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SOME PROPERTIES OF A SOLUTION AND FINITE DIFFERENCE SCHEME FOR ONE NONLINEAR PARTIAL DIFFERENTIAL MODEL BASED ON THE MAXWELL SYSTEM

Abstract. Linear stability and Hoph bifurcation of a solution of the initial-boundary value problem as well as the finite difference scheme for one system of nonlinear partial differential equations are investigated. The blow up case is fixed. The mentioned system is based on the Maxwell equations which describe the process of electromagnetic field penetration into a substance. Numerous computer experiments are carried out and relying on the obtained results, some graphical illustrations are presented.

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რეზიუმე. ერთი არაწრფივი კერძოწარმოებულებიან დიფერენციალურ განტოლებათა სისტემისთვის გამოკვლეულია საწყის-სასაზღვრო ამოცანის ამონახსნის წრფივი მდგრადობა და ჰოფის ბიფურკაცია. დაფიქსირებულია ფეთქებადი ამონახსნის შემთხვევა. აღნიშნული სისტემა დაფუძნებულია მაქსველის განტოლებებზე, რომელიც აღწერს ელექტრომაგნიტური ველის გარემოში გავრცელების პროცესს. ჩატარებულია მრავალი რიცხვითი ექსპერიმენტი და მიღებულ შედეგებზე დაყრდნობით წარმოდგენილია გრაფიკული ილუსტრაციები.

1 Introduction

The aim of the present paper is to study the linear stability and Hoph bifurcation of a solution of the initial-boundary value problem and finite difference scheme for one diffusion system of nonlinear partial differential equations. Such systems arise in mathematical modeling of the process of penetration of an electromagnetic field into a substance. Upon penetrating through a material, the variable magnetic field induces in it a variable electronic field, which causes the appearance of currents. The currents lead to the heating of the material and arise its temperature that affects the diffusion process. For large oscillations of temperature, the dependence on it should be taken into consideration. In a quasistationary case, the corresponding system of Maxwell equations has the form [10]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1.1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is the vector of a magnetic field, θ is temperature, c_ν and ν_m characterize the thermal heat capacity and electroconductivity of the substance. System (1.1) defines the process of diffusion of the magnetic field and equation (1.2) describes the change of temperature. As a rule, the coefficients c_ν and ν_m depend on temperature θ, $c_\nu = c_\nu(\theta)$, $\nu_m = \nu_m(\theta)$.

Many authors are studying models (1.1), (1.2) and their different variations and generalizations (see, e.g., [1–3, 8, 14, 16, 17] and the references therein). In [7], the reduction to the integro-differential model of system (1.1), (1.2) was proposed and investigated. As for the investigation and approximation solution of various versions of Maxwell system and the corresponding to it integro-differential models, one can find, for example, in [8] (see also the references therein). The existence of the corresponding initial-boundary value problems for such kind of integro-differential models can be proved by using Galerkin's modified method and compactness arguments as in [11, 15] for nonlinear parabolic equations and, as it is carried out in [5–7], for the case of one-component magnetic field.

The rest of the present paper is organized as follows. In Section 2, the problem is stated and the linear stability of a solution of the initial-boundary value problem with nonhomogeneous boundary conditions on the right side of the lateral boundary is studied. The possibility of appearance of Hoph bifurcation and the blow up case are fixed, as well. In Section 3, the finite difference scheme for the problem considered in Section 2 is constructed and its convergence is investigated. At the end of this section, some graphical illustrations, confirming theoretical findings are given. The final Section 4 contains brief conclusion.

2 Linear stability and Hoph bifurcation

The model of Maxwell equations (1.1), (1.2) is complex enough for theoretical investigations and practical applications.

In some physical assumptions, if the vector of a magnetic field has the form $H = (0, U, V)$, where $U = U(x,t)$ and $V = V(x,t)$, then in the cylinder $[0,1] \times [0,\infty)$ we consider the initial-boundary value problem

$$\begin{gathered}
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(S^\alpha \frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big(S^\alpha \frac{\partial V}{\partial x}\Big), \\
\frac{\partial S}{\partial t} = -aS^\beta + bS^\gamma \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big], \\
U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \\
S(x,0) = S_0(x) > s_0 = const > 0,
\end{gathered} \tag{2.1}$$

where a, b, ψ_1, ψ_2 are positive constants and α, β, γ are real numbers which will be specified later; $U_0(x)$, $V_0(x)$, $S_0(x)$ are the known functions of their arguments.

