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**Ahlème Bouakkaz, Abdelouaheb Ardjouni,
Rabah Khemis, Ahcene Djoudi**

GREEN'S FUNCTIONS FOR THE FOURTH-ORDER DIFFERENTIAL EQUATIONS

Abstract. The purpose of this work is to establish and study some useful properties of Green's functions of the fourth-order linear differential equation before using them together with the Guo–Krasnosel'skiĭ's fixed point theorem for proving the existence of positive periodic solutions of the fourth-order nonlinear differential equation.

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1 Introduction

In this work we are essentially interested in studding the existence of positive periodic solutions for certain classes of fourth-order nonlinear differential equations which are ubiquitous in different scientific disciplines and arise especially in the beam theory, viscoelastic and inelastic flows and electric circuits.

There is a vast literature related to this topic, for instance, in the middle of the past century, the existence and uniqueness of solutions for higher-order differential equations have been extensively studied by many researches (see, e.g., [1–7]). During the last two decades, there has been increasing activity in the study of periodic problems of higher-order nonlinear differential equations (see [12] and the references therein).

Some mathematicians used transformation in order to reduce the equation to a more simple one, or to a system of equations, or used synthetic division, others gave the solution in a form of series which converges to the exact solution and some of them dealt with the fourth-order differential equations by using numerical techniques such as the Ritz, finite difference, finite element, cubic spline and multi derivative methods. In this paper, these usual methods may seem inefficient to establish the existence of positive periodic solutions for the fourth-order nonlinear differential equations. For this, inspired by the method presented in [9], we convert the ordinary differential equation to an integral equation in which the kernel is a Green's function, before using the fixed point theorem in cones.

The paper is organized as follows.

The main goal of the next section is to give the Green's functions of the fourth-order constant-coefficient linear differential equation

$$u'''' + au''' + bu'' + cu' + du = h(t), \tag{1.1}$$

where $a, b, c, d \in \mathbb{R}$ and $h \in C(\mathbb{R}, (0, +\infty))$ is a w-periodic function with the period $w > 0$.

The associated homogeneous equation of (1.1) is

$$u'''' + au''' + bu'' + cu' + du = 0, \tag{1.2}$$

where its characteristic equation is

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0. \tag{1.3}$$

In this work we assume that $d \neq 0$ and we will study only the situation when the roots λ_1, λ_2, λ_3, λ_4 are real numbers. These roots satisfy one of the following five cases:

(1) $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$;

(2) $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$;

(3) $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$;

(4) $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$;

(5) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

In the third section, some useful properties of the obtained Green's functions are established. Finally, in the last part, by using the fixed point theorem in cones, we establish the existence of positive periodic solutions of the fourth-order nonlinear differential equation

$$u'''' + au''' + bu'' + cu' + du = f(t, u(t)), \tag{1.4}$$

where $f \in C(\mathbb{R}\times[0, +\infty), [0, +\infty))$ and $f(t, u) > 0$, for $u > 0$.

2 Green's functions

Theorem 2.1. *If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$, then equation* (1.1) *has a unique w-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds,$$

where $s \in [t, t+w]$ and

$$\begin{aligned} G_1(t,s) = {} & \frac{e^{\lambda_1(w+t-s)}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{\lambda_2(w+t-s)}}{(1-e^{w\lambda_2})(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ & + \frac{e^{\lambda_3(w+t-s)}}{(1-e^{w\lambda_3})(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{e^{\lambda_4(w+t-s)}}{(1-e^{w\lambda_4})(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}. \end{aligned}$$

Proof. For $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$, it is easy to see that the general solution of the homogeneous equation (1.2) is

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} + c_4 e^{\lambda_4 t},$$

and that $u(t) \equiv 0$ is its unique solution. Applying the method of variation of parameters, we obtain

$$c_1'(t) = h(t)\,\frac{e^{-t\lambda_1}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}, \quad c_2'(t) = -h(t)\,\frac{e^{-t\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$

$$c_3'(t) = h(t)\frac{e^{-t\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}, \quad c_4'(t) = -h(t)\,\frac{e^{-t\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},$$

whence

$$c_1(t+w) = c_1(t) + \int_t^{t+w} h(s)\,\frac{e^{-s\lambda_1}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\,ds,$$

$$c_2(t+w) = c_2(t) - \int_t^{t+w} h(s)\,\frac{e^{-s\lambda_2}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\,ds,$$

$$c_3(t+w) = c_3(t) + \int_t^{t+w} h(s)\,\frac{e^{-s\lambda_3}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\,ds,$$

$$c_4(t+w) = c_4(t) - \int_t^{t+w} h(s)\,\frac{e^{-s\lambda_4}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\,ds.$$

Since we are looking for w-periodic solutions of (1.1), we have

$$c_1(t) = \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\,ds,$$

$$c_2(t) = -\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_2}}{(1-e^{w\lambda_2})(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\,ds,$$

$$c_3(t) = \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_3}}{(1-e^{w\lambda_3})(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\,ds,$$

$$c_4(t) = -\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\,ds.$$

Therefore,

$$u(t+w) = \int\limits_t^{t+w} G_1(t+w,\theta+w)h(\theta+w)\,d\theta = \int\limits_t^{t+w} G_1(t,s)h(s)\,ds = u(t),$$

which proves the periodicity of u.

Assume that u_1 and u_2 are two w-periodic solutions of (1.1), then $v(t) = u_1(t) - u_2(t)$ is a w-periodic solution of (1.2), i.e., $v(t) = 0$, hence the uniqueness of the w-periodic solution for (1.1) is guaranteed. □

Theorem 2.2. *If $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$, then equation* (1.1) *has a unique w-periodic solution of the form*

$$u(t) = \int\limits_t^{t+w} G_2(t,s)h(s)\,ds,$$

where $s \in [t, t+w]$ and

$$\begin{aligned} G_2(t,s) = {} & \frac{e^{(t+w-s)\lambda_1}\big(w(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)\big(\lambda_3 - 2\lambda_1 + \lambda_4 - s(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \\ & + t\,\frac{e^{(t+w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1^2 - \lambda_1\lambda_3 - \lambda_1\lambda_4 + \lambda_3\lambda_4)} + \frac{e^{(t+w-s)\lambda_3}}{(1-e^{w\lambda_3})(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)} \\ & - \frac{e^{(t+w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}\,. \end{aligned}$$

Proof. For $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$, it is easy to see that the general solution of the homogeneous equation (1.2) is

$$u(t) = c_1e^{\lambda_1 t} + c_2te^{\lambda_1 t} + c_3e^{\lambda_3 t} + c_4e^{\lambda_4 t}.$$

Applying the method of variation of parameters, we obtain

$$c_1'(t) = h(t)\,\frac{e^{-t\lambda_1}}{(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}\,\big(\lambda_3 - 2\lambda_1 + \lambda_4 - t(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big),$$

$$c_2'(t) = \frac{h(t)e^{-t\lambda_1}}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\,, \quad c_3'(t) = \frac{h(t)e^{-t\lambda_3}}{(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}\,, \quad c_4'(t) = -\frac{h(t)e^{-t\lambda_4}}{(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}\,.$$

Since $u(t)$, $u'(t)$, $u''(t)$ and $u'''(t)$ are supposed to be continuous functions, we get

$$c_1(t) = \int\limits_t^{t+w} h(s)\,\frac{e^{\lambda_1(w-s)}\big(w(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)\big(\lambda_3 - 2\lambda_1 + \lambda_4 - s(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}\,ds,$$

$$c_2(t) = \int\limits_t^{t+w} \frac{h(s)e^{(w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1^2 - \lambda_1\lambda_3 - \lambda_1\lambda_4 + \lambda_3\lambda_4)}\,ds,$$

$$c_3(t) = \int\limits_t^{t+w} \frac{h(s)e^{(w-s)\lambda_3}}{(1-e^{w\lambda_3})(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}\,ds,$$

$$c_4(t) = -\int\limits_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}\,ds.$$

Therefore,

$$u(t) = c_1(t)e^{\lambda_1 t} + c_2(t)te^{\lambda_1 t} + c_3(t)e^{\lambda_3 t} + c_4(t)e^{\lambda_4 t} = \int\limits_t^{t+w} G_2(t,s)h(s)\,ds.$$

In the same way as in the proof of Theorem 2.1, we can prove the uniqueness and periodicity of the solution. □

Theorem 2.3. *If $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$, then equation* (1.1) *has a unique w-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds,$$

where $s \in [t, t+w]$ and

$$\begin{aligned} G_3(t,s) = &- \frac{e^{(w+t-s)\lambda_1}\big((1-e^{w\lambda_1})(s\lambda_1 - s\lambda_4 + 2) + w\lambda_4 - w\lambda_1\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3} + t\,\frac{e^{(w+t-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_4)^2} \\ &- \frac{e^{(w+t-s)\lambda_4}\big((e^{w\lambda_4}-1)(s\lambda_4 - s\lambda_1 + 2) + w\lambda_4 - w\lambda_1\big)}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3} + t\,\frac{e^{(w+t-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2}. \end{aligned}$$

Proof. For $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$, (1.2) has the general solution

$$u(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + c_3 e^{\lambda_4 t} + c_4 t e^{\lambda_4 t}.$$

Applying the method of variation of parameters, we obtain

$$\begin{aligned} c_1'(t) &= -h(t)\,\frac{e^{-t\lambda_1}}{(\lambda_1-\lambda_4)^3}\,(t\lambda_1 - t\lambda_4 + 2), \quad c_2'(t) = h(t)\,\frac{e^{-t\lambda_1}}{(\lambda_1-\lambda_4)^2}, \\ c_3'(t) &= h(t)\,\frac{e^{-t\lambda_4}}{(\lambda_1-\lambda_4)^3}\,(t\lambda_4 - t\lambda_1 + 2), \quad c_4'(t) = h(t)\,\frac{e^{-t\lambda_4}}{(\lambda_1-\lambda_4)^2}. \end{aligned}$$

Since $u(t)$, $u'(t)$, $u''(t)$ and $u'''(t)$ are continuous, we have

$$\begin{aligned} c_1(t) &= \int_t^{t+w} -h(s)\,\frac{e^{-\lambda_1(s-w)}\big((1-e^{w\lambda_1})(s\lambda_1 - s\lambda_4 + 2) + w\lambda_4 - w\lambda_1\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,ds, \\ c_2(t) &= \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_4)^2}\,ds, \\ c_3(t) &= \int_t^{t+w} -h(s)\,\frac{e^{-\lambda_4(s-w)}\big((e^{w\lambda_4}-1)(s\lambda_4 - s\lambda_1 + 2) + w\lambda_4 - w\lambda_1\big)}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3}\,ds, \\ c_4(t) &= \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2}\,ds. \end{aligned}$$

Therefore,

$$u(t) = c_1(t)e^{\lambda_1 t} + c_2(t)te^{\lambda_1 t} + c_3(t)e^{\lambda_4 t} + c_4(t)te^{\lambda_4 t} = \int_t^{t+w} G_3(t,s)h(s)\,ds.$$

