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# A VARIANT OF THE METHOD OF STEP ALGORITHM FOR A DELAY DIFFERENTIAL EQUATION

**Abstract.** In this paper we develop a new method to obtain explicit solutions for a first order linear delay differential equation based upon the generating function concept. The advantage of this new method as regards the traditional Method of Step Algorithm (MSA) is also showed through an example.[1]

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**Key words and phrases.** Delay differential equation, method of step algorithm (MSA), generating function, polynomials with a single delay.

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# 1 Introduction

In this work[2] we present the detailed proofs of the results reached in [5,6], as referred in [3], concerning the solutions of the Basic Initial Problem (BIP),

$$\begin{cases} x'(t) = f(\phi(t-r)), \\ x(0) = \phi(0), \end{cases}$$

where $B$ and $r$ are constants, $r > 0$ is the delay, and $\phi(t)$ is a given continuous function on $[-r, 0]$.

Delay differential equations (DDEs) are well studied in [1,4,7] from a point of view of the existence, uniqueness and properties of solutions. Here we point out that when the MSA is applied to the BIP, there appears combinatorial structure on the solutions.

This kind of structure that we designated by tree combinatorial structure led us to conjecture that there is a generating function defined over a specific class of polynomials with a single delay that solves the initial problem. As far as we know, the approach via a generating function is new to the relevant literature.

Assuming $\phi(t)$ is constant on $[-r, 0]$, and applying the MSA to the BIP, the solutions $x_n(t)$ defined on $A_n = ((n-1)r, nr]$, $n \geq 1$, showed one tree structure effect for the solution $x(t)$ of the problem.

That $x(t)$ *is the generating function for a sequence of polynomials with a single delay,* $P_j^n(rB)$, is our starting point.

Using the MSA, each solution $x_n(t)$ would depend upon the solution $x_{n-1}(t)$ defined on the previous intervals $A_{n-1}$. In order to provide an explicite formula for $x_n(t)$ on the interval $A_n$ without the knowledge of all back solutions $x_{n-1}(t)$ defined on the previous intervals $A_{n-1}$, we introduce the polynomials $P_j^n(rB)$, which we refer to as delay polynomials, and the main theorem proves that this is possible.

The present paper is organized as follows. Section 2 describes the MSA and presents the conjecture. Section 3 constructs the alternative method to obtain the BIP's solution. Section 4 contains the two fundamental propositions allowing us to obtain the calculating formulas for any solution $x_n(t)$ defined on $A_n$ with $n \geq 2$. Section 5 is devoted to the Main Theorem. Section 6 consists of the proof of the lemma, which is the basis of the new solution's method. An example is given to illustrate the theorem in Section 7.

# 2 Preliminaries

## 2.1 The method of step algorithm

Consider the Basic Initial Problem

$$\begin{cases} x'(t) = Bx(t-r), & t \geq 0, \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{2.1}$$

where $B \in \Re$, $r > 0$ is the delay, and $\phi(t)$ is a given continuous function on $[-r, 0]$.

The Method of Step Algorithm (MSA) can be described as follows.

### Step 1

Consider $x'(t) = f(x(t-r))$. Given $\phi(t)$ on $[-r, 0]$, we can determine $x(t)$ on the interval $[0, r]$ by solving the ODE

$$\begin{cases} x'(t) = f(\phi(t-r)), \\ x(0) = \phi(0). \end{cases}$$

Denote its solution by $x_1(t)$.

[2]See [2] for a previous version of this paper

**Step n**

For each integer $n \geq 2$, given the solution $x_{n-1}(t)$ on $[(n-2)r, (n-1)r]$, we can determine $x(t)$ on the interval $[(n-1)r, nr]$ by solving the ODE

$$\begin{cases} x'(t) = f(x_{n-1}(t-r)), \\ x((n-1)r) = x_{n-1}((n-1)r). \end{cases}$$

Denote its solution by $x_n(t)$.

**Conclusion**

We can define the solution of

$$\begin{cases} x'(t) = f(x(t-r)), & t \geq 0, \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases}$$

on each interval $A_n = [(n-1)r, nr]$, $n \geq 1$, by

$$x_n(t) = x_{n-1}((n-1)r) + \int_{(n-1)r}^{t} f(x_{n-1}(s-r))\, ds,$$

where $x_0(\cdot) \equiv \phi(\cdot)$.

For $j = 1, \ldots, 5$, let $x_j(t)$ be the solution of (2.1) defined on the interval $A_j$ obtained with the MSA. Assuming $\phi(t) = \phi(0)$ is a constant,

$$\begin{aligned}
x_1(t) &= \phi(0)[1 + Bt], \\
x_2(t) &= \phi(0)\Big[\frac{(Bt)^2}{2!} + Bt(1 - Br) + \frac{(Br)^2}{2!} + 1\Big], \\
x_3(t) &= \phi(0)\Big[\frac{(Bt)^3}{3!} + \frac{(Bt)^2}{2!}\Big(1 - \frac{3.2}{3}Br\Big) + Bt\Big(1 - Br + \frac{3.2^2}{3!}(Br)^2\Big) - \frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1\Big], \\
x_4(t) &= \phi(0)\Big[\frac{(Bt)^4}{4!} + \frac{(Bt)^3}{3!}\Big(1 - \frac{4.3}{4}Br\Big) + \frac{(Bt)^2}{2!}\Big(1 - \frac{3.2}{3}Br + \frac{6.3^2}{4.3}(Br)^2\Big) \\
&\qquad + Bt\Big(1 - Br + \frac{3.2^2}{3!}(Br)^2 - \frac{4.3^3}{4!}(Br)^3\Big) + \frac{(Br)^4}{4!}3^4 - \frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1\Big], \\
x_5(t) &= \phi(0)\Big[\frac{(Bt)^5}{5!} + \frac{(Bt)^4}{4!}\Big(1 - \frac{5.4}{5}Br\Big) + \frac{(Bt)^3}{3!}\Big(1 - \frac{4.3}{4}Br + \frac{10.4^2}{5.4}(Br)^2\Big) \\
&\qquad + \frac{(Bt)^2}{2!}\Big(1 - \frac{3.2}{3}Br + \frac{6.3^2}{4.3}(Br)^2 - \frac{10.4^3}{5.4.3}(Br)^3\Big) \\
&\qquad + Bt\Big(1 - Br + \frac{3.2^2}{3!}(Br)^2 - \frac{4.3^3}{4!}(Br)^3 + \frac{5.4^4}{5!}(Br)^4\Big) \\
&\qquad - \frac{(Br)^5}{5!}4^5 + \frac{(Br)^4}{4!}3^4 - \frac{(Br)^3}{3!}2^3 + \frac{(Br)^2}{2!} + 1\Big].
\end{aligned}$$

Analysing the form of these first iterates, we observe a tree structure effect, which allow us to formulate the following conjecture.