Stabilization of the stationary solution and the finite difference scheme for the special cases of the above model were investigated in [4, 6, 9].

It is not difficult to show that if $\beta \neq \gamma$, then the stationary solution of problem (2.1) has the form

$$U_s = \psi_1 x, \quad V_s = \psi_2 x, \quad S_s = \Big(\frac{(\psi_1^2 + \psi_2^2)b}{a}\Big)^{\frac{1}{\beta-\gamma}}.$$

The following statement holds.

Theorem 2.1. *Let $2\alpha + \beta - \gamma > 0$, $\beta \neq \gamma$, then the stationary solution $(U_s, V_s S_s)$ of problem (2.1) is linearly stable if and only if the inequality*

$$a(\gamma - \beta)\Big[\frac{b}{a}\,(\psi_1^2 + \psi_2^2)\Big]^{\frac{\beta-\alpha-1}{\beta-\gamma}} < \pi^2$$

is fulfilled.

Proof. Assume that a solution of problem (2.1) has the form

$$U(x,t) = U_s + u(x,t), \quad V(x,t) = V_s + v(x,t), \quad S(x,t) = S_s + s(x,t), \tag{2.2}$$

where $u(x,t)$, $v(x,t)$, $s(x,t)$ are small perturbations.

Introducing the notations

$$\begin{aligned}
\alpha_s &= \alpha\psi_1\Big(\frac{(\psi_1^2 + \psi_2^2)b}{a}\Big)^{-\frac{\alpha-1}{\beta-\gamma}}, & \beta_s &= \Big(\frac{(\psi_1^2 + \psi_2^2)b}{a}\Big)^{\frac{\alpha}{\beta-\gamma}},\\
\gamma_s &= \alpha\psi_2\Big(\frac{(\psi_1^2 + \psi_2^2)b}{a}\Big)^{\frac{\alpha-1}{\beta-\gamma}}, & \nu_s &= (\gamma - \beta)\,\frac{b^{\frac{\beta-1}{\beta-\gamma}}}{a^{\frac{\gamma-1}{\beta-\gamma}}}\,(\psi_1^2 + \psi_2^2)^{\frac{\beta-1}{\beta-\gamma}},\\
\eta_s &= 2\psi_1 b\Big[\frac{b}{a}\,(\psi_1^2 + \psi_2^2)\Big]^{\frac{\gamma}{\beta-\gamma}}, & \mu_s &= 2\psi_2 b\Big[\frac{b}{a}(\psi_1^2 + \psi_2^2)\Big]^{\frac{\gamma}{\beta-\gamma}},
\end{aligned}$$

after linearization of the system of problem (2.1) we get the following system of partial differential equations:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha_s\,\frac{\partial s}{\partial x} + \beta_s\,\frac{\partial^2 u}{\partial x^2},\\
\frac{\partial v}{\partial t} &= \gamma_s\,\frac{\partial s}{\partial x} + \beta_s\,\frac{\partial^2 v}{\partial x^2},\\
\frac{\partial s}{\partial t} &= \nu_s s + \eta_s\,\frac{\partial u}{\partial x} + \mu_s\,\frac{\partial v}{\partial x}\,.
\end{aligned} \tag{2.3}$$

We seek for a solution of system (2.3) in the form

$$u(x,t) = u(x)e^{\omega t}, \quad v(x,t) = v(x)e^{\omega t}, \quad s(x,t) = s(x)e^{\omega t}, \tag{2.4}$$

and get the problem on eigenvalues for the following system of ordinary differential equations:

$$\begin{gathered}
\omega u = \alpha_s\,\frac{ds}{dx} + \beta_s\,\frac{d^2u}{dx^2}\,, \quad \omega v = \gamma_s\,\frac{ds}{dx} + \beta_s\,\frac{d^2v}{dx^2}\,,\\
\omega s = \nu_s s + \eta_s\,\frac{du}{dx} + \mu_s\,\frac{dv}{dx}\,.
\end{gathered} \tag{2.5}$$