The uniqueness and periodicity of the solution can again be shown in the same way as in the proof of Theorem 2.1. □

Theorem 2.4. *If $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$, then equation* (1.1) *has a unique w-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds,$$

where $s \in [t, t+w]$ *and*

$$G_4(t,s) = e^{(t+w-s)\lambda_1} \frac{(1-e^{w\lambda_1})\big((e^{w\lambda_1}-1)\big(((s-t)(\lambda_1-\lambda_4)+1)^2+1\big)\big)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3} + \frac{e^{(t+w-s)\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}$$
$$+ \frac{e^{(t+w-s)\lambda_1}\big((1-e^{w\lambda_1})\big(w(\lambda_1-\lambda_4)(2(s-t)(\lambda_1-\lambda_4)+2)\big)\big)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}$$
$$- \frac{w^2 e^{(t+w-s)\lambda_1}(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}.$$

Proof. For $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$, (1.2) has the general solution

$$u(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + c_3 t^2 e^{\lambda_1 t} + c_4 t e^{\lambda_4 t}.$$

The application of the method of variation of parameters gives

$$c_1'(t) = h(t)\, \frac{e^{-t\lambda_1}(t^2\lambda_1^2 - 2t^2\lambda_1\lambda_4 + t^2\lambda_4^2 + 2t\lambda_1 - 2t\lambda_4 + 2)}{2(\lambda_1-\lambda_4)^3},$$
$$c_2'(t) = -h(t)\, \frac{e^{-t\lambda_1}(t\lambda_1 - t\lambda_4 + 1)}{(\lambda_1-\lambda_4)^2}, \quad c_3'(t) = h(t)\, \frac{e^{-t\lambda_1}}{2(\lambda_1-\lambda_4)}, \quad c_4'(t) = -h(t)\, \frac{e^{-t\lambda_4}}{(\lambda_1-\lambda_4)^3}.$$

Since $u(t)$, $u'(t)$, $u''(t)$ and $u'''(t)$ are continuous functions, we have

$$c_1(t) = \int_t^{t+w} h(s)\, \frac{e^{(w-s)\lambda_1}(s^2\lambda_1^2 - 2s^2\lambda_1\lambda_4 + s^2\lambda_4^2 + 2s\lambda_1 - 2s\lambda_4 + 2)}{2(1-e^{w\lambda_1})(\lambda_1-\lambda_4)^3}\, ds$$
$$+ w\, \frac{1}{(1-e^{w\lambda_1})} \int_t^{t+w} h(s)\, \frac{e^{(w-s)\lambda_1}\big((e^{w\lambda_1}-1)(s(\lambda_1-\lambda_4)+1) + w(\lambda_1-\lambda_4)\big)}{(1-e^{w\lambda_1})^2(\lambda_1-\lambda_4)^2}\, ds$$
$$- w^2\, \frac{1}{(1-e^{w\lambda_1})} \int_t^{t+w} h(s)\, \frac{e^{(w-s)\lambda_1}}{2(\lambda_1-\lambda_4)(1-e^{w\lambda_1})}\, ds,$$
$$c_2(t) = \int_t^{t+w} h(s)\, \frac{e^{(w-s)\lambda_1}\big((e^{w\lambda_1}-1)(s(\lambda_1-\lambda_4)+1) + w(\lambda_1-\lambda_4)\big)}{(1-e^{w\lambda_1})^2(\lambda_1-\lambda_4)^2}\, ds,$$
$$c_3(t) = \int_t^{t+w} h(s)\, \frac{e^{(w-s)\lambda_1}}{2(\lambda_1-\lambda_4)(1-e^{w\lambda_1})}\, ds,$$
$$c_4(t) = \int_t^{t+w} h(s)\, \frac{e^{(w-s)\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\, ds.$$

Therefore,

$$u(t) = c_1(t)e^{\lambda_1 t} + c_2(t) t e^{\lambda_1 t} + c_3(t) t^2 e^{\lambda_1 t} + c_4(t) t e^{\lambda_4 t} = \int_t^{t+w} G_4(t,s) h(s)\, ds.$$

In the same way as in the proof of Theorem 2.1 we can prove the uniqueness and periodicity of the solution. □

Theorem 2.5. *If* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, *then equation* (1.1) *has a unique* w*-periodic solution of the form*

$$u(t) = \int_t^{t+w} G_5(t,s) h(s)\, ds,$$

where $s \in [t, t+w]$ *and*

$$G_5(t,s) = e^{(t+w-s)\lambda_1}\,\frac{(s-t)^3(e^{w\lambda_1}-1)^3 + 3w(s-t)^2(e^{w\lambda_1}-1)^2}{6(e^{w\lambda_1}-1)^4} + e^{(t+w-s)\lambda_1}\,\frac{3w^2(s-t)(e^{2w\lambda_1}-1) + w^3(e^{2(w\lambda_1)}+4e^{w\lambda_1}+1)}{6(e^{w\lambda_1}-1)^4}.$$

Proof. For $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, (1.2) has the general solution

$$u(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + c_3 t^2 e^{\lambda_1 t} + c_4 t^3 e^{\lambda_1 t}.$$

By the method of variation of parameters, we arrive at

$$c_1'(t) = -\frac{1}{6}\,ht^3e^{-t\lambda_1},\quad c_2'(t) = \frac{1}{2}\,ht^2e^{-t\lambda_1},\quad c_3'(t) = -\frac{1}{2}\,hte^{-t\lambda_1},\quad c_4'(t) = \frac{1}{6}\,he^{-t\lambda_1}.$$

Since $u(t)$, $u'(t)$, $u''(t)$ and $u'''(t)$ are continuous functions, we get

$$\begin{aligned}
c_1(t) = {} & \int_t^{t+w} -h(s)\,\frac{s^3e^{(w-s)\lambda_1}}{6(1-e^{w\lambda_1})}\,ds \\
& + \int_t^{t+w} -h(s)\,\frac{we^{(w-s)\lambda_1}\big(s^2e^{2(w\lambda_1)} - 2s^2e^{w\lambda_1} + s^2 + 2swe^{w\lambda_1} - 2sw + w^2e^{w\lambda_1} + w^2\big)}{2(1-e^{w\lambda_1})(e^{w\lambda_1}-1)^3}\,ds \\
& - \frac{w^2}{(1-e^{w\lambda_1})}\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}(w-s+se^{w\lambda_1})}{2(1-e^{w\lambda_1})^2}\,ds \\
& + \frac{w^3}{(1-e^{w\lambda_1})}\int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{6(1-e^{w\lambda_1})}\,ds, \\
c_2(t) = {} & \int_t^{t+w} -h(s)\,\frac{e^{(w-s)\lambda_1}\big(s^2e^{2(w\lambda_1)} - 2s^2e^{w\lambda_1} + s^2 + 2swe^{w\lambda_1} - 2sw + w^2e^{w\lambda_1} + w^2\big)}{2(e^{w\lambda_1}-1)^3}\,ds, \\
c_3(t) = {} & \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}(w-s+se^{w\lambda_1})}{2(1-e^{w\lambda_1})^2}\,ds, \\
c_4(t) = {} & \int_t^{t+w} h(s)\,\frac{e^{(w-s)\lambda_1}}{6(1-e^{w\lambda_1})}\,ds.
\end{aligned}$$

Therefore,

$$u(t) = c_1(t)e^{\lambda_1 t} + c_2(t)te^{\lambda_1 t} + c_3(t)t^2e^{\lambda_1 t} + c_3(t)t^3e^{\lambda_1 t} = \int_t^{t+w} G_5(t,s)h(s)\,ds.$$

In the same way as in the proof of Theorem 2.1, we can prove the uniqueness and the periodicity of the solution. □

3 Properties of the Green's functions

We denote

$$\mathcal{C}_w^+ = \big\{u \in \mathcal{C}(\mathbb{R},(0,+\infty)):\ \ u(t+w) = u(t)\big\},$$
$$\mathcal{C}_w^- = \big\{u \in \mathcal{C}(\mathbb{R},(-\infty,0)):\ \ u(t+w) = u(t)\big\}.$$

Case 1. *If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$. For ease of exposition, we use the following abbreviations:*

$$\begin{aligned}
g_{1,1}(t,s) &= \frac{e^{(w+t-s)\lambda_1}}{(1-e^{(\lambda_1)w})(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},\\
g_{1,2}(t,s) &= \frac{e^{(w+t-s)\lambda_2}}{(1-e^{(\lambda_2)w})(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},\\
g_{1,3}(t,s) &= \frac{e^{(w+t-s)\lambda_3}}{(1-e^{(\lambda_3)w})(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)},\\
g_{1,4}(t,s) &= \frac{e^{(w+t-s)\lambda_4}}{(1-e^{(\lambda_4)w})(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)};
\end{aligned}$$

$$\begin{aligned}
A_{1,1} = &-\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\\
&+\frac{1}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},\\
A_{1,2} = &-\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\\
&-\frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)} + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)};
\end{aligned}$$

$$\begin{aligned}
B_{1,1} = &-\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\\
&+\frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},\\
B_{1,2} = &-\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{1}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\\
&-\frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)};
\end{aligned}$$

$$\begin{aligned}
n_{1,1} = &\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)},\\
n_{1,2} = &+\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\\
&-\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},\\
n_{1,3} = &+\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\\
&-\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},\\
n_{1,4} = &\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)},\\
n_{1,5} = &+\frac{1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)};
\end{aligned}$$

$$\begin{aligned}
p_{1,1} &= \frac{1}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},\\
p_{1,2} &= \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},
\end{aligned}$$

$$p_{1,3} = \frac{1}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$
$$p_{1,4} = \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},$$
$$p_{1,5} = \frac{e^{w\lambda_2}}{(e^{w\lambda_2}-1)(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}.$$

Theorem 3.1. *For all $t \in [0, w]$ and $s \in [t, t+w]$, we have*

$$\int_t^{t+w} G_1(t,s)\,ds = \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

Proof. We have

$$\int_t^{t+w} g_{1,1}(t,s)\,ds = -\frac{1}{\lambda_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$
$$\int_t^{t+w} g_{1,2}(t,s)\,ds = \frac{1}{\lambda_2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$
$$\int_t^{t+w} g_{1,3}(t,s)\,ds = -\frac{1}{\lambda_3(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)},$$
$$\int_t^{t+w} g_{1,4}(t,s)\,ds = \frac{1}{\lambda_4(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)},$$

and

$$\int_t^{t+w} G_1(t,s)\,ds = \int_t^{t+w} g_{1,1}(t,s)\,ds + \int_t^{t+w} g_{1,2}(t,s)\,ds + \int_t^{t+w} g_{1,3}(t,s)\,ds + \int_t^{t+w} g_{1,4}(t,s)\,ds = \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$ □

We have four different roots satisfying one of the five cases:

- All roots are positive.
- Three roots are positive and one root is negative.
- Three roots are negative and one root is positive.
- Two roots are positive and two roots are negative.
- All roots are negative.