## 2.2 The conjecture

**Definition 2.1** (Rainville, [8])**.** Let $c_j$, $j \in \mathbb{N}_0$, be a specified sequence independent of $r$ and $t$. We say that $X(r,t)$ is a generating function of the set $g_j(r)$ if

$$X(r,t) = \sum_{j \geq 0} c_j g_j(r) t^j.$$

**Conjecture 2.2.** *If $x(t)$, $t \geq 0$, is a solution of BIP, then*

$$x(t) \equiv X(r,t) = \sum_{j\geq 0} v_j(r)t^j,$$

*i.e., $x(t)$ is a generating function for some sequence $(v_j(r))_{j\geq 0}$ in the delay $r$.*

# 3 Construction of a new solution's method

## 3.1 A new solution's formalization

In order to prove our claim, we will proceed in the following way. Consider the decomposition of $(0,\infty)$ in disjoint subintervals of equal length $r$. We will consider the restriction of the solution to each of these subintervals, as a generating function of some family of polynomials in $r$. That is,

$$(0,\infty) = \bigcup_{n\geq 1} A_n, \quad \text{where for each } \ n \geq 1, \ \ A_n = ((n-1)r, nr],$$

$$\varphi(t) = \begin{cases} \phi(t) & \text{if } \ t \in A_0 = [-r, 0], \\ \sum_{j\geq 0} w_j^1(r)t^j & \text{if } \ t \in A_1 = (0, r], \\ \sum_{j\geq 0} w_j^2(r)t^j & \text{if } \ t \in A_2 = (r, 2r], \\ \quad \vdots & \\ \sum_{j\geq 0} w_j^n(r)t^j & \text{if } \ t \in A_n = ((n-1)r, nr], \\ \quad \vdots & \end{cases}$$

Hence, we have $\varphi(t)$ defined on each interval $A_n$, $n \geq 1$, as

$$\varphi(t)\big|_{t\in A_n} \equiv x_n(t) = \sum_{j\geq 0} w_j^n(r)t^j. \tag{3.1}$$

If our conjecture is valid, we must have $\varphi'(t) = B\varphi(t-r)$ for $t \geq 0$, where the derivative at $t = 0$ represents the right-hand derivative. Two different types of conditions must hold. On the one hand, we are concerned with the differentiability at each point $t = nr$, which will guarantee the continuity of the solution. This will be treated in conditions **(2.A)**.

On the other hand, we want $x'(t) = Bx(t-r)$ to be satisfied at any interior point of $A_n$. This will be treated in conditions **(2.B)**. To do this, we determine which of the conditions should the iterates $w_j^n(r)$ in equation (3.1) satisfy in terms of $\varphi(t)$. Meaning

$$\varphi'(0) = B\varphi(-r), \tag{3.2}$$

$$x_n'(t) = Bx_{n-1}(t-r) \ \text{ for } \ t \in A_n, \ \ n \geq 1, \ \text{ where } \ x_0 \equiv \phi. \tag{3.3}$$

## 3.2 The constructive process

**(2.A.1)** $\varphi'(0) = B\phi(-r)$**.** At $t = 0$, we have

$$\varphi'(0) = \lim_{h\to 0^+} \frac{\sum_{j\geq 0} w_j^1(r)h^j - \phi(0)}{h} = \lim_{h\to 0^+} \frac{w_0^1(r) + w_1^1(r)h + w_2^1(r)h^2 + \cdots - \phi(0)}{h}.$$

A sufficient condition for (2.A.1) to hold, is

**(2.a.1)** $\boxed{w_0^1(r) = \phi(0) \text{ and } w_1^1(r) = B\phi(-r) \text{ and } w_j^1(r) \text{ takes any value for } j \geq 2\,.}$

**(2.B.1)** $\varphi'(t) = B\phi(t-r)$**,** $t \in (0,r)$**.** Since

$$\varphi'(t) = \sum_{j\geq 0}(j+1)w^1_{j+1}(r)t^j,$$

we can establish the following statement. A sufficient condition for having **(2.B.1)** is

**(2.b.1)** $$\boxed{B\phi(-r) + \sum_{j\geq 2} j w^1_j(r)t^{j-1} = B\phi(t-r)\,.}$$

We want to emphasize an important statement that later will lead us to the Main Theorem. If the initial function $\phi(t)$ is constant, combining **(2.a.1)** and **(2.b.1)** we can choose $w^1_j(r) = (\phi(0), B\phi(-r), 0, 0, \dots)$. In fact, condition **(2.b.1)** implies $\phi(-r) = \phi(t-r)$ since $t-r \in (-r, 0)$, and on this interval the function is constant.

Therefore, the solution on the interval $A_1$ can be defined as

$$x_1(t) = \sum_{j\geq 0} w^1_j(r)t^j = \phi(0)[1 + Bt],$$

where $\phi(t) = \phi(0)$ for $t \in [-r, 0]$, which is exactly the solution obtained by the MSA.

Returning to a continuous initial function $\phi(t)$, we can state the following proposition.

**Proposition 3.1.** *If there exists* $w^1_j(r)$ *satisfying* **(2.a.1)** *and* **(2.b.1)***, then* $x_1(t) = \sum_{j\geq 0} w^1_j(r)t^j$ *satisfies* (3.3) *on the interval* $(0,r)$*.*

*Proof.* For $t \in (0,r)$, let

$$x_1(t) = \sum_{j\geq 0} w^1_j(r)t^j = w^1_0(r) + w^1_1(r)t + \sum_{j\geq 2} w^1_j(r)t^j.$$

If $w^1_j(r)$ satisfies **(2.a.1)**, then

$$x_1(t) = \phi(0) + B\phi(-r)t + \sum_{j\geq 2} w^1_j(r)t^j.$$

By differentiation we obtain

$$x_1'(t) = B\phi(-r) + \sum_{j\geq 2} j w^1_j(r)t^{j-1},$$

and if **(2.b.1)** holds, then

$$x_1'(t) = B\phi(t-r).$$ □

From now on we will use the following lemma whose proof can be seen in Section 6.

**Lemma 3.2.** *For* $t \neq 0$ *and* $t \neq r$*,*

$$\sum_{j\geq 0} f_j(r)(t-r)^j = \sum_{j\geq 0} \frac{t^j}{j!}\Big(\sum_{i\geq 0} f_{j+i}(r)\,\frac{(-r)^i}{i!}\,(j+i)!\Big).$$

**(2.A.2)** $\varphi'(r) = B\phi(0)$**.** This equality requires: the existence of the derivative at $t = r$, the derivative has the value $B\phi(0)$.

