ \ t\notin\bigcup\limits_{i=1}^{2k^2}]t_{2i-1\,k},t_{2i\,k}],\end{cases}$$

where $t_{ik}=(2k^2+1)^{-1}\,i$ $(i=0,\dots,2k^2)$ for every natural k. Then all conditions of Corollary 1.2.7 are fulfilled except (1.2.69). It is evident that $x_0(t)\equiv 1$. On the other hand, the initial problem $(1.2.1_k),(1.2.2_k)$ has the unique solution x_k and, in addition, $x_k(1)=(1-\frac{1}{k^2})^{k^2}$. Therefore, condition (1.2.3) is not valid, since

$$\lim_{k\to+\infty}x_k(1)=\exp(-1)\neq x_0(1).$$

Remark 1.2.12. In some results given above, the matrix-functions A_k $(k=1,2,\dots)$ and the vector-functions f_k $(k=1,2,\dots)$ have really bounded total variations on the whole interval I.

The examples concerning the importance of some conditions given in the above results, one can find in Section 3.1. See Examples 3.1.1, 3.1.2, 3.1.3, etc.

Now, we give the results concerning the set $\mathcal{S}_\Delta(A_0,f_0;t_0)$.

In the case under consideration, we use the following notation. Let the matrix-functions $A_{\Delta k}$ $(k=0,1,\dots)$ and the vector-functions $f_{\Delta k}$ $(k=0,1,\dots)$ be defined by the equalities

$$A_{\Delta k}(t_k)=O_{n\times n},\quad f_{\Delta k}(t_k)=0_n; \tag{1.2.74}$$

$$A_{\Delta k}(t)=\int\limits_{t_k-}^{t} dA_k(\tau+)\cdot(I_n+d_2A_k(\tau))^{-1}-d_1A(t_k \ \text{ and}$$

$$f_{\Delta k}(t)=f_k(t+)-f(t_k)+\int\limits_{t_k-}^{t} dA_k(\tau)\cdot d_2f_k(\tau) \ \text{ for } \ t<t_k; \tag{1.2.75}$$

$$A_{\Delta k}(t)=\int\limits_{t_k+}^{t} dA_k(\tau-)\cdot(I_n-d_1A_k(\tau))^{-1}+d_2A_k(t_k) \ \text{ and}$$

$$f_{\Delta k}(t)=f_k(t-)-f(t_k)+\int\limits_{t_k+}^{t} dA_k(\tau)\cdot d_1f_k(\tau) \ \text{ for } \ t>t_k. \tag{1.2.76}$$

Theorem 1.2.4. *Let $A_0\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in \mathrm{BV}(I;\mathbb{R}^n)$, $t_0\in I$, and a sequence of points $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.8) *and* (1.2.9) *hold. Let, moreover, the sequence of matrix-functions $A_k\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ be such that*

$$\det\big(I_n+(-1)^jd_jA_k(t)\big)\neq 0 \ \textit{ for } \ t\in I, \quad (-1)^j(t-t_k)<0 \ \ (j=1,2) \tag{1.2.77}$$

for every sufficiently large k. Then

$$((A_k,f_k;t_k))_{k=1}^{+\infty}\in \mathcal{S}_\Delta(A_0,f_0;t_0) \tag{1.2.78}$$

if and only if there exists a sequence of matrix-functions $H_k\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that condition (1.2.11) *holds, and conditions* (1.2.12),

$$\lim_{k\to+\infty}\bigg\{\Big\|\mathcal{I}(H_k,A_{\Delta\,k})(\tau)\Big|_{t_k}^{t}-\mathcal{I}(H_0,A_{\Delta\,0})(\tau)\Big|_{t_0}^{t}\Big\|\bigg(1+\Big|\bigvee_{t_k}^{t}(\mathcal{I}(H_k,A_{\Delta\,k}))\Big|\bigg)\bigg\}=0 \tag{1.2.79}$$

and

$$\lim_{k\to+\infty}\bigg\{\Big\|\mathcal{B}(H_k,f_{\Delta\,k})(\tau)\Big|_{t_k}^{t}-\mathcal{B}(H_0,f_{\Delta\,0})(\tau)\Big|_{t_0}^{t}\Big\|\bigg(1+\Big|\bigvee_{t_k}^{t}(\mathcal{I}(H_k,A_{\Delta\,k}))\Big|\bigg)\bigg\}=0 \tag{1.2.80}$$

hold uniformly on I, where the matrix- and the vector-functions $A_{\Delta\,k}$ and $f_{\Delta\,k}$ $(k=0,1,\dots)$ are defined by (1.2.74)–(1.2.76)*, respectively.*

Theorem 1.2.5. *Let $A_0\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in \mathrm{BV}(I;\mathbb{R}^n)$, $t_0\in I$, and a sequence of points $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.8) *and* (1.2.9) *hold. Let, moreover, the sequences $A_k\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $f_k\in \mathrm{BV}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ be such that condition* (1.2.77) *holds for every sufficiently large k, and the conditions*

$$\lim_{k\to+\infty}\bigg(\|A_{\Delta\,k}(t)-A_{\Delta\,0}(t)\|+\Big|\bigvee_{t_k}^{t}\big(\mathcal{B}(A_{\Delta\,k}-A_{\Delta\,0},A_{\Delta\,k})\big)\Big|\bigg)=0 \tag{1.2.81}$$

and

$$\lim_{k\to+\infty}\big\|f_{\Delta\,k}(t)-f_{\Delta\,0}(t)-\mathcal{B}(A_{\Delta\,k}-A_{\Delta\,0},f_{\Delta\,k})(t)+\mathcal{B}(A_{\Delta\,k}-A_{\Delta\,0},f_{\Delta\,k})(t_k)\big\|=0 \tag{1.2.82}$$

hold uniformly on I. Then inclusion (1.2.78) *holds.*

Theorem 1.2.4 is analogous to Theorem 1.2.1 for the matrix-functions $A_{\Delta\,k}$ $(k=0,1,\dots)$ and the vector-functiosn $f_{\Delta\,k}$ $(k=0,1,\dots)$ defined above.

As for Theorem 1.2.5, it is a particular case of Theorem 1.2.4 when $H_k(t)\equiv I_n$ $(k=0,1,\dots)$.

It is evident that the results, analogous to those given above for the considered case, are true.

Remark 1.2.13. In Theorem 1.2.2, under condition $(1.2.5_j)$, for sufficiently large natural k, we have the following three cases: a) $t_k < t_0$ for $k \in \mathbb{N}_-$; b) $t_k = t_0$ for $k \in \mathbb{N}_0$ or c) $t_k > t_0$ for $k \in \mathbb{N}_+$, where $\mathbb{N}_-$, $\mathbb{N}_0$ and $\mathbb{N}_+$ are some infinite subsets of natural numbers. It follows from the proof of theorem that in addition to the statement of theorem we have the following propositions:

1) if $\mathbb{N}_-=\varnothing$ and $\mathbb{N}_0=\varnothing$, then condition (1.2.3) is valid uniformly on the set $\{t\in I : t\leq t_0\}$, as well;
2) if $\mathbb{N}_0=\varnothing$ and $\mathbb{N}_+=\varnothing$, then condition (1.2.3) is valid uniformly on the set $\{t\in I : t\geq t_0\}$, as well;
3) if $\mathbb{N}_- = \varnothing$ and $\mathbb{N}_+ = \varnothing$, then condition (1.2.3) is valid uniformly into $I \setminus \{t_0\}$, i.e., on the every closed interval from I.

1.2.2 Auxiliary propositions

Lemma 1.2.1. *Let $a \in I$ be fixed. Then:*

(a) *if $X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$, $Y \in \mathrm{BV}_{loc}(I;\mathbb{R}^{m\times l})$ and $Z \in \mathrm{BV}_{loc}(I;\mathbb{R}^{l\times k})$, then*

$$\mathcal{B}(X,\mathcal{B}(Y,Z))(t) = \mathcal{B}(X\,Y,Z)(t) \ \ \textit{for} \ \ t \in I \tag{1.2.83}$$

and

$$\mathcal{B}\Big(X, \int_a^{\cdot} dY(s)\cdot Z(s)\Big)(t) = \int_a^t d\mathcal{B}(X,Y)(s)\cdot Z(s) \ \ \textit{for} \ \ t \in I; \tag{1.2.84}$$

(b) *if $X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $Y \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $Z \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, then*

$$\mathcal{I}(X,\mathcal{I}(Y,Z))(t) = \mathcal{I}(X\,Y,Z)(t) \ \ \textit{for} \ \ t \in I; \tag{1.2.85}$$

where the operators $\mathcal{B}$ and $\mathcal{I}$ are defined by (0.0.2) *and* (0.0.3), *respectively.*

Proof. Consider the case $t \geq a$. Let us show that (1.2.83) is valid. According to equalities (0.0.9)–(0.0.12) and (0.0.13), we have

$$\begin{aligned}
\mathcal{B}(X,\mathcal{B}(Y,Z))(t) &= X(t)\mathcal{B}(Y,Z)(t) - \int_a^t dX(s)\cdot\mathcal{B}(Y,Z)(s) \\
&= X(t)\cdot\Big(Y(t)Z(t) - Y(a)Z(a) - \int_a^t dY(s)\cdot Z(s)\Big) \\
&\quad - \int_a^t dX(s)\cdot\Big(Y(s)Z(s) - Y(a)Z(a) - \int_a^s dY(\tau)\cdot Z(\tau)\Big) \\
&= X(t)Y(t)Z(t) - X(a)Y(a)Z(a) - X(t)\int_a^t dY(s)\cdot Z(s) \\
&\quad - \int_a^t dX(s)\cdot Y(s)Z(s) + \int_a^t dX(s)\cdot\int_a^s dY(\tau)\cdot Z(\tau) \\
&= X(t)Y(t)Z(t) - X(a)Y(a)Z(a) - \int_a^t dX(s)\cdot Y(s)Z(s) \\
&\quad - \int_a^t X(s)\,dY(s)\cdot Z(s) + \sum_{a<s\leq t} d_1X(s)\cdot d_1Y(s)\cdot Z(s) - \sum_{a\leq t<s} d_2X(s)\cdot d_2Y(s)\cdot Z(s)
\end{aligned}$$

$$= X(t)Y(t)Z(t) - X(a)Y(a)Z(a) - \int_a^t d\Big(\int_a^s dX(\tau)\cdot Y(\tau)$$

$$+ \int_a^s X(\tau)\,dY(\tau) - \sum_{a<\tau\le s} d_1X(\tau)\cdot d_1Y(\tau) + \sum_{a\le\tau<s} d_2X(\tau)\cdot d_2Y(\tau)\Big)\cdot Z(s)$$

$$= X(t)Y(t)Z(t) - X(a)Y(a)Z(a) - \int_a^t d(X(s)Y(s))\cdot Z(s) = \mathcal{B}(XY,Z)(t).$$

Let us verify (1.2.84). By (0.0.12) and (1.2.83), it can be easily shown that

$$\mathcal{B}\Big(X, \int_a^{\cdot} dY(s)\cdot Z(s)\Big)(t) = \mathcal{B}(X, YZ - \mathcal{B}(Y,Z))(t) = \mathcal{B}(X,YZ)(t) - \mathcal{B}(XY,Z)(t)$$

$$= \int_a^t d(X(s)Y(s))\cdot Z(s) - \int_a^t dX(s)\cdot Y(s)Z(s) = \int_a^t d\mathcal{B}(X,Y)(s)\cdot Z(s).$$

Finally, using (0.0.12), (1.2.83) and (1.2.84), we have

$$\mathcal{I}(X,\mathcal{I}(Y,Z))(t) = \int_a^t d\big[X(\tau) + \mathcal{B}(X,\mathcal{I}(Y,Z))(\tau)\big]\cdot X^{-1}(\tau)$$

$$= \int_a^t d\Big(X(\tau) + \mathcal{B}\Big(X, \int_a^{\cdot} d\big[Y(s) + \mathcal{B}(Y,Z)(s)\big]\cdot Y^{-1}(s)\Big)(\tau)\Big)\cdot X^{-1}(\tau)$$

$$= \int_a^t d\Big(X(\tau) + \int_a^\tau d\mathcal{B}(X, Y + \mathcal{B}(Y,Z))(s)\cdot Y^{-1}(s)\Big)\cdot X^{-1}(\tau)$$

$$= \int_a^t d\Big(X(\tau) + \int_a^\tau d\mathcal{B}(X,Y)(s)\cdot Y^{-1}(s) + \int_a^\tau d\mathcal{B}(X,\mathcal{B}(Y,Z))(s)\cdot Y^{-1}(s)\Big)\cdot X^{-1}(\tau)$$

$$= \int_a^t d\Big(X(\tau) + \int_a^\tau d\Big(X(s)Y(s) - \int_a^s dX(\sigma)\cdot Y(\sigma)\Big)\cdot Y^{-1}(s) + \int_a^\tau d\mathcal{B}(XY,Z)(s)\cdot Y^{-1}(s)\Big)\cdot X^{-1}(\tau)$$

$$= \int_a^t d\Big(\int_a^\tau d(X(s)Y(s))\cdot Y^{-1}(s) + \int_a^\tau d\mathcal{B}(XY,Z)(s)\cdot Y^{-1}(s)\Big)\cdot X^{-1}(\tau)$$

$$= \int_a^t d\big[X(\tau)Y(\tau) + \mathcal{B}(XY,Z)(\tau)\big]\cdot Y^{-1}(\tau)X^{-1}(\tau) = \mathcal{I}(XY,Z)(t).$$

Equalities (1.2.83), (1.2.84) and (1.2.85) for $t < a$ can be proved similarly. □

Lemma 1.2.2. *Let $h \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$, and $H \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ be a nonsingular matrix-function. Then the mapping*

$$x \to y = Hx + h$$

establishes a one-to-one correspondence between the solutions x and y of systems (1.1.1) *and*

$$dy = dA_*(t)\cdot y + df_*(t), \tag{1.2.86}$$

respectively, where the matrix- and vector-functions A_* and f_* are defined, respectively, by

$$A_*(t) \equiv \mathcal{I}(H,A)(t) \ \ \text{and} \ \ f_*(t) \equiv h(t) - h(a) + \mathcal{B}(H,f)(t) - \int_a^t dA_*(s)\cdot h_k(s),$$

and $a \in I$ is a fixed point. Besides,

$$I_n + (-1)^j d_j A_*(t) = (H(t) + (-1)^j d_j H(t))(I_n + (-1)^j d_j A(t))H^{-1}(t) \ \ \text{for} \ \ t \in I \ \ (j=1,2). \quad (1.2.87)$$

Proof. Let x be a solution of system (1.1.1) and let $y(t) \equiv H(t)x(t) + h(t)$. In view of (1.2.84) and the definition of a solution, we have

$$\int_a^t d\mathcal{B}(H,A)(s)\cdot x(s) = \mathcal{B}(H, x-f)(t) \ \ \text{for} \ \ t \in I.$$

In view of this and (0.0.12), we obtain

$$\begin{aligned}
\int_a^t dA_*(s)\cdot y(s) + f_*(t) - f_*(a) &= \int_a^t dA_*(s)\cdot (y(s)-h(s)) + \mathcal{B}(H,f)(t) + h(t) - h(a) \\
&= \int_a^t d\bigg(\int_a^t d\big[H(\tau) + \mathcal{B}(H,A)(\tau)\big]\cdot H^{-1}(\tau)\bigg)\cdot H(s)x(s) + \mathcal{B}(H,f)(t) + h(t) - h(a) \\
&= \int_a^t d\big[H(s) + \mathcal{B}(H,A)(s)\big]\cdot x(s) + \mathcal{B}(H,f)(t) + h(t) - h(a) \\
&= \int_a^t dH(s)\cdot x(s) + \mathcal{B}(H, x-f)(t) + \mathcal{B}(H,f)(t) + h(t) - h(a) \\
&= \int_a^t dH(s)\cdot x(s) + \mathcal{B}(H,x)(t) + h(t) - h(a) \\
&= H(t)x(t) - H(a)x(a) + h(t) - h(a) = y(t) - y(a) \ \ \text{for} \ \ t \in I,
\end{aligned}$$

i.e., y is a solution of system (1.2.86).

Let us prove the converse assertion. It suffices to show that

$$\mathcal{I}(H^{-1}, A_*)(t) = A(t) - A(a) \ \ \text{for} \ \ t \in I \quad (1.2.88)$$

and

$$\begin{aligned}
&- H^{-1}(t)h(t) + H^{-1}(a)h(a) + \mathcal{I}(H^{-1}, f^*)(t) \\
&\qquad + \int_a^t d\mathcal{I}(H^{-1}, A^*)(\tau)\cdot H^{-1}(\tau)h(\tau) = f(t) - f(a) \ \ \text{for} \ \ t \in I. \quad (1.2.89)
\end{aligned}$$

Indeed, by (1.2.85), we have

$$\begin{aligned}
\mathcal{I}(H^{-1}, A_*)(t) &= \mathcal{I}(H^{-1}, \mathcal{I}(H,A))(t) = \mathcal{I}(I,A)(t) \\
&= \int_a^t d[I_n + \mathcal{B}(I_n, A)(s)] = \mathcal{B}(I_n, A)(t) = A(t) - f(a) \ \ \text{for} \ \ t \in I.
\end{aligned}$$

Therefore, equality (1.2.88) is proved.

Let us show that (1.2.89) is valid. Let $\mathcal{R}(t)$ be the left-hand side of the equality. In view of (1.2.83) and (1.2.84), it is easy to verify that

$$\mathcal{B}\Big(H^{-1},\int\limits_a^{\cdot} d\mathcal{B}(H,A)(s)\cdot H^{-1}(s)h(s)\Big)(t)=\int\limits_a^t dA(s)\cdot H^{-1}(s)h(s)\ \text{ for }\ t\in I$$

and

$$\mathcal{B}\Big(H^{-1},\int\limits_a^{\cdot} dH(s)\cdot H^{-1}(s)h(s)\Big)(t)=-\int\limits_a^t dH(s)\cdot h(s)\ \text{ for }\ t\in I.$$

Taking these equalities, (0.0.12), (1.2.83), (1.2.84) and (1.2.88) into account, we obtain

$$\begin{aligned}\mathcal{R}(t)&=-H^{-1}(t)h(t)+H^{-1}(a)h(a)+\mathcal{B}(H^{-1},h)(t)+\mathcal{B}(H^{-1},\mathcal{B}(H,f))(t)\\&\quad-\mathcal{B}\Big(H^{-1},\int\limits_a^{\cdot} dA_*(s)\cdot h(s)\Big)(t)+\int\limits_a^t dA(s)\cdot H^{-1}(s)h(s)\\&=\mathcal{B}(I_n,f)(t-\int\limits_a^t dH^{-1}(s)\cdot h(s)-\mathcal{B}\Big(H^{-1},\int\limits_a^{\cdot} d\mathcal{I}(H,A)\cdot h(s)\Big)(t)\\&\quad+\int\limits_a^t dA(s)\cdot H^{-1}(s)h(s)=f(t)-f(a)-\int\limits_a^t dH^{-1}(s)\cdot h(s)\\&\quad-\mathcal{B}\Big(H^{-1},\int\limits_a^{\cdot} dH(s)\cdot H^{-1}h(s)\Big)(t)-\mathcal{B}\Big(H^{-1},\int\limits_a^{\cdot} d\mathcal{B}(H,A)(s)\cdot H^{-1}(s)h(s)\Big)(t)\\&\quad+\int\limits_a^t dA(s)\cdot H^{-1}(s)h(s)=f(t)-f(a)\ \text{ for }\ t\in I.\end{aligned}$$

Hence (1.2.89) is valid.

Equality (1.2.87) follows from the equalities

$$d_jA^*(t)=d_j(H(t)+\mathcal{B}(H,A)(t))\cdot H^{-1}(t)\ \text{ for }\ t\in I\ \ (j=1,2)$$

and

$$d_j\mathcal{B}(H,A)(t)=d_j(H(t)A(t))\cdot d_jH(t)\cdot A(t)\ \text{ for }\ t\in I\ \ (j=1,2).$$ □

Let ε be an arbitrary positive number and let $g:[a,b]\to\mathbb{R}$ be a non-decreasing function. We denote

$$\mathcal{D}_j(a,b,\varepsilon;g)=\big\{t\in[a,b]:\ d_jg(t)\geq\varepsilon\big\}\ \ (j=1,2).$$

Let $\mathcal{R}(a,b,\varepsilon;g)$ be the set of all subdivisions $\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\}$ of $[a,b]$ such that

(a) $a=\alpha_0<\alpha_1<\cdots<\alpha_m=b$, $\alpha_0\leq\tau_1\leq\alpha_1\leq\cdots\leq\tau_m\leq\alpha_m$;

(b) if $\tau_i\notin\mathcal{D}_1(a,b,\varepsilon;g)$ then $g(\tau_i)-g(\alpha_{i-1})<\varepsilon$;
if $\tau_i\in\mathcal{D}_1(a,b,\varepsilon;g)$, then $\alpha_{i-1}<\tau_i$ and $g(\tau_i-)-g(\alpha_{i-1})<\varepsilon$;

(c) if $\tau_i\notin\mathcal{D}_2(a,b,\varepsilon;g)$, then $g(\alpha_i)-g(\tau_i)<\varepsilon$;
if $\tau_i\in\mathcal{D}_2(a,b,\varepsilon;g)$, then $\tau_i<\alpha_i$ and $g(\alpha_i)-g(\tau_i+)<\varepsilon$.

Lemma 1.2.3. *The set $\mathcal{R}(a,b,\varepsilon;g)$ is not empty for an arbitrary positive number ε and a non-decreasing function $g:[a,b]\to\mathbb{R}$.*

We omit the proof of the lemma because it is analogous to that of Lemma 1.1.1 from [37].

Lemma 1.2.4. *Let $\alpha_k, \beta_k \in \mathrm{BV}([a,b];\mathbb{R})$ $(k=0,1,\dots)$ be such that*

$$\lim_{k\to+\infty} \|\beta_k-\beta_0\|_s = 0, \tag{1.2.90}$$

$$\lim_{k\to+\infty} \sup \bigvee_a^b(\alpha_k) < +\infty, \tag{1.2.91}$$

and let the condition

$$\lim_{k\to+\infty} (\alpha_k(t)-\alpha_k(a)) = \alpha_0(t)-\alpha_0(a) \tag{1.2.92}$$

be fulfilled uniformly on $[a,b]$. Then

$$\lim_{k\to+\infty} \int_a^t \beta_k(\tau)\,d\alpha_k(\tau) = \int_a^t \beta_0(\tau)\,d\alpha_0(\tau) \tag{1.2.93}$$

is fulfilled uniformly on $[a,b]$, as well.

Proof. Let ε be an arbitrary positive number. By Lemma 1.2.3,

$$\mathcal{R}\Big(a,b,\frac{\varepsilon}{5}\Big) \neq \varnothing,$$

where $g(t) \equiv V(\beta_0)(t)$.

Let

$$\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\} \in \mathcal{R}\Big(a,b,\frac{\varepsilon}{5}\Big)$$

be an arbitrary fixed subdivision. We set

$$\eta(t) = \begin{cases} \beta_0(t) & \text{for } t\in\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\},\\ \beta_0(\tau_i-) & \text{for } t\in\,]\alpha_{i-1},\tau_i[\,,\ \tau_i\in\mathcal{D}_1(a,b,\varepsilon;g),\\ \beta_0(\tau_i) & \text{for } t\in\,]\alpha_{i-1},\tau_i[\,,\ \tau_i\notin\mathcal{D}_1(a,b,\varepsilon;g) \text{ or for } t\in\,]\tau_i,\alpha_i[\,,\ \tau_i\notin\mathcal{D}_2(a,b,\varepsilon;g),\\ \beta_0(\tau_i+) & \text{for } t\in\,]\tau_i,\alpha_i[\,,\ \tau_i\in\mathcal{D}_1(a,b,\varepsilon;g)\\ & \qquad (i=1,\dots,m). \end{cases}$$

It can be easily shown that $\eta\in\mathrm{BV}(a,b;\mathbb{R})$ and

$$|\beta_0(t)-\eta(t)| < 2\varepsilon \ \text{ for } \ t\in[a,b]. \tag{1.2.94}$$

For every natural k and $t\in[a,b]$, we assume

$$\gamma_k(t) = \int_a^t \beta_k(\tau)\,d\alpha_k(t) - \int_a^t \beta_0(\tau)\,d\alpha_0(\tau)$$

and

$$\delta_k(t) = \int_a^t \eta(t)\,d(\alpha_k(\tau)-\alpha_0(\tau)).$$

It follows from (1.2.92) that

$$\lim_{k\to+\infty} \|\delta_k\|_\infty = 0. \tag{1.2.95}$$

On the other hand, by (1.2.92) and (1.2.94), we have

$$\|\gamma_k\|_\infty \le 4r\varepsilon + r\|\beta_k-\beta_0\|_\infty + \|\delta\|_\infty \ \ (k=1,2,\dots).$$

Hence, in view of (1.2.91) and (1.2.95), we obtain $\lim\limits_{k\to+\infty}\|\gamma_k\|_\infty = 0$, since ε is arbitrary. □

Lemma 1.2.5. *Let condition* (1.2.8) *hold and let*

$$\lim_{k\to+\infty} X_k(t) = X_0(t) \tag{1.2.96}$$

uniformly on $[a,b]\subset I$, *where* X_0 *and* X_k $(k=1,2,\dots)$ *are the fundamental matrices, respectively, of the homogeneous systems* $(1.2.1_0)$ *and* $(1.2.1_{k0})$ $(k=1,2,\dots)$. *Then*

$$\inf\big\{|\det(X_0(t))|:\ t\in[a,b]\big\}>0, \tag{1.2.97}$$
$$\inf\big\{|\det(X_0^{-1}(t))|:\ t\in[a,b]\big\}>0 \tag{1.2.98}$$

and condition (1.2.37) *holds uniformly on* $[a,b]$, *as well.*

Proof. According to equalities (1.1.6), we have

$$d_jX_0(t)=d_jA_0(t)\cdot X_0(t) \ \text{ for } \ t\in[a,b] \ \ (j=1,2). \tag{1.2.99}$$

From this, by (1.2.5), we find

$$\det(X_0(t-)\cdot X_0(t+)) \\ =\big[\det(X_0(t))\big]^2\cdot\prod_{j=1}^{2}\det\big(I_n+(-1)^jd_jA_0(t)\big)\neq 0 \ \text{ for } \ t\in[a,b] \ \ (j=1,2). \tag{1.2.100}$$

Let us show that (1.2.97) is valid. Assume the contrary. Then it can be easily shown that there exists a point $t_0\in[a,b]$ such that

$$\det\big(X_0(t_0-)\cdot X_0(t_0+)\big)=0.$$

But this equality contradicts (1.2.100). Thus inequality (1.2.97) is proved.

The proof of inequality (1.2.98) is analogous.

In view of (1.2.96) and (1.2.97), there exists a positive number r such that

$$\inf\big\{|\det(X_k(t))|:\ t\in[a,b]\big\}>r>0$$

for any sufficiently large k. From this and (1.2.96), we obtain (1.2.37). □

Lemma 1.2.6. *Let the sequences of the matrix-functions* $B_k\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *and of the points* $t_k\in I$ *be such that conditions* (1.2.9),

$$\det\big(I_n+(-1)^jd_jB_0(t)\big)\neq 0 \ \text{ for } \ t\in I, \ \ (-1)^j(t-t_0)<0 \ \ (j=1,2) \tag{1.2.101}$$

and

$$\lim_{k\to+\infty}\sup\Big\{\|d_jB_k(t)-d_jB_0(t)\|:\ t\in I,\ (-1)^j(t-t_k)<0\Big\}=0 \ \ (j=1,2) \tag{1.2.102}$$

hold. Then there exists a positive number r_0 *such that*

$$\det\big(I_n+(-1)^jd_jB_k(t)\big)\neq 0 \ \text{ for } \ t\in I, \ \ (-1)^j(t-t_k)<0 \ \ (j=1,2), \tag{1.2.103}$$

$$\big\|\big(I_n+(-1)^jd_jB_0(t)\big)^{-1}\big\|\le r_0 \ \text{ for } \ t\in I, \ \ (-1)^j(t-t_0)<0 \ \text{ and} \\ \big\|\big(I_n+(-1)^jd_jB_k(t)\big)^{-1}\big\|\le r_0 \ \text{ for } \ t\in I, \ \ (-1)^j(t-t_k)<0 \ \ (j=1,2) \tag{1.2.104}$$

for every sufficiently large k.

Proof. Since $\bigvee_I B_0<+\infty$, the series $\sum_{t\in I}\|d_jB_0(t)\|$ $(j=1,2)$ converge. Thus for any $j\in\{1,2\}$ the inequality

$$\|d_jB_0(t)\|\ge\frac{1}{2}$$

may hold only for some finite number of points $t_{j1}, \dots, t_{j\,m_j}$ in I. Therefore,

$$\|d_j B_0(t)\| < \frac{1}{2} \text{ for } t \in I, \ \ t \neq t_{ji} \ \ (i = 1, \dots, m_j). \tag{1.2.105}$$

First, let us consider the case where $j = 2$ and $t_k \geq t_0$ for every sufficiently large k. We may assume that $t_{2i} \geq t_k$ $(i = 1, \dots, m_2)$ for every sufficiently large k.

It follows from (1.2.101), (1.2.102) and (1.2.105) that

$$\det(I_n + d_2 B_k(t_{2i})) \neq 0 \ \ (i = 1, \dots, m_2)$$

and

$$\|d_2 B_k(t)\| < \frac{1}{2} \text{ for } t \in I_{t_k}, \ \ t \neq t_{2i} \ \ (i = 1, \dots, m_2)$$

for every sufficiently large k. The latter inequalities imply that the matrices $I_n + d_2 B_k(t)$ are invertible for $t \in I_{t_0}$, $t \neq t_{ji}$ $(i = 1, \dots, m_j)$, too. From this, it is evident that condition (1.2.103) is fulfilled and there exists a positive number r_0 for which estimates (1.2.104) hold. Analogously we prove this estimate for the other cases. □

Lemma 1.2.7. *Let $A \in \mathrm{BV}_{loc}(I; \mathbb{R}^{n\times n})$, $f \in \mathrm{BV}_{loc}(I; \mathbb{R}^n)$ and $a \in I$ be such that*

$$\det\big(I_n + (-1)^j d_j A(t)\big) \neq \ 0 \ \ \textit{for} \ \ (-1)^j (t - a) < 0 \ \ (j = 1, 2). \tag{1.2.106}$$

Let, moreover, the vector-function $x \in \mathrm{BV}_{loc}(I; \mathbb{R}^n)$ be a solution of the initial problem

$$dx = dA(t) \cdot x + df(t), \quad x(a) = c_0.$$

Then the vector-function $y \in \mathrm{BV}_{loc}(I; \mathbb{R}^n)$, defined by $y(t) = x(t+)$ for $t < a$, $y(t) = x(t-)$ for $t > a$ and $y(a) = x(a)$, is a solution of the initial problem

$$dy = d\,\overline{A}(t) \cdot y + d\,\overline{f}(t), \ \ y(a) = c_0, \tag{1.2.107}$$

where $\overline{A}(a) = O_{n\times n}$, $\overline{f}(a) = 0_n$;

$$\overline{A}(t) = \int\limits_{a-}^{t} dA(\tau+) \cdot (I_n + d_2 A(\tau))^{-1} - d_1 A(a) \ \ \textit{and}$$

$$\overline{f}(t) = f(t+) - f(a) + \int\limits_{a-}^{t} d\overline{A}(\tau) \cdot d_2 f(\tau) \ \ \textit{for} \ \ t < a;$$

$$\overline{A}(t) = \int\limits_{a+}^{t} dA(\tau-) \cdot (I_n - d_1 A(\tau))^{-1} + d_2 A(a) \ \ \textit{and}$$

$$\overline{f}(t) = f(t-) - f(a) + \int\limits_{a}^{t} d\overline{A}(\tau) \cdot d_1 f(\tau) \ \ \textit{for} \ \ t > a.$$

Proof. Let $j = 1$ and $a < s < t$; $s, t \in I$. Then $y(t) \equiv x(t-)$ and by the definition of the solution of linear systems we have

$$y(t) = y(s) + \int\limits_{s-}^{t-} dA(\tau) \cdot x(\tau) + f(t-) - f(s-).$$

From this, according to (0.0.8),

$$y(t) = y(s) + \int\limits_{s}^{t} dA(\tau) \cdot x(\tau) - d_1 A(t) \cdot x(t) + d_1 A(s) \cdot x(s) + f(t-) - f(s-).$$

Using now equalities (0.0.7) and

$$d_1x(t)\equiv d_1A(t)\cdot x(t)+d_1f(t),$$

due to (1.2.106), we get

$$x(t)\equiv (I_n-d_1A(t))^{-1}(x(t-)+d_1f(t))$$

and

$$y(t)=y(s)+\int\limits_s^t dA(\tau-)\cdot (I_n-d_1A(\tau))^{-1}(y(\tau)+d_1f(\tau))+f(t-)-f(s-).$$

So, by (0.0.13), the vector-function y is the solution of system (1.2.107) for $t>a$.

Similarly, we can show that y is also the solution of system (1.2.107) for $t<a$.

In addition, it is not difficult to verify the validity of the equalities

$$d_j\overline{A}(a)=d_jA(a) \ \text{ and } \ d_j\overline{f}(a)=d_jf(a) \ \ (j=1,2),$$

whence $d_jy(a)=d_jx(a)$ $(j=1,2)$. Therefore, the vector-function y is the solution of system (1.2.107) on the whole interval I. □

1.2.3 Proof of the results

Proof of Theorem 1.2.2. By (1.2.15),

$$\lim_{k\to+\infty}\ \sup_{t\in I,\,t\neq t_k}\|A_{kj}(t)-A_{0j}(t)\|=0 \ \ (j=1,2)$$

and, therefore,

$$\lim_{k\to+\infty}\ \sup_{t\in I,\,t\neq t_k}\|d_jA_k(t)-d_jA_0(t)\|=0 \ \ \ (j=1,2) \tag{1.2.108}$$

if $j\in\{1,2\}$ is such that $(-1)^j(t-t_k)\geq 0$ for every $k\in\{1,2,\dots\}$. So, according to Lemma 1.2.6, there exists a positive number r_0 such that

$$\det\big(I_n+(-1)^ld_lA_k(t)\big)\neq 0 \ \text{ for } \ t\in I, \ \ (-1)^l(t-t_k)<0 \ \ (l=1,2)$$

and

$$\big\|\big(I_n+(-1)^ld_lA_k(t)\big)^{-1}\big\|\leq r_0 \ \text{ for } \ t\in I, \ \ (-1)^l(t-t_k)<0 \ \ (l=1,2) \tag{1.2.109}$$

for every sufficiently large k.

Therefore, there exists a natural number k_0 such that problem $(1.2.1_k),(1.2.2_k)$ has a unique solution x_k for every $k\geq k_0$.

Let $z_k(t)\equiv x_k(t)-x_0(t)$ for $k\in\{k_0,k_0+1,\dots\}$.

First, consider the case $t_k>t_0$ $(k=k_0,k_0+1,\dots)$, i.e., $j=2$.

Let ε be an arbitrarily small positive number.

It is not difficult to check that

$$z_k(t)=z_k(t_k+\varepsilon)+\int\limits_{t_k+\varepsilon}^t dA_0(s)\cdot z_k(s)+\int\limits_{t_k+\varepsilon}^t d\overline{A}_{k2}(s)\cdot x_k(s)+\overline{f}_{k2}(t)-\overline{f}_{k2}(t_k+\varepsilon) \ \text{ for } \ t\geq t_k+\varepsilon,$$

where

$$\overline{A}_{kj}(t)\equiv A_{kj}(t)-A_{0j}(t), \quad \overline{f}_{kj}(t)\equiv f_{kj}(t)-f_{0j}(t) \ \ (j=1,2;k=0,1,\dots).$$

Using the integration-by-parts formula (0.0.9), the equalities

$$d_lx_k(t)=d_lA_k(t)\cdot x_k(t)+d_lf(t) \ \text{ for } \ t\in I \ \ (j=1,2), \tag{1.2.110}$$

and general integration-by-parts formulas (0.0.10) and (0.0.12), we conclude that

$$\int_{t_k+\varepsilon}^{t} d\overline{A}_{k2}(s)\cdot x_k(s) = \overline{A}_{k2}(t)\cdot x_k(t) - \overline{A}_{k2}(t_k+\varepsilon)\cdot x_k(t_k+\varepsilon)$$

$$- \int_{t_k+\varepsilon}^{t} \overline{A}_{k2}(s)dx_k(s) + \sum_{t_k+\varepsilon<s\le t} d_1\overline{A}_{k2}(s)\cdot d_1x_k(s) - \sum_{t_k+\varepsilon\le s<t} d_2\overline{A}_{k2}(s)\cdot d_2x_k(s)$$

$$= \overline{A}_{k2}(t)\cdot x_k(t) - \overline{A}_{k2}(t_k+\varepsilon)\cdot x_k(t_k+\varepsilon) - \int_{t_k+\varepsilon}^{t} \overline{A}_{k2}(s)\big(dA_k(s)\cdot x_k(s)+df_k(s)\big)$$

$$+ \sum_{t_k+\varepsilon<s\le t} d_1\overline{A}_{k2}(s)\cdot\big(d_1A_k(s)\cdot x_k(s)+d_1f_k(s)\big)$$

$$- \sum_{t_k+\varepsilon\le s<t} d_2\overline{A}_{k2}(s)\cdot\big(d_2A_k(s)\cdot x_k(s)+d_2f_k(s)\big) \ \text{ for } \ t\ge t_k+\varepsilon.$$

Therefore,

$$z_k(t) = z_k(t_k+\varepsilon) + \mathcal{J}_{k2}(t,t_k+\varepsilon) + \mathcal{Q}_{k2}(t,t_k+\varepsilon) + \int_{t_k+\varepsilon}^{t} dA_0(s)\cdot z_k(s) \ \text{ for } \ t\ge t_k+\varepsilon. \tag{1.2.111}$$

where

$$\mathcal{J}_{kj}(t,\tau) = \overline{A}_{kj}(t)\cdot x_k(t) - \overline{A}_{kj}(\tau)\cdot x_k(\tau) - \int_{\tau}^{t} \overline{A}_{kj}(s)dA_k(s)\cdot x_k(s)$$

$$+ \sum_{s\in]\tau,t]} d_1\overline{A}_{kj}(s)\cdot d_1A_k(s)\cdot x_k(s) - \sum_{s\in[\tau,t[} d_2\overline{A}_{kj}(s)\cdot d_2A_k(s)\cdot x_k(s) \ \text{ for } \ \tau<t \ \ (j=1,2),$$

$$\mathcal{J}_{kj}(t,t)\equiv 0 \ \ (j=1,2) \ \text{ and } \ \mathcal{J}_{kj}(t,\tau) = -\mathcal{J}_{kj}(\tau,t) \ \text{ for } \ t<\tau \ \ (j=1,2),$$

and

$$\mathcal{Q}_{kj}(t,\tau)\equiv \overline{f}_{kj}(t) - \overline{f}_{kj}(\tau) - \mathcal{B}(\overline{A}_{kj},f_k)(t) + \mathcal{B}(\overline{A}_{kj},f_k)(\tau) \ \ (j=1,2).$$

Let B_0 be a matrix-function defined by $B_0(t_k+\varepsilon)=A_0(t_k+\varepsilon)$ and $B_0(s)=A_0(s-)$ for $s>t_k+\varepsilon$. Obviously,

$$d_2B_0(t_k+\varepsilon) = d_2A_0(t_k+\varepsilon) \ \text{ and } \ d_1(B_0(s)-A_0(s)) = -d_1A_0(s) \ \text{ for } \ s>t_k+\varepsilon.$$

Therefore, according to (0.0.7),

$$\int_{t_k+\varepsilon}^{t} dA_0(s)\cdot z_k(s) = \int_{t_k+\varepsilon}^{t} dB_0(s)\cdot z_k(s) + d_1A_0(t)\cdot z_k(t) \ \text{ for } \ t>t_k+\varepsilon.$$

Consequently, by (1.2.8), from (1.2.111) it follows that

$$z_k(t) = (I_n - d_1A_0(t))^{-1}\bigg(z_k(t_k+\varepsilon) + \mathcal{J}_{k2}(t,t_k+\varepsilon) + \mathcal{Q}_{k2}(t,t_k+\varepsilon) + \int_{t_k+\varepsilon}^{t} dB_0(s)\cdot z_k(s)\bigg) \ \text{ for } \ t>t_k+\varepsilon.$$

From this, due to (1.2.108) and estimate (1.2.109), without loss of generality, for $k>k_0$, we get

$$\|z_k(t)\| \le r_1\bigg(\|z_k(t_k+\varepsilon)\| + \|\mathcal{J}_{k2}(t,t_k+\varepsilon)\|$$

$$+ \|\mathcal{Q}_{k2}(t,t_k+\varepsilon)\| + \int_{t_k+\varepsilon}^{t} \|z_k(\tau)\|\, d\|V(B_0)(\tau)\|\bigg) \ \text{ for } \ t\ge t_k+\varepsilon, \tag{1.2.112}$$

where $r_1 = r_0 + 1$.

Let

$$\rho_0 = \bigvee_I (A_0), \quad \varrho_0 = \bigvee_I (f_0),$$
$$\alpha_k = \sup_{t\in I,\, t\neq t_k} \|\overline{A}_{k2}(t)\|, \quad \beta_k = \sup_{t\in I,\, t\neq t_k} \|\overline{f}_{k2}(t)\| \ \ (k=1,2,\dots)$$

and

$$\gamma_k = \sup_{t\in I,\, t\neq t_k} \Big| \bigvee_{t_k}^{t} (A_k - A_0) \Big| \ \ (k=1,2,\dots).$$

In view of the conditions $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$, (1.2.15) and (1.2.16), we have

$$\lim_{k\to+\infty} \alpha_k(1+\gamma_k) = \lim_{k\to+\infty} \beta_k(1+\varrho_0+\gamma_k) = 0. \tag{1.2.113}$$

By the inequalities

$$\bigvee_{t_k}^{t}(A_k) \leq \bigvee_{t_k}^{t}(A_k - A_0) + \bigvee_{t_k}^{t}(A_0) \ \text{ for } \ t\in I \ \ (k=1,2,\dots),$$

we find

$$\begin{aligned}\|\mathcal{J}_{k2}(t,t_k+\varepsilon)\| &\leq 2\alpha_k\|x_k\|_{k2} + \alpha_k(\gamma_k+\rho_0)\|x_k\|_{k2}\\ &\quad + 2\alpha_k\|x_k\|_{k2}\bigg(\sum_{t_k+\varepsilon<s\leq t}\Big(\big\|d_1(A_k(s)-A_0(s))\big\| + \|d_1A_0(s)\|\Big)\\ &\qquad + \sum_{t_k+\varepsilon\leq s<t}\Big(\big\|d_2(A_k(s)-A_0(s))\big\| + \|d_2A_0(s)\|\Big)\bigg)\end{aligned}$$

and, therefore,

$$\big\|\mathcal{J}_{k2}(t,t_k+\varepsilon)\big\| \leq \varepsilon_k\|x_k\|_{k2} \ \text{ for } \ t\geq t_k+\varepsilon, \tag{1.2.114}$$

where

$$\|x\|_{k2} = \sup\big\{\, \|x(t)\| : \ t\in I, \ \ t>t_k\big\} \ \text{ and } \ \varepsilon_k = \alpha_k(2+3\rho_0+3\gamma_k) \ \ (k=1,2,\dots).$$

Moreover, if we take into account the fact that the operator $\mathcal{B}$ is linear with respect to every its variable and equals zero if the second variable is a constant function, then we conclude that

$$\begin{aligned}&\big\|\mathcal{B}(\overline{A}_{k2},f_k)(t) - \mathcal{B}(\overline{A}_{k2},f_k)(t_k+\varepsilon)\big\|\\ &\leq \big\|\mathcal{B}(\overline{A}_{k2},\overline{f}_{k2})(t) - \mathcal{B}(\overline{A}_{k2},\overline{f}_{k2})(t_k+\varepsilon)\big\| + \big\|\mathcal{B}(\overline{A}_{k2},f_0)(t) - \mathcal{B}(\overline{A}_{k2},f_0)(t_k+\varepsilon)\big\| \ \text{ for } \ t\geq t_k+\varepsilon.\end{aligned}$$

By the definition of the operator $\mathcal{B}$, we have

$$\big\|\mathcal{B}(\overline{A}_{k2},\overline{f}_{k2})(t) - \mathcal{B}(\overline{A}_{k2},\overline{f}_{k2})(t_k+\varepsilon)\big\| \leq \beta_k(2\alpha_k+\gamma_k) \ \text{ for } \ t\geq t_k+\varepsilon.$$

Using the integration-by-part formula, we find

$$\begin{aligned}&\big\|\mathcal{B}(\overline{A}_{k2},f_0)(t) - \mathcal{B}(\overline{A}_{k2},f_0)(t_k+\varepsilon)\big\|\\ &\qquad \leq \alpha_k\bigvee_{t_k+\varepsilon}^{t}(f_0) + 2\alpha_k\bigg(\sum_{t_k+\varepsilon<s\leq t}\|d_1f_0(s)\| + \sum_{t_k+\varepsilon\leq s<t}\|d_2f_0(s)\|\bigg) \ \text{ for } \ t\geq t_k+\varepsilon\end{aligned}$$

and, therefore,

$$\left\|\mathcal{B}(\overline{A}_{k2},f_0)(t)-\mathcal{B}(\overline{A}_{k2},f_0)(t_k+\varepsilon)\right\|\le 3\varrho_0\alpha_k \ \text{ for } \ t\ge t_k+\varepsilon.$$

So,

$$\|\mathcal{Q}_{k2}(t,t_k+\varepsilon)\|\le\delta_k \ \text{ for } \ t\ge t_k+\varepsilon, \tag{1.2.115}$$

where $\delta_k=\beta_k(2+2\alpha_k+\gamma_k)+3\varrho_0\alpha_k$.

From (1.2.112), by (1.2.114) and (1.2.115), we find

$$\|z_k(t)\|\le r_1\Big(\|z_k(t_k+\varepsilon)\|+\varepsilon_k\|x_k\|_{k2}+\delta_k+\int\limits_{t_k+\varepsilon}^{t}\|z_k(\tau)\|\,d\|V(B_0)(\tau)\|\Big) \ \text{ for } \ t\ge t_k+\varepsilon. \tag{1.2.116}$$

Hence, according to Gronwall's inequality (see Lemma 1.1.4′ (b)),

$$\begin{aligned}\|z_k(t)\|&\le r_1\big(\|z_k(t_k+\varepsilon)\|+\varepsilon_k\|x_k\|_{k2}+\delta_k\big)\exp\big(r_1\|V(B_0)(t)-V(B_0)(t_k)\|\big)\\&\le r_1\big(\|z_k(t_k+\varepsilon)\|+\varepsilon_k\|x_k\|_{k2}+\delta_k\big)\exp(\rho_0r_1) \ \text{ for } \ t\ge t_k+\varepsilon.\end{aligned}$$

Now, passing to the limit as $\varepsilon\to0$ in the last inequality, we conclude that

$$\|z_k\|_{k2}\le r_1\Big(\|z_k(t_k+)\|+\varepsilon_k\|x_k\|_{k2}+\delta_k\Big)\exp(\rho_0r_1). \tag{1.2.117}$$

Due to (1.2.113), we have

$$\lim_{k\to+\infty}\varepsilon_k=0. \tag{1.2.118}$$

Therefore, there exists a natural $k_1>k_0$ such that

$$r_1\varepsilon_k\exp(\rho_0r_1)<\frac12 \ \text{ for } \ k>k_1.$$

By this, from (1.2.117) it follows that

$$\|x_k\|_{k2}\le\|x_0\|_{k2}+\|z_k\|_{k2}\le\|x_0\|_{k2}+\frac12\|x_k\|_{k2}+r_1\big(\|z_k(t_k+)\|+\delta_k\big)\exp(\rho_0r_1) \ \ (k>k_1)$$

and, therefore,

$$\|x_k\|_{k2}\le 2\Big(\|x_0\|_{k2}+r_1\big(\|z_k(t_k+)\|+\delta_k\big)\exp(\rho_0r_1)\Big) \ \ (k>k_1).$$

which, due to (1.2.5$_2$), implies that the sequence $\|x_k\|_{k2}$ $(k=1,2,\dots)$ is bounded.

In view of conditions (1.2.15) and (1.2.16),

$$\lim_{k\to+\infty}\delta_k=0. \tag{1.2.119}$$

Moreover, using (1.2.5$_2$), we conclude

$$\begin{aligned}\lim_{k\to+\infty}z_k(t_k+)&=\lim_{k\to+\infty}\big(x_k(t_k+)-x_0(t_k+)\big)=\lim_{k\to+\infty}\big(x_k(t_k+)-x_0(t_0+)\big)\\&=\lim_{k\to+\infty}\Big(\big[(I_n+d_2A_k(t_k))x_k(t_k)+d_2f_k(t_k)\big]-\big[(I_n+d_2A(t_0))x_0(t_0)+d_2f_0(t_0)\big]\Big)\\&=\lim_{k\to+\infty}(c_{k2}-c_{02})=0.\end{aligned}$$

Therefore, by this, (1.2.118) and (1.2.119), it follows from (1.2.117) that

$$\lim_{k\to+\infty}\|z_k\|_{k2}=0.$$

Analogously to (1.2.111), we show that

$$z_k(t)=z_k(t_k-\varepsilon)-\mathcal{J}_{k2}(t_k-\varepsilon,t)-\mathcal{Q}_{k2}(t_k-\varepsilon,t)-\int\limits_{t}^{t_k-\varepsilon}dA_0(s)\cdot z_k(s) \ \text{ for } \ t\le t_k-\varepsilon. \tag{1.2.120}$$

Let now the matrix-function B_0 be defined by $B_0(t_k-\varepsilon)=A_0(t_k-\varepsilon)$ and $B_0(s)=A_0(s+)$ for $s<t_k-\varepsilon$. Obviously,

$$d_1B_0(t_k-\varepsilon)=d_1A_0(t_k-\varepsilon) \ \text{ and } \ d_2(B_0(s)-A_0(s))=-d_2A_0(s) \ \text{ for } \ s<t_k-\varepsilon.$$

Therefore, according to (0.0.7),

$$\int\limits_t^{t_k-\varepsilon} dA_0(s)\cdot z_k(s)=\int\limits_t^{t_k-\varepsilon} dB_0(s)\cdot z_k(s)+d_2A_0(t)\cdot z_k(t) \ \text{ for } \ t<t_k-\varepsilon.$$

Using these equalities, from (1.2.120) we obtain

$$z_k(t)=(I_n+d_2A_0(t))^{-1}\bigg(z_k(t_k-\varepsilon)-\mathcal{J}_{k2}(t_k-\varepsilon,t) \\ -\mathcal{Q}_{k2}(t_k-\varepsilon,t)-\int\limits_t^{t_k-\varepsilon} dA_0(s)\cdot z_k(s)\bigg) \ \text{ for } \ t\le t_k-\varepsilon.$$

From this, analogously as above, we get

$$\|z_k\|_{k1}\le r_1\big(\|z_k(t_k-)\|+\varepsilon_k\|x_k\|_{k1}+\delta_k\big)\exp(\rho_0r_1) \tag{1.2.121}$$

and, in addition, the sequence $\|x_k\|_{k1}$ $(k=1,2,\dots)$ is bounded.

By (1.2.15) and (1.2.16),

$$\lim_{k\to+\infty}\Big(\|d_1A_k(t_k)+d_2A_k(t_k)\|+\|d_1f_k(t_k)+d_2f_k(t_k)\|\Big)=0.$$

Using the latter equality, $(1.2.5_2)$ and take into account that the sequence c_k $(k=1,2,\dots)$ is bounded, we can conclude that

$$\begin{aligned}\lim_{k\to+\infty} z_k(t_k-)&=\lim_{k\to+\infty}(x_k(t_k-)-x_0(t_k-))=\lim_{k\to+\infty}(x_k(t_k-)-x_0(t_0+))\\ &=\lim_{k\to+\infty}\Big(\big[(I_n-d_1A_k(t_k))x_k(t_k)-d_1f_k(t_k)\big]-\big[(I_n+d_2A(t_0))x_0(t_0)+d_2f_0(t_0)\big]\Big)\\ &=\lim_{k\to+\infty}\Big(\big[(I_n+d_2A_k(t_k))c_k+d_2f_k(t_k)\big]-\big[(I_n+d_2A(t_0))x_0(t_0)+d_2f_0(t_0)\big]\Big)\\ &\quad-\lim_{k\to+\infty}\big(d_1A_k(t_k)+d_2A_k(t_k))c_k-(d_1f_k(t_k)+d_2f_k(t_k)\big)=\lim_{k\to+\infty}(c_{k2}-c_{02})=0.\end{aligned}$$

Therefore, due to (1.2.121), taking into account (1.2.118) and (1.2.119), we get

$$\lim_{k\to+\infty}\|z_k\|_{k1}=0.$$

Thus condition (1.2.17) holds for $t_k>t_0$ $(k=1,2,\dots)$.

In a similar way we can prove the statement of the theorem for another cases, as well, i.e., when $t_k<t_0$ $(k=1,2,\dots)$ or $t_k=t_0$ $(k=1,2,\dots)$. □

Proof of Theorem 1.2.3. Due to condition (1.2.20), analogously to the proof of Theorem 1.2.2, we show that the initial problem $(1.2.21_k),(1.2.22_k)$ has the unique solution x_k^* for every sufficiently large k. Moreover, according to Lemma 1.2.2, the mapping $x\to x^*$, $x^*=H_kx+h_k$, establishes a one-to-one correspondence between the solutions of problem $(1.2.1_k),(1.2.2_k)$ and those of the initial problem $(1.2.21_k),(1.2.22_k)$ for every natural k. Thus problem $(1.2.1_k),(1.2.2_k)$ has the unique solution x_k and

$$x_k^*(t)\equiv H_k(t)x_k+h_k(t)$$

for every sufficiently large k.

Conditions (1.2.20), (1.2.23)–(1.2.26) guarantee the fulfillment of the conditions of Theorem 1.2.2 for the initial problem (1.2.21), (1.2.22) and for the sequence of the initial problems $(1.2.21_k), (1.2.22_k)$ $(k = 1, 2, \dots)$. Therefore, according to Theorem 1.2.2,

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \|x_k^*(t) - x_0^*(t)\| = 0.$$

So, condition (1.2.27) holds. □

Proof of Corollary 1.2.1. Verify the conditions of Theorem 1.2.3. From (1.2.11), (1.2.12) it follows that condition (1.2.23) holds, and the condition

$$\lim_{k\to+\infty} \|H_k^{-1}(t) - H_0^{-1}(t)\| = 0 \tag{1.2.122}$$

holds uniformly on I.

Put

$$h_k(t) \equiv -H_k(t)\varphi_k(t) \ \ (k = 1, 2, \dots).$$

Due to (1.2.9) and (1.2.12), we get

$$\lim_{k\to+\infty} H_k(t_k) = Q_0,$$

where $Q_0 = H_0(t_0-)$ if $t_k < t_0$, $Q_0 = H_0(t_0)$ if $t_k = t_0$ and $Q_0 = H_0(t_0+)$ if $t_k > t_0$ for sufficiently large k. By this and (1.2.31), condition (1.2.28) is fulfilled for $c_0^* = Q_0 c_0$.

Moreover, by (1.2.13) and (1.2.32), conditions (1.2.29) and (1.2.30) hold uniformly on I, where

$$h_k(t) \equiv -H_k(t)\varphi_k(t), \quad A_k^*(t) \equiv \mathcal{I}(H_k, A_k)(t) - \mathcal{I}(H_k, A_k)(t_k) \ \ (k = 0, 1, \dots);$$
$$f_0^*(t) \equiv \mathcal{B}(H_0, f_0)(t) - \mathcal{B}(H_0, f_0)(t_0),$$
$$f_k^*(t) \equiv \mathcal{B}(H_k, f_k - \varphi_k)(t) - \mathcal{B}(H_k, f_k - \varphi)(t_k) + \int_{t_k}^{t} d\mathcal{I}(H_k, A_k)(s) \cdot \varphi_k(s) \ \ (k = 1, 2, \dots).$$

Taking into account Lemma 1.2.2, it is not difficult to see that problem (1.2.21), (1.2.22) has the unique solution

$$x_0^*(t) \equiv H_0(t)x_0(t).$$

By Theorem 1.2.3 and Remark 1.2.2, we have

$$\lim_{k\to+\infty} \|H_k(t)x_k(t) - H_k(t)\varphi_k(t) - x_0^*(t)\| = 0$$

uniformly on I. Therefore, owing to (1.2.12) and (1.2.122), condition (1.2.33) holds uniformly on I. □

Proof of Theorem 1.2.1. The sufficiency follows from Corollary 1.2.1 if we assume $\varphi_k(t) \equiv 0$ $(k = 1, 2, \dots)$ therein.

