Memoirs on Differential Equations and Mathematical Physics

Volume 73, 2018, 29–44

Rim Bourguiba, Faten Toumi

EXISTENCE RESULTS OF A SINGULAR FRACTIONAL DIFFERENTIAL EQUATION WITH PERTURBED TERM

Abstract. The boundary value problem

$$D^\alpha u(t) + \mu a(t) f(t, u(t)) - q(t) = 0,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\, ds$$

is studied, where μ is a positive parameter, $f : [0,1] \times [0;+\infty) \to [0;+\infty)$ and $a : (0,1) \to [0,+\infty)$ are continuous functions, while $q : (0,1) \to [0,+\infty)$ is a measurable function. The case, where the function a has singularities at the points $t = 0$ and $t = 1$, is admissible.

Conditions are found guaranteeing, respectively, the existence of at least one and at least two positive solutions. Examples are gives.[1]

2010 Mathematics Subject Classification. 34A08, 34B18, 35G60.

Key words and phrases. Fractional differential equation, positive solution, integral boundary conditions, Green's function, dependence on a parameter, perturbed term.

რეზიუმე. შესწავლილია დადებით μ პარამეტრზე დამოკიდებული სასაზღვრო ამოცანა

$$D^\alpha u(t) + \mu a(t) f(t, u(t)) - q(t) = 0,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\, ds$$

სადაც $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$, $0 < \lambda < \alpha$, D^α არის α რიგის რიმან-ლიუვილის წარმოებული, $f : [0,1] \times [0;+\infty) \to [0;+\infty)$ და $a : (0,1) \to [0,+\infty)$ უწყვეტი, ხოლო $q : (0,1) \to [0,+\infty)$ ზომადი ფუნქციებია. დასაშვებია შემთხვევა, როცა a ფუნქციას გააჩნია სინგულარობები $t = 0$ და $t = 1$ წერტილებში.

ნაპოვნია პირობები, რომლებიც სათანადოდ უზრუნველყოფენ ერთი მაინც და ორი მაინც დადებითი ამონახსნის არსებობას. მოყვანილია მაგალითები.

[1]Reported on Conference "Differential Equation and Applications", September 4–7, 2017, Brno

1 Introduction

Fractional differential equations have applications in various fields of science and engineering and have been a focus of research for decades (see [6,9,10,12] and the references therein). There is a large number of important subjects in various fields of fractional calculus and related applications such as the solvability, existence and multiplicity of positive solutions for the given boundary value problems of fractional differential equations. For more details see [1,3,4,11].

Namely, A. Cabada and Z. Hamdi [3] presented the existence results for the following boundary value problem

$$\begin{cases} D^\alpha u(t) + \mu g(t) f(u(t)) = 0 \ \text{ in } \ [0,1], \\ u(0) = u'(0) = 0, \quad u(1) = \lambda \int\limits_0^1 u(s)\,ds, \end{cases}$$

where μ is a positive parameter, $2 < \alpha \le 3$, $0 < \lambda < \alpha$ and f, g are continuous functions. Under the conditions $g \in L^1([0,1])$ and $\int\limits_{1/2}^1 g(t)\,dt > 0$, they derived various existence and multiplicity results of positive solutions depending on the parameter $\mu > 0$.

However, all of the above mentioned works are based on a key assumption, that is, the nonlinear term is required to be nonnegative. When nonlinear fractional differential equations involve a sign-changing term, J. Henderson and R. Luca [5] investigated the existence of a positive solution for the nonlinear fractional problem, and then under the similar conditions X. Zhang, L. Liu and Y. Wu [13] studied the existence of positive solutions of the boundary value problem for a singular fractional differential equation with a negatively perturbed term. More precisely, the authors considered the following problem

$$\begin{cases} -D^\alpha u(t) = p(t) f(t, u(t)) - q(t) \ \text{ in } \ (0,1), \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where $2 < \alpha \le 3$. The function p is continuous nonnegative on $(0,1)$ and f is in $\mathcal{C}([0,1] \times [0,+\infty), [0,+\infty))$. The perturbed term $q : (0,1) \to [0,+\infty)$ is Lebesgue integrable and may be singular at some zero measure sets of $[0,1]$.