If all roots are positive, we suppose that $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.2. *If $p_{1,1} > n_{1,1}$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, then*

$$0 < A_{1,1} \le G_1(t,s) \le B_{1,1}.$$

Proof. If $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s, gives $\frac{\partial}{\partial s} g_{1,1}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) > 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) < 0$. This implies that

$$\begin{aligned} g_{1,1}(t,t) + g_{1,2}(t,t+w) + g_{1,3}(t,t) + g_{1,4}(t,t+w) \\ \le G_1(t,s) \le g_{1,1}(t,t+w) + g_{1,2}(t,t) + g_{1,3}(t,t+w) + g_{1,4}(t,t). \end{aligned}$$

From the above double inequality and the assumption $p_{1,1} > n_{1,1}$, we obtain $0 < A_{1,1} \le G_1(t,s) \le B_{1,1}$. □

Corollary 3.1. *If $h \in \mathcal{C}_w^+$ and $p_{1,1} > n_{1,1}$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.1. Consider the equation

$$u'''' - 0.56u''' + 0.0311u'' - 5.56 \times 10^{-4}u' + 3 \times 10^{-6}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda - 0.5)(\lambda - 0.03)(\lambda - 0.02)(\lambda - 0.01) = 0$ has four roots $\lambda_1 = 0.5$, $\lambda_2 = 0.03$, $\lambda_3 = 0.02$, $\lambda_4 = 0.01$.

Since $p_{1,1} = 2.086\,4 \times 10^5 > n_{1,1} = 1.764\,3 \times 10^5$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$, with $\int_t^{t+w} G_1(t,s)\,ds = 3.3333 \times 10^5$ and $0 < 32210 < G_1(t,s) < 73894$.

If three roots are positive and one root is negative, we suppose that $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_4 < 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.3. *If $p_{1,2} < n_{1,2}$, $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_4 < 0$, then*

$$A_{1,1} \le G_1(t,s) \le B_{1,1} < 0.$$

Proof. If $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_4 < 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s} g_{1,1}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) > 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) < 0$. Similarly, as in the proof of Theorem 3.2, we obtain $A_{1,1} \le G_1(t,s) \le B_{1,1} < 0$. □

Corollary 3.2. *If $h \in \mathcal{C}_w^-$, $p_{1,2} < n_{1,2}$, $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and $\lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.2. We consider the equation

$$u'''' - 0.59u''' + 0.104u'' - 0.0049u' - 0.00006u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda - 0.3)(\lambda - 0.2)(\lambda - 0.1)(\lambda + 0.01) = 0$ has the roots $\lambda_1 = 0.3$, $\lambda_2 = 0.2$, $\lambda_3 = 0.1$, $\lambda_4 = -0.01$.

Since $p_{1,2} = 665.64 < n_{1,2} = 2702.1$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$, with $\int_t^{t+w} G_1(t,s)\,ds = -16667$ and $-3268.1 < G_1(t,s) < -2036.5 < 0$.

If three roots are negative and one root is positive, we suppose that $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and $\lambda_4 > 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.4. *If $p_{1,3} < n_{1,3}$, $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and $\lambda_4 > 0$, then*

$$B_{1,1} \le G_1(t,s) \le A_{1,1} < 0.$$

Proof. If $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and $\lambda_4 > 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s} g_{1,1}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) < 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) > 0$. Similarly, as in the proof of Theorem 3.2, we obtain $B_{1,1} \le G_1(t,s) \le A_{1,1} < 0 < 0$. □

Corollary 3.3. *If $h \in \mathcal{C}_w^-$, $p_{1,3} < n_{1,3}$, $\lambda_1 < \lambda_2 < \lambda_3 < 0$ and $\lambda_4 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.3. Consider the equation

$$u'''' + 0.59u''' + 0.104u'' + 0.0049u' - 0.00006u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.3)(\lambda+0.2)(\lambda+0.1)(\lambda-0.01) = 0$ has the roots $\lambda_1 = -0.3$, $\lambda_2 = -0.2$, $\lambda_3 = -0.1$, $\lambda_4 = 0.01$.

Since $p_{1,3} = 665.64 < n_{1,3} = 2702.1$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$ with $\int_t^{t+w} G_1(t,s)\,ds = -16667$ and $-3268.1 < G_1(t,s) < -2036.5 < 0$.

If two roots are negative and two roots are positive, we suppose that $\lambda_1 < \lambda_2 < 0$ and $\lambda_3 > \lambda_4 > 0$ (the other situations can be proved by using the same method) and have

Theorem 3.5. *If $p_{1,4} > n_{1,4}$, $\lambda_1 < \lambda_2 < 0$ and $\lambda_3 > \lambda_4 > 0$, then*

$$0 < A_{1,2} \le G_1(t,s) \le B_{1,2}.$$

Proof. If $\lambda_1 < \lambda_2 < 0$ and $\lambda_3 > \lambda_4 > 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s} g_{1,1}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) > 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) < 0$. Similarly, as in the proof of Theorem 3.2, we obtain $0 < A_{1,2} \le G_1(t,s) \le B_{1,2}$. □

Corollary 3.4. *If $h \in \mathcal{C}_w^+$, $p_{1,4} > n_{1,4}$, $\lambda_1 < \lambda_2 < 0$ and $\lambda_3 > \lambda_4 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.4. Consider the equation

$$u'''' - 0.054u'' - 4.9304 \times 10^{-32}u' + 0.0004u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.2)(\lambda+0.1)(\lambda-0.2)(\lambda-0.1) = 0$ has the roots $\lambda_1 = -0.2$, $\lambda_2 = -0.1$, $\lambda_3 = 0.2$, $\lambda_4 = 0.1$.

Since $p_{1,4} = 381.19 > n_{1,4} = 232.97$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$ with $\int_t^{t+w} G_1(t,s)\,ds = 2500$ and $0 < 148.22 < G_1(t,s) < 648.22$.

If all roots are negative, we suppose that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.6. *If $p_{1,5} > n_{1,5}$ and $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$, then*

$$0 < B_{1,1} \le G_1(t,s) \le A_{1,1}.$$

Proof. If $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s} g_{1,1}(t,s) < 0$, $\frac{\partial}{\partial s} g_{1,2}(t,s) > 0$, $\frac{\partial}{\partial s} g_{1,3}(t,s) < 0$ and $\frac{\partial}{\partial s} g_{1,4}(t,s) > 0$. Similarly, as in the proof of Theorem 3.2, we obtain $0 < B_{1,1} \le G_1(t,s) \le A_{1,1}$. □

Corollary 3.5. *If $h \in \mathcal{C}_w^+$, $p_{1,5} > n_{1,5}$ and $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds.$$

Example 3.5. Consider the equation

$$u'''' + 0.56u''' + 0.0311u'' + 5.56 \times 10^{-4}u' + 3.0 \times 10^{-6}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.5)(\lambda+0.03)(\lambda+0.02)(\lambda+0.01) = 0$ has the roots $\lambda_1 = -0.5$, $\lambda_2 = -0.03$, $\lambda_3 = -0.02$, $\lambda_4 = -0.01$. Since $p_{1,5} = 2.086\,4 \times 10^5 > n_{1,5} = 1.764\,3 \times 10^5$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_1(t,s)h(s)\,ds$, with $\int_t^{t+w} G_1(t,s)\,ds = 3.333 \times 10^5$ and $0 < 32210 < G_1(t,s) < 73894$.

Case 2. *If $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$, $\lambda_1, \lambda_3, \lambda_4 \in \mathbb{R}$. We use the following abbreviations:*

$$g_{2,1}(t,s) = \frac{e^{(t+w-s)\lambda_1}\big(w(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)\big(\lambda_3-2\lambda_1+\lambda_4-s(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2},$$

$$g_{2,2}(t,s) = t\,\frac{e^{(t+w-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1^2-\lambda_1\lambda_3-\lambda_1\lambda_4+\lambda_3\lambda_4)},$$

$$g_{2,3}(t,s) = \frac{e^{(t+w-s)\lambda_3}}{(1-e^{w\lambda_3})(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)},$$

$$g_{2,4}(t,s) = -\frac{e^{(t+w-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)};$$

$$h_{2,1}(s,t) = \frac{\big((\lambda_1^2-\lambda_3\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)e^{\lambda_1(t-s+w)}}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2},$$

$$h_{2,2}(s,t) = \frac{(s\lambda_1+1)e^{\lambda_1(t-s+w)}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)};$$

$$\begin{aligned}
A_{2,1} = {} & \frac{we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \\
& - w\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)} \\
& + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},
\end{aligned}$$

$$\begin{aligned}
A_{2,2} = {} & \frac{1}{\lambda_1}\,\frac{\big((\lambda_1^2-\lambda_3\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \\
& + \frac{1}{\lambda_1}\,\frac{2w\lambda_1+1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} - \frac{we^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} \\
& - \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)};
\end{aligned}$$

$$\begin{aligned}
B_{2,1} = {} & \frac{\big(we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)\big)e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \\
& - \frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)} + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},
\end{aligned}$$

$$B_{2,2} = \frac{(\lambda_1^2-\lambda_3\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$

$$+\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}+\frac{e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}$$
$$-\frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)};$$

$$n_{2,1}=w\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}+\frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)},$$
$$n_{2,2}=\frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}-\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},$$
$$n_{2,3}=+\frac{e^{w\lambda_3}}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}-\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},$$
$$n_{2,4}=\frac{1}{(e^{w\lambda_3}-1)(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)}-\frac{1}{\lambda_1}\frac{2w\lambda_1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)};$$

$$p_{2,1}=\frac{we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$
$$+\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)},$$
$$p_{2,2}=\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$
$$\times\Big(we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)\Big),$$
$$p_{2,3}=\frac{1}{\lambda_1}\,\frac{\big((\lambda_1)(\lambda_1)-\lambda_3\lambda_4\big)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$
$$+\frac{e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$
$$p_{2,4}=\frac{e^{w\lambda_1}\big((\lambda_1^2-\lambda_3\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)\big)}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}$$
$$+\frac{1}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}$$
$$+\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}-w\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}.$$

Theorem 3.7. *For all $t\in[0,w]$ and $s\in[t,t+w]$, we have*

$$\int_t^{t+w} G_2(t,s)\,ds=\frac{1}{\lambda_1^2\lambda_3\lambda_4}.$$

Proof. We have

$$\int_t^{t+w} g_{2,1}(t,s)\,ds=\frac{3\lambda_1^2-2\lambda_1\lambda_3-2\lambda_1\lambda_4+\lambda_3\lambda_4+t\lambda_1^3-t\lambda_1^2\lambda_3-t\lambda_1^2\lambda_4+t\lambda_1\lambda_3\lambda_4}{\lambda_1^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2},$$
$$\int_t^{t+w} g_{2,2}(t,s)\,ds=-\frac{t}{\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$
$$\int_t^{t+w} g_{2,3}(t,s)\,ds=-\frac{1}{\lambda_3(\lambda_1-\lambda_3)^2(\lambda_3-\lambda_4)},$$

$$\int_t^{t+w} g_{2,4}(t,s)\,ds = \frac{1}{\lambda_4(\lambda_1-\lambda_4)^2(\lambda_3-\lambda_4)}\,,$$

and

$$\int_t^{t+w} G_2(t,s)\,ds = \int_t^{t+w} g_{2,1}(t,s)\,ds + \int_t^{t+w} g_{2,2}(t,s)\,ds + \int_t^{t+w} g_{2,3}(t,s)\,ds + \int_t^{t+w} g_{2,4}(t,s)\,ds = \frac{1}{\lambda_1^2\lambda_3\lambda_4}\,. \qquad \square$$

We have three different roots satisfying one of the following four cases:

- All roots are positive.
- Two roots are positive and one root is negative.
- Two roots are negative and one root is positive.
- All roots are negative.