Let us show the necessity. Let $c_k \in \mathbb{R}^n$ $(k = 0, 1, \dots)$ be an arbitrary sequence of constant vectors satisfying (1.2.5) and let $e_j = (\delta_{ij})_{i=1}^n$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$ $(i, j = 1, \dots, n)$ (the Kroneker symbol).

In view of (1.2.10), without loss of generality, we may assume that problem $(1.2.1_k), (1.2.2_k)$ has a unique solution x_k for every natural k.

For any $k \in \{0, 1, \dots\}$ and $j \in \{1, \dots, n\}$, let us denote

$$z_{kj}(t) \equiv x_k(t) - x_{kj}(t),$$

where x_{kj} is a unique solution of system $(1.2.1_k)$ under the initial condition

$$x(t_k) = c_k - e_j.$$

Moreover, let $X_k(t)$ be a matrix-function whose columns are $z_{k1}(t), \dots, z_{kn}(t)$.

It can be easily shown that X_0 and X_k $(k=1,2,\dots)$ satisfy, respectively, the homogeneous systems $(1.2.1_0)$ and $(1.2.1_{k0})$ $(k=1,2,\dots)$ and

$$z_{kj}(t_k)=e_j \ \ (k=0,1,\dots) \tag{1.2.123}$$

for every $j\in\{1,\dots,n\}$.

If we assume

$$\sum_{j=1}^{n}\alpha_j z_{kj}(t)\equiv 0$$

for some natural k and $\alpha_j\in\mathbb{R}$ $(j=1,\dots,n)$, then using (1.2.123) we get

$$\sum_{j=1}^{n}\alpha_j e_j=0$$

and, therefore,

$$\alpha_1=\cdots=\alpha_n=0,$$

i.e., X_0 and X_k $(k=1,2,\dots)$ are the fundamental matrices, respectively, of the homogeneous systems $(1.2.1_0)$ and $(1.2.1_{k0})$ $(k=1,2,\dots)$. Without loss of generality, we assume that

$$X_k(t_k)=I_n \ \ (k=0,1,\dots).$$

Owing to (1.2.10), condition (1.2.96) holds uniformly on every closed interval $[a,b]$ from I. Therefore, due to Lemma 1.2.5, conditions (1.2.97) and (1.2.98) hold, and condition (1.2.37) holds uniformly on every closed interval $[a,b]$ from I, as well.

Let us verify conditions (1.2.11)–(1.2.14) of the theorem for

$$H_k(t)\equiv X_k^{-1}(t) \ \ (k=0,1,\dots).$$

Conditions (1.2.11) and (1.2.12) coincide with conditions (1.2.98) and (1.2.37), respectively.

According to Proposition 1.1.5 (see equality (1.1.17)), we have

$$X_k^{-1}(t)=I_n-\mathcal{B}(X_k^{-1},A_k)(t) \ \text{ for } \ t\in I \ \ (k=0,1,\dots). \tag{1.2.124}$$

Therefore,

$$H_k(t)+\mathcal{B}(H_k,A_k)(t)\equiv I_n \ \ (k=0,1,\dots) \tag{1.2.125}$$

and, by the definition of the operator $\mathcal{I}$ (see (0.0.3)), we conclude

$$\mathcal{I}(H_k,A_k)(t)\equiv O \ \ (k=0,1,\dots). \tag{1.2.126}$$

Thus condition (1.2.13) is evident.

On the other hand, by (1.2.125) and equalities $H_k(t_k)=I_n$ $(k=0,1,\dots)$, according to Lemma 1.2.1 and the definition of the solutions of system $(1.2.1_k)$, we have

$$\begin{aligned}
\mathcal{B}(H_k,f_k)(t)-\mathcal{B}(H_k,f_k)(t_k)&=\mathcal{B}\Big(H_k,x_k-\int_{t_k}^{\cdot}dA_k(s)\cdot x_k(s)\Big)(t)-\mathcal{B}\Big(H_k,x_k-\int_{t_k}^{\cdot}dA_k(s)\cdot x_k(s)\Big)(t_k)\\
&=\mathcal{B}(H_k,x_k)(t)-\mathcal{B}(H_k,x_k)(t_k)-\mathcal{B}\Big(H_k,\int_{t_k}^{\cdot}dA_k(s)\cdot x_k(s)\Big)(t)+\mathcal{B}\Big(H_k,\int_{t_k}^{\cdot}dA_k(s)\cdot x_k(s)\Big)(t_k)\\
&=\mathcal{B}(H_k,x_k)(t)-\mathcal{B}(H_k,x_k)(t_k)-\int_{t_k}^{t}d\mathcal{B}(H_k,A_k)(s)\cdot x_k(s)\\
&=H_k(t)x_k(t)-H_k(t_k)x_k(t_k)-\int_{t_k}^{t}dH_k(s)\cdot x_k(s)-\int_{t_k}^{t}d(I_n-H_k(s))\cdot x_k(s)\\
&=H_k(t)x_k(t)-H_k(t_k)x_k(t_k) \ \text{ for } \ t\in I \ \ (k=0,1,\dots).
\end{aligned}$$

Hence

$$\mathcal{B}(H_k,f_k)(t)-\mathcal{B}(H_k,f_k)(t_k)-(\mathcal{B}(H_0,f_0)(t)-\mathcal{B}(H_0,f_0)(t_0))\\ =H_k(t)x_k(t)-H_0(t)x_0(t)-(x_k(t_k)-x_0(t_0)) \ \text{ for } \ t\in I \ \ (k=0,1,\dots). \quad (1.2.127)$$

By this, (1.2.5) and (1.2.126), if we take into account the fact that due to the necessity of the theorem condition (1.2.3) holds uniformly on I, we conclude that condition (1.2.14) holds uniformly on I, as well. □

Proof of Theorem 1.2.2′. It is evident that in view of conditions (1.2.38), (1.2.39) and (1.2.40), conditions (1.2.18) and (1.2.19) hold uniformly on I. So, the theorem follows from Theorem 1.2.2 and Remark 1.2.1. □

Proof of Theorem 1.2.3′. In this case, condition (1.2.41) is equivalent to condition (1.2.24). Moreover, by conditions (1.2.42), (1.2.43) and (1.2.44), conditions (1.2.29) and (1.2.30) hold uniformly on I. So, the theorem follows from Theorem 1.2.3 and the remark analogous to Remark 1.2.1. □

Proof of Corollary 1.2.1′. Verify the conditions of Theorem 1.2.3′. The validity of conditions (1.2.23), (1.2.41) and (1.2.122) we show as in the proof of Corollary 1.2.1. In addition, by (1.2.122), there exists a positive number M such that

$$\|H_k^{-1}(t)\|\le M \ \text{ for } \ t\in I \ \ (k=0,1,\dots).$$

Using Lemma 1.2.1, from this estimate and also from (1.2.12), (1.2.34), (1.2.35), (1.2.45) and (1.2.122) we find that condition (1.2.42) holds, and conditions (1.2.43) and (1.2.44) are fulfilled uniformly on I, where

$$h_k(t)\equiv -H_k(t)\varphi_k(t),\quad A_k^*(t)\equiv \mathcal{I}(H_k,A_k)(t)-\mathcal{I}(H_k,A_k)(t_k) \ \ (k=0,1,\dots);$$
$$f_0^*(t)\equiv \mathcal{B}(H_0,f_0)(t)-\mathcal{B}(H_0,f_0)(t_0),$$
$$f_k^*(t)\equiv \mathcal{B}(H_k,f_k-\varphi_k)(t)-\mathcal{B}(H_k,f_k-\varphi)(t_k)+\int\limits_{t_k}^{t} d\mathcal{B}(H_k,A_k)(s)\cdot\varphi_k(s) \ \ (k=1,2,\dots).$$

Further, the proof coincides with that of Corollary 1.2.1. □

Proof of Theorem 1.2.1′. Sufficiency follows from Corollary 1.2.1′ if we assume $\varphi_k(t)\equiv 0$ $(k=1,2,\dots)$ therein. The proof of necessity is the same as in the proof of Theorem 1.2.1. We only note that by condition (1.2.12) and equality (1.2.125), condition (1.2.34) is valid, and condition (1.2.35) holds uniformly on I. □

Proof of Theorem 1.2.1″. As it follows from the proof of Theorem 1.2.1, we may assume that $H_k(t)\equiv X_k^{-1}(t)$. In this case, Theorem 1.2.1′ has the form of Theorem 1.2.1″. We only note that by (1.2.37) and (1.2.124), condition (1.2.35) holds uniformly on I. □

Proof of Corollary 1.2.2. By (1.2.48), (1.2.49) and (1.2.50) (or (1.2.51)), we have

$$\lim_{k\to+\infty}\sum_{s\le t;\, s,t\in I}\big(d_1H_k(s)\cdot d_1A_k(s)-d_1H_0(s)\cdot d_1A_0(s)\big)=O_{n\times n},$$
$$\lim_{k\to+\infty}\sum_{s\le t;\, s,t\in I}\big(d_1H_k(s)\cdot d_1f_k(s)-d_1H_0(s)\cdot d_1f_0(s)\big)=o_n,$$
$$\lim_{k\to+\infty}\sum_{s\le t;\, s,t\in I}\big(d_2H_k(s)\cdot d_2A_k(s)-d_2H_0(s)\cdot d_2A_0(s)\big)=O_{n\times n},$$
$$\lim_{k\to+\infty}\sum_{s\le t;\, s,t\in I}\big(d_2H_k(s)\cdot d_2f_k(s)-d_2H_0(s)\cdot d_2f_0(s)\big)=o_n$$

uniformly on I. From this, the integration-by-parts formula, (1.2.46) and (1.2.47), we find that conditions (1.2.35) and (1.2.36) are fulfilled uniformly on I. Condition (1.2.36) coincides with (1.2.45) for $\varphi_k(t)\equiv 0$ $(k=1,2,\dots)$.

Therefore, the corollary follows from Corollary 1.2.1′. □

Proof of Corollary 1.2.3. Using (1.2.12), (1.2.39) and (1.2.52), we conclude

$$d_j A^*(t)\equiv O_{n\times n}\ \ (j=1,2).$$

Hence, in view of (1.2.8), we have

$$\det\big(I_n+(-1)^j d_j A_0^*(t)\big)\neq\ 0\ \ \text{for}\ \ t\in I,\ \ (-1)^j(t-t_0)<0$$
$$\text{and for}\ \ t=t_0\ \ \text{if}\ \ j\in\{1,2\}\ \ \text{is such that}\ \ (-1)^j(t_k-t_0)>0\ \ \text{for every}\ \ k\in\{1,2,\dots\}.$$

On the other hand, from (1.2.12), (1.2.39), (1.2.40), (1.2.52) and (1.2.53) we obtain that the conditions

$$\lim_{k\to+\infty}\big(\mathcal{B}(H_k,A_k)(t)-\mathcal{B}(H_k,A_k)(t_k)\big)=\mathcal{B}(I_n,A_0^*)(t)-\mathcal{B}(I_n,A_0^*)(t_0)$$

and

$$\lim_{k\to+\infty}\big(\mathcal{B}(H_k,f_k)(t)-\mathcal{B}(H_k,f_k)(t_k)\big)=\mathcal{B}(I_n,f_0^*)(t)-\mathcal{B}(I_n,f_0^*)(t_0)$$

hold uniformly on I. Thus, Corollary 1.2.3 is a direct consequence of Theorem 1.2.1′. □

Proof of Corollary 1.2.4. By virtue of (1.2.9) and (1.2.55), we have

$$\lim_{k\to+\infty}B_j(t_k)=B_j(t_0)\ \ (j=1,\dots,m-1)$$

and, therefore,

$$\lim_{k\to+\infty}C_{kj}(t)=I_n\ \ \text{and}\ \ \lim_{k\to+\infty}H_{kj}(t)=I_n\ \ (j=1,\dots,m-1)$$

uniformly on I, where

$$C_{kj}(t)\equiv I_n-(A_{kj}(t)-A_{kj}(t_k))+(B_j(t)-B_j(t_k))\ \ (j=1,\dots,m;\ k=1,2,\dots).$$

Thus, without loss of generality, we can assume that the matrix-functions H_{kj} $(j=1,\dots,m)$ and C_{kj} $(j=1,\dots,m)$ are nonsingular for every natural k. Using now Lemma 1.2.1, we find that

$$\mathcal{B}(C_{kj},\mathcal{B}(H_{k\,j-1},A_k))(\tau)\big|_{t_k}^{t}\equiv\mathcal{B}(H_{kj},A_k)(\tau)\big|_{t_k}^{t},$$
$$\mathcal{B}(C_{kj},\mathcal{B}(H_{k\,j-1},f_k))(\tau)\big|_{t_k}^{t}\equiv\mathcal{B}(H_{kj},f_k)(\tau)\big|_{t_k}^{t}$$

and

$$\mathcal{I}(C_{kj},\mathcal{I}(H_{k\,j-1},A_k))(\tau)\big|_{t_k}^{t}\equiv\mathcal{I}(H_{kj},A_k)(\tau)\big|_{t_k}^{t}\ \ (j=1,\dots,m;\ k=1,2,\dots).$$

In addition, by conditions (1.2.54)–(1.2.57), according to Lemma 1.2.4 and the definition of the operator $\mathcal{I}$, we find that conditions (1.2.12), (1.2.35) and (1.2.36) hold uniformly on I, where $H_0(t)\equiv I_n$ and $H_k(t)\equiv H_{k\,m-1}(t)$ $(k=1,2,\dots)$. The corollary follows from Theorem 1.2.1′. □

Proof of Corollary 1.2.5. Let us show the sufficiency. Let $H_k(t)=Z_k^{-1}(t)$ $(k=0,1,\dots)$ in Theorem 1.2.1′. In view of (1.2.60), there exists a positive number r such that

$$\|Z_k^{-1}(t)\|\leq r\ \ \text{for}\ \ t\in I\ \ (k=0,1,\dots).$$

Using this estimate, by (1.1.17), the definition of the operator $\mathcal{B}$ and the integration-by-parts formula, we have

$$\begin{aligned}
&\big\|Z_k^{-1}(t)+\mathcal{B}(Z_k^{-1},A_k)(t)-Z_k^{-1}(s)-\mathcal{B}(Z_k^{-1},A_k)(s)\big\| \\
&\quad=\big\|\mathcal{B}(Z_k^{-1},A_k-B_k)(t)-\mathcal{B}(Z_k^{-1},A_k-B_k)(s)\big\|=\bigg\|\int_s^t Z_k^{-1}(\tau)\,d(A_k(\tau)-B_k(\tau)) \\
&\qquad-\sum_{s<\tau\le t} d_1Z_k^{-1}(\tau)\cdot d_1(A_k(\tau)-B_k(\tau))+\sum_{s\le\tau<t} d_2Z_k^{-1}(\tau)\cdot d_2(A_k(\tau)-B_k(\tau))\bigg\| \\
&\quad\le r\bigvee_s^t(A_k-B_k)+2r\sum_{s<\tau\le t}\|d_1(A_k(\tau)-B_k(\tau))\|+2r\sum_{s\le\tau<t}\|d_2(A_k(\tau)-B_k(\tau))\| \\
&\quad\le 5r\bigvee_s^t(A_k-B_k)\ \text{ for }\ s<t\ \ (k=0,1,\dots).
\end{aligned}$$

Consequently,

$$\bigvee_I(H_k+\mathcal{B}(H_k,A_k))\le 5r\bigvee_I(A_k-B_k)\ \ (k=0,1,\dots)$$

and due to (1.2.58), estimate (1.2.34) holds. Conditions (1.2.35) and (1.2.36) coincide with (1.2.61) and (1.2.62), respectively. Hence the sufficiency follows from Theorem 1.2.1′.

Let us show the necessity. Let $B_k(t)=A_k(t)$ $(k=0,1,\dots)$. Then $Z_k(t)\equiv X_k(t)$ $(k=0,1,\dots)$, where X_0 and X_k $(k=1,2,\dots)$ are the fundamental matrices of systems $(1.1.1_0)$ and $(1.1.1_{k0})$, respectively. Analogously, just as in the proof of Theorem 1.2.1, conditions (1.2.60) and (1.2.127) are valid. In addition, condition (1.2.61) coincides with (1.2.35), and condition (1.2.62) follows from (1.2.127). □

Proof of Corollary 1.2.6. Due to conditions (1.2.64) and (1.2.65), without loss of generality, we may assume that condition (1.2.59) holds for every natural k. Condition (1.2.60) follows from (1.2.65) by representation (1.2.68).

Let us verify condition (1.2.61). Using the integration-by-parts formula, we find that

$$\begin{aligned}
\mathcal{B}(Z_k^{-1},A_k)(t)-\mathcal{B}(Z_k^{-1},A_k)(s)&=\int_s^t Z_k^{-1}(\tau)\,dA_k(\tau) \\
&-\sum_{s<\tau\le t} d_1Z_k^{-1}(\tau)\cdot d_1A_k(\tau)+\sum_{s\le\tau<t} d_2Z_k^{-1}(\tau)\cdot d_2A_k(\tau)\ \text{ for }\ s<t\ \ (k=0,1,\dots).
\end{aligned}$$

In addition, by equalities (1.1.18), we have

$$d_jZ_k^{-1}(t)\equiv -Z_k^{-1}(t)\,d_jB_k(t)\cdot(I_n+(-1)^jd_jB_k(t))^{-1}\ \ (j=1,2;\ k=0,1,\dots).$$

Consequently, due to (0.0.1), we get

$$\mathcal{B}(Z_k^{-1},A_k)(t)-\mathcal{B}(Z_k^{-1},A_k)(s)=\int_s^t Z_k^{-1}(\tau)\,d\mathcal{A}(B_k,A_k)(\tau)\ \ (k=0,1,\dots)$$

for $s<t$. In the same way we establish the last equalities for the case $t<s$.

Analogously, we check the equalities

$$\mathcal{B}(Z_k^{-1},f_k)(t)-\mathcal{B}(Z_k^{-1},f_k)(s)=\int_s^t Z_k^{-1}(\tau)\,d\mathcal{A}(B_k,f_k)(\tau)\ \text{ for }\ s,t\in I\ \ (k=0,1,\dots).$$

Therefore, equalities (1.2.61) and (1.2.62) coincide with equalities (1.2.66) and (1.2.67), respectively. The corollary follows from Corollary 1.2.5. □

Proof of Corollary 1.2.7. The corollary follows from Corollary 1.2.6 if we assume that $B_k(t) \equiv S_c(A_k)(t)$ $(k = 0, 1, \dots)$ therein. In addition, we note that condition (1.2.58) has form (1.2.69), condition (1.2.65) is equivalent to conditions (1.2.70) and (1.2.71), and by (1.2.68), conditions (1.2.66) and (1.2.67) coincide with (1.2.72) and (1.2.73), respectively. □

Proof of Corollary 1.2.8. The corollary follows from Corollary 1.2.6 if we assume that $B_k(t) \equiv \operatorname{diag}(A_k(t))$ $(k = 0, 1, \dots)$ therein. □

1.3 The stability in the Liapunov sense

1.3.1 Statement of the problem and formulation of the results

In this section, we investigate the question on the stability of the solutions of the system

$$dx = dA(t) \cdot x + df(t) \ \text{ for } \ t \in \mathbb{R}_+ \tag{1.3.1}$$

with respect to the small perturbation of initial data under the assumptions

$$A \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n}), \ \ f \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^n).$$

We consider, mainly, the case where $A \notin \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$.

As above, we assume that condition (1.1.10) holds, i.e.,

$$\det\big(I_n + (-1)^j d_j A(t)\big) \neq 0 \ \text{ for } \ t \in \mathbb{R}_+ \ \ (j = 1, 2),$$

and, without loss of generality, $A(0) = O_{n\times n}$. This condition guarantees the unique solvability of the initial problem for system (1.3.1) (see Section 1.1).

Definition 1.3.1. A solution x_0 of system (1.3.1) is called stable if for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists a positive number $\delta = \delta(\varepsilon, t_0)$ such that an arbitrary solution x of system (1.3.1), satisfying the inequality

$$\|x(t_0) - x_0(t_0)\| < \delta, \tag{1.3.2}$$

admits the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \ \text{ for } \ t \geq t_0. \tag{1.3.3}$$

Definition 1.3.2. A solution x_0 of system (1.3.1) is called uniformly stable if for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that for every $t_0 \in \mathbb{R}_+$ an arbitrary solution x of system (1.3.1), satisfying inequality (1.3.2), admits estimate (1.3.3).

Definition 1.3.3. A solution x_0 of system (1.3.1) is called asymptotically stable if it is stable and for every $t_0 \in \mathbb{R}_+$ there exists a positive number $\delta_0 = \delta_0(t_0)$ such that an arbitrary solution x of system (1.3.1), satisfying the inequality

$$\|x(t_0) - x_0(t_0)\| < \delta_0,$$

has the property

$$\lim_{t\to+\infty} \|x(t) - x_0(t)\| = 0.$$

Definition 1.3.4. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty. \tag{1.3.4}$$

A solution x_0 of system (1.3.1) is called ξ-exponentially asymptotically stable if there exists a positive number η such that for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that an arbitrary solution x of system (1.3.1), satisfying inequality (1.3.2) for some $t_0 \in \mathbb{R}_+$, admits the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp\big(-\eta(\xi(t) - \xi(t_0))\big) \ \text{ for } \ t \geq t_0.$$

We assume that $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function satisfying condition (1.3.4) when we consider the question of ξ-exponential asymptotic stability.

Note that the exponentially asymptotic stability is a particular case of the ξ-exponentially asymptotic stability if we assume $\xi(t) \equiv t$.

Definition 1.3.5. System (1.3.1) is called stable in one or another sense if every its solution is stable in the same sense.

Let x_0 be some solution of system (1.3.1). Then every solution x of the system has the form $x(t) \equiv y(t) + x_0(t)$, where y is a solution of the homogeneous system

$$dx = dA(t) \cdot x \ \text{ for } \ t \in \mathbb{R}_+. \tag{1.3.1$_0$}$$

From this and Definitions 1.3.1–1.3.5 we have the following propositions.

Proposition 1.3.1. *System* (1.3.1) *is stable in one or another sense if and only if the zero solution of the homogeneous system* (1.3.1$_0$) *is stable in the same sense.*

Proposition 1.3.2. *System* (1.3.1) *is stable in one or another sense if and only if some of its solutions is stable in the same sense.*

Therefore, the stability is not the property of any solution of system (1.3.1). It is the common property of all solutions of the system, and the vector-function f does not affect this property. Hence it is the property only of the matrix-function A. Thus the following definition is natural.

Definition 1.3.6. The matrix-function A is called stable in one or another sense if system (1.3.1$_0$) is stable in the same sense.

By Theorem 1.1.4, it is not difficult to verify the following propositions.

Proposition 1.3.3. *The matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is stable if and only if every solution of system* (1.3.1$_0$) *is bounded on $\mathbb{R}_+$.*

Proposition 1.3.4. *The matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is uniformly stable if and only if the Cauchy matrix U of system* (1.3.1$_0$) *admits the estimate*

$$\sup\big\{\|U(t,t_0)\| : \ t \geq t_0 \geq 0\big\} < +\infty. \tag{1.3.5}$$

Proposition 1.3.5. *The matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is asymptotically stable if and only if every solution x of system* (1.3.1$_0$) *has the property*

$$\lim_{t\to+\infty} \|x(t)\| = 0. \tag{1.3.6}$$

Proposition 1.3.6. *The matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is ξ-exponentially asymptotically stable if and only if there exists a positive number η such that the Cauchy matrix U of system* (1.3.1$_0$) *admits the estimate*

$$\sup\Big\{\exp\big(\eta(\xi(t) - \xi(t_0))\big) \cdot \|U(t,t_0)\| : \ t \geq t_0 \geq 0\Big\} < +\infty. \tag{1.3.7}$$

In connection with Propositions 1.3.3–1.3.6, we present some results (see Theorems 1.3.1–1.3.3) concerning the necessary and sufficient conditions for the stability in one or another sense of the matrix-function A.

Below, we assume that $H(0) = I_n$ in each statement where the matrix-function H appears.

Theorem 1.3.1. *The matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is stable if and only if there exists a nonsingular matrix-function $H \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that*

$$\sup\big\{\|H^{-1}(t)\| : \ t \in \mathbb{R}_+\big\} < +\infty \tag{1.3.8}$$

and

$$\bigvee_0^{+\infty} (H + \mathcal{B}(H, A)) < +\infty. \tag{1.3.9}$$

Theorem 1.3.2. *The matrix-function $A\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is uniformly stable if and only if there exists a nonsingular matrix-function $H\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that conditions* (1.3.9) *and*

$$\sup\big\{\|H^{-1}(t)H(\tau)\|:\ t\geq\tau\geq 0\big\}<+\infty \tag{1.3.10}$$

hold.

Theorem 1.3.3. *The matrix-function $A\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is asymptotically stable if and only if there exists a nonsingular matrix-function $H\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that conditions* (1.3.9) *and*

$$\lim_{t\to+\infty}\|H^{-1}(t)\|=0 \tag{1.3.11}$$

hold.

Theorem 1.3.4. *The matrix-function $A\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is ξ-exponentially asymptotically stable if and only if there exist a positive number η and a nonsingular matrix-function $H\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that*

$$\sup\Big\{\exp\big(\eta(\xi(t)-\xi(\tau))\big)\cdot\|H^{-1}(t)H(\tau)\|:\ t\geq\tau\geq 0\Big\}<+\infty \tag{1.3.12}$$

and

$$\bigvee_0^{+\infty}\mathcal{B}_\eta(H,A)<+\infty, \tag{1.3.13}$$

where

$$\mathcal{B}_\eta(H,A)(t)\equiv\int_0^t\exp(-\eta\xi(\tau))\,d(H(\tau)+\mathcal{B}(H,A)(\tau)). \tag{1.3.14}$$

Remark 1.3.1. In Theorem 1.3.3, if the function $\xi:\mathbb{R}_+\to\mathbb{R}_+$ is continuous, then condition (1.3.13) can be rewritten in the form

$$\bigg\|\int_0^{+\infty}d\Big(V(\mathcal{I}(H,A))(t)+\eta\,\mathrm{diag}(\xi(t),\dots,\xi(t))\Big)\cdot|H(t)|\bigg\|<+\infty.$$

Corollary 1.3.1. *Let the matrix-function $Q\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\det\big(I_n+(-1)^jd_jQ(t)\big)\neq 0\ \ for\ \ t\in\mathbb{R}_+\ \ (j=1,2) \tag{1.3.15}$$

and

$$\bigvee_0^{+\infty}\mathcal{B}(Y^{-1},A-Q)<+\infty, \tag{1.3.16}$$

where Y $(Y(0)=I_n)$ is the fundamental matrix of the system

$$dy=dQ(t)\cdot y\ \ for\ \ t\in\mathbb{R}_+. \tag{1.3.17}$$

Then the stability in one or another sense of the matrix-function Q guarantees the stability of the matrix-functions A in the same sense.

Theorem 1.3.5. *Let the matrix-function $A_0\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be uniformly stable and*

$$\det\big(I_n+(-1)^jd_jA_0(t)\big)\neq 0\ \ for\ \ t\in\mathbb{R}_+\ \ (j=1,2). \tag{1.3.18}$$

Let, moreover, the matrix-function $A\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that

$$\bigvee_0^{+\infty}\mathcal{A}\big(A_0,\mathcal{I}(H,A)-A_0\big)<+\infty, \tag{1.3.19}$$

where $H\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is a nonsingular matrix-function satisfying condition (1.3.10)*. Then the matrix-function A is uniformly stable, as well.*

Remark 1.3.2. In Theorem 1.3.5, if $H(t) \equiv I_n$, then condition (1.3.19) has the form

$$\bigvee_{0}^{+\infty} \mathcal{A}(A_0, A - A_0) < +\infty. \tag{1.3.20}$$

If the matrix-function A_0 is stable, but not uniformly stable, then condition (1.3.20) does not guarantee the uniform stability of A. We give a corresponding example which is a simple modification of the analogous one from [35]. Let

$$A_0(t) = \begin{pmatrix} -\dfrac{1}{8}\,t & 0 \\ 0 & -t(1+\cos^2 t)i \end{pmatrix}, \quad A(t) = \begin{pmatrix} -\dfrac{1}{8}\,t & 0 \\ -8\exp\Big(-\dfrac{t}{8}\Big) & -t(1+\cos^2 t) \end{pmatrix}.$$

It is not difficult to verify that A_0 is not only stable, but also asymptotically stable. As to the matrix-function A, it is not uniformly stable.

Theorem 1.3.6. *Let the matrix-function $A_0 \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable and condition* (1.3.18) *hold. Let, moreover, the matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\lim_{t\to+\infty} \bigvee_{t}^{\nu(\xi)(t)} \mathcal{A}(A_0, A - A_0) = 0, \tag{1.3.21}$$

where

$$\nu(\xi)(t) = \sup\big\{\tau \ge t:\ \xi(\tau) \le \xi(t+) + 1\big\}. \tag{1.3.22}$$

Then the matrix-function A is also ξ-exponentially asymptotically stable.

Corollary 1.3.2. *Let the components of the matrix-function $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$1 + (-1)^j d_j a_{ii}(t) \neq 0 \ \ \textit{for}\ \ t \in \mathbb{R}_+ \ \ (j=1,2;\ i=1,\dots,n),$$

$$\lim_{t\to+\infty} \bigvee_{t}^{\nu(\xi)(t)} \mathcal{A}(a_{ii}, a_{ik}) = 0 \ \ (i,k=1,\dots,n) \tag{1.3.23}$$

and

$$a_{ii}(t) - a_{ii}(\tau) \le -\eta(\xi(t) - \xi(\tau)) \ \ \textit{for}\ \ 0 \le \tau < t \ \ (i=1,\dots,n), \tag{1.3.24}$$

where $\eta > 0$, and the function $\nu(\xi) : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by (1.3.22)*. Then the matrix-function A is ξ-exponentially asymptotically stable.*

Corollary 1.3.3. *Let a matrix-function $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable, and the matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\lim_{t\to+\infty} \bigvee_{t}^{\nu_c(\xi)(t)} (A - A_0) = 0,$$

where

$$\nu_c(\xi)(t) = \max\big\{\tau \ge t:\ \xi(\tau) = \xi(t) + 1\big\}, \quad A_0(t) \equiv \int_0^t P(\tau)\,d\tau,$$

and $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function satisfying condition (1.3.4)*. Then the matrix-function A is ξ-exponentially asymptotically stable, as well.*

Remark 1.3.3. If the function ξ is strongly increasing, then

$$\nu_c(\xi)(t) = \xi^{-1}(\xi(t) + 1).$$

In particular, if $\xi(t) = t$, then $\nu_c(\xi)(t) = t + 1$, and the obtained results coincide with ones given in [35] for the case of ordinary differential equations.

Proposition 1.3.7. *Let the matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable, and the vector-function $f \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ be such that*

$$\lim_{t\to+\infty} \bigvee_{t}^{\nu(\xi)(t)} \mathcal{A}(A,f) = 0, \tag{1.3.25}$$

where the function $\nu(\xi) : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by (1.3.22)*. Then each solution x of system* (1.3.1) *satisfies condition* (1.3.6)*.*

Proposition 1.3.8. *Let the matrix-function $A_0 \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable and condition* (1.3.18) *hold. Let, moreover, the matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\lim_{t\to+\infty} \frac{1}{\xi(t)} \bigvee_{0}^{t} \mathcal{A}(A_0, A-A_0) = 0 \tag{1.3.26}$$

and

$$\big\|d_1\mathcal{A}(A_0, A-A_0)(t)\big\| < 1 \ \text{ for } \ t \in \mathbb{R}_+. \tag{1.3.27}$$

Then the matrix-function A is ξ-exponentially asymptotically stable, as well.

Theorem 1.3.7. *Let the matrix-function $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$1 + (-1)^j d_j a_{ii}(t) \neq 0 \ \text{ for } \ t \geq t^* \ \ (j=1,2;\ i=1,\dots,n), \tag{1.3.28}$$

$$\int_{t^*}^{t} \exp\big(J(a_{ii})(t) - J(a_{ii})(\tau)\big)\, dv(b_{ik})(\tau) \leq h_{ik} \ \text{ for } \ t \geq t^* \ \ (i\neq k;\ i,k=1,\dots,n) \tag{1.3.29}$$

and

$$\sup\big\{J(a_{ii})(t) :\ t \in \mathbb{R}_+\big\} < +\infty \ \ (i=1,\dots,n), \tag{1.3.30}$$

where $b_{ik}(t) \equiv \mathcal{A}(a_{ii}, a_{ik})(t)$ $(i\neq k;\ i,k=1,\dots,n)$, t^* and $h_{ik} \in \mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$. Let, moreover, the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0$ $(i=1,\dots,n)$, be such that

$$r(H) < 1. \tag{1.3.31}$$

Then the matrix-function A is stable.

Theorem 1.3.8. *Let the matrix-function $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that conditions* (1.3.28), (1.3.29) *and*

$$\sup\big\{J(a_{ii})(t) - J(a_{ii})(\tau) :\ t \geq \tau \geq t^*\big\} < +\infty \ \ (i=1,\dots,n) \tag{1.3.32}$$

hold, where $b_{ik}(t) \equiv \mathcal{A}(a_{ii}, a_{ik})(t)$ $(i\neq k;\ i,k=1,\dots,n)$, t^* and $h_{ik} \in \mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$. Let, moreover, the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0$ $(i=1,\dots,n)$, be such that condition (1.3.31) *holds. Then the matrix-function A is uniformly stable.*

Remark 1.3.4. In Theorem 1.3.8, condition (1.3.32) cannot be replaced by (1.3.30). We give the corresponding example from [35]. Let $n = 1$, $A(t) = -t(1+\cos^2 t)$. Then every solution of system $(1.3.1_0)$ has the form

$$x(t) = \exp\big(-t(1+\cos^2 t) + t_0(1+\cos^2 t_0)\big)x(t_0).$$

Therefore, the matrix-function A is asymptotically stable, since each solution of $(1.3.1_0)$ satisfies condition (1.3.6). On the other hand,

$$x(t) = \exp\Big(k\pi - \frac{\pi}{2}\Big)x(t_0)$$

for $t = k\pi + \frac{\pi}{2}$ and $t_0 = k\pi$ for every natural k. From this, it is evident that for every $\rho > 0$, condition (1.3.5) is violated for some $t_0 \in \mathbb{R}_+$ and $t > t_0$. So, the matrix-function A is not uniformly stable.

Corollary 1.3.4. *Let the components of the matrix-function $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that the conditions*

$$1 + (-1)^j d_j a_{ii}(t) > 0 \ \text{ for } \ t \geq t_* \ \ (j = 1, 2; \ i = 1, \dots, n) \tag{1.3.33}$$

and

$$\big|\mathcal{A}(a_{ii}, a_{ik})(t) - \mathcal{A}(a_{ii}, a_{ik})(\tau)\big| \\ \leq -h_{ik}\big(\mathcal{A}(a_{ii}, a_{ii})(t) - \mathcal{A}(a_{ii}, a_{ii})(\tau)\big) \ \text{ for } \ t \geq \tau \geq t_* \ \ (i \neq k; \ i, k = 1, \dots, n) \tag{1.3.34}$$

hold, where a_{ii} $(i = 1, \dots, n)$ are non-increasing functions, t_* and $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i, k = 1, \dots, n)$. Let, moreover, the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0$ $(i = 1, \dots, n)$, be such that condition (1.3.31) *holds. Then the matrix-function A is uniformly stable.*

Theorem 1.3.9. *Let the matrix-function $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that conditions* (1.3.28),

$$J(a_{ii})(t) - J(a_{ii})(t^*) \leq -\eta(t) + \eta(t^*) \ \text{ for } \ t \geq t^* \ \ (i = 1, \dots, n), \tag{1.3.35}$$

and

$$\int_{t^*}^{t} \exp\big(\eta(t)-\eta(\tau)+J(a_{ii})(t)-J(a_{ii})(\tau)\big)\, dv(b_{ik})(\tau) \leq h_{ik} \ \text{ for } \ t\geq t^* \ \ (i\neq k; \ i,k=1,\dots,n) \tag{1.3.36}$$

hold, where $b_{ik}(t) \equiv \mathcal{A}(a_{ii}, a_{ik})(t)$ $(i \neq k;\ i, k = 1, \dots, n)$, t_* and $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i, k = 1, \dots, n)$, and the function $\eta \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})$ satisfies condition (1.3.4)*. Let, moreover, the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0$ $(i = 1, \dots, n)$, be such that condition* (1.3.31) *holds. Then the matrix-function A is asymptotically stable.*

Corollary 1.3.5. *Let the components of the matrix-function $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that the conditions*

$$\big|s_c(a_{ik})(t) - s_c(a_{ik})(\tau)\big| \\ \leq -h_{ik}\big(s_c(a_{ii})(t) - s_c(a_{ii})(\tau)\big) \ \text{ for } \ t \geq \tau \geq t^* \ \ (i \neq k; \ i, k = 1, \dots, n), \tag{1.3.37}$$

$$(-1)^j d_j a_{ii}(t) > 0 \ \text{ or } \ -1 < (-1)^j d_j a_{ii}(t) < \exp^{-1}(1) - 1 \\ \text{ for } \ t \geq t^* \ \ (j = 1, 2; \ i = 1, \dots, n), \tag{1.3.38}$$

and

$$|d_j a_{ik}(t)| \leq h_{ik}\Big(1 + \ln\big(1 + (-1)^j d_j a_{ii}(t)\big)\Big)^{-1} \\ \times \ln\big(1 + (-1)^j d_j a_{ii}(t)\big) \ \text{ for } \ t \geq t^* \ \ (j = 1, 2; \ i \neq k; \ i, k = 1, \dots, n) \tag{1.3.39}$$

hold, where $s_c(a_{ii})$ $(i = 1, \dots, n)$ are non-increasing functions, t^* and $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i, k = 1, \dots, n)$ are such that the matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0$ $(i = 1, \dots, n)$, satisfies condition (1.3.31)*. Let, moreover, there exists a function $a_0 \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})$ such that*

$$a_0(t) - a_0(\tau) \leq \min\Big\{\big|J(a_{ii})(t) - J(a_{ii})(\tau)\big| : \ i = 1, \dots, n\Big\} \ \text{ for } \ t \geq \tau \geq t^* \tag{1.3.40}$$

and

$$\lim_{t\to+\infty} a_0(t) = +\infty. \tag{1.3.41}$$

Then the matrix-function A is asymptotically, as well as uniformly stable.

Corollary 1.3.6. *Let the components of the matrix-function $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that conditions* (1.3.37)–(1.3.39) *hold, where a_{ii} $(i = 1, \dots, n)$ are non-increasing functions such that $s_c(a_{ii}) \in AC_{loc}(\mathbb{R}_+;\mathbb{R})$ $(i = 1, \dots, n)$, t^* and $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i, k = 1, \dots, n)$ are such that* *the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfies condition* (1.3.31). *Let, moreover, condition* (1.3.41) *hold, where*

$$a_0(t)\equiv\int\limits_0^t\eta_0(s)\,ds+\sum_{0<s\le t}\ln(1-\eta_1(s))-\sum_{0\le s<t}\ln(1+\eta_2(s)), \tag{1.3.42}$$

$$\eta_0(t)\equiv\min\big\{|(s_c(a_{ii})(t))'|:\ i=1,\dots,n\big\} \tag{1.3.43}$$

and

$$\eta_1(t)\equiv\max\big\{d_1a_{ii}(t):\ i=1,\dots,n\big\},\quad \eta_2(t)\equiv\min\big\{d_2a_{ii}(t):\ i=1,\dots,n\big\}. \tag{1.3.44}$$

Then the matrix-function A is asymptotically, as well as uniformly stable.

Theorem 1.3.10. *Let the matrix-function* $A=(a_{ik})_{i,k=1}^n\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that conditions* (1.3.28),

$$\sup\Big\{\frac{J(a_{ii})(t)-J(a_{ii})(\tau)}{\xi(t)-\xi(\tau)}\ :\ t\ge\tau\ge t^*,\ \xi(t)\ne\xi(\tau)\Big\}<-\gamma\ \ (i=1,\dots,n) \tag{1.3.45}$$

and

$$\int\limits_{t^*}^t\exp\big(\gamma(\xi(t)-\xi(\tau))+J(a_{ii})(t)-J(a_{ii})(\tau)\big)\,dv(b_{ik})(\tau)$$
$$\le h_{ik}\ \ for\ \ t\ge t^*\ \ (i\ne k;\ i,k=1,\dots,n) \qquad *$$

hold, where $\gamma>0$, t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k;\ i,k=1,\dots,n)$, $b_{ik}(t)\equiv\mathcal{A}(a_{ii},a_{ik})(t)$ $(i,k=1,\dots,n)$. *Let, moreover, the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *be such that condition* (1.3.31) *holds. Then the matrix-function A is ξ-exponentially asymptotically stable.*

Corollary 1.3.7. *Let the components of the matrix-function* $A=(a_{ik})_{i,k=1}^n\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that conditions* (1.3.33), (1.3.37), (1.3.39) *and* (1.3.45) *hold, where* $\gamma>0$, t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k;\ i,k=1,\dots,n)$, a_{ii} $(i=1,\dots,n)$ *are non-decreasing functions. Let, moreover, the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *be such that condition* (1.3.31) *holds. Then the matrix-function A is ξ-exponentially asymptotically stable.*

Theorem 1.3.11. *Let* $A=(a_{ik})_{i,k=1}^n\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, *and let the matrix-function* $A_0=(a_{0ik})_{i,k=1}^n\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, *with non-decreasing components* a_{0ik} $(i\ne k;\ i,k=1,\dots,n)$, *be such that*

$$\|d_jA(t)\|<1\ \ for\ \ t\in\mathbb{R}_+\ \ (j=1,2),$$
$$s_c(a_{ii})(t)-s_c(a_{ii})(\tau)\le s_c(a_{0ii})(t)-s_c(a_{0ii})(\tau)\ \ for\ \ t>\tau\ge0\ \ (i=1,\dots,n),$$
$$\big|s_c(a_{ik})(t)-s_c(a_{ik})(\tau)\big|\le s_c(a_{0ik})(t)-s_c(a_{0ik})(\tau)\ \ for\ \ t>\tau\ge0\ \ (i\ne k;\ i,k=1,\dots,n)$$

and

$$d_ja_{ii}(t)|\le d_ja_{0ii}(t)\ \ and\ \ |d_ja_{ik}(t)|\le d_ja_{0ik}(t)\ \ for\ \ t\in\mathbb{R}_+\ \ (j=1,2;\ i,k=1,\dots,n).$$

Let, moreover, the matrix-function A_0 be stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable). Then the matrix-function A is also stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable).

Theorem 1.3.12. *Let* $\alpha_{ik}\in\mathbb{R}$ $(i,k=1,\dots,n)$ *and* $\mu_i:\mathbb{R}_+\to\mathbb{R}$ $(i=1,\dots,n)$ *be the functions such that* $s_c(\mu_i)$ $(i=1,\dots,n)$ *are absolutely continuous and non-decreasing functions, conditions* (1.3.41) *and*

$$(-1)^j\alpha_{ii}d_j\mu_i(t)>0\ \ or\ \ -1<(-1)^j\alpha_{ii}d_j\mu_i(t)<\exp^{-1}(1)-1$$
$$for\ \ t\in\mathbb{R}_+\ \ (j=1,2;\ i=1,\dots,n) \tag{1.3.46}$$

hold, where the function $a_0(t)$ *is defined by* (1.3.42),

$$\eta_0(t) \equiv \min\big\{|\alpha_{ii}|(s_c(\mu_i)(t))' : \ i=1,\dots,n\big\}$$

and

$$\eta_1(t) \equiv \max\big\{\alpha_{ii} d_1\mu_i(t) : i=1,\dots,n\big\}, \quad \eta_2(t) \equiv \min\big\{\alpha_{ii} d_2\mu_i(t) : i=1,\dots,n\big\}.$$

Then conditions (1.3.31) *and*

$$\alpha_{ii} < 0 \ \ (i=1,\dots,n), \tag{1.3.47}$$

where

$$H = \Big((1-\delta_{ik})\,\frac{|\alpha_{ik}|}{|\alpha_{ii}|}\Big)_{i,k=1}^{n}, \tag{1.3.48}$$

are sufficient for the matrix-function $A(t) = (\alpha_{ik}\mu_i(t))_{i,k=1}^n$ *to be asymptotically stable; and if the conditions*

$$\alpha_{ik} \geq 0 \ \ (i \neq l;\ i,k=1,\dots,n), \tag{1.3.49}$$

$$\sum_{l=1;\, l\neq i}^{n} \alpha_{il} d_j\mu_i(t) < |1-\alpha_{ii}d_j\mu_i(t)| \ \ or$$

$$\sum_{l=1;\, l\neq k}^{n} \alpha_{lk} d_j\mu_i(t) < |1-\alpha_{kk}d_j\mu_k(t)| \ \ for\ \ t\in\mathbb{R}_+ \ \ (j=1,2;\ i,k=1,\dots,n), \tag{1.3.50}$$

$$\alpha_{ii}d_j\mu_i(t) < 1 \ \ for\ \ t\in\mathbb{R}_+ \ \ (j=1,2;\ i=1,\dots,n) \tag{1.3.51}$$

and

$$\big((\delta_{ik}-\alpha_{ik}d_j\mu_i(t))_{i,k=1}^n\big)^{-1} \geq O_{n\times n} \ \ for\ \ t\in\mathbb{R}_+ \ \ (j=1,2) \tag{1.3.52}$$

hold, then conditions (1.3.31) *and* (1.3.47) *are necessary, as well.*

Corollary 1.3.8. *Let the matrix-function* $Q \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *satisfying condition* (1.3.15) *be uniformly stable and there exist a number* $\eta > 0$ *such that*

$$\Big\|\int_0^{+\infty} |Y^{-1}(t)|\, dV(G_\eta(\xi,Q,A))(t)\Big\| < +\infty, \tag{1.3.53}$$

where Y $(Y(0)=I_n)$ *is the fundamental matrix of system* (1.3.17), *and*

$$\begin{aligned} G_\eta(\xi,Q,A)(t) \equiv\ & \mathcal{A}(Q,A-Q)(t) + \eta s_c(\xi)(t) I_n \\ &+ \sum_{0<\tau\leq t} \exp(-\eta\xi(\tau))\, d_1\exp(\eta\xi(\tau))(I_n - d_1Q(\tau))^{-1}(I_n - d_1A(\tau)) \\ &+ \sum_{0\leq\tau<t} \exp(-\eta\xi(\tau))\, d_2\exp(\eta\xi(\tau))(I_n + d_2Q(\tau))^{-1}(I_n + d_2A(\tau)). \end{aligned} \tag{1.3.54}$$

Then the matrix-function A *is* ξ*-exponentially asymptotically stable.*

Remark 1.3.5. In Corollary 1.3.8, if the function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, then

$$G_\eta(\xi,Q,A)(t) \equiv \mathcal{A}(Q,A-Q)(t) + \eta\xi(t) I_n.$$

Remark 1.3.6. As

$$\mathcal{B}(Z^{-1},A-Q)(t) = \int_0^t Z^{-1}(\tau)\, d\mathcal{A}(Q,A-Q)(\tau) \ \ \text{for}\ \ t\in\mathbb{R}_+,$$

then the condition

$$\bigg\| \int\limits_0^{+\infty} |Y^{-1}(t)|\, dV(\mathcal{A}(Q, A-Q))(t) \bigg\| < +\infty$$

guarantees the fulfillment of condition (1.3.16) in Corollary 1.3.1. On the other hand,

$$\lim_{\eta\to 0+} G_\eta(\xi, Q, A)(t) = \mathcal{A}(Q, A-Q)(t) \ \text{ for } \ t \in \mathbb{R}_+,$$

where $G_\eta(\xi, Q, A)(t)$ is defined by (1.3.54). Therefore, Corollary 1.3.8 is true for the limit case ($\eta = 0$), as well, if require the ξ-exponentially asymptotic stability of the matrix-function Q instead of the uniform stability of that matrix-function.

Corollary 1.3.9. *Let $Q \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ be a continuous matrix-function satisfying the Lappo–Danilevskiĭ condition. Let, moreover, there exist a nonnegative number η such that*

$$\bigg\| \int\limits_0^{+\infty} \exp(-Q(t))\, dV(A - Q + \eta\xi I_n)(t) \bigg\| < +\infty.$$

Then:

(a) *the uniform stability of the matrix-function Q guarantees the ξ-exponentially asymptotically stability of the matrix-function A if $\eta > 0$;*

(b) *the ξ-exponentially asymptotic stability of the matrix-function Q guarantees the ξ-exponentially asymptotically stability of the matrix-function A if $\eta = 0$.*

Corollary 1.3.10. *Let there exist a nonnegative number η such that the components a_{ik} $(i,k = 1,\dots,n)$ of the matrix-function A satisfy the conditions*

$$1 + (-1)^j d_j a_{ii}(t) \neq 0 \ \textit{ for } \ t \in \mathbb{R}_+ \ \ (i = 1,\dots,n), \tag{1.3.55}$$

$$\begin{aligned} s_c(a_{ii})(t) - s_c(a_{ii})(\tau) - \sum_{\tau<s\le t} \ln|1 - d_1 a_{ii}(s)| + \sum_{\tau\le s<t} \ln|1 + d_2 a_{ii}(s)| \\ \le -\eta\big(s_c(\xi)(t) - s_c(\xi)(\tau)\big) - \mu(\xi(t) - \xi(\tau)) \ \textit{ for } \ t \ge \tau \ge 0 \ \ (i = 1,\dots,n), \end{aligned} \tag{1.3.56}$$

$$(-1)^j \sum_{0\le t<+\infty} |y_i^{-1}(t)|\big(\exp\big((-1)^j \eta d_j \xi(\tau)\big) - 1\big) < +\infty \ \ (j = 1,2; \ i = 1,\dots,n) \tag{1.3.57}$$

and

$$\int\limits_0^{+\infty} |y_i^{-1}(t)|\, dv(g_{ik})(t) < +\infty \ \ (i \neq k; \ i,k = 1,\dots,n),$$

where $\mu = 0$ if $\eta > 0$ and $\mu > 0$ if $\eta = 0$,

$$y_i(t) \equiv \exp\big(s_c(a_{ii})(t) + \eta s_c(\xi)(t)\big) \prod_{0<\tau\le t} (1 - d_1 a_{ii}(\tau))^{-1} \prod_{0\le\tau<t} (1 + d_2 a_{ii}(\tau)) \ \ (i = 1,\dots,n),$$

$$\begin{aligned} g_{ik}(t) \equiv s_c(a_{ik})(t) + \sum_{0<\tau\le t} \exp(-\eta d_1 \xi(\tau))\, d_1 a_{ik}(\tau) \cdot (1 - d_1 a_{ii}(\tau))^{-1} \\ + \sum_{0\le\tau<t} \exp(\eta d_2 \xi(\tau))\, d_2 a_{ik}(\tau) \cdot (1 + d_2 a_{ii}(\tau))^{-1} \ \ (i \neq k; \ i,k = 1,\dots,n). \end{aligned}$$

Then the matrix-function A is ξ-exponentially asymptotically stable.

Remark 1.3.7. In Corollary 1.3.10, if the components a_{ik} $(i,k = 1,\dots,n)$ of the matrix-function A satisfy the condition

$$(-1)^j d_j a_{ik}(t) \cdot \big(1 + (-1)^j d_j a_{ii}(t)\big)^{-1} > 0 \ \text{ for } \ t \in \mathbb{R}_+ \ \ (i \neq k; \ i,k = 1,\dots,n; \ j = 1,2)$$

together with condition (1.3.55), then we can assume without loss of generality that $\eta > 0$ and $\mu = 0$ in Corollary 1.3.10.

Theorem 1.3.13. *Let the matrix-function* $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that*

$$S_c(A)(t) = \sum_{k=1}^{m} s_c(\alpha_k)(t)\cdot B_k \ \ \text{for}\ \ t\in\mathbb{R}_+ \tag{1.3.58}$$

and

$$I_n + (-1)^j d_j A(t) = \exp\Big((-1)^j \sum_{k=1}^{m} d_j\alpha_k(t)\cdot B_k\Big) \ \ \text{for}\ \ t\in\mathbb{R}_+ \ \ (j=1,2), \tag{1.3.59}$$

where $\alpha_k \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ $(k=1,\dots,m)$, *and* $B_k\in\mathbb{R}^{n\times n}$ $(k=1,\dots,m)$ *are pairwise permutable constant matrices. Let, moreover,* $(\lambda-\lambda_{ki})^{n_{ki}}$ $(i=1,\dots,m_k;\ \sum\limits_{i=1}^{m_k} n_{ki}=n)$ *be the elementary divisors of the matrix* B_k *for every* $k\in\{1,\dots,m\}$. *Then*

(a) *the matrix-function* A *is stable if and only if*

$$\sup\bigg\{\prod_{k=1}^{m}\Big(\sum_{i=1}^{m_k}(1+\alpha_k(t))^{n_{ki}-1}\exp(\alpha_k(t)\mathrm{Re}\lambda_{ki})\Big):\ t\in\mathbb{R}_+\bigg\}<+\infty; \tag{1.3.60}$$

(b) *the matrix-function* A *is asymptotically stable if and only if*

$$\lim_{t\to+\infty}\prod_{k=1}^{m}\Big(\sum_{i=1}^{m_k}(1+\alpha_k(t))^{n_{ki}-1}\exp(\alpha_k(t)\mathrm{Re}\lambda_{ki})\Big)=0. \tag{1.3.61}$$

Corollary 1.3.11. *Let conditions* (1.3.58) *and* (1.3.59) *hold, where* $B_k\in\mathbb{R}^{n\times n}$ $(k=1,\dots,m)$ *are pairwise permutable constant matrices and* $\alpha_k\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ $(k=1,\dots,m)$ *are such that*

$$\lim_{t\to+\infty} a_k(t)=+\infty \ \ (k=1,\dots,m). \tag{1.3.62}$$

Then

(a) *the matrix-function* A *is stable if and only if every eigenvalue of the matrices* B_k $(k=1,\dots,m)$ *has the nonpositive real part; in addition, every elementary divisor corresponding to the eigenvalues with the zero real part is simple;*

(b) *the matrix-function* A *is asymptotically stable if and only if every eigenvalue of the matrices* B_k $(k=1,\dots,m)$ *has a negative real part.*

If the matrix-function $A\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ has at most a finite number of discontinuity points in $[a,t]$ for every $t>0$, then by $\nu_1(t)$ and $\nu_2(t)$ we denote, respectively, a number of points $\tau\in]0,t]$ for which $\|d_1A(\tau)\|\neq 0$ and a number of points $\tau\in[0,t[$ for which $\|d_2A(\tau)\|\neq 0$.

Corollary 1.3.12. *Let the matrix-function* $A\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that*

$$S_0(A)(t)=\alpha(t)A_0 \ \ \text{for}\ \ t\in\mathbb{R}_+$$

and

$$d_jA(t)=A_j \ \ \text{if}\ \ \|d_jA(t)\|\neq 0 \ \ (t\in\mathbb{R}_+;\ j=1,2),$$

where $\alpha\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ *is a continuous function such that*

$$\lim_{t\to+\infty}\alpha(t)=+\infty, \tag{1.3.63}$$

and A_0, A_1 *and* $A_2\in\mathbb{R}^{n\times n}$ *are the pairwise permutable constant matrices. Let, moreover, there exist numbers* $\beta_1,\beta_2\in\mathbb{R}_+$ *such that*

$$\limsup_{t\to+\infty}|\nu_j(t)-\beta_j\alpha(t)|<+\infty \ \ (j=1,2). \tag{1.3.64}$$

Then

(a) *the matrix-function A is stable if and only if every eigenvalue of the matrix $P = A_0 - \beta_1 \ln(I_n - A_1) + \beta_2 \ln(I_n + A_2)$ has the nonpositive real part; in addition, every elementary divisor corresponding to the eigenvalue with the zero real part is simple;*

(b) *the matrix-function A is asymptotically stable if and only if every eigenvalue of the matrix P has the negative real part.*

Corollary 1.3.13. *Let the matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$S_c(A)(t) = C \operatorname{diag}\big(S_c(G_1)(t),\dots,S_c(G_m)(t)\big)C^{-1} \quad \text{for } t\in\mathbb{R}_+$$

and

$$I_n + (-1)^j d_j A(t) = C \operatorname{diag}\Big(\exp\big((-1)^j d_j G_1(t)\big),\dots,\exp\big((-1)^j d_j G_m(t)\big)\Big)C^{-1} \quad \text{for } t\in\mathbb{R}_+ \ \ (j=1,2),$$

where $C \in \mathbb{C}^{n\times n}$ is a nonsingular complex matrix, $G_k(t) = \sum\limits_{i=0}^{n_k-1} \alpha_{ki}(t) Z_{n_k}^i$ $(k=1,\dots,m;\ \sum\limits_{k=1}^{m} n_k = n)$, $\alpha_{ki} \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ $(k=1,\dots,m;\ i=1,\dots,n_k-1)$, and α_{k0} is a complex-valued function such that $\mathrm{Re}(\alpha_{k0})$ and $\mathrm{Im}(\alpha_{k0}) \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})$. Then

(a) *the matrix-function A is stable if and only if*

$$\sup\bigg\{\exp\big(\mathrm{Re}(\alpha_{k0}(t))\big)\prod_{i=1}^{n_k-1}(1+\alpha_{ki}(t))^{[\frac{n_k-1}{i}]} :\ t\in\mathbb{R}_+\bigg\} < +\infty \ \ (k=1,\dots,m); \tag{1.3.65}$$

(b) *the matrix-function A is asymptotically stable if and only if*

$$\lim_{t\to+\infty}\exp\big(\mathrm{Re}(\alpha_{k0}(t))\big)\prod_{i=1}^{n_k-1}(1+\alpha_{ki}(t))^{[\frac{n_k-1}{i}]} = 0 \ \ (k=1,\dots,m). \tag{1.3.66}$$

Later, we will use the following notation.