Assume now that a solution of system (2.5) is of the form

$$u(x) = u_0 e^{ikx}, \quad v(x) = v_0 e^{ikx}, \quad s(x) = s_0 e^{ikx}.$$

Substituting these functions in (2.5), we get

$$\begin{gathered}
\omega u_0 e^{ikx} = \alpha_s ike^{ikx}s_0 - \beta_s e^{ikx}k^2u_0, \quad \omega v_0 e^{ikx} = \gamma_s ike^{ikx}s_0 - \beta_s e^{ikx}k^2 v_0,\\
\omega s_0 e^{ikx} = \nu_s s_0 e^{ikx} + \eta_s iku_0 e^{ikx} + \mu_s ikv_0 e^{ikx}
\end{gathered}$$

from which we obtain

$$u_0(\omega+\beta_s k^2)-\alpha_s iks_0=0,\quad v_0(\omega+\beta_s k^2)-\gamma_s iks_0=0,$$
$$u_0ik\eta_s+v_0\mu_s ik+s_0(\nu_s-\omega)=0.$$

It is clear that this system has a nontrivial solution if and only if the condition

$$\Delta(\omega,k)=\begin{vmatrix}\omega+\beta_s k^2 & 0 & -ik\alpha_s\\ 0 & \omega+\beta_s k^2 & -ik\gamma_s\\ ik\eta_s & ik\mu_s & \nu_s-\omega\end{vmatrix}$$
$$=(\omega+\beta_s k^2)^2(\nu_s-\omega)-(\omega+\beta_s k^2)k^2\alpha_s\eta_s-(\omega+\beta_s k^2)\mu_s\gamma_s k^2=0,$$

or

$$(\omega+\beta_s k^2)[(\omega+\beta_s k^2)(\nu_s-\omega)-k^2\alpha_s\eta_s-k^2\mu_s\gamma_s]=0$$

is fulfilled. This implies that

$$k^2(\beta_s\nu_s-\beta_s\omega-\alpha_s\eta_s-\mu_s\gamma_s)-\omega^2+\omega\nu_s=0. \tag{2.6}$$

Since the case $\omega+\beta_s k^2=0$ is trivial, the latest equality gives two values of the parameter k such as $k_1=-k_2$.

It is easy to show that the solution of system (2.5) has the form

$$u(x)=\frac{ik_1\alpha_s}{\omega+\beta_s k_1^2}\,(S_1e^{ik_1x}-S_2e^{-ik_1x}),\quad v(x)=\frac{ik_1\gamma_s}{\omega+\beta_s k_1^2}\,(S_1e^{ik_1x}-S_2e^{-ik_1x}),$$
$$s(x)=S_1e^{ik_1x}+S_2e^{-ik_1x}, \tag{2.7}$$

where S_1 and S_2 are the constants.

Taking into account the boundary conditions (2.1), from (2.2) and (2.4) we get

$$u(0)=u(1)=0.$$

From this, taking into account (2.7), we get the following system:

$$S_1-S_2=0,$$
$$S_1e^{ik_1}-S_2e^{-ik_1}=0,$$

which above has a nontrivial solution when

$$\Delta=\begin{vmatrix}1 & -1\\ e^{ik_1} & -e^{-ik_1}\end{vmatrix}=e^{ik_1}-e^{-ik_1}=2i\sin k_1=0,$$

or

$$k_{1n}=\pi n,\ \ n\in Z.$$

Let us rewrite equation (2.6) in the form

$$\omega_n^2+P_n(\beta_s,k_n,\nu_s)\omega_n+L_n(\beta_s,k_n,\nu_s,\eta_s,\mu_s,\gamma_s)=0,$$

where

$$P_n(\beta_s,k_n,\nu_s)=\beta_s k_n^2-\nu_s,$$
$$L_n(\beta_s,k_n,\nu_s,\eta_s,\mu_s,\gamma_s)=-\beta_s\nu_s k_n^2+\alpha_s\eta_s k_n^2+\mu_s\gamma_s k_n^2.$$

It should be noted that the solution of problem (2.1) is linearly stable if and only if for all n the inequality $\mathrm{Re}(\omega_n)<0$ holds. It is easy to show that if $2\alpha+\beta-\gamma>0$, then $L_n(\beta_s,k_n,\nu_s,\eta_s,\mu_s,\gamma_s)>0$.