**(ai)** To prove the existence of $\varphi'(r)$, notice that

$$\begin{aligned}
\varphi'(r^-) &= \lim_{h\to 0^-} \frac{\varphi(r+h)-\varphi(r)}{h} \\
&= \lim_{h\to 0^-} \frac{\sum\limits_{j\geq 0} w_j^1(r)(r+h)^j - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h} \\
&= \lim_{h\to 0^-} \frac{\sum\limits_{j\geq 0} \frac{h^j}{j!}\big(\sum\limits_{i\geq 0} w_{j+i}^1(r)\frac{r^i}{i!}(j+i)!\big) - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h} \\
&= \lim_{h\to 0^-} \frac{\sum\limits_{i\geq 0} w_i^1(r)r^i + h\sum\limits_{i\geq 0} w_{1+i}^1(r)r^i(1+i) + \sum\limits_{j\geq 2}\frac{h^j}{j!}\big(\sum\limits_{i\geq 0} w_{j+i}^1(r)\frac{r^i}{i!}(j+i)!\big) - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h} \\
&= \sum_{i\geq 0} w_{1+i}^1(r)r^i(1+i) + \lim_{h\to 0^-}\sum_{j\geq 2}\frac{h^{j-1}}{j!}\Big(\sum_{i\geq 0} w_{j+i}^1(r)\,\frac{r^i}{i!}\,(j+i)!\Big).
\end{aligned}$$

If convergence of the series is ensured, then the left-hand derivative at $t = r$ is equal to

$$\varphi'(r^-) = \sum_{i\geq 0} w_{1+i}^1(r)r^i(1+i).$$

Proceeding in a similar way, we have

$$\begin{aligned}
\varphi'(r^+) &= \lim_{h\to 0^+} \frac{\varphi(r+h)-\varphi(r)}{h} \\
&= \lim_{h\to 0^+} \frac{\sum\limits_{j\geq 0} w_j^2(r)(r+h)^j - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h} \\
&= \lim_{h\to 0^+} \frac{\sum\limits_{j\geq 0} \frac{h^j}{j!}\big(\sum\limits_{i\geq 0} w_{j+i}^2(r)\frac{r^i}{i!}(j+i)!\big) - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h} \\
&= \lim_{h\to 0^+} \frac{\sum\limits_{i\geq 0} w_i^2(r)r^i + h\sum\limits_{i\geq 0} w_{1+i}^2(r)r^i(1+i) + \sum\limits_{j\geq 2}\frac{h^j}{j!}\big(\sum\limits_{i\geq 0} w_{j+i}^2(r)\frac{r^i}{i!}(j+i)!\big) - \sum\limits_{j\geq 0} w_j^1(r)r^j}{h}.
\end{aligned}$$

The right-hand derivative of $\varphi$ at $t = r$ exists, if

$$\sum_{i\geq 0} w_i^2(r)r^i = \sum_{j\geq 0} w_j^1(r)r^j,$$
$$\sum_{i\geq 0} w_{1+i}^2(r)r^i(1+i) = \sum_{i\geq 0} w_{1+i}^1(r)r^i(1+i),$$

and the series $\sum\limits_{i\geq 0} w_{j+i}^k(r)\,\frac{r^i}{i!}\,(j+i)!$, $k = 1, 2$, converge.

**(aii)** $\varphi'(r) = B\phi(0)$.

We notice that the second condition

$$\sum_{i\geq 0} w_{1+i}^2(r)r^i(1+i) = \sum_{i\geq 0} w_{1+i}^1(r)r^i(1+i)$$

represents the equality between, respectively, $\varphi'(r^+)$ and $\varphi'(r^-)$. In order to have $\varphi'(r) = B\phi(0)$, it suffices to have

$$\sum_{i\geq 0} w_{1+i}^2(r)r^i(1+i) = B\phi(0).$$

The next proposition tells us the behaviour $w_j^2(r)$ must be such that the delay differential equation is satisfied at $t = r$. We note that in Proposition 3.1, we have established an equivalent result for the interior points of $A_1$.

**Proposition 3.3.** *If there exists $w_j^2(r)$ satisfying*

$$(2.2i)\qquad \boxed{\begin{cases} \sum_{j\geq 0} w_j^2(r)r^j = \sum_{j\geq 0} w_j^1(r)r^j, \\ \sum_{j\geq 0} w_{1+j}^2(r)r^j(1+j) = \sum_{j\geq 0} w_{1+j}^1(r)r^j(1+j) \end{cases}}$$

*and*

$$(2.2ii)\qquad \boxed{\sum_{j\geq 0} w_{1+j}^2(r)r^j(1+j) = B\phi(0),}$$

*then $\varphi'(r) = B\phi(0)$ and equality* (2.b.1) *holds at $t = r$.*

*Proof.* We have already seen that (2.2$i$) and (2.2$ii$) imply $\varphi'(r) = B\phi(0)$. We have to prove

$$B\phi(-r) + \sum_{j\geq 2} j w_j^1(r)r^{j-1} - B\phi(0) = 0.$$

If (2.2$ii$) holds, then we can write the first term as

$$\begin{aligned} B\phi(-r) &+ \sum_{i\geq 0}(2+i)w_{2+i}^1(r)r^{i+1} - \sum_{j\geq 0} w_{1+j}^2(r)r^j(1+j) \\ &= B\phi(-r) + \sum_{i\geq 0}(2+i)w_{2+i}^1(r)r^{i+1} - \sum_{j\geq 0} w_{1+j}^1(r)r^j(1+j) \\ &= \sum_{j\geq 0} w_{1+j}^1(r)r^j(1+j) - \sum_{j\geq 0} w_{1+j}^1(r)r^j(1+j), \end{aligned}$$

taking into account that $B\phi(-r) = w_1^1(r)$ and associating the terms in an appropriate way. □

The procedure we have just described, can be repeated in an inductive manner. Hence we can proceed in the following way;

**(2.B.2)** $\boldsymbol{\varphi'(t) = B\varphi(t-r),\ t \in (r, 2r).}$ Since

$$\varphi'(t) = \sum_{j\geq 0}(j+1)w_{j+1}^2(r)t^j$$

and

$$\varphi(t-r) = \sum_{j\geq 0} w_j^1(r)(t-r)^j = \sum_{j\geq 0} \frac{t^j}{j!}\Big(\sum_{i\geq 0} w_{j+i}^1(r)\,\frac{(-r)^i}{i!}\,(j+i)!\Big),$$

we can state that a sufficient condition for having **(2.B.2)** is

$$\textbf{(2.b.2)}\qquad \boxed{w_{j+1}^2(r) = \frac{B}{(j+1)!}\sum_{i\geq 0} w_{j+i}^1(r)\,\frac{(-r)^i}{i!}\,(j+i)! \text{ for each } j \geq 0.}$$

Thus we have finished the analysis of the solution defined on interior points of $A_2$.