If all roots are positive, we suppose that $\lambda_1 > \lambda_3 > \lambda_4 > 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.8. *If $p_{2,1} > n_{2,1}$ and $\lambda_1 > \lambda_3 > \lambda_4 > 0$, then*

$$0 < A_{2,1} \le G_2(t,s) \le B_{2,1}.$$

Proof. If $\lambda_1 > \lambda_3 > \lambda_4 > 0$, the study of the derivatives of $g_{1,i}$, $i = \overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s}\, g_{2,1}(s,t) < 0$, $\frac{\partial}{\partial s}\, g_{2,2}(s,t) > 0$, $\frac{\partial}{\partial s}\, g_{2,3}(s,t) > 0$ and $\frac{\partial}{\partial s}\, g_{2,4}(s,t) < 0$. This implies that

$$g_{2,1}(t,t+w) + g_{2,2}(t,t) + g_{2,3}(t,t) + g_{2,4}(t,t+w) \le G_2(t,s) \le g_{2,1}(t,t) + g_{2,2}(t,t+w) + g_{2,3}(t,t+w) + g_{2,4}(t,t).$$

It is easy to check that

$$0 < \frac{we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2} \le g_{2,1}(t,t+w),$$

$$0 < g_{2,1}(t,t) \le \frac{\big(we^{w\lambda_1}(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)(\lambda_3-2\lambda_1+\lambda_4)\big)e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2}\,,$$

$$-w\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} \le g_{2,2}(t,t) \le 0, \quad g_{2,2}(t,t+w) \le 0.$$

By using the last double inequality together with the assumption $p_{2,1} > n_{2,1}$, we arrive at $0 < A_{2,1} \le G_1(t,s) \le B_{2,1}$. $\square$

Corollary 3.6. *If $h \in \mathcal{C}_w^+$, $p_{2,1} > n_{2,1}$ and $\lambda_1 > \lambda_3 > \lambda_4 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_2(t,s)h(s)\,ds.$$

Example 3.6. Consider the equation

$$u'''' - 0.51u''' + 0.085u'' - 0.0048u' + 0.00004u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda-0.2)^2(\lambda-0.1)(\lambda-0.01)=0$ has the roots $\lambda_1=0.2$, $\lambda_3=0.1$, $\lambda_4=0.01$. Since $p_{2,1}=5249.8>n_{2,1}=2844$, the equation has a unique 2π-periodic solution $u(t)=\int_t^{t+w} G_2(t,s)h(s)\,ds$ with $\int_t^{t+w} G_2(t,s)\,ds=25000$ and $0<2405.8<G_2(t,s)<5552.5$.

If two roots are positive and one root is negative, we suppose that $\lambda_1>\lambda_3>0$ and $\lambda_4<0$ (the other situations can be proved by using the same method), and we have

Theorem 3.9. *If $p_{2,2}<n_{2,2}$, $\lambda_1>\lambda_3>0$ and $\lambda_4<0$, then*

$$A_{2,1}\le G_2(t,s)\le B_{2,1}<0.$$

Proof. If $\lambda_1>\lambda_3>0$ and $\lambda_4<0$, the study of the derivatives of $g_{1,i}$, $i=\overline{1,4}$, with respect to s gives $\frac{\partial}{\partial s}g_{2,1}(s,t)<0$, $\frac{\partial}{\partial s}g_{2,2}(s,t)>0$, $\frac{\partial}{\partial s}g_{2,3}(s,t)>0$ and $\frac{\partial}{\partial s}g_{2,4}(s,t)<0$. Similarly, as in the proof of Theorem 3.8, we obtain $A_{2,1}\le G_2(t,s)\le B_{2,1}<0$. □

Corollary 3.7. *If $h\in\mathcal{C}_w^-$, $p_{2,2}<n_{2,2}$, $\lambda_1>\lambda_3>0$ and $\lambda_4<0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t)=\int_t^{t+w} G_2(t,s)h(s)\,ds.$$

Example 3.7. Consider the equation

$$u''''-0.49u'''+0.075u''-0.0032u'-0.00004u=h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda-0.2)^2(\lambda-0.1)(\lambda+0.01)=0$ has the roots $\lambda_1=0.2$, $\lambda_3=0.1$, $\lambda_4=-0.01$. Since $p_{2,2}=1567.2<n_{2,2}=4218.5$, the equation has a unique 2π-periodic solution $u(t)=\int_t^{t+w} G_2(t,s)h(s)\,ds$ with $\int_t^{t+w} G_2(t,s)\,ds=-25000$ and $-5305.9<G_2(t,s)<-2651.3<0$.

If two roots are negative and one root is positive, we suppose that $\lambda_1<\lambda_3<0$ and $\lambda_4>0$ (the other situations can be proved by using the same method), and we have

Theorem 3.10. *If $p_{2,3}<n_{2,3}$, $\lambda_1<\lambda_3<0$ and $\lambda_4>0$, then*

$$A_{2,2}\le G_2(t,s)\le B_{2,2}<0.$$

Proof. We have $g_{2,1}(s,t)=h_{2,1}(s,t)+h_{2,2}(s,t)$. If $\lambda_1<\lambda_3<0$ and $\lambda_4>0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s}h_{2,1}(s,t)>0$, $\frac{\partial}{\partial s}h_{2,2}(s,t)<0$, $\frac{\partial}{\partial s}g_{2,2}(s,t)>0$, $\frac{\partial}{\partial s}g_{2,3}(s,t)<0$ and $\frac{\partial}{\partial s}g_{2,4}(s,t)>0$. This implies that

$$\begin{aligned}&h_{2,1}(t,t)+h_{2,2}(t,t+w)+g_{2,2}(t,t)+g_{2,3}(t,t+w)+g_{2,4}(t,t)\\&\qquad\le G_2(t,s)\le h_{2,1}(t,t+w)+h_{2,2}(t,t)+g_{2,2}(t,t+w)+g_{2,3}(t,t)+g_{2,4}(t,t+w).\end{aligned}$$

It is easy to check that

$$\frac{e^{w\lambda_1}(w\lambda_1+1)}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\le h_{2,2}(t,t)\le\frac{e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$
$$\frac{2w\lambda_1+1}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\le h_{2,2}(t,t+w)\le\frac{w\lambda_1+1}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}.$$

The above double inequality and the assumption $p_{2,3}>n_{2,3}$ lead to $0<A_{2,2}\le G_2(t,s)\le B_{2,2}$. □

Corollary 3.8. *If $h\in\mathcal{C}_w^-$, $p_{2,3}<n_{2,3}$, $\lambda_1<\lambda_3<0$ and $\lambda_4>0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t)=\int_t^{t+w} G_2(t,s)h(s)\,ds.$$

Example 3.8. Consider the equation

$$u'''' + 0.49u''' + 0.075u'' + 0.0032u' - 0.00004u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.2)^2(\lambda+0.1)(\lambda-0.01) = 0$ has the roots $\lambda_1 = -0.2$, $\lambda_3 = -0.1$, $\lambda_4 = 0.01$. Since $p_{2,3} = 1329.1 < n_{2,3} = 4218.5$, the equation has a unique 2π-periodic solution $u(t) = \int\limits_t^{t+w} G_2(t,s)h(s)\,ds$ with $\int\limits_t^{t+w} G_2(t,s)\,ds = -25000$, $-5367.0 < G_2(t,s) < -2889.4 < 0$.

If all roots are negative, we suppose that $\lambda_1 < \lambda_3 < 0 < \lambda_4 < 0$ (the other situations can be proved by using the same method), and we have

Theorem 3.11. *If $p_{2,4} > n_{2,4}$ and $\lambda_1 < \lambda_3 < \lambda_4 < 0$, then*

$$0 < A_{2,2} \le G_2(t,s) \le B_{2,2}.$$

Proof. The study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{2,1}(s,t) > 0$, $\frac{\partial}{\partial s} h_{2,2}(s,t) < 0$, $\frac{\partial}{\partial s} g_{2,2}(s,t) > 0$, $\frac{\partial}{\partial s} g_{2,3}(s,t) < 0$ and $\frac{\partial}{\partial s} g_{2,4}(s,t) > 0$. Similarly, as in the proof of Theorem 3.8, we obtain $0 < A_{2,2} \le G_2(t,s) \le B_{2,2}$. □

Corollary 3.9. *If $h \in \mathcal{C}_w^+$, $p_{2,4} > n_{2,4}$ and $\lambda_1 < \lambda_3 < \lambda_4 < 0$, then equation (1.1) has a unique positive periodic solution*

$$u(t) = \int\limits_t^{t+w} G_2(t,s)h(s)\,ds.$$

Example 3.9. Consider the equation

$$u'''' + 0.51u''' + 0.085u'' + 0.0048u' + 0.00004u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.2)^2(\lambda+0.1)(\lambda+0.01) = 0$ has the roots $\lambda_1 = -0.2$, $\lambda_3 = -0.1$, $\lambda_4 = -0.01$. Since $p_{2,4} = 5644.5 > n_{2,4} = 3306.3$, the equation has a unique 2π-periodic solution $u(t) = \int\limits_t^{t+w} G_2(t,s)h(s)\,ds$ with $\int\limits_t^{t+w} G_2(t,s)\,ds = 25000$ and $0 < 2338.3 < G_2(t,s) < 5289.4$.