Therefore, for the solution to be linearly stable, it is necessary and sufficient that the inequality

$$P_n=\beta_s k_n^2-\nu_s=\Big(\frac{(\psi_1^2+\psi_2^2)b}{a}\Big)^{\frac{\alpha}{\beta-\gamma}}\pi^2n^2-(\gamma-\beta)\,\frac{b^{\frac{\beta-1}{\beta-\gamma}}}{a^{\frac{\gamma-1}{\beta-\gamma}}}(\psi_1^2+\psi_2^2)^{\frac{\beta-1}{\beta-\gamma}}>0,$$

or

$$a(\gamma-\beta)\Big[\frac{b}{a}\,(\psi_1^2+\psi_2^2)\Big]^{\frac{\beta-\alpha-1}{\beta-\gamma}}<\pi^2\ \ (n=1)$$

holds. Thus, the proof of Theorem 2.1 is complete. □

Remark. As we can see from the above inequality, when $\gamma < \beta$, the solution of problem (2.1) is always linearly stable.

Assume that $\gamma > \beta$, $\beta - \alpha - 1 \neq 0$ and consider the value

$$\psi_s = \Big[\frac{\pi^2}{\gamma-\beta} a^{\frac{\gamma-\alpha-1}{\beta-\gamma}} b^{\frac{\alpha-\beta+1}{\beta-\gamma}}\Big]^{\frac{\beta-\gamma}{\beta-\alpha-1}},$$

for which

$$P_1(\psi_s,\alpha,\beta,\gamma) = 0, \quad P_n(\psi_s,\alpha,\beta,\gamma) > 0, \quad n = 2,3,\dots .$$

In addition, if we assume that $\beta - \alpha - 1 < 0$, then for $\psi \in (0,\psi_s)$, $\psi = \psi_1^2 + \psi_2^2$, we have $P_n(\psi,\alpha,\beta,\gamma) > 0$, $n \in Z_0$.

Therefore, if $\psi \in (0,\psi_s)$, then the solution of problem (2.1) is linearly stable, and if $\psi > \psi_s$, then it is unstable. For $\psi = \psi_s$, we have $\mathrm{Re}(\omega_1) = 0$ and $\mathrm{Im}(\omega_1) \neq 0$, i.e., there appears the possibility of Hoph bifurcation. The small perturbations may cause transformation of a solution into a periodic oscillations [12].

Consider the problem

$$\begin{gathered}
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(S^\alpha \frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big(S^\alpha \frac{\partial V}{\partial x}\Big), \\
\frac{\partial S}{\partial t} = S^\alpha\Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big], \\
U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad S(x,0) = S_0(x) \geq s_0 = const > 0.
\end{gathered} \tag{2.8}$$

It is not difficult to verify that if $\alpha \neq 1$ and $S_0(x) = s_0$, then the functions

$$\begin{gathered}
U(x,t) = \psi_1 x, \quad V(x,t) = \psi_2 x, \\
S(x,t) = \big[s_0^{1-\alpha} + (1-\alpha)(\psi_1^2+\psi_2^2)t\big]^{\frac{1}{1-\alpha}}
\end{gathered}$$

are the solutions of problem (2.8). But if $\alpha > 1$ at a finite time $t_0 = s_0^{1-\alpha}/[(\psi_1^2+\psi_2^2)(\alpha-1)]$, the function $S(x,t)$ becomes infinity. This example shows that the solution of problem (2.8) with smooth initial and boundary conditions can be blown up at a finite time.

3 Convergence of finite difference scheme

In the rectangle $[0,1]\times[0,T]$, where T is a positive number, let us consider the initial-boundary value problem (2.1).

Now, we study a numerical approximation of problem (2.1). If we introduce the notation $W = S^{1/2}$, then problem (2.1) takes the form

$$\begin{gathered}
\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\Big(W^{2\alpha} \frac{\partial U}{\partial x}\Big) = 0, \quad \frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\Big(W^{2\alpha} \frac{\partial V}{\partial x}\Big) = 0, \\
\frac{\partial W}{\partial t} = -\frac{a}{2} W^{2\beta-1} + \frac{b}{2} W^{2\gamma-1}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big], \\
U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad W(x,0) = [S_0(x)]^{1/2}.
\end{gathered} \tag{3.1}$$