Let $H = (h_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n};\alpha)$, where $\alpha:\mathbb{R}_+\to\mathbb{R}$ is a non-decreasing function. Then by $\mathcal{Q}(H;\alpha)$ we denote the set of all matrix-functions $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that

$$b_{ik}(t) = \int_0^t h_{ik}(\tau)\,d\alpha(\tau) \ \ \text{for } t\in\mathbb{R}_+ \ \ (i,k=1,\dots,n), \tag{1.3.67}$$

where

$$b_{ik} \equiv a_{ik}(t) - \frac{1}{2}\sum_{l=1}^{n}\Big(\sum_{0<\tau\le t} d_1 a_{li}(\tau)\cdot d_1 a_{lk}(\tau) - \sum_{0\le\tau<t} d_2 a_{li}(\tau)\cdot d_2 a_{lk}(\tau)\Big) \ \ (i,k=1,\dots,n). \tag{1.3.68}$$

If $\beta \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})$ is such that

$$1 + (-1)^j d_j\beta(t) \neq 0 \ \ \text{for } t\in\mathbb{R}_+ \ \ (j=1,2),$$

then by γ_β we denote the unique solution of the initial problem

$$d\gamma(t) = \gamma(t)\,d\beta(t), \quad \gamma(0)=1.$$

By (1.1.5),

$$\gamma_\beta(t) = \exp\big(s_c(\beta)(t) - s_c(\beta)(0)\big)\prod_{0<\tau\le t}(1-d_1\beta(\tau))^{-1}\prod_{0\le\tau<t}(1+d_2\beta(\tau)) \ \ \text{for } t\in\mathbb{R}_+. \tag{1.3.69}$$

Theorem 1.3.14. *Let* $A = (a_{ik})_{i,k=1}^n \in \mathcal{Q}(H;\alpha)$,

$$\sum_{i,k=1}^n h_{ik}(t)x_i x_k \le p(t)\sum_{i=1}^n x_i^2 \ \ \textit{for}\ \ t \in \mathbb{R}_+, \ \ (x_i)_{i=1}^n \in \mathbb{R}^n \tag{1.3.70}$$

and

$$1 + 2(-1)^j p(t)\, d_j\alpha(t) > 0 \ \ \textit{for}\ \ t \in \mathbb{R}_+ \ \ (j=1,2), \tag{1.3.71}$$

where $H = (h_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n};\alpha)$, $\alpha : \mathbb{R}_+ \to \mathbb{R}$ *is a non-decreasing function, and* $p \in L_{loc}(\mathbb{R}_+;\mathbb{R};\alpha)$. *Let, moreover,* $\beta(t) \equiv 2\int\limits_0^t p(\tau)\, d\alpha(\tau)$, *and* $\gamma_\beta(t)$ *be defined by* (1.3.69). *Then*

(a) *the condition*

$$\limsup_{t\to+\infty} \gamma_\beta^{1/2}(t) < +\infty \tag{1.3.72}$$

guarantees the stability of the matrix-function A*;*

(b) *the condition*

$$\sup\left\{\gamma_\beta^{1/2}(t)\gamma_\beta^{-1/2}(\tau) : \ t \ge \tau \ge 0\right\} < +\infty \tag{1.3.73}$$

guarantees the uniform stability of the matrix-function A*;*

(c) *the condition*

$$\lim_{t\to+\infty} \gamma_\beta^{1/2}(t) = 0 \tag{1.3.74}$$

guarantees the asymptotic stability of the matrix-function A*;*

(d) *the condition*

$$\sup\left\{\frac{\ln\gamma_\beta(t) - \ln\gamma_\beta(\tau)}{2(\xi(t)-\xi(\tau))} \ : \ t \ge \tau \ge t^*, \ \xi(t) \ne \xi(\tau)\right\} < 0, \tag{1.3.75}$$

where $t^* \in \mathbb{R}_+$, *guarantees the* ξ*-exponentially asymptotic stability of the matrix-function* A*;*

(e) *if the inequality, opposite to inequality* (1.3.70), *holds and*

$$\limsup_{t\to+\infty} \gamma_\beta^{1/2}(t) = +\infty, \tag{1.3.76}$$

then the matrix-function A *is nonstable.*

Corollary 1.3.14. *Let* $A \in \mathcal{Q}(H;\alpha)$ *and*

$$(-1)^j \lambda^0(C(t))\, d_j\alpha(t) > -\frac{1}{2} \ \ \textit{for}\ \ t \in \mathbb{R}_+ \ \ (j=1,2), \tag{1.3.77}$$

where

$$C(t) \equiv \frac{1}{2}\left(H(t) + H^T(t)\right),$$

$H = (h_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n};\alpha)$, *and* $\alpha : \mathbb{R}_+ \to \mathbb{R}$ *is a non-decreasing function. Then*

(a) *the condition*

$$\limsup_{t\to+\infty}\Bigg(\int\limits_0^t \lambda^0(C(\tau))\, ds_c(\alpha)(\tau) - \frac{1}{2}\sum_{0<\tau\le t}\ln\left(1 - 2\lambda^0(C(\tau))\, d_1\alpha(\tau)\right) + \frac{1}{2}\sum_{0\le\tau<t}\ln\left(1 + 2\lambda^0(C(\tau))\, d_2\alpha(\tau)\right)\Bigg) < +\infty$$

guarantees the stability of the matrix-function A*;*

(b) *the condition*

$$\sup\bigg\{\int\limits_{\tau}^{t}\lambda^0(C(s))\,ds_c(\alpha)(s)-\frac{1}{2}\sum_{\tau<s\le t}\ln\big(1-2\lambda^0(C(s))\,d_1\alpha(s)\big)+\frac{1}{2}\sum_{\tau\le s<t}\ln\big(1+2\lambda^0(C(s))\,d_2\alpha(s)\big):\ t\ge\tau\ge 0\bigg\}<+\infty$$

guarantees the uniform stability of the matrix-function A;

(c) *the condition*

$$\lim_{t\to+\infty}\bigg(\int\limits_{0}^{t}\lambda^0(C(\tau))\,ds_c(\alpha)(\tau)-\frac{1}{2}\sum_{\tau<s\le t}\ln\big(1-2\lambda^0(C(\tau))\,d_1\alpha(\tau)\big)+\frac{1}{2}\sum_{\tau\le s<t}\ln\big(1+2\lambda^0(C(\tau))\,d_2\alpha(\tau)\big)\bigg)=-\infty$$

guarantees the asymptotic stability of the matrix-function A;

(d) *the condition*

$$\sup\bigg\{\frac{1}{\xi(t)-\xi(\tau)}\bigg(\int\limits_{\tau}^{t}\lambda^0(C(s))\,ds_c(\alpha)(s)-\frac{1}{2}\sum_{\tau<s\le t}\ln\big(1-2\lambda^0(C(s))\,d_1\alpha(s)\big)+\frac{1}{2}\sum_{\tau\le s<t}\ln\big(1+2\lambda^0(C(s))\,d_2\alpha(s)\big)\bigg):\ t\ge\tau\ge t^*,\ \xi(t)\neq\xi(\tau)\bigg\}<0,$$

where $t^*\in\mathbb{R}_+$, guarantees the ξ-exponentially asymptotic stability of the matrix-function A;

(e) *if instead of condition* (1.3.77), *the condition*

$$(-1)^j\lambda_0(C(t))\,d_j\alpha(t)>-\frac{1}{2}\ \ \text{for}\ \ t\in\mathbb{R}_+\ \ (j=1,2)$$

holds and

$$\limsup_{t\to+\infty}\bigg(\int\limits_{0}^{t}\lambda_0(C(\tau))\,ds_c(\alpha)(\tau)-\frac{1}{2}\sum_{0<\tau\le t}\ln\big(1-2\lambda_0(C(\tau))\,d_1\alpha(\tau)\big)+\frac{1}{2}\sum_{0\le\tau<t}\ln\big(1+2\lambda_0(C(\tau))\,d_2\alpha(\tau)\big)\bigg)=+\infty,$$

then the matrix-function A is nonstable.

1.3.2 The well-posedness of the initial problem on infinite intervals and stability

In this section, we consider the question on the well-posedness of problem (1.2.1), (1.2.2) for the case $I=\mathbb{R}_+$, $A_0=A\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $f_0=f\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^n)$.

Definition 1.3.7. The initial problem (1.2.1), (1.2.2), where $A_0\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ is the matrix-function satisfying condition (1.2.8), and $f_0\in\mathrm{BV}_{loc}(I;\mathbb{R}^n)$, is said to be well-posed if condition (1.2.10) holds for every sequences A_k $(k=1,2,\dots)$, f_k $(k=1,2,\dots)$, t_k $(k=0,1,\dots)$ and c_k $(k=0,1,\dots)$ for which there exists a sequence H_k $(k=0,1,\dots)$ such that conditions (1.2.5), (1.2.9) and (1.2.11) hold, and conditions (1.2.12)–(1.2.14) are fulfilled uniformly on I.

It is evident that the statements of Theorems 1.2.1, 1.2.1′ and Corollary 1.2.2 imply that the initial problem (1.2.1), (1.2.2) is well-posed.

Definition 1.3.8. The initial problem (1.2.1), (1.2.2), where $A_0 \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ is the matrix-function satisfying condition (1.2.8), and $f_0 \in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$, is said to be weakly well-posed if condition (1.2.10) holds for every sequences A_k $(k = 1, 2, \dots)$, f_k $(k = 1, 2, \dots)$, t_k $(k = 0, 1, \dots)$ and c_k $(k = 0, 1, \dots)$ for which there exists a sequence H_k $(k = 0, 1, \dots)$ such that conditions (1.2.5), (1.2.9) and (1.2.11) hold, and the conditions (1.2.12),

$$\lim_{k\to+\infty} \bigvee_a^t \big(\mathcal{I}(H_k, A_k) - \mathcal{I}(H_0, A_0)\big) = 0 \tag{1.3.78}$$

and

$$\lim_{k\to+\infty} \bigvee_a^t \big(\mathcal{B}(H_k, f_k) - \mathcal{B}(H_0, f_0)\big) = 0 \tag{1.3.79}$$

hold uniformly on I, where $a \in I$ is a fixed point.

Theorem 1.3.15. *Let $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ be such that*

$$\limsup_{t\to+\infty} \bigvee_t^{\nu(\xi)(t)} \mathcal{A}(A, A) < +\infty \tag{1.3.80}$$

and

$$\lim_{t\to+\infty} \bigvee_t^{\nu(\xi)(t)} \mathcal{A}(A, f) = 0,$$

where the function $\nu(\xi)$ is defined by (1.3.22). *Then ξ-exponentially asymptotic stability of the matrix-function A guarantees the well-posedness of problem* (1.2.1), (1.2.2) *on the $\mathbb{R}_+$.*

Theorem 1.3.16. *Let $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and*

$$f \in \mathrm{BV}(\mathbb{R}_+;\mathbb{R}^n). \tag{1.3.81}$$

Then uniform stability of the matrix-function A guarantees the weakly well-posedness of problem (1.2.1), (1.2.2) *on the $\mathbb{R}_+$.*

1.3.3 Auxiliary propositions

Lemma 1.3.1. *Let the matrix-function $A_0 \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfy condition* (1.3.18). *Let, moreover, the following conditions hold:*

(a) *the Cauchy matrix U_0 of the system*

$$dx = dA_0(t)\cdot x \tag{1.3.82}$$

satisfies the inequality

$$|U_0(t, t_0)| \le \Omega \exp\big(-\eta(t) + \eta(t_0)\big) \;\text{ for }\; t \ge t_0 \tag{1.3.83}$$

for some $t_0 \in \mathbb{R}_+$, where $\Omega = (\rho_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$, and $\eta \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})$;

(b) *there exists a matrix $H \in \mathbb{R}_+^{n\times n}$ such that conditions* (1.3.31) *and*

$$\int_{t_0}^t \exp(\eta(t) - \eta(\tau))\cdot |U_0(t, \tau)|\, dV(\mathcal{A}(A_0, A - A_0))(\tau) \le H \;\text{ for }\; t \ge t_0 \tag{1.3.84}$$

hold.

Then an arbitrary solution x of system $(1.3.1_0)$ admits the estimate

$$|x(t)| \le R|x(t_0)| \exp\big(-\eta(t)+\eta(t_0)\big) \ \ \text{for} \ \ t \ge t_0, \tag{1.3.85}$$

where $R=(I_n-H)^{-1}\Omega$.

Proof. Let $A=(a_{ik})_{i,k=1}^n$, $A_0=(a_{0ik})_{i,k=1}^n$, $U_0=(u_{0ik})_{i,k=1}^n$, and $x=(x_i)_{i=1}^n$ be an arbitrary solution of system $(1.3.1_0)$.

It is evident that

$$dx(t) \equiv dA_0(t)\cdot x(t) + d(A(t)-A_0(t))\cdot x(t).$$

From this, according to the variation-of-constant formula (see (1.1.14)), the integration-by-parts formula and properties of the Cauchy matrix U_0 (see Theorem 1.1.6(d)), we have

$$\begin{aligned}
x(t) &= U_0(t,t_0)x(t_0) + \int_{t_0}^{t} d(A(\tau)-A_0(\tau))\cdot x(\tau) - \int_{t_0}^{t} dU_0(t,\tau)\Big(\int_{t_0}^{\tau} d(A(s)-A_0(s))\cdot x(s)\Big)\\
&= U_0(t,t_0)x(t_0) + \int_{t_0}^{t} U_0(t,\tau)\, d(A(\tau)-A_0(\tau))\cdot x(\tau)\\
&\qquad - \sum_{t_0<\tau\le t} d_1U_0(t,\tau)\cdot d_1(A(\tau)-A_0(\tau))\cdot x(\tau)\\
&\qquad + \sum_{t_0\le\tau<t} d_2U_0(t,\tau)\cdot d_2(A(\tau)-A_0(\tau))\cdot x(\tau)\\
&= U_0(t,t_0)x(t_0) + \int_{t_0}^{t} U_0(t,\tau)\, d(A(\tau)-A_0(\tau))\cdot x(\tau)\\
&\qquad + \sum_{t_0<\tau\le t} U_0(t,\tau)\, d_1A_0(\tau)\cdot (I_n-d_1A_0(\tau))^{-1}\cdot d_1(A(\tau)-A_0(\tau))\cdot x(\tau)\\
&\qquad - \sum_{t_0\le\tau<t} U_0(t,\tau)\, d_2A_0(\tau)\cdot (I_n+d_2A_0(\tau))^{-1}\cdot d_2\big(A(\tau)-A_0(\tau)\big)\cdot x(\tau).
\end{aligned}$$

Therefore,

$$x(t) = U_0(t,t_0)x(t_0) + \int_{t_0}^{t} U_0(t,\tau)\, d\mathcal{A}(A_0,A-A_0)(\tau)\cdot x(\tau) \ \ \text{for} \ \ t\ge t_0. \tag{1.3.86}$$

Let

$$\begin{aligned}
y_k(t) &= \sup\big\{\exp\big(\eta(\tau)-\eta(t_0)\big)\cdot |x_k(\tau)| : \ t_0\le\tau\le t\big\},\\
y(t) &= (y_k(t))_{k=1}^n.
\end{aligned}$$

Then

$$\begin{aligned}
\Big|\sum_{k,j=1}^{n} \int_{t_0}^{t} u_{0ij}(t,\tau)x_k(\tau)\, db_{jk}(\tau)\Big| &\le \sum_{k,j=1}^{n} y_k(t) \int_{t_0}^{t} |u_{0ij}(t,\tau)||x_k(\tau)|\, dv(b_{jk})(\tau)\\
&\le \sum_{k,j=1}^{n} \int_{t_0}^{t} \exp(-\eta(\tau)+\eta(t_0))|u_{0ij}(t,\tau)|\, dv(b_{jk})(\tau) \ \ \text{for} \ \ t\ge t_0,
\end{aligned}$$

where $b_{ik}(t) \equiv \mathcal{A}(a_{0ik}, a_{ik}-a_{0ik})(t)$ $(i,k=1,\dots,n)$. From this and (1.3.86) we find

$$\exp(\eta(t)-\eta(t_0))\cdot|x_i(t)|\le\sum_{k=1}^{n}\exp(\eta(t)-\eta(t_0))|u_{0ik}(t,t_0)|\,|x_k(t_0)|$$
$$+\sum_{k,j=1}^{n}y_k(t)\int_{t_0}^{t}\exp(\eta(t)-\eta(\tau))|u_{0ij}(t,\tau)|\,dv(b_{jk})(\tau)\ \text{ for }\ t\ge t_0\ \ (i=1,\dots,n).$$

Therefore, by (1.3.83) and (1.3.84), we obtain

$$y(t)\le\Omega|x(t_0)|+Hy(t)\ \text{ for }\ t\ge t_0.$$

Hence,

$$(I_n-H)y(t)\le\Omega|x(t_0)|\ \text{ for }\ t\ge t_0. \tag{1.3.87}$$

On the other hand, due to (1.3.31), the matrix I_n-H is nonsingular and the matrix $(I_n-H)^{-1}$ is nonnegative, since H is nonnegative. From this, by (1.3.87) and the definition of y, we get

$$y(t)\le(I_n-H)^{-1}\Omega|x(t_0)|\ \text{ for }\ t\ge t_0$$

and

$$|x(t)|\le(I_n-H)^{-1}\Omega|x(t_0)|\exp(-\eta(t)+\eta(t_0))\ \text{ for }\ t\ge t_0.$$

Thus estimate (1.3.85) is proved. □

Lemma 1.3.2. *Let* $h\in\mathrm{BV}([a,b];\mathbb{R}^n)$ *and* $\beta\in\mathrm{BV}([a,b];\mathbb{R})$. *Then*

$$\int_a^b h(t)\exp(-\beta(t))\,d\exp(\beta(t))=\int_a^b h(t)\,d\beta(t)$$
$$+\sum_{a<t\le b}h(t)\big(1-d_1\beta(t)-\exp(-d_1\beta(t))\big)+\sum_{a\le t<b}h(t)\big(\exp(d_2\beta(t))-d_2\beta(t)-1\big).$$

Proof. Let $\xi(t)\equiv s_1(\beta)(t)+s_2(\beta)(t)$. Using (0.0.10), (0.0.11) and (0.0.12), we have

$$\int_a^b h(t)\exp(-\beta(t))\,d\exp(\beta(t))=\int_a^b h(t)\exp(-s_c(\beta)(t)-\xi(t))\,d\exp(s_c(\beta)(t)+\xi(t))$$
$$=\int_a^b h(t)\exp(-s_c(\beta)(t))\,d\exp(s_c(\beta)(t))+\int_a^b h(t)\exp(-\xi(t))\,d\exp(\xi(t))$$
$$-\sum_{a<t\le b}h(t)\exp(-s_c(\beta)(t)-\xi(t))\,d_1\exp(s_c(\beta)(t))\cdot d_1\exp(\xi(t))$$
$$+\sum_{a\le t<b}h(t)\exp(-s_c(\beta)(t)-\xi(t))\,d_2\exp\big(s_c(\beta)(t)\big)\cdot d_2\exp(\xi(t))$$
$$=\int_a^b h(t)\exp(-s_c(\beta)(t))\,d\exp(s_c(\beta)(t))+\sum_{j=1}^{2}\int_a^b h(t)\exp(-s_j(\beta)(t))\,d\exp(s_j(\beta)(t))$$
$$=\int_a^b h(t)\,ds_c(\beta)(t)+\sum_{a<t\le b}h(t)(1-\exp(-d_1\beta(t)))+\sum_{a\le t<b}h(t)(\exp(d_2\beta(t))-1)$$
$$=\int_a^b h(t)\,d\beta(t)+\sum_{a<t\le b}h(t)\big(1-d_1\beta(t)-\exp(-d_1\beta(t))\big)$$
$$+\sum_{a\le t<b}h(t)\big(\exp(d_2\beta(t))-d_2\beta(t)-1\big).$$

□

Lemma 1.3.3. *Let the matrix-function $B \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfy the Lappo–Danilevskiĭ condition. Then*

$$\int_a^b d\exp(B(t))\cdot\exp(-B(t)) = S_c(B)(b) - S_c(B)(a) + \sum_{a<t\le b}\big(I_n - \exp(-d_1B(t))\big) + \sum_{a\le t<b}(\exp(d_2B(t)) - I_n) \ \ \textit{for} \ \ 0\le a<b. \tag{1.3.88}$$

Proof. Since $S_c(B)(t)$, $S_1(B)(t)$ and $S_2(B)(t)$ $(t\in\mathbb{R}_+)$ are pairwise permutable matrices, we, in addition, have

$$S_c(B)(t)\cdot d_jB(t) = d_jB(t)\cdot S_c(B)(t) \ \text{ for } \ t\in\mathbb{R}_+ \ \ (j=1,2)$$

and

$$S_j(B)(t)\cdot d_{3-j}B(t) = d_{3-j}B(t)\cdot S_j(B)(t) \ \text{ for } \ t\in\mathbb{R}_+ \ \ (j=1,2).$$

Therefore, according to the general integration-by-parts formula (0.0.10) and (0.0.11), we find that

$$\begin{aligned}\int_a^b d\exp(B(t))\cdot\exp(-B(t)) &= \int_a^b d\exp(S_c(B)(t))\cdot\exp\big(S_1(B)(t)+S_2(B)(t)\big)\cdot\exp(-B(t))\\ &\quad + \int_a^b \exp(S_c(B)(t))\,d\exp\big(S_1(B)(t)+S_2(B)(t)\big)\cdot\exp(-B(t))\\ &= \int_a^b d\exp(S_c(B)(t))\cdot\exp(-S_c(B)(t))\\ &\quad + \sum_{a<t\le b}\exp(S_c(B)(t))\,d_1\exp\big(S_1(B)(t)+S_2(B)(t)\big)\cdot\exp(-B(t))\\ &\quad + \sum_{a\le t<b}\exp(S_c(B)(t))\,d_2\exp\big(S_1(B)(t)+S_2(B)(t)\big)\cdot\exp(-B(t)).\end{aligned}$$

Hence

$$\int_a^b d\exp(B(t))\cdot\exp(-B(t)) = \int_a^b d\exp(S_c(B)(t))\cdot\exp(-S_c(B)(t)) + \sum_{a<t\le b}\big(I_n-\exp(-d_1B(t))\big) + \sum_{a\le t<b}\big(\exp(d_2B(t)) - I_n\big). \tag{1.3.89}$$

Due to the Lappo–Danilevskiĭ condition, we easily get

$$\int_a^b dS_c^k(B)(t)\cdot S_c^m(B)(t) = \frac{k}{k+m}\big(S_c^{k+m}(B)(b) - S_c^{k+m}(B)(a)\big)$$

for every natural k and m.

By this and the definition of the exponential matrix, we obtain

$$\int_a^b d\exp(S_c(B)(t))\cdot\exp(-S_c(B)(t))$$

$$= \exp(S_c(B)(b)) - \exp(S_c(B)(a)) + \sum_{m=1}^{\infty}\sum_{k=1}^{m}\frac{(-1)^{m-k+1}}{k!(m-k+1)!}\int_a^b dS_c^k(B)(t)\cdot S_c^{m-k+1}(B)(t)$$

$$= \exp(S_c(B)(b)) - \exp(S_c(B)(a)) + \sum_{m=1}^{\infty}\frac{S_c^{m+1}(B)(b) - S_c^{m+1}(B)(a)}{m+1}\sum_{k=0}^{m-1}\frac{(-1)^{m-k}}{k!(m-k)!}$$

$$= \exp(S_c(B)(b)) - \exp(S_c(B)(a)) - \sum_{m=1}^{\infty}\frac{S_c^{m+1}(B)(b) - S_c^{m+1}(B)(a)}{(m+1)!}.$$

Thus

$$\int_a^b d\exp(S_c(B)(t))\cdot\exp(-S_c(B)(t)) = S_c(B)(b) - S_c(B)(a). \tag{1.3.90}$$

By (1.3.89) and (1.3.90), equality (1.3.88) holds. □

Lemma 1.3.4. *Let the matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$S_c(A)(t) \equiv S_c(B)(t) \ \text{ and } \ I_n + (-1)^j d_j A(t) \equiv \exp\big((-1)^j d_j B(t)\big) \ \ (j=1,2),$$

where the matrix-function $B \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition. Then the matrix-function $\exp(B(t))$ is a solution of system $(1.3.1_0)$.

Proof. By (1.3.88),

$$\int_s^t d\exp(B(\tau))\cdot\exp(-B(\tau)) = A(t) - A(s) \ \text{ for } \ 0 \le t < s.$$

Consequently, using the substitution formula (0.0.12), we get

$$\int_s^t dA(\tau)\cdot\exp(B(\tau)) = \int_s^t d\bigg(\int_s^\tau d\exp(B(\sigma))\cdot\exp(-B(\sigma))\bigg)\cdot\exp(B(\tau))$$

$$= \exp(B(t)) - \exp(B(s)) \ \text{ for } \ 0 \le t < s.$$ □

Remark 1.3.8. Let the function $\beta \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})$ be such that

$$1 + (-1)^j d_j\beta(t) > 0 \ \text{ for } \ t \in \mathbb{R}_+ \ \ (j=1,2).$$

Then if one of the functions β, $J(\beta)$ and $\mathcal{A}(\beta,\beta)$ is non-decreasing (non-increasing), then all the others will be the same. This fact immediately follows from equalities (1.1.4), (1.1.5) and (1.1.19).

For completeness, we give the following lemma from [35].

Lemma 1.3.5. *Let $P = (p_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$ be the symmetric matrix. Then*

$$\lambda_0(P)(x*x) \le Px*x \le \lambda^0(P)(x*x).$$

1.3.4 Proof of the results

Proof of Theorem 1.3.1. First, we show the sufficiency. According to Lemma 1.2.2, the mapping

$$x \to y = Hx$$

establishes a one-to-one correspondence between the solutions of systems $(1.3.1_0)$ and

$$dy = dA^*(t)\cdot y, \tag{1.3.91}$$

respectively, where

$$A^*(t) \equiv \mathcal{I}(H,A)(t).$$

On the other hand, by (1.3.8),

$$\inf\big\{\det(H(t)):\ t\in\mathbb{R}_+\big\} > 0$$

and so,

$$\det H(t-)\neq 0 \ \text{ and } \ \det H(t+)\neq 0 \ \text{ for } \ t\in\mathbb{R}_+. \tag{1.3.92}$$

Therefore, by (1.2.87), we have

$$\det\big(I_n + (-1)^j d_j A^*(t)\big)\neq 0 \ \text{ for } \ t\in\mathbb{R}_+ \ \ (j=1,2).$$

Let X $(X(0)=I_n)$ and Y $(Y(0)=I_n)$ be the fundamental matrices of systems $(1.3.1_0)$ and (1.3.91), respectively. Then

$$X(t) = H^{-1}(t)Y(t) = H^{-1}(t)\bigg(I_n + \int_0^t dA^*(\tau)\cdot Y(\tau)\bigg)$$

$$= H^{-1}(t)\bigg(I_n + \int_0^t d(H(\tau)+\mathcal{B}(H,A))(\tau)\cdot X(\tau)\bigg) \ \text{ for } \ t\in\mathbb{R}_+.$$

From this, by virtue of (1.3.8), we have

$$u(t)\le r + \int_0^t u(\tau)\,da(\tau) \ \text{ for } \ t\in\mathbb{R}_+, \tag{1.3.93}$$

where

$$u(t)\equiv\|X(t)\|, \quad a(t)\equiv r\|V(H+\mathcal{B}(H,A))(t)\|,$$

and

$$r=\sup\big\{\|H^{-1}(t)\|:\ t\in\mathbb{R}_+\big\}.$$

It is evident that $a(t)$ $(t\in\mathbb{R}_+)$ is the non-decreasing function and the series $\sum\limits_{t\in\mathbb{R}_+} d_j a(t)$ $(j=1,2)$ contain at most countable nonzero terms. On the other hand, in view of (1.3.9), these series converge. Therefore, there exists $t^*\in\mathbb{R}_+$ such that

$$0\le d_j a(t)\le \frac{1}{2} \ \text{ for } \ t\ge t^* \ \ (j=1,2). \tag{1.3.94}$$

Due to (1.3.93), (1.3.94) and equality (0.0.7), we get

$$\int_0^t u(\tau)\,da(\tau) = u(t)\,d_1 a(t) + \int_0^t u(\tau)\,db(\tau) \ \text{ for } \ t\ge 0,$$

where $b(t) \equiv a(t-)$. So,

$$u(t) \le (1 - d_1 a(t))^{-1} \bigg(r + \int\limits_0^{t^*} u(\tau)\, da(\tau) + \int\limits_{t^*}^{t} u(\tau)\, db(\tau) \bigg) \ \text{ for } \ t \ge t^*$$

and

$$u(t) \le r_1 + 2 \int\limits_{t^*}^{t} u(\tau)\, db(\tau) \ \text{ for } \ t \ge t^*, \tag{1.3.95}$$

where

$$r_1 = 2\bigg(r + \int\limits_0^{t^*} u(\tau)\, da(\tau) \bigg).$$

From (1.3.95), according to the Gronwall's inequality (see Lemma 1.1.4), we get

$$u(t) \le r_1 \exp\Big(2 \bigvee_{t^*}^{t}(b) \Big) \le r_1 \exp\Big(\bigvee_{0}^{+\infty} (H + \mathcal{B}(H,A)) \Big) \ \text{ for } \ t \ge t^*.$$

Hence, by (1.3.9),

$$\sup\big\{ \|X(t)\| : \ t \in \mathbb{R}_+ \big\} < +\infty.$$

In view of Proposition 1.3.3, the stability of the matrix-function A is proved.

Let us show the necessity. Let the matrix-function A be stable. Then, due to Proposition 1.3.2, there exists $r > 0$ such that

$$\|X(t)\| < r \ \text{ for } \ t \in \mathbb{R}_+,$$

where X ($X(0) = I_n$) is the fundamental matrix of system ($1.3.1_0$).

If we assume $H(t) \equiv X^{-1}(t)$, then by (1.1.17) we conclude that

$$H(t) + \mathcal{B}(H,A)(t) = X^{-1}(t) + \mathcal{B}(X^{-1},A)(t) = X^{-1}(t) + I_n - X^{-1}(t) = I_n \ \text{ for } \ t \in \mathbb{R}_+.$$

Therefore, estimates (1.3.8) and (1.3.9) hold. □

Proof of Theorem 1.3.2. Let us show the sufficiency. Let U^* be the Cauchy matrices of system (1.3.91). Then, by Lemma 1.2.2, for every fixed $s \in \mathbb{R}_+$, we have

$$U(t,s) = H^{-1}(t) U^*(t,s) H(s) = H^{-1}(t) \bigg(I_n + \int\limits_s^t d\mathcal{L}(H,A)(\tau) \cdot U^*(\tau,s) \bigg) H(s)$$

$$= H^{-1}(t)H(s) + H^{-1}(t) \int\limits_s^t d\big(H(\tau) + \mathcal{B}(H,A)(\tau) \big) \cdot H^{-1}(\tau) U^*(\tau,s) H(s) \ \text{ for } \ t \in \mathbb{R}_+.$$

Therefore,

$$U(t,s) = H^{-1}(t)H(s) + H^{-1}(t) \int\limits_s^t d\big(H(\tau) + \mathcal{B}(H,A)(\tau) \big) \cdot U(\tau,s) \ \text{ for } \ t \ge s.$$

From this, if we take into account (1.3.10), we find that

$$\|U(t,s)\| \le r + \int\limits_s^t \|U(\tau,s)\|\, da(\tau) \ \text{ for } \ t \ge s,$$

where
$$a(t)\equiv r\big\|V(H+\mathcal{B}(H,A))(t)\big\|$$
and
$$r=\sup\big\{\|H^{-1}(t)H(s)\|:\ t\geq s\geq 0\big\}.$$

Let $t^*\in\mathbb{R}_+$ be such that estimates (1.3.94) hold and let $s\geq t^*$ be fixed. Analogously, as in the proof of Theorem 1.3.1, we get
$$\|U(t,s)\|\leq 2r\exp\Big(2\bigvee_s^t(b)\Big)\leq 2r\exp\Big(2\bigvee_0^{+\infty}(H+\mathcal{B}(H,A))\Big)\ \text{ for }\ t\geq s\geq t^*,$$
where $b(t)\equiv a(t-)$. Thus, estimate (1.3.5) is valid. Therefore, due to Proposition 1.3.4, the matrix-function A is uniformly stable.

The proof of the necessity is analogous to that of Theorem 1.3.2, but with the use of Proposition 1.3.4. □

Proof of Theorem 1.3.3. Let $\varepsilon>0$ be an arbitrary positive number. According to (1.3.11), there exists $t^*\in\mathbb{R}_+$ such that estimates (1.3.94) and
$$\|H^{-1}(t)\|<\varepsilon\ \text{ for }\ t\geq t^*$$
hold. From the last estimate, due to Theorem 1.3.1, the matrix-function A is stable. Therefore, there exists $r>0$ such that
$$\|X(t)\|<r\ \text{ for }\ t\in\mathbb{R}_+,$$
where X $(X(0)=I_n)$ is the fundamental matrix of system $(1.3.1_0)$.

As in the proof of Theorem 1.3.1, we obtain
$$u(t)\leq\varepsilon r+\int_{t^*}^t u(\tau)\,da_\varepsilon(\tau)\ \text{ for }\ t\geq t^*,$$
where
$$u(t)\equiv\|X(t)\|$$
and
$$a_\varepsilon(t)\equiv\varepsilon\big\|V(H+\mathcal{B}(H,A))(t)\big\|.$$
Therefore, by Gronwall's inequality, we have
$$\|X(t)\|\leq\varepsilon\exp\Big(\varepsilon\bigvee_0^{+\infty}\big(H+\mathcal{B}(H,A)\big)\Big)\ \text{ for }\ t\geq t^*.$$
Consequently, with regard to (1.3.9), we have
$$\lim_{t\to+\infty}\|X(t)\|=0.$$
Hence, in view of Proposition 1.3.5, the matrix-function A is asymptotically stable.

The proof of the necessity is analogous to that of Theorem 1.3.1, but with the use of Proposition 1.3.5. □

Proof of Theorem 1.3.4. Let U and U^* be the Cauchy matrices of systems $(1.3.1_0)$ and (1.3.93), respectively, where
$$A^*(t)\equiv\mathcal{I}(H,A)(t).$$
According to Lemma 1.2.2,
$$U^*(t,s)=H(t)U(t,s)H^{-1}(s)\ \text{ for }\ t,s\in\mathbb{R}_+.$$

From this, by definition of the operator $\mathcal{I}$, we conclude that

$$\begin{aligned}\exp\big(\eta(\xi(t)-\xi(s))\big)\cdot U(t,s) &= H_1^{-1}(t)U^*(t,s)H_1(s)\\ &= H_1^{-1}(t)\Big(I_n+\int\limits_s^t d\mathcal{I}(H,A)(\tau)\cdot U^*(\tau,s)\Big)H_1(s)\\ &= H_1^{-1}(t)H_1(s)+H_1^{-1}(t)\int\limits_s^t \exp\big(\eta(\xi(\tau)-\xi(s))\big)\,d\mathcal{B}_\eta(H,A)(\tau)\cdot U(\tau,s)\ \text{ for } t,s\in\mathbb{R}_+,\end{aligned}$$

where

$$H_1(t)\equiv \exp(-\eta\xi(t))\cdot H(t).$$

As in the proof of Theorem 1.3.1, we get

$$\begin{aligned}W(t,s) = H_1^{-1}(t)H_1(s)+H_1^{-1}(t)\,d_1\mathcal{B}_\eta(H,A)(t)\cdot W(t,s)&\\ +H_1^{-1}(t)\int\limits_s^t dG(\tau)\cdot W(\tau,s)\ \text{ for } t,s\in\mathbb{R}_+,&\end{aligned}\tag{1.3.96}$$

where

$$\begin{aligned}W(t,s)&\equiv \exp\big(\eta(\xi(t)-\xi(s))\big)\cdot U(t,s),\\ G(t)&\equiv \mathcal{B}_\eta(H,A)(t-).\end{aligned}$$

On the other hand, as above, by (1.3.12), inequalities (1.3.92) hold. Therefore, taking into account this and the equalities

$$I_n+(-1)^jH_1^{-1}(t)\,d_j\mathcal{B}_\eta(H,A)(t)=H^{-1}(t)\big(I_n+(-1)^jd_jA^*(t)\big)H(t)\ \text{ for } t,s\in\mathbb{R}_+\ \ (j=1,2),$$

by (1.1.10) and (1.2.87), we have

$$\det\big(I_n+(-1)^jH_1^{-1}(t)\,d_j\mathcal{B}_\eta(H,A)(t)\big)\neq 0\ \text{ for } t\in\mathbb{R}_+\ \ (j=1,2).\tag{1.3.97}$$

Moreover, according to conditions (1.3.12) and (1.3.13), there exists a positive number r_0 such that

$$\Big\|\big(I_n+(-1)^jH_1^{-1}(t)\,d_j\mathcal{B}_\eta(H,A)(t)\big)^{-1}\Big\|<r_0\ \text{ for } t\in\mathbb{R}_+\ \ (j=1,2).\tag{1.3.98}$$

From (1.3.96), by (1.3.12), (1.3.97) and (1.3.98), we find that

$$\|W(t,s)\|\le r_0\Big(\rho+\rho_1\int\limits_s^t\|W(\tau,s)\|\,d\|V(G)(\tau)\|\Big)\ \text{ for } t\ge s\ge 0,$$

where

$$\begin{aligned}\rho&=\sup\Big\{\exp\big(\eta(\xi(t)-\xi(\tau))\big)\cdot\|H^{-1}(t)H(\tau)\|:\ t\ge\tau\ge\ 0\Big\},\\ \rho_1&=\rho\exp(\eta(\xi(0))).\end{aligned}$$

Hence, according to Gronwall's inequality,

$$\|W(t,s)\|\le r<+\infty\ \text{ for } t\ge s\ge 0,$$

where

$$r=r_0\rho\exp\Big(r_0\rho_1\bigvee_0^{+\infty}\mathcal{B}_\eta(H,A)\Big).$$

Therefore,

$$\|U(t,s)\| \le r\exp\big(-\eta(\xi(t)-\xi(s))\big) \ \text{ for } \ t \ge s \ge 0.$$

So, estimate (1.3.7) is valid and, due to Proposition 1.3.6, the matrix-function A is ξ-exponentially asymptotically stable.

Let us show the necessity. Let the matrix-function A be ξ-exponentially asymptotically stable. Then, due to Proposition 1.3.6, there exist a positive numbers η and ρ such that

$$\|X(t)X^{-1}(s)\| \le \rho\exp\big(-\eta(\xi(t)-\xi(s))\big) \ \text{ for } \ t \ge s \ge 0,$$

where X $(X(0)=I_n)$ is the fundamental matrix of system $(1.3.1_0)$.

Let

$$H(t) \equiv X^{-1}(t).$$

Then, due to the definition of $\mathcal{B}_\eta(H,A)$, using equality (1.1.17), we have

$$\exp\big(\eta(\xi(t)-\xi(s))\big)\cdot\|H^{-1}(t)H(s)\| \le \rho \ \text{ for } \ t \ge s \ge 0$$

and

$$\mathcal{B}_\eta(H,A)(t) = \mathcal{B}_\eta(X^{-1},A)(t) = 0 \ \text{ for } \ t \in \mathbb{R}_+.$$

Consequently, conditions (1.3.12) and (1.3.13) are fulfilled. □

Proof of Corollary 1.3.1. The cases of stability, uniform stability and asymptotic stability of the matrix-function A follow from Theorems 1.3.1–1.3.3, respectively, if we assume that $H(t) \equiv Y^{-1}(t)$ in these theorems. Indeed, by definition of the operator $\mathcal{B}$, (1.1.17) and (1.3.16), it is easy to verify that

$$\begin{aligned} Y^{-1}(t) + \mathcal{B}(Y^{-1},A)(t) &= Y^{-1}(t) + \mathcal{B}(Y^{-1},A-Q)(t) + \mathcal{B}(Y^{-1},Q)(t) \\ &= \mathcal{B}(Y^{-1},A-Q)(t) + I_n \ \text{ for } \ t \in \mathbb{R}_+ \end{aligned}$$

and

$$\bigvee_0^{+\infty}(H+\mathcal{B}(H,A)) = \bigvee_0^{+\infty}\mathcal{B}(Z^{-1},A-Q) < +\infty.$$

Let now the matrix-function Q be ξ-exponentially asymptotically stable. Then there exist positive numbers η and ρ such that

$$\|Y(t)Y^{-1}(s)\| \le \rho\exp\big(-\eta(\xi(t)-\xi(s))\big) \ \text{ for } \ t \ge s \ge 0.$$

Therefore, estimate (1.3.12) is valid, where $H(t) \equiv Y^{-1}(t)$. On the other hand, by (1.1.17),

$$Y^{-1}(t) = I_n + \mathcal{B}(Y^{-1},-Q)(t) \ \text{ for } \ t \in \mathbb{R}_+.$$

Then

$$\mathcal{B}_\eta(H,A)(t) = \int_0^t \exp(-\eta\xi(\tau))\,d\mathcal{B}(Y^{-1},A-Q)(\tau) \ \text{ for } \ t \in \mathbb{R}_+,$$

where $\mathcal{B}_\eta(H,A)$ is the matrix-function defined by (1.3.14). From this, by (1.3.16), we conclude that condition (1.3.13) holds. Hence, due to Theorem 1.3.4, the matrix-function A is ξ-exponentially asymptotically stable, as well. □

Proof of Theorem 1.3.5. According to Lemma 1.2.2, the mapping

$$x \to y = Hx$$

establishes a one-to-one correspondence between the solutions of systems $(1.3.1_0)$ and (1.3.91), respectively, where $A^*(t) \equiv \mathcal{I}(H,A)(t)$. On the other hand, by the uniform stability of the matrix-function

A_0 there exists a constant matrix $\Omega \in \mathbb{R}_+^{n\times n}$ such that the Cauchy matrix U_0 of system (1.3.82) admits the estimate

$$|U_0(t,t_0)| \le \Omega \ \text{ for } \ t \ge t_0 \ge 0.$$

Taking into account the latter estimate, we conclude that

$$\int_{t_0}^{t} |U_0(t,\tau)|\, dV(\mathcal{A}(A_0, A^* - A_0))(\tau) \le \Omega \int_{t_0}^{t} dV\big(\mathcal{A}(A_0, A^* - A_0)\big)(\tau)$$
$$= \Omega \cdot \big(V(\mathcal{A}(A_0, A^* - A_0))(t) - V(\mathcal{A}(A_0, A^* - A_0))(t_0)\big) \ \text{ for } \ t \ge t_0 \ge 0. \quad (1.3.99)$$

Moreover, by inequality (1.3.19), the constant matrix

$$Q = \Omega \bigvee_{t^*}^{+\infty} \mathcal{A}(A_0, A^* - A_0) \quad (1.3.100)$$

admits estimate (1.3.31), i.e., $r(Q) < 1$ for some sufficiently large $t^* \in \mathbb{R}_+$.

According to (1.3.99) and (1.3.100),

$$\int_{t_0}^{t} |U_0(t,\tau)|\, dV\big(\mathcal{A}(A_0, A^* - A_0)\big)(\tau) \le Q \ \text{ for } \ t \ge t_0 \ge t^*.$$

Therefore, by Lemma 1.3.1, every solution y of system (1.3.91) admits the estimate

$$\|y(t)\| \le \rho \|y(t_0)\| \ \text{ for } \ t \ge t_0 \ge t^*,$$

where $\rho > 0$ is a number independent of t_0. The latter estimate guarantees the uniform stability of the matrix-function A^*. Hence there exist a positive number ρ_1 such that

$$\|U^*(t,t_0)\| \le \rho_1 \ \text{ for } \ t \ge t_0 \ge t^*, \quad (1.3.101)$$

where U^* is the Cauchy matrix of system (1.3.91).

Let now U be the Cauchy matrix of system $(1.3.1_0)$. Then, according to Lemma 1.2.2,

$$U(t,t_0) = H^{-1}(t) U^*(t,t_0) H(t_0) \ \text{ for } \ t \ge t_0 \ge 0.$$

From this, in view of (1.3.10) and (1.3.101), we get

$$\|U(t,t_0)\| \le \rho_1 \rho_2 \ \text{ for } \ t \ge t_0 \ge t^*,$$

where

$$\rho_2 = \sup\big\{\|H^{-1}(t)H(\tau)\| : \ t \ge \tau \ge 0\big\}.$$

Consequently, the matrix-function A is uniformly stable, as well. □

Proof of Theorem 1.3.6. By the ξ-exponentially asymptotic stability of the matrix-function A_0 and Proposition 1.3.6, there exist positive numbers η and ρ_0 such that the Cauchy matrix U_0 of system (1.3.82) satisfies the estimate

$$|U_0(t,\tau)| \le R_0 \exp\big(-\eta(\xi(t) - \xi(\tau))\big) \ \text{ for } \ t \ge \tau \ge 0, \quad (1.3.102)$$

where R_0 is an $n \times n$ matrix whose every component equals ρ_0.

Let

$$\varepsilon = (4n\rho_0)^{-1}(\exp(\eta) - 1)\exp(-2\eta). \quad (1.3.103)$$

Due to (1.3.21), there exists $t^* \in \mathbb{R}_+$ such that

$$\bigvee_{t}^{\nu(\xi)(t)} \mathcal{A}(A_0, A - A_0) < \varepsilon \ \text{ for } \ t \ge t^*. \quad (1.3.104)$$

On the other hand, by (1.3.102), we have

$$\int\limits_{t_0}^{t} \exp\big(\eta(\xi(t)-\xi(\tau))\big)|U_0(t,\tau)|\,dV(B)(\tau) \le \mathcal{J}(t)\ \ (t\ge t_0) \tag{1.3.105}$$

for every $t_0 \ge 0$, where $B(t) \equiv \mathcal{A}(A_0, A-A_0)(t)$ and

$$\mathcal{J}(t) \equiv R_0 \int\limits_{t_0}^{t} \exp\big(-\eta\big(\xi(t)-\xi(\tau)\big)\big)\,dV(B)(\tau). \tag{1.3.106}$$

Let $k(t)$ be the integer part of $\xi(t)-\xi(t_0)$ for every $t\ge t_0$, where t_0 is an arbitrary fixed point.

We put

$$T_i = \big\{\tau \ge t_0:\ \xi(t_0)+i \le \xi(\tau) < \xi(t_0)+i+1\big\}\ \ (i=0,\dots,k(t)).$$

Let the points $\tau_0, \tau_1, \dots, \tau_{k(t)}$ be defined as follows:

$$\tau_0 = \sup T_0, \quad \tau_i = \begin{cases} \tau_{i-1} & \text{if } T_i = \varnothing, \\ \sup T_i & \text{if } T_i \ne \varnothing \end{cases} \quad (i=1,\dots,k(t)).$$

Let us show that

$$\tau_i \le \nu(\xi)(\tau_{i-1})\ \ (i=1,\dots,k(t)). \tag{1.3.107}$$

If $T_i=\varnothing$, then (1.3.107) is evident.

Let now $T_i \ne \varnothing$. It suffices to show that

$$T_i \subset Q_i\ \ (i=1,\dots,k(t)),$$

where

$$Q_i = \big\{\tau \ge t_0:\ \xi(\tau) < \xi(\tau_{i-1}+)+1\big\}.$$

It is easy to verify that

$$\xi(\tau_{i-1}+) \ge \xi(\tau_0)+i\ \ (i=1,\dots,k(t)). \tag{1.3.108}$$

Indeed, otherwise there exist $i_0 \in \{1,\dots,k(t)\}$ and $\delta>0$ such that

$$\xi(\tau_{i_0-1}+s) < \xi(\tau_0)+i_0 \ \text{ for } \ 0\le s\le\delta.$$

On the other hand, by the definition of τ_{i_0-1}, we have

$$\xi(\tau_0)+i_0-1 \le \xi(\tau_{i_0-1}-)$$

and, therefore,

$$\xi(\tau_0)+i_0-1 \le \xi(\tau_{i_0-1}+s) < \xi(\tau_0)+i_0 \ \text{ for } \ 0\le s\le\delta.$$

But this contradicts the definition of τ_{i_0-1}.

Let $\tau\in T_i$ $(i=1,\dots,k(t))$. Then from (1.3.108) and the inequality $\xi(\tau)<\xi(\tau_0)+i+1$ it follows that $\xi(\tau)<\xi(\tau_{i-1}+)+1$, $\tau_i\in Q_i$ $(i=1,\dots,k(t))$. Hence (1.3.107) is proved.

Let now $t_0\ge t^*$ and let $k_i=k(\tau_i)$ $(i=1,\dots,k(t))$. Then, according to (1.3.104) and (1.3.107), we get

$$\mathcal{J}(t) \le R_0 \exp\big(-\eta(\xi(t)-\xi(t_0))\big) \sum_{i=1}^{1+k(t)} \int\limits_{\tau_{i-1}}^{\tau_i} \exp\big(\eta\big(\xi(\tau)-\xi(t_0)\big)\big)\,dV(B)(\tau)$$

$$= R_0 \exp\big(-\eta(\xi(t)-\xi(t_0))\big)\Bigg(\sum_{i=1,\,i=1+k_i}^{1+k(t)} \int\limits_{\tau_{i-1}}^{\tau_i} \exp\big(\eta(\xi(\tau)-\xi(t_0))\big)\,dV(B)(\tau)$$

$$+\sum_{i=1,\,i\neq 1+k_i}^{1+k(t)}\int\limits_{\tau_{i-1}}^{\tau_i}\exp\big(\eta(\xi(\tau)-\xi(t_0))\big)\,dV(B)(\tau)\Bigg)$$

$$\le R_0\exp\big(-\eta(\xi(t)-\xi(t_0))\big)\Bigg(\sum_{i=1,\,i=1+k_i}^{1+k(t)}\exp(\eta\,i)\big(V(B)(\tau_i)-V(B)(\tau_{i-1})\big)$$

$$+\sum_{i=1,\,i\neq 1+k_i}^{1+k(t)}\exp(\eta\,i)\big(V(B)(\tau_i)-V(B)(\tau_{i-1})\big)+\sum_{i=1,\,i\neq 1+k_i}^{1+k(t)}\exp\big((1+k_i)\eta\big)\,d_1B(\tau_i)\Bigg)$$

$$\le \varepsilon R_0\exp\big(-\eta(\xi(t)-\xi(t_0))\big)\Bigg(\sum_{i=1}^{1+k(t)}\exp(\eta\,i)+\sum_{i=1,\,i\neq 1+k_i}^{1+k(t)}\exp((1+k_i)\eta)\Bigg)$$

$$\le 2\varepsilon R_0\exp\big(-\eta(\xi(t)-\xi(t_0))\big)\sum_{i=1}^{1+k(t)}\exp(\eta\,i)$$

$$=2\varepsilon R_0\exp\big(-\eta(\xi(t)-\xi(t_0))\big)\exp(\eta)\big(\exp((1+k(t))\eta)-1\big)(\exp(\eta)-1)^{-1}$$

$$\le 2\varepsilon R_0\exp(-\eta k(t))\exp\big((2+k(t))\eta\big)(\exp(\eta)-1)\ \text{ for }\ t\ge t_0$$

and, therefore,

$$\mathcal{J}(t)\le 2\varepsilon R_0\exp(2\eta)\big(\exp(\eta)-1\big)^{-1}\ \text{ for }\ t\ge t_0. \tag{1.3.109}$$

From (1.3.103), (1.3.105) and (1.3.109), it follows that inequality (1.3.84) holds for $t_0\ge t^*$, where $H\in\mathbb{R}^{n\times n}$ is the matrix whose every component equals $1/2n$. On the other hand, it can be easily shown that

$$r(H)<\frac{1}{2}\,.$$

Consequently, by Lemma 1.3.1, an arbitrary solution x of system $(1.3.1_0)$ admits the estimate

$$\|x(t)\|\le\rho\exp\big(-\eta(\xi(t)-\xi(t_0))\big)\ \text{ for }\ t\ge t_0\ge t^*,$$

where $\rho>0$ is a constant independent of t_0. □

Proof of Corollary 1.3.2. Corollary 1.3.2 follows from Theorem 1.3.6 if we assume that

$$A_0(t)\equiv\operatorname{diag}\big(a_{11}(t),\dots,a_{nn}(t)\big).$$

Indeed, by the definition of the operator $\mathcal{A}$, we have

$$\big[\mathcal{A}(A_0,A-A_0)(t)\big]_{ik}=a_{ik}(t)+\sum_{0<\tau\le t}\frac{d_1a_{ii}(\tau)}{1-d_1a_{ii}(\tau)}\,d_1a_{ik}(\tau)$$

$$-\sum_{0\le\tau<t}\frac{d_2a_{ii}(\tau)}{1+d_2a_{ii}(\tau)}\,d_2a_{ik}(\tau)=\mathcal{A}(a_{ii},a_{ik})(t)\ \text{ for }\ t\in\mathbb{R}_+\ \ (i\neq k;\ i,k=1,\dots,n)$$

and

$$\big[\mathcal{A}(A_0,A-A_0)(t)\big]_{ii}=0\ \text{ for }\ t\in\mathbb{R}_+\ \ (i=1,\dots,n).$$

Let now $U_0(t,\tau)$ be the Cauchy matrix of system (1.3.82). Then

$$U_0(t,\tau)=\operatorname{diag}\big(u_{011}(t,\tau),\dots,u_{0nn}(t,\tau)\big),$$

where

$$u_{0ii}(t,\tau)\equiv\exp\big(s_c(a_{ii})(t)-s_c(a_{ii})(\tau)\big)\prod_{\tau<s\le t}\big(1-d_1a_{ii}(s)\big)^{-1}\prod_{\tau\le s<t}\big(1+d_2a_{11}(s)\big)$$

is the Cauchy function of the equation

$$dx=x\,da_{ii}(t)$$

for every $i \in \{1,\dots,n\}$.

In view of (1.3.24), we find

$$|u_{0ii}(t,\tau)| \le \exp\big(-\eta(\xi(t)-\xi(\tau))\big) \ \text{ for } \ t \ge \tau \ge 0 \ \ (i=1,\dots,n).$$

Therefore, inequality (1.3.7) is valid. Thus, due to Proposition 1.3.6, the matrix-function A_0 is ξ-exponentially asymptotically stable. Consequently, by (1.3.23), using Theorem 1.3.6, we find that the matrix-function A is ξ-exponentially asymptotically stable, as well. □

Proof of Corollary 1.3.3. Corollary 1.3.3 immediately follows from Theorem 1.3.6 if we observe that

$$\mathcal{A}(A_0, A-A_0)(t) = A(t) - A_0(t) \ \text{ for } \ t \in \mathbb{R}_+$$

in this case and, moreover, $\nu(\xi)(t) \equiv \nu_c(\xi)(t)$, because ξ is the non-decreasing continuous function. □

Proof of Proposition 1.3.7. By the ξ-exponentially asymptotic stability of the matrix-function A and Proposition 1.3.6, there exist positive numbers η and ρ_0 such that the Cauchy matrix U of system $(1.3.1_0)$ satisfies estimate (1.3.102), where R_0 is an $n\times n$ matrix whose every component equals ρ_0.

Let ε be a positive number. Then, by (1.3.25), there exists $t_0 > 0$ such that

$$\bigvee_{t}^{\nu(\xi)(t)} \mathcal{A}(A,f) < \varepsilon \ \text{ for } \ t \ge t_0.$$

Let x be an arbitrary solution of system (1.3.1). Then by integration-by-parts and variation-of-constants formulas, we easily show that

$$\begin{aligned} x(t) &= U(t,t_0)x(t_0) + \int_{t_0}^{t} U(t,\tau)\,df(\tau) - \sum_{t_0<\tau\le t} d_1U(t,\tau)\cdot d_1f(\tau) + \sum_{t_0\le\tau<t} d_2U(t,\tau)\cdot d_2f(\tau) \\ &= U(t,t_0)x(t_0) + \int_{t_0}^{t} U(t,\tau)\,d\mathcal{A}(A,f)(\tau) \ \text{ for } \ t \ge t_0. \end{aligned}$$

From this, due to (1.3.102), we have

$$|x(t)| \le R_0\exp\big(-\eta(\xi(t)-\xi(t_0))\big)|x(t_0)| + J(t) \ \text{ for } \ t \ge t_0,$$

where $B(t) \equiv \mathcal{A}(A_0, A-A_0)(t)$ and $\mathcal{J}(t)$ is defined by (1.3.106).