Case 3. *If $\lambda_1 = \lambda_2 \ne \lambda_3 = \lambda_4$. We use the following abbreviations:*

$$g_{3,1}(t,s) = -\frac{\big((1-e^{w\lambda_1})(s\lambda_1 - s\lambda_4 + 2) - w(\lambda_1-\lambda_4)\big)e^{(w+t-s)\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3} + t\,\frac{e^{(w+t-s)\lambda_1}}{(1-e^{w\lambda_1})(\lambda_1-\lambda_4)^2},$$

$$g_{3,2}(t,s) = -\frac{\big((e^{w\lambda_4}-1)(s\lambda_4 - s\lambda_1 + 2) - w(\lambda_1-\lambda_4)\big)e^{(w+t-s)\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3} + t\,\frac{e^{(w+t-s)\lambda_4}}{(1-e^{w\lambda_4})(\lambda_1-\lambda_4)^2};$$

$$h_{3,1}(s,t) = \frac{e^{\lambda_4(t-s+w)}\big(\lambda_4(s-t)(e^{w\lambda_4}-1) + e^{w\lambda_4} + w\lambda_4 - 1\big)}{\lambda_4(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2},$$

$$h_{3,2}(s,t) = -\frac{e^{\lambda_4(t-s+w)}(\lambda_1+\lambda_4)}{\lambda_4(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},$$

$$h_{3,3}(t,s) = \frac{1}{\lambda_1}\,\frac{e^{\lambda_1(t-s+w)}\big(\lambda_1(s-t)(e^{w\lambda_1}-1)(\lambda_1-\lambda_4) + w\lambda_1(\lambda_1-\lambda_4) - (e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\big)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3},$$

$$h_{3,4}(t,s) = \frac{1}{\lambda_1}\,\lambda_4\,\frac{e^{\lambda_1(t-s+w)}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^3},$$

$$h_{3,5}(t,s) = -\frac{e^{\lambda_4(t-s+w)}(1-(s-t)(\lambda_1-\lambda_4))}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},$$

$$
\begin{aligned}
h_{3,6}(t,s) &= -\frac{e^{\lambda_4(t-s+w)}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3}\,(e^{w\lambda_4}-w\lambda_1+w\lambda_4-1),\\
h_{3,7}(t,s) &= \frac{e^{\lambda_1(t-s+w)}\big(w\lambda_1(\lambda_1-\lambda_4)+(\lambda_1+\lambda_4)(e^{w\lambda_1}-1)\big)}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3},\\
h_{3,8}(t,s) &= \frac{e^{\lambda_1(t-s+w)}(s\lambda_1-t\lambda_1+1)}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2},\\
h_{3,9}(t,s) &= w\,\frac{e^{\lambda_4(t-s+w)}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2},\\
h_{3,10}(t,s) &= -\frac{e^{\lambda_4(t-s+w)}(2-(s-t)(\lambda_1-\lambda_4))}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3};
\end{aligned}
$$

$$
\begin{aligned}
A_{3,1} &= \frac{2e^{w\lambda_1}+w\lambda_1e^{w\lambda_1}-w\lambda_4e^{w\lambda_1}-2}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}-\frac{e^{w\lambda_4}(\lambda_1+\lambda_4)}{\lambda_4(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{(e^{w\lambda_4}+w\lambda_4e^{w\lambda_4}-1)}{\lambda_4(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2},\\
A_{3,2} &= \frac{\lambda_4-2\lambda_1+2\lambda_1e^{w\lambda_1}-\lambda_4e^{w\lambda_1}+w\lambda_1^2e^{w\lambda_1}-w\lambda_1\lambda_4e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{\lambda_4e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^3}-\frac{(w\lambda_4-w\lambda_1+1)}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad-\frac{(e^{w\lambda_4}-w\lambda_1+w\lambda_4-1)e^{w\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3},\\
A_{3,3} &= \frac{\big((\lambda_1+\lambda_4)(e^{w\lambda_1}-1)+w\lambda_1(\lambda_1-\lambda_4)\big)e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}+\frac{we^{w\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2}\\
&\quad+\frac{1}{\lambda_1}\,\frac{w\lambda_1+1}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2}-\frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,(w\lambda_4-w\lambda_1+2);\\
B_{3,1} &= \frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,(2e^{w\lambda_1}+w\lambda_1-w\lambda_4-2)-\frac{1}{\lambda_4}\,\frac{\lambda_1+\lambda_4}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{1}{\lambda_4}\,\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2}\,(e^{w\lambda_4}+w\lambda_4-1),\\
B_{3,2} &= \frac{1}{\lambda_1}\,\frac{e^{w\lambda_1}(\lambda_4-2\lambda_1+w\lambda_1^2+2\lambda_1e^{w\lambda_1}-\lambda_4e^{w\lambda_1}-w\lambda_1\lambda_4)}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{1}{\lambda_1}\,\frac{\lambda_4}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^3}-\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad-\frac{1}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3}\,(e^{w\lambda_4}-w\lambda_1+w\lambda_4-1),\\
B_{3,3} &= -\frac{1}{\lambda_1(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,(\lambda_1+\lambda_4-w\lambda_1^2-\lambda_1e^{w\lambda_1}-\lambda_4e^{w\lambda_1}+w\lambda_1\lambda_4)\\
&\quad+\frac{1}{\lambda_1}\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2}+\frac{w}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^2}-2\,\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3};
\end{aligned}
$$

$$
\begin{aligned}
n_{3,1} &= \frac{e^{w\lambda_4}(\lambda_1+\lambda_4)}{\lambda_4(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},\\
n_{3,2} &= -\frac{\lambda_4e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^3}-\frac{w(\lambda_1-\lambda_4)}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3},\\
n_{3,3} &= \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}(w\lambda_4-w\lambda_1+2)-\frac{w}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2};
\end{aligned}
$$

$$p_{3,1} = \frac{2e^{w\lambda_1} + w\lambda_1 e^{w\lambda_1} - w\lambda_4 e^{w\lambda_1} - 2}{(e^{w\lambda_1} - 1)^2(\lambda_1 - \lambda_4)^3} + \frac{e^{w\lambda_4} + w\lambda_4 e^{w\lambda_4} - 1}{\lambda_4(e^{w\lambda_4} - 1)^2(\lambda_1 - \lambda_4)^2},$$

$$p_{3,2} = \frac{\lambda_4 - 2\lambda_1 + 2\lambda_1 e^{w\lambda_1} - \lambda_4 e^{w\lambda_1} + w\lambda_1^2 e^{w\lambda_1} - w\lambda_1\lambda_4 e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1} - 1)^2(\lambda_1 - \lambda_4)^3} - \frac{1}{(e^{w\lambda_4} - 1)(\lambda_1 - \lambda_4)^3} - \frac{(e^{w\lambda_4} - w\lambda_1 + w\lambda_4 - 1)e^{w\lambda_4}}{(e^{w\lambda_4} - 1)^2(\lambda_1 - \lambda_4)^3},$$

$$p_{3,3} = \frac{\big((\lambda_1 + \lambda_4)(e^{w\lambda_1} - 1) + w\lambda_1(\lambda_1 - \lambda_4)\big)e^{w\lambda_1}}{\lambda_1(e^{w\lambda_1} - 1)^2(\lambda_1 - \lambda_4)^3} + \frac{1}{\lambda_1(e^{w\lambda_1} - 1)(\lambda_1 - \lambda_4)^2} + w\,\frac{e^{w\lambda_4}}{(e^{w\lambda_4} - 1)^2(\lambda_1 - \lambda_4)^2}.$$

Theorem 3.12. *For all $t \in [0, w]$ and $s \in [t, t + w]$, we have*

$$\int_t^{t+w} G_3(t,s)\,ds = \frac{1}{\lambda_1^2\lambda_4^2}.$$

Proof. We have

$$\int_t^{t+w} G_3(t,s)\,ds = \int_t^{t+w} g_{3,1}(t,s)\,ds + \int_t^{t+w} g_{3,2}(t,s)\,ds = -\frac{\lambda_4 - 3\lambda_1}{\lambda_1^2(\lambda_1 - \lambda_4)^3} + \frac{\lambda_1 - 3\lambda_4}{\lambda_4^2(\lambda_1 - \lambda_4)^3} = \frac{1}{\lambda_1^2\lambda_4^2}. \quad \square$$

We have two different roots satisfying one of the following three cases:

- Two positive roots.
- One positive root and one negative root.
- Two negative roots.

If all roots are positive, we suppose that $\lambda_1 > \lambda_4 > 0$ (the situation when $\lambda_4 > \lambda_1 > 0$ can be proved by using the same method), and we have

Theorem 3.13. *If $p_{3,1} > n_{3,1}$ and $\lambda_1 > \lambda_2 > 0$, then*

$$0 < A_{3,1} \le G_3(t,s) \le B_{3,1}.$$

Proof. We write $g_{3,2}(t,s) = h_{3,1}(t,s) + h_{3,2}(t,s)$. If $\lambda_1 > \lambda_2 > 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} g_{3,1}(t,s) < 0$, $\frac{\partial}{\partial s} h_{3,1}(t,s) < 0$ and $\frac{\partial}{\partial s} h_{3,2}(t,s) > 0$. This implies that

$$g_{3,1}(t,t+w) + h_{3,1}(t,t+w) + h_{3,2}(t,t) \le G_3(t,s) \le g_{3,1}(t,t) + h_{3,1}(t,t) + h_{3,2}(t,t+w).$$

This double inequality together with the assumption $p_{3,1} > n_{3,1}$ lead to $0 < A_{3,1} \le G_3(t,s) \le B_{3,1}$. $\square$

Corollary 3.10. *If $h \in \mathcal{C}_w^+$, $p_{3,1} > n_{3,1}$ and $\lambda_1 > \lambda_2 > 0$, then equation (1.1) has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds.$$

Example 3.10. Consider the equation

$$u'''' - 0.06\,u''' + 0.0013u'' - 1.2 \times 10^{-5}u' + 4.0 \times 10^{-8}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation is $(\lambda - 0.02)^2(\lambda - 0.01)^2 = 0$ has two roots $\lambda_1 = 0.02$ and $\lambda_4 = 0.01$. Since $p_{3,1} = 5.0241 \times 10^7 > n_{3,1} = 4.9262 \times 10^7$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds$ with $\int_t^{t+w} G_3(t,s)\,ds = 2.5 \times 10^7$ and $0 < 9.7887 \times 10^5 < G_3(t,s) < 6.9789 \times 10^6$.

If one root is positive and one root is negative, we suppose that $\lambda_1 > 0$ and $\lambda_4 < 0$ (the situation when $\lambda_1 < 0$ and $\lambda_4 > 0$ can be proved by using the same method), and we have

Theorem 3.14. *If $p_{3,2} > n_{3,2}$, $\lambda_1 > 0$ and $\lambda_4 < 0$, then*

$$0 < A_{3,2} \leq G_3(t,s) \leq B_{3,2}.$$

Proof. We write $g_{3,1}(s,t) = h_{3,3}(s,t) + h_{3,4}(s,t)$ and $g_{3,2}(s,t) = h_{3,5}(s,t) + h_{3,6}(s,t)$. If $\lambda_1 > 0$ and $\lambda_4 < 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{3,3}(t,s) < 0$, $\frac{\partial}{\partial s} h_{3,4}(t,s) > 0$, $\frac{\partial}{\partial s} h_{3,5}(s,t) < 0$ and $\frac{\partial}{\partial s} h_{3,6}(s,t) > 0$. Similarly, as in the proof of Theorem 3.13, we obtain $0 < A_{3,2} \leq G_3(t,s) \leq B_{3,2}$. □

Corollary 3.11. *If $h \in \mathcal{C}_w^+$, $p_{3,2} > n_{3,2}$, $\lambda_1 > 0$ and $\lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds.$$

Example 3.11. Consider the equation

$$u'''' - 0.02u'' + 0.0001u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation

$$(\lambda - 0.1)^2(\lambda + 0.1)^2 = 0$$

has two roots $\lambda_1 = 0.1$ and $\lambda_4 = -0.1$. Since $p_{3,2} = 1609.8 > n_{3,2} = 604.66$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds$ with $\int_t^{t+w} G_3(t,s)\,ds = 10000$, $1005.2 < G_3(t,s) < 2178.6$.