# 4 The fundamental propositions

From a structural point of view, conditions (2.2$i$), (2.2$ii$) and **(2.b.2)** are identical in each interval $A_n$, for $n \geq 2$. Then we can state two fundamental propositions which establish sufficient conditions on $w_j^n(r)$, $n \geq 2$, in order for (3.3) to hold. The first one concerns with interior points, and the second one concerns with end points.

**Proposition 4.1.** *If for each $n \geq 2$ and $j \geq 0$, there exist a sequence $w_j^n(r)$ satisfying*

$$w_{j+1}^n(r) = \frac{B}{(j+1)!} \sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)!, \tag{4.1}$$

*then $x_n'(t) = Bx_{n-1}(t-r)$ for $t \in int\ A_n$.*

*Proof.* Let $x_n(t) = \sum_{j \geq 0} w_j^n(r) t^j$. If $t \in int\, A_n$ and $n \geq 2$, then

$$\begin{aligned} x_n'(t) &= \sum_{j \geq 0} (1+j) w_{1+j}^n(r) t^j = \sum_{j \geq 0} (1+j) t^j \Big( \frac{B}{(j+1)!} \sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)! \Big) \\ &= B \sum_{j \geq 0} \frac{t^j}{j!} \Big( \sum_{i \geq 0} w_{j+i}^{n-1}(r) \frac{(-r)^i}{i!} (j+i)! \Big) = B \sum_{j \geq 0} w_j^{n-1}(r)(t-r)^j = Bx_{n-1}(t-r), \end{aligned}$$

where we have considered (4.1) and Lemma 3.2. □

**Proposition 4.2.** *If for each $n \geq 2$ there exist a sequence $w_j^n(r)$ satisfying the conditions*

$$\begin{cases} \sum_{j \geq 0} (nr)^j w_j^{n+1}(r) = \sum_{j \geq 0} (nr)^j w_j^n(r), \\ \sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^{n+1}(r) = \sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^n(r) \end{cases} \tag{4.2}$$

*and*

$$\sum_{j \geq 0} (1+j)(nr)^j w_{1+j}^{n+1}(r) = B \sum_{j \geq 0} w_j^{n-1}(r)[(n-1)r]^j, \tag{4.3}$$

*then $x_n'(nr) = Bx_{n-1}((n-1)r)$.*

*Proof.* Let $n \geq 2$ and $t = nr$.

We start by showing the existence of a derivative at the points $t = nr$ for $n \geq 2$.

- The left-hand derivative:

$$\begin{aligned} x_n'(nr^-) &= \lim_{h \to 0^-} \frac{x_n(nr+h) - x_n(nr)}{h} \\ &= \lim_{h \to 0^-} \frac{\sum_{j \geq 0} w_j^n(r)(nr+h)^j - \sum_{j \geq 0} w_j^n(r)(nr)^j}{h} \\ &= \lim_{h \to 0^-} \frac{\sum_{j \geq 0} \frac{h^j}{j!} \big( \sum_{i \geq 0} w_{j+i}^n(r) \frac{(nr)^i}{i!} (j+i)! \big) - \sum_{j \geq 0} w_j^n(r)(nr)^j}{h} \\ &= \lim_{h \to 0^-} \left\{ \frac{\sum_{i \geq 0} w_i^n(r)(nr)^i + h \sum_{i \geq 0} w_{1+i}^n(r)(nr)^i(1+i)}{h} \right. \\ &\qquad \left. + \frac{\sum_{j \geq 2} \frac{h^j}{j!} \big( \sum_{i \geq 0} w_{j+i}^n(r) \frac{(nr)^i}{i!} (j+i)! \big) - \sum_{j \geq 0} w_j^n(r)(nr)^j}{h} \right\} \\ &= \sum_{i \geq 0} w_{1+i}^n(r)(nr)^i(1+i), \end{aligned}$$

assuming that $\sum_{i\geq 0} w_{j+i}^n(r)\,\frac{(nr)^i}{i!}\,(j+i)!$ converges for $j \geq 1$.

- The right-hand derivative:

$$
\begin{aligned}
x_n'(nr^+) &= \lim_{h\to 0^+} \frac{x_n(nr+h) - x_n(nr)}{h} \\
&= \lim_{h\to 0^+} \frac{\sum_{j\geq 0} w_j^{n+1}(r)(nr+h)^j - \sum_{j\geq 0} w_j^n(r)(nr)^j}{h} \\
&= \lim_{h\to 0^+} \frac{\sum_{j\geq 0} \frac{h^j}{j!}\left(\sum_{i\geq 0} w_{j+i}^{n+1}(r)\,\frac{(nr)^i}{i!}\,(j+i)!\right) - \sum_{j\geq 0} w_j^n(r)(nr)^j}{h} \\
&= \lim_{h\to 0^+} \left\{ \frac{\sum_{i\geq 0} w_i^{n+1}(r)(nr)^i + h\sum_{i\geq 0} w_{1+i}^{n+1}(r)(nr)^i(1+i)}{h} \right. \\
&\qquad\qquad \left. + \frac{\sum_{j\geq 2} \frac{h^j}{j!}\left(\sum_{i\geq 0} w_{j+i}^{n+1}(r)\,\frac{(nr)^i}{i!}\,(j+i)!\right) - \sum_{j\geq 0} w_j^n(r)(nr)^j}{h} \right\} \\
&= \sum_{i\geq 0} w_{1+i}^{n+1}(r)(nr)^i(1+i),
\end{aligned}
$$

assuming that (4.2) holds, and $\sum_{i\geq 0} w_{j+i}^{n+1}(r)\,\frac{(nr)^i}{i!}\,(j+i)!$ converges for $n \geq 1$ and $j \geq 1$.

We have proved the existence of derivative of $x_n(t)$ at $t = nr$, $n \geq 2$, and

$$x_n'(nr) = \sum_{j\geq 0} w_{1+j}^{n+1}(r)(nr)^j(1+j) = \sum_{j\geq 0} w_{1+j}^{n}(r)(nr)^j(1+j).$$

Next, we will show that $x_n'(t) = Bx_{n-1}(t-r)$ at $t = nr$, $n \geq 2$.

$$x_n'(nr) = \sum_{j\geq 0} w_{1+j}^{n+1}(r)(nr)^j(1+j) = B\sum_{j\geq 0} w_j^{n-1}(r)[(n-1)r]^j = Bx_{n-1}((n-1)r),$$

where we have considered (4.3). □

We point out that equalities (4.1), (4.2) and (4.3) provide calculating formulas for all terms of the sequences $w_j^n(r)$ for $n \geq 2$.