As in the proof of Theorem 1.3.6, we get

$$\mathcal{J}(t) \le 2\varepsilon R_0 \exp(2\eta)(\exp(\eta)-1)^{-1} \ \text{ for } \ t \ge t_0.$$

Therefore,

$$|x(t)| \le R_0\exp\big(-\eta(\xi(t)-\xi(t_0))\big)|x(t_0)| + 2\varepsilon R_0\exp\big(2\eta\big)\big(\exp(\eta)-1\big)^{-1} \ \text{ for } \ t \ge t_0.$$

where ε is an arbitrary positive number, and the function ξ satisfies condition (1.3.4). Thus condition (1.3.6) holds. □

Proof of Proposition 1.3.8. Let U_0 be the Cauchy matrix of system (1.3.82). Then, as above, by the integration-by-parts and variation-of-constants formulas we have

$$x(t) = U_0(t,0)x(0) + \int_0^t U_0(t,\tau)\,d\mathcal{A}(A_0, A-A_0)(\tau)\cdot x(\tau) \ \text{ for } \ t \in \mathbb{R}_+.$$

On the other hand, by the exponentially ξ-asymptotic stability of the matrix-function A_0, there exist positive numbers η and ρ such that

$$\|U_0(t,\tau)\| \le \rho\exp\big(-\eta(\xi(t)-\xi(\tau))\big) \ \text{ for } \ t \ge \tau \ge 0. \tag{1.3.110}$$

Let now $\varepsilon \in (0, \eta\rho^{-1})$. Then, due to (1.3.26), there exists $t_0 \in \mathbb{R}_+$ such that

$$\bigvee_0^t \mathcal{A}(A_0, A - A_0) < \varepsilon\xi(t) \ \text{ for } \ t \geq t_0.$$

From this estimate, if we take into account (1.3.27) and (1.3.110), according to Lemma 1.3.1, we conclude that

$$\|x(t)\| \exp\big(\eta\xi(t)\big) \leq \rho\|x(0)\| \exp\big(\rho\varepsilon\xi(t) + \eta\xi(0)\big) \ \text{ for } \ t \geq t_0.$$

So,

$$\|x(t)\| \leq \rho\|x(0)\| \exp\big((\rho\varepsilon - \eta)\xi(t) + \eta\xi(0)\big) \ \text{ for } \ t \geq t_0.$$

Therefore, by (1.3.4), condition (1.3.6) holds. In view of Proposition 1.3.5, the matrix-function A is asymptotically stable. □

Proof of Theorem 1.3.7. Let

$$A_0(t) = \operatorname{diag}\big(a_{11}(t), \dots, a_{nn}(t)\big) \ \text{ for } \ t \in \mathbb{R}_+. \tag{1.3.111}$$

Then the Cauchy matrix U_0 of system (1.3.82) has the form

$$U_0(t,\tau) = \operatorname{diag}\Bigg(\exp\big(s_c(a_{11})(t) - s_c(a_{11})(\tau)\big) \prod_{\tau<s\leq t} \big(1 - d_1 a_{11}(s)\big)^{-1} \prod_{\tau\leq s<t} \big(1 + d_2 a_{11}(s)\big), \dots,$$
$$\exp\big(s_c(a_{nn})(t) - s_c(a_{nn})(\tau)\big) \prod_{\tau<s\leq t} \big(1 - d_1 a_{nn}(s)\big)^{-1} \prod_{\tau\leq s<t} \big(1 + d_2 a_{nn}(s)\big)\Bigg).$$

So, by the definition of the operator J, we get

$$|U_0(t,\tau)| = \operatorname{diag}\Big(\exp\big(J(a_{11})(t) - J(a_{11})(\tau)\big), \dots, \exp\big(J(a_{nn})(t) - J(a_{nn})(\tau)\big)\Big). \tag{1.3.112}$$

Therefore, due to (1.3.29), (1.3.30) and (1.3.112), estimates (1.3.83) and (1.3.84) are fulfilled for every $t_0 \in [t^*, +\infty[$, where

$$\Omega = \operatorname{diag}(\rho_0, \dots, \rho_0),$$
$$\rho_0 = \sup\Big\{\sum_{i=1}^n \exp\big(J(a_{ii})(t) - J(a_{ii})(t_0)\big) : \ t \geq t_0\Big\},$$

and $\eta(t) \equiv 0$. According to Lemma 1.3.1, every solution x of system (1.3.1$_0$) admits the estimate

$$\|x(t)\| \leq \rho(t_0)\|x(t_0)\| \ \text{ for } \ t \geq t_0,$$

where

$$\rho(t_0) = \|(I_n - H)^{-1}\|\rho_0(t_0).$$

So, every solution of system (1.3.1$_0$) is bounded on $\mathbb{R}_+$ and, by Proposition 1.3.2, the matrix-function A is stable. □

Proof of Theorem 1.3.8. In view of (1.3.32),

$$\rho_0 = \sup\Big\{\sum_{i=1}^n \exp\big(J(a_{ii})(t) - J(a_{ii})(\tau)\big) : \ t \geq \tau \geq 0\Big\} < +\infty. \tag{1.3.113}$$

Let the matrix-function A_0 be defined by (1.3.111). Then by (1.3.28), (1.3.29), (1.3.112) and (1.3.113), conditions (1.3.83) and (1.3.84) are fulfilled for every $t_0 \in [t^*, +\infty[$, where $\Omega = \operatorname{diag}(\rho_0, \dots, \rho_0)$, and $\eta(t) \equiv 0$. Hence, according to Lemma 1.3.1, every solution x of system (1.3.1$_0$) admits the estimate

$$\|x(t)\| \leq \rho_1\|x(t_0)\| \ \text{ for } \ t \geq t_0 \geq t^*,$$

where $\rho_1 = \|(I_n - H)^{-1}\|\rho_0$ is the number not depending on x and t_0. On the other hand, by Gronwall's inequality (see Lemma 1.1.4′), we get

$$\|x(t)\| \le \rho_2\|x(t_0)\| \text{ for } 0 \le t_0 \le t \le t^*,$$

where

$$\rho_2 = r_0 \exp\Big(r_0 \bigvee_0^t A\Big)$$

and r_0 is such that

$$\big\|(I_n - d_1 A(t))^{-1}\big\| \le r_0 \text{ for } t \in [0, t^*].$$

The latter two estimates imply estimate (1.3.5) for every $t \in \mathbb{R}_+$, where $\rho = \rho_1\rho_2$. Therefore, according to Proposition 1.3.3, the matrix-function is uniformly stable. □

Proof of Corollary 1.3.4. According to Remark 1.3.8, the functions $J(a_{ii})$ and $\mathcal{A}(a_{ii}, a_{ii})$ $(i = 1, \dots, n)$ are non-increasing. So, with regard (1.3.33), (1.3.34) and (1.1.19), we have

$$\int_{t^*}^{t} \exp\big(J(a_{ii})(t) - J(a_{ii})(\tau)\big)\, dv(b_{ik})(\tau) \le -h_{ik} \int_{t^*}^{t} \exp\big(J(a_{ii})(t) - J(a_{ii})(\tau)\big)\, d\mathcal{A}(a_{ii}, a_{ii})(\tau)$$

$$= -h_{ik} \int_{t^*}^{t} u_{0ii}(t,\tau)\, d\mathcal{A}(a_{ii}, a_{ii})(\tau) = h_{ik}\big(1 - u_{0ii}(t, t^*)\big) \le h_{ik} \text{ for } t \ge t^* \ (i = 1, \dots, n),$$

where $b_{ik}(t) \equiv \mathcal{A}(a_{ii}, a_{ik})(t)$ $(i \ne k;\ i, k = 1, \dots, n)$, and

$$u_{0ii}(t,\tau) = \exp\big(J(a_{ii})(t) - J(a_{ii})(\tau)\big) > 0 \ (i = 1, \dots, n).$$

Therefore, (1.3.29) is valid. Moreover, since $J(a_{ii})$ $(i = 1, \dots, n)$ are non-increasing functions, it is evident that

$$u_{0ii}(t,\tau) \le 1 \text{ for } t \ge \tau \ge t^* \ (i = 1, \dots, n).$$

From this we have (1.3.32). So, the corollary follows from Theorem 1.3.8. □

Proof of Theorem 1.3.9. Let the matrix-function A_0 be defined by (1.3.111) and U_0 be the Cauchy matrix of system (1.3.82). Then by (1.3.35), (1.3.36) and (1.3.112), conditions (1.3.83) and (1.3.84) are fulfilled for every $t_0 \in [t^*, +\infty[$. Then, according to Lemma 1.3.1, estimate (1.3.85) is true for every solution x of system $(1.3.1_0)$, where $\rho = \|(I_n - H)^{-1}\|n$. Hence, due to Proposition 1.3.5, the matrix-function A is asymptotically stable, since the function η satisfies condition (1.3.4). □

Proof of Corollary 1.3.5. By (1.3.31), there exists $\varepsilon \in]0, 1[$ such that

$$r(H_\varepsilon) < 1, \tag{1.3.114}$$

where

$$H_\varepsilon = \Big(\frac{h_{ik}}{1 - \varepsilon}\Big)_{i,k=1}^{n}.$$

Let

$$\eta(t) \equiv \varepsilon a_0(t) \text{ and } b_{ik}(t) \equiv \mathcal{A}(a_{ii}, a_{ik})(t) \ (i, k = 1, \dots, n).$$

Due to Remark 1.3.8, the functions $J(a_{ii})$ $(i = 1, \dots, n)$ are non-increasing. From this, by (1.3.40) and (1.3.41), we have inequalities (1.3.35), and the function η satisfies condition (1.3.4).

On the other hand, due to (1.3.37)–(1.3.40), we find

$$\int_{t^*}^{t} \exp\big(\eta(t) - \eta(\tau) + J(a_{ii})(t) - J(a_{ii})(\tau)\big)\, dv(b_{ik})(\tau)$$

$$\le \int_{t^*}^{t} \exp\big((1-\varepsilon)\big(J(a_{ii})(t) - J(a_{ii})(\tau)\big)\big)\, dv(b_{ik})(\tau) \text{ for } t \ge t^* \ (i \ne k;\ i, k = 1, \dots, n) \tag{1.3.115}$$

and

$$\big|s_c(b_{ik})(t) - s_c(b_{ik})(\tau)\big| \le -\frac{h_{ik}}{1-\varepsilon}\Big(s_c\big(\mathcal{A}\big((1-\varepsilon)J(a_{ii}),(1-\varepsilon)J(a_{ii})\big)(t) - s_c\big(\mathcal{A}\big((1-\varepsilon)J(a_{ii}),(1-\varepsilon)J(a_{ii})\big)(\tau)\Big) \text{ for } t \ge \tau \ge t^* \ (i \ne k;\ i,k = 1,\dots,n). \quad (1.3.116)$$

Let now $j = 1$. Then $d_1 a_{ii}(t) \le 0$ $(i = 1,\dots,n)$ for $t \ge t^*$. It is not difficult to verify that

$$h_{ik}\big(1 + \ln\big(1 - d_1 a_{ii}(t)\big)\big)^{-1} \ln\big(1 - d_1 a_{ii}(t)\big) \le -\frac{h_{ik}}{1-\varepsilon}\, d_1\mathcal{A}\big((1-\varepsilon)J(a_{ii}),(1-\varepsilon)J(a_{ii})\big)(t) \text{ for } t \ge t^* \ (i \ne k;\ i,k = 1,\dots,n).$$

Hence, by (1.3.39) and the equality $|d_1 b_{ik}(t)| = |d_1 a_{ik}(t)|$, the estimates

$$|d_1 b_{ik}(t)| \le -\frac{h_{ik}}{1-\varepsilon}\, d_1\mathcal{A}\big((1-\varepsilon)J(a_{ii}),(1-\varepsilon)J(a_{ii})\big)(t) \text{ for } t \ge t^* \ (i \ne k;\ i,k = 1,\dots,n) \quad (1.3.117)$$

are valid.

Analogously, we show that

$$|d_2 b_{ik}(t)| \le -\frac{h_{ik}}{1-\varepsilon}\, d_2\mathcal{A}\big((1-\varepsilon)J(a_{ii}),(1-\varepsilon)J(a_{ii})\big)(t) \text{ for } t \ge t^* \ (i \ne k;\ i,k = 1,\dots,n). \quad (1.3.118)$$

From (1.3.115), by (1.3.116)–(1.3.118), we get

$$\begin{aligned}
&\int_{t^*}^{t} \exp\big((1-\varepsilon)\big(J(a_{ii})(t) - J(a_{ii})(\tau)\big)\big)\, dv(b_{ik})(\tau) \\
&\quad \le -\frac{h_{ik}}{1-\varepsilon}\int_{t^*}^{t} \exp\big((1-\varepsilon)\big(J(a_{ii})(t) - J(a_{ii})(\tau)\big)\big)\, d\mathcal{A}\big((1-\varepsilon)J(a_{ii}),(1-\varepsilon)J(a_{ii})\big)(\tau) \\
&\quad = \frac{h_{ik}}{1-\varepsilon}\exp\big((1-\varepsilon)J(a_{ii})(t)\big)\Big(\exp\big((\varepsilon-1)J(a_{ii})(t)\big) - \exp\big((\varepsilon-1)J(a_{ii})(t^*)\big)\Big) \\
&\quad \le \frac{h_{ik}}{1-\varepsilon} \text{ for } t \ge t^* \ (i \ne k;\ i,k = 1,\dots,n). \qquad (1.3.119)
\end{aligned}$$

According to Theorem 1.3.9, conditions (1.3.4), (1.3.35), (1.3.114) and (1.3.119) guarantee the asymptotic stability of the matrix-function A. As to the uniform stability of A, it follows from Corollary 1.3.4. □

Proof of Corollary 1.3.6. Immediately follows from Corollary 1.3.5, since the functions defined by (1.3.42)–(1.3.44) satisfy condition (1.3.40). □

Proofs of Theorem 1.3.10 and Corollary 1.3.7 are analogous to those of Theorem 1.3.9 and Corollary 1.3.5.

Proof of Theorem 1.3.11. Let x and $t_0 \in \mathbb{R}_+$ be an arbitrary solution of system $(1.3.1_0)$ and a point, respectively. It is not difficult to verify that the conditions of the theorem guarantee the fulfilment of the corresponding conditions of Lemma 1.1.3 for $t_0 = t_1 = \cdots = t_n$ and of Theorem 1.1.10 on the set $\mathbb{R}_+$. So, using the lemma, we have

$$d|x(t)| \le dA_0(t)\cdot|x(t)| \text{ for } t \in \mathbb{R}_+.$$

In addition, from this, by Theorem 1.1.10, we obtain

$$\|x(t)\| \le \|U_0(t,t_0)\|\,\|x(t_0)\| \text{ for } t \ge t_0 \ge 0,$$

where U_0 is the Cauchy matrix of system (1.3.82).

Therefore, by Propositions 1.3.3–1.3.6, we conclude that the stability of the matrix-function A_0 in one or another sense guarantees the stability of the matrix-function A in the same sense. □

Proof of Theorem 1.3.12. Let us prove the necessity. Let

$$a_{ii}(t)\equiv \alpha_{ii}\mu_i(t)\ \ (i=1,\dots,n);$$
$$a_{ik}(t)\equiv \alpha_{ik}\Big(s_c(\mu_i)(t)+|\alpha_{ii}|^{-1}\sum_{0<\tau\le t}\zeta_{i1}(\tau)+|\alpha_{ii}|^{-1}\sum_{0\le\tau<t}\zeta_{i2}(\tau)\Big)\ \ (i\neq k;\ i,k=1,\dots,n)$$

and

$$h_{ik}=(1-\delta_{ik})|\alpha_{ik}|\,|\alpha_{ii}|^{-1}\ \ (i,k=1,\dots,n),$$

where

$$\zeta_{ij}(t)\equiv \big(1+\ln(1+(-1)^j\alpha_{ii}d_j\mu_i(t))\big)^{-1}\ln\big(1+(-1)^j\alpha_{ii}d_j\mu_i(t)\big)\ \ (j=1,2;\ i=1,\dots,n).$$

Due to (1.3.46), we conclude that the functions $s_c(a_{ii})$ $(i=1,\dots,n)$ are non-increasing, since the functions $s_c(\mu_i)$ $(i=1,\dots,n)$ are non-decreasing. Moreover, by (1.3.46), the functions ζ_{ij} $(j=1,2;$ $i=1,\dots,n)$ are nonnegative. Thus it is not difficult to verify that conditions (1.3.37) and (1.3.39) are fulfilled.

From condition (1.3.46) it follows that $1+(-1)^j d_j a_{ii}(t)>0$ $(j=1,2;\ i=1,\dots,n)$. Consequently, by virtue of Remark 1.3.8, the functions $J(a_{ii})$ $(i=1,\dots,n)$ are non-increasing and

$$-J(a_{ii})(t)+J(a_{ii})(\tau)\ge a_0(t)-a_0(\tau)\ \text{ for }\ t\ge\tau\ \ (i=1,\dots,n).$$

So, condition (1.3.40) is valid. By virtue of Corollary 1.3.5, the sufficiency is proved.

Let us show the necessity. Assume the contrary. Let conditions (1.3.49), (1.3.50), (1.3.51) and (1.3.52) hold, A be asymptotically stable, but condition (1.3.47) be violated. Then either

$$\alpha_{i_0i_0}\ge 0 \tag{1.3.120}$$

for some $i_0\in\{1,\dots,n\}$, or

$$\alpha_{ii}<0\ \ (i=1,\dots,n), \tag{1.3.121}$$

but

$$r(H)\ge 1. \tag{1.3.122}$$

If condition (1.3.120) holds, then, in view of (1.3.49), the non-diagonal components of the matrix-function A are non-decreasing. By this and (1.3.120), the vector-function $x(t)\equiv(\delta_{ii_0})_{i=1}^n$ satisfies the system of generalized differential inequalities

$$dx(t)\le dA(t)\cdot x(t)\ \text{ for }\ t\in\mathbb{R}_+. \tag{1.3.123}$$

Moreover, with regard to (1.3.50), (1.3.51) and (1.3.52), taking into consideration Hadamard's condition on the non-singularity of matrices (see [28, p. 382]), it is not difficult to verify that conditions (1.1.29), (1.1.30) and (1.1.31) of Theorem 1.1.10 are fulfilled for the matrix-function A. By this theorem,

$$x(t)\le U(t,0)x(0)\ \text{ for }\ t\in\mathbb{R}_+,$$

where $U(t,\tau)$ is the Cauchy matrix of system $(1.3.1_0)$. Hence, due to the asymptotic stability of A, we have

$$\|x(t)\|\le \big\|U(t,0)x(0)\big\|\longrightarrow 0\ \text{ as }\ t\to+\infty. \tag{1.3.124}$$

But this is impossible, since $\|x(t)\|\equiv 1$. Therefore, (1.3.121) holds.

Assume now that (1.3.122) is fulfilled. Then there exist a complex vector $(c_i)_{i=1}^n$ and a complex number λ such that

$$\sum_{k=1}^n |c_k|=1,\ \ |\lambda|=r(H)\ge 1$$

and

$$\sum_{k=1}^n(1-\delta_{ik})|\alpha_{ik}|\,|\alpha_{ii}|^{-1}c_k=\lambda c_i\ \ (i=1,\dots,n).$$

Therefore,

$$|\alpha_{ii}|\,|c_i| \leq \sum_{k=1,\,k\neq i}^{n} |\alpha_{ik}|\,|c_k| \quad (i=1,\dots,n).$$

Since $\alpha_{ii}<0$ and $\alpha_{ik}\geq 0$ $(i\neq k:\ i,k=1,\dots,n)$, we find

$$0 \leq \sum_{k=1}^{n} \alpha_{ik}\,|c_k| \quad (i=1,\dots,n).$$

Consequently, the vector-function $x(t)\equiv(|c_k|)_{k=1}^{n}$ is a solution of the system of differential inequalities (1.3.123), since $\mu_1,\dots,\mu_n$ are non-decreasing functions. As above, we can show that (1.3.124) holds. But this is impossible, since $\|x(t)\|\equiv 1$. The obtained contradiction proves the theorem. □

Proof of Corollary 1.3.8. Let $B_\eta(H,A)$ be the matrix-function defined by (1.3.14), where $H(t)\equiv\eta\xi(t)Y^{-1}(t)$. Using the formulae of integration-by-parts, the properties of the operator $\mathcal{B}$ and equality (1.1.17), we conclude that

$$\begin{aligned}
B_\eta(H,A)(t) &= \int_0^t \exp(-\eta\xi(\tau))\,d\Big(\exp(\eta\xi(\tau))Y^{-1}(\tau)+\mathcal{B}(\exp(\eta\xi)Y^{-1},A)(\tau)\Big)\\
&= \int_0^t \exp(-\eta\xi(\tau))\,d(\exp(\eta\xi(\tau))Y^{-1}(\tau)) + \int_0^t \exp(-\eta\xi(\tau))\,d\mathcal{B}\Big(\exp(\eta\xi(\tau))Y^{-1},A)(\tau)\Big)\\
&= \int_0^t \exp(-\eta\xi(\tau))\,d(\exp(\eta\xi(\tau))Y^{-1}(\tau))\\
&\qquad + \int_0^t \exp(-\eta\xi(\tau))\,d\mathcal{B}\Big(\exp(\eta\xi)I_n,\mathcal{B}(Y^{-1},A)\Big)(\tau) \quad \text{for } t\in\mathbb{R}_+,
\end{aligned} \tag{1.3.125}$$

$$\begin{aligned}
&\int_0^t \exp(-\eta\xi(\tau))\,d(\exp(\eta\xi(\tau))Y^{-1}(\tau))\\
&= \int_0^t Y^{-1}(\tau)\,d\Bigg(\eta s_c(\xi)(\tau)I_n-\mathcal{A}(Q,Q)(\tau)+\sum_{0<s\leq t}\exp(-\eta\xi(s))\,d_1\exp(\eta\xi(s))\cdot(I_n-d_1Q(s))^{-1}\\
&\qquad + \sum_{0\leq s<t}\exp(-\eta\xi(s))\,d_2\exp(\eta\xi(s))\cdot(I_n+d_2Q(s))^{-1}\Bigg) \quad \text{for } t\in\mathbb{R}_+,
\end{aligned} \tag{1.3.126}$$

$$\begin{aligned}
\mathcal{B}(Y^{-1},A)(t) &= \int_0^t Y^{-1}(\tau)\,dA(\tau) - \sum_{0<\tau\leq t} d_1Y^{-1}(\tau)\cdot d_1A(\tau) + \sum_{0\leq\tau<t} d_2Y^{-1}(\tau)\cdot d_1A(\tau)\\
&= \int_0^t Y^{-1}(\tau)\,d\mathcal{A}(Q,A-Q)(\tau) \quad \text{for } t\in\mathbb{R}_+,
\end{aligned} \tag{1.3.127}$$

$$\mathcal{B}\Big(\exp(\eta\xi)I_n,\mathcal{B}(Y^{-1},A)\Big)(t) = \int_0^t Y^{-1}(\tau)\,d\mathcal{B}\Big(\exp(\eta\xi)I_n,\mathcal{A}(Q,A)\Big)(\tau) \quad \text{for } t\in\mathbb{R}_+ \tag{1.3.128}$$

and

$$\int_0^t \exp(-\eta\xi(\tau))\,d\mathcal{B}\big(\exp(\eta\xi)I_n,\mathcal{A}(Q,A)\big)(\tau)$$
$$= \mathcal{A}(Q,A)(t) - \sum_{0<\tau\le t} \exp(-\eta\xi(\tau))\,d_1\exp(\eta\xi(\tau))(I_n - d_1Q(\tau))^{-1}\,d_1A(\tau)$$
$$+ \sum_{0\le\tau<t} \exp(-\eta\xi(\tau))\,d_2\exp(\eta\xi(\tau))(I_n + d_2Q(\tau))^{-1}\,d_2A(\tau) \ \text{ for } \ t\in\mathbb{R}_+. \qquad (1.3.129)$$

From (1.3.125), by (1.3.126)–(1.3.129), we get

$$B_\eta(H,A)(t) = \int_0^t \exp(-\eta\xi(\tau))\,d(\exp(\eta\xi(\tau))Y^{-1}(\tau))$$
$$+ \int_0^t Y^{-1}(\tau)\,d\bigg(\int_0^\tau \exp(-\eta\xi(s))\,d\mathcal{B}\big(\exp(\eta\xi)I_n,\mathcal{A}(Q,A)\big)(s)\bigg)$$
$$= \int_0^t Y^{-1}(\tau)\,dG_\eta(\xi,Q,A)(\tau) \ \text{ for } \ t\in\mathbb{R}_+$$

and

$$\bigvee_0^{+\infty} B_\eta(H,A) \le \bigg\|\int_0^{+\infty} |Y^{-1}(t)\,dVG_\eta(\xi,Q,A)(t)\bigg\|.$$

Therefore, from (1.3.53) and the fact that the matrix-function Q is ξ-exponentially asymptotically stable, it follows that the conditions of Theorem 1.3.4 are fulfilled. □

Proof of Corollary 1.3.9. The corollary follows immediately from Corollaries 1.3.1 and 1.3.6 and Remark 1.3.6 if we note that

$$Y(t) \equiv \exp(Q(t)) \ \text{ and } \ G_\eta(\xi,Q,A)(t) \equiv A(t) - Q(t) + \eta\xi(t)I_n$$

in this case. □

Proof of Corollary 1.3.10. For $\eta > 0$, the corollary follows from Corollary 1.3.8 if we assume that

$$Q(t) \equiv \operatorname{diag}\big(a_{11}(t) + \eta s_c(\xi)(t), \dots, a_{nn}(t) + \eta s_c(\xi)(t)\big).$$

Indeed, let $\mathcal{A}(Q,A-Q)(t) \equiv (\beta_{ik}(t))_{i,k=1}^n$ and $G_\eta(\xi,Q,A)(t) \equiv (\gamma_{ik}(t))_{i,k=1}^n$. Then, by the definition of the operator $\mathcal{A}$, we have

$$\beta_{ik}(t) = a_{ik}(t) + \sum_{0<\tau\le t} d_1a_{ii}(\tau)\cdot(1 - d_1a_{ii}(\tau))^{-1}d_1a_{ik}(\tau)$$
$$- \sum_{0\le\tau<t} d_2a_{ii}(\tau)\cdot(1 + d_2a_{ii}(\tau))^{-1}\,d_2a_{ik}(\tau) \ \text{ for } \ t\in\mathbb{R}_+ \ \ (i\ne k;\ \ i,k=1,\dots,n)$$

and

$$\beta_{ii} = -\eta s_c(\xi)(t) \ \text{ for } \ t\in\mathbb{R}_+ \ \ (i=1,\dots,n).$$

From the above relations, using (1.3.54), we obtain

$$\gamma_{ik}(t) = a_{ik}(t) + \sum_{0<\tau\le t} d_1a_{ii}(\tau)\cdot(1 - d_1a_{ii}(\tau))^{-1}d_1a_{ik}(\tau)$$
$$- \sum_{0\le\tau<t} d_2a_{ii}(\tau)\cdot(1 + d_2a_{ii}(\tau))^{-1}d_2a_{ik}(\tau) - \sum_{0<\tau\le t} d_1a_{ik}(\tau)\cdot(1 - d_1a_{ii}(\tau))^{-1}\big(1 - \exp(-\eta d_1\xi(\tau))\big)$$

$$-\sum_{0\le\tau<t} d_2a_{ik}(\tau)\cdot(1+d_2a_{ii}(\tau))^{-1}\big(1-\exp(\eta d_2\xi(\tau))\big)=g_{ik}(t)\quad\text{for}\quad t\in\mathbb{R}_+\ \ (i\ne k;\ \ i,k=1,\dots,n)$$

and

$$\gamma_{ii}=\sum_{0<\tau\le t}\big(1-\exp(-\eta d_1\xi(\tau))\big)+\sum_{0\le\tau<t}\big(1-\exp(\eta d_2\xi(\tau))\big)\ \text{ for }\ t\in\mathbb{R}_+\ \ (i=1,\dots,n).$$

On the other hand, the matrix-function $Y(t)=\operatorname{diag}(y_1(t),\dots,y_n(t))$ is the fundamental matrix of system (1.3.17) satisfying the condition $Y(0)=I_n$. Therefore, by (1.3.55)–(1.3.57), the conditions of Corollary 1.3.6 are valid. For $\eta=0$, the corollary follows from Corollary 1.3.1 and Remark 1.3.5. □

Proof of Theorem 1.3.13. It is evident that the matrix-function

$$B(t)\equiv\sum_{k=1}^{m}\alpha_k(t)B_k$$

satisfies the Lappo–Danilevskiĭ condition. Therefore, in view of Lemma 1.3.4, the matrix-function

$$X(t)=\prod_{k=1}^{m}\exp(\alpha_k(t)B_k)\ \text{ for }\ t\in\mathbb{R}_+ \tag{1.3.130}$$

is a fundamental matrix of system $(1.3.1_0)$.

According to the Jordan theorem,

$$B_k=C_k\operatorname{diag}\big(J_{n_{k1}}(\lambda_{k1}),\dots,J_{n_{lm_k}}(\lambda_{km_k})\big)C_k^{-1}\ \ (k=1,\dots,m),$$

where $J_{n_{ki}}(\lambda_{ki})=\lambda_{ki}I_{n_{ki}}+Z_{n_{ki}}$ is the Jordan box corresponding to the elementary divisor $(\lambda-\lambda_{ki})^{n_{ki}}$ for every $k\in\{1,\dots,m\}$ and $i\in\{1,\dots m_k\}$, and $C_k\in\mathbb{C}^{n\times n}$ $(k=1,\dots,m)$ are nonsingular complex matrices. Hence

$$\begin{aligned}\exp(\alpha_k(t)B_k)=C_k\operatorname{diag}\Big(&\exp\big(\alpha_k(t)J_{n_{k1}}(\lambda_{k1})\big),\dots,\\ &\exp\big(\alpha_k(t)J_{n_{km_k}}(\lambda_{km_k})\big)\Big)C_k^{-1}\ \text{ for }\ t\in\mathbb{R}_+\ \ (k=1,\dots,m),\end{aligned} \tag{1.3.131}$$

where

$$\exp(\alpha_k(t)J_{n_{ki}}(\lambda_{ki}))=\exp(\lambda_{ki}\alpha_k(t))\sum_{j=0}^{n_{ki}-1}\frac{\alpha_k^j(t)}{j!}\,Z_{n_{ki}}^j\ \text{ for }\ t\in\mathbb{R}_+\ \ (k=1,\dots,m). \tag{1.3.132}$$

In view of (1.3.131) and (1.3.132), it is evident that

$$\exp(\alpha_k(t)B_k)=\bigg(\sum_{i=1}^{m_k}p_{kijl}(\alpha_k(t))\exp(\lambda_{ki}\alpha_k(t))\bigg)_{i,l=1}^{n}\ \text{ for }\ t\in\mathbb{R}_+\ \ (k=1,\dots,m), \tag{1.3.133}$$

where $p_{kijl}(s)$ is a polynomial with respect to the variable s, whose degree is at most $n_{ki}-1$ $(i,l=1,\dots,n;\ k=1,\dots,m)$.

Substituting (1.3.133) in (1.3.130), we find

$$\begin{aligned}\beta_1\prod_{k=1}^{m}\bigg(\sum_{i=1}^{m_k}(1+\alpha_k(t))^{n_{ki}-1}\exp(\alpha_k(t)\mathrm{Re}\lambda_{ki})\bigg)&\le\|X(t)\|\\ &\le\beta_2\prod_{k=1}^{m}\bigg(\sum_{i=1}^{m_k}(1+\alpha_k(t))^{n_{ki}-1}\exp(\alpha_k(t)\mathrm{Re}\lambda_{ki})\bigg)\ \text{ for }\ t\in\mathbb{R}_+,\end{aligned}$$

where β_1 and β_2 are some positive numbers.

The latter estimates imply the validity of the theorem. □

Proof of Corollary 1.3.11. The corollary follows immediately from Theorem 1.3.13, since conditions (1.3.60) and (1.3.61) are equivalent to the conditions imposed on the real parts of the eigenvalues λ_{ki} $(k=1,\dots,m;\ i=1,\dots,m_k)$ of the matrices B_k $(k=1,\dots,m)$. □

Proof of Corollary 1.3.12. Let

$$\alpha_1(t)\equiv\alpha(t),\ \ \alpha_2(t)\equiv\beta_1\alpha(t)-\nu_1(t),\ \ \alpha_3(t)\equiv\nu_2(t)-\beta_2\alpha(t)$$

and

$$B_1=A_0-\beta_1\ln(I_n-A_1)+\beta_2\ln(I_n+A_2),\ \ B_2=\ln(I_n-A_1),\ \ B_3=\ln(I_n+A_2).$$

Then we have

$$S_c(A)(t)=\sum_{k=1}^{3}s_c(\alpha_k)(t)\cdot B_k\ \text{ for }\ t\in\mathbb{R}_+\ \ (j=1,2)$$

and

$$\begin{aligned}\exp\bigg((-1)^j\sum_{k=1}^{3}d_j\alpha_k(t)\cdot B_k\bigg)&\exp\big((-1)^j\ln(I_n+(-1)^jA_j)\big)\\ &=I_n+(-1)^jA_j=I_n+(-1)^jd_jA(t)\ \text{ if }\ \|d_jA(t)\|\neq0\ \text{ for }\ t\in\mathbb{R}_+\ \ (j=1,2),\end{aligned}$$

since the function α is continuous, and $d_j\nu_i(t)\equiv\delta_{ij}$ $(i,j=1,2)$, in the case.

Hence the conditions of Theorem 1.3.13 are fulfilled. Moreover, due to (1.3.64), the functions α_2 and α_3 are bounded on $\mathbb{R}_+$. So, conditions (1.3.60) and (1.3.61) of Theorem 1.3.13 are equivalent to the conditions applied to the matrix P in the cases (a) and (b) of the corollary, respectively. □

Proof of Corollary 1.3.13. The corollary follows from Theorem 1.3.13 if we choose the functions α_l $(l=1,\dots,m)$ and the matrices B_l $(l=1,\dots,m)$ in a suitable way. But the proof of the corollary is more easy if we use the same way as in the proof of Theorem 1.3.13.

By Lemma 1.3.4, the matrix-function

$$X(t)\equiv C\operatorname{diag}\big(\exp(G_1(t)),\dots,\exp(G_m(t))\big)C^{-1}$$

is a fundamental matrix of system $(1.3.1_0)$. Moreover, obviously,

$$\begin{aligned}\exp(G_k(t))&=\prod_{i=0}^{n_k-1}\exp\big(\alpha_{ki}(t)Z^i_{n_k}\big)\\ &=\exp(\alpha_{k0}(t))\prod_{i=1}^{n_k-1}\sum_{j=1}^{[(n_k-1)/i]}\frac{\alpha^j_k(t)}{j!}\,Z^{ij}_{n_k}\ \text{ for }\ t\in\mathbb{R}_+\ \ (k=1,\dots,m).\end{aligned}$$

Hence, the statement of the corollary follows as in Theorem 1.3.13. □

Proof of Theorem 1.3.14. Let $x=(x_i)_{i=1}^n$ be a solution of system $(1.3.1_0)$ and let

$$u(t)\equiv\sum_{i=1}^{n}x_i^2(t).$$

Then, by (0.0.11), we have

$$\begin{aligned}u(t)-u(s)&=\sum_{i=1}^{n}\bigg(2\int_s^t x_i(\tau)\,dx_i(\tau)-\sum_{s<\tau\le t}(d_1x_i(\tau))^2+\sum_{s\le\tau<t}(d_2x_i(\tau))^2\bigg)\\ &=\sum_{i=1}^{n}\bigg(2\sum_{k=1}^{n}\int_s^t x_i(\tau)x_k(\tau)\,da_{ik}(\tau)+\sum_{s<\tau\le t}\big(x_i^2(\tau)-x_i^2(\tau-)-2x_i(\tau)d_1x_i(\tau)\big)\end{aligned}$$

$$+\sum_{s\le\tau<t}\big(x_i^2(\tau+)-x_i^2(\tau)-2x_i(\tau)d_2x_i(\tau)\big)\Big)$$

$$=2\sum_{i,k=1}^{n}\Big(\int_s^t x_i(\tau)x_k(\tau)\,da_{ik}(\tau)-\sum_{s<\tau\le t}x_i(\tau)x_k(\tau)\,d_1a_{ik}(\tau)-\sum_{s\le\tau<t}x_i(\tau)x_k(\tau)\,d_2a_{ik}(\tau)\Big)$$

$$+\sum_{j=1}^{2}(s_j(u)(t)-s_j(u)(\tau))\ \text{ for }\ 0\le s\le t<+\infty.$$

Hence

$$u(t)-u(s)=2\sum_{i,k=1}^{n}\int_s^t x_i(\tau)x_k(\tau)\,ds_c(a_{ik})(\tau)+\sum_{j=1}^{2}(s_j(u)(t)-s_j(u)(s))\ \text{ for }\ 0\le s\le t<+\infty.$$

On the other hand,

$$\sum_{j=1}^{2}(s_j(u)(t)-s_j(u)(s))$$

$$=\sum_{i=1}^{n}\Big(\sum_{s<\tau\le t}d_1x_i(\tau)(2x_i(\tau)-d_1x_i(\tau))+\sum_{s\le\tau<t}d_2x_i(\tau)(2x_i(\tau)+d_2x_i(\tau))\Big)$$

$$=2\sum_{i,k=1}^{n}\Big(\sum_{s<\tau\le t}x_i(\tau)x_k(\tau)\Big(d_1a_{ik}(\tau)-\frac{1}{2}\sum_{l=1}^{n}d_1a_{li}(\tau)\cdot d_1a_{lk}(\tau)\Big)$$

$$+\sum_{s\le\tau<t}x_i(\tau)x_k(\tau)\Big(d_2a_{ik}(\tau)-\frac{1}{2}\sum_{l=1}^{n}d_2a_{li}(\tau)\cdot d_2a_{lk}(\tau)\Big)\Big)\ \text{ for }\ 0\le s\le t<+\infty.$$

From this, taking into account (1.3.67) and (1.3.68), we find

$$u(t)-u(s)=2\sum_{i,k=1}^{n}\int_s^t x_i(\tau)x_k(\tau)\,db_{ik}(\tau)=2\sum_{i,k=1}^{n}\int_s^t h_{ik}(\tau)x_i(\tau)x_k(\tau)\,d\alpha(\tau)\ \text{ for }\ 0\le s\le t<+\infty.$$

Therefore, due to (1.3.70),

$$u(t)-u(s)\le 2\int_s^t p(\tau)\sum_{i=1}^{n}x_i^2(\tau)\,d\alpha(\tau)=\int_s^t u(\tau)\,d\beta(\tau)\quad\text{for}\quad 0\le s\le t<+\infty.$$

Using now Lemma 1.1.4 (or Lemma 1.1.5), for every $t_0\ge 0$ we get

$$u(t)\le u(t_0)\gamma_\beta(t)\gamma_\beta^{-1}(t_0)\ \text{ for }\ t\ge t_0\ge 0.$$

In addition, it is evident that

$$u(t)\le\|x(t)\|^2\le nu(t)\ \text{ for }\ t\in\mathbb{R}_+.$$

So,

$$\|x(t)\|\le n^{1/2}\big(\gamma_\beta(t)\gamma_\beta^{-1}(t_0)\big)^{1/2}\|x(t_0)\|\ \text{ for }\ t\ge t_0\ge 0.$$

From the last estimate and conditions (1.3.72)–(1.3.75) there immediately follow the conclusions (a)–(d) of the theorem.

As to the proof of the conclusion (e), it suffices to note that, by the last inequality, the estimate

$$\|x(t)\|\ge n^{-1/2}\gamma_\beta(t)\|x(0)\|\ \text{ for }\ t\in\mathbb{R}_+$$

is valid. □

Proof of Corollary 1.3.14. If we take into account the equality

$$\sum_{i,k=1}^{n} p_{ik}(t)x_ix_k \equiv \frac{1}{2}\sum_{i,k=1}^{n} \big(h_{ik}(t)+h_{ki}(t)\big)x_ix_k,$$

then, according to Lemma 1.3.5, the estimates

$$\lambda_0(C(t))\sum_{i=1}^{n} x_i^2 \leq \sum_{i,k=1}^{n} h_{ik}(t)x_ix_k \leq \lambda^0(C(t))\sum_{i=1}^{n} x_i^2$$

hold. From this and (1.3.77) follow estimates (1.3.70) and (1.3.71), where $p(t) \equiv \lambda^0(C(t))$. Hence, the conclusions (a)–(d) immediately follow from conclusions (a)–(d) of Theorem 1.3.13, respectively.

As for the conclusion (e) of the corollary, it also follows from the conclusion (e) of Theorem 1.3.13 if we take $p(t) \equiv \lambda_0(C(t))$. □

Proof of Theorem 1.3.15. We assume $A_0(t) \equiv A(t)$ and $f_0(t) \equiv f(t)$. Let $(A_k, f_k; t_k)$ $(k = 1, 2, \dots)$ and c_k $(k = 0, 1, \dots)$ be the sequences satisfying the conditions appearing in Definition 1.3.7, where $I = \mathbb{R}_+$. First, consider the case $H_k(t) \equiv I_n$ $(k = 0, 1, \dots)$. Then conditions (1.2.11) and (1.2.12) are obviously satisfied, and according to Remark 1.2.1, conditions (1.2.13) and (1.2.14) coincide with (1.2.18) and (1.2.19), respectively.

Let $t_0 \in \mathbb{R}_+$ and $c_0 \in \mathbb{R}^n$ be arbitrarily fixed, and let x be a solution of the initial problem $(1.1.1), (1.1.2)$. By the ξ-exponentially asymptotic stability of the matrix-function A, condition (1.3.80) and Proposition 1.3.7, we have

(i) the solution x satisfies condition (1.3.6);

(ii) there exist $\rho_0 > 0$ and $\eta > 0$ such that the Cauchy matrix U of system $(1.3.1_0)$ admits estimate (1.3.110) for $\rho = \rho_0$, where $U_0(t,\tau) \equiv U(t,\tau)$;

(iii) there exists $\rho_1 > 0$ such that (see the proof of estimate (1.3.109) in Theorem 1.3.6)

$$\int_{t_0}^{t} \exp\big(-\eta(\xi(t)-\xi(\tau))\big)\, d\big\|V(\mathcal{A}(A,A))(\tau)\big\| \leq \rho_1 \ \text{ for } \ t \in \mathbb{R}_+. \tag{1.3.134}$$

In view of (1.3.110) and the equality $U(t,t-) = (I_n - d_1A(t))^{-1}$ (see Theorem 1.1.6(d)), we have

$$\big\|(I_n - d_1A(t))^{-1}\big\| < \rho_0 + 1 \ \text{ for } \ t \in \mathbb{R}_+. \tag{1.3.135}$$

Let r_1, r_2 and r_3 be sufficiently small positive numbers such that

$$r^* < \frac{1}{4}\exp(-1), \tag{1.3.136}$$

where

$$r^* = \big(\rho_0r_2 + (r_1(2r_4+r_5)+2r_3)(1+\rho_0\rho_1)\big), \ \ r_4 = \frac{1}{4(1+\rho_0\rho_1)} \ \text{ and } \ r_5 = \frac{1}{12\rho_0}\,.$$

Let $\varepsilon \in]0,1[$ be an arbitrary number. Due to (1.3.6), there exists $t^* = t^*(\varepsilon,\rho) > t_0+1$ such that

$$\|x(t)\| < r_1\varepsilon \ \text{ for } \ t \geq t^*. \tag{1.3.137}$$

Let $I = [0,t^*]$. According to Theorem 1.2.2 and Remark 1.2.1, there exists a natural number $k_1(\varepsilon)$ such that problem $(1.2.1_k), (1.2.2_k)$ has the unique solution x_k on I, satisfying the condition

$$\|x(t) - x_k(t)\| < r_2\varepsilon \ \text{ for } \ t \in I \ \ (k = k_1(\varepsilon), k_1(\varepsilon)+1, \dots). \tag{1.3.138}$$

For the proof it suffices to show that there exists a natural number $k_*(\varepsilon)\ge k_1(\varepsilon)$ such that problem $(1.2.1_k),(1.2.2_k)$ has the unique solution x_k on $\mathbb{R}_+$ for sufficiently large k and

$$\|x(t)-x_k(t)\|<\varepsilon \ \text{ for } \ t>t_* \ \ (k=k_*(\varepsilon),k_*(\varepsilon)+1,\dots). \tag{1.3.139}$$

First, we show the uniqueness. By conditions (1.2.18) and (1.2.19), there exists a natural number $k_*(\varepsilon)\ge k_1(\varepsilon)$ such that

$$\|A_k(t)-A(t)\|<r_4\varepsilon \ \text{ for } \ t\in\mathbb{R}_+ \ \ (k=k_*(\varepsilon),k_*(\varepsilon)+1,\dots), \tag{1.3.140}$$

$$\Big|\bigvee_{t_k}^{t}\big(\mathcal{B}(A_k-A,A_k)\big)\Big|<r_5\varepsilon \ \text{ for } \ t\in\mathbb{R}_+ \ \ (k=k_*(\varepsilon),k_*(\varepsilon)+1,\dots) \tag{1.3.141}$$

and

$$\big\|f_k(t)-f(t)-\mathcal{B}(A_k-A,f_k)(t)+\mathcal{B}(A_k-A,f_k)(t_k)\big\| \\ <r_3\varepsilon \ \text{ for } \ t\in\mathbb{R}_+ \ \ (k=k_*(\varepsilon),k_*(\varepsilon)+1,\dots). \tag{1.3.142}$$

According to (1.3.135) and (1.3.140), we have

$$\|d_1A(t)-d_1A_k(t)\|<2r_4\varepsilon<\frac{1}{2(\rho_0+1)} \ \text{ for } \ t\in\mathbb{R}_+ \ \ (k=k_*(\varepsilon),k_*(\varepsilon)+1,\dots)$$

and

$$\big\|(I_n-d_1A(t))^{-1}\cdot(d_1A(t)-d_1A_k(t))\big\|<\frac{1}{2} \ \text{ for } \ t\in\mathbb{R}_+ \ \ (k=k_*(\varepsilon),k_*(\varepsilon)+1,\dots).$$

From this, by the equality

$$I_n-d_1A_k(t)=(I_n-d_1A(t))\cdot\big(I_n+(I_n-d_1A(t))^{-1}(d_1A(t)-d_1A_k(t))\big) \ \text{ for } \ t\in\mathbb{R}_+,$$

we obtain that

$$\det(I_n-d_1A_k(t))\ne 0 \ \text{ for } \ t\in\mathbb{R}_+ \ \ (k=k_*(\varepsilon),k_*(\varepsilon)+1,\dots).$$

Therefore, problem $(1.2.1_k),(1.2.2_k)$ has the unique solution x_k on $\mathbb{R}_+$ for $k\ge k_*(\varepsilon)$.

Let now $k\in\{k_*(\varepsilon),k_*(\varepsilon)+1,\dots\}$ be a fixed natural number and $z(t)\equiv x_k(t)-x(t)$. Then, due to the definition of solutions, we easily show that

$$z(t)=c_{k0}+A(t_k)c_k+f_0(t_k)+\int_{t_k}^{t}dA(s)\cdot z(s)(A_k(t)-A(t))\cdot x_k(t) \\ -\int_{t_k}^{t}d\mathcal{B}(A_k-A,A_k)(s)\cdot x_k(s)+f_k(t)-f(t)-\mathcal{B}(A_k-A,f_k)(t)+\mathcal{B}(A_k-A,f_k)(t_k) \ \text{ for } \ t\ge t_k,$$

where $c_{k0}=z(t_k)=c_k-x(t_k)$.

So, z will be the solution of the system

$$dz(t)=dA(t)\cdot z(t)+dg(t) \ \text{ for } \ t\ge t^*, \tag{1.3.143}$$

where

$$g(t)=g_{01}(t)+g_{02}(t)+g_1(t)+g_2(t),$$

$$g_{01}(t)=-\int_{t^*}^{t}d\mathcal{B}(A_k-A,A_k)(s)\cdot z_k(s),$$

$$g_{02}(t)=(A_k(t)-A(t))\cdot z_k(t)-(A_k(t^*)-A(t^*))\cdot z_k(t*),$$

$$g_1(t)=\int_{t^*}^{t} d\mathcal{B}(A_k-A,A_k)(s)\cdot x(s)-(A_k(t)-A(t))\cdot x(t)+(A_k(t^*)-A(t^*))\cdot x(t^*),$$

$$g_2(t)=f_k(t)-f_k(t^*)-f(t)+f(t^*)-\mathcal{B}(A_k-A,f_k)(t)+\mathcal{B}(A_k-A,f_k)(t^*).$$

According to the variation-of-constants formula, we have

$$z(t)=U(t,t^*)z(t^*)+g(t)-\int_{t^*}^{t} d_\tau U(t,\tau)\cdot g(\tau)\ \text{ for }\ t\geq t^*, \tag{1.3.144}$$

where $U(t,\tau)$ is the Cauchy matrix of system $(1.3.1_0)$.

By the integration-by-part formula and equality (1.1.19), for every $t\geq t*$, we have

(a)

$$\begin{aligned}
&\bigg\|g_{01}(t)-\int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_{01}(\tau)\bigg\| \\
&\quad=\bigg\|\int_{t^*}^{t} U(t,\tau)\,dg_{01}(\tau)-\sum_{t^*<\tau\leq t} d_{1\tau}U(t,\tau)\cdot d_1 g_{01}(\tau)+\sum_{t^*\leq\tau<t} d_{2\tau}U(t,\tau)\cdot d_2 g_{01}(\tau)\bigg\| \\
&\leq\int_{t^*}^{t}\|U(t,\tau)\|\,\|z(\tau)\|d\bigvee\big(\mathcal{B}(A_k-A,A_k)\big)(\tau)+\sum_{t^*<\tau\leq t}\|d_{1\tau}U(t,\tau)\|\,\|z(\tau)\|\,\big\|d_1\mathcal{B}(A_k-A,A_k)(\tau)\big\| \\
&\qquad+\sum_{t^*\leq\tau<t}\|d_{2\tau}U(t,\tau)\|\,\|z(\tau)\|\,\big\|d_2\mathcal{B}(A_k-A,A_k)(\tau)\big\| \\
&\qquad\leq\rho_0\int_{t^*}^{t}\|z(\tau)\|\,d\Big\|\bigvee(\mathcal{B}(A_k-A,A_k))(\tau)\Big\| \\
&\quad+2\rho_0\sum_{t^*<\tau\leq t}\|z(\tau)\|\,\big\|d_1\mathcal{B}(A_k-A,A_k)(\tau)\big\|+2\rho_0\sum_{t^*\leq\tau<t}\|z(\tau)\|\,\big\|d_2\mathcal{B}(A_k-A,A_k)(\tau)\big\|
\end{aligned}$$

and, therefore, by (1.3.110), we conclude

$$\bigg\|g_{01}(t)-\int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_{01}(\tau)\bigg\|\leq 3\rho_0\int_{t^*}^{t}\|z(\tau)\|\,d\alpha(\tau), \tag{1.3.145}$$

where

$$\alpha(t)=\bigvee_{t^*}^{t}\big(\mathcal{B}(A_k-A,A_k)\big);$$

(b)

$$\begin{aligned}
&\bigg\|g_{02}(t)-\int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_{02}(\tau)\bigg\|=\bigg\|g_{02}(t)+\int_{t^*}^{t} U(t,\tau)\,d\mathcal{A}(A,A)(\tau)\cdot g_{02}(\tau)\bigg\| \\
&\quad\leq\|A_k(t)-A(t)\|\cdot\|z(t)\|+\int_{t^*}^{t}\|U(t,\tau)\|\,\|(A_k(\tau)-A(\tau))\|\,\|z(\tau)\|\,d\Big\|\bigvee(\mathcal{A}(A,A))(\tau)\Big\|
\end{aligned}$$

and, therefore, by (1.3.110) and (1.3.140),

$$\bigg\|g_{02}(t)-\int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_{02}(\tau)\bigg\|\leq r_4\varepsilon\|z(t)\|+r_4\rho_0\rho_1\varepsilon\sup_{t^*\leq\tau\leq t}\|z(\tau)\|; \tag{1.3.146}$$

(c)

$$\left\| g_1(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_1(\tau) \right\| \le \|g_1(t)\| + \int_{t^*}^{t} \|U(t,\tau)\|\,\|g_1(\tau)\|\, d\Big\|\bigvee(\mathcal{A}(A,A))(\tau)\Big\|$$

and, therefore, by (1.3.134), (1.3.137), (1.3.140) and (1.3.141),

$$\left\| g_1(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_1(\tau) \right\| \le r_1(2r_4 + r_5)(1+\rho_0\rho_1)\varepsilon, \tag{1.3.147}$$

since

$$\|g_1(t)\| \le \|A_k(t) - A(t)\|\,\|x(t)\| + \|A_k(t^*) - A(t^*)\|\,\|x(t^*)\| + \int_{t^*}^{t} \|x(s)\|\, d\Big\|\bigvee(\mathcal{B}(A_k - A, A_k))(s)\Big\|;$$

(d)

$$\left\| g_2(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_2(\tau) \right\| \le \|g_2(t)\| + \int_{t^*}^{t} \|U(t,\tau)\|\,\|g_2(\tau)\|\, d\Big\|\bigvee(\mathcal{A}(A,A))(\tau)\Big\|$$

and, therefore, by (1.3.110),(1.3.134) and (1.3.142),

$$\left\| g_2(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_2(\tau) \right\| \le 2r_3(1+2\rho_0\rho_1)\varepsilon. \tag{1.3.148}$$

Moreover, it follows from (1.3.110) and (1.3.138) that

$$\|U(t,t^*)z(t^*)\| \le \rho_0\|z(t^*)\| < r_2\varepsilon \ \text{ for } \ t \ge t^*.$$

By this and (1.3.145)–(1.3.148), it follows from (1.3.144) that

$$\|z(t)\| \le \|U(t,t^*)z(t^*)\| + \sum_{j=1}^{2}\left(\left\| g_{0j}(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_{0j}(\tau)\right\| + \left\| g_j(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_j(\tau)\right\| \right)$$

$$\le r^*\varepsilon + r_3(1+\rho_0\rho_1)\varepsilon \sup_{t^*\le\tau\le t}\|z(\tau)\| + 3\rho_0\int_{t^*}^{t}\|z(\tau)\|\, d\Big\|\bigvee(\mathcal{B}(A_k - A, A_k))(\tau)\Big\| \ \text{ for } \ t \ge t^*.$$

From this, by the estimate $r_4(1+\rho_0\rho_1) < 1/2$, we get

$$\varphi(t) \le 2r^*\varepsilon + 6\rho_0\int_{t^*}^{t}\varphi(\tau)\, d\alpha(\tau) \ \text{ for } \ t \ge t^*, \tag{1.3.149}$$

where

$$\varphi(t) = \sup_{t_*\le\tau\le t}\|z(\tau)\|.$$

Let now $\beta : [t^*, +\infty) \to \mathbb{R}_+$ be the function defined by $\beta(t^*) = \alpha(t^*)$ and $\beta(t) = \alpha(t-)$ for $t > t^*$. Then due to (0.0.7) and (1.3.149),

$$\varphi(t) \le 2r^*\varepsilon + 6\rho_0\varphi(t)d_1\alpha(t) + 6\rho_0\int_{t^*}^{t}\varphi(\tau)\, d\alpha(\tau) \ \text{ for } \ t \ge t^*. \tag{1.3.150}$$

By the choice of r_5, we have $16\rho_0 d_1\alpha(t) < 1/2$ and $(I_n - 12\rho_0 d_1\alpha(t))^{-1} < 2$. Hence, by (1.3.150),

$$\varphi(t) \le 4r^*\varepsilon + 12\rho_0 \int_{t^*}^{t} \varphi(\tau)\, d\beta(\tau) \ \text{ for } \ t \ge t^*.$$

From this, using Gronwall's inequality (see Lemma 1.1.4′), by estimate (1.3.136), we get

$$\|z(t)\| \le 4r^*\varepsilon \exp(12\rho_0\beta(t)) \le 4r^*\varepsilon \exp(\varepsilon) < \varepsilon \ \text{ for } \ t \ge t^*.$$

So, condition (1.3.139) holds.

Consider now the general case, i.e., when the matrix-functions H_k $(k = 0, 1, \dots)$ are not identically equal I_n. In this case, without loss of generality, we may assume that $H_0(t) \equiv I_n$.