If all roots are negative, we suppose that $\lambda_1 < \lambda_4 < 0$ (the situation when $\lambda_4 < \lambda_1 < 0$ can be proved by using the same method), and we have

Theorem 3.15. *If $p_{3,3} > n_{3,3}$ and $\lambda_1 < \lambda_4 < 0$, then*

$$0 < A_{3,3} \leq G_3(t,s) \leq B_{3,3}.$$

Proof. We write $g_{3,1}(s,t) = h_{3,7}(s,t) + h_{3,8}(s,t)$, $g_{3,2}(s,t) = h_{3,9}(s,t) + h_{3,10}(s,t)$. If $\lambda_1 < \lambda_4 < 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{3,7}(t,s) < 0$, $\frac{\partial}{\partial s} h_{3,8}(t,s) < 0$, $\frac{\partial}{\partial s} h_{3,9}(t,s) > 0$ and $\frac{\partial}{\partial s} h_{3,10}(t,s) < 0$. Similarly, as in the proof of Theorem 3.13, we obtain $0 < A_{3,3} \leq G_3(t,s) \leq B_{3,3}$. □

Corollary 3.12. *If $h \in \mathcal{C}_w^+$, $p_{3,3} > n_{3,3}$, $\lambda_1 < \lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds.$$

Example 3.12. Consider the equation

$$u'''' + 0.22u''' + 0.0141u'' + 0.00022u' + 1.0 \times 10^{-6}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation

$$(\lambda + 0.1)^2(\lambda + 0.01)^2 = 0$$

has two roots $\lambda_1 = -0.1$ and $\lambda_4 = -0.01$. Since $2.027 \times 10^5 > n_{3,3} = 59450$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_3(t,s)h(s)\,ds$ with $\int_t^{t+w} G_3(t,s)\,ds = 1000000$ and $1.4325 \times 10^5 < G_3(t,s) < 1.7506 \times 10^5$.

Case 4. *If $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$. We use the following abbreviations:*

$$\begin{aligned} g_{4,1}(t,s) &= e^{(t+w-s)\lambda_1}\,\frac{(1-e^{w\lambda_1})\big((e^{w\lambda_1}-1)(((s-t)(\lambda_1-\lambda_4)+1)^2+1)\big)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3} \\ &\quad + \frac{e^{(t+w-s)\lambda_1}(1-e^{w\lambda_1})\big(w(\lambda_1-\lambda_4)(2(s-t)(\lambda_1-\lambda_4)+2)\big)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3} \\ &\quad - \frac{w^2 e^{(t+w-s)\lambda_1}(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\,, \end{aligned}$$

$$g_{4,2}(t,s) = \frac{e^{(t+w-s)\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,;$$

$$\begin{aligned} h_{4,1}(t,s) &= -\frac{\lambda_1(s-t)(e^{w\lambda_1}-1)e^{\lambda_1(t-s+w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)}\,\Big(\lambda_1(s-t)(e^{w\lambda_1}-1)+2(e^{w\lambda_1}+w\lambda_1-1)\Big) \\ &\quad - \frac{e^{\lambda_1(t-s+w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)}\,\Big(w^2\lambda_1^2(e^{w\lambda_1}+1)+2(e^{w\lambda_1}-1)(e^{w\lambda_1}+w\lambda_1-1)\Big), \end{aligned}$$

$$\begin{aligned} h_{4,2}(t,s) &= -\frac{1}{\lambda_1^2}\,\lambda_4\,\frac{e^{\lambda_1(t-s+w)}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,\Big(w\lambda_1(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\Big) \\ &\quad - \frac{1}{\lambda_1}\,\lambda_4 e^{\lambda_1(t-s+w)}\,\frac{s-t}{(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)^2}\,, \end{aligned}$$

$$\begin{aligned} h_{4,3}(t,s) &= -\frac{\big(2w\lambda_1\lambda_4(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+w^2\lambda_1^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2\big)e^{\lambda_1(t-s+w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3} \\ &\quad - \frac{e^{\lambda_1(t-s+w)}\big(\lambda_1(s-t)(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2(s\lambda_1-t\lambda_1+2)\big)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3} \\ &\quad - \frac{e^{\lambda_1(t-s+w)}\big(2(e^{w\lambda_1}-1)^2(\lambda_1^2-\lambda_1\lambda_4+\lambda_4^2)\big)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\,, \end{aligned}$$

$$h_{4,4}(t,s) = \frac{1}{\lambda_1^2}\,\frac{e^{\lambda_1(t-s+w)}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}\,(s\lambda_1-t\lambda_1+1)(\lambda_4-w\lambda_1^2-\lambda_4 e^{w\lambda_1}+w\lambda_1\lambda_4);$$

$$\begin{aligned} A_{4,1} &= -e^{w\lambda_1}\,\frac{2(e^{w\lambda_1}-1)^2+w^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2+2w(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3} \\ &\quad + \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,, \end{aligned}$$

$$\begin{aligned} A_{4,2} &= \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3} - \frac{e^{w\lambda_1}\big(w^2\lambda_1^2(e^{w\lambda_1}+1)+2(e^{w\lambda_1}-1)^2+2w\lambda_1(e^{w\lambda_1}-1)\big)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)} \\ &\quad - \frac{\lambda_4}{\lambda_1^2(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,\Big(w\lambda_1 e^{w\lambda_1}(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\Big), \end{aligned}$$

$$\begin{aligned} A_{4,3} &= -\frac{1}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\,\Big((\lambda_1-\lambda_4)^2\big(w^2\lambda_1^2(e^{w\lambda_1}+1)+w\lambda_1(w\lambda_1+2)(e^{w\lambda_1}-1)^2\big)\Big) \\ &\quad - \frac{1}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\,\Big(2w\lambda_1\lambda_4(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+2(e^{w\lambda_1}-1)^2(\lambda_1^2-\lambda_1\lambda_4+\lambda_4^2)\Big) \\ &\quad + \frac{1}{\lambda_1^2}\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}\,(\lambda_4-w\lambda_1^2-\lambda_4 e^{w\lambda_1}+w\lambda_1\lambda_4) \\ &\quad + \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,; \end{aligned}$$

$$
\begin{aligned}
B_{4,1} &= -\frac{2we^{w\lambda_1}(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+w^2e^{w\lambda_1}(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2+2(e^{w\lambda_1}-1)^2}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,,\\
B_{4,2} &= \frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad-\frac{w^2\lambda_1^2e^{w\lambda_1}(e^{w\lambda_1}+1)+2(e^{w\lambda_1}-1)^2+2w\lambda_1e^{w\lambda_1}(e^{w\lambda_1}-1)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)}\\
&\quad-\frac{1}{\lambda_1^2}\,\frac{\lambda_4e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,\big(w\lambda_1(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\big),\\
B_{4,3} &= -\frac{e^{\lambda_1(w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(2w\lambda_1\lambda_4(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+w^2\lambda_1^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2\Big)\\
&\quad-\frac{e^{\lambda_1(w)}}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(2(e^{w\lambda_1}-1)^2(\lambda_1^2-\lambda_1\lambda_4+\lambda_4^2)\Big)\\
&\quad+\frac{(w\lambda_1+1)(\lambda_4-w\lambda_1^2-\lambda_4e^{w\lambda_1}+w\lambda_1\lambda_4)}{\lambda_1^2(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}-\frac{2e^{w\lambda_4}-w\lambda_1e^{w\lambda_4}+w\lambda_4e^{w\lambda_4}-2}{(e^{w\lambda_4}-1)^2(\lambda_1-\lambda_4)^3}\,;
\end{aligned}
$$

$$
\begin{aligned}
n_{4,1} &= e^{w\lambda_1}\,\frac{2(e^{w\lambda_1}-1)^2+w^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2+2w(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)}{2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\,,\\
n_{4,2} &= -\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\\
&\quad+\frac{w^2\lambda_1^2e^{w\lambda_1}(e^{w\lambda_1}+1)+2(e^{w\lambda_1}-1)^2+2w\lambda_1e^{w\lambda_1}(e^{w\lambda_1}-1)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)}\,,\\
n_{4,3} &= \frac{1}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(2w\lambda_1\lambda_4(e^{w\lambda_1}-1)(\lambda_1-\lambda_4)+w^2\lambda_1^2(e^{w\lambda_1}+1)(\lambda_1-\lambda_4)^2\Big)\\
&\quad+\frac{2(e^{w\lambda_1}-1)^2(\lambda_1^2-\lambda_1\lambda_4+\lambda_4^2)}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}+\frac{w\lambda_1(w\lambda_1)(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\\
&\quad-\frac{1}{\lambda_1^2}\,\frac{e^{w\lambda_1}}{(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2}\,(\lambda_4-w\lambda_1^2-\lambda_4e^{w\lambda_1}+w\lambda_1\lambda_4);
\end{aligned}
$$

$$
\begin{aligned}
p_{4,1} &= \frac{1}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,,\\
p_{4,2} &= -\frac{\lambda_4e^{w\lambda_1}\big(w\lambda_1(\lambda_1-\lambda_4)-(e^{w\lambda_1}-1)(\lambda_4-2\lambda_1)\big)}{\lambda_1^2(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^3}\,,\\
p_{4,3} &= -\frac{1}{2\lambda_1^2(e^{w\lambda_1}-1)^3(\lambda_1-\lambda_4)^3}\Big(w\lambda_1(2)(e^{w\lambda_1}-1)^2(\lambda_1-\lambda_4)^2\Big)\\
&\quad+\frac{e^{w\lambda_4}}{(e^{w\lambda_4}-1)(\lambda_1-\lambda_4)^3}\,.
\end{aligned}
$$

Theorem 3.16. *For all $t\in[0,w]$ and $s\in[t,t+w]$, we have*

$$\int_t^{t+w} G_4(t,s)\,ds=\frac{1}{\lambda_1^3\lambda_4}\,.$$

Proof. We have

$$\int_t^{t+w} g_{4,1}(s,t)\,ds + \int_t^{t+w} g_{4,2}(s,t)\,ds = -\frac{3\lambda_1^2 - 3\lambda_1\lambda_4 + \lambda_4^2}{\lambda_1^3(\lambda_1-\lambda_4)^3} + \frac{1}{\lambda_4(\lambda_1-\lambda_4)^3} = \frac{1}{\lambda_1^3\lambda_4}\,. \qquad \square$$

We have two different roots satisfying one of the three cases:

- Two positive roots.
- One positive root and one negative root.
- Two negative roots.