**Corollary 4.3.** *Equality* (4.3) *is a direct consequence of* (4.1) *and* (4.2).

*Proof.*

$$
\begin{aligned}
\sum_{j\geq 0}(1+j)(nr)^j w_{1+j}^{n+1}(r) &\underset{(4.2)}{=} \sum_{j\geq 0}(1+j)(nr)^j w_{1+j}^{n}(r) \\
&= \sum_{j\geq 0}(1+j)(nr)^j \left(\frac{B}{(j+1)!}\sum_{i\geq 0} w_{i+j}^{n-1}(r)\,\frac{(-r)^i}{i!}\,(i+j)!\right) \\
&= B\sum_{j\geq 0}\frac{(nr)^j}{j!}\left(\sum_{i\geq 0} w_{i+j}^{n-1}(r)\,\frac{(-r)^i}{i!}\,(i+j)!\right) = B\sum_{j\geq 0} w_j^{n-1}(r)(nr-r)^j,
\end{aligned}
$$

where we have considered (4.1) and Lemma 3.2. □

We also have a correspondent result to (2.2ii), which refers to $n = 1$.

**Corollary 4.4.** *Equality* (2.2*ii*) *is a direct consequence of* (2.2*i*) *and* **(2.b.1)**, *being the last one applied to* $t = r$.

*Proof.* As a consequence of Proposition 3.3, **(2.b.1)** is verified at $t = r$, so

$$B\phi(-r) + \sum_{j\geq 2} j w_j^1(r) r^{j-1} = B\phi(0).$$

Then

$$\begin{aligned}\sum_{j\geq 0} w_{1+j}^2(r) r^j (1+j) &= \sum_{j\geq 0} w_{1+j}^1(r) r^j (1+j) \\ &= w_1^1(r) + \sum_{j\geq 1} w_{1+j}^1(r) r^j (1+j) = w_1^1(r) + B\phi(0) - B\phi(-r) = B\phi(0),\end{aligned}$$

since $w_1^1(r) = B\phi(-r)$. □

These two corollaries suggest that during the constructive process of the solution, some conditions with distinct functions emerge.

## 5 The main theorem

From now on, we consider $\phi(t) = \phi(0) = C$ for $t \in [-r, 0]$, where $C$ is a real constant.

**Proposition 5.1.** *If*

$$w_0^1(r) = C, \quad w_1^1(r) = BC \ \text{ and } \ w_j^1(r) = 0 \ \text{ for } \ j \geq 2 \tag{5.1}$$

*then*

$$x_1(t) = \sum_{j\geq 0} w_j^1(r) t^j = C(1 + Bt)$$

*is the solution of problem* (2.1) *defined on* $A_1 = (0, r]$.

*Proof.* Equalities (5.1) verify **(2.a.1)** and **(2.b.1)**, implying $x_1'(t) = BC$ for $t \in [0, r)$. According to (3.1), we can then write

$$x_1(t) = \sum_{j\geq 0} w_j^1(r) t^j = C(1 + Bt) \ \text{ for } \ t \in (0, r).$$

By Proposition 3.3, the result is also true at $t = r$. □

The main result of this paper is the following theorem.

**Theorem 5.2.** *The solution of problem* (2.1) *with* $\phi(t) = C$ *if* $t \in [-r, 0]$ *can be written as*

$$X(r, t) = \sum_{j\geq 0} v_j(r) t^j$$

*for* $t \geq 0$. *The sequence* $v_j(r)$ *is defined by*

$$v_j(r) = C\,\frac{B^j}{j!}\,P_j^n(rB),$$

*where the polynomials* $P_j^n(rB)$ *are defined by*

$$P_j^n(rB) = \begin{cases} 1 + \displaystyle\sum_{i=0}^{n-(j+1)} \frac{(-rB)^{i+1}}{(i+1)!}\,(i+j)^{i+1} & \text{if } j \leq n-1, \\ 1 & \text{if } j = n, \\ 0 & \text{if } j \geq n+1. \end{cases}$$

The proof of this Theorem is divided into four stages: Propositions 5.3 and 5.4, and Corollaries 5.5 and 5.6. In Proposition 5.3, we will obtain the calculating formulas to get all terms of the sequences, $w_j^n(r)$, $n \geq 2$. Moreover, we will show that these formulas do not depend on the fact that the initial function is constant. This fact makes this procedure an alternative method to solve problem (2.1). In Proposition 5.4 and its Corollaries, we will show the consequences of taking the initial function as a constant one.

**Proposition 5.3.** *For $n \geq 2$, the solution $x_n(t) = \sum_{j\geq 0} w_j^n(r)t^j$, defined on each interval $A_n$, is obtained through the application of the following formulas, in the following order*

$$w_{j+1}^n(r) = \frac{B}{(j+1)!}\sum_{i\geqslant 0} w_{i+j}^{n-1}(r)\,\frac{(-r)^i}{i!}\,(i+j)! \tag{5.2}$$

*and*

$$w_0^{n+1}(r) = w_0^n(r) - \sum_{j\geqslant 1}\left[w_j^{n+1}(r) - w_j^n(r)\right](nr)^j. \tag{5.3}$$

*Proof.* Equality (5.2) is the sufficient condition (4.1) in Proposition 4.1, and equality (5.3) is obtained from the first equality of (4.2). □

From now on, we consider $w_j^1(r) = 0$ for $j \geq 2$.

Combining equalities (5.2) and (5.3), we obtain the sequence $w_j^2(r)$.

According to (5.2), for $n \geq 2$,

$$w_1^2(r) = B\sum_{i\geq 0} w_i^1(r)(-r)^i = B\sum_{i=0}^{1} w_i^1(r)(-r)^i,$$

and since $w_j^1(r) = 0$ for $j \geq 2$,

$$w_1^2(r) = B(C - rBC).$$

On the other hand,

$$w_2^2(r) = \frac{B}{2!}\sum_{i\geq 0} w_{i+1}^1(r)(-r)^i(i+1) = \frac{B}{2!}\,BC.$$

It is easy to check that $w_j^2(r) = 0$ for $j \geq 3$. This will lead us to assume that for $j \geq n+1$, $w_j^n(r) = 0$. We will prove this fact in the next proposition.

It remains to calculate the first term. According to (5.3),

$$w_0^2(r) = w_0^1(r) - \sum_{j\geq 1}(w_j^2(r) - w_j^1(r))r^j = C - \sum_{j=1}^{2}(w_j^2(r) - w_j^1(r))r^j,$$

and since $w_j^2(r) = 0$ for $j \geq 3$,

$$w_0^2(r) = C - r(w_1^2(r) - w_1^1(r)) - r^2 w_2^2(r) = C\Big(1 + \frac{r^2B^2}{2}\Big).$$

Finally, we obtain the solution $x_2(t)$ defined on $A_2 = (r, 2r]$,

$$x_2(t) = \sum_{i=0}^{2} w_j^2(r)t^j = C\Big(1 + \frac{(rB)^2}{2} + t(B - rB^2) + \frac{t^2B^2}{2!}\Big).$$

We can verify that

$$x_1(r) = x_2(r) = C(1 + rB).$$

Also, notice that the form of $x_2(t)$, obtained with our calculating formulas, has exactly the same form like the one obtained with MSA.