Under other boundary conditions, X. Zhou, J.-G. Peng and Y.-D. Chu [14] studied the following problem

$$\begin{cases} D^\alpha u(t) = p(t) f(t, u(t)) - q(t) \ \text{ in } \ (0,1), \\ u(0) = u(1) = u'(1) = 0, \end{cases}$$

where $2 < \alpha \le 3$. The functions p and q are Lebesgue integrable on $(0,1)$ and f is in $\mathcal{C}([0,1] \times [0,+\infty), [0,+\infty))$.

The existence of positive solutions of a fractional differential equation with a perturbed term, integral boundary and parametric dependence, however, has not been studied previously. In this paper, motivated by [2,3,13,14], we give sufficient conditions for the existence and multiplicity of positive solutions for problem

$$\begin{cases} D^\alpha u(t) + \mu a(t) f(t, u(t)) - q(t) = 0 \ \text{ in } \ (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \lambda \int\limits_0^1 u(s)\,ds. \end{cases} \tag{1.1}$$

The results derived depend on the positive parameter μ.

The outline of this paper is as follows. In Section 2, we present some preliminaries and lemmas that will be used for the proofs of our main results. The main theorems are presented in Section 3. The final section of the paper contains examples to illustrate our results.

2 Preliminaries and lemmas

In this section, we introduce definitions and preliminary facts that will be used throughout this paper. We refer the reader to [2, 6, 8] for more details.

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha > 0$ for a measurable function $f : (0, +\infty) \to \mathbb{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0,$$

where Γ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a measurable function $f : (0, +\infty) \to \mathbb{R}$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds = \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t),$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$. Here $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. *Let $\alpha > 0$. Let $u \in \mathcal{C}(0,1) \cap L^1(0,1)$. Then*

(i) $D^\alpha I^\alpha u = u$.

(ii) *For $\delta > \alpha - 1$, $D^\alpha t^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta-\alpha+1)} t^{\delta-\alpha}$. Moreover, we have $D^\alpha t^{\alpha-i} = 0$, $i = 1, 2, \dots, n$.*

(iii) *$D^\alpha u(t) = 0$ if and only if $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$.*

(iv) *Assume that $D^\alpha u \in \mathcal{C}(0,1) \cap L^1(0,1)$, then we have*

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R}, \quad i = 1, 2, \dots, n.$$

Now, we give the explicit expression of the Green function for the linear fractional differential equation associated to the problem (1.1).

Lemma 2.4 ([2])**.** *Let $n \geq 3$, $n - 1 < \alpha \leq n$ and $\lambda \in (0, \alpha)$. Let $y \in \mathcal{C}([0,1])$. Then the unique solution of the linear fractional differential problem*

$$\begin{cases} D^\alpha u(t) + y(t) = 0 \ \text{ in } \ (0,1), \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\, ds, \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,s) y(s)\, ds,$$

where for all $t, s \in [0,1]$,

$$G(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha - \lambda + \lambda s) - (\alpha-\lambda)((t-s)^*)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, \tag{2.2}$$

$G(t,s)$ is called the Green function of the boundary value problem (2.1). Here, for $x \in \mathbb{R}$, $x^* = \max(x, 0)$.

Now we recall some properties of the Green function.