If all roots are positive, we suppose that $\lambda_1 > \lambda_4 > 0$ (the situation when $\lambda_4 > \lambda_1 > 0$ can be proved by using the same method), and we have

Theorem 3.17. *If $p_{4,1} > n_{4,1}$ and $\lambda_1 > \lambda_4 > 0$, then*

$$0 < A_{4,1} \le G_4(t,s) \le B_{4,1}.$$

Proof. If $\lambda_1 > \lambda_4 > 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} g_{4,1}(s,t) > 0$ and $\frac{\partial}{\partial s} g_{4,2}(s,t) < 0$. So $g_{4,1}(t,t) + g_{4,2}(t,t+w) \le g_{4,1}(t,s) \le g_{4,1}(t,t+w) + g_{4,2}(t,t)$. This double inequality together with the assumption $p_{4,1} > n_{4,1}$ give $0 < A_{4,1} \le G_4(t,s) \le B_{4,1}$. $\square$

Corollary 3.13. *If $h \in \mathcal{C}_w^+$, $p_{4,1} > n_{4,1}$ and $\lambda_1 > \lambda_4 > 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds.$$

Example 3.13. Consider the equation

$$u'''' - 0.61u''' + 0.126u'' - 0.0092u' + 0.00008u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda-0.2)^3(\lambda-0.01) = 0$ has the roots $\lambda_1 = 0.2$ and $\lambda_4 = 0.01$. We compute $p_{4,1} = 2248.2 > n_{4,1} = 404.33$, and hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds$ with $\int_t^{t+w} G_4(t,s)\,ds = 2.5\times 10^5$ and $0 < 1843.9 < G_4(t,s) < 2135.5$.

If one root is positive and one root is negative, we suppose that $\lambda_1 > 0$ and $\lambda_4 < 0$ (the situation when $\lambda_1 < 0$ and $\lambda_4 > 0$ can be proved by using the same method), and we have

Theorem 3.18. *If $p_{4,2} < n_{4,2}$, $\lambda_1 > 0$ and $\lambda_4 < 0$ then*

$$A_{4,2} \le G_4(t,s) \le B_{4,2} < 0.$$

Proof. We have $g_{4,1}(t,s) = h_{4,1}(t,s) + h_{4,2}(t,s)$. If $\lambda_1 > 0$ and $\lambda_4 < 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{4,1}(s,t) > 0$, $\frac{\partial}{\partial s} h_{4,2}(s,t) < 0$ and $\frac{\partial}{\partial s} g_{4,2}(s,t) < 0$. Similarly, as in the proof of Theorem 3.17, we obtain $A_{4,2} \le G_4(t,s) \le B_{4,2} < 0$. $\square$

Corollary 3.14. *If $h \in \mathcal{C}_w^-$, $p_{4,2} < n_{4,2}$, $\lambda_1 > 0$ and $\lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds.$$

Example 3.14. Consider the equation

$$u'''' - 0.29u''' + 0.027u'' - 0.0007u' - 0.00001u = h(t),$$

here h is the given continuous and 2π-periodic function. The characteristic equation $(\lambda - 0.1)^3(\lambda + 0.01) = 0$ has the roots $\lambda_1 = 0.1$ and $\lambda_4 = -0.01$. Since $p_{4,2} = 465.49 < n_{4,2} = 15472$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds$ with $\int_t^{t+w} G_4(t,s)\,ds = 10^6$ and $-16824 < G_4(t,s) < -15006 < 0$.

If all roots are negative, we suppose that $\lambda_1 < \lambda_4 < 0$ (the situation when $\lambda_4 < \lambda_1 < 0$ can be proved by using the same method), and we have

Theorem 3.19. *If $p_{4,3} > n_{4,3}$ and $\lambda_1 < \lambda_4 < 0$, then*

$$0 < A_{4,3} \le G_4(t,s) \le B_{4,3}.$$

Proof. We have $g_{4,1}(t,s) = h_{4,3}(t,s) + h_{4,4}(t,s)$. If $\lambda_1 < \lambda_4 < 0$, the study of the derivatives with respect to s gives $\frac{\partial}{\partial s} h_{4,3}(s,t) < 0$, $\frac{\partial}{\partial s} h_{4,4}(s,t) > 0$ and $\frac{\partial}{\partial s} g_{4,2}(s,t) > 0$. Similarly, as in the proof of Theorem 3.17, we obtain $0 < A_{4,3} \le G_4(t,s) \le B_{4,3}$. □

Corollary 3.15. *If $h \in \mathcal{C}_w^+$, $p_{4,3} > n_{4,3}$ and $\lambda_1 < \lambda_4 < 0$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds.$$

Example 3.15. Consider the equation

$$u'''' + 0.601u''' + 0.1206u'' + 0.00812u' + 8.0 \times 10^{-6}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda+0.2)^3(\lambda+0.001) = 0$ has the roots $\lambda_1 = -0.2$, $\lambda_4 = -0.001$. Since $p_{4,3} = 20353 > n_{4,3} = 748.34$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_4(t,s)h(s)\,ds$ with $\int_t^{t+w} G_4(t,s)\,ds = 2.5 \times 10^7$, $0 < 20134 < G_4(t,s) < 3.9784 \times 10^6$.

Case 5. *If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. We use the following abbreviations:*

$$\begin{aligned}
A_{5,1} = {} & \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(6w\lambda_1 e^{w\lambda_1}(e^{w\lambda_1}-1)^2 + w^3\lambda_1^3 e^{w\lambda_1}(e^{2(w\lambda_1)} + 4e^{w\lambda_1} + 1)\Big) \\
& + \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(3w^2\lambda_1^2 e^{w\lambda_1}(e^{2(w\lambda_1)} - 1) + 6(e^{w\lambda_1}-1)^3\Big) \\
& - \frac{e^{w\lambda_1}}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\Big(2(e^{w\lambda_1}-1)(e^{w\lambda_1} + w\lambda_1 - 1) + w^2\lambda_1^2(e^{w\lambda_1}+1)\Big),
\end{aligned}$$

$$A_{5,2} = w^3\,\frac{e^{2(w\lambda_1)} + e^{w\lambda_1} + 4}{6(e^{w\lambda_1}-1)^3} + w^3 e^{w\lambda_1}\,\frac{e^{2(w\lambda_1)} + 4e^{w\lambda_1} + 1}{6(e^{w\lambda_1}-1)^4}\,;$$

$$\begin{aligned}
B_{5,1} = {} & \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(6w\lambda_1 e^{w\lambda_1}(e^{w\lambda_1}-1)^2 + 3w^2\lambda_1^2 e^{w\lambda_1}(e^{2(w\lambda_1)} - 1)\Big) \\
& + \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(w^3\lambda_1^3 e^{w\lambda_1}(e^{2(w\lambda_1)} + 4e^{w\lambda_1} + 1) + 6e^{w\lambda_1}(e^{w\lambda_1}-1)^3\Big) \\
& - \frac{2(e^{w\lambda_1}-1)(e^{w\lambda_1}(w\lambda_1+1) - 1) + w^2\lambda_1^2 e^{w\lambda_1}(e^{w\lambda_1}+1)}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\,,
\end{aligned}$$

$$B_{5,2} = w^3\,\frac{2e^{2(w\lambda_1)} - e^{w\lambda_1} + 2}{3(e^{w\lambda_1}-1)^4}\,;$$

$$n_{5,1} = \frac{e^{w\lambda_1}\big(2(e^{w\lambda_1}-1)(e^{w\lambda_1}+w\lambda_1-1)+w^2\lambda_1^2(e^{w\lambda_1}+1)\big)}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\,,$$

$$n_{5,2} = -w^3\,\frac{e^{2(w\lambda_1)}+e^{w\lambda_1}+4}{6(e^{w\lambda_1}-1)^3}\,;$$

$$p_{5,1} = \frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(6w\lambda_1 e^{w\lambda_1}(e^{w\lambda_1}-1)^2+w^3\lambda_1^3 e^{w\lambda_1}\big(e^{2(w\lambda_1)}+4e^{w\lambda_1}+1\big)\Big)$$
$$+\frac{1}{6\lambda_1^3(e^{w\lambda_1}-1)^4}\Big(3w^2\lambda_1^2 e^{w\lambda_1}(e^{2(w\lambda_1)}-1)+6(e^{w\lambda_1}-1)^3\Big),$$

$$p_{5,2} = w^3 e^{w\lambda_1}\,\frac{e^{2(w\lambda_1)}+4e^{w\lambda_1}+1}{6(e^{w\lambda_1}-1)^4}\,;$$

$$h_{5,1}(s,t) = \frac{e^{\lambda_1(t-s+w)}(s-t)^2(e^{w\lambda_1}+w\lambda_1-1)}{2\lambda_1(e^{w\lambda_1}-1)^2}-\frac{e^{\lambda_1(t-s+w)}(\lambda_1^3(s-t)^3)}{6\lambda_1^3-6\lambda_1^3e^{w\lambda_1}}$$
$$+e^{\lambda_1(t-s+w)}\,\frac{s\lambda_1-t\lambda_1+1}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\Big(2(e^{w\lambda_1}-1)^2+2w\lambda_1(e^{w\lambda_1}-1)+w^2\lambda_1^2(e^{w\lambda_1}+1)\Big)$$
$$+\frac{w^3e^{\lambda_1(t-s+w)}(e^{2(w\lambda_1)}+4e^{w\lambda_1}+1)}{6(e^{w\lambda_1}-1)^4}\,,$$

$$h_{5,2}(s,t) = -\frac{e^{\lambda_1(t-s+w)}}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\Big((e^{w\lambda_1}-1)^2(\lambda_1^2(s-t)^2+2(\lambda_1(s-t)+1))\Big)$$
$$-\frac{e^{\lambda_1(t-s+w)}}{2\lambda_1^3(e^{w\lambda_1}-1)^3}\Big(2w\lambda_1(e^{w\lambda_1}-1)(\lambda_1(s-t)+1)+w^2\lambda_1^2(e^{w\lambda_1}+1)\Big),$$

$$h_{5,3}(s,t) = e^{\lambda_1(t-s+w)}\,\frac{s-t}{6(e^{w\lambda_1}-1)^3}\Big((s-t)^2(e^{w\lambda_1}-1)^2+3w^2(e^{w\lambda_1}+1)\Big),$$

$$h_{5,4}(s,t) = w\frac{e^{\lambda_1(t-s+w)}}{6(e^{w\lambda_1}-1)^4}\Big(3(s-t)^2(e^{w\lambda_1}-1)^2+w^2\big(e^{2(w\lambda_1)}+4e^{w\lambda_1}+1\big)\Big).$$

Theorem 3.20. *For all $t\in[0,w]$ and $s\in[t,t+w]$, we have*

$$\int_t^{t+w} G_5(t,s)\,ds = \frac{1}{\lambda_1^4}\,.$$

Proof. We have

$$\int_t^{t+w} G_5(t,s)\,ds = \int_t^{t+w} h_{5,1}(t,s)\,ds + \int_t^{t+w} h_{5,2}(t,s)\,ds.$$