Let us discretize the domain $[0,1]\times[0,T]$ and apply the following known notations [13]

$$\begin{gathered}
h = \frac{1}{M}, \quad \tau = \frac{T}{N}, \quad x_i = ih, \quad t_j = j\tau, \quad r(x_i,t_j) = r_i^j, \\
\overline{\omega}_h = \big\{x_i, \ i = 0,1,\dots,M\big\}, \quad \omega_h^* = \Big\{x_i = \Big(i - \frac{1}{2}\Big)h, \ i = 1,2,\dots,M\Big\}, \\
\omega_\tau = \big\{t_j = j\tau, \ j = 0,1,\dots,N\big\}, \quad \overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau, \quad \omega_{h\tau}^* = \omega_h^* \times \omega_\tau, \\
r_{x,i}^j = \frac{r_{i+1}^j - r_i^j}{h}, \quad r_{\overline{x},i}^j = \frac{r_i^j - r_{i-1}^j}{h}, \quad r_{t,i}^j = \frac{r_i^{j+1} - r_i^j}{\tau}
\end{gathered}$$

and the corresponding inner products and norms

$$(r^j, g^j) = h\sum_{i=1}^{M-1} r_i^j g_i^j, \quad (r^j, g^j] = h\sum_{i=1}^{M} r_i^j g_i^j,$$
$$\|r^j\| = (r^j, r^j)^{1/2}, \quad \|r^j]| = (r^j, r^j]^{1/2}.$$

For problem (3.1), consider the following finite difference scheme:

$$\begin{gathered}
u_t^j = (w^{2\alpha} u_{\overline{x}})_x, \quad v_t^j = (w^{2\alpha} v_{\overline{x}})_x, \\
w_t^j = -\frac{a}{2}\, w^{2\beta-1} + \frac{b}{2}\, w^{2\gamma-1}(u_{\overline{x}}^2 + v_{\overline{x}}^2), \\
u_0^j = v_0^j = 0, \quad u_M^j = \psi_1, \quad v_M^j = \psi_2, \quad j = 0,1,\dots,N, \\
u_i^0 = U_0(x_i), \quad v_i^0 = V_0(x_i), \quad w_i^0 = \big[S_0(x_{i+1/2})\big]^{1/2}, \ \ i = 0,1,\dots,M-1,
\end{gathered} \tag{3.2}$$

where the grid functions u and v are defined on $\overline{\omega}_{h\tau}$, while the grid function w is defined on $\omega_{h\tau}^*$. Note that here and below, if the grid functions are taken without indices of time level, it assumed that they are considered at t_{j+1}.

It is not difficult to show that an approximation error of scheme (3.2) on smooth solutions of problem (3.1) is $O(\tau + h^2)$.

The following statement holds.

Theorem 3.1. *An approximation error of scheme* (3.2) *on smooth solutions of problem* (3.1) *is* $O(\tau + h^2)$ *and if* $\beta \ge 1/2$, $\alpha = \gamma$, $|\alpha| \le 1/2$, *then a solution of scheme* (3.2) *converges to the solution of problem* (3.1) *in discrete analogues of the norms of the space* $L_2(0,1)$ *and the rate of convergence is the same as an approximation error.*

Proof. For the errors $X = u - U$, $Y = v - V$ and $Z = w - W$, we have

$$X_t^j = (w^{2\alpha} u_{\overline{x}} - W^{2\alpha} U_{\overline{x}})_x + \varphi_1, \tag{3.3}$$

$$Y_t^j = (w^{2\alpha} v_{\overline{x}} - W^{2\alpha} V_{\overline{x}})_x + \varphi_2, \tag{3.4}$$

$$Z_t^j = -\frac{a}{2}\,(w^{2\beta-1} - W^{2\beta-1}) + \frac{b}{2}\,(w^{2\gamma-1}u_{\overline{x}}^2 - W^{2\gamma-1}U_{\overline{x}}^2 + w^{2\gamma-1}v_{\overline{x}}^2 - W^{2\gamma-1}V_{\overline{x}}^2) + \varphi_3, \tag{3.5}$$

where $\varphi_k = O(\tau + h^2)$, $k = 1,2,3$.