Let x_k be a solution of the initial problem $(1.2.1_k), (1.2.2_k)$ for every natural k. Then according to Lemma 1.2.2, for every natural k, the function $y_k = H_k x_k$ will be a solution of the problem

$$dy(t) = dA_{k*}(t) \cdot y(t) + df_{k*}(t),$$
$$y(t_k) = c_{k*},$$

where

$$A_{k*}(t) \equiv \mathcal{I}(H_k, A_k)(t), \ \ f_{k*}(t) \equiv \mathcal{B}(H_k, f_k)(t) \ \text{ and } \ c_{k*} = H_k c_k.$$

In addition,

$$I_n + (-1)^j d_j A_{k*}(t) \equiv \big(H_k(t) + (-1)^j d_j H_k(t)\big) \cdot \big(I_n + (-1)^j d_j A_k(t)\big) H_k^{-1}(t) \ \ (j = 1, 2).$$

Obviously, the conditions of Theorem 1.2.3 and Remark 1.2.2 are valid for the sequences $A_{k*}(t)$ $(k = 1, 2, \dots)$, $f_{k*}(t)$ $(k = 1, 2, \dots)$, c_{k*} $(k = 1, 2, \dots)$ and t_k $(k = 1, 2, \dots)$. Consequently,

$$\lim_{k\to+\infty} \|x_k(t) - x(t)\| = 0 \ \text{ uniformly on } \ \mathbb{R}_+.$$

From this, in view of the inequalities

$$\|x_k(t) - x(t)\| \le \big\|H_k^{-1}(t) y_k(t) - y_k(t)\big\| + \|y_k(t) - x(t)\| \ \text{ for } \ t \in \mathbb{R}_+,$$

we, as above, have

$$\lim_{k\to+\infty} \|x_k(t) - x(t)\| = 0 \ \text{ uniformly on } \ \mathbb{R}_+. \qquad \square$$

Proof of Theorem 1.3.16. Let $(A_k, f_k; t_k)$ $(k = 1, 2, \dots)$ and c_k $(k = 0, 1, \dots)$ be the sequences satisfying the conditions appearing in Definition 1.3.8, where $I = \mathbb{R}_+$, $A_0(t) \equiv A(t)$ and $f_0(t) \equiv f(t)$.

We first consider the case $H_k(t) \equiv I_n$ $(k = 0, 1, \dots)$. Then conditions (1.2.11) and (1.2.12) are obviously satisfied, and according to Remark 1.2.1, condition (1.2.13) coincides with (1.2.18), and the fulfillment of conditions (1.3.78) and (1.3.79) uniformly on $\mathbb{R}_+$ implies, respectively, that

$$\lim_{k\to+\infty} \bigvee_0^t (A_k - A) = 0 \tag{1.3.151}$$

and

$$\lim_{k\to+\infty} \bigvee_0^t (f_k - f) = 0 \tag{1.3.152}$$

uniformly on $\mathbb{R}_+$.

Let $t_0 \in \mathbb{R}_+$ and $c_0 \in \mathbb{R}^n$ be arbitrarily fixed, x be a solution of the initial problem (1.1.1), (1.1.2), and U be the Cauchy matrix of system $(1.3.1_0)$. By the uniform stability of the matrix-function A and condition (1.3.81), based on the integration-by-part formula, there exists a number $\rho_0 > 1$ such that

$$\|x(t)\| \le \rho_0 \ \text{ for and } \ \|U(t,\tau)\| \le \rho_0 \ \text{ for } \ t \ge \tau \ge 0. \tag{1.3.153}$$

Let r_1, r_2 and r_3 be sufficiently small positive numbers such that

$$r_1 < \frac{1}{6(\rho_0+1)} \ \text{ and } \ r^* < \frac{1}{2}\exp(-1), \tag{1.3.154}$$

where

$$r^* = \rho_0(3r_1 + r_2 + 3r_3).$$

Let $\varepsilon \in]0,1[$ be an arbitrary number. Let $t^* = t_0 + 1$ and $I = [0, t^*]$. By (1.3.151) and (1.3.152), it is not difficult to verify that conditions (1.2.18) and (1.2.19) of Remark 1.2.1 hold.

According to Theorem 1.2.2 and Remark 1.2.1, there exists a natural number $k_1(\varepsilon)$ such that problem $(1.2.1_k), (1.2.2_k)$ has the unique solution x_k on I for $k \ge k_1(\varepsilon)$, satisfying condition (1.3.138).

To prove the theorem, it suffices to show that there exists a natural number $k_*(\varepsilon) \ge k_1(\varepsilon)$ such that problem $(1.2.1_k), (1.2.2_k)$ has the unique solution x_k on $\mathbb{R}_+$ and estimate (1.3.139) holds for $k \ge k_*(\varepsilon)$.

By (1.3.151) and (1.3.152), there exists a natural number $k_*(\varepsilon) \ge k_1(\varepsilon)$ such that

$$\bigvee_0^t (A_k - A) < r_1\varepsilon \ \text{ for } \ t \ge 0 \ \ (k = k_*(\varepsilon), k_*(\varepsilon)+1, \dots) \tag{1.3.155}$$

and

$$\bigvee_0^t (f_k - f) < r_3\varepsilon \ \text{ for } \ t \ge 0 \ \ (k = k_*(\varepsilon), k_*(\varepsilon)+1, \dots). \tag{1.3.156}$$

The existence and uniqueness of a solution of the problem for $k \ge k_*(\varepsilon)$ can be shown as above, in the proof of Theorem 1.3.15.

Let now $k \in \{k_*(\varepsilon), k_*(\varepsilon)+1, \dots\}$ be a fixed natural number and $z(t) \equiv x_k(t) - x(t)$. Then z will be the solution of system (1.3.143), and presentation (1.3.144) is valid, where

$$g(t) = g_0(t) + g_1(t) + g_2(t), \quad g_0(t) = \int_{t^*}^{t} d(A_k(\tau) - A(\tau)) \cdot z(\tau),$$

$$g_1(t) = \int_{t^*}^{t} d(A(\tau) - A_k(\tau)) \cdot x(\tau), \quad g_2(t) = f(t) - f_k(t) - f(t^*) + f_k(t^*).$$

By the integration-by-part formula, equality (1.1.19) and estimate (1.3.153), for every $t \ge t*$ we have:

(a)

$$\begin{aligned} g_0(t) - \int_{t^*}^{t} d_\tau U(t,\tau) \cdot g_0(\tau) &= \int_{t^*}^{t} U(t,\tau)\, d(A_k(\tau) - A(\tau)) \cdot z(\tau) \\ &- \sum_{t^* < \tau \le t} d_{1\tau} U(t,\tau)\, d_1(A_k(\tau) - A(\tau)) \cdot z(\tau) + \sum_{t^* \le \tau < t} d_{2\tau} U(t,\tau)\, d_2(A_k(\tau) - A(\tau)) \cdot z(\tau) \end{aligned}$$

and, therefore,

$$\left\| g_0(t) - \int_{t^*}^{t} d_\tau U(t,\tau) \cdot g_0(\tau) \right\| \le 3\rho_0 \int_{t^*}^{t} \|z(\tau)\|\, d\alpha(\tau), \tag{1.3.157}$$

where

$$\alpha(t) = \bigvee_{t^*}^{t} (A_k - A);$$

(b)

$$g_1(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_1(\tau) = \int_{t^*}^{t} U(t,\tau)\,d(A(\tau)-A_k(\tau))\cdot x(\tau)$$
$$- \sum_{t^*<\tau\le t} d_1U(t,\tau)\cdot d_1(A(\tau)-A_k(\tau))\cdot x(\tau) + \sum_{t^*\le\tau<t} d_2U(t,\tau)\cdot d_2(A(\tau)-A_k(\tau))\cdot x(\tau)$$

and, therefore,

$$\Big\|g_1(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_1(\tau)\Big\| \le 3\rho_0^2\alpha(t);$$

(c)

$$g_2(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_2(\tau) = \int_{t^*}^{t} U(t,\tau)\,d(f(\tau)-f_k(\tau))$$
$$- \sum_{t^*<\tau\le t} d_1U(t,\tau)\cdot d_1(f(\tau)-f_k(\tau)) + \sum_{t^*\le\tau<t} d_2U(t,\tau)\cdot d_2(f(\tau)-f_k(\tau))$$

and, therefore,

$$\Big\|g_2(t) - \int_{t^*}^{t} d_\tau U(t,\tau)\cdot g_2(\tau)\Big\| \le 3\rho_0 \bigvee_{t^*}^{t}(f-f_k). \tag{1.3.158}$$

From (1.3.144), according to (1.3.153)–(1.3.158), we obtain

$$\|z(t)\| \le \rho_0\|z(t^*)\| + 3\rho_0^2\alpha(t) + 3\rho_0\bigvee_{t^*}^{t}(f-f_k) + 3\rho_0\int_{t^*}^{t}\|z(\tau)\|\,d\alpha(\tau) \ \text{ for } \ t\ge t^*.$$

Consequently, taking into account (1.3.138), (1.3.155) and (1.3.156), we get

$$\|z(t)\| \le r^*\varepsilon + 3\rho_0\int_{t^*}^{t}\|z(\tau)\|\,d\alpha(\tau) \ \text{ for } \ t\ge t^*. \tag{1.3.159}$$

Let now $\beta : [t^*,+\infty) \to \mathbb{R}_+$ be the function defined as in the proof of Theorem 1.3.15. Then, due to (0.0.7) and (1.3.159),

$$\|z(t)\| \le r^*\varepsilon + 3\rho_0\|z(t)\|\,d_1\alpha(t) + 3\rho_0\int_{t^*}^{t}\|z(\tau)\|\,d\beta(\tau) \ \text{ for } \ t\ge t^*.$$

By the choice of r_1, we have $3\rho_0 d_1\alpha(t) < 1/2$ and $(I_n - 3\rho_0 d_1\alpha(t))^{-1} < 2$. Hence

$$\|z(t)\| \le 2r^*\varepsilon + 6\rho_0\int_{t^*}^{t}\|z(\tau)\|\,d\beta(\tau) \ \text{ for } \ t\ge t^*.$$

From this, using Gronwall's inequality, by estimate (1.3.154), we get

$$\|z(t)\| \le 2r^*\varepsilon\exp(6\rho_0\beta(t)) \le 2r^*\varepsilon\exp(6\rho_0 r_1) < \varepsilon \ \text{ for } \ t\ge t^*.$$

Thus condition (1.3.139) holds.

The theorem in a general case is proved as Theorem 1.3.15 in the same case. □

Chapter 2

Systems of linear impulsive differential equations

2.1 The initial problem

In this chapter, we realize the results of Chapter 1 for the initial problem for the following impulsive differential systems

$$\frac{dx}{dt} = P(t)x + q(t) \ \text{ for a.a. } \ t \in I \setminus T, \tag{2.1.1}$$

$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \ \ (l = 1, 2, \dots); \tag{2.1.2}$$

$$x(t_0) = c_0, \tag{2.1.3}$$

where $P \in L_{loc}(I; \mathbb{R}^{n\times n})$, $q \in L_{loc}(I; \mathbb{R}^n)$, $G \in B_{loc}(T; \mathbb{R}^{n\times n})$, $u \in B_{loc}(T; \mathbb{R}^n)$, $T = \{\tau_1, \tau_2, \dots\}$, $\tau_l \in I$ $(l = 1, 2, \dots)$, $\tau_l \neq \tau_k$ if $l \neq k$ $(l, k = 1, 2, \dots)$, $t_0 \in I$, and $c_0 \in \mathbb{R}^n$.

Definition 2.1.1. By a solution of the impulsive differential system (2.1.1), (2.1.2) we understand a continuous from the left vector-function $x \in \mathrm{ACV}_{loc}(I, T; \mathbb{R}^n)$ satisfying both the system

$$x'(t) = P(t)x(t) + q(t) \ \text{ for a.a. } \ t \in I \setminus T$$

and relation (2.1.2) for every $l \in \{1, 2, \dots\}$.

Quite a number of issues of the theory of linear systems of differential equations with impulsive effect have been studied sufficiently well (for survey of the results on impulsive systems see the references in the introduction). But the above-mentioned works do not contain the results analogous to those obtained in [3, 35] for ordinary differential equations. Using the theory of generalized ordinary differential equations, we extend these results to the systems of impulsive differential equations.

We assume that the condition

$$\det(I_n + G(\tau_l)) \neq 0 \ \text{ for } \ \tau_l < t_0 \ \ (l = 1, 2, \dots) \tag{2.1.4}$$

holds.

To establish the results dealing with the initial and other problems for the impulsive differential system (2.1.1), (2.1.2), we use the following conception.

Remark 2.1.1. A vector-function x is a solution of the impulsive system (2.1.1), (2.1.2) if and only if it is a solution of system (1.1.1), where

$$\begin{aligned} A(t) &= \int_a^t P(\tau)\,d\tau + \operatorname{sgn}(t-a) \sum_{\tau_l \in T_{a,t}} G(\tau_l) \ \text{ for a.a. } \ t \in I, \\ f(t) &= \int_a^t q(\tau)\,d\tau + \operatorname{sgn}(t-a) \sum_{\tau_l \in T_{a,t}} u(\tau_l) \ \text{ for a.a. } \ t \in I, \end{aligned} \tag{2.1.5}$$

and $a\in I$ is an arbitrary fixed point.

It is evident that these matrix- and vector-functions A and f have the following properties:

$$\begin{gathered}
d_1A(t)=O_{n\times n},\quad d_1f(t)=0 \ \text{ for a.a. } \ t\in I,\\
d_2A(t)=O_{n\times n},\quad d_2f(t)=0 \ \text{ for a.a. } \ t\in I\setminus T,\\
d_2A(\tau_l)=G(\tau_l),\quad d_2f(\tau_l)=u(\tau_l) \ \ (l=1,2,\dots);
\end{gathered}\tag{2.1.6}$$

$$\begin{gathered}
S_c(A)(t)-S_c(A)(s)=\int\limits_s^t P(\tau)\,d\tau,\ \ s_c(f)(t)-s_c(f)(s)=\int\limits_s^t q(\tau)\,d\tau \ \text{ for a.a. } \ s,t\in I\setminus T,\\
S_1(A)(t)=O_{n\times n},\ \ s_1(f)(t)=0 \ \text{ for a.a. } \ t\in I\setminus T,\\
S_2(A)(t)=S_2(A)(s)+\sum_{s\le\tau_l<t}G(\tau_l),\ \ s_2(f)(t)=s_2(f)(s)+\sum_{s\le\tau_l<t}u(\tau_l) \ \text{ for a.a. } \ s,t\in I;\ s<t
\end{gathered}\tag{2.1.7}$$

(in particular, they are continuous from the left everywhere).

So, condition (2.1.4) is equivalent to condition (1.1.3). Moreover, due to the conditions imposed on P, G, q and u, we have $A\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $f\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n})$.

Along with problem (2.1.1)–(2.1.3), we consider the corresponding homogeneous problem

$$\frac{dx}{dt}=P(t)x \ \text{ for a.a. } \ t\in I\setminus T, \tag{2.1.1$_0$}$$

$$x(\tau_l+)-x(\tau_l-)=G(l)x(\tau_l) \ \ (l=1,2,\dots); \tag{2.1.2$_0$}$$

$$x(t_0)=0. \tag{2.1.3$_0$}$$

We say that the pair (X,Y) consisting of the matrix-functions $X\in L_{loc}(I;\mathbb{R}^{n\times n})$ and $Y\in B_{loc}(T;\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ conditions if there exists $t_*\in I$ such that

$$X(t)\int\limits_{t_*}^t X(\tau)\,d\tau=\int\limits_{t_*}^t X(\tau)\,d\tau\cdot X(t) \quad \text{and}$$

$$\int\limits_{t_*}^t X(\tau)\,d\tau\cdot\sum_{\tau_l\in T_{t_*,t}}Y(\tau_l)=\sum_{\tau_l\in T_{t_*,t}}Y(\tau_l)\cdot\int\limits_{t_*}^t X(\tau)\,d\tau \ \text{ for } \ t\in I.$$

Remark 2.1.2. By Definition 2.1.1, under a solution of the impulsive system (2.1.1), (2.1.2) we understand the continuous from the left vector-function. If under a solution we understand the continuous from the right vector-function, then we have to require the condition

$$\det(I_n-G(\tau_l))\neq 0 \ \text{ for } \ \tau_l>t_0 \ \ (l=1,2,\dots)$$

instead of (2.1.4). In this case, the matrix $A(t)$ and vector $f(t)$ will be defined such that

$$\begin{gathered}
d_1A(t)=O_{n\times n},\quad d_1f(t)=0 \ \text{ for a.a. } \ t\in I\setminus T,\\
d_1A(\tau_l)=G(\tau_l),\quad d_1f(\tau_l)=u(\tau_l) \ \ (l=1,2,\dots),\\
d_2A(t)=O_{n\times n},\quad d_2f(t)=0 \ \text{ for a.a. } \ t\in I
\end{gathered}$$

instead of (2.1.6). In particular, $A(t)$ and $f(t)$ can be defined as in (2.1.5), where the set $T_{s,t}$ will be defined by the equality $T_{s,t}=]\min\{s,t\},\max\{s,t\}]$ for $s,t\in I$. The results corresponding to this case are analogous to the results corresponding to the first case given in Sections 2.1–2.3 below, if we replace the expressions of type $I_n+G(\tau_l)$ by $I_n-G(\tau_l)$, the intervals $[s,t[$ by $]s,t]$, and the right limits by the left ones.

Basing on the results of Section 1.1, we obtain the following results. Some of them are already well-known.

Theorem 2.1.1. *Let $t_0 \in I$. The initial value problem* (2.1.1)–(2.1.3) *possesses a unique solution x defined on I for any $q \in L_{loc}(I;\mathbb{R}^n)$ and $u \in B_{loc}(T;\mathbb{R}^n)$ if and only if condition* (2.1.4) *holds.*

Proposition 2.1.1. *Let $s \in I$, $p \in L_{loc}(I;\mathbb{R})$, and $g \in B_{loc}(T;\mathbb{R})$ be such that*

$$g(\tau_l) \neq -1 \quad (l = 1, 2, \dots).$$

Then the initial problem

$$\begin{gathered}\frac{d\gamma}{dt} = p(t)\gamma \;\; \text{for a.a. } t \in I \setminus T,\\ \gamma(\tau_l+) - \gamma(\tau_l-) = g(\tau_l)\gamma(\tau_l) \;\; (l = 1, 2, \dots);\\ \gamma(s) = 1\end{gathered}$$

has the unique solution $\gamma_p(\cdot, s)$ defined by

$$\gamma_p(t,s) = \begin{cases} \exp\Big(\int\limits_s^t p(\tau)\,d\tau\Big) \prod\limits_{s \le \tau_l < t} (1 + g(\tau_l)) & \text{for } t > s,\\ \exp\Big(\int\limits_s^t p(\tau)\,d\tau\Big) \prod\limits_{t \le \tau_l < s} (1 + g(\tau_l))^{-1} & \text{for } t < s,\\ 1 & \text{for } t = s. \end{cases}$$

Theorem 2.1.2. *Let the matrix-functions $P \in L([a,b];\mathbb{R}^{n\times n})$ and $G \in B(T;\mathbb{R}^{n\times n})$ be such that condition* (2.1.4) *hold. Then there exist a constant $r \in \mathbb{R}_+$ such that*

$$\|x(t)\| \le r\bigg(\|x(t_0)\| + \int\limits_a^{t_0} \|q(\tau)\|\,d\tau + \sum_{a \le \tau_l \le t_0} \|u(\tau_l)\|\bigg) \times \exp\bigg\{r\bigg(\int\limits_a^{t_0} \|P(\tau)\|\,d\tau + \sum_{a \le \tau_l \le t_0} \|G(\tau_l)\|\bigg)\bigg\} \;\; \text{for } a \le t \le t_0$$

and

$$\|x(t)\| \le r\bigg(\|x(t_0)\| + \int\limits_{t_0}^{b} \|q(\tau)\|\,d\tau + \sum_{t_0 \le \tau_l \le b} \|u(\tau_l)\|\bigg) \times \exp\bigg\{r\bigg(\int\limits_{t_0}^{b} \|P(\tau)\|\,d\tau + \sum_{t_0 \le \tau_l \le b} \|G(\tau_l)\|\bigg)\bigg\} \;\; \text{for } t_0 \le t \le b$$

for every $q \in L([a,b];\mathbb{R}^n)$, $u \in B(T;\mathbb{R}^n)$, where x is a solution of the impulsive system (2.1.1), (2.1.2).

Theorem 2.1.3. *The set of all solutions of the homogeneous system* $(2.1.1_0), (2.1.2_0)$ *is an n-dimensional subset of* $\mathrm{ACV}_{loc}(I,T;\mathbb{R}^n)$.

We have the theorem on the existence of the Cauchy matrix.

Theorem 2.1.4. *Let the matrix-functions $P \in L_{loc}(I;\mathbb{R}^{n\times n})$ and $G \in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that condition* (2.1.4) *holds. Then there exists a unique $n \times n$ matrix-function $U(t,s)$ defined for $a \le t \le s \le t_0$ and $t_0 \le s \le t \le b$ such that the matrix function $X(t) = U(t,s)$ satisfies the matrix initial value problem*

$$\frac{dX}{dt} = P(t)X \;\; \text{for a.a. } t \in I \setminus T, \tag{2.1.8}$$

$$X(\tau_l+)-X(\tau_l-)=G(\tau_l)X(\tau_l)\ \ (l=1,2,\dots); \tag{2.1.9}$$

$$X(s)=I_n. \tag{2.1.10}$$

In addition, relation (1.1.12) *holds, and every solution of the homogeneous system* $(2.1.1_0),(2.1.2_0)$ *defined on* $\{t\in I, t\le s\}$ *if* $s\le t_0$ *and on* $\{t\in I, t\ge s\}$ *if* $t_0\le s$ *is given by relation* (1.1.13) *on the intervals of definition.*

Theorem 2.1.5 (Variation-of-constants formula)**.** *Let the matrix-functions* $P\in L_{loc}(I;\mathbb{R}^{n\times n})$ *and* $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that condition* (2.1.4) *holds. Then every solution of the impulsive system* (2.1.1), (2.1.2) *admits the representation*

$$x(t)=U(t,t_0)x(t_0)+\int\limits_{t_0}^{t}U(t,\tau)\,q(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}U(t,\tau_l)u(\tau_l)\ \ \textit{for}\ \ t\in I$$

for every $u\in B_{loc}(T;\mathbb{R}^n)$*, where* $U(t,s)$ *is the matrix-function appearing in Theorem* 2.1.4*.*

Proposition 2.1.2. *Let the pair* (P,G) *consisting of the matrix-functions* $P\in L_{loc}(I;\mathbb{R}^{n\times n})$ *and* $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ *satisfy the Lappo–Danilevskiĭ condition and inequality* (2.1.4) *hold for every* τ_l $(l=1,2,\dots)$*. Then the impulsive matrix-system* (2.1.8), (2.1.9)*, under the condition* $X(a)=I_n$*, has the unique solution* X *defined by*

$$X(t)=\begin{cases}\exp\Big(\int\limits_a^t P(\tau)\,d\tau\Big)\prod\limits_{t\le\tau_l<a}(I_n+G(\tau_l))^{-1} & \textit{for}\ \ t<a,\\ \exp\Big(\int\limits_a^t P(\tau)\,d\tau\Big)\prod\limits_{a\le\tau_l<t}(I_n+G(\tau_l)) & \textit{for}\ \ t>a,\\ \qquad I_n & \textit{for}\ \ t=a.\end{cases} \tag{2.1.11}$$

Representation (2.1.11) follows from (1.2.68), due to (2.1.7).

Theorem 2.1.6. *Let the matrix-functions* $P\in L_{loc}(I;\mathbb{R}^{n\times n})$ *and* $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that inequality* (2.1.4) *holds for every* τ_l $(l=1,2,\dots)$*. Then there exists a unique* $n\times n$ *matrix-function* $U: I\times I\to\mathbb{R}^{n\times n}$ *such that the matrix function* $X(t)=U(t,s)$ *satisfies the matrix impulsive problem* (2.1.8), (2.1.9); (2.1.10) *for every* $s\in[a,b]$*. In addition, the matrix-function* $U(t,s)$ *has the following properties:*

(a) $U(t,t)=I_n$ *for* $t\in I$*;*

(b) *relation* (1.1.12) *holds for* $r,s,t\in I$*;*

(c) $U(t-,s)=U(t,s)$ *for* $t,s\in I$*;* $U(t+,s)=U(t,s)$ *for* $t\in I\setminus T$ *and* $U(\tau_l+,s)=(I_n+G(\tau_l))U(\tau_l,s)$ *for* $s\in I$ $(l=1,2,\dots)$*;*

(d) $\det(U(t,s))\neq 0$ *for* $s,t\in I$*;*

(e) *the matrices* $U(t,s)$ *and* $U(s,t)$ *are mutually reciprocal, i.e.,* $U^{-1}(t,s)=U(t,s)$ *for* $s,t\in I$*;*

(f) $U(t,s)=X(t)X^{-1}(s)$*, where* $X(t)=U(t,a)$ *for* $s,t\in I$*.*

The matrix-function U defined in the theorem is called the Cauchy matrix of impulsive system $(2.1.1_0),(2.1.2_0)$.

The matrix-function $X(t)=U(t,a)$ is called a fundamental matrix of the impulsive system $(2.1.1_0),(2.1.2_0)$.

Proposition 2.1.3. *Let inequality* (2.1.4) *hold for every* τ_l $(l=1,2,\dots)$*. Then*

$$\frac{dX^{-1}}{dt}=-X^{-1}\,P(t)\ \ \textit{for a.a.}\ \ t\in I\setminus T, \tag{2.1.12}$$

$$X^{-1}(\tau_l+)-X^{-1}(\tau_l-)=-X^{-1}(\tau_l)G(\tau_l)(I_n+G(\tau_l))^{-1}\ \ (l=1,2,\dots). \tag{2.1.13}$$

where $X\in L_{loc}(I;\mathbb{R}^{n\times n})$ *is a fundamental matrix of system* $(2.1.1_0),(2.1.2_0)$*.*

Equality (2.1.12) follows from the definition of the operator $\mathcal{A}$ (see (0.0.1)), equality (1.1.19) and (2.1.5).

Note that equality (2.1.13) follows immediately from condition (c) of Theorem 2.1.6, since

$$X(\tau_l+) = (I_n + G(\tau_l))X(\tau_l) \ \ (l = 1, 2, \dots).$$

Theorem 2.1.7 (Variation-of-constants formula)**.** *Let inequality* (2.1.4) *hold for every* τ_l $(l=1,2,\dots)$*. Then every solution of the impulsive system* (2.1.1), (2.1.2) *admits the representation*

$$x(t) = U(t,s)x(s) + \int\limits_s^t U(t,\tau)q(\tau)\,d\tau + \operatorname{sgn}(t-s) \sum_{\tau_l \in T_{s,t}} U(t,\tau_l)u_l \ \ \text{for} \ \ s,t \in I,$$

where U *is the Cauchy matrix of the homogeneous system* $(2.1.1_0), (2.1.2_0)$.

The equality follows from (1.1.16) if we take into account the integration-by-parts formula (see (0.0.9)) and equalities (0.0.11) and (2.1.5).

We give here a method of successive approximations for constructing the solution of the initial problem (2.1.1)–(2.1.3).

Theorem 2.1.8. *Let* x *be a unique solution of the initial problem* (2.1.1)–(2.1.3)*. Then condition* (1.1.20) *holds for every* $[a,b] \subset I$*, where*

$$x_k(t_0) = c_0 \ \ (k = 0, 1, \dots);$$

$$x_0(t) = c_0 \ \ \text{and} \ \ x_k(t) = c_0 + \int\limits_{t_0}^t \big(P(\tau)x_{k-1}(\tau) + q(\tau)\big)\,d\tau$$
$$+ \operatorname{sgn}(t-t_0) \sum_{\tau_l \in T_{t_0,t}} \big(G(\tau_l)x_{k-1}(\tau_l) + u(\tau_l)\big) \ \ \text{for} \ \ t < t_0, \ \ t \notin T \ \ \text{or} \ \ t > t_0 \ \ (k = 1, 2, \dots)$$

and

$$x_0(\tau_m) = (I_n + G(\tau_m))^{-1}c_0 \ \ \text{and} \ \ x_k(\tau_m) = (I_n + G(\tau_l))^{-1}\bigg\{c_0 - \int\limits_{\tau_l}^{t_0} \big(P(\tau)x_{k-1}(\tau) + q(\tau)\big)\,d\tau$$
$$- \sum_{\tau_m \le \tau_l < t_0} \big(G(\tau_l)x_{k-1}(\tau_l) + u(\tau_l)\big) + G(\tau_m)x_{k-1}(\tau_m)\bigg\} \ \ \text{for} \ \ \tau_m < t_0 \ \ (m = 1, 2, \dots; \ k = 1, 2, \dots).$$

2.1.1 Nonnegativity of the Cauchy matrix. The systems of linear differential and integral impulsive inequalities

In this subsection, we establish the sufficient conditions guaranteeing the nonnegativity of the Cauchy matrix of system $(2.1.1_0), (2.1.2_0)$. Moreover, we investigate the question of the estimates of solutions of linear systems of differential and integral inequalities.

Theorem 2.1.9. *Let* $t_0 \in I$, $P = (p_{ik})_{i,k=1}^n \in L_{loc}(I;\mathbb{R}^{n\times n})$, $G = (g_{ik})_{i,k=1}^n \in B_{loc}(T;\mathbb{R}^{n\times n})$ *and* $Q = \operatorname{diag}(\alpha_1,\dots,\alpha_n) \in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that conditions* (2.1.4),

$$1 + \alpha_i(\tau_l) > 0 \ \ \text{for} \ \ \tau_l \le t_0 \ \ (l = 1, 2, \dots),$$
$$\det\big(I_n + \widetilde{G}(\tau_l) + Q(\tau_l)\big) \ne 0 \ \ \text{for} \ \ \tau_l < t_0 \ \ (l = 1, 2, \dots) \tag{2.1.14}$$

and

$$\big(I_n + \widetilde{G}(\tau_l) + Q(\tau_l)\big)^{-1} \ge 0 \ \ \text{for} \ \ \tau_l < t_0 \ \ (l = 1, 2, \dots)$$

hold, where $\widetilde{G}(\tau_l) = G(\tau_l) - \operatorname{diag}(G(\tau_l))$*. Let, moreover,*

$$p_{ik}(t)\operatorname{sgn}(t - t_0) \ge 0 \ \ \text{for a.a.} \ \ t \in I \setminus T \ \ (i \ne k; \ i,k = 1,\dots,n)$$

and

$$g_{ik}(l)\,\mathrm{sgn}(\tau_l-t_0)\geq 0 \;\; (i\neq k;\; i,k=1,\dots,n;\; l=1,2,\dots).$$

Then condition (1.1.25) *holds, where U is the Cauchy matrix of system* $(2.1.1_0),(2.1.2_0)$.

Remark 2.1.3. The condition

$$\|G(\tau_l)\|<1 \;\text{ for }\; \tau_l<t_0 \;\; (l=1,2,\dots) \tag{2.1.15}$$

guarantees the validity of condition (2.1.4). If

$$\big\|(I_n+G(\tau_l))^{-1}\big(Q(\tau_l)-\mathrm{diag}(G(\tau_l))\big)\big\|<1 \;\text{ for }\; \tau_l<t_0 \;\; (l=1,2,\dots),$$

then condition (2.1.14) follows from (2.1.4). If the condition

$$\widetilde{G}(\tau_l)+Q(\tau_l)\leq O_{n\times n} \;\text{ for }\; \tau_l<t_0 \;\; (l=1,2,\dots)$$

holds together with (2.1.14), then condition (1.1.27) holds, as well. If $Q(\tau_l)=\mathrm{diag}(G(\tau_l))$ $(l=1,2,\dots)$, then condition (2.1.14) coincides with (2.1.4).

Theorem 2.1.10. *Let $t_0\in I$, $P=(p_{ik})_{i,k=1}^n\in L_{loc}(I;\mathbb{R}^{n\times n})$, $G=(g_{ik})_{i,k=1}^n\in B_{loc}(T;\mathbb{R}^{n\times n})$, $q\in L_{loc}(I;\mathbb{R}^{n\times n})$ and $u\in B_{loc}(T;\mathbb{R}^n)$ be such that the conditions*

$$\det(I_n-G(\tau_l))\neq 0 \;\textit{ for }\; \tau_l<t_0 \;\; (l=1,2,\dots), \tag{2.1.16}$$

$$1-g_{ii}(\tau_l)>0 \;\textit{ for }\; \tau_l\leq t_0 \;\; (l=1,2,\dots),$$

$$(I_n-G(\tau_l))^{-1}\geq O_{n\times n} \;\textit{ for }\; \tau_l<t_0 \;\; (l=1,2,\dots), \tag{2.1.17}$$

$$p_{ik}(t)\geq 0 \;\textit{ for a.a. }\; t\in I\setminus\{t_0\}\setminus T \;\; (i\neq k;\; i,k=1,\dots,n)$$

and

$$g_{ik}(\tau_l)\geq 0 \;\; (i\neq k;\; i,k=1,\dots,n;\; l=1,2,\dots)$$

hold. Let, moreover, a vector-function x satisfy the system of linear impulsive inequalities

$$\mathrm{sgn}(t-t_0)\frac{dx}{dt}\leq P(t)x+q(t) \;\textit{ for a.a. }\; t\in I\setminus T,$$

$$\mathrm{sgn}(\tau_l-t_0)(x(\tau_l+)-x(\tau_l-))\leq G(\tau_l)x(\tau_l)+u(\tau_l) \;\; (l=1,2,\dots)$$

on the intervals J_1 and J_2 and

$$\begin{gathered} x(t_0)\leq c_0 \;\textit{ if }\; t_0\notin\{\tau_1,\tau_2,\dots\},\\ x(\tau_{l_0})\leq (I_n+G(\tau_{l_0}))\,c_0+u(\tau_{l_0}) \;\textit{ if }\; t_0=\tau_{l_0} \;\textit{ for some natural }\; l_0, \end{gathered} \tag{2.1.18}$$

where $c_0\in\mathbb{R}^n$. Then the estimate

$$x(t)\leq y(t) \;\textit{ for a.a. }\; t\in I\setminus\{t_0\} \tag{2.1.19}$$

holds, where y is a solution of the impulsive system

$$\mathrm{sgn}(t-t_0)\frac{dy}{dt}=P(t)y+q(t) \;\textit{ for a.e. }\; t\in I\setminus T,$$

$$\mathrm{sgn}(\tau_l-t_0)\big(y(\tau_l+)-y(\tau_l-)\big)=G(\tau_l)y(\tau_l)+u(\tau_l) \;\; (l=1,2,\dots)$$

on the intervals J_1 and J_2 satisfying the conditions

$$y(t_0)=c_0 \;\textit{ if }\; t_0\notin\{\tau_1,\tau_2,\dots\},$$

and

$$y(\tau_{l_0})=(I_n+G(\tau_{l_0}))\,c_0+u(\tau_{l_0}) \;\textit{ if }\; t=\tau_{l_0}.$$

Remark 2.1.4. It is evident that if in Theorem 2.1.10 we assume

$$x(t_0) \le c_0,$$

then inequality (2.1.19) is fulfilled on the whole I.

Remark 2.1.5. If estimate (2.1.15) holds and

$$G(\tau_l) \ge O_{n\times n} \ \ (l = 1, 2, \dots),$$

then condition (2.1.17) holds, as well.

Theorem 2.1.11. *Let $t_0 \in I$, $P = (p_{ik})_{i,k=1}^n \in L_{loc}(I; \mathbb{R}^{n\times n})$, $G = (g_{ik})_{i,k=1}^n \in B_{loc}(T; \mathbb{R}^{n\times n})$, $q \in L_{loc}(I; \mathbb{R}^{n\times n})$ and $u \in B_{loc}(T; \mathbb{R}^n)$ be such that conditions* (2.1.16), (2.1.17) *hold. Let, moreover, x be a solution of linear impulsive integral inequality*

$$x(t) \le c_0 + \left(\int\limits_{t_0}^{t} \big(P(\tau)x(\tau) + q(\tau)\big)\, d\tau + \sum_{\tau_l \in T_{[t_0,t[}} \big(G(\tau_l)x(\tau_l) + u(\tau_l)\big) \right) \cdot \operatorname{sgn}(t - t_0)$$

on the sets J_1 and J_2, under condition (2.1.18)*. Then the conclusion of Theorem* 2.1.10 *is true.*

2.2 The well-posedness of the initial problem

2.2.1 Statement of the problem and formulation of the results

Let $P_0 \in L_{loc}(I; \mathbb{R}^{n\times n})$, $q_0 \in L_{loc}(I; \mathbb{R}^n)$, $G_0 \in B_{loc}(T; \mathbb{R}^{n\times n})$, $u_0 \in B_{loc}(T; \mathbb{R}^n)$, $\tau_l \ne \tau_k$ if $l \ne k$ $(l, k = 1, \dots, n)$, $t_0 \in I$, where $I \subset \mathbb{R}$ is an arbitrary interval non-degenerated at a point. Consider the system

$$\frac{dx}{dt} = P_0(t)x + q_0(t) \ \text{ for a.a. } \ t \in I \setminus \{\tau_l\}_{l=1}^{\infty}, \tag{2.2.1}$$

$$x(\tau_l+) - x(\tau_l-) = G_0(\tau_l)x(\tau_l) + u_0(\tau_l) \ \ (l = 1, 2, \dots) \tag{2.2.2}$$

under the initial condition

$$x(t_0) = c_0, \tag{2.2.3}$$

where $c_0 \in \mathbb{R}^n$ is an arbitrary constant vector.

Let x_0 be a unique solution of problem (2.2.1)–(2.2.3).

Along with the initial problem (2.2.1)–(2.2.3), consider the sequence of the initial problems

$$\frac{dx}{dt} = P_k(t)x + q_k(t) \ \text{ for a.a. } \ t \in I \setminus \{\tau_l\}_{l=1}^{\infty}, \tag{2.2.1$_k$}$$

$$x(\tau_l+) - x(\tau_{l-}) = G_k(\tau_l)x(\tau_l) + u_k(\tau_l) \ \ (l = 1, 2, \dots), \tag{2.2.2$_k$}$$

$$x(t_k) = c_k \tag{2.2.3$_k$}$$

$(k = 1, 2, \dots)$, where $P_k \in L_{loc}(I; \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$, $q_k \in L_{loc}(I; \mathbb{R}^n)$ $(k = 1, 2, \dots)$, $G_k \in B_{loc}(T; \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$, $u_k \in B_{loc}(T; \mathbb{R}^n)$, $t_k \in I$ $(k = 1, 2, \dots)$, $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$.

We assume that $P_k = (p_{kij})_{i,j=1}^n$ $(k = 0, 1, \dots)$, $q_k = (q_{ki})_{i=1}^n$ $(k = 0, 1, \dots)$; $G_k = (g_{kij})_{i,j=1}^n$ $(k = 0, 1, \dots)$, $u_k = (u_{ki})_{i=1}^n$ $(k = 0, 1, \dots)$, and, without loss of generality, either $t_k < t_0$ $(k = 1, 2, \dots)$, or $t_k = t_0$ $(k = 1, 2, \dots)$, or $t_k > t_0$ $(k = 1, 2, \dots)$.

In this section, we establish the necessary and sufficient and effective sufficient conditions for the initial problem (2.2.1$_k$)–(2.2.3$_k$) to have a unique solution x_k for any sufficiently large k and condition (1.2.3) would be satisfied uniformly on I.

Remark 2.2.1. If we consider the case where for every natural k, the impulses points depend on k in the impulsive system $(2.2.1_k), (2.2.2_k)$, in particular, the linear algebraic system $(2.2.2_k)$ has the form

$$x(\tau_{kl}+) - x(\tau_{kl}-) = G_k(\tau_{kl})x(\tau_{kl}) + u_k(\tau_{kl}) \ \ (l = 1, 2, \dots),$$

where $\tau_{kl} \in I$ $(l = 1, 2, \dots)$, then the last general case will be reduced to case $(2.2.2_k)$ by using the following conception.

Let $T = T_0 \cup T_1 \cup T_2 \cup \dots$, where $T_k = \{\tau_{k1}, \tau_{k2}, \dots\}$ $(k = 0, 1, \dots)$, and $\tau_{0l} = \tau_l$ $(l = 1, 2, \dots)$. The set T is countable. Therefore $T = \{\tau_1^*, \tau_2^*, \dots\}$, where $\tau_l^* \in I$ $(l = 1, 2, \dots)$. For every $k \in \{0, 1, \dots\}$ and $l \in \{1, 2, \dots\}$, we set $G_k^*(\tau_l^*) = G_k^*(\tau_{kl})$ and $u_k^*(\tau_l^*) = u_k^*(\tau_{kl})$ if $\tau_l^* \in T_k$, where $l_k \in \mathbb{N}$ is such that $\tau_l^* = \tau_{kl_k}$, and $G_k^*(\tau_l^*) = O_{n\times n}$ and $u_k^*(\tau_l^*) = 0$ if $\tau_l^* \notin T_k$. So, the last general case is equivalent to the impulsive system $(2.2.1_k), (2.2.2_k)$, where $\tau_l = \tau_l^*$ $(l = 1, 2, \dots)$, $G_k(\tau_l) = G_k^*(\tau_l^*)$ $(l = 1, 2, \dots)$ and $u_k(\tau_l) = u_k^*(\tau_l^*)$ $(l = 1, 2, \dots)$.

Below, as in Section 2.1, we assume that $T = \{\tau_1, \tau_2, \dots\}$.

Along with systems $(2.2.1), (2.2.2)$ and $(2.2.1_k), (2.2.2_k)$, we consider the corresponding homogeneous systems

$$\frac{dx}{dt} = P_0(t)x \ \text{ for a.a. } \ t \in I \setminus T, \tag{$2.2.1_0$}$$

$$x(\tau_l+) - x(\tau_l-) = G_0(\tau_l)x(\tau_l) \ \ (l = 1, 2, \dots) \tag{$2.2.2_0$}$$

and

$$\frac{dx}{dt} = P_k(t)x \ \text{ for a.a. } \ t \in I \setminus T, \tag{$2.2.1_{k0}$}$$

$$x(\tau_l+) - x(\tau_l-) = G_k(\tau_l)x(\tau_l) \ \ (l = 1, 2, \dots) \tag{$2.2.2_{k0}$}$$

$(k = 1, 2, \dots)$.

Definition 2.2.1. We say that the sequence $(P_k, q_k; G_k, u_k; t_k)$ $(k = 1, 2, \dots)$ belongs to the set $\mathcal{S}(P_0, q_0; G_0, u_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ satisfying the condition (1.2.5), problem $(2.2.1_k)$–$(2.2.3_k)$ has a unique solution x_k for any sufficiently large k and condition (1.2.3) holds uniformly on I.

As in the previous section, the impulsive systems $(2.2.1), (2.2.2)$ and $(2.2.1_k), (2.2.2_k)$ $(k = 1, 2, \dots)$ are the particular case, respectively, of the general systems (1.2.1) and $(1.2.1_k)$ $(k = 1, 2, \dots)$ if we set

$$\begin{aligned} A_k(t) &= \int\limits_{t_k}^{t} P_k(\tau)\,d\tau + \operatorname{sgn}(t - t_k) \sum_{\tau_l \in T_{t_k,t}} G_k(\tau_l) \ \text{ for } \ t \in I \ \ (k = 0, 1, \dots), \\ f_k(t) &= \int\limits_{t_k}^{t} q_k(\tau)\,d\tau + \operatorname{sgn}(t - t_k) \sum_{\tau_l \in T_{t_k,t}} u_k(\tau_l) \ \text{ for } \ t \in I \ \ (k = 0, 1, \dots), \end{aligned} \tag{2.2.4}$$

where $a \in I$ is an arbitrary fixed point.

Let

$$\begin{aligned} &c_{k1} = c_k \ \ (k = 0, 1, \dots), \\ &c_{k2} = c_k \ \text{ if } \ t_k \geq t_0, \ \ t_k \notin \{\tau_1, \tau_2, \dots\} \ \ (k = 0, 1, \dots), \\ &c_{k2} = (I_n + G_k(\tau_l))\,c_k + u_k(\tau_l) \ \text{ if } l \text{ is such that } \ t_k = \tau_l \ \ (k = 0, 1, \dots). \end{aligned} \tag{2.2.5}$$

To realize and formulate the well-posed results of Section 1.2, we use the following forms of the operators $\mathcal{B}(X, Y)$ and $\mathcal{I}(X, Y)$ (see (0.0.2) and (0.0.3)) for the impulsive case, in particular, when the matrix-functions X and Y are continuous from the left on I. Using integration-by-parts formula (0.0.9), (0.0.11) and the definition of the Kurzweil integral, we find

$$\begin{aligned} &\mathcal{B}(X, Y)(t) - \mathcal{B}(X, Y)(s) \\ &\quad = \int\limits_{s}^{t} X(\tau)Y'(\tau)\,d\tau + \operatorname{sgn}(t - s) \sum_{\tau_l \in T_{s,t}} X(\tau_l+)\,d_2Y(\tau_l) \ \text{ for } \ s < t, \ s, t \in I, \end{aligned} \tag{2.2.6}$$

if $X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times j})$ and $Y \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{j\times m})$, and

$$\mathcal{I}(X,Y)(t) - \mathcal{I}(X,Y)(s) = \int\limits_s^t \big(X'(\tau) + X(\tau)Y'(\tau)\big)X^{-1}(\tau)\,d\tau \\ + \operatorname{sgn}(t-s)\sum_{\tau_l\in T_{s,t}} \big(d_2X(\tau_l) + X(\tau_l+)d_2Y(\tau_l)\big)X^{-1}(\tau_l) \ \text{ for } \ s<t, \ \ s,t\in I, \tag{2.2.7}$$

if $X, Y \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$, $\det X(t) \neq 0$. In addition, if

$$Q(t) - Q(s) = \int\limits_s^t Y(\tau)\,d\tau + \operatorname{sgn}(t-s)\sum_{\tau_l\in T_{s,t}} Z(\tau_l) \ \text{ for } \ s<t; \ s,t\in I,$$

where $Y \in L_{loc}(I;\mathbb{R}^{n\times m})$ and $Z \in B_{loc}(T;\mathbb{R}^{n\times n})$, we set

$$\mathcal{B}_\iota(X;Y,Z)(t) \equiv \mathcal{B}(X,Q)(t) \ \text{ and } \ \mathcal{I}_\iota(X;Y,Z)(t) \equiv \mathcal{I}(X,Q)(t).$$

Consequently,

$$\begin{aligned} \mathcal{B}_\iota(X;Y,Z)(t) &\equiv \int\limits_a^t X(\tau)Y(\tau)\,d\tau + \operatorname{sgn}(t-a)\sum_{\tau_l\in T_{a,t}} X(\tau_l+)\,Z(\tau_l) \ \text{ and} \\ \mathcal{I}_\iota(X;Y,Z)(t) &\equiv \int\limits_a^t \big(X'(\tau) + X(\tau)Y(\tau)\big)X^{-1}(\tau)\,d\tau \\ &\quad + \operatorname{sgn}(t-a)\sum_{\tau_l\in T_{a,t}} \big(d_2X(\tau_l) + X(\tau_l+)d_2Z(\tau_l)\big)X^{-1}(\tau_l), \end{aligned} \tag{2.2.8}$$

where $a \in I$ is a fixed point.

Analogously to equalities (0.0.5) and (0.0.4), we introduce the operators

$$\mathcal{D}_{\mathcal{B}_\iota}(Y_1,Z_1,X_1;Y_2,Z_2,X_2)(t) \equiv \mathcal{B}_\iota(X_1,Y_1,Z_1)(t) - \mathcal{B}_\iota(X_2,Z_2,Z_2)(t)$$

and

$$\mathcal{D}_{\mathcal{I}_\iota}(Y_1,Z_1,X_1;Y_2,Z_2,X_2)(t) \equiv \mathcal{I}_\iota(X_1,Y_1,Z_1)(t) - \mathcal{I}_\iota(X_2,Z_2,Z_2)(t).$$

Note that if $X(t) \equiv I_n$, then

$$\mathcal{B}_\iota(I_n;Y,Z)(t) \equiv \int\limits_a^t Y(\tau)\,d\tau + \operatorname{sgn}(t-a)\sum_{\tau_l\in T_{a,t}} Z(\tau_l).$$

Therefore, by (2.2.4),

$$A_k(t) \equiv \mathcal{B}_\iota(I_n;P_k,G_k)(t) - B_i(I_n;P_k,G_k)(t_k) \ \text{ and} \\ f_k(t) \equiv \mathcal{B}_\iota(I_n;q_k,u_k)(t) - \beta_i(I_n;q_k,u_k)(t_k) \ \ (k=0,1,\dots).$$

Theorem 2.2.1. *Let $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $G_0 \in B(T;\mathbb{R}^{n\times n})$, $u_0 \in B(T;\mathbb{R}^n)$, $\tau_l \in I$ $(l=1,2,\dots)$, $t_0 \in I$ and a sequence of points $t_k \in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.9) *and*

$$\det(I_n + G_0(\tau_l)) \neq 0 \ \ \textit{if} \ \ \tau_l < t_0 \ \ \textit{and} \\ \textit{for} \ \ l=l_0 \ \ \textit{if} \ \ t_0=\tau_{l_0} \ \ \textit{and} \ \ t_k>t_0 \ \ \textit{for every} \ \ k\in\{1,2,\dots\} \ \ (l=1,2,\dots) \tag{2.2.9}$$

hold. Then

$$\big((P_k,q_k;G_k,u_k;t_k)\big)_{k=1}^{\infty} \in \mathcal{S}(P_0,q_0;G_0,u_0;t_0) \tag{2.2.10}$$

if and only if there exists a sequence of matrix-functions $H_k \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that condition (1.2.11) *holds, and conditions* (1.2.12),

$$\lim_{k\to+\infty}\bigg\{\bigg\|\mathcal{D}_{\mathcal{I}_\iota}(P_k,G_k,H_k;P_0,G_0,H_0)(\tau)\bigg|_{t_k}^{t}\bigg\|\bigg(1+\bigg|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}_\iota}(P_k,G_k,H_k;P_0,G_0,H_0)\big)\bigg|\bigg)\bigg\}=0 \quad (2.2.11)$$

and

$$\lim_{k\to+\infty}\bigg\{\bigg\|\mathcal{D}_{\mathcal{B}_\iota}(q_k,u_k,H_k;q_0,u_0,H_0)(\tau)\bigg|_{t_k}^{t}\bigg\|\bigg(1+\bigg|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}_\iota}(P_k,G_k,H_k;P_0,G_0,H_0)\big)\bigg|\bigg)\bigg\}=0 \quad (2.2.12)$$

hold uniformly on I.

Note that in Theorem 2.2.1, due to (2.2.6), (2.2.7) and (2.2.8), we have

$$\begin{aligned}\mathcal{I}_\iota(H_k;P_k,G_k)(t)\equiv\int_a^t\big(H_k'(\tau)+H_k(\tau)P_k(\tau)\big)H_k^{-1}(\tau)\,d\tau\\ +\operatorname{sgn}(t-a)\sum_{\tau_l\in T_{a,t}}\big(d_2H_k(\tau_l)+H_k(\tau_l+)G_k(\tau_l)\big)H_k^{-1}(\tau_l)\ \ (k=0,1,\dots)\end{aligned} \quad (2.2.13)$$

and

$$\mathcal{B}_\iota(H_k;q_k,u_k)(t)\equiv\int_a^t H_k(\tau)\,q_k(\tau)\,d\tau+\operatorname{sgn}(t-a)\sum_{\tau_l\in T_{a,t}}H_k(\tau_l+)u_k(\tau_l\ \ (k=0,1,\dots). \quad (2.2.14)$$

Theorem 2.2.2. *Let $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)$, $\tau_l\in I$ $(l=1,2,\dots)$, $t_0\in I$ and a sequence of points $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.9) *and* (2.2.9) *hold. Let, moreover, the sequences $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $G_k\in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $u_k\in B_{loc}(T;\mathbb{R}^n)$ $(k=1,2,\dots)$, and bounded sequence of constant vectors $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ be such that conditions* $(1.2.5_j)$, (1.2.15) *and* (1.2.16) *hold if $j\in\{1,2\}$ is such that $(-1)^j(t_k-t_0)\ge 0$ for every $k\in\{1,2,\dots\}$, where c_{kj} $(k=0,1,\dots)$ are defined by* (2.2.5),

$$\begin{gathered}A_{k1}(t)\equiv -A_k(t),\ \ f_{k1}(t)\equiv -f_k(t)\ \ (k=0,1,\dots);\\ A_{k2}(t)\equiv A_k(t),\ \ \ f_{k2}\equiv f_k(t)\ \ \textit{if}\ t_k\notin\{\tau_1,\tau_2,\dots\}\ \ (k=0,1,\dots);\\ A_{k2}(t)\equiv A_k(t)-G_k(\tau_l),\ \ f_{k2}(t)\equiv f_k(t)-u_k(\tau_l)\ \ \textit{if}\ t_k=\tau_l\ \textit{for some}\ l\ \ (k=0,1,\dots),\end{gathered} \quad (2.2.15)$$

and $A_k(t)$ and $f_k(t)$ are defined by (2.2.4). *Then the initial problem* $(2.2.1_k)$–$(2.2.3_k)$ *has the unique solution x_k for any sufficiently large k and* (1.2.17) *holds.*

Note that in Theorem 2.2.2, we have

$$\bigg|\bigvee_{t_k}^{t}(A_k)\bigg|\equiv\bigg|\int_{t_k}^{t}\|P_k(\tau)\|\,d\tau\bigg|+\sum_{\tau_l\in T_{t_k,t}}\|G_k(\tau_l)\|\ \ (k=0,1,\dots).$$

Remark 2.2.2. In Theorem 2.2.2, it is evident that the sequence $x_k(t)$ $(k=1,2,\dots)$ converges to x_0 uniformly on the set $\{t\in I, t\le t_0\}$ if $t_k>t_0$ $(k=1,2,\dots)$, and on the set $\{t\in I, t\ge t_0\}$ if $t_k<t_0$ $(k=1,2,\dots)$. Moreover, in Theorem 2.2.2, if conditions (1.2.15) and (1.2.16) hold uniformly on the set I instead of sets I_{t_k} $(k=1,2,\dots)$, then these conditions are equivalent, respectively, to the limit equalities (1.2.18) and (1.2.19) uniformly on I, since, due to (2.2.4), the matrix- and vector-functions A_k $(k=0,1,\dots)$ and f_k $(k=0,1,\dots)$ satisfy the equalities given in condition (2.1.6). In addition, equalities (1.2.7) hold and, therefore, in view of (1.2.5) and (1.2.6), conditions $(1.2.5_j)$ $(j=1,2)$ hold, as well. Thus, in this case, condition (1.2.3) holds uniformly on I.