**Proposition 5.4.** *Consider problem* (2.1), *where* $\phi(t) = C$ *for* $t \in [-r, 0]$. *If* $w_j^1(r) = 0$ *for* $j \geq 2$, *then*

$$w_j^n(r) = 0 \ \text{ for } \ j \geq n+1. \tag{5.4}$$

*Proof.* We will use the induction on $n$. The case $n = 1$ is obviously true. Assuming $w_j^n(r) = 0$ for $j \geq n+1$, consider $j \geq n+2$. As a consequence of (5.2),

$$w_j^{n+1}(r) = \frac{B}{j!} \sum_{i \geq 0} w_{i+j-1}^n(r) \, \frac{(-r)^i}{i!} \, (i+j-1)!\,.$$

By the induction step, $w_j^{n+1}(r) = 0$ if $i+j-1 \geq n+1$. Since $i \geq 0$, we can conclude $w_j^{n+1}(r) = 0$ for $j \geq n+2$ as required. □

**Corollary 5.5.** *In the above conditions, if* $n \geq 1$, *then*

$$w_n^n(r) = C \, \frac{B^n}{n!}\,. \tag{5.5}$$

*Proof.* We will use the induction on $n$. Since $w_1^1(r) = BC$, the result holds for $n = 1$. Assuming $w_n^n(r) = C \, \frac{B^n}{n!}$, we have

$$\begin{aligned} w_{n+1}^{n+1}(r) &= \frac{B}{(n+1)!} \sum_{i \geq 0} w_{i+n}^n(r) \, \frac{(-r)^i}{i!} \, (i+n)! \\ &= \frac{B}{(n+1)!} \, w_n^n(r) n! = \frac{B}{(n+1)!} \, C \, \frac{B^n}{n!} \, n! = C \, \frac{B^{n+1}}{(n+1)!}\,, \end{aligned}$$

where we used (5.2), (5.4) and the induction step. □

**Corollary 5.6.** *In the conditions of Proposition 5.4, if* $j \leq n-1$, *then*

$$w_j^n(r) = C \, \frac{B^j}{j!} \Big( 1 + \sum_{i=0}^{n-(j+1)} \frac{(-rB)^{i+1}}{(i+1)!} \, (i+j)^{i+1} \Big). \tag{5.6}$$

*Proof.* To prove (5.6), we will use the induction reasoning applied to $j = n-k$ for the successive values $k = 1, 2, \ldots, n$. So, we will do it, first considering $k = 1$, then $k = 2$ and, finally, by an induction reasoning.

1. If $j = n-1$, $n \geq 1$, we have to prove that

$$w_{n-1}^n(r) = C \, \frac{B^{n-1}}{(n-1)!} \big[ 1 + (-rB)(n-1) \big].$$

Using the induction on $n$, the case $n = 1$ is valid as a consequence of **(2.a.1)**. Assuming $w_{n-1}^n(r) = C \, \frac{B^{n-1}}{(n-1)!} \, [1 + (-rB)(n-1)]$, we have

$$\begin{aligned} w_n^{n+1}(r) &= \frac{B}{n!} \sum_{i \geq 0} w_{i+n-1}^n(r) \, \frac{(-r)^i}{i!} \, (i+n-1)! \\ &= \frac{B}{n!} \sum_{i=0}^{1} w_{i+n-1}^n(r) \, \frac{(-r)^i}{i!} \, (i+n-1)! = \frac{B}{n!} \, \big\{ w_{n-1}^n(r)(n-1)! + w_n^n(r)(-r)n! \big\} \\ &= B \Big\{ \frac{1}{n} \, \frac{CB^{n-1}}{(n-1)!} \big[ 1 + (-rB)(n-1) \big] + (-r) C \, \frac{B^n}{n!} \Big\} \\ &= C \, \frac{B^n}{n!} \, [1 - rBn + rB - rB] = C \, \frac{B^n}{n!} \, [1 + (-rB)n], \end{aligned}$$

where we have used (5.2) for $j = n-1$, (5.4), (5.5) and the induction step.

2. If $j = n-2$ for $n \geq 2$, we have to prove that

$$w^n_{n-2}(r) = C\,\frac{B^{n-2}}{(n-2)!}\left[1 + (-rB)(n-2) + \frac{(-rB)^2}{2!}\,(n-1)^2\right].$$

For $n = 2$, using (5.3), we have

$$\begin{aligned} w^2_0(r) = w^1_0(r) - \sum_{j\geq 1}\left[w^2_j(r) - w^1_j(r)\right]r^j &= C - [w^2_1(r) - w^1_1(r)]r - [w^2_2(r) - 0]r^2 - 0 \\ &= C - \left[B(C - rBC) - CB\right]r - C\,\frac{B^2}{2!}\,r^2 = C\Big(1 + \frac{(-rB)^2}{2!}\Big). \end{aligned}$$

Assuming

$$w^n_{n-2}(r) = C\,\frac{B^{n-2}}{(n-2)!}\left[1 + (-rB)(n-2) + \frac{(-rB)^2}{2!}\,(n-1)^2\right],$$

we have

$$\begin{aligned} w^{n+1}_{n-1}(r) &= \frac{B}{(n-1)!}\sum_{i\geq 0} w^n_{i+n-2}(r)\,\frac{(-r)^i}{i!}\,(i+n-2)! \\ &= \frac{B}{(n-1)!}\Big\{w^n_{n-2}(r)(n-2)! + w^n_{n-1}(r)(-r)(n-1)! + w^n_n(r)\,\frac{(-r)^2}{2!}\,n!\Big\} \\ &= C\,\frac{B^{n-1}}{(n-1)!}\left[1 + (-rB)(n-1) + \frac{(-rB)^2}{2!}\,n^2\right], \end{aligned}$$

where we have used (5.2) for $j = n-2$, (5.4), (5.5) and the induction step.

It can be assumed by induction that

$$\text{for } \; k = 1, 2, \ldots, n \; \text{ and } \; n \geq k,$$
$$w^n_{n-k}(r) = C\,\frac{B^{n-k}}{(n-k)!}\Big(1 + \sum_{i=0}^{k-1}\frac{(-rB)^{i+1}}{(i+1)!}\,(i+n-k)^{i+1}\Big).$$

The proof of the theorem is now complete. □

Hence, if we fix $n \in \mathbb{N}$, we can calculate $x_n(t)$ with $t \in A_n = ((n-1)r, nr]$ using

$$x_n(t) = \sum_{j\geq 0} w^n_j(r)t^j = C\sum_{j\geq 0}\frac{B^j}{j!}\,P^n_j(rB)t^j, \tag{5.7}$$

where $P^n_j(rB)$ are defined in Theorem 5.2.