Assume $\alpha = \gamma$ and $|\alpha| \le \frac{1}{2}$. Let us multiply scalarly equations (3.3)–(3.5) by $2\tau X$, $2\tau Y$ and $\frac{2}{b}\,\tau Z$, respectively. Using the discrete analogue of integration by parts and the identities [13]

$$2\tau(X_t, X) = \|X\|^2 - \|X^j\|^2 + \tau^2\|X_t\|^2, \quad 2\tau(Y_t, Y) = \|Y\|^2 - \|Y^j\|^2 + \tau^2\|Y_t\|^2,$$
$$2\tau(Z_t, Z] = ||Z]|^2 - ||Z^j]|^2 + \tau^2||Z_t]|^2,$$

we get

$$\begin{aligned}
\|X\|^2 - \|X^j\|^2 + \tau^2\|X_t\|^2 &= -\,2\tau\big[(w^\delta, u_{\overline{x}}^2] - (w^\delta + W^\delta, u_{\overline{x}}U_{\overline{x}}] + (W^\delta, U_{\overline{x}}^2] - (\varphi_1, X)\big], \\
\|Y\|^2 - \|Y^j\|^2 + \tau^2\|Y_t\|^2 &= -\,2\tau[(w^\delta, v_{\overline{x}}^2] - (w^\delta + W^\delta, v_{\overline{x}}V_{\overline{x}}] + (W^\delta, V_{\overline{x}}^2] - (\varphi_2, Y)], \\
\frac{1}{b}\big(||Z]|^2 - ||Z^j]|^2 + \tau^2||Z_t]|^2\big) &= -\,\frac{a}{b}\,\tau(w^{2\beta-1} - W^{2\beta-1})(w - W) \\
&\quad + \tau\Big((w^\delta - w^{\delta-1}W, u_{\overline{x}}^2] - (W^{\delta-1}w - W^\delta, U_{\overline{x}}^2] \\
&\quad + (w^\delta - w^{\delta-1}W, v_{\overline{x}}^2] - (W^{\delta-1}w - W^\delta, V_{\overline{x}}^2]\Big) + \frac{2\tau}{b}\,(\varphi_3, Z].
\end{aligned}$$

Here we introduced the notation $2\alpha = \delta$.

Adding the above equalities and assuming $\beta \geq 1/2$, we get

$$\begin{aligned}
&\|X\|^2-\|X^j\|^2+\|Y\|^2-\|Y^j\|^2+\frac{\tau}{b}\left(||Z]|^2-||Z^j]|^2\right)\\
&\qquad\leq -2\tau\bigg[\Big(\frac{w^\delta+w^{\delta-1}W}{2}\,u_{\overline{x}}^2+\frac{W^\delta+W^{\delta-1}w}{2}\,U_{\overline{x}}^2,1\Big]-(w^\delta+W^\delta,u_{\overline{x}}U_{\overline{x}}]\\
&\quad+\Big(\frac{w^\delta+w^{\delta-1}W}{2}v_{\overline{x}}^2+\frac{W^\delta+W^{\delta-1}w}{2}V_{\overline{x}}^2,1\Big]-(w^\delta+W^\delta,v_{\overline{x}}V_{\overline{x}}]-(\varphi_1,X)-(\varphi_2,Y)-\frac{1}{b}\,(\varphi_3,Z]\bigg]\\
&\qquad\leq -2\tau\Big\{\Big(\big[(w^\delta+w^{\delta-1}W)(W^\delta+W^{\delta-1}w)\big]^{\frac12}-w^\delta-W^\delta,|u_{\overline{x}}|\,|U_{\overline{x}}|\Big]\\
&\quad+\Big(\big[(w^\delta+w^{\delta-1}W)(W^\delta+W^{\delta-1}w)\big]^{\frac12}-w^\delta-W^\delta,|v_{\overline{x}}|\,|V_{\overline{x}}|\Big]-(\varphi_1,X)-(\varphi_2,Y)-\frac{1}{b}\,(\varphi_3,Z]\Big\}. \qquad (3.6)
\end{aligned}$$

Note that

$$\begin{aligned}
&(w^\delta-w^{\delta-1}W)(W^\delta-W^{\delta-1}w)-(w^\delta+W^\delta)^2\\
&\qquad=2w^\delta W^\delta+w^{\delta+1}W^{\delta-1}+w^{\delta-1}W^{\delta+1}-w^{2\delta}-2w^\delta W^\delta-W^{2\delta}\\
&\qquad\qquad=(w^{\delta+1}-W^{\delta+1})(W^{\delta-1}-w^{\delta-1}). \qquad (3.7)
\end{aligned}$$