Theorem 2.2.3. *Let $P_0^* \in L(I;\mathbb{R}^{n\times n})$, $q_0^* \in L(I;\mathbb{R}^n)$, $G_0^* \in B(T;\mathbb{R}^{n\times n})$, $u_0^* \in B(T;\mathbb{R}^n)$, $\tau_l \in I$ $(l=1,2,\dots)$, $c_0^* \in \mathbb{R}^n$, $t_0 \in I$ and a sequence of points $t_k \in I$ $(k=1,2,\dots)$ be such that condition* (1.2.9) *and*

$$\det(I_n + G_0^*(\tau_l)) \neq 0 \ \text{ if } \tau_l < t_0 \ \text{ and } \text{ for } l = l_0 \ \text{ if } t_0 = \tau_{l_0} \ \text{ and } t_k > t_0 \ \text{ for every } k \in \{1,2,\dots\} \ (l=1,2,\dots)$$

hold and the initial problem

$$\frac{dx}{dt} = P_0^*(t)x + q_0^*(t) \ \text{ for a.a. } t \in I\setminus T, \tag{2.2.16}$$

$$x(\tau_l+) - x(\tau_l-) = G_0^*(\tau_l)x(\tau_l) + u_0^*(\tau_l) \ \ (l=1,2,\dots); \tag{2.2.17}$$

$$x(t_0) = c_0^*$$

has a unique solution x_0^*. Let, moreover, the sequences of matrix-functions $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $G_k \in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $H_k \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$, of vector-functions $q_k \in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $u_k \in B_{loc}(T;\mathbb{R}^n)$ $(k=1,2,\dots)$ and $h_k \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^n)$ $(k=0,1,\dots)$, and the bounded sequence of constant vectors $c_k^* \in \mathbb{R}^n$ $(k=1,2,\dots)$ be such that conditions (1.2.23) *and* (1.2.24) *hold, and* (1.2.25) *and* (1.2.26) *be fulfilled for the matrix-functions A_{kj}^*, A_k^* $(k=0,1,\dots)$ and vector-functions f_{kj}^* $(k=0,1,\dots)$ if $j\in\{1,2\}$ is such that $(-1)^j(t_k-t_0)\geq 0$ for every $k\in\{1,2,\dots\}$, where c_{kj}^* $(j=1,2;\ k=0,1,\dots)$ and A_{kj}^* and f_{kj}^* $(j=1,2;\ k=0,1,\dots)$ are defined, respectively, analogously to* (2.2.5) *and* (2.2.15),

$$c_k^* = H_k(t_k)c_k + h_k(t_k), \quad A_k^*(t) \equiv \mathcal{I}_\iota(H_k;P_k,G_k)(t) \ \ (k=1,2,\dots),$$

$$f_k^*(t) = h_k(t) - h_k(t_k) + \mathcal{B}_\iota(H_k;q_k,u_k)(t) - \mathcal{B}_\iota(H_k;q_k,u_k)(t_k) - \int_{t_k}^{t} dA_k^*(\tau)\cdot h_k(\tau) \ \ (k=1,2,\dots),$$

and $\mathcal{I}_\iota(H_k;P_k,G_k)(t)$ and $\mathcal{B}_\iota(H_k;q_k,u_k)(t)$ are defined by (2.2.13) *and* (2.2.14)*, respectively. Then problem* $(2.2.1_k)$–$(2.2.3_k)$ *has the unique solution x_k for any sufficiently large k and* (1.2.27) *holds.*

Remark 2.2.3. In Theorem 2.2.3, the vector-function $x_k^*(t) \equiv H_k(t)x_k(t) + h_k(t)$ is a solution of the problem

$$\frac{dx}{dt} = P_k^*(t)x + q_k^*(t) \ \text{ for a.a. } t \in I\setminus T,$$

$$x(\tau_l+) - x(\tau_l-) = G_k^*(\tau_l)x(\tau_l) + u_k^*(\tau_l) \ \ (l=1,2,\dots);$$

$$x(t_k) = c_k^*$$

for every sufficiently large k, where

$$P_k^*(t) \equiv \big(H_k'(t) + H_k(t)P_k^*(t)\big)H_k^{-1}(t),$$

$$G_{kl}^*(\tau_l) = \big(d_2H_k(\tau_l) + H_k(\tau_l+)G_{kl}(\tau_l)\big)H_k^{-1}(\tau_l) \ \ (k=0,1,\dots;\ l=1,2,\dots);$$

$$q_k^*(t) \equiv h_k'(t) + H_k(t)\,q_k(t) - P_k^*(t)h_k(t) \ \ (k=1,2,\dots),$$

$$g_{kl}^*(\tau_l) = d_2h_k(\tau_l) + H_k(\tau_l+)u_{kl}(\tau_l) - G_{kl}^*(\tau_l)h_k(\tau_l) \ \ (k=1,2,\dots;\ l=1,2,\dots).$$

Below, as in Chapter 2, we consider, mainly, the question on the well-posedness only on the whole interval I. For the last case, instead of conditions (1.2.25) and (1.2.26), we assume that the limit equalities (1.2.29) and (1.2.30), where the circumscribed matrix- and vector-functions are defined as above in this section, hold uniformly on the whole interval I.

Corollary 2.2.1. *Let $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $G_0 \in B(T;\mathbb{R}^{n\times n})$, $u_0 \in B(T;\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$, $t_0 \in I$ and the sequences $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k \in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $G_k \in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $u_k \in B_{loc}(T;S\mathbb{R}^n)$ $(k=1,2,\dots)$, $c_k \in \mathbb{R}^n$ $(k=1,2,\dots)$ and*

$t_k \in I$ $(k = 1, 2, \dots)$ *be such that conditions* (1.2.9), (1.2.11), (1.2.24) *and* (2.2.9) *hold, and conditions* (1.2.12), (2.2.11) *and*

$$\lim_{k\to+\infty}\bigg\{\bigg\|\mathcal{D}_{\mathcal{B}_\iota}(q_k-\varphi_k',u_k-s_2(\varphi_k),H_k;q_0,u_0,H_0)(\tau)\Big|_{t_k}^{t} + \int_{t_k}^{t} d\mathcal{I}_\iota(H_k;P_k,G_k)(\tau)\cdot\varphi_k(\tau)\bigg\|\Big(1+\Big|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}_\iota}(P_k,G_k,H_k;P_0,G_0,H_0)\big)\Big|\Big)\bigg\}=0$$

hold uniformly on I*, where* $H_k \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$, $\varphi_k \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^n)$ $(k = 1, 2, \dots)$. *Then problem* $(2.2.1_k)$–$(2.2.3_k)$ *has the unique solution* x_k *for any sufficiently large* k *and condition* (1.2.33) *holds uniformly on* I.

Below, as in Section 2.1, we will give certain sufficient conditions guaranteeing inclusion (2.2.10). Towards this end, we state a theorem, different from Theorem 2.2.1, concerning the necessary and sufficient conditions for the inclusion, as well.

Theorem 2.2.1′. *Let* $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $G_0 \in B(T;\mathbb{R}^{n\times n})$, $u_0 \in B(T;\mathbb{R}^n)$, $t_0 \in I$ *and a sequence of points* $t_k \in I$ $(k = 1, 2, \dots)$ *be such that conditions* (1.2.9) *and* (2.2.9) *hold. Then inclusion* (2.2.10) *holds if and only if there exists a sequence of matrix-functions* $H_k \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$ *such that conditions* (1.2.11) *and*

$$\limsup_{k\to+\infty}\bigg(\int_I \big\|H_k'(t)+H_k(t)P_k(t)\big\|\,dt+\sum_{\tau_l\in T}\big\|H_k(\tau_l+)-H_k(\tau_l)+H_k(\tau_l+)G_k(\tau_l)\big\|\bigg)<+\infty \tag{2.2.18}$$

hold, and conditions (1.2.12),

$$\lim_{k\to+\infty}\bigg(\int_{t_k}^{t} H_k(\tau)P_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}H_k(\tau_l+)\,G_k(\tau_l)\bigg) = \int_{t_0}^{t} H_0(\tau)P_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}H_0(\tau_l+)G_0(\tau_l)$$

and

$$\lim_{k\to+\infty}\bigg(\int_{t_k}^{t} H_k(\tau)q_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}H_k(\tau_l+)\,u_k(\tau_l)\bigg) = \int_{t_0}^{t} q_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}H_0(\tau_l+)u_0(\tau_l)$$

hold uniformly on I.

Remark 2.2.4. As in Remark 1.2.4, conditions (2.2.11) and (2.2.12) in Theorem 2.2.1 are fulfilled uniformly on I.

Theorem 2.2.1″. *Let* $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $G_0 \in B(T;\mathbb{R}^{n\times n})$, $u_0 \in B(T;\mathbb{R}^n)$, $\tau_l \in I$ $(l = 1, 2, \dots)$, $t_0 \in I$ *and the sequence of points* $t_k \in I$ $(k = 1, 2, \dots)$ *be such that conditions* (1.2.9) *and*

$$\det(I_n+G_k(\tau_l))\neq 0 \ \ (l,k=1,2,\dots)$$

hold. Then inclusion (2.2.10) *holds if and only if conditions* (1.2.37) *and*

$$\lim_{k\to+\infty}\big(\mathcal{B}_\iota(X_k^{-1};q_k,u_k)(t)-\mathcal{B}_\iota(X_k^{-1};q_k,u_k)(t_k)\big)=\mathcal{B}_\iota(X_0^{-1};q_0,u_0)(t)-\mathcal{B}_\iota(X_0^{-1};q_0,u_0)(t_0)$$

hold uniformly on I, where X_k is the fundamental matrix of the homogeneous system $(2.2.1_{k0}), (2.2.1_{k0})$ *for every $k \in \{0,1,\dots\}$, and $\mathcal{B}_\iota(X_k^{-1}; q_k, u_k)(t)$ $(k = 0,1,\dots)$ are defined by* (2.2.14).

Theorem 2.2.2′. *Let $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $G_0 \in B(T;\mathbb{R}^{n\times n})$, $u_0 \in B(T;\mathbb{R}^n)$, $\tau_l \in I$ $(l = 1,2,\dots)$, $c_0 \in \mathbb{R}^n$, $t_0 \in I$ and a sequence of points $t_k \in I$ $(k = 1,2,\dots)$ be such that conditions* (1.2.9) *and* (2.2.9) *hold. Let, moreover, the sequences $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $q_k \in L_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$, $G_k \in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $u_k \in B_{loc}(T;\mathbb{R}^n)$ $(k = 1,2,\dots)$ and $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ be such that conditions* (1.2.5) *and*

$$\limsup_{k\to+\infty}\bigg(\int\limits_I \|P_k(t)\|\,dt + \sum_{l=1}^{\infty}\|G_k(\tau_l)\|\bigg) < +\infty$$

hold, and conditions

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t} P_k(\tau)\,d\tau + \operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}} G_k(\tau_l)\bigg) = \int\limits_{t_0}^{t} P_0(\tau)\,d\tau + \operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}} G_0(\tau_l)$$

and

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t} q_k(\tau)\,d\tau + \operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}} u_k(\tau_l)\bigg) = \int\limits_{t_0}^{t} q_0(\tau)\,d\tau + \operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}} u_0(\tau_l)$$

hold uniformly on I. Then problem $(1.2.1_k), (1.2.2_k)$ *has the unique solution x_k for any sufficiently large k and condition* (1.2.3) *holds uniformly on I.*

Corollary 2.2.2. *Let $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $G_0 \in B(T;\mathbb{R}^{n\times n})$, $u_0 \in B(T;\mathbb{R}^n)$, $\tau_l \in I$ $(l = 1,2,\dots)$, $c_0 \in \mathbb{R}^n$, $t_0 \in I$ and the sequences $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $q_k \in L_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$, $G_k \in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $u_k \in B_{loc}(T;\mathbb{R}^n)$ $(k = 1,2,\dots)$ and $t_k \in I$ $(k = 1,2,\dots)$ be such that conditions* (1.2.9) *and* (2.2.9) *hold, conditions* (1.2.12),

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} H_k(\tau)P_k(\tau)\,d\tau = \int\limits_{t_0}^{t} H_0(\tau)P_0(\tau)\,d\tau$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} H_k(\tau)\,q_k(\tau)\,d\tau = \int\limits_{t_0}^{t} H_0(\tau)\,q_0(\tau)\,d\tau$$

hold uniformly on I, and

$$\lim_{k\to+\infty} G_k(\tau_l) = G_0(\tau_l) \quad \text{and} \quad \lim_{k\to+\infty} u_k(\tau_l) = u_0(\tau_l)$$

hold uniformly on T, where $H_k \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(k = 0,1,\dots)$. Let, moreover, either

$$\limsup_{k\to+\infty}\sum_{l=1}^{\infty}\big(\|G_k(\tau_l)\| + \|u_k(\tau_l)\|\big) < +\infty \quad \text{or} \quad \limsup_{k\to+\infty}\sum_{l=1}^{\infty}\|H_k(\tau_l+) - H_k(\tau_l)\| < +\infty.$$

Then inclusion (2.2.10) *holds.*

Corollary 2.2.3. *Let $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $G_0 \in B(T;\mathbb{R}^{n\times n})$, $u_0 \in B(T;\mathbb{R}^n)$, $\tau_l \in I$ $(l = 1,2,\dots)$, $t_0 \in I$ and the sequences $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $q_k \in L_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$, $G_k \in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $u_k \in B_{loc}(T;\mathbb{R}^n)$ $(k = 1,2,\dots)$ and $t_k \in I$ $(k = 1,2,\dots)$ be such* *that conditions* (1.2.9), (2.2.9) *and* (2.2.18) *hold, and conditions* (1.2.12),

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t} H_k(\tau)P_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}} H_k(\tau_l+)G_k(\tau_l)\bigg) =\int\limits_{t_0}^{t} P^*(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}} G^*(\tau_l)$$

and

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t} H_k(\tau)q_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}} H_k(\tau_l+)u_k(\tau_l)\bigg) =\int\limits_{t_0}^{t} q^*(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}} u^*(\tau_l)$$

hold uniformly on I, *where* $H_k\in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $P^*\in L(I;\mathbb{R}^{n\times n})$, $q^*\in L(I;\mathbb{R}^n)$, $G^*\in B(T;\mathbb{R}^{n\times n})$, $u^*\in B(T;\mathbb{R}^n)$. *Let, moreover, system* (2.2.16), (2.2.17), *where* $P_0^*(t)\equiv P_0(t)-P^*(t)$, $q_0^*(t)\equiv q_0(t)-q^*(t)$, $G_0^*(\tau_l)\equiv G_0(\tau_l)-G^*(\tau_l)$, $u_0^*(\tau_l)\equiv u_0(\tau_l)-u^*(\tau_l)$, *have a unique solution satisfying condition* (2.2.3). *Then*

$$\big((P_k,q_k;G_k,u_k;t_k)\big)_{k=1}^{\infty}\in\mathcal{S}(P_0^*,q_0^*;G_0^*,u_0^*;t_0).$$

Corollary 2.2.4. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)$, $\tau_l\in I$ $(l=1,2,\dots)$, $t_0\in I$ *and the sequences* $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $G_k\in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $u_k\in B_{loc}(T;\mathbb{R}^n)$ $(k=1,2,\dots)$ *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.9) *and* (2.2.9) *hold. Let, moreover, there exist a natural number* m *and matrix-functions* $B_j\in\mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(j=1,\dots,m-1)$ *such that the condition*

$$\limsup_{k\to+\infty}\bigvee_I\big(H_{k\,m-1}+\mathcal{B}_\iota(H_{k\,m-1};P_k,G_k)\big)<+\infty$$

holds, and conditions

$$\lim_{k\to+\infty}\Big(H_{k\,j-1}(\tau)\Big|_{t_k}^{t}+\mathcal{B}_\iota(H_{k\,j-1};P_k,G_k)(\tau)\Big|_{t_k}^{t}\Big)=B_j(t)-B_j(t_0)\ \ (j=1,\dots,m-1),$$

$$\lim_{k\to+\infty}\Big(H_{k\,m-1}(\tau)\Big|_{t_k}^{t}+\mathcal{B}_\iota(H_{k\,m-1};P_k,G_k)(\tau)\Big|_{t_k}^{t}\Big)=\int\limits_{t_0}^{t}P_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}G_0(\tau_l)$$

and

$$\lim_{k\to+\infty}\mathcal{B}_\iota(H_{k\,m-1};q_k,u_k)(\tau)\Big|_{t_k}^{t}=\int\limits_{t_0}^{t}q_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}u_0(\tau_l)$$

hold uniformly on I, *where*

$$H_{k0}(t)=I_n,\ \ H_{kj}(t)=\bigg(I_n-H_{k\,j-1}(\tau)\Big|_{t_k}^{t}-\mathcal{B}_\iota(H_{k\,j-1};P_k,G_k)(\tau)\Big|_{t_k}^{t}+B_j(\tau)\Big|_{t_k}^{t}\bigg)H_{k\,j-1}(t)$$
$$\text{for }\ t\in I\ \ (j=1,\dots,m-1;\ k=1,2,\dots).$$

Then inclusion (2.2.10) *holds.*

If $m=1$, then Corollary 2.2.4 coincides with Theorem 2.2.2′.

If $m=2$, then Corollary 2.2.4 has the following form.

Corollary 2.2.4′. *Let $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)$, $\tau_l\in I$ $(l=1,2,\dots)$, $t_0\in I$, and the sequences $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $G_k\in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $u_k\in B_{loc}(T;\mathbb{R}^n)$ $(k=1,2,\dots)$ and $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.9), (2.2.9) *and* (2.2.18) *hold, and the conditions*

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t}P_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}G_k(\tau_l)\bigg)=B(t)-B(t_0),$$

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t}H_k(\tau)P_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}H_k(\tau_l+)G_k(\tau_l)\bigg)$$
$$=\int\limits_{t_0}^{t}P_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}G_0(\tau_l)$$

and

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t}H_k(\tau)\,q_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}H_k(\tau_l+)u_k(\tau_l)\bigg)$$
$$=\int\limits_{t_0}^{t}q_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}u_0(\tau_l)$$

hold uniformly on I, where $B\in\mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ and

$$H_k(t)=I_n-\int\limits_{t_k}^{t}P_k(\tau)\,d\tau-\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}G_k(\tau_l)+B(t)-B(t_0)\ \ \textit{for}\ \ t\in I\ \ (k=1,2,\dots).$$

Then inclusion (2.2.10) *holds.*

Corollary 2.2.5. *Let $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)$, $\tau_l\in I$ $(l=1,2,\dots)$ and $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.9) *and* (2.2.9) *hold. Then inclusion* (2.2.10) *holds if and only if there exist matrix-functions $Q_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ and constant matrices $W_k\in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that*

$$\limsup_{k\to+\infty}\bigg(\int\limits_{I}\|P_k(t)-Q_k(t)\|\,dt+\sum_{l=1}^{\infty}\|G_k(\tau_l)-W_k(\tau_l)\|\bigg)<+\infty \tag{2.2.19}$$

and

$$\det(I_n+W_k(\tau_l))\neq0\ \ (k=0,1,\dots;\ l=1,2,\dots), \tag{2.2.20}$$

and the conditions

$$\lim_{k\to+\infty}Z_k^{-1}(t)=Z_0^{-1}(t),$$
$$\lim_{k\to+\infty}\big(\mathcal{B}_\iota(Z_k^{-1};P_k,G_k)(t)-\mathcal{B}_\iota(Z_k^{-1};P_k,G_k)(t_k)\big)=\mathcal{B}_\iota(Z_0^{-1};P_0,G_0)(t)-\mathcal{B}_\iota(Z^{-1};P_0,G_0)(t_0) \tag{2.2.21}$$

and

$$\lim_{k\to+\infty}\big(\mathcal{B}_\iota(Z_k^{-1};q_k,u_k)(t)-\mathcal{B}_\iota(Z_k^{-1};q_k,u_k)(t_k)\big)=\mathcal{B}_\iota(Z_0^{-1};q_0,u_0)(t)-\mathcal{B}_\iota(Z_0^{-1};q_0,u_0)(t_0) \tag{2.2.22}$$

hold uniformly on I, where Z_k $(Z_k(t_k)=I_n)$ is a fundamental matrix of the homogeneous system

$$\frac{dx}{dt}=Q_k(t)\ \text{ for a.a. }\ t\in I\setminus T, \tag{2.2.23}$$

$$x(\tau_l+)-x(\tau_l-)=W_k(\tau_l)x(\tau_l)\ \ (l=1,2,\dots) \tag{2.2.24}$$

for every $k\in\{0,1\dots\}$.

Corollary 2.2.6. *Let $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)$, $\tau_l\in I$ $(l=1,2,\dots)$ and the sequences $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $G_k\in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $u_k\in B_{loc}(T;\mathbb{R}^n)$ $(k=1,2,\dots)$ and $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.9) *and* (2.2.9) *hold. Let, moreover, there exist matrix-functions $Q_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ and $W_k\in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=0,1,\dots;\ l=1,2,\dots)$ such that the pairs (Q_k,W_k) $(k=1,2,\dots)$ satisfy the Lappo–Danilevskiĭ condition, conditions* (2.2.19) *and*

$$\det(I_n+W_0(\tau_l))\neq 0\ \ (l=1,2,\dots) \tag{2.2.25}$$

hold, and the conditions

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t}Q_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}W_k(\tau_l)\bigg)=\int\limits_{t_0}^{t}Q_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}W_{0l}, \tag{2.2.26}$$

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t}Z_k^{-1}(\tau)P_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}Z_k^{-1}(\tau_l)(I_n+W_k(\tau_l))^{-1}G_k(\tau_l)\bigg)$$
$$=\int\limits_{t_0}^{t}Z_0^{-1}(\tau)P_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}Z_0^{-1}(\tau_l)(I_n+W_0(\tau_l))^{-1}G_0(\tau_l) \tag{2.2.27}$$

and

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t}Z_k^{-1}(\tau)q_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}Z_k^{-1}(\tau_l)(I_n+W_k(\tau_l))^{-1}u_k(\tau_l)\bigg)$$
$$=\int\limits_{t_0}^{t}Z_0^{-1}(\tau)q_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}Z_0^{-1}(\tau_l)(I_n+W_0(\tau_l))^{-1}u_0(\tau_l) \tag{2.2.28}$$

hold uniformly on I, where Z_k $(Z_k(t_k)=I_n)$ is a fundamental matrix of the homogeneous system (2.2.23), (2.2.24) *for every $k\in\{0,1\dots\}$. Then inclusion* (2.2.10) *holds.*

Remark 2.2.5. In Corollary 2.2.6, due to (2.2.26), it follows from (2.2.25) that condition (2.2.20) holds for every sufficiently large k and, therefore, conditions (2.2.27) and (2.2.28) of the corollary are correct.

Remark 2.2.6. In Corollaries 2.2.5 and 2.2.6, if we assume that the constant matrices $W_{kl}=O_{n\times n}$ $(k=0,1,\dots;\ l=1,2,\dots)$, then conditions (2.2.20) and (2.2.25) are valid, obviously. Moreover, due to the definition of the operator $\mathcal{B}_\iota$, each of conditions (2.2.21) and (2.2.27) has the form

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^{t}Z_k^{-1}(\tau)P_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}Z_k^{-1}(\tau_l)G_k(\tau_l)\bigg)$$
$$=\int\limits_{t_0}^{t}Z_0^{-1}(\tau)P_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}Z_0^{-1}(\tau_l)G_0(\tau_l)$$

and each of conditions (2.2.22) and (2.2.28) has the form

$$\lim_{k\to+\infty}\Big(\int_{t_k}^{t} Z_k^{-1}(\tau)q_k(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}} Z_k^{-1}(\tau_l)u_k(\tau_l)\Big)$$
$$=\int_{t_0}^{t} Z_0^{-1}(\tau)q_0(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}} Z_0^{-1}(\tau_l)u_0(\tau_l).$$

Remark 2.2.7. If a pair (P,G), satisfied the Lappo–Danilevskiĭ condition, and $s\in I$ are such that $\det(I_n+G(\tau_l))\neq 0$ for $\tau_l<s$, then, due to (2.1.11), the fundamental matrix Z ($Z(s)=I_n$) of the homogeneous system

$$\frac{dx}{dt}=P(t)\ \text{ for a.a. }\ t\in I\setminus T,$$
$$x(\tau_l+)-x(\tau_{l-})=G(\tau_l)x(\tau_l)\ \ (l=1,2,\dots)$$

has the form

$$Z(t)=\begin{cases}\exp\Big(\int_t^s P(\tau)\,d\tau\Big)\prod\limits_{s\le\tau_l<t}(I_n+G(\tau_l)) & \text{for }\ t>s,\\ \exp\Big(\int_t^s P(\tau)\,d\tau\Big)\prod\limits_{t\le\tau_l<s}(I_n+G(\tau_l))^{-1} & \text{for }\ t<s,\\ I_n & \text{for }\ t=s.\end{cases}\tag{2.2.29}$$

Corollary 2.2.7. *Let $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)$, $\tau_l\in I$ ($l=1,2,\dots$), $t_0\in I$ and the sequences $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ ($k=1,2,\dots$), $q_k\in L_{loc}(I;\mathbb{R}^n)$ ($k=1,2,\dots$), $G_k\in B_{loc}(T;\mathbb{R}^{n\times n})$ ($k=1,2,\dots$), $u_k\in B_{loc}(T;\mathbb{R}^n)$ ($k=1,2,\dots$) and $t_k\in I$ ($k=1,2,\dots$) be such that conditions* (1.2.9), (2.2.9) *and*

$$\limsup_{k\to+\infty}\sum_{l=1}^{\infty}\|G_k(\tau_l)\|<+\infty$$

hold. Let, moreover, the matrix-functions P_k ($k=0,1,\dots$) satisfy the Lappo–Danilevskiĭ condition and the conditions

$$\lim_{k\to+\infty}\int_{t_k}^{t}P_k(\tau)\,d\tau=\int_{t_0}^{t}P_0(\tau)\,d\tau,\tag{2.2.30}$$

$$\lim_{k\to+\infty}\sum_{\tau_l\in T_{t_k,t}}G_k(\tau_l)=\sum_{\tau_l\in T_{t_0,t}}G_0(\tau_l),\tag{2.2.31}$$

$$\lim_{k\to+\infty}\int_{t_k}^{t}\exp\Big(-\int_{t_k}^{\tau}P_k(s)\,ds\Big)P_k(\tau)\,d\tau=\int_{t_0}^{t}\exp\Big(-\int_{t_0}^{\tau}P_0(s)\,ds\Big)P_0(\tau)\,d\tau,\tag{2.2.32}$$

$$\lim_{k\to+\infty}\int_{t_k}^{t}\exp\Big(-\int_{t_k}^{\tau}P_k(s)\,ds\Big)q_k(\tau)\,d\tau=\int_{t_0}^{t}\exp\Big(-\int_{t_0}^{\tau}P_0(s)\,ds\Big)q_0(\tau)\,d\tau\tag{2.2.33}$$

and

$$\lim_{k\to+\infty}\sum_{\tau_l\in T_{t_k,t}}\exp\Big(-\int_{t_k}^{\tau_l}P_k(s)\,ds\Big)u_k(\tau_l)=\sum_{\tau_l\in T_{t_0,t}}\exp\Big(-\int_{t_0}^{\tau_l}P_0(s)\,ds\Big)u_0(\tau_l)$$

hold uniformly on I. Then inclusion (1.2.10) *holds.*

Corollary 2.2.8. *Let* $P_0=(p_{0ij})_{i,j=1}^n\in L(I;\mathbb{R}^{n\times n})$, $q_0=(q_{0i})_{i=1}^n\in L(I;\mathbb{R}^n)$, $G_0=(g_{0ij})_{i,j=1}^n\in B(T;\mathbb{R}^{n\times n})$, $u_0=(g_{0i})_{i=1}^n\in B(T;\mathbb{R}^n)$, $\tau_l\in I$ $(l=1,2,\dots)$, $t_0\in I$, *and the sequences* $P_k=(p_{kij})_{i,j=1}^n\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k=(q_{ki})_{i=1}^n\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $G_k=(g_{kij})_{i,j=1}^n\in B_{loc}(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $u_k=(g_{ki})_{i=1}^n\in B_{loc}(T;\mathbb{R}^n)$ $(k=1,2,\dots)$, *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.9), (2.2.9),

$$\limsup_{k\to+\infty}\sum_{i,j=1;\,i\neq j}^n\bigg(\int\limits_I|p_{kij}(t)|\,dt+\sum_{l=1}^{\infty}|g_{kij}(\tau_l)|\bigg)<+\infty$$

and

$$1+g_{0ii}(\tau_l)\neq 0\ \ (i=1,\dots,n;\ l=1,2,\dots)$$

hold, and the conditions

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^t p_{kii}(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}g_{kii}(\tau_l)\bigg)=\int\limits_{t_0}^t p_{0ii}(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}g_{0ii}(\tau_l)\ \ (i=1,\dots,n),$$

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^t z_{kii}^{-1}(\tau)p_{kij}(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}z_{kii}^{-1}(\tau_l)(1+g_{kii}(\tau_l))^{-1}g_{kij}(\tau_l)\bigg)$$
$$=\int\limits_{t_0}^t z_{0ii}^{-1}(\tau)p_{0ij}(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_0,t}}z_{0ii}^{-1}(\tau_l)(1+g_{0ii}(\tau_l))^{-1}g_{0ij}(\tau_l)\ \ (i\neq j;\ i,j=1,\dots,n)$$

and

$$\lim_{k\to+\infty}\bigg(\int\limits_{t_k}^t z_{kii}^{-1}(\tau)q_{ki}(\tau)\,d\tau+\operatorname{sgn}(t-t_k)\sum_{\tau_l\in T_{t_k,t}}z_{kii}^{-1}(\tau_l)(1+g_{kii}(\tau_l))^{-1}u_{ki}(\tau_l)\bigg)$$
$$=\int\limits_{t_k}^t z_{0ii}^{-1}(\tau)q_{0i}(\tau)\,d\tau+\operatorname{sgn}(t-t_0)\sum_{\tau_l\in T_{t_k,t}}z_{0ii}^{-1}(\tau_l)(1+g_{0ii}(\tau_l))^{-1}u_{0i}(\tau_l)\ \ (i=1,\dots,n)$$

hold uniformly on I, *where* z_{kii} *is a unique solution of the initial problem*

$$\frac{dz}{dt}=p_{kii}(t)z\ \ \text{for a.a. } t\in I\setminus T,$$
$$z(\tau_l+)-z(\tau_{l-})=g_{kii}(\tau_l)\,z(\tau_l)\ \ (l=1,2,\dots);$$
$$z(t_k)=1$$

for $i\in\{1,\dots,n\}$ *and any sufficiently large* k (*it is defined according to* (2.2.29)). *Then inclusion* (2.2.10) *holds.*

Remark 2.2.8. For Corollary 2.2.8, the remark analogous to Remark 2.2.5, is true, i.e.,

$$1+g_{kii}(\tau_l)\neq 0\ \ (i=1,\dots,n;\ l=1,2,\dots)$$

for every sufficiently large k and, therefore, all conditions of the corollary are correct.

Remark 2.2.9. In Theorem 2.2.3 and Corollaries 2.2.1, 2.2.2, we can assume without loss of generality that $H_0(t)=I_n$.

Remark 2.2.10. In some results given above, we really have $P_k\in L(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $G_k\in B(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $u_k\in B(T;\mathbb{R}^n)$ $(k=1,2,\dots)$.

2.3 The stability in Liapunov sense

2.3.1 Statement of the problem and formulation of the results

In this section, we realize the results of Section 1.3 for the stability in the Liapunov sense of the following impulsive system

$$\frac{dx}{dt} = P(t)x + q(t) \;\text{ for a.a. }\; t \in \mathbb{R}_+ \setminus T, \tag{2.3.1}$$

$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \;\; (l = 1, 2, \dots), \tag{2.3.2}$$

where $P \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$, $q \in L_{loc}(\mathbb{R}_+; \mathbb{R}^n)$, $G \in B_{loc}(T; \mathbb{R}^{n\times n})$, $u \in B_{loc}(T; \mathbb{R}^n)$, $T = \{\tau_1, \tau_2, \dots\}$, $\tau_l \in \mathbb{R}_+$ $(l = 1, 2, \dots)$, $0 \le \tau_l < \tau_2 < \cdots$ and $\lim\limits_{l\to+\infty} \tau_l = +\infty$.

Below we will mainly consider the case $P \notin L(\mathbb{R}_+; \mathbb{R}^{n\times n})$ or $G \notin B(T; \mathbb{R}^{n\times n})$, i.e.,

$$\sum_{\tau_l \in T} G(\tau_l) = +\infty.$$

In this section, we assume that inequality (2.1.4) holds for every $l \in \{1, 2, \dots\}$, and the function ξ appearing in Section 1.3, in addition, is continuous from the left and belongs to $\mathrm{ACV}_{loc}(\mathbb{R}_+, T; \mathbb{R}_+)$.

Definition 2.3.1. The pair (P, G) is said to be stable in one or another sense if the matrix-function A defined by (2.1.5) is stable in the same sense, according to Definition 1.3.6.

It is evident that the stability of the pair (P, G) is equivalent to that of the corresponding homogeneous impulsive system

$$\frac{dx}{dt} = P(t)x \;\text{ for a.a. }\; t \in \mathbb{R}_+ \setminus T, \tag{2.3.1$_0$}$$

$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) \;\; (l = 1, 2, \dots). \tag{2.3.2$_0$}$$

Theorem 2.3.1. *The pair (P, G) is stable if and only if there exists a nonsingular continuous from the left matrix-function $H \in \mathrm{ACV}_{loc}(\mathbb{R}_+, T; \mathbb{R}^{n\times n})$ such that conditions (1.3.8) and*

$$\int_0^{+\infty} \|H'(t) + H(t)P(t)\|\, dt + \sum_{l=1}^{+\infty} \big\|H(\tau_l+) - H(\tau_l) + H(\tau_l+)G(\tau_l)\big\| < +\infty \tag{2.3.3}$$

hold.

Theorem 2.3.2. *The pair (P, G) is uniformly stable if and only if there exists a nonsingular continuous from the left matrix-function $H \in \mathrm{ACV}_{loc}(\mathbb{R}_+, T; \mathbb{R}^{n\times n})$ such that conditions (1.3.10) and (2.3.3) hold.*

Theorem 2.3.3. *The pair (P, G) is asymptotically stable if and only if there exists a nonsingular continuous from the left matrix-function $H \in \mathrm{ACV}_{loc}(\mathbb{R}_+, T; \mathbb{R}^{n\times n})$ such that conditions (1.3.11) and (2.3.3) hold.*

Theorem 2.3.4. *The pair (P, G) is ξ-exponentially asymptotically stable if and only if there exists a nonsingular continuous from the left matrix-function $H \in \mathrm{ACV}_{loc}(\mathbb{R}_+, T; \mathbb{R}^{n\times n})$ such that conditions (1.3.12) and*

$$\int_0^{+\infty} \exp(-\eta\xi(\tau))\|H'(t) + H(t)P(t)\|\, dt$$
$$+ \sum_{l=1}^{+\infty} \exp(-\eta\xi(\tau_l))\big\|H(\tau_l+) - H(\tau_l) + H(\tau_l+)G(\tau_l)\big\| < +\infty$$

hold.

Corollary 2.3.1. *Let the matrix-functions $Q \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $W \in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that*

$$\det(I_n + W(\tau_l)) \neq 0 \quad (l = 1,2,\dots) \tag{2.3.4}$$

and

$$\int\limits_0^{+\infty} \big\|Y^{-1}(t)(P(t)-Q(t))\big\|\,dt + \sum_{l=1}^{+\infty} \big\|Y^{-1}(\tau_l)(I_n+W(\tau_l))^{-1}(G(\tau_l)-W(\tau_l))\big\| < +\infty,$$

where Y $(Y(0) = I_n)$ is the fundamental matrix of the system

$$\frac{dx}{dt} = Q(t) \ \text{ for a.a. } t \in \mathbb{R}_+ \setminus T, \tag{2.3.5}$$

$$x(\tau_l+) - x(\tau_l-) = W(\tau_l)x(\tau_l) \quad (l = 1,2,\dots). \tag{2.3.6}$$

Then the stability in one or another sense of the pair (Q,W) guarantees the stability of the pair (P,G) in the same sense.

Theorem 2.3.5. *Let the pair (P_0,G_0), consisting of matrix-functions $P_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G_0 \in B_{loc}(T;\mathbb{R}^{n\times n})$, be uniformly stable and*

$$\det(I_n + G_0(\tau_l)) \neq 0 \quad (l = 1,2,\dots). \tag{2.3.7}$$

Let, moreover, the matrix-functions $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G \in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that

$$\begin{aligned} &\int\limits_0^{+\infty} \big\|\big(H'(t) + H(t)P(t) - P_0(t)H(t)\big)H^{-1}(t)\big\|\,dt \\ &\quad + \sum_{l=1}^{+\infty} \Big\|(I_n + G_0(\tau_l)^{-1}\big(H(\tau_l+) - H(\tau_l) + H(\tau_l+)G(\tau_l) - G_0(\tau_l)H(\tau_l)\big)H^{-1}(\tau_l)\Big\| < +\infty, \end{aligned} \tag{2.3.8}$$

where $H \in \mathrm{ACV}_{loc}(\mathbb{R}_+,T;\mathbb{R}^{n\times n})$ is a nonsingular continuous from the left matrix-function satisfying condition (1.3.10). Then the pair (P,G) is uniformly stable, as well.

Remark 2.3.1. In Theorem 2.3.5, if $H(t) \equiv I_n$, then condition (2.3.8) has the form

$$\int\limits_0^{+\infty} \|P(t) - P_0(t)\|\,dt + \sum_{l=1}^{+\infty} \big\|(I_n + G_0(\tau_l))^{-1}(G(\tau_l) - G_0(\tau_l))\big\| < +\infty.$$

Theorem 2.3.6. *Let the pair (P_0,G_0), consisting of matrix-functions $P_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G_0 \in B_{loc}(T;\mathbb{R}^{n\times n})$, be ξ-exponentially asymptotically stable and condition (2.3.7) hold. Let, moreover, the matrix-functions $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G \in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that*

$$\lim_{t\to+\infty}\Bigg(\int\limits_0^{\nu(\xi)(t)} \|P(\tau) - P_0(\tau)\|\,d\tau + \sum_{0\le\tau_l<\nu(\xi)(t)} \big\|(I_n + G_0(\tau_l))^{-1}(G(\tau_l) - G_0(\tau_l))\big\|\Bigg) = 0,$$

where the function $\nu(\xi)$ is defined by (1.3.22), and $\xi \in \mathrm{ACV}_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ is a continuous from the left nondecreasing function satisfying condition (1.3.4). Then the pair (P,G) is ξ-exponentially asymptotically stable, as well.

Corollary 2.3.2. *Let the components of the matrix-functions $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G = (g_{ik})_{i,k=1}^n \in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that*

$$1 + g_{ii}(\tau_l) \neq 0 \quad (i = 1,\dots,n;\ \ l = 1,2,\dots), \tag{2.3.9}$$

$$\lim_{t\to+\infty}\Bigg(\int\limits_t^{\nu(\xi)(t)} \|p_{ik}(\tau)\|\,d\tau + \sum_{t\le\tau_l<\nu(\xi)(t)} \big\|(1 + g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)\big\|\Bigg) = 0 \quad (i \neq k;\ i,k = 1,\dots,n)$$

and

$$\int_{\tau}^{t} p_{ii}(s)\,ds + \sum_{\tau\le\tau_l<t} g_{ii}(\tau_l) \le -\eta(\xi(t)-\xi(\tau)) \ \ for \ \ t>\tau\ge 0 \ \ (i=1,\dots,n),$$

where $\eta>0$, *and the function* $\nu(\xi)$ *is defined by* (1.3.22). *Then the pair* (P,G) *is* ξ*-exponentially asymptotically stable.*

Corollary 2.3.3. *Let a matrix-function* $P_0\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be* ξ*-exponentially asymptotically stable, and the matrix-functions* $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that*

$$\lim_{t\to+\infty}\Bigg(\int_{t}^{\xi(t)+1}\|P(\tau)-P_0(\tau)\|\,d\tau + \sum_{t\le\tau_l<\xi(t)+1}\|G(\tau_l)\|\Bigg)=0,$$

where $\xi:\mathbb{R}_+\to\mathbb{R}_+$ *is a continuous nondecreasing function satisfying condition* (1.3.4). *Then the pair* (P,G) *is* ξ*-exponentially asymptotically stable, as well.*

Proposition 2.3.1. *Let the pair* (P,G) *be* ξ*-exponentially asymptotically stable, and the vector-functions* $q\in L_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ *and* $u\in B_{loc}(T;\mathbb{R}^n)$ *be such that*

$$\lim_{t\to+\infty}\Bigg(\int_{t}^{\xi(t)+1}\|q(\tau)\|\,d\tau + \sum_{t\le\tau_l<\xi(t)+1}\big\|(I_n+G(\tau_l))^{-1}u(\tau_l)\big\|\Bigg)=0,$$

where the function $\nu(\xi):\mathbb{R}_+\to\mathbb{R}_+$ *is defined by* (1.3.22). *Then every solution of system* (2.3.1), (2.3.2) *satisfies condition* (1.3.6).

Proposition 2.3.2. *Let the pair* (P_0,G_0), *consisting of matrix-functions* $P_0\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G_0\in B_{loc}(T;\mathbb{R}^{n\times n})$, *be* ξ*-exponentially asymptotically stable and condition* (2.3.7) *hold. Let, moreover, the matrix-functions* $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that*

$$\lim_{t\to+\infty}\frac{1}{\xi(t)}\Bigg(\int_{0}^{t}\|P(\tau)-P_0(\tau)\|\,d\tau + \sum_{0\le\tau_l<t}\big\|(I_n+G_0(\tau_l))^{-1}(G(\tau_l)-G_0(\tau_l))\big\|\Bigg)=0.$$

Then the pair (P,G) *is* ξ*-exponentially asymptotically stable, as well.*

Theorem 2.3.7. *Let the components of the matrix-functions* $P=(p_{ik})_{i,k=1}^n\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G=(g_{ik})_{i,k=1}^n\in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that inequality* (2.3.9) *for* $i\in\{1,\dots,n\}$ *and* $l\in\{1,2,\dots\}$ *holds if* $\tau_l\ge t^*$ *and the conditions*

$$\sup\Bigg\{\int_{0}^{t}p_{ii}(s)\,ds + \sum_{0\le\tau_l<t}\ln|1+g_{ii}(\tau_l)| :\ t\ge t^*\Bigg\}<+\infty \ \ (i=1,\dots,n)$$

and

$$\begin{aligned}\int_{t^*}^{t}\exp\Bigg(\int_{\tau}^{t}p_{ii}(s)\,ds\Bigg)|p_{ik}(\tau)|\prod_{\tau\le\tau_l<t}|1+g_{ii}(\tau_l)|\,d\tau \\ +\sum_{t^*\le\tau_j<t}\exp\Bigg(\int_{\tau_j}^{t}p_{ii}(s)\,ds\Bigg)|g_{ik}(\tau_j)|\prod_{\tau_j\le\tau_l<t}|1+g_{ii}(\tau_l)| \\ \le h_{ik} \ \ for \ \ t\ge t^* \ \ (i\ne k;\ i,k=1,\dots,n) \qquad (2.3.10)\end{aligned}$$

hold, where t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k;\ i,k=1,\dots,n)$. *Let, moreover, the matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *be such that condition* (1.3.31) *holds. Then the pair* (P,G) *is stable.*

Theorem 2.3.8. *Let the components of the matrix-functions $P=(p_{ik})_{i,k=1}^n\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G=(g_{ik})_{i,k=1}^n\in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that inequality* (2.3.9) *for $i\in\{1,\dots,n\}$ and $l\in\{1,2,\dots\}$ if $\tau_l\geq t^*$, conditions* (2.3.10) *and*

$$\sup\bigg\{\int\limits_{\tau}^{t}p_{ii}(s)\,ds+\sum_{\tau\leq\tau_l<t}\ln|1+g_{ii}(\tau_l)|:\ t>\tau\geq t^*\bigg\}<+\infty\ \ (i=1,\dots,n)$$

hold, where t^* and $h_{ik}\in\mathbb{R}_+$ ($i\neq k$; $i,k=1,\dots,n$). Let, moreover, the matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ ($i=1,\dots,n$), be such that condition (1.3.31) *holds. Then the pair (P,G) is uniformly stable.*

Corollary 2.3.4. *Let the components of the matrix-functions $P=(p_{ik})_{i,k=1}^n\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G=(g_{ik})_{i,k=1}^n\in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that*

$$p_{ii}(t)\leq 0,\ \ |p_{ik}(t)|\leq -h_{ik}p_{ii}(t)\ \ \text{for}\ \ t\geq t^*\ \ (i\neq k;\ i,k=1,\dots,n) \tag{2.3.11}$$

and

$$-1<g_{ii}(\tau_l)\leq 0,\ \ |g_{ik}(\tau_l)|\leq -h_{ik}g_{ii}(\tau_l)\ \ \text{if}\ \ \tau_l>t^*\ \ (i\neq k;\ i,k=1,\dots,n;\ l=1,2,\dots),$$

where t_* and $h_{ik}\in\mathbb{R}_+$ ($i\neq k$; $i,k=1,\dots,n$). Let, moreover, the matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ ($i=1,\dots,n$), be such that condition (1.3.31) *holds. Then the pair (P,G) is uniformly stable.*

Theorem 2.3.9. *Let the components of the matrix-functions $P=(p_{ik})_{i,k=1}^n\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G=(g_{ik})_{i,k=1}^n\in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that inequality* (2.3.9) *for $i\in\{1,\dots,n\}$ and $l\in\{1,2,\dots\}$ if $\tau_l>t^*$, the conditions*

$$\int\limits_{t^*}^{t}p_{ii}(s)\,ds+\sum_{t^*\leq\tau_l<t}\ln|1+g_{ii}(\tau_l)|\leq-\xi(t)+\xi(t^*)\ \ \text{for}\ \ t\geq t^*\ \ (i=1,\dots,n)$$

and

$$\begin{aligned}\int\limits_{t^*}^{t}\exp\bigg(\xi(t)-\xi(\tau)+\int\limits_{\tau}^{t}p_{ii}(s)\,ds\bigg)|p_{ik}(\tau)|\prod_{\tau\leq\tau_l<t}|1+g_{ii}(\tau_l)|\,d\tau\\ +\sum_{t^*\leq\tau_j<t}\exp\bigg(\xi(t)-\xi(\tau_j)+\int\limits_{\tau_j}^{t}p_{ii}(s)\,ds\bigg)|g_{ik}(\tau_j)|\prod_{\tau_j\leq\tau_l<t}|1+g_{ii}(\tau_l)|\\ \leq h_{ik}\ \ \text{for}\ \ t\geq t^*\ \ (i\neq k;\ i,k=1,\dots,n)\end{aligned}$$

hold, where t_* and $h_{ik}\in\mathbb{R}_+$ ($i\neq k$; $i,k=1,\dots,n$). Let, moreover, the matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ ($i=1,\dots,n$), be such that condition (1.3.31) *holds. Then the pair (P,G) is ξ-asymptotically stable.*

Corollary 2.3.5. *Let the components of the matrix-functions $P=(p_{ik})_{i,k=1}^n\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G=(g_{ik})_{i,k=1}^n\in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that conditions* (2.3.11),

$$g_{ii}(\tau_l)>0\ \ \text{or}\ \ -1<g_{ii}(\tau_l)\leq\exp(-1)-1\ \ \text{if}\ \ \tau_l>t^*\ \ (i\neq k;\ i,k=1,\dots,n;\ l=1,2,\dots), \tag{2.3.12}$$

$$|g_{ik}(\tau_l)|\leq-h_{ik}\big(1+\ln(1+g_{ii}(\tau_l))\big)^{-1}\ln(1+g_{ii}(\tau_l))\ \ \text{if}\ \tau_l>t^*\ \ (i\neq k;\ i,k=1,\dots,n;\ l=1,2,\dots) \tag{2.3.13}$$

hold, where t_* and $h_{ik}\in\mathbb{R}_+$ ($i\neq k$; $i,k=1,\dots,n$) are such that the matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ ($i=1,\dots,n$), satisfies condition (1.3.31). *Let, moreover, there exist a function $a_0\in\mathrm{ACV}_{loc}(\mathbb{R}_+,T;\mathbb{R})$, satisfying condition* (1.3.41), *such that*

$$a_0(t)-a_0(\tau)\leq\min\bigg\{\bigg|\int\limits_{\tau}^{t}p_{ii}(s)\,ds+\sum_{\tau\leq\tau_l<t}\ln(1+g_{ii}(\tau_l))\bigg|:\ (i=1,\dots,n)\bigg\}\ \ \text{for}\ \ t\geq\tau\geq t^*.$$

Then the pair (P,G) is asymptotically and also uniformly stable.

Corollary 2.3.6. *Let the components of the matrix-functions* $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G = (g_{ik})_{i,k=1}^n \in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that conditions* (2.3.11), (2.3.12) *and* (2.3.13) *hold, where* t_* *and* $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i,k = 1,\dots,n)$ *are such that the matrix* $H = (h_{ik})_{i,k=1}^n$, *where* $h_{ii} = 0$ $(i = 1,\dots,n)$, *satisfies condition* (1.3.31). *Let, moreover,*

$$\int\limits_0^{+\infty} \eta_0(s)\,ds + \sum_{l=1}^{+\infty} \ln(1+\eta(\tau_l)) = -\infty, \tag{2.3.14}$$

where

$$\begin{aligned} &\eta_0(t) \equiv \max\big\{p_{ii}(t):\ i = 1,\dots,n\big\} \\ &\eta(\tau_l) = \max\big\{g_{ii}(\tau_l):\ i = 1,\dots,n\big\}\ \ (l = 1,2,\dots). \end{aligned} \tag{2.3.15}$$

Then the pair (P,G) *is asymptotically and also uniformly stable.*

Theorem 2.3.10. *Let the components of the matrix-functions* $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G = (g_{ik})_{i,k=1}^n \in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that conditions* (2.3.9),

$$\sup\Bigg\{(\xi(t)-\xi(\tau))^{-1}\bigg(\int\limits_\tau^t p_{ii}(s)\,ds + \sum_{\tau\le\tau_l<t} \ln|1+g_{ii}(\tau_l)|\bigg):\\ t>\tau\ge t^*,\ \xi(t)\neq\xi(\tau);\ t,\tau\in\mathbb{R}_+\setminus T\Bigg\} \le -\gamma\ \ (i=1,\dots,n) \tag{2.3.16}$$

and

$$\int\limits_{t^*}^t \exp\bigg(\gamma(\xi(t)-\xi(\tau)) + \int\limits_\tau^t p_{ii}(s)\,ds\bigg)|p_{ik}(\tau)| \prod_{\tau\le\tau_l<t} |1+g_{ii}(\tau_l)|\,d\tau \\ + \sum_{t^*\le\tau_j<t} \exp\bigg(\gamma(\xi(t)-\xi(\tau_j)) + \int\limits_{\tau_j}^t p_{ii}(s)\,ds\bigg)|g_{ik}(\tau_j)| \prod_{\tau_j\le\tau_l<t} |1+g_{ii}(\tau_l)| \\ \le h_{ik}\ \text{ for }\ t\ge t^*\ \ (i\neq k;\ i,k=1,\dots,n)$$

hold, where $\gamma > 0$, t_* *and* $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i,k = 1,\dots,n)$. *Let, moreover, the matrix* $H = (h_{ik})_{i,k=1}^n$, *where* $h_{ii} = 0$ $(i = 1,\dots,n)$, *be such that condition* (1.3.31) *holds. Then the pair* (P,G) *is* ξ*-asymptotically stable.*

Corollary 2.3.7. *Let the components of the matrix-functions* $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G = (g_{ik})_{i,k=1}^n \in B_{loc}(T;\mathbb{R}^{n\times n}$ *be such that conditions* (2.3.11), (2.3.12), (2.3.13) *and* (2.3.16) *hold, where* $\gamma > 0$, t_* *and* $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i,k = 1,\dots,n)$. *Let, moreover, the matrix* $H = (h_{ik})_{i,k=1}^n$, *where* $h_{ii} = 0$ $(i = 1,\dots,n)$, *be such that condition* (1.3.31) *holds. Then the pair* (P,G) *is* ξ*-asymptotically stable.*

Theorem 2.3.11. *Let the matrix-functions* $P = (p_{ik})_{i,k=1}^n$ *and* $P_0 = (p_{0ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, $G = (g_{ik})_{i,k=1}^n$ *and* $G_0 = (g_{0ik})_{i,k=1}^n \in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that*

$$\begin{gathered} \|G_l(\tau_l)\| < 1\ \ (l = 1,2,\dots), \\ p_{ii}(t) \le p_{0ii}(t)\ \text{ and }\ |p_{ik}(t)| \le p_{0ik}(t)\ \text{ for a.a. }\ t\in\mathbb{R}_+\ \ (i\neq k;\ i,k=1,\dots,n), \\ g_{ii}(\tau_l) \le g_{0ii}(\tau_l)\ \text{ and }\ |g_{ik}(\tau_l)| \le g_{0ik}(\tau_l)\ \ (i,k=1,\dots,n;\ l=1,2,\dots). \end{gathered} \tag{2.3.17}$$

Let, moreover, the pair (P_0,G_0) *be stable (uniformly stable, asymptotically stable or* ξ*-exponentially asymptotically stable). Then the pair* (P,G_l) *will be stable (uniformly stable, asymptotically stable or* ξ*-exponentially asymptotically stable), as well.*

Theorem 2.3.12. *Let $\alpha_{ik}\in\mathbb{R}$ $(i,k=1,\dots,n)$, $\nu_i\in B_{loc}(T;R)$ $(i=1,\dots,n)$, and $\mu_i\in AC_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ $(i,\dots,n)$ be nondecreasing functions such that conditions* (2.3.14) *and*

$$\alpha_{ii}\nu_i(\tau_l)>0 \ \text{ or } \ -1<\alpha_{ii}\nu_i(\tau_l)<\exp(-1)-1 \ \ (i=1,\dots,n;\ l=1,2,\dots)$$

hold, where

$$\eta_0(t)\equiv\min\big\{\alpha_{ii}\mu_i(t):\ i=1,\dots,n\big\},$$
$$\eta(\tau_l)=\max\big\{\alpha_{ii}\nu_i(\tau_l):\ i=1,\dots,n\big\}\ \ (l=1,2,\dots).$$

Then conditions (1.3.31) *and* (1.3.47)*, where the constant matrix H is defined by* (1.3.48)*, are sufficient for the pair (P,G), where $P(t)\equiv(\alpha_{ik}\,\mu_i(t))_{i,k=1}^n$ and $G(\tau_l)=(\alpha_{ik}\nu_i(\tau_l))_{i,k=1}^n$, to be asymptotically stable; and if conditions* (1.3.49)*,*

$$\sum_{l=1;\,l\neq i}^{n}\alpha_{il}\nu_i(\tau_l)<|1-\alpha_{ii}\nu_i(\tau_l)| \ \text{ or }$$
$$\sum_{l=1;\,l\neq k}^{n}\alpha_{lk}\nu_i(\tau_l)<|1-\alpha_{kk}\nu_k(\tau_l)| \ \ (i,k=1,\dots,n;\ l=1,2,\dots),$$

$$\alpha_{ii}\nu_i(\tau_l)<1 \ \text{ for } \ (i=1,\dots,n;\ l=1,2,\dots)$$

and

$$\big((\delta_{ik}-\alpha_{ik}\nu_i(\tau_l))_{i,k=1}^n\big)^{-1}\geq O_{n\times n}\ \ (j=1,2;\ l=1,2,\dots)$$

hold, then conditions (1.3.31) *and* (1.3.47) *are necessary, as well.*

Corollary 2.3.8. *Let the pair (Q,W) be uniformly stable, where $Q\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $W\in B_{loc}(T;\mathbb{R}^{n\times n})$ are such that conditions* (2.3.4) *and*

$$\bigg\|\int_0^{+\infty}|Y^{-1}(t)|\,\big|P(t)-Q(t)+\eta\xi'(t)I_n\big|\,dt\bigg\|+\bigg\|\sum_{l=1}^{+\infty}|Y^{-1}(\tau_l)|\,\big|(I_n+W(\tau_l))^{-1}\,(G(\tau_l)-W(\tau_l))\big|\bigg\|$$
$$+\bigg\|\sum_{l=1}^{+\infty}\big(\exp(\eta d_2\xi(\tau_l))-1\big)\cdot|Y^{-1}(\tau_l)|\,\big|(I_n+W(\tau_l))^{-1}\,(I_n+G(\tau_l))\big|\bigg\|<+\infty \qquad (2.3.18)$$

hold, where Y $(Y(0)=I_n)$ is the fundamental matrix of system (2.3.5), (2.3.6)*, and η is a positive number. Then the pair (P,G) is ξ-exponentially asymptotically stable.*

Remark 2.3.2. In Corollary 2.3.8, if the function $\xi:\mathbb{R}_+\to\mathbb{R}_+$ is continuous, then

$$\exp(\eta d_2\xi(\tau_l))-1=0.$$

So, the last term in the left-hand sides of conditions (2.3.18) vanishes.

Moreover, Corollary 2.3.8 is true for the limit case $(\eta=0)$, too, if instead of the uniform stability we require the ξ-exponentially asymptotic stability of the matrix-function Q.

Corollary 2.3.9. *Let $Q\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be a continuous matrix-function satisfying the Lappo–Danilevskiĭ condition. Let, moreover, there exist a nonnegative number η such that*

$$\bigg\|\int_0^{+\infty}\exp\bigg(-\int_0^t Q(\tau)\,d\tau\bigg)\big|P(t)-Q(t)+\eta\xi'(t)I_n\big|\,dt\bigg\|+\bigg\|\sum_{l=1}^{+\infty}\exp\bigg(-\int_0^{\tau_l}Q(\tau)\,d\tau\bigg)|G(\tau_l)|\bigg\|<+\infty,$$

where $\xi:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous nondecreasing function satisfying condition (1.3.4)*. Then*

(a) *the uniform stability of the matrix-function Q guarantees the ξ-exponentially asymptotic stability of the pair (P,G) if $\eta>0$;*

(b) *the ξ-exponentially asymptotic stability of the matrix-function Q guarantees the ξ-exponentially asymptotic stability of the pair (P,G) if $\eta=0$.*

Corollary 2.3.10. *Let there exist a nonnegative number η such that the components of the matrix-functions $P=(p_{ij})_{i,j=1}^n\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G=(g_{ij})_{i,j=1}^n\in B_{loc}(T;\mathbb{R}^{n\times n})$ satisfy the conditions* (2.3.9),

$$\int\limits_\tau^t p_{ii}(s)\,ds+\sum_{\tau\le\tau_l<t}\ln|1+g_{ii}(\tau_l)|$$
$$\le-\eta(s_c(\xi)(t)-s_c(\xi)(\tau))-\mu(\xi(t)-\xi(\tau))\ \ \text{for}\ \ t>\tau\ge0\ \ (i=1,\dots,n),$$
$$\sum_{l=1}^{+\infty}|y_i^{-1}(\tau_l)|\big(\exp(\eta d_2\xi(\tau_l))-1\big)<+\infty\ \ (j=1,2;\ i=1,\dots,n)$$

and

$$\int\limits_0^{+\infty}|y_i^{-1}(t)|\,|p_{ik}(t)|\,dt+\sum_{l=1}^{+\infty}\exp(\eta d_2\xi(\tau_l))g_{ik}(\tau_l)\cdot(1+g_{ii}(\tau_l))^{-1}<+\infty\ \ (i\ne k;\ i,k=1,\dots,n),$$

where $\mu=0$ if $\eta>0$ and $\mu>0$ if $\eta=0$,

$$y_i(t)\equiv\exp\bigg(\int\limits_0^t(p_{ii}(s)+\eta\xi'(s))\,ds\bigg)\cdot\prod_{0\le\tau_l<t}(1+g_{ii}(\tau_l))\ \ (i=1,\dots,n).$$

Then the pair (P,G) is ξ-exponentially asymptotically stable.