The solution found by this new method coincides with the one obtained by the method of steps, the recurrences formulas (5.2) and (5.3) can be replaced by (5.7), whenever $\phi(t)$ is a constant, and, finally, the solution of problem (2.1) is the generating function for $\{w^n_j(r)\}_{j=0,1,\ldots}$.

# 6 Proof of Lemma 3.2

For $r > 0$, $t \neq 0$ and $t \neq r$,

$$\sum_{j\geq 0} f_j(r)(t-r)^j = \sum_{j\geq 0}\frac{t^j}{j!}\Big(\sum_{i\geq 0} f_{j+i}(r)\,\frac{(-r)^i}{i!}(j+i)!\Big).$$

*Proof.*

$$
\begin{aligned}
&\sum_{j\geq 0} f_j(r)(t-r)^j \\
&\quad = f_0(r)+f_1(r)(t-r)+f_2(r)(t-r)^2+f_3(r)(t-r)^3+\cdots+f_p(r)(t-r)^p+f_{p+1}(r)(t-r)^{p+1}+\cdots \\
&\quad = \Big[f_0(r)+f_1(r)(-r)+f_2(r)(-r)^2+f_3(r)(-r)^3+\cdots+f_p(r)(-r)^p+f_{p+1}(r)(-r)^{p+1}+\cdots\Big] \\
&\qquad + t\Big[f_1(r)-2rf_2(r)+3r^2f_3(r)-4r^3f_4(r)+\cdots+\binom{p}{p-1}(-r)^{p-1}f_p(r) \\
&\qquad\qquad + f_{p+1}(r)(-r)^p\Big(1+\binom{p}{p-1}\Big)+\cdots\Big] \\
&\qquad + t^2\Big[f_2(r)-3rf_3(r)+6r^2f_4(r)+\cdots+\binom{p}{p-2}(-r)^{p-2}f_p(r) \\
&\qquad\qquad + f_{p+1}(r)(-r)^{p-1}\Big(\binom{p}{p-1}+\binom{p}{p-2}\Big)+\cdots\Big] \\
&\qquad + t^3\Big[f_3(r)-4rf_4(r)+\cdots+\binom{p}{p-3}(-r)^{p-3}f_p(r) \\
&\qquad\qquad + f_{p+1}(r)(-r)^{p-2}\Big(\binom{p}{p-2}+\binom{p}{p-3}\Big)+\cdots\Big] \\
&\qquad + \cdots + t^{p-2}\Big[\binom{p}{2}(-r)^2f_p(r)+f_{p+1}(r)(-r)^3\Big(\binom{p}{3}+\binom{p}{2}\Big)+\cdots\Big] \\
&\qquad + t^{p-1}\Big[\binom{p}{1}(-r)f_p(r)+f_{p+1}(r)(-r)^2\Big(\binom{p}{2}+\binom{p}{1}\Big)+\cdots\Big] \\
&\qquad + t^p\Big[f_p(r)+f_{p+1}(r)(-r)\Big(\binom{p}{1}+1\Big)+\cdots\Big]+\cdots.
\end{aligned}
$$

Define $g(z)=\sum_{j\geqslant 0}(-1)^j f_j(r)z^j$, where $g(r)=\sum_{j\geqslant 0} f_j(r)(-r)^j$ represents

$$f_0(r)+f_1(r)(-r)+f_2(r)(-r)^2+f_3(r)(-r)^3+\cdots+f_p(r)(-r)^p+f_{p+1}(r)(-r)^{p+1}+\cdots$$

**Claim 6.1.**

$$
\begin{aligned}
&t\Big[f_1(r)-2rf_2(r)+3r^2f_3(r)-4r^3f_4(r)+\cdots \\
&\qquad + \binom{p}{p-1}(-r)^{p-1}f_p(r)+f_{p+1}(r)(-r)^p\Big(1+\binom{p}{p-1}\Big)+\cdots\Big]
\end{aligned}
$$

*can be written as*

$$-tg'(z)\big|_{z=r}.$$

Indeed, $g'(z)=\sum_{j\geqslant 0}(-1)^{j+1}(j+1)f_{j+1}(r)z^j$. So,

$$
\begin{aligned}
-tg'(z)_{z=r} &= t\Big[\sum_{j\geqslant 0}(-z)^j(j+1)f_{j+1}(r)\Big]_{z=r} \\
&= t\big[f_1(r)-2rf_2(r)+3r^2f_3(r)+\cdots+p(-r)^{p-1}f_p(r)+\cdots\big].
\end{aligned}
$$

**Claim 6.2.**

$$
\begin{aligned}
&t^2\Big[f_2(r)-3rf_3(r)+6r^2f_4(r)+\cdots \\
&\qquad + \binom{p}{p-2}(-r)^{p-2}f_p(r)+f_{p+1}(r)(-r)^{p-1}\Big(\binom{p}{p-1}+\binom{p}{p-2}\Big)+\cdots\Big]
\end{aligned}
$$

*can be written as*

$$\frac{t^2}{2!}\, g''(z)\Big|_{z=r}.$$

Indeed, $g''(z) = \sum_{j\geqslant 0}(-1)^{j+2}(j+1)(j+2)f_{j+2}(r)z^j$. So,

$$\begin{aligned}\frac{t^2}{2!}\, g''(z)_{z=r} = \frac{t^2}{2!}\Big[\sum_{j\geqslant 0}(-z)^j(j+1)(j+2)f_{j+2}(r)\Big]_{z=r}\\ = t^2\Bigg[f_2(r) - 3rf_3(r) + 6r^2 f_4(r) + \cdots + \binom{p}{2}(-r)^{p-2}f_p(r) + \cdots\Bigg].\end{aligned}$$

Repeating the process,

**Claim 6.3.**

$$t^n\Bigg[f_n(r) + f_{n+1}(r)(-r)\Bigg(\binom{n}{1}+1\Bigg) + \cdots\Bigg]$$

*can be written as*

$$(-1)^n\,\frac{t^n}{n!}\, g^{(n)}(z)\Big|_{z=r},$$

*where*

$$g^{(n)}(z) = \sum_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)\cdots(j+n)f_{j+n}(r)z^j.$$

We prove this fact by the induction on $n$.

*Proof.* As we have already seen,

$$g'(z) = \sum_{j\geqslant 0}(-1)^{j+1}(j+1)f_{j+1}(r)z^j,$$

so the case $n = 1$ is verified.