Since $|\delta|\leq 1$, we have

$$(w^{\delta+1}-W^{\delta+1})(W^{\delta-1}-w^{\delta-1})\geq 0.$$

Using relations (3.6) and (3.7) and taking into account the last inequality, we arrive at

$$\|X\|^2+\|Y\|^2+\frac{1}{b}\,||Z]|^2\leq \|X^j\|^2+\|Y^j\|^2+\frac{1}{b}\,\|Z^j]|^2+2\tau\Big((\varphi_1,X)+(\varphi_2,Y)+\frac{1}{b}\,(\varphi_3,Z]\Big),$$

which yields

$$\|X\|+\|Y\|+||Z]|=O(\tau+h^2).$$

Thus, the proof of Theorem 3.1 is complete. □

Using the approach on proving Theorem 3.1, it is not difficult to prove the stability of scheme (3.2).

Applying scheme (3.2) given in this section and the Newton iterative method, various numerical experiments have been carried out which fully agree with theoretical findings. Using the results obtained in Section 2, we get graphical illustrations for the stability of solution (see Fig. 1) and fix the bifurcation phenomena (see Fig. 2).

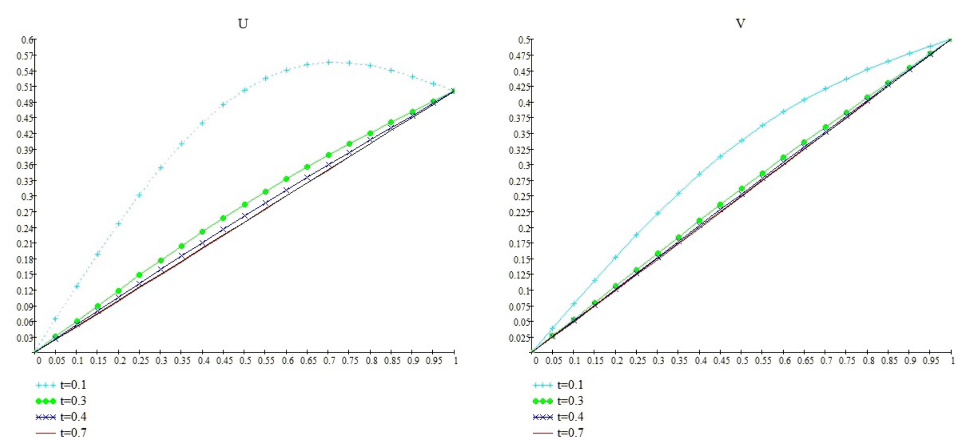

Figure 1. Stabilization of solution.

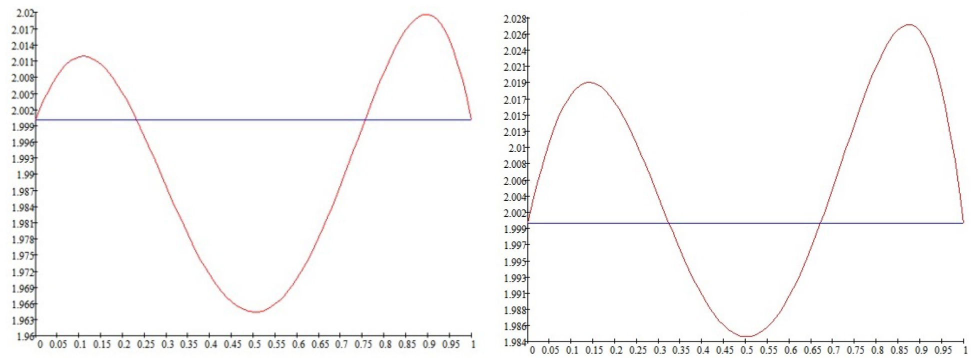

Figure 2. Hoph bifurcation.

4 Conclusion

For the system of nonlinear partial differential equations, which is based on the Maxwell equations describing the process of penetration of an electromagnetic field into a substance, the linear stability of a solution, as well as the possibility of Hoph bifurcation are studied. The blow up case is fixed, too. The corresponding finite difference scheme is constructed and its convergence is proved. The carried out various numerical experiments show the linear stability of a solution of the corresponding initial-boundary value problem and also Hopf type bifurcation for certain boundary data.

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