Remark 2.3.3. In Corollary 2.3.10, if the condition

$$g_{ik}(\tau_l)\,(1+g_{ii}(\tau_l))^{-1}>0\ \ (i\ne k;\ i,k=1,\dots,n;\ l=1,2,\dots)$$

holds together with condition (2.3.9), then without loss of generality we can assume that $\eta>0$ and $\mu=0$.

Theorem 2.3.13. *Let the matrix-functions $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that*

$$P(t)=\sum_{k=1}^m\alpha_k'(t)\,B_k\ \ \text{for a.a.}\ \ t\in\mathbb{R}_+ \tag{2.3.19}$$

and

$$I_n+G(\tau_l)=\exp\Big(\sum_{k=1}^m(\alpha_k(\tau_l+)-\alpha_k(\tau_l))B_k\Big)\ \ (l=1,2,\dots), \tag{2.3.20}$$

where $\alpha_k\in\mathrm{ACV}_{loc}(\mathbb{R}_+,T;\mathbb{R})$ $(k=1,\dots,m)$, and $B_k\in\mathbb{R}^{n\times n}$ $(k=1,\dots,m)$ are pairwise permutable constant matrices. Let, moreover, $(\lambda-\lambda_{ki})^{n_{ki}}$ $(i=1,\dots,m_k;\ \sum\limits_{i=1}^{m_l}n_{ki}=n)$ be elementary divisors of the matrix B_k for every $k\in\{1,\dots,m\}$. Then

(a) *the pair (P,G) is stable if and only if condition* (1.3.60) *holds;*

(b) *the pair (P,G) is asymptotically stable if and only if condition* (1.3.61) *holds.*

Corollary 2.3.11. *Let conditions* (2.3.19) *and* (2.3.20) *hold, where $B_k\in\mathbb{R}^{n\times n}$ $(k=1,\dots,m)$ are pairwise permutable constant matrices, and $\alpha_k\in\mathrm{ACV}_{loc}(\mathbb{R}_+,T;\mathbb{R})$ $(k=1,\dots,m)$ are such that condition* (1.3.62) *holds. Then*

(a) *the pair* (P,G) *is stable if and only if every eigenvalue of the matrices* B_k $(k=1,\dots,m)$ *has the nonpositive real part; in addition, every elementary divisor corresponding to the eigenvalue with the zero real part is simple;*

(b) *the pair* (P,G) *is asymptotically stable if and only if every eigenvalue of the matrices* B_k $(k=1,\dots,m)$ *has the negative real part.*

By $\nu(t)$ we denote a number of points τ_l $(l=1,2,\dots)$ belonging to $[0,t[$ for every $t\in\mathbb{R}_+$. It is evident that $\nu(t)$ is finite for every $t\in\mathbb{R}_+$.

Corollary 2.3.12. *Let the matrix-functions* $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that*

$$P(t)=\alpha'(t)P_0 \ \text{ for a.a. } \ t\in\mathbb{R}_+$$

and

$$G(\tau_l)=G_0 \ \text{ if } \ G(\tau_l)\neq O_{n\times n} \ \ (j=1,2,\dots),$$

where $\alpha\in\mathrm{ACV}_{loc}(\mathbb{R}_+;\mathbb{R})$ *is a function satisfying condition* (1.3.63), *and* P_0 *and* $G_0\in\mathbb{R}^{n\times n}$ *are permutable constant matrices. Let, moreover, there exist a number* $\beta\in\mathbb{R}_+$ *such that condition* (1.3.64) *holds. Then*

(a) *the pair* (P,G) *is stable if and only if every eigenvalue of the matrix* $A=P_0+\beta\ln(I_n+G_0)$ *has the nonpositive real part; in addition, every elementary divisor corresponding to the eigenvalue with the zero real part is simple;*

(b) *the pair* (P,G) *is asymptotically stable if and only if every eigenvalue of the matrix* A *has the negative real part.*

If $\alpha(t)\equiv t$, then Corollary 2.3.12 has the following form.

Corollary 2.3.13. *Let* $P(t)\equiv P_0$ *and* $G(\tau_l)=G_0$ $(l=1,2,\dots)$, *where* P_0 *and* G_0 *are permutable constant matrices. Let, moreover, there exist a number* $\beta\in\mathbb{R}_+$ *such that*

$$\limsup_{t\to+\infty}|\nu(t)-\beta t|<+\infty.$$

Then the conclusion of Corollary 2.3.12 *is true.*

Corollary 2.3.14. *Let* P_0 *and* G_0 *be constant* $n\times n$*-matrices circumscribed in Corollary* 2.3.13, *and*

$$\tau_{l+1}-\tau_l=\theta=\text{constant} \ \ (l=1,2,\dots).$$

Then the conclusion of Corollary 2.3.12, *where* $A=P_0+\theta^{-1}\ln(I_n+G_0)$, *is true.*

Corollary 2.3.15. *Let the matrix-functions* $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ *be such that*

$$P(t)=C\,\mathrm{diag}(J_1(t),\dots,J_m(t))C^{-1} \ \text{ for } \ t\in\mathbb{R}_+$$

and

$$I_n+G(\tau_l)=C\,\mathrm{diag}\big(\exp(J_{1l}),\dots,\exp(J_{ml})\big)C^{-1} \ \ (l=1,2,\dots),$$

where $C\in\mathbb{C}^{n\times n}$ *is a nonsingular complex matrix,* $J_k(t)=\sum\limits_{i=0}^{n_k-1}\alpha_{ki}(t)Z^i_{n_k}$ $(k=1,\dots,m;\ \sum\limits_{k=1}^{m}n_k=n)$, $J_{kl}=\sum\limits_{i=0}^{n_k-1}d_2\alpha_{ki}(\tau_l)Z^i_{n_k}$ $(k=1,\dots,m;\ l=1,2,\dots)$, $\alpha_{ki}\in\mathrm{ACV}_{loc}(\mathbb{R}_+,T;\mathbb{R})$ $(k=1,\dots,m;\ i=1,\dots,n_k-1)$, *and* α_{k0} *is a complex-valued function such that* $\mathrm{Re}(\alpha_{k0})$ *and* $\mathrm{Im}(\alpha_{k0})\in\mathrm{ACV}_{loc}(\mathbb{R}_+,T;\mathbb{R}_+)$. *Then*

(a) *the pair* (P,G) *is stable if and only if condition* (1.3.65) *holds;*

(b) *the pair* (P,G) *is asymptotically stable if and only if condition* (1.3.66) *holds.*

We use the following notation.

Let $\alpha \in \mathrm{ACV}_{loc}(\mathbb{R}_+, T; \mathbb{R})$ and $g \in B_{loc}(T; \mathbb{R})$ be nondecreasing functions and $H = (h_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$. Then by $\mathcal{Q}(H; \alpha, g)$ we denote a set of all pairs (P, G) consisting of matrix-functions $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $G = (g_{ik})_{i,k=1}^n \in B_{loc}(T; \mathbb{R}^{n\times n})$ such that

$$p_{ik}(t) = \alpha'(t) h_{ik}(t) \ \text{ for a.a. } \ t \in \mathbb{R}_+ \setminus T \ \ (i,k = 1,\dots,n;\ l = 1,2,\dots)$$

and

$$g_{ik}(\tau_l) + \frac{1}{2}\sum_{j=1}^n g_{ji}(\tau_l) g_{jk}(\tau_l) = g(\tau_l) h_{ik}(\tau_l) \ \ (i,k = 1,\dots,n;\ l = 1,2,\dots).$$

Theorem 2.3.14. *Let $(P, G) \in \mathcal{Q}(H; \alpha, g)$, inequality (1.3.70) hold for a.a. $t \in \mathbb{R}_+ \setminus T$ and $(x_i)_{i=1}^n \in \mathbb{R}^n$ and*

$$1 + 2g(\tau_l)\, p(\tau_l) > 0 \ \ (l = 1,2,\dots),$$

where $p \in L_{loc}(\mathbb{R}_+; \mathbb{R})$. Let, moreover,

$$\gamma(t) = \exp\left(2\int_0^t p(\tau)\alpha'(\tau)\,d\tau\right)\prod_{0\le\tau_l<t}\left(1 + 2g(\tau_l)p(\tau_l)\right) \ \ \text{for} \ \ t \in \mathbb{R}_+.$$

Then

(a) *condition (1.3.72) guarantees the stability of the pair (P, G);*

(b) *condition (1.3.73) guarantees the uniform stability of the pair (P, G);*

(c) *condition (1.3.74) guarantees the asymptotic stability of the pair (P, G);*

(d) *condition (1.3.75), where $t^* \in \mathbb{R}_+$ is some point, guarantees the ξ-exponentially asymptotic stability of the pair (P, G);*

(e) *if the inequality opposite to inequality (1.3.70) and condition (1.3.76) hold, then the pair (P, G) is nonstable.*

Here in conditions (1.3.72)–(1.3.75) we take $\gamma_\beta(t) \equiv \gamma(t)$.

Corollary 2.3.16. *Let $(P, G) \in \mathcal{Q}(H; \alpha, g)$ and*

$$g(\tau_l)\,\lambda^0(C(\tau_l)) > -\frac{1}{2} \ \ (l = 1,2,\dots), \tag{2.3.21}$$

where

$$C(t) \equiv \frac{1}{2}\left(H(t) + H^T(t)\right).$$

Then

(a) *the condition*

$$\limsup_{t\to+\infty}\left(\int_0^t \alpha'(s)\,\lambda^0(C(s))\,ds + \frac{1}{2}\sum_{0\le\tau_l<t}\ln\left(1 + 2g_l\lambda^0(C(\tau_l))\right)\right) < +\infty$$

guarantees the stability of the pair (P, G);

(b) *the condition*

$$\sup\left\{\int_\tau^t \alpha'(s)\lambda^0(C(s))\,ds + \frac{1}{2}\sum_{0\le\tau_l<t}\ln\left(1 + 2g_l\lambda^0(C(\tau_l))\right):\ t \ge \tau \ge 0\right\} < +\infty$$

guarantees the uniform stability of the pair (P, G);

(c) *the condition*

$$\lim_{t\to+\infty}\bigg(\int\limits_0^t \alpha'(s)\lambda^0(C(s))\,ds+\frac{1}{2}\sum_{0\le\tau_l<t}\ln\big(1+2g_l\lambda^0(C(\tau_l))\big)\bigg)=-\infty$$

guarantees the asymptotically stability of the pair (P,G)*;*

(d) *the condition*

$$\sup\bigg\{\frac{1}{\xi(t)-\xi(\tau)}\bigg(\int\limits_\tau^t \alpha'(s)\lambda^0(C(s))\,ds+\frac{1}{2}\sum_{0\le\tau_l<t}\ln\big(1+2g_l\lambda^0(C(\tau_l))\big)\bigg):\\ t\ge\tau\ge t^*,\ \xi(t)\ne\xi(\tau)\bigg)\bigg\}<0,$$

where $t^*\in\mathbb{R}_+$ *is some point, guarantees the* ξ*-exponentially asymptotic stability of the pair* (P,G)*;*

(e) *if, instead of condition* (2.3.21)*, the condition*

$$g(\tau_l)\,\lambda^0(C(\tau_l))<-\frac{1}{2}\ \ (l=1,2,\dots)$$

hold and

$$\limsup_{t\to+\infty}\bigg(\int\limits_0^t \alpha'(s)\lambda_0(C(s))\,ds+\frac{1}{2}\sum_{0\le\tau_l<t}\ln\big(1+2g_l\lambda_0(C(\tau_l)\big)\bigg)=+\infty,$$

then the pair (P,G) *is nonstable.*

2.3.2 The well-posedness of the initial problem on infinite intervals and stability

In this section, we consider the question of the well-posedness of problem (2.2.1)–(2.2.3) for the case $I=\mathbb{R}_+$, $P_0(t)\equiv P(t)$, $q_0(t)\equiv q(t)$, $G_0(\tau_l)\equiv G(\tau_l)$ $(l=1,2,\dots)$, $u_0(\tau_l)\equiv u(\tau_l)$, $0\le\tau_l<\tau_2\dots$ and $\lim\limits_{l\to+\infty}\tau_l=+\infty$.

Definition 2.3.2. Let $P_0\in L_{loc}(I;\mathbb{R}^{n\times n})$, $q_0\in L_{loc}(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)$ and $\tau_l\in I$ $(l=1,2,\dots)$, $\tau_l\ne\tau_m$ if $l\ne m$, be such that condition (2.2.9) holds. Then the initial problem (2.2.1), (2.2.2); (2.2.3) is said to be well-posed if condition (2.2.10) holds for every sequences $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$; $G_k\in B(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $u_k\in B(T;\mathbb{R}^n)$ $(k=1,2,\dots)$, t_k $(k=0,1,\dots)$ and c_k $(k=0,1,\dots)$ for which there exists a sequence $H_k\in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that conditions (1.2.5), (1.2.9) and (1.2.11) hold, and conditions (1.2.12), (2.2.11) and (2.2.12) are fulfilled uniformly on I.

It is evident that the statements of Theorems 2.2.1, 2.2.1′ and Corollaries 2.2.2 imply that the initial problem (2.2.1), (2.2.2); (2.2.3) is well-posed.

Definition 2.3.3. Let $P_0\in L_{loc}(I;\mathbb{R}^{n\times n})$, $q_0\in L_{loc}(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)n$ and $\tau_l\in I$ $(l=1,2,\dots)$, $\tau_l\ne\tau_m$ if $l\ne m$, be such that condition (2.2.9) holds. Then the initial problem (2.2.1), (2.2.2); (2.2.3) is said to be weakly well-posed if condition (2.2.10) holds for every sequences $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$; $G_k\in B(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $u_k\in B(T;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, t_k $(k=0,1,\dots)$ and c_k $(k=0,1,\dots)$ for which there exists a sequence $H_k \in \mathrm{ACV}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that conditions (1.2.5), (1.2.9) and (1.2.11) hold, and conditions (1.2.12),

$$\lim_{k\to+\infty} \bigvee_{a_*}^{t} \big(\mathcal{I}_\iota(H_k;P_k,G_k) - \mathcal{I}_\iota(H_0;P_0,G_0)\big) = 0$$

and

$$\lim_{k\to+\infty} \bigvee_{a_*}^{t} \big(\mathcal{B}_\iota(H_k;q_k,u_k) - \mathcal{B}_\iota(H_0;q_0,u_0)\big) = 0$$

hold uniformly on I, where the operators $\mathcal{I}_\iota$ and $\mathcal{B}_\iota$ are defined by (2.2.13) and (2.2.14), respectively.

Theorem 2.3.15. *Let $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, $q\in L_{loc}(\mathbb{R}_+;\mathbb{R}^n)$, $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ and $u\in B_{loc}(T;\mathbb{R}^n)$ be such that inequality* (2.1.4) *holds for every $l\in\{1,2,\dots\}$,*

$$\limsup_{t\to+\infty}\bigg(\int\limits_{t}^{\nu(\xi)(t)} \|P(\tau)\|\,d\tau + \sum_{t\le\tau_l<\nu(\xi)(t)} \big\|(I_n+G(\tau_l))^{-1}G_l\big\|\bigg) < +\infty,$$

and

$$\lim_{t\to+\infty}\bigg(\int\limits_{t}^{\nu(\xi)(t)} \|q(\tau)\|\,d\tau + \sum_{t\le\tau_l<\nu(\xi)(t)} \big\|(I_n+G(\tau_l))^{-1}u_l\big\|\bigg) = 0,$$

where the function ν is defined by (1.3.22). *Then the ξ-exponentially asymptotic stability of the pair (P,G) guarantees the well-posedness of problem* (2.2.1)–(2.2.3) *on the $\mathbb{R}_+$.*

Theorem 2.3.16. *Let $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G\in B_{loc}(T;\mathbb{R}^{n\times n})$ be such that inequality* (2.1.4) *holds for every $l\in\{1,2,\dots\}$. Let, moreover,*

$$q\in L(\mathbb{R}_+;\mathbb{R}^n) \ \ \textit{and} \ \ \sum_{l=1}^{+\infty}\|u_l\| < +\infty.$$

Then uniform stability of the pair (P,G) guarantees a weak well-posedness of problem (2.2.1)–(2.2.3) *on the $\mathbb{R}_+$.*

Chapter 3

Systems of ordinary differential equations

3.1 The well-posedness and stability of systems of ordinary differential equations

3.1.1 The well-posedness of the initial problem

In this section, we use the results of Section 2.1 for the initial problem

$$\frac{dx}{dt} = P_0(t)\,x + q_0(t) \ \text{ for } \ t \in I, \tag{3.1.1}$$

$$x(t_0) = c_0, \tag{3.1.2}$$

where $P_0 \in L_{loc}(I;\mathbb{R}^{n\times n})$, $q_0 \in L_{loc}(I;\mathbb{R}^n)$, $t_0 \in I$ and $c_0 \in \mathbb{R}^n$.

The results given below are the particular cases of analogous ones established for impulsive systems if we assume that $G_(\tau_l) \equiv O_{n\times n}$ $(l = 1,2,\dots,n)$ and the set of impulsive points is empty therein.

We formulate the results in a clear form because they differ from the earlier known results.

As above, let $x_0 \in \mathrm{AC}_{loc}(I;\mathbb{R}^n)$ be a unique solution of the initial problem (3.1.1), (3.1.2).

Along with the initial problem (3.1.1), (3.1.2), consider the sequence of initial problems

$$\frac{dx}{dt} = P_k(t)\,x + q_k(t) \ \text{ for } \ t \in I, \tag{3.1.1$_k$}$$

$$x(t_k) = c_k, \tag{3.1.2$_k$}$$

$(k = 1,2,\dots)$, where $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $q_k \in L_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$, $t_k \in I$ $(k = 1,2,\dots)$ and $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$.

Definition 3.1.1. We say that the sequence $(P_k, q_k; t_k)$ $(k = 1,2,\dots)$ belongs to the set $\mathcal{S}(P_0, q_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$, satisfying condition (1.2.5), condition (1.2.3) holds uniformly on I, where x_k is a unique solution of the initial problem (3.1.1$_k$), (3.1.2$_k$) for any sufficiently large k.

In this case, the operators $\mathcal{B}$ and $\mathcal{I}$ have the forms:

$$\mathcal{B}(X,Y)(t) = \int_{a_*}^{t} X(\tau)\,Y'(\tau)\,d\tau \ \text{ for } \ \ t \in I \tag{3.1.3}$$

if $X \in L_{loc}(I;\mathbb{R}^{n\times l})$ and $Y \in \mathrm{AC}_{loc}(I;\mathbb{R}^{l\times m})$, and

$$\mathcal{I}(X,Y)(t) = \int_{a_*}^{t} (X'(\tau) + X(\tau)Y'(\tau))X^{-1}(\tau)\,d\tau \ \text{ for } \ t \in I \tag{3.1.4}$$

if $X, Y \in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$, $\det X(t) \neq 0$, where a_* is some fixed point from I.

Due to (2.2.8), it is evident that

$$\mathcal{B}_\iota(X,Y)(t) \equiv \mathcal{B}(X,Y)(t) \ \text{ and } \ \mathcal{I}_\iota(X,Y)(t) \equiv \mathcal{I}(X,Y)(t). \tag{3.1.5}$$

Note that if $X(t) \equiv I_n$, then

$$\mathcal{B}(I_n,Y)(t) = \mathcal{I}(I_n,Y)(t) \equiv Y(t) - Y(a_*).$$

Theorem 3.1.1. *Let $P_0 \in L(I;\mathbb{R}^{n\times n})$ $q_0 \in L(I;\mathbb{R}^n)$, $t_0 \in I$ and a sequence of points $t_k \in I$ $(k = 1,2,\dots)$ be such that condition* (1.2.9) *holds. Then*

$$((P_k, q_k; t_k))_{k=1}^{\infty} \in \mathcal{S}(P_0, q_0; t_0) \tag{3.1.6}$$

if and only if there exists a sequence of matrix-functions $H_k \in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 0,1,\dots)$ such that condition (1.2.11) *holds and conditions* (1.2.12),

$$\lim_{k\to+\infty}\left\{\Big\|\mathcal{D}_{\mathcal{I}}(H_k,P_k;H_0,P_0)(\tau)\Big|_{t_k}^{t}\Big\|\left(1+\Big|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}}(H_k,P_k;H_0,P_0)\big)\Big|\right)\right\} = 0 \tag{3.1.7}$$

and

$$\lim_{k\to+\infty}\left\{\Big\|\mathcal{D}_{\mathcal{B}}(H_k,P_k;H_0,P_0)(\tau)\Big|_{t_k}^{t}\Big\|\left(1+\Big|\bigvee_{t_k}^{t}\big(\mathcal{D}_{\mathcal{I}}(H_k,P_k;H_0,P_0)\big)\Big|\right)\right\} = 0 \tag{3.1.8}$$

hold uniformly on I, where the operators $\mathcal{D}_{\mathcal{I}}$ and $\mathcal{D}_{\mathcal{B}}$ are defined, respectively, analogously to (0.0.5) *and* (0.0.4).

Note that, in Theorem 3.1.1, due to (3.1.3), (3.1.4) and (3.1.5), we have

$$\mathcal{I}(H_k;P_k)(t) \equiv \int_a^t \big(H_k'(\tau) + H_k(\tau)\,P_k(\tau)\big)H_k^{-1}(\tau)\,d\tau \ \ (k = 0,1,\dots)$$

and

$$\mathcal{B}(H_k;q_k)(t) \equiv \int_a^t H_k(\tau)q_k(\tau)\,d\tau \ \ (k = 0,1,\dots).$$

Theorem 3.1.2. *Let $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $t_0 \in I$, $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, $q_k \in L_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$, $t_k \in I$ $(k = 1,2,\dots)$ and $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ be such that conditions* (1.2.5), (1.2.9),

$$\lim_{k\to+\infty}\ \sup_{t\in I,\,t\neq t_k}\left\{\Big\|\int_{t_k}^{t}(P_k(\tau)-P_0(\tau))\,d\tau\Big\|\left(1+\Big|\int_{t_k}^{t}\|P_k(\tau)-P_0(\tau)\|\,d\tau\Big|\right)\right\} = 0$$

and

$$\lim_{k\to+\infty}\ \sup_{t\in I,\,t\neq t_k}\left\{\Big\|\int_{t_k}^{t}(q_k(\tau)-q_0(\tau))\,d\tau\Big\|\left(1+\Big|\int_{t_k}^{t}\|P_k(\tau)-P_0(\tau)\|\,d\tau\Big|\right)\right\} = 0$$

hold. Then condition (1.2.17) *holds, where x_k is the unique solution of the initial problem* ($3.1.1_k$), ($3.1.2_k$) *for any natural k.*

Theorem 3.1.3. *Let $P_0^* \in L(I;\mathbb{R}^{n\times n})$, $q_0^* \in L(I;\mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$ and x_0^* be a unique solution of the initial problem*

$$\frac{dx}{dt} = P_0^*(t)\,x + q_0^*(t) \ \ \textit{for} \ \ t \in I,$$

$$x(t_0) = c_0^*.$$

Let, moreover, the sequences $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $H_k \in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$, $q_k \in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $h_k \in \mathrm{AC}_{loc}(I;\mathbb{R}^n)$ $(k=0,1,\dots)$, $t_k \in I$ $(k=1,2,\dots)$ *and* $c_k^* \in \mathbb{R}^n$ $(k=1,2,\dots)$ *be such that conditions* (1.2.9),

$$\lim_{k\to+\infty} c_k^* = c_0^*,$$

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\bigg\{\bigg\|\int_{t_k}^{t}(P_k^*(\tau)-P_0^*(\tau))\,d\tau\bigg\|\bigg(1+\bigg|\int_{t_k}^{t}\|P_k^*(\tau)-P_0^*(\tau)\|\,d\tau\bigg|\bigg)\bigg\}=0 \tag{3.1.9}$$

and

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\bigg\{\bigg\|\int_{t_k}^{t}(q_k^*(\tau)-q_0^*(\tau))\,d\tau\bigg\|\bigg(1+\bigg|\int_{t_k}^{t}\|P_k^*(\tau)-P_0^*(\tau)\|\,d\tau\bigg|\bigg)\bigg\}=0$$

hold, where

$$c_k^* = H_k(t_k)c_k + h_k(t_k),\quad P_k^*(t)\equiv (H_k'(t)+H_k(t)P_k(t))H_k^{-1}(t)\ \ (k=1,2,\dots),$$
$$q_k^*(t) = h_k'(t) - P_k^*(t)h_k(t) + H_k(t)q_k(t)\ \ (k=1,2,\dots).$$

Then condition (1.2.27) *holds, where* x_k *is a unique solution of the initial problem* $(3.1.1_k), (3.1.2_k)$ *for any natural* k.

Remark 3.1.1. In Theorem 3.1.3, the vector-function $x_k^*(t)\equiv H_k(t)\,x_k(t)+h_k(t)$ is a solution of the problem

$$\frac{dx}{dt} = P_k^*(t)\,x + q_k^*(t)\ \text{ for }\ t\in I,$$
$$x(t_k) = c_k^*$$

for every natural k.

Corollary 3.1.1. *Let* $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $c_0\in\mathbb{R}^n$, $t_0\in I$ *and the sequences* $P_k \in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.9), (1.2.11) *and* (1.2.31) *hold, and conditions* (1.2.12), (3.1.9) *and*

$$\lim_{k\to+\infty}\bigg\{\bigg\|\int_{t_k}^{t}\Big(H_k(\tau)q_k(\tau)-H_0(\tau)q_0(\tau)-H_k(\tau)\varphi_k'(\tau)+P_k^*(\tau)\varphi_k(\tau)\Big)\,d\tau\bigg\|$$
$$\times\bigg(1+\bigg|\int_{t_k}^{t}\|P_k^*(\tau)-P_0^*(\tau)\|\,d\tau\bigg|\bigg)\bigg\}=0$$

hold uniformly on I, *where* $H_k\in \mathrm{AC}\,`_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$, $\varphi_k\in\mathrm{AC}_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $P_k^*(t)\equiv(H_k'(t)+H_k(t)\,P_k(t))H_k^{-1}(t)$ $(k=0,1,\dots)$. *Then condition* (1.2.33) *holds uniformly on* I, *where* x_k *is a unique solution of the initial problem* $(2.2.1_k), (2.2.3_k)$ *for any natural* k.

Below, as in Section 2.1, we give some sufficient conditions guaranteeing inclusion (3.1.6). To this end, we present a theorem, different from Theorem 3.1.1, concerning the necessary and sufficient conditions for inclusion (3.1.6), as well as corresponding propositions.

Theorem 3.1.1′. *Let* $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $c_0\in\mathbb{R}^n$, $t_0\in I$ *and the sequence of points* $t_k\in I$ $(k=1,2,\dots)$ *be such that condition* (1.2.9) *is satisfied. Then inclusion* (3.1.6) *holds if and only if there exists a sequence of matrix-functions* $H_k\in\mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *such that conditions* (1.2.11) *and*

$$\limsup_{k\to+\infty}\int_I \|H_k'(t)+H_k(t)P_k(t)\|\,dt < +\infty \tag{3.1.10}$$

hold, and conditions (1.2.12),

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} H_k(\tau)P_k(\tau)\,d\tau=\int\limits_{t_0}^{t} H_0(\tau)P_0(\tau)\,d\tau \tag{3.1.11}$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} H_k(\tau)q_k(\tau)\,d\tau=\int\limits_{t_0}^{t} H_0(\tau)q_0(\tau)\,d\tau \tag{3.1.12}$$

hold uniformly on I.

Theorem 3.1.1″. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $t_0\in I$ *and the sequence* $t_k\in I$ $(k=1,2,\dots)$ *be such that condition* (1.2.9) *is satisfied. Then inclusion* (3.1.6) *holds if and only if conditions* (1.2.37) *and*

$$\lim_{k\to+\infty}\int\limits_{a_*}^{t} X_k^{-1}(\tau)q_k(\tau)\,d\tau=\int\limits_{a_*}^{t} X_0^{-1}(\tau)q_0(\tau)\,d\tau$$

hold uniformly on I, *where* X_k *is the fundamental matrix of the homogeneous system*

$$\frac{dx}{dt}=P_k(t)\,x \ \ \text{for} \ \ t\in I \tag{3.1.1$_{k0}$}$$

for every $k\in\{0,1,\dots\}$.

Theorem 3.1.2′. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $c_0\in\mathbb{R}^n$, $t_0\in I$, *the sequences* $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $t_k\in I$ $(k=1,2,\dots)$ *and* $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ *be such that conditions* (1.2.5), (1.2.9) *and*

$$\limsup_{k\to+\infty}\int\limits_{I}\|P_k(t)\|\,dt<+\infty \tag{3.1.13}$$

hold, and the conditions

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} P_k(\tau)\,d\tau=\int\limits_{t_0}^{t} P_0(\tau)\,d\tau \tag{3.1.14}$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} q_k(\tau)\,d\tau=\int\limits_{t_0}^{t} q_0(\tau)\,d\tau \tag{3.1.15}$$

hold uniformly on I. *Then condition* (1.2.3) *holds uniformly on* I, *where* x_k *is a unique solution of the initial problem* $(3.1.1_k),(3.1.2_k)$ *for any natural* k.

Corollary 3.1.2. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $t_0\in I$, *the sequences* $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.9), (1.2.11) *and* (3.1.10) *hold, and conditions* (1.2.12), (3.1.11) *and* (3.1.12) *hold uniformly on* I, *where* $H_k\in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$. *Then inclusion* (3.1.6) *is valid.*

Corollary 3.1.3. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $t_0\in I$, *the sequences* $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.9) *and* (3.1.10) *hold, and conditions* (1.2.12),

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} H_k(\tau)P_k(\tau)\,d\tau=\int\limits_{t_0}^{t} P^*(\tau)\,d\tau$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} H_k(\tau)q_k(\tau)\,d\tau=\int\limits_{t_0}^{t} q^*(\tau)\,d\tau$$

hold uniformly on I, *where* $H_0(t)\equiv I_n$, $H_k\in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $P^*\in L(I;\mathbb{R}^{n\times n})$, $q^*\in L(I;\mathbb{R}^n)$. *Then*

$$((P_k,q_k;t_k))_{k=1}^{\infty}\in\mathcal{S}(P_0^*,q_0^*;t_0),$$

where $P_0^*(t)\equiv P_0(t)-P^*(t)$ *and* $q_0^*(t)\equiv q_0(t)-q^*(t)$.

Corollary 3.1.4. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $t_0\in I$, *the sequences* $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that condition* (1.2.9) *hold and let there exist a natural number* m *and the matrix-functions* $B_j\in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ $(j=1,\dots,m-1)$ *such that condition*

$$\limsup_{k\to+\infty}\int\limits_{I}\|H'_{k\,m-1}(t)+H_{k\,m-1}(t)P_k(t)\|\,dt<+\infty$$

holds, and conditions

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}(H'_{k\,j-1}(\tau)+H_{k\,j-1}(\tau)P_k(\tau))\,d\tau=B_j(t)-B_j(t_0)\ \ (j=1,\dots,m-1),$$

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}(H'_{k\,m-1}(\tau)+H_{k\,m-1}(\tau)P_k(\tau))\,d\tau=\int\limits_{t_0}^{t}P_0(\tau)\,d\tau$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}H_{k\,m-1}(\tau)q_k(\tau)\,d\tau=\int\limits_{t_0}^{t}q_0(\tau)\,d\tau$$

hold uniformly on I, *where*

$$H_{k0}(t)=I_n,$$

$$H_{kj}(t)=\bigg(I_n-\int\limits_{t_k}^{t}\big(H'_{k\,j-1}(\tau)+H_{k\,j-1}(\tau)P_k(\tau)\big)\,d\tau+B_j(t)-B_j(t_k)\bigg)H_{k\,j-1}(t)$$
$$\text{for }\ t\in I\ \ (j=1,\dots,m-1;\ k=1,2,\dots).$$

Then inclusion (3.1.6) *is valid.*

If $m=1$, then Corollary 3.1.4 coincides with Theorem 3.1.2′.
If $m=2$, then Corollary 3.1.4 has the following form.

Corollary 3.1.4′. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $t_0\in I$, *the sequences* $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that conditions* (1.2.9) *and* (3.1.10) *hold, and the conditions*

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}P_k(\tau)\,d\tau=B(t)-B(t_0),$$

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}H_k(\tau)P_k(\tau)\,d\tau=\int\limits_{t_0}^{t}P_0(\tau)\,d\tau$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} H_k(\tau)q_k(\tau)\,d\tau=\int\limits_{t_0}^{t} q_0(\tau)\,d\tau$$

hold uniformly on I, *where* $B\in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ *and*

$$H_k(t)=I_n-\int\limits_{t_k}^{t} P_k(\tau)\,d\tau+B(t)-B(t_k)\ \ \text{for}\ \ t\in I\ \ (k=1,2,\dots).$$

Then inclusion (3.1.6) *is valid.*

Corollary 3.1.5. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that the condition* (1.2.9) *hold. Then inclusion* (3.1.6) *holds if and only if there exist the matrix-functions* $Q_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *such that*

$$\limsup_{k\to+\infty}\int\limits_{I}\|P_k(t)-Q_k(t)\|\,dt<+\infty \tag{3.1.16}$$

and the conditions

$$\lim_{k\to+\infty} Z_k^{-1}(t)=Z_0^{-1}(t),$$

$$\int\limits_{t_k}^{t} Z_k^{-1}(\tau)\,P_k(\tau)\,d\tau=\int\limits_{t_0}^{t} Z_0^{-1}(\tau)P_0(\tau)\,d\tau$$

and

$$\int\limits_{t_k}^{t} Z_k^{-1}(\tau)\,q_k(\tau)\,d\tau=\int\limits_{t_0}^{t} Z_0^{-1}(\tau)q_0(\tau)\,d\tau$$

hold uniformly on I, *where* Z_k $(Z_k(t_k)=I_n)$ *is the fundamental matrix of the homogeneous system*

$$\frac{dx}{dt}=Q_k(t)\,x\ \ \text{for a.a.}\ \ t\in I$$

for every $k\in\{0,1\dots\}$.

Corollary 3.1.6. *Let* $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, *and* $t_k\in I$ $(k=1,2,\dots)$ *be such that condition* (1.2.9) *is satisfied. Let, moreover, there exist matrix-functions* $Q_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$, *satisfying the Lappo–Danilevskiĭ condition, such that condition* (3.1.16) *hold, and the conditions*

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t} Q_k(\tau)\,d\tau=\int\limits_{t_0}^{t} Q_0(\tau)\,d\tau,$$

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}\exp\bigg(-\int\limits_{t_k}^{\tau} Q_k(s)\,ds\bigg)P_k(\tau)\,d\tau=\int\limits_{t_0}^{t}\exp\bigg(-\int\limits_{t_k}^{\tau} Q_0(s)\,ds\bigg)P_0(\tau)\,d\tau$$

and

$$\lim_{k\to+\infty}\int\limits_{t_k}^{t}\exp\bigg(-\int\limits_{t_k}^{\tau} Q_k(s)\,ds\bigg)q_k(\tau)\,d\tau=\int\limits_{t_0}^{t}\exp\bigg(-\int\limits_{t_k}^{\tau} Q_0(s)\,ds\bigg)q_0(\tau)\,d\tau$$

hold uniformly on I. *Then inclusion* (3.1.6) *is valid.*

Remark 3.1.2. In Corollaries 3.1.5 and 3.1.6, if $Q_k(t)\equiv P_k(t)$ for any sufficiently large k, then condition (3.1.16) vanishes.

Corollary 3.1.7. *Let $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $t_0\in I$, $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, and $t_k\in I$ $(k=1,2,\dots)$ be such that condition* (1.2.9) *is satisfied. Let, moreover, the matrix-functions P_k $(k=0,1,\dots)$ satisfy the Lappo–Danilevskiĭ condition and conditions* (2.2.30), (2.2.32) *and* (2.2.33) *hold uniformly on I. Then inclusion* (3.1.6) *is valid.*

Corollary 3.1.8. *Let $P_0=(p_{0ij})_{i,j=1}^n\in L(I;\mathbb{R}^{n\times n})$, $q_0=(q_{0i})_{i=1}^n\in L(I;\mathbb{R}^n)$, $t_0\in I$, $P_k=(p_{kij})_{i,j=1}^n\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k=(q_{ki})_{i=1}^n\in L_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ and $t_k\in I$ $(k=1,2,\dots)$ be such that conditions* (1.2.9) *and*

$$\limsup_{k\to+\infty}\sum_{i,j=1,\,i\neq j}^n\int_I|p_{kij}(t)|\,dt<+\infty$$

hold, and conditions

$$\lim_{k\to+\infty}\int_{t_k}^t p_{kii}(\tau)\,d\tau=\int_{t_0}^t p_{0ii}(\tau)\,d\tau,$$

$$\lim_{k\to+\infty}\bigg(\int_{t_k}^t z_{kii}^{-1}(\tau)p_{kij}(\tau)\,d\tau\bigg)=\int_{t_0}^t z_{0ii}^{-1}(\tau)p_{0ij}(\tau)\,d\tau\ \ (i\neq j;\,i,j=1,\dots,n)$$

and

$$\lim_{k\to+\infty}\bigg(\int_{t_k}^t z_{kii}^{-1}(\tau)q_{ki}(\tau)\,d\tau\bigg)=\int_{t_k}^t z_{0ii}^{-1}(\tau)q_{0i}(\tau)\,d\tau\ \ (i=1,\dots,n)$$

hold uniformly on I, where z_{kii} is a unique solution of the initial problem

$$\frac{dz}{dt}=p_{kii}(t),\quad z(t_k)=1\ \ (i=1,\dots,n;\ k=1,2,\dots).$$

Then inclusion (3.1.6) *is valid.*

Remark 3.1.3. In Theorem 3.1.3 and Corollaries 3.1.1, 3.1.2, without loss of generality, we can assume that $H_0(t)=I_n$.

Remark 3.1.4. Theorem 3.1.2′ has been obtained in earlier works (see [34, 40]). In this theorem, condition (3.1.13) is essential and it cannot be neglected. So, if condition (3.1.13) is not satisfied, the statement of the theorem is not true. In this connection, we give an example from [34, 37, 40, 46].

Example 3.1.1. Let $I=[0,2\pi]$, $n=1$, $c_k=c_0=0$ $(k=1,2,\dots)$, $P_0(t)=q(t)\equiv0$, $P_k(t)=k\cos k^2t$ $(k=1,2,\dots)$, $q_k(t)=-k\sin k^2t$ $(k=1,2,\dots)$. Then

$$x_0(t)\equiv0,\quad x_k(t)\equiv-k\int_0^t\exp\Big(\frac{\sin k^2t}{k}-\frac{\sin k^2\tau}{k}\Big)\sin k^2\tau\ \ (k=1,2,\dots)$$

and

$$\lim_{k\to+\infty}\Big(x_k(t)-\frac{t}{2}\Big)=0$$

uniformly on $[0,\pi]$. In this case, all conditions of Theorem 3.1.2′, except condition (3.1.13), are fulfilled. On the other hand, this case is consistent with Corollary 3.1.3, since all its conditions are satisfied for $P^*(t)\equiv0$, $q^*(t)\equiv1/2$ and

$$H_k(t)\equiv\exp\Big(-\frac{\sin k^2t}{k}\Big)\ \ (k=1,2,\dots),$$

and the function $x^*(t) \equiv t/2$ is the solution of the initial problem

$$\frac{dx}{dt} = P^*(t)x + q^*(t), \quad x(0) = 0.$$

Below, based on the above example, we construct a homogeneous system ($n = 2$) for which condition (3.1.13) is violated, but the situation analogous to the given above is explained in Theorem 3.1.1′.

Example 3.1.2. Let $I = [0, 2\pi]$, $n = 2$,

$$c_k = \begin{pmatrix} 1 \\ 1 \\ \frac{1}{k} \end{pmatrix}, \quad c_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_k(t) = \begin{pmatrix} k\cos k^2 t & 0 \\ -k\sin k^2 t & 0 \end{pmatrix}), \quad P_0(t) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix},$$

$$q_k(t) = q_0(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (k = 1, 2, \dots).$$

Then $x_0(t) \equiv \begin{pmatrix} 1 \\ -\frac{t}{2} \end{pmatrix}$. In this case, condition (3.1.13) does not hold, as well. But all the conditions of Theorem 3.1.1′ are satisfied for $H_k(t) \equiv Y_0(t)\, Y_k^{-1}(t)$, where Y_k, $Y_k(0) = I_2$, is the fundamental matrix of system (3.1.1$_{k0}$) for every $k \in \{0, 1, \dots\}$.

Remark 3.1.5. In Theorem 3.1.1′, as opposed to Theorem 3.1.2′, it was not assumed that equalities (3.1.14) and (3.1.15) hold uniformly on I. Below we will give an example of a sequence of the initial problems for which inclusion (3.1.6) holds but condition (3.1.14) is not fulfilled uniformly on I.

Example 3.1.3. Let $I = [0, 2\pi]$, $n = 2$, and for every natural k and $t \in [0, 2\pi]$,

$$P_k(t) = \begin{pmatrix} 0 & p_{k1}(t) \\ 0 & p_{k2}(t) \end{pmatrix}, \quad P_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi_k(t) = q_k(t) = q_0(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$p_{k1}(t) = \begin{cases} \left(\sqrt{k} + \sqrt[4]{k}\right)\sin kt & \text{for } t \in I_k, \\ \sqrt{k}\sin kt & \text{for } t \in [0, 2\pi] \setminus I_k; \end{cases}$$

$$p_{k2}(t) = \begin{cases} -\alpha_k'(t) \cdot (1 - \alpha_k(t))^{-1} & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0, 2\pi] \setminus I_k; \end{cases}$$

$$\beta_k(t) = \int_0^t (1 - \alpha_k(\tau)) \cdot p_{k1}(\tau)\, d\tau;$$

$$\alpha_k(t) = \begin{cases} 4\pi^{-1}\left(\sqrt[4]{k} + 1\right)^{-1}\sin kt & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0, 2\pi] \setminus I_k, \end{cases}$$

where $I_k = \bigcup\limits_{m=0}^{k-1}]2mk^{-1}\pi, (2m+1)k^{-1}\pi[$. Let, moreover, $Y_k(t)$, $Y_k(0) = I_n$, be a fundamental matrix of system (3.1.1$_{k0}$) for every $k \in \{0, 1, \dots\}$.

It can easily be shown that for every natural k we have

$$Y_0(t) = I_n, \quad Y_k(t) = \begin{pmatrix} 1 & \beta_k(t) \\ 0 & 1 - \alpha_k(t) \end{pmatrix} \quad \text{for } t \in [0, 2\pi]$$

and

$$\lim_{k\to+\infty} Y_k(t) = Y_0(t)$$

uniformly on $[0, 2\pi]$, since

$$\lim_{k\to+\infty} \|\alpha_k\|_c = \lim_{k\to+\infty} \|\beta_k\|_c = 0.$$

Note that

$$\lim_{k\to+\infty} \int_0^{2\pi} p_{k1}(t)\, dt = 2 \lim_{k\to+\infty} \sqrt[4]{k} = +\infty$$

and, in addition,

$$\lim_{k\to+\infty}\int_0^{2\pi} p_{k1}(t)\,dt = 2\lim_{k\to+\infty}\sqrt[4]{k} = +\infty.$$

Therefore, the conditions of Theorem 3.1.2′ are not satisfied.

On the other hand, if we assume that

$$H_k(t) = Y_k^{-1}(t) \ \text{ for } \ t\in[0,2\pi] \ \ (k=1,2,\dots),$$

then the conditions of Theorems 3.1.1′ are fulfilled.

3.1.2 The stability in the Liapunov sense

In this section, we use the results of Section 2.1 for the stability in the Liapunov sense of the ordinary differential system (2.3.1), where $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $q\in L_{loc}(\mathbb{R}_+;\mathbb{R}^n)$.

The results of this section are the particular cases of the corresponding results of Section 2.1 if we assume that $G(l)\equiv O_{n\times n}$ and $u_(l)\equiv 0$ for $l\in\{1,2,\dots\}$.

Mainly, we consider the case $P\notin L(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and assume that the function ξ that appears in Section 1.3 is nondecreasing continuous and belongs to $\mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R}_+)$. So we have the following definition.

In this subsection, we will assume that ξ, in addition, is from $\mathrm{AC}_{loc}(|\mathbb{R}_+;\mathbb{R}_+)$.

Definition 3.1.2. The matrix-function $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is said to be stable in one or other sense if the zero solution of the homogeneous system $(2.3.1_0)$ is stable in the same sense.

Theorem 3.1.4. *The matrix-function $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is stable if and only if there exists a nonsingular matrix-function $H\in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that conditions* (1.3.8) *and*

$$\int_0^{+\infty} \|H'(t)+H(t)P(t)\|\,dt < +\infty \tag{3.1.17}$$

hold.

Theorem 3.1.5. *The matrix-function $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is uniformly stable if and only if there exists a nonsingular matrix-function $H\in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that conditions* (1.3.10) *and* (3.1.17) *hold.*

Theorem 3.1.6. *The matrix-function $P\in L_{loc}(\mathbb{R}_+:\mathbb{R}^{n\times n})$ is asymptotically stable if and only if there exists a nonsingular matrix-function $H\in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that conditions* (1.3.11) *and* (3.1.17) *hold.*

Theorem 3.1.7. *The matrix-function $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is ξ-exponentially asymptotically stable if and only if there exists a nonsingular $H\in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that conditions* (1.3.12) *and*

$$\int_0^{+\infty} \exp(-\eta\xi(\tau))\|H'(t)+H(t)P(t)\|\,dt < +\infty$$

hold.

Corollary 3.1.9. *Let a matrix-function $Q\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\int_0^{+\infty} \big\|Y^{-1}(t)(P(t)-Q(t))\big\|\,dt < +\infty,$$

where Y $(Y(0)=I_n)$ is the fundamental matrix of system (2.3.5). *Then the stability in one or another sense of the matrix-function Q guarantees the stability of the matrix-function P in the same sense.*

Theorem 3.1.8. *Let a matrix-function $P_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be uniformly stable. Let, moreover, the matrix-function $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\int_0^{+\infty} \Big\| \big(H'(t) + H(t)P(t) - P_0(t)H(t)\big)H^{-1}(t)\Big\|\, dt < +\infty, \tag{3.1.18}$$

where $H \in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is a nonsingular matrix-function satisfying condition (1.3.10)*. Then the matrix-function P is also uniformly stable.*

Remark 3.1.6. In Theorem 3.1.8, if $H(t) \equiv I_n$, then condition (3.1.18) has the form

$$\int_0^{+\infty} \|P(t) - P_0(t)\|\, dt < +\infty.$$

Theorem 3.1.9. *Let a matrix-function $P_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable. Let, moreover, the matrix-function $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\lim_{t\to+\infty} \int_0^{\nu(\xi)(t)} \|P(\tau) - P_0(\tau)\|\, d\tau = 0,$$

where the function $\nu(\xi)(t)$ is defined by (1.3.22)*. Then the matrix-function P is also ξ-exponentially asymptotically stable.*

Corollary 3.1.10. *Let the components of the matrix-function $P = (p_{ij})_{i,j=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\lim_{t\to+\infty} \int_t^{\nu(\xi)(t)} \|p_{ik}(\tau)\|\, d\tau = 0 \ \ (i \neq k;\ i,k = 1,\dots,n) \tag{3.1.19}$$

and

$$\int_\tau^t p_{ii}(s)\, ds \leq -\eta\big(\xi(t) - \xi(\tau)\big) \ \ \textit{for} \ \ t > \tau \geq 0 \ \ (i = 1,\dots,n),$$

where $\eta > 0$, and the function $\nu(\xi)(t)$ is defined by (1.3.22)*. Then the matrix-function P is ξ-exponentially asymptotically stable.*

Corollary 3.1.11. *Let a matrix-function $P_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable, and the matrix-function $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\lim_{t\to+\infty} \int_t^{\xi(t)+1} \|P(\tau) - P_0(\tau)\|\, d\tau = 0.$$

Then the matrix-function P is also ξ-exponentially asymptotically stable.

Proposition 3.1.1. *Let a matrix-function $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable and a vector-function $q \in L_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ be such that*

$$\lim_{t\to+\infty} \int_t^{\xi(t)+1} \|q(\tau)\|\, d\tau = 0,$$

where the function $\nu(\xi)(t)$ is defined by (1.3.22)*. Then every solution x of system* (3.1.1) *satisfies condition* (1.3.6)*.*

Proposition 3.1.2. *Let a matrix-function $P_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable. Let, moreover, the matrix-function $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\lim_{t\to+\infty}\frac{1}{\xi(t)}\int\limits_0^t \|P(\tau)-P_0(\tau)\|\,d\tau=0.$$

Then the matrix-function P is also ξ-exponentially asymptotically stable.

Theorem 3.1.10. *Let the components of the matrix-function $P=(p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfy the conditions*

$$\sup\left\{\int\limits_0^t p_{ii}(s)\,ds:\ t\geq t^*\right\}<+\infty\ \ (i=1,\dots,n),$$

and

$$\int\limits_{t^*}^t \exp\bigg(\int\limits_\tau^t p_{ii}(s)\,ds\bigg)|p_{ik}(\tau)|\leq h_{ik}\ \ \text{for}\ \ t\geq t^*\ \ (i\neq k;\ i,k=1,\dots,n),$$

where t^* and $h_{ik}\in\mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$. Let, moreover, the matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ $(i=1,\dots,n)$, be such that condition (1.3.31) *holds. Then the matrix-function P is stable.*

Theorem 3.1.11. *Let the components of the matrix-function $P=(p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that conditions* (3.1.19) *and*

$$\sup\left\{\int\limits_\tau^t p_{ii}(s)\,ds:\ t>\tau\geq t^*\right\}<+\infty\ \ (i=1,\dots,n)$$

hold, where t^* and $h_{ik}\in\mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$. Let, moreover, a matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ $(i=1,\dots,n)$, be such that condition (1.3.31) *holds. Then the matrix-function P is uniformly stable.*

Corollary 3.1.12. *Let the components of the matrix-function $P=(p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that condition* (2.3.11) *holds, where t_* and $h_{ik}\in\mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$. Let, moreover, the matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ $(i=1,\dots,n)$, be such that condition* (1.3.31) *holds. Then the matrix-function P is uniformly stable.*

Theorem 3.1.12. *Let the components of the matrix-function $P=(p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that the conditions*

$$\int\limits_{t^*}^t p_{ii}(s)\,ds\leq -\xi(t)+\xi(t^*)\ \ \text{for}\ \ t\geq t^*\ \ (i=1,\dots,n)$$

and

$$\int\limits_{t^*}^t \exp\bigg(\xi(t)-\xi(\tau)+\int\limits_\tau^t p_{ii}(s)\,ds\bigg)|p_{ik}(\tau)|\,d\tau\leq h_{ik}\ \ \text{for}\ \ t\geq t^*\ \ (i\neq k;\ i,k=1,\dots,n)$$

hold, where t_* and $h_{ik}\in\mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$. Let, moreover, a matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ $(i=1,\dots,n)$, be such that condition (1.3.31) *holds. Then the matrix-function P is ξ-asymptotically stable.*

Corollary 3.1.13. *Let the components of the matrix-function $P=(p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that condition* (2.3.11) *holds, where t_* and $h_{ik}\in\mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$, and let $h_{ik}\in\mathbb{R}_+$ $(i\neq k;\ i,k=1,\dots,n)$ be such that the matrix $H=(h_{ik})_{i,k=1}^n$, where $h_{ii}=0$ $(i=1,\dots,n)$, satisfies* *condition* (1.3.31). *Let, moreover, there exists a function* $a_0 \in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R})$ *satisfying condition* (1.3.41) *such that*

$$a_0(t) - a_0(\tau) \le \min\left\{\left|\int_\tau^t p_{ii}(s)\,ds\right| : \ (i=1,\dots,n)\right\} \ \text{for } \ t \ge \tau \ge t^*.$$

Then the matrix-function P *is asymptotically and also uniformly stable.*

Corollary 3.1.14. *Let the components of the matrix-function* $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that condition* (2.3.11) *holds, where* t_* *and* $h_{ik} \in \mathbb{R}_+$ $(i \ne k;\ i,k=1,\dots,n)$, *and let* $h_{ik} \in \mathbb{R}_+$ $(i \ne k;\ i,k=1,\dots,n)$ *be such that the matrix* $H = (h_{ik})_{i,k=1}^n$, *where* $h_{ii} = 0$ $(i=1,\dots,n)$, *satisfies condition* (1.3.31). *Let, moreover,*

$$\int_0^{+\infty} \eta_0(s)\,ds = -\infty,$$

where the function $\eta_0(t)$ *is defined by* (2.3.15). *Then the matrix-function* P *is asymptotically and also uniformly stable.*

Theorem 3.1.13. *Let the components of the matrix-function* $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that conditions* (2.3.9),

$$\sup\left\{(\xi(t)-\xi(\tau))^{-1}\left(\int_\tau^t p_{ii}(s)\,ds\right) : \ t>\tau\ge t^*,\ \xi(t)\ne\xi(\tau);\ t,\tau\in\mathbb{R}_+\right\} \le -\gamma \ \ (i=1,\dots,n) \tag{3.1.20}$$

and

$$\int_{t^*}^t \exp\left(\gamma(\xi(t)-\xi(\tau)) + \int_\tau^t p_{ii}(s)\,ds\right)|p_{ik}(\tau)|\,d\tau \le h_{ik} \ \ \text{for } \ t \ge t^* \ \ (i \ne k;\ i,k=1,\dots,n)$$

hold, where $\gamma > 0$, t_* *and* $h_{ik} \in \mathbb{R}_+$ $(i \ne k;\ i,k=1,\dots,n)$. *Let, moreover, the matrix* $H = (h_{ik})_{i,k=1}^n$, *where* $h_{ii} = 0$ $(i=1,\dots,n)$, *be such that condition* (1.3.31) *holds. Then the matrix-function* P *is* ξ*-asymptotically stable.*

Corollary 3.1.15. *Let the components of the matrix-function* $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that conditions* (2.3.11) *and* (3.1.20) *hold, where* $\gamma > 0$, t_* *and* $h_{ik} \in \mathbb{R}_+$ $(i \ne k;\ i,k=1,\dots,n)$. *Let, moreover, the matrix* $H = (h_{ik})_{i,k=1}^n$, *where* $h_{ii} = 0$ $(i=1,\dots,n)$, *be such that condition* (1.3.31) *hold. Then the matrix-function* P *is* ξ*-asymptotically stable.*

Theorem 3.1.14. *Let the matrix-functions* $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *and* $P_0 = (p_{0ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that condition* (2.3.17) *holds. Let, moreover, the matrix-function* P_0 *be stable (uniformly stable, asymptotically stable or* ξ*-exponentially asymptotically stable). Then the matrix-function* P *is also stable (uniformly stable, asymptotically stable or* ξ*-exponentially asymptotically stable).*

Theorem 3.1.15. *Let* $\alpha_{ik} \in \mathbb{R}$ $(i,k=1,\dots,n)$ *and* $\mu_i \in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ $(i,\dots,n)$ *be nondecreasing functions such that*

$$\int_0^{+\infty} \eta_0(s)\,ds = -\infty,$$

where

$$\eta_0(t) \equiv \min\left\{|\alpha_{ii}|\mu_i'(t) : \ i=1,\dots,n\right\}.$$

Then conditions (1.3.31) *and* (1.3.47), *where the constant matrix* H *is defined by* (1.3.48), *are sufficient for the matrix-function* $P(t) \equiv (\alpha_{ik}\,\mu_i(t))_{i,k=1}^n$ *to be asymptotically stable; and if condition* (1.3.49) *holds, then conditions* (1.3.31) *and* (1.3.47) *are necessary, as well.*

Corollary 3.1.16. *Let a matrix-function $Q \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be uniformly stable and the condition*

$$\bigg\| \int_0^{+\infty} |Y^{-1}(t)|\,|P(t)-Q(t)+\eta\xi'(t)I_n|\,dt \bigg\| < +\infty$$

hold, where Y $(Y(0)=I_n)$ is the fundamental matrix of system (2.3.5), *and η is a positive number. Then the matrix-function P is ξ-exponentially asymptotically stable.*

Remark 3.1.7. Corollary 3.1.16 is likewise true for the limit case $(\eta = 0)$, if we require the ξ-exponentially asymptotically stability of the matrix-function Q instead of the uniform stability.