Assuming

$$g^{(n-1)}(z) = \sum_{j\geqslant 0}(-1)^{j+n-1}(j+1)(j+2)\cdots(j+n-1)f_{j+n-1}(r)z^j,$$

we have to prove that

$$g^{(n)}(z) = \sum_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)\cdots(j+n)f_{j+n}(r)z^j.$$

We have

$$\begin{aligned}g^{(n)}(z) &= \frac{d}{dz}\Big(\sum_{j\geqslant 0}(-1)^{j+n-1}(j+1)(j+2)\cdots(j+n-1)f_{j+n-1}(r)z^j\Big)\\ &= \sum_{j\geqslant 0}(-1)^{j+n}(j+1)(j+2)\cdots(j+n)f_{j+n}(r)z^j.\end{aligned}$$

Hence, we can write

$$\begin{aligned}\sum_{j\geq 0} f_j(r)(t-r)^j &= g(r) - tg'(r) + \frac{t^2}{2!}\, g''(r) - \frac{t^3}{3!}\, g'''(r) + \cdots + (-1)^p\,\frac{t^p}{p!}\, g^{(p)}(r) + \cdots\\ &= \sum_{j\geqslant 0}(-1)^j\,\frac{t^j}{j!}\, g^{(j)}(r) = \sum_{j\geqslant 0}(-1)^j\,\frac{t^j}{j!}\Big(\sum_{i\geqslant 0}(-1)^{i+j}(i+1)\cdots(i+j)f_{i+j}(r)r^i\Big)\\ &= \sum_{j\geqslant 0}\frac{t^j}{j!}\Big(\sum_{i\geqslant 0}(-r)^i\,\frac{(i+j)!}{i!}\, f_{i+j}(r)\Big) = \sum_{j\geqslant 0}\frac{t^j}{j!}\Big(\sum_{i\geqslant 0} f_{i+j}(r)\,\frac{(-r)^i}{i!}\,(i+j)!\Big),\end{aligned}$$

where we have used the equality $(i+1)(i+2)(i+3)\cdots(i+j) = \frac{(i+j)!}{i!}$. □

# 7 An application

This example is presented in [5].

Suppose we wish to study a population $P(t) = (x(t), y(t))$, where $x(t)$ denotes the average height and $y(t)$ the average weight. It was observed that $x(t)$ depends on the height of the previous generation through $x'(t) = Bx(t-r)$, where $r$ is the size (per units of time) of a generation.

We can determine explicitly the behaviour of this variable regarding the fourth generation. This means that we wish to compute $x_4(t)$, given $x(t) = C$ for $t \in [-r, 0]$, where $C$ is the average height.

Using equation (5.7), we can determine it directly, without having to compute the height for the previous generations,

$$x_4(t) = \sum_{j\geq 0} w_j^4(r)t^j = C \sum_{j\geq 0} \frac{B^j}{j!}\, P_j^4(rB)t^j,$$

where $P_j^4(rB)$ are computed by applying Theorem 5.2. From this theorem, since

$$\begin{aligned}
&P_4^4(rB) = 1,\\
&P_3^4(rB) = 1 + \sum_{i=0}^{0} \frac{(-rB)^{i+1}}{(i+1)!}\,(i+3)^{i+1} = 1 + 3(-rB),\\
&P_2^4(rB) = 1 + \sum_{i=0}^{1} \frac{(-rB)^{i+1}}{(i+1)!}\,(i+2)^{i+1} = 1 + 2(-rB) + \frac{3^2}{2!}\,(-rB)^2,\\
&P_1^4(rB) = 1 + \sum_{i=0}^{2} \frac{(-rB)^{i+1}}{(i+1)!}\,(i+1)^{i+1} = 1 + (-rB) + \frac{2^2}{2!}\,(-rB)^2 + \frac{3^3}{3!}\,(-rB)^3,\\
&P_0^4(rB) = 1 + \sum_{i=0}^{3} \frac{(-rB)^{i+1}}{(i+1)!}\,i^{i+1} = 1 + \frac{(-rB)^2}{2!} + \frac{2^3}{3!}\,(-rB)^3 + \frac{3^4}{4!}\,(-rB)^4,
\end{aligned}$$

for $j = 0, 1, 2, 3, 4$, we have

$$\begin{aligned}
x_4(t) &= C\Big\{P_0^4(rB) + BP_1^4(rB)t + \frac{B^2}{2!}\,P_2^4(rB)t^2 + \frac{B^3}{3!}\,P_3^4(rB)t^3 + \frac{B^4}{4!}\,P_4^4(rB)t^4\Big\}\\
&= C\Big\{1 + \frac{(-rB)^2}{2!} + \frac{2^3}{3!}\,(-rB)^3 + \frac{3^4}{4!}\,(-rB)^4 + Bt\Big(1 - rB + \frac{2^2}{2!}\,(-rB)^2 + \frac{3^3}{3!}\,(-rB)^3\Big)\\
&\qquad + \frac{B^2}{2!}\,t^2\Big(1 - 2rB + \frac{3^2}{2!}\,(-rB)^2\Big) + \frac{B^3}{3!}\,t^3(1 - 3rB) + \frac{B^4}{4!}\,t^4\Big\}.
\end{aligned}$$

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# References

[1] R. Bellman and K. L. Cooke, *Differential-Difference Equations*. Academic Press, New York–London, 1963.

[2] P. B. Brito, M. F. Fabião, and A. G. St.Aubyn, A new approach on the solution of a delay differential equation. `http://cemapre.iseg.ulisboa.pt/archive/preprints/ICNAAM2008.pdf`.

[3] P. B. Brito, M. F. Fabião and A. G. St.Aubyn, The Lambert function on the solution of a delay differential equation. *Numer. Funct. Anal. Optim.* **32** (2011), no. 11, 1116–1126.

[4] O. Diekmann, S. M. Verduyn Lunel, S. A. Gils and H.-O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*. Springer, New York, 1995.

[5] M. F. Fabião, *Equação diferencial com atraso, das funções geradoras até à função W-Lambert: contributo para uma aplicação à economia, introdução do efeito de atraso no modelo de Solow.* Ph.D. Thesis, Instituto Superior de Economia e Gestão, Universidade Técnica de Lisboa, Portugal, 2005.

[6] M. F. Fabião, P. B. Brito and A. St.Aubyn, A tree combinatorial structure on the solution of a delay differential equation: a generating function approach. *AIP Conf. Proc.* **1048** (2008), 118–121.

[7] J. Hale, *Theory of Functional Differential Equations.* Applied Mathematical Sciences, Vol. 3. Springer-Verlag, New York–Heidelberg, 1977.

[8] E. D. Rainville, *Special Functions.* Reprint of 1960 first edition. Chelsea Publishing Co., Bronx, N.Y., 1971.

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