So

$$\int_t^{t+w} G_5(t,s)\,ds = -\frac{3w^2\lambda_1^2(e^{w\lambda_1}+1)+w^3\lambda_1^3(e^{w\lambda_1}+2)-6(e^{w\lambda_1}-1)^2}{6\lambda_1^4(e^{w\lambda_1}-1)^2}$$
$$+\frac{w^3\lambda_1(e^{w\lambda_1}+2)}{6\lambda_1^2(e^{w\lambda_1}-1)^2}+\frac{3w^2(e^{w\lambda_1}+1)}{6\lambda_1^2(e^{w\lambda_1}-1)^2}=\frac{1}{\lambda_1^4}\,. \qquad \square$$

Theorem 3.21. *If $\lambda_1>0$ and $p_{5,1}>n_{5,1}$, then*

$$0<A_{5,1}\le G_5(s,t)\le B_{5,1}.$$

Proof. We have $G_5(s,t) = h_{5,1}(t,s)+h_{5,2}(t,s)$. If $\lambda_1>0$, the study of the derivatives gives $\frac{\partial}{\partial s}h_{5,1}(t,s)<0$ and $\frac{\partial}{\partial s}h_{5,2}(t,s)>0$, so $h_{5,1}(t,t+w)+h_{5,2}(t,t)\le G_5(s,t)\le h_{5,1}(t,t)+h_{5,2}(t,t+w)$. If we use this double inequality together with the assumption $p_{5,1}>n_{5,1}$, we arrive at $0<A_{5,1}\le G_5(s,t)\le B_{5,1}$. $\square$

Corollary 3.16. *If $h \in \mathcal{C}_w^+$, $\lambda_1 > 0$ and $p_{5,1} > n_{5,1}$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_5(t,s)h(s)\,ds.$$

Example 3.16. Consider the equation

$$u'''' - 0.4u''' + +0.06u'' - 0.004u' + 0.0001u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda - 0.1)^4 = 0$ has the root $\lambda_1 = 0.1$. Since $p_{5,1} = 5866.2 > n_{5,1} = 5274.3$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_5(t,s)h(s)\,ds$ with $\int_t^{t+w} G_5(t,s)\,ds = 10^5$ and $0 < 591.86 < G_5(t,s) < 2591.9$.

Theorem 3.22. *If $\lambda_1 < 0$ and $p_{5,2} > n_{5,2}$, then*

$$0 < A_{5,2} \le G_5(s,t) \le B_{5,2}.$$

Proof. We have $G_5(s,t) = h_{5,3}(t,s) + h_{5,4}(t,s)$. If $\lambda_1 < 0$, the study of the derivatives gives $\frac{\partial}{\partial s} h_{5,3}(t,s) < 0$ and $\frac{\partial}{\partial s} h_{5,4}(t,s) > 0$. Similarly, as in the proof of Theorem 3.21, we obtain $0 < A_{5,2} \le G_5(s,t) \le B_{5,2}$. □

Corollary 3.17. *If $h \in \mathcal{C}_w^+$, $\lambda_1 < 0$ and $p_{5,2} > n_{5,2}$, then equation* (1.1) *has a unique positive periodic solution*

$$u(t) = \int_t^{t+w} G_5(t,s)h(s)\,ds.$$

Example 3.17. Consider the equation

$$u'''' + 0.04u''' + 0.0006u'' + 4.0 \times 10^{-6}u' + 1.0 \times 10^{-8}u = h(t),$$

here h is a given 2π-periodic continuous function. The characteristic equation $(\lambda + 0.01)^4 = 0$ has the root $\lambda_1 = -0.01$. Since $p_{5,2} = 1.5915 \times 10^7 > n_{5,2} = 1.0655 \times 10^6$, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+w} G_5(t,s)\,h(s)\,ds$ with $\int_t^{t+w} G_5(t,s)\,ds = 10^8$ and $0 < 1.4850 \times 10^7 < G_5(t,s) < 1.6981 \times 10^7$.

4 Positive periodic solutions

Lemma 4.1 ([10,11])**.** *Let X be a Banach space and let $K \subset X$ be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that either

(i) $\|Tu\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_2$,

or

(ii) $\|Tu\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Denote

$$f_0 = \lim_{u\to 0^+} \sup_{t\in[0,w]} \frac{f(t,u)}{u} \quad\text{and}\quad f_\infty = \lim_{u\to\infty} \inf_{t\in[0,w]} \frac{f(t,u)}{u}.$$

Theorem 4.1. *If $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, then equation* (1.4) *has at least one positive periodic solution in the cases*

(i) $f_0 = 0$ *and* $f_\infty = \infty$,

or

(ii) $f_0 = \infty$ *and* $f_\infty = 0$.

Proof. To apply the Guo–Krasnosel'skiĭ's theorem, let

$$X = \big\{u \in \mathcal{C}(\mathbb{R},\mathbb{R}):\ \ u(t+w) = u(t),\ \ t \in \mathbb{R}\big\}$$

with the norm $\|u\| = \sup\limits_{t\in[0,w]} |u(t)|$. Then $(X, \|\cdot\|)$ is a Banach space and we define the cone K by

$$K = \Big\{u \in X:\ \ u(t) \geq \frac{A_{1,1}}{B_{1,1}}\,\|u\|\ \text{ for all }\ t \in [0,w]\Big\}.$$

For $u \in K$, we define

$$Tu(t) = \int\limits_t^{t+w} G_1(t,s) f(s,u(s))\,ds.$$

In view of Theorem 3.2, we have

$$0 < Tu(t) = \int\limits_t^{t+w} G_1(t,s) f(s,u(s))\,ds \leq B_{1,1} \int\limits_t^{t+w} f(s,u(s))\,ds.$$

So $\|Tu\| \leq B_{1,1} \int\limits_t^{t+w} f(s,u(s))\,ds$. Also, we have

$$Tu(t) = \int\limits_t^{t+w} G_1(t,s) f(s,u(s))\,ds \geq A_{1,1} \int\limits_t^{t+w} f(s,u(s))\,ds \geq \frac{A_{1,1}}{B_{1,1}}\,\|Tu\|,$$

which shows that $T(K) \subset K$. Moreover, $T: K \to K$ is a completely continuous operator and the fixed point of T is a solution of (1.4).

(i) If $f_0 = 0$ and $f_\infty = \infty$.

Since $f_0 = 0$, we may choose $0 < r_1 < 1$ such that $f(t,u) \leq \varepsilon u$, for $0 \leq u \leq r_1$ and $t \in [0,w]$, where $\varepsilon > 0$ satisfies $w\varepsilon B_{1.1} \leq 1$.

Thus, if $u \in K$ and $\|u\| = r_1$, we have

$$Tu(t) = \int\limits_t^{t+w} G_1(t,s) f(s,u(s))\,ds \leq B_{1,1} \int\limits_t^{t+w} f(s,u(s))\,ds \leq w\varepsilon B_{1.1}\|u\| \leq r_1. \tag{4.1}$$

Now, if we set $\Omega_1 = \{u \in X:\ \|u\| < r_1\}$, then (4.1) shows that $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$.

Since $f_\infty = \infty$, there exists $r > r_1$ such that $f(t,u) \geq \eta u$ for $u \geq r$ and $t \in [0,w]$, where $\eta > 0$, so $\frac{A_{1,1}^2 w\eta}{B_{1,1}} \geq 1$.

Let

$$r_2 = \max\Big\{2r_1, \frac{B_{1,1} r}{A_{1,1}}\Big\},$$

and $\Omega_2 = \{u \in X:\ \|u\| < r_2\}$, then $u \in K$ and $\|u\| = r_2$ imply that

$$u(t) \geq \frac{A_{1,1}}{B_{1,1}}\,\|u\| = \frac{A_{1,1}}{B_{1,1}}\,r_2 \geq r,$$

and hence

$$Tu(t) = \int_t^{t+w} G_1(t,s) f(s,u(s))\, ds \geq A_{1.1} \int_t^{t+w} f(s,u(s))\, ds \geq \frac{(A_{1,1}^2 w\eta)}{B_{1.1}} \|u\| \geq \|u\|. \tag{4.2}$$

Thus (4.2) shows that $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$.

It follows from Lemma 4.1 that T has a fixed point $u^* \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Consequently, the equation has a positive w-periodic solution $0 < r_1 < u(t) < r_2$.

(ii) If $f_0 = \infty$ and $f_\infty = 0$.

We choose $r_3 > 0$ such that $f(u) \geq \lambda u$ for $0 \leq u \leq r_3$, where $\lambda > 0$ satisfies $\frac{\lambda A_{1,1}^2 w}{B_{1,1}} \geq 1$. Then for $u \in K$ and $\|u\| = r_3$, we have

$$Tu(t) = \int_t^{t+w} G_1(t,s) f(s,u(s))\, ds \geq A_{1,1} \int_t^{t+w} f(s,u(s))\, ds \geq \frac{\lambda A_{1,1}^2 w}{B_{1,1}} \|u\| \geq \|u\|. \tag{4.3}$$

If we put $\Omega_3 = \{u \in X : \|u\| < r_3\}$, (4.3) shows that $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_3$.

Since $f_\infty = 0$, there exists $M > 0$ such that $f(t,u) \leq \xi u$ for $u \geq M$ and $\xi > 0$ satisfies $\xi B_{1,1} w < 1$. We choose

$$r_4 = \max\Big\{2r_3, \frac{B_{1,1} M}{A_{1,1}}\Big\},$$

then $u \in K$ and $\|u\| = r_4$, this implies that $u(t) \geq \frac{A_{1,1}}{B_{1,1}} \|u\| \geq M$, and so

$$\begin{aligned} Tu(t) &= \int_t^{t+w} G_1(t,s) f(s,u(s))\, ds \leq B_{1,1} \int_t^{t+w} f(s,u(s)) rm\, ds \\ &\leq B_{1,1} \xi \int_t^{t+w} u(s)\, ds \leq B_{1,1} w \xi \|u\| \leq \|u\|. \end{aligned} \tag{4.4}$$

We set $\Omega_4 = \{u \in X : \|u\| < r_4\}$, then for $u \in K \cap \partial\Omega_4$ we have $\|Tu\| \leq \|u\|$.

In view of Lemma 4.1, equation (1.4) has at least one positive solution $0 < r_3 < u(t) < r_4$. □

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Authors' addresses:

Ahlème Bouakkaz, Rabah Khemis

LAMAHIS Lab, Faculty of Sciences, Department of Mathematics, Univ Skikda, P.O. Box 26, Skikda 21000, Algeria.

E-mail: ahlmekholode@yahoo.fr; kbra28@yahoo.fr

Abdelouaheb Ardjouni

Faculty of Sciences and Technology, Department of Mathematics and Informatics, Univ Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria.

E-mail: abd_ardjouni@yahoo.fr

Ahcene Djoudi

Applied Mathematics Lab, Faculty of Sciences, Department of Mathematics, Univ Annaba, P.O. Box 12, Annaba 23000, Algeria.

E-mail: adjoudi@yahoo.com