Corollary 3.1.17. *Let $Q \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be a continuous matrix-function satisfying the Lappo–Danilevskiĭ condition and*

$$\bigg\| \int_0^{+\infty} \exp\Big(-\int_0^t Q(\tau)\,d\tau\Big)|P(t)-Q(t)+\eta\xi'(t)I_n|\,dt \bigg\| < +\infty, \tag{3.1.21}$$

where η is a nonnegative number. Then:

(a) *the uniform stability of the matrix-function Q guarantees ξ-exponentially asymptotical stability of the matrix-function P if $\eta > 0$;*

(b) *ξ-exponentially asymptotical stability of the matrix-function Q guarantees ξ-exponentially asymptotical stability of the matrix-function P if $\eta = 0$.*

Corollary 3.1.18. *Let there exist a positive number η such that the components of the matrix-function $P = (p_{ik})_{i,k=1}^n \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfy the conditions*

$$p_{ii}(t) \le -\eta\xi'(t) \quad \text{for a.a.} \quad t \in \mathbb{R}_+ \ \ (i=1,\dots,n)$$

and

$$\int_0^{+\infty} |y_i^{-1}(t)|\,|p_{ik}(t)|\,dt < +\infty \ \ (i \ne k;\ i,k=1,\dots,n),$$

where

$$y_i(t) \equiv \exp\bigg(\int_0^t (p_{ii}(s)+\eta\xi'(s))\,ds\bigg) \ \ (i=1,\dots,n).$$

Then the matrix-function P is ξ-exponentially asymptotical stable.

Corollary 3.1.19. *Let a matrix-function $Q \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be stable and*

$$\int_0^{+\infty} \exp\big(-\operatorname{tr}(Q(t))\big)\|P(t)-Q(t)\|\,dt < +\infty.$$

Then the matrix-function P is also stable.

Theorem 3.1.16. *Let the matrix-function $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$P(t) = \sum_{k=1}^m \alpha_k'(t)\,B_k \ \ \text{for a.a.} \ \ t \in \mathbb{R}_+, \tag{3.1.22}$$

where $\alpha_k \in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R})$ $(k=1,\dots,m)$ and $B_k \in \mathbb{R}^{n\times n}$ $(k=1,\dots,m)$ are pairwise permutable constant matrices. Let, moreover, $(\lambda-\lambda_{ki})^{n_{ki}}$ $(i=1,\dots,m_k;\ \sum\limits_{i=1}^{m_l} n_{ki} = n)$ be elementary divisors of the matrix B_k for every $k \in \{1,\dots,m\}$. Then:

(a) *the matrix-function P is stable if and only if condition* (1.3.60) *holds;*

(b) *the matrix-function P is asymptotically stable if and only if condition* (1.3.61) *holds.*

Corollary 3.1.20. *Let the matrix-function $P \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that conditions* (3.1.22) *hold, where $B_k \in \mathbb{R}^{n\times n}$ $(k = 1,\dots,m)$ are pairwise permutable constant matrices, and $\alpha_k \in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R})$ are such that condition* (1.3.62) *holds. Then:*

(a) *the matrix-function P is stable if and only if every eigenvalue of the matrices B_k $(k = 1,\dots,m)$ has a nonpositive real part; in addition, every elementary divisor corresponding to the eigenvalue with a zero real part is simple;*

(b) *the matrix-function P is asymptotically stable if and only if every eigenvalue of the matrices B_k $(k = 1,\dots,m)$ has a negative real part.*

Theorem 3.1.17. *Let $P(t) \equiv \alpha'(t)\,H(t)$ and*

$$(H(t)x * x) \le p(t)(x * x) \ \text{ for a.a. } \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R}^n, \tag{3.1.23}$$

where $\alpha \in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R})$ is a nondecreasing function, $H \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $p \in L_{loc}(\mathbb{R}_+;\mathbb{R})$. Let, moreover, $\gamma(t) \equiv \exp(\int\limits_0^t p(\tau)\alpha'(\tau)\,d\tau)$. Then:

(a) *condition* (1.3.72) *guarantees the stability of the matrix-function P;*

(b) *condition* (1.3.73) *guarantees the uniform stability of the matrix-function P;*

(c) *condition* (1.3.74) *guarantees the asymptotical stability of the matrix-function P;*

(d) *condition* (1.3.75), *where $t^* \in \mathbb{R}_+$ is some point, guarantees the ξ-exponentially asymptotical stability of the matrix-function P;*

(e) *if the inequality, opposite to inequality* (3.1.23), *and condition* (1.3.76) *hold, then the matrix-function P is nonstable.*

Here in conditions (1.3.72)–(1.3.75) *we take $\gamma_\beta(t) \equiv \gamma(t)$.*

Corollary 3.1.21. *Let $P(t) \equiv \alpha'(t)\,H(t)$, where $\alpha \in \mathrm{AC}_{loc}(\mathbb{R}_+;\mathbb{R})$ is a nondecreasing function and $H \in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, and let*

$$C(t) \equiv \frac{1}{2}\big(H(t) + H^T(t)\big).$$

Then:

(a) *the condition*

$$\limsup_{t\to+\infty} \int\limits_0^t \alpha'(s)\lambda^0(C(s))\,ds < +\infty$$

guarantees the stability of the matrix-function P;

(b) *the condition*

$$\sup\bigg\{\int\limits_\tau^t \alpha'(s)\lambda^0(C(s))\,ds : \ t \ge \tau \ge 0\bigg\} < +\infty$$

guarantees the uniform stability of the matrix-function P;

(c) *the condition*

$$\lim_{t\to+\infty} \int\limits_0^t \alpha'(s)\lambda^0(C(s))\,ds = -\infty$$

guarantees the asymptotical stability of the matrix-function P;

(d) *the condition*

$$\sup\left\{\frac{1}{\xi(t)-\xi(\tau)}\int\limits_{\tau}^{t}\alpha'(s)\lambda^0(C(s))\,ds:\ t\geq\tau\geq t^*,\ \xi(t)\neq\xi(\tau)\right\}<0,$$

where $t^*\in\mathbb{R}_+$ is some point, guarantees the ξ-exponentially asymptotical stability of the matrix-function P;

(e) *if*

$$\limsup_{t\to+\infty}\left(\int\limits_{0}^{t}\alpha'(s)\lambda_0(C(s))\,ds\right)=+\infty,$$

then the matrix-function P is nonstable.

3.1.3 The well-posedness of the initial problem on infinite intervals and stability

In this section, we consider the question on the well-posedness of problem (3.1.1), (3.1.2) for the case $I=\mathbb{R}_+$, $P_0\equiv P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, $q_0\equiv q\in L_{loc}(\mathbb{R}_+;\mathbb{R}^n)$.

It is evident that the statements of Theorems 2.2.1, 2.2.1′ and Corollary 2.2.2 mean that the initial problem (3.1.1), (3.1.2) is well-posed.

Definition 3.1.3. The initial problem (3.1.1), (3.1.2), where $P_0\in L_{loc}(I;\mathbb{R}^{n\times n})$, $q_0\in L_{loc}(I;\mathbb{R}^n)$, is said to be well-posed if inclusion (3.1.6) holds for every sequences $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, t_k $(k=0,1,\dots)$ and c_k $(k=0,1,\dots)$ for which there exists a sequence $H_k\in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that conditions (1.2.5), (1.2.9) and (1.2.11) hold, and conditions (1.2.12), (3.1.7) and (3.1.8) are fulfilled uniformly on I.

Definition 3.1.4. The initial problem (3.1.1), (3.1.2), where $P_0\in L_{loc}(I;\mathbb{R}^{n\times n})$, $q_0\in L_{loc}(I;\mathbb{R}^n)$, is said to be weakly well-posed if condition (3.1.6) holds for every sequences $P_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $q_k\in L_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, t_k $(k=0,1,\dots)$ and c_k $(k=0,1,\dots)$ for which there exists a sequence $H_k\in \mathrm{AC}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that conditions (1.2.5), (1.2.9) and (1.2.11) hold, and conditions (1.2.12),

$$\lim_{k\to+\infty}\vee_a^t(\mathcal{I}(H_k,P_k)-\mathcal{I}(H_0,P_0)=0$$

and

$$\lim_{k\to+\infty}\vee_a^t(\mathcal{B}(H_k,q_k)-\mathcal{B}(H_0,q_0)=0,$$

where $a\in\mathbb{R}_+$ is a fixed point, are fulfilled uniformly on I.

Theorem 3.1.18. *Let $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $q\in L_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ be such that*

$$\limsup_{t\to+\infty}\int\limits_{t}^{\nu(\xi)(t)}\|P(\tau)\|\,d\tau<+\infty,$$

and

$$\lim_{t\to+\infty}\int\limits_{t}^{\nu(\xi)(t)}\|q(\tau)\|\,d\tau=0,$$

where the function $\nu(\xi)$ is defined by (1.3.22). *Then ξ-exponentially asymptotical stability of the matrix-function P guarantees the well-posedness of problem* (3.1.1), (3.1.2) *on $\mathbb{R}_+$.*

Theorem 3.1.19. *Let $P\in L_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and*

$$q\in L(\mathbb{R}_+;\mathbb{R}^n).$$

Then uniform stability of the matrix-function P guarantees the weak well-posedness of problem (3.1.1), (3.1.2) *on $\mathbb{R}_+$.*

3.2 The numerical solvability of the initial problem for the linear systems of ordinary differential equations

In this section, we investigate the question of numerical solvability of the initial problem for the system of ordinary differential equations

$$\frac{dx}{dt} = P(t)\,x + q(t), \tag{3.2.1}$$

$$x(t_0) = c_0, \tag{3.2.2}$$

where P and q are, respectively, real matrix- and vector-functions with the Lebesque integrable components defined on a closed interval $[a,b]$, $t_0 \in [a,b]$, $c_0 \in \mathbb{R}^n$.

We assume that the absolutely continuous vector-function $x_0 : [a,b] \to \mathbb{R}^n$ is the unique solution of problem (3.2.1), (3.2.2).

Along with problem (3.2.1), (3.2.2), we consider the difference scheme

$$\Delta y(k-1) = \frac{1}{m}\Big(G_{1m}(k)y(k) + G_{2m}(k-1)y(k-1) + g_{1m}(k) + g_{2m}(k-1)\Big) \ \ (k=1,\dots,m), \tag{3.2.1$_m$}$$

$$y(k_m) = \gamma_m \tag{3.2.2$_m$}$$

$(m = 1, 2, \dots)$, where G_{jm} $(j = 1,2)$ and g_{jm} $(j = 1,2)$ are, respectively, discrete real matrix- and vector-functions acting from the set $\{1,\dots,m\}$ into $\mathbb{R}^{n\times n}$, $\gamma_m \in \mathbb{R}^n$, and $k_m \in \{0,1,\dots,m\}$ for every natural m.

In this section, we establish the effective necessary and sufficient and effective sufficient conditions for the convergence of difference scheme $(3.2.1_m), (3.2.2_m)$ $(m = 1,2,\dots)$ to the solution x_0 of problem (3.2.1), (3.2.2). Moreover, the stability criteria are obtained for the difference scheme $(3.2.1_m), (3.2.2_m)$.

The question of the numerical solvability is classical. There are a lot of papers where the problem has been investigated (see, for example, the references in the introduction, and the references therein). In the earlier papers, the sufficient conditions for the convergence and stability of difference schemes were established for the linear and nonlinear cases with continuous right sides. In addition, it should be noted that the method of investigation of the convergence of difference schemes depended on the type of the right-hand side of system (3.2.1).

Let

$$\tau_{0m} = a, \ \ \tau_{km} = a + k\tau_m \ \text{ and } \ I_{km} =]\tau_{k-1\,m}, \tau_{km}[\ \ (k = 1,\dots,m;\ m = 1,2,\dots),$$

where $\tau_m = \frac{b-a}{m}$.

Let ν_m $(m = 1,2,\dots)$ be the functions defined by equalities

$$\nu_m(t) = \Big[\frac{t-a}{b-a}\,m\Big] \ \text{ for } \ t \in [a,b] \ \ (m = 1,2,\dots).$$

It is evident that

$$\nu_m(\tau_{km}) = k \ \ (k = 0,\dots,m;\ m = 1,2,\dots).$$

We introduce the operators $p_m : \mathrm{BV}([a,b];\mathbb{R}^n) \to \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ and $q_m : \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n) \to \mathrm{BV}([a,b];\mathbb{R}^n)$ defined as follows:

$$p_m(x)(k) = x(\tau_{km}) \ \text{ for } \ x \in \mathrm{BV}([a,b];\mathbb{R}^n) \ \ (k = 0,\dots,m)$$

and

$$q_m(y)(t) = \begin{cases} y(k) & \text{for } \ t = \tau_{km} \ \ (k = 0,\dots,m), \\ y(k) - \dfrac{1}{m}\big(G_{1m}(k)y(k) + g_{1m}(k)\big) & \text{for } \ t \in]\tau_{k-1\,m}, \tau_{k\,m}[\ \ (k = 0,\dots,m) \end{cases}$$

for every $m \in \{1,2,\dots\}$.

We assume that $P \in L([a,b];\mathbb{R}^{n\times n})$, $q \in L([a,b];\mathbb{R}^n)$; $G_{jm} \in \mathrm{E}(N_m;\mathbb{R}^{n\times n})$ $(j = 1,2)$, $g_{jm} \in \mathrm{E}(N_m;\mathbb{R}^n)$. In addition, if necessary, we assume

$$G_{1m}(0) = G_{2m}(m) = O_{n\times n}, \ \ g_{1m}(0) = g_{2m}(m) = 0_n \ \ (m = 1,2,\dots).$$

Definition 3.2.1. We say that a sequence $(G_{1m}, G_{2m}, g_{1m}, g_{2m}; k_m)$ $(m = 1, 2, \dots)$ belongs to the set $\mathcal{CS}(P, q, t_0)$ if for every $c_0 \in \mathbb{R}^n$ and the sequence $\gamma_m \in \mathbb{R}^n$ $(m = 1, 2, \dots)$, satisfying the condition

$$\lim_{m\to+\infty} \gamma_m = c_0, \tag{3.2.3}$$

the difference problem $(3.2.1_m), (3.2.2_m)$ has a unique solution $y_m \in \mathrm{E}(\widetilde{N}_m; \mathbb{R}^n)$ for any sufficiently large m and the condition

$$\lim_{m\to+\infty} \|y_m - p_m(x_0)\|_{\widetilde{N}_m} = 0 \tag{3.2.4}$$

holds.

The proofs of the results given below are based on the following concept.

We rewrite both problems $(3.2.1), (3.2.2)$ and $(3.2.1_m), (3.2.2_m)$ $(m = 1, 2, \dots)$ as the initial problem for the systems of generalized ordinary differential equations considered in Section 1.1. So, the continuous system $(3.2.1)$ and the discrete systems $(3.2.1_m)$ $(m = 1, 2, \dots)$ are really the equations of the same type. Therefore, the convergence of the difference scheme $(3.2.1_m), (3.2.2_m)$ $(m = 1, 2, \dots)$ to the solution of problem $(3.2.1), (3.2.2)$ is equivalent to the question of the well-posedness of the initial problem for the systems of the latter type. So, using the results of Section 1.1, we establish the results presented in this section.

As above, it is evident that problem $(3.2.1), (3.2.2)$ is equivalent to problem $(1.2.1), (1.2.2)$, where

$$A_0(t) \equiv \int_a^t P(\tau)\,d\tau, \quad f_0(t) \equiv \int_a^t q(\tau)\,d\tau.$$

Consider now the difference initial problem $(3.2.1_m), (3.2.2_m)$, where $m \in \{1, 2, \dots\}$.

Let the matrix-function A_m and the vector-function f_m be defined by the equalities

$$A_m(a) = A_m(\tau_{0m}) = O_{n\times n}, \quad A_m(\tau_{km}) = \frac{1}{m}\Big(\sum_{i=0}^{k} G_{1m}(i) + \sum_{i=1}^{k} G_{2m}(i-1)\Big),$$
$$A_m(t) = \frac{1}{m}\Big(\sum_{i=0}^{k-1} G_{1m}(i) + \sum_{i=1}^{k} G_{2m}(i-1)\Big) \ \text{ for } \ t \in]\tau_{k-1\,m}, \tau_{k\,m}[\ \ (k = 1, \dots, m); \tag{3.2.5}$$

$$f_m(a) = f(\tau_{0m}) = 0_n, \quad f_m(\tau_{km}) = \frac{1}{m}\Big(\sum_{i=0}^{k} g_{1m}(i) + \sum_{i=1}^{k} g_{2m}(i-1)\Big),$$
$$f_m(t) = \frac{1}{m}\Big(\sum_{i=0}^{k-1} g_{1m}(i) + \sum_{i=1}^{k} g_{2m}(i-1)\Big) \ \text{ for } \ t \in]\tau_{k-1\,m}, \tau_{k\,m}[\ \ (k = 1, \dots, m) \tag{3.2.6}$$

for every natural m.

Due to $(3.2.5)$ and $(3.2.6)$, the matrix- and the vector functions have the following properties:

$$\begin{gathered} d_1 A_m(\tau_{k\,m}) = \frac{1}{m}\,G_{1m}(k), \quad d_2 A_m(\tau_{km}) = \frac{1}{m}\,G_{2m}(k) \ \ (k = 1, \dots, m), \\ d_j A_m(t) = O_{n\times n} \ \text{ for } \ t \in [a, b] \setminus \{\tau_{1m}, \dots, \tau_{km}\} \ \ (j = 1, 2); \end{gathered} \tag{3.2.7}$$

$$\begin{gathered} d_1 f_m(\tau_{k\,m}) = \frac{1}{m}\,g_{1m}(k), \quad d_2 f_m(\tau_{km}) = \frac{1}{m}\,g_{2m}(k) \ \ (k = 1, \dots, m), \\ d_j f_m(t) = 0_n \ \text{ for } \ t \in [a, b] \setminus \{\tau_{1m}, \dots, \tau_{km}\} \ \ (j = 1, 2). \end{gathered} \tag{3.2.8}$$

Lemma 3.2.1. *Let m be fixed. Then the discrete vector-function $y \in E(\widetilde{N}_m; \mathbb{R}^n)$ is a solution of problem $(3.2.1_m), (3.2.2_m)$ if and only if the vector-function $x = q_m(y) \in \mathrm{BV}([a, b]; \mathbb{R}^n)$ is a solution of the generalized problem $(1.2.1_m), (1.2.2_m)$, where the matrix-function A_m and the vector-function f_m are defined by $(3.2.5)$ and $(3.2.6)$, respectively, $t_m = a + \frac{b-a}{m} k_m$ and $c_m = \gamma_m$.*

Proof of Lemma 3.2.1. In view of equalities (0.0.11), (3.2.7) and (3.2.8), if we take into account the fact that by the definition of the operator q_m we have $x(\tau_{km}) = q_m(y)(\tau_{km}) = y(k)$ $(k = 1, \dots, m)$, we can find

$$\int\limits_{\tau_{k-1\,m}}^{\tau_{km}} dA_m(\tau)x_m(\tau) + f(\tau_{km}) - f(\tau_{k-1\,m})$$
$$= \frac{1}{m}\, G_{1m}(k)x_m(\tau_{km}) + \frac{1}{m}\, G_{2m}(k-1)x_m(\tau_{k-1\,m}) + \frac{1}{m}\, g_{1m}(k) + \frac{1}{m}\, g_{2m}(k-1)$$
$$= \frac{1}{m}\, G_{1m}(k)y(k) + \frac{1}{m}\, G_{2m}(k-1)y(k-1) + \frac{1}{m}\, g_{1m}(k) + \frac{1}{m}\, g_{2m}(k-1)$$
$$= \Delta y(k-1) = x_m(\tau_{km}) - x_m(\tau_{k-1\,m}) \ \ (k = 1, \dots, m)$$

and

$$\begin{aligned} d_1x_m(\tau_{km}) &= x_m(\tau_{km}) - x_m(\tau_{km}-) = \frac{1}{m}\, G_{1m}(k)y(k) + \frac{1}{m}\, g_{1m}(k) \\ &= d_1A_m(\tau_{km}) + d_1f_m(\tau_{km}) \ \ (k = 1, \dots, m); \\ d_2x_m(\tau_{k-1\,m}) &= x_m(\tau_{k-1\,m}+) - x_m(\tau_{k-1\,m}) = y(k) - y(k-1) - \frac{1}{m}\, G_{1m}(k)y(k) - \frac{1}{m}\, g_{1m}(k) \\ &= \frac{1}{m}\, G_{2m}(k-1)y(k-1) + \frac{1}{m}\, g_{2m}(k-1) \\ &= d_2A_m(\tau_{k-1\,m}) + d_2f_m(\tau_{k-1\,m}) \ \ (k = 1, \dots, m) \end{aligned}$$

for every $m \in \{1, 2, \dots\}$.

Analogously, we show that if the vector-function $x \in \mathrm{BV}([a, b]; \mathbb{R}^n)$ is a solution of the generalized problem $(1.2.1_m), (1.2.2_m)$ defined above, then the vector-function $y(k) = p_m(x)(k)$ $(k = 1, \dots, m)$ will be a solution of the difference problem $(3.2.1_m), (3.2.2_m)$ for every natural m.

So, we have shown that the convergence of the difference scheme is equivalent to the question of the well-posedness of the initial problem (1.2.1), (1.2.2). Therefore, the inclusion

$$\big((G_{1m}, G_{2m}, g_{1m}, g_{2m}; k_m)\big)_{m=1}^{+\infty} \in \mathcal{CS}(P, q; l) \tag{3.2.9}$$

is equivalent to inclusion (1.2.10). □

Remark 3.2.1. In view of (3.2.5) and (3.2.6), we have $A_m(t) = const$ and $f_m(t) = const$ for $t \in]\tau_{k-1\,m}, \tau_{k\,m}[$ $(k = 1, \dots, m;\ m = 1, 2, \dots)$, i.e., they are the break matrix- and vector-functions. Therefore, all solutions of system $(1.2.1_m)$ $(m = 1, 2, \dots)$ have the same property. Such property have also the matrix-functions H_m $(m = 1, 2, \dots)$ appearing in the results of Section 1.2 (see Remark 1.2.10). So they are also break matrix-functions, and hence

$$H_m(\tau_{k-1\,m}+) = H_m(\tau_{km}-) \ \ (k = 0, \dots, m;\ m = 1, 2, \dots). \tag{3.2.10}$$

Here we use some results of Chapter 2. For this, we give the following lemma.

Lemma 3.2.2. *Let the matrix-functions* $A_m \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n})$ $(m = 1, 2, \dots)$ *and the vector-functions* $f_m \in \mathrm{BV}([a, b]; \mathbb{R}^n)$ $(m = 1, 2, \dots)$ *be defined by* (3.2.5) *and* (3.2.6)*, respectively, and* $Q_m \in \mathrm{BV}([a, b]; \mathbb{R}^{n\times n})$ $(m = 1, 2, \dots)$*. Then there exist discrete matrix-functions* $Q_{1m}, Q_{2m} \in \mathrm{E}(\widetilde{\mathbb{N}}_m; \mathbb{R}^{n\times n})$ $(m = 1, 2, \dots)$ *such that*

$$\mathcal{B}(Q_m, A_m)(t) \equiv \frac{1}{m} \sum_{k=1}^{\nu_m(t)} \Big(Q_{1m}(k)G_{1m}(k) + Q_{2m}(k)G_{2m}(k-1)\Big) \ \ (m = 1, 2, \dots) \tag{3.2.11}$$

and

$$\mathcal{B}(Q_m, f_m)(t) \equiv \frac{1}{m} \sum_{k=1}^{\nu_m(t)} \Big(Q_{1m}(k)g_{1m}(k) + Q_{2m}(k)+)g_{2m}(k-1)\Big) \ \ (m = 1, 2, \dots). \tag{3.2.12}$$

Proof. By the definition of operator $\mathcal{B}(H,A)$, the integration-by-parts formulae and equalities (0.0.11) we have

$$\begin{aligned}\mathcal{B}(Q_m,A_m)(t)&=\int\limits_a^t Q_m(\tau)\,dA_m(\tau)-\sum_{a<\tau\le t}d_1Q_m(\tau)\,d_1A_m(\tau)+\sum_{0\le\tau<t}d_2Q(\tau)\,d_2A_m(\tau)\\&=\sum_{a<\tau_{km}\le t}Q_m(\tau_{km}-)\,d_1A_m(\tau_{km})+\sum_{a\le\tau_{km}<t}Q_m(\tau_{km}+)\,d_2A_m(\tau_{km})\\&=\sum_{k=1}^{\nu_m(t)}Q_m(\tau_{km}-)\,d_1A_m(\tau_{km})+\sum_{k=0}^{\nu_m(t)-1}Q_m(\tau_{km}+)\,d_2A_m(\tau_{km})\end{aligned}$$

$$=\sum_{k=1}^{\nu_m(t)}\Big(Q_m(\tau_{km}-)\,d_1A_m(\tau_{km})+Q_m(\tau_{k-1\,m}+)\,d_2A_m(\tau_{k-1\,m})\Big)\ \text{ for }\ t\in[a,b]\ (m=1,2,\dots).\qquad(3.2.13)$$

Owing to (3.2.7), from (3.2.13) we get (3.2.11), where $Q_{1m}(k)\equiv Q_m(\tau_{km}-)$ and $Q_{2m}(k)\equiv Q_m(\tau_{k-1\,m}+)$ $(m=1,2\dots)$. Analogously, using (3.2.8), we obtain presentation (3.2.12). □

Theorem 3.2.1. *Let*

$$\lim_{m\to+\infty}\frac{k_m}{m}=\frac{t_0-a}{b-a}\,.\qquad(3.2.14)$$

Then inclusion (3.2.9) *holds if and only if there exist a matrix-function* $H\in\mathrm{AC}([a,b];\mathbb{R}^{n\times n})$ *and the sequences of discrete matrix-functions* $H_{jm}\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(j=1,2;\ m=1,2,\dots)$ *such that*

$$\begin{aligned}\lim_{m\to+\infty}\sup\sum_{k=1}^{m}\bigg(\Big\|H_{2m}(k)-H_{1m}(k)+\frac{1}{m}\,H_{1m}(k)G_{1m}(k)\Big\|\\+\Big\|H_{1m}(k)-H_{2m}(k-1)+\frac{1}{m}\,H_{1m}(k)G_{2m}(k-1)\Big\|\bigg)<+\infty,\end{aligned}\qquad(3.2.15)$$

$$\inf\big\{|\det(H(t))|:\ t\in[a,b]\big\}>0,\qquad(3.2.16)$$

and the conditions

$$\lim_{m\to+\infty}H_m(t)=H(t),\qquad(3.2.17)$$

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int\limits_a^t H(\tau)P(\tau)\,d\tau,\qquad(3.2.18)$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int\limits_a^t H(\tau)q(\tau)\,d\tau\qquad(3.2.19)$$

hold uniformly on $[a,b]$*, where the matrix-functions* $H_m\in\mathrm{BV}([a,b];\mathbb{R}^n)$ $(m=1,2,\dots)$ *are defined by equalities*

$$H_m(t)=H_{1m}(k)\ \text{ for }\ \tau_{k-1\,m}<t<\tau_{km},\ \ H_m(\tau_{km})=H_{2m}(k)\ \ (k=1,\dots,m;\ m=1,2,\dots).$$

Proof. To prove the theorem, we use Theorem 1.2.1′.

Let us show the *sufficiency*. It is evident that H_m $(m=1,2,\dots)$ are break matrix-functions that are constants on the intervals $]\tau_{k-1\,m},\tau_{km}[$, respectively. Hence equalities (3.2.10) hold, and

$$d_1H_m(\tau_{k\,m})=H_{2m}(k)-H_{1m}(k),\ \ d_2H_m(\tau_{km})=H_{1m}(k+1)-H_{2m}(k)\ \ (k=1,\dots,m;\ m=1,2,\dots).$$

Owing to Lemma 3.2.2, Remark 3.2.1 and equalities (3.2.10), we get

$$\mathcal{B}(H_m,A_m)(t)\equiv\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\big(G_{1m}(k)+G_{2m}(k-1)\big)\ \ (m=1,2,\dots). \tag{3.2.20}$$

and

$$\mathcal{B}(H_m,f_m)(t)\equiv\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)\big(g_{1m}(k)+g_{2m}(k-1)\big)\ \ (m=1,2,\dots).$$

On the other hand, condition (1.2.34) is equivalent to condition (3.2.15). Thus, conditions (3.2.14), (3.2.15), (3.2.16), (3.2.17), (3.2.18) and (3.2.19) guarantee the fulfilment of the condition of Theorem 1.2.1′.

Let us show the *necessity*. Inclusion (3.2.9) is equivalent to inclusion (1.2.10), where A_m and f_m $(m=1,2,\dots)$ are defined as above. Due to Theorem 1.2.1′, there exists a sequence $H_m\in\mathrm{BV}([a,b];\mathbb{R}^n)$ $(m=1,2,\dots)$ satisfying the conditions given in the theorem. Let

$$H_{1m}(k)\equiv H_m(\tau_{km}-),\quad H_{2m}(k)\equiv H_m(\tau_{km})\ \ (m=1,2,\dots).$$

According to Remark 3.2.1, equality (3.2.10) holds. Using Lemma 3.2.1, we can easily show that the above-defined discrete matrix-functions H_{1m} and H_{2m} $(m=1,2,\dots)$ satisfy the condition of Theorem 3.2.1. □

Remark 3.2.2. The limit equality (3.2.17) holds uniformly on $[a,b]$ if and only if

$$\lim_{m\to+\infty}\max\big\{\|H_{1m}(k)-H(t)\|:\ t\in\,]\tau_{k-1\,m},\tau_{km}[\,,\ k=1,\dots,m\big\}=0$$

and

$$\lim_{m\to+\infty}\max\big\{\|H_{2m}(k)-H(\tau_{km})\|:\ k=0,\dots,m\big\}=0.$$

The limit equalities (3.2.18) and (3.2.19) hold uniformly on $[a,b]$ if and only if the conditions

$$\lim_{m\to+\infty}\max\bigg\{\frac{1}{m}\sum_{i=1}^{l}H_{1m}(i)(G_{1m}(i)+G_{2m}(i-1))-\int_a^{\tau_{lm}}H(\tau)P(\tau)\,d\tau:\ l=1,\dots,m\bigg\}=O_{n\times n}$$

and

$$\lim_{m\to+\infty}\max\bigg\{\frac{1}{m}\sum_{i=1}^{l}H_{1m}(i)(g_{1m}(i)+g_{2m}(i-1))-\int_a^{\tau_{lm}}H(\tau)q(\tau)\,d\tau:\ l=1,\dots,m\bigg\}=0_n$$

hold, respectively. Moreover, in Theorem 3.2.1, without loss of generality, we can assume that $H(t)\equiv I_n$.

Let X, $X(a)=I_n$, be the fundamental matrix of the homogeneous system

$$\frac{dx}{dt}=P(t)x \tag{3.2.1$_0$}$$

and let Y_m, $Y_m(0)=I_n$, be the fundamental matrix of the homogeneous difference system

$$\Delta y(k-1)=\frac{1}{m}\big(G_{1m}(k)y(k)+G_{2m}(k-1)y(k-1)\big)\ \ (k=1,\dots,m) \tag{3.2.22$_{m_0}$}$$

for every natural m.

Theorem 3.2.2. *Let conditions* (3.2.14) *and*

$$\det\Big(I_n+(-1)^j\,\frac{1}{m}\,G_{jm}(k)\Big)\neq 0\ \ (j=1,2;\ k=1,\dots,m;\ m=1,2,\dots) \tag{3.2.23}$$

hold. Then inclusion (3.2.9) *holds if and only if the conditions*

$$\lim_{m\to+\infty}\max\big\{\|Y_m^{-1}(k)-X^{-1}(\tau_{km})\|:\ k=0,\dots,m\big\}=0 \tag{3.2.24}$$

and

$$\lim_{m\to+\infty}\max\bigg\{\frac{1}{m}\sum_{i=1}^{l}Y_m^{-1}(i)(g_{1m}(i)+g_{2m}(i-1))-\int_a^{\tau_{lm}}X^{-1}(\tau)q(\tau)\,d\tau:\ l=1,\dots,m\bigg\}=0_n \tag{3.2.25}$$

are satisfied.

Proof. The theorem is a realization of Theorem 1.2.1″ for this case. □

Remark 3.2.3.

(a) By the evident equality $(X^{-1}(t))'\equiv -X^{-1}(t)\,P(t)$, the right-hand side of (3.2.18) equals $I_n-X^{-1}(t)$ if $H(t)\equiv X^{-1}(t)$; moreover, in view of (1.1.17) and (3.2.20), the left-hand side of equality (3.2.18) equals $I_n-Y_m^{-1}(k)$ if $H_{1m}(k)\equiv Y_m^{-1}(k)$ for every natural m. Hence condition (3.2.18) is equivalent to (3.2.24);

(b) if
$$P(t)\int_{t_0}^{t}P(\tau)\,d\tau\equiv\int_{t_0}^{t}P(\tau)\,d\tau\,P(t),$$
then
$$X(t)\equiv\exp\bigg(\int_{t_0}^{t}P(\tau)\,d\tau\bigg);$$

(c) by (3.2.23), we conclude that
$$Y_m(k)=\prod_{i=k}^{1}\Big(I_n-\frac{1}{m}\,G_{1m}(i)\Big)^{-1}\Big(I_n+\frac{1}{m}\,G_{2m}(i-1)\Big)\ \ (k=1,\dots,m) \tag{3.2.26}$$
for every natural m;

(d) in Theorem 3.2.2, condition (3.2.15) is automatically satisfied, since Y_m is the fundamental matrix of the homogeneous system $(3.2.22_{m_0})$ for every natural m.

Now we present a method of constructing discrete real matrix- and vector-functions G_{jm} $(j=1,2)$ and g_{jm} $(j=1,2)$ $(m=1,2,\dots)$, respectively, for which the conditions of Theorem 3.2.2 are satisfied.

To this end, we use the inductive method. Let $\mathcal{E}_m:\widetilde{\mathbb{N}}_m\to\mathbb{R}^{n\times n}$ and $\xi_m:\widetilde{\mathbb{N}}_m\to\mathbb{R}^n$ $(m=1,2,\dots)$ be discrete matrix- and vector-functions, respectively, such that

$$\lim_{m\to+\infty}\|\mathcal{E}_m\|_{\widetilde{\mathbb{N}}_m}=0\ \text{ and }\ \lim_{m\to+\infty}m\|\xi_m\|_{\widetilde{\mathbb{N}}_m}=0.$$

Let
$$P_{lm}=X(\tau_{lm})+\mathcal{E}_m(l)\ \ (l=0,\dots,m;\ m=1,2,\dots).$$

Let m be an arbitrary natural number and $G_{1m}(1)$ and $G_{2m}(0)$ be such that

$$Y_m(1)=P_{1m}.$$

According to (3.2.26) we get

$$\Big(I_n - \frac{1}{m}\,G_{1m}(1)\Big)^{-1}\Big(I_n + \frac{1}{m}\,G_{2m}(0)\Big) = P_{1m}.$$

Therefore, $G_{1m}(1)$ and $G_{2m}(0)$ will be arbitrary matrices such that

$$G_{1m}(1) = m(I_n - P_{1m}^{-1}) - G_{2m}(0)\,P_{1m}^{-1}.$$

Assume now that $G_{1m}(k)$, $G_{2m}(k-1)$ and $Y_m(k)$ $(k = 1,\dots,l-1)$ are constructed. For the construction of $G_{1m}(l)$ and $G_{2m}(l-1)$ we use the equalities

$$Y_m(l) = P_{lm}$$

and

$$Y_m(l) = \Big(I_n - \frac{1}{m}\,G_{1m}(l)\Big)^{-1}\Big(I_n + \frac{1}{m}\,G_{2m}(l-1)\Big)Y_m(l-1).$$

As above, we obtain the relation

$$G_{1m}(l) = m(I_n - P_{l-1\,m}P_{lm}^{-1}) - G_{2m}(l-1)P_{l-1\,m}P_{lm}^{-1}.$$

So, $G_{1m}(l)$ and $G_{2m}(l-1)$ will be arbitrary matrices satisfying the latter equality.

Let now construct the discrete vector-functions g_{1m} and g_{2m} $(m = 1,2,\dots)$. As $g_{1m}(l)$ and $g_{2m}(l-1)$, we choose arbitrary vectors satisfying the equalities

$$\frac{1}{m}\,Y_m^{-1}(l)\big(g_{1m}(l) + g_{2m}(l-1)\big) = q_{lm} \ \ (l = 1,\dots,m),$$

where

$$q_{lm} = \xi_m(l) + \int\limits_a^{\tau_{lm}} X^{-1}(\tau)q(\tau)\,d\tau \ \ (l = 1,\dots,m)$$

for every natural m. Therefore, we have the equalities

$$g_{1m}(l) + g_{2m}(l-1) = mY_m(l)q_{lm} \ \ (l = 1,\dots,m;\ m = 1,2,\dots)$$

for the definition of the vector-functions g_{1m} and g_{2m} $(m = 1,2,\dots)$.

It is evident that the above-constructed vector-functions satisfy condition (3.2.25).

We realize the above-constructed discrete matrix- and vectors-functions by the following example.

Example 3.2.1. Let

$$X(t) \equiv \exp\bigg(\int\limits_a^t P(\tau)\,d\tau\bigg)$$

be the fundamental matrix of system $(3.2.1_0)$ and let $\mathcal{E}_m \equiv O_{n\times n}$ and $\xi_m \equiv 0_n$ $(m = 1,2,\dots)$. Then

$$P_{lm} = \exp\bigg(\int\limits_a^{\tau_{lm}} P(\tau)\,d\tau\bigg) \ \ (l = 0,\dots,m;\ m = 1,2,\dots).$$

If we choose

$$G_{2m}(l-1) = P_{lm}P_{l-1\,m}^{-1} = \exp\bigg(\int\limits_{\tau_{l-1\,m}}^{\tau_{lm}} P(\tau)\,d\tau\bigg) \ \ (l = 1,\dots,m;\ m = 1,2,\dots),$$

then

$$G_{1m}(l) = (m-1)I_n - m\exp\bigg(-\int\limits_{\tau_{l-1\,m}}^{\tau_{lm}} P(\tau)\,d\tau\bigg) \ \ (l = 1,\dots,m;\ m = 1,2,\dots).$$

For the definition of the discrete vector-functions g_{1m} and g_{2m} $(m=1,2,\dots)$ we have the relations

$$g_{1m}(l)+g_{2m}(l-1)=m\int\limits_a^{\tau_{lm}}U(\tau_{lm},\tau)q(\tau)\,d\tau\ \ (l=1,\dots,m;\ m=1,2,\dots),$$

where $U(t,\tau)$ is the Cauchy matrix of system (3.2.1).

In particular, we can take

$$g_{1m}(l)=\alpha m\int\limits_a^{\tau_{lm}}C(\tau_{lm},\tau)q(\tau)\,d\tau\ \text{ and }\ g_{2m}(l-1)=(1-\alpha)m\int\limits_a^{\tau_{lm}}C(\tau_{lm},\tau)q(\tau)\,d\tau$$
$$(l=1,\dots,m;\ m=1,2,\dots),$$

where α is some number.

Moreover, we can choose these discrete vector-functions in connection with the Cauchy formulae for system (3.2.1).

Theorem 3.2.3. *Let condition* (3.2.14) *be satisfied. Let, moreover, the sequences* $G_{jm}\in E(\widetilde{N}_m;\mathbb{R}^{n\times n})$ $(j=1,2;\ m=1,2,\dots)$, $g_{jm}\in E(\widetilde{N}_m;\mathbb{R}^n)$ $(j=1,2;\ m=1,2,\dots)$ *and* γ_m $(m=1,2,\dots)$ *be such that conditions* (3.2.3) *and*

$$\limsup_{m\to+\infty}\Big(\frac{1}{m}\sum_{k=0}^{m}\big(\|G_{jm}(k)\|+\|g_{jm}(k)\|\big)\Big)<+\infty\ \ (j=1,2) \tag{3.2.27}$$

hold, and the conditions

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int\limits_a^t P(\tau)\,d\tau,$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int\limits_a^t q(\tau)\,d\tau$$

hold uniformly on $[a,b]$*. Then the difference initial problem* $(3.2.1_m),(3.2.2_m)$ *has the unique solution* y_m *for any sufficiently large* m *and condition* (3.2.4) *holds.*

Proof. The validity of the theorem follows from the sufficiency of Theorem 3.2.1 if we assume $H_{1m}(k)=H_{2m}(k)\equiv I_n$ and $H(t)\equiv I_n$ therein. □

Proposition 3.2.1. *Let conditions* (3.2.14)–(3.2.16) *and*

$$\lim_{m\to+\infty}\frac{1}{m}\max\big\{\|G_{jm}(k)\|+\|g_{jm}(k)\|:\ k=0,\dots,m\big\}=0\ \ (j=1,2) \tag{3.2.28}$$

hold and let conditions (3.2.17)–(3.2.19) *be fulfilled uniformly on* $[a,b]$*, where* $H\in\mathrm{AC}([a,b];\mathbb{R}^{n\times n})$, $H_{1m},H_{2m}\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$*. Let, moreover, either condition* (3.2.27) *or the condition*

$$\limsup_{m\to+\infty}\sum_{k=0}^{m}\Big(\|H_{2m}(k)-H_{1m}(k)\|+\|H_{1m}(k)-H_{2m}(k-1)\|\Big)<+\infty$$

be satisfied. Then inclusion (3.2.9) *holds.*

Proof. The proposition is a realization of Corollary 1.2.2 for this case. □

Theorem 3.2.4. *Let conditions* (3.2.14), (3.2.15) *and* (3.2.28) *hold and let conditions* (3.2.17),

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m}(k)+G_{2m}(k-1)\big)=\int_a^t P(\tau)\,d\tau, \tag{3.2.29}$$

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(g_{1m}(k)+g_{2m}(k-1)\big)=\int_a^t q(\tau)\,d\tau, \tag{3.2.30}$$

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)(G_{1m}(k)+G_{2m}(k-1))=\int_a^t P_*(\tau)\,d\tau$$

and

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}H_{1m}(k)(g_{1m}(k)+g_{2m}(k-1))=\int_a^t q_*(\tau)\,d\tau$$

hold uniformly on $[a,b]$*, where* $P_*\in L([a,b];\mathbb{R}^{n\times n})$*,* $q_*\in L([a,b];\mathbb{R}^n)$*;* $H_{1m},H_{2m}\in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$*. Let, moreover, the system*

$$\frac{dx}{dt}=(P(t)-P_*(t))x+q(t)-q_*(t)$$

have a unique solution under the initial condition (3.2.2)*. Then*

$$\big((G_{1m},G_{2m},g_{1m},g_{2m};\tau_{k_m})\big)_{m=1}^{+\infty}\in\mathcal{CS}(P-P_*,q-q_*;l).$$

Proof. The theorem is a realization of Corollary 1.2.3 for this case. □

Proposition 3.2.2. *Let conditions* (3.2.14) *hold and there exist a natural* μ *and matrix-functions* $B_{jl}\in\mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$*,* $B_{jl}(a)=O_{n\times n}$ $(j=1,2;\ l=0,\dots,\mu-1)$ *such that*

$$\lim_{m\to+\infty}\sup\sum_{k=1}^{m}\Big(\big\|H_{2m\mu}(k)-H_{1m\mu}(k)+\frac{1}{m}\,H_{1m\mu}(k)\,G_{1m\mu}(i)\big\|$$
$$+\big\|H_{1m\mu}(k)-H_{2m\mu}(k-1)+\frac{1}{m}\,H_{1m\mu}(k)\,G_{2m\mu}(k-1)\big\|\Big)<+\infty,$$
$$\lim_{m\to+\infty}\max\big\{\|H_{jm\mu}(k)-I_n\|:\ k=0,\dots,m\big\}=0\ \ (j=1,2),$$

and the conditions

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(G_{1m\mu}(k)+G_{2m\mu}(k-1)\big)=\int_a^t P(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{k=1}^{\nu_m(t)}\big(g_{1m\mu}(k)+g_{2m\mu}(k-1)\big)=\int_a^t q(\tau)\,d\tau$$

are fulfilled uniformly on $[a,b]$, *where*

$$H_{1m0}(k)=H_{2m0}(k)\equiv I_n,$$
$$H_{1m\,l+1}(k)\equiv\Big(\frac{1}{m}\,H_{1ml}(k)G_{1m}(k)+\mathcal{Q}_1(H_{1ml},G_{1m},G_{2m})(k)+B_{1\,l+1}(k)\Big)H_{1ml}(k),$$
$$H_{2m\,l+1}(k)\equiv\big(\mathcal{Q}_2(H_{1ml},G_{1m},G_{2m})(k)+B_{2\,l+1}(k)\big)H_{2ml}(k),$$
$$G_{1m\,l+1}(k)\equiv H_{1ml}(k)G_{1m}(k),\quad G_{2m\,l+1}(k)\equiv H_{1ml}(k+1)G_{2m}(k),$$
$$g_{1m\,l+1}(k)\equiv H_{ml}(k)g_{1m}(k),\quad g_{2m\,l+1}(k)\equiv H_{ml}(k+1)g_{2m}(k),$$
$$\mathcal{Q}_j(H_{1ml},G_{1m},G_{2m})(k)\equiv 2I_n-H_{jml}(k)-\frac{1}{m}\sum_{i=1}^{k}H_{1ml}(i)(G_{1m}(i)+G_{2m}(i-1))$$
$$(j=1,2;\ l=0,\dots,\mu-1;\ m=1,2,\dots).$$

Then inclusion (3.2.9) *holds.*

Proof. The proposition is a realization of Corollary 1.2.4 for this case. □

If $\mu=1$ and $B_{j0}(t)\equiv O_{n\times n}$ $(j=1,2)$, then Proposition 3.2.2 takes the following form.

Proposition 3.2.3. *Let conditions* (3.2.14) *and*

$$\lim_{m\to+\infty}\sup\Big(\frac{1}{m}\sum_{k=1}^{m}(\|G_{1m}(k)\|+\|G_{2m}(k)\|)\Big)<+\infty$$

hold and conditions (3.2.29) *and* (3.2.30) *be fulfilled uniformly on* $[a,b]$. *Then inclusion* (3.2.9) *holds.*

Remark 3.2.4. In Theorem 3.2.3 and Propositions 3.2.1–3.2.3, if condition (3.2.23) holds, we can assume that $H_m(t)\equiv Y_m^{-1}(t)$, where Y_m is the fundamental matrix of the homogeneous system (3.2.22$_{m_0}$) defined by (3.2.26) for every natural m. Moreover, condition (3.2.15) and the analogous conditions hold automatically everywhere in the results described above.

Consider now the question on the stability of a solution of the difference initial problem

$$\Delta y(k-1)=G_{10}(k)y(k)+G_{20}(k-1)y(k-1)+g_{10}(k)+g_{20}(k-1)\ \ (k=1,\dots,m_0),\tag{3.2.31}$$
$$y(k_0)=\gamma_0,\tag{3.2.32}$$

where $m_0\ge 2$ is a fixed natural number, $G_{j0}\in\mathrm{E}(N_{m_0};\mathbb{R}^{n\times n})$ $(j=1,2)$, $g_{j0}\in\mathrm{E}(N_{m_0};\mathbb{R}^n)$ $(j=1,2)$, $k_0\in\{0,\dots,m_0\}$ and $\gamma_0\in\mathbb{R}^n$.

Along with problem (3.2.31), (3.2.32), consider the sequence of problems

$$\Delta y(k-1)=G_{1m}(k)y(k)+G_{2m}(k-1)y(k-1)+g_{1m}(k)+g_{2m}(k-1)\ \ (k=1,\dots,m_0),\tag{3.2.31$_m$}$$
$$y(k_0)=\gamma_m\tag{3.2.32$_m$}$$

$(m=1,2,\dots)$, where $G_{jm}\in\mathrm{E}(N_{m_0};\mathbb{R}^{n\times n})$ $(j=1,2)$, $g_m\in\mathrm{E}(N_{m_0};\mathbb{R}^n)$, $B_m\in\mathrm{E}(N_{m_0};\mathbb{R}^n)$, and $\gamma_m\in\mathbb{R}^n$ for every natural m.

As above, if necessary, we assume that

$$G_{1m}(0)=O_{n\times n},\ \ g_{1m}(0)=0_n\ \ (m=0,1,\dots),$$
$$G_{2m}(m_0)=O_{n\times n},\ \ g_{2m}(m_0)=0_n\ \ (m=0,1,\dots)$$

and problem (3.2.31), (3.2.32) has a unique solution $y_0\in\mathrm{E}(\widetilde{N}_{m_0};\mathbb{R}^n)$.

In (3.2.32$_m$), if instead of k_0 we take k_{0m}, $k_{0m}\ne k_0$ $(m=1,2,\dots)$, then it follows from the condition $\lim\limits_{m\to+\infty}k_{0m}=k_0$ that $k_{0m}=k_0$ for any sufficiently large m.

Definition 3.2.2. We say that a sequence $(G_{1m}, G_{2m}, g_{1m}, g_{2m}; k_0)$ $(m = 1, 2, \dots)$ belongs to the set $\mathcal{S}(G_{10}, G_{20}, g_{10}, g_{20}; k_0)$ if for every $\gamma_0 \in \mathbb{R}^n$ and the sequence $\gamma_m \in \mathbb{R}^n$ $(m = 1, 2, \dots)$ satisfying the condition

$$\lim_{m\to+\infty} \gamma_m = \gamma_0,$$

the difference problem $(3.2.31_m), (3.2.32_m)$ has a unique solution $y_m \in \mathrm{E}(\widetilde{N}_{m_0}; \mathbb{R}^n)$ for any sufficiently large m and the condition

$$\lim_{m\to+\infty} y_m(k) = y_0(k) \quad (k = 0, \dots, m_0)$$

holds.

Theorem 3.2.5. *Let*

$$\det\big(I_n + (-1)^j G_{j0}(k)\big) \neq 0 \quad (j = 1, 2;\ k = 0, \dots, m_0). \tag{3.2.33}$$

Then

$$\big((G_{1m}, G_{2m}, g_{1m}, g_{2m}; k_0)\big)_{m=1}^{+\infty} \in \mathcal{S}(G_{10}, G_{20}, g_{10}, g_{20}; k_0) \tag{3.2.34}$$

if and only if there exist the sequences of matrix-functions $H_{jm} \in \mathrm{E}(\widetilde{\mathrm{N}}_m; \mathbb{R}^{n\times n})$ $(j = 1, 2;\ m = 1, 2, d \dots)$ *such that*

$$\lim_{m\to+\infty} \sup \sum_{k=1}^{m_0} \Big(\big\|H_{2m}(k) - H_{1m}(k) + H_{1m}(k)\, G_{1m}(k)\big\| + \big\|H_{1m}(k) - H_{2m}(k-1) + H_{1m}(k)\, G_{2m}(k-1)\big\|\Big) < +\infty,$$

$$\lim_{m\to+\infty} H_{jm}(k) = I_n \quad (j = 1, 2,\ k = 0, \dots, m_0),$$

$$\lim_{m\to+\infty} H_{1m}(k)\big(G_{1m}(k) + G_{2m}(k-1)\big) = G_{10}(k) + G_{20}(k-1) \quad (k = 1, \dots, m_0)$$

and

$$\lim_{m\to+\infty} H_{1m}(k)\big(g_{1m}(k) + g_{2m}(k-1)\big) = g_{10}(k) + g_{20}(k-1) \quad (k = 1, \dots, m_0).$$

Proof. The theorem is a realization of Theorem 1.2.1′, where $H_0(t) \equiv I_n$, for this case. □

Theorem 3.2.6. *Let the condition*

$$\det(I_n + (-1)^j G_{jm}(k)) \neq 0 \quad (j = 1, 2;\ k = 0, \dots, m_0;\ m = 1, 2, \dots)$$

be satisfied. Then inclusion (3.2.34) *holds if and only if*

$$\lim_{m\to+\infty} G_{jm}(k) = G_{j0}(k) \quad (j = 1, 2;\ k = 0, \dots, m_0)$$

and

$$\lim_{m\to+\infty} g_{jm}(k) = g_{j0}(k) \quad (j = 1, 2;\ k = 0, \dots, m_0).$$

Proof. The theorem is a realization of Theorem 1.2.1″ for this case. □

Proposition 3.2.4. *Let conditions* (3.2.33),

$$\lim_{m\to+\infty} G_{jm}(k) = G_{j0}(k) \quad (j = 1, 2;\ k = 0, \dots, m_0)$$

and

$$\lim_{m\to+\infty} g_{jm}(k) = g_{j0}(k) \quad (j = 1, 2;\ k = 0, \dots, m_0)$$

be satisfied. Then inclusion (3.2.34) *holds.*

Proof. The theorem is a realization of Theorem 1.2.2′ for this case. □

Proposition 3.2.5. *Let condition* (3.2.33) *hold and there exist a natural μ and matrix-functions $B_{jl}\in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0};\mathbb{R}^{n\times n})$, $B_{jl}(a)=O_{n\times n}$ $(j=1,2;\ l=0,\dots,\mu-1)$ such that the conditions*

$$\lim_{m\to+\infty}\sup\sum_{k=0}^{m_0}\Big(\big\|H_{2m\mu}(k)-H_{1m\mu}(k)+H_{1m\mu}(k)\,G_{1m\mu}(i)\big\|$$
$$+\big\|H_{1m\mu}(k)-H_{2m\mu}(k-1)+H_{1m\mu}(k)\,G_{2m\mu}(k-1)\big\|\Big)<+\infty,$$
$$\lim_{m\to+\infty}H_{jm\mu}(k)=I_n\ \ (j=1,2;\ k=0,\dots,m_0),$$
$$\lim_{m\to+\infty}H_{1m\mu}(k)\big(G_{1m\mu}(k)+G_{2m\mu}(k-1)\big)=G_{10}(k)+G_{20}(k-1)\ \ (k=1,\dots,m_0)$$

and

$$\lim_{m\to+\infty}H_{1m\mu}(k)\big(g_{1m\mu}(k)+g_{2m\mu}(k-1)\big)=g_{10}(k)+g_{20}(k-1)\ \ (k=1,\dots,m_0)$$

are satisfied, where

$$H_{1m0}(k)=H_{2m0}(k)\equiv I_n,$$
$$H_{1m\,l+1}(k)\equiv\big(H_{1ml}(k)G_{1m}(k)+\mathcal{Q}_1(H_{1ml},G_{1m},G_{2m})(k)+B_{1\,l+1}(k)\big)H_{1ml}(k),$$
$$H_{2m\,l+1}(k)\equiv\big(\mathcal{Q}_2(H_{1ml},G_{1m},G_{2m})(k)+B_{2\,l+1}(k)\big)H_{2ml}(k),$$
$$G_{1m\,l+1}(k)\equiv H_{1ml}(k)G_{1m}(k),\quad G_{2m\,l+1}(k)\equiv H_{1ml}(k+1)G_{2m}(k),$$
$$g_{1m\,l+1}(k)\equiv H_{ml}(k)g_{1m}(k),\quad g_{2m\,l+1}(k)\equiv H_{ml}(k+1)g_{2m}(k),$$
$$\mathcal{Q}_j(H_{1ml},G_{1m},G_{2m})(k)\equiv 2I_n-H_{jml}(k)-\sum_{i=1}^{k}H_{1ml}(i)\,(G_{1m}(i)+G_{2m}(i-1))$$
$$(j=1,2;\ l=0,\dots,\mu-1;\ m=1,2,\dots).$$

Then inclusion (3.2.34) *holds.*

Proof. The proposition is a realization of Corollary 1.2.4 for this case. □

If $\mu=1$ and $B_{j0}(t)=O_{n\times n}$ $(j=1,2)$, then Proposition 3.2.5 takes the following form.

Proposition 3.2.6. *Let conditions* (3.2.33),

$$\lim_{m\to+\infty}\sup\sum_{k=1}^{m_0}\big(\|G_{1m}(k)\|+\|G_{2m}(k)\|\big)<+\infty,$$
$$\lim_{m\to+\infty}\big(G_{1m}(k)+G_{2m}(k-1)\big)=G_{10}(k)+G_{20}(k-1)\ \ (k=1,\dots,m_0)$$

and

$$\lim_{m\to+\infty}\big(g_{1m}(k)+g_{2m}(k-1)\big)=g_{10}(k)+g_{20}(k-1)\ \ (k=1,\dots,m_0)$$

be satisfied. Then inclusion (3.2.34) *holds.*

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(Received 10.12.2018)

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