Proposition 2.5. *Let $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$, and $\lambda \in [0,\alpha)$. Then the function G defined by (2.2) satisfies the following properties:*

(i) *G is a nonnegative continuous function on $[0,1] \times [0,1]$ and $G(t,s) > 0$ for all $t, s \in (0,1)$.*

(ii) *$G(t,s) \leq \eta K(s)$ for all $t, s \in [0,1]$, where $K(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ and $\eta = \frac{\alpha}{\alpha-\lambda}$.*

(iii) *$G(t,s) \leq \eta t^{\alpha-1} K_1(s)$ for all $t, s \in [0,1]$, where $K_1(s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$.*

(iv) *$G(t,s) \geq \eta \lambda^* t^{\alpha-1} K(s)$ $\forall\, t, s \in [0,1]$, where $\lambda^* = \dfrac{\lambda}{\alpha}$.*

(v) *If $\theta \in (0,\frac{1}{2})$, $s \in [0,1]$, then $\min\limits_{t \in [\theta, 1-\theta]} G(t,s) \geq \gamma K(s)$, where $\gamma = (\frac{\theta}{\alpha-1} + \frac{\lambda}{\alpha-\lambda})\theta^{\alpha-1}$.*

Proof. The proofs of (i), (ii) and (v) are given in [2]. To prove (iii), we use Lemmas 2.5 and 2.6 in [2]. Assertion (iv) follows immediately from Proposition 2.7 in [2]. □

Using assertion (ii) of Proposition 2.5, we have the following

Proposition 2.6. *Let q be a nonnegative measurable function on $(0,1)$. Then $w(t) = \int_0^1 G(t,s) q(s)\, ds$ is continuous on $[0,1]$ if and only if $\int_0^1 (1-t)^{\alpha-1} q(t)\, dt$ converges.*

Now we state the following key lemma.

Lemma 2.7. *Let $n \geq 3$, $n-1 < \alpha \leq n$ and $0 < \lambda < \alpha$. Assume that $(1-t)^{\alpha-1} q(t) \in \mathcal{C}(0,1) \cap L(0,1)$. Then the boundary value problem*

$$\begin{cases} D^\alpha w(t) + q(t) = 0 \ \ in \ \ (0,1), \\ w(0) = w'(0) = \cdots = w^{(n-2)}(0) = 0, \quad w(1) = \lambda \displaystyle\int_0^1 w(s)\, ds, \end{cases} \tag{2.3}$$

has a unique nonnegative solution $w(t) = \int_0^1 G(t,s) q(s)\, ds \in \mathcal{C}([0,1])$ satisfying

$$w(t) \leq \eta \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |q(s)|\, ds$$

on $[0,1]$.

Proof. First, we will prove that $D^\alpha w(t) + q(t) = 0$ on $(0,1)$. By Proposition 2.6, we have that w is continuous on $[0,1]$ and so $I^{n-\alpha}|w|$ is bounded on $[0,1]$. Thus, using Fubini's theorem, for each $t \in (0,1)$ we obtain

$$I^{n-\alpha} w(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^1 \int_0^t (t-s)^{n-\alpha-1} G(s,\xi) q(\xi)\, ds\, d\xi = \int_0^1 H(t,\xi) q(\xi)\, d\xi, \tag{2.4}$$

where

$$H(t,\xi) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} G(s,\xi)\, ds.$$

Now, let us find an explicit form of $H(t,\xi)$. Let $t,\xi \in (0,1)$ and $c=(\alpha-\lambda)\Gamma(\alpha)\Gamma(n-\alpha)$, then

$$cH(t,\xi)=\begin{cases}(\alpha-\lambda+\lambda\xi)(1-\xi)^{\alpha-1}\displaystyle\int_0^t(t-s)^{n-\alpha-1}s^{\alpha-1}\,ds, & 0<t\le\xi<1,\\ (\alpha-\lambda+\lambda\xi)(1-\xi)^{\alpha-1}\displaystyle\int_0^t(t-s)^{n-\alpha-1}s^{\alpha-1}\,ds & \\ \qquad -(\alpha-\lambda)\displaystyle\int_\xi^t(t-s)^{n-\alpha-1}(s-\xi)^{\alpha-1}\,ds, & 0<\xi\le t<1.\end{cases}$$

Using the fact that for each $a,b\ge 0$ and $p,q>0$,

$$\int_a^b(b-\theta)^p(\theta-a)^q\,d\theta=\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\,(b-a)^{p+q+1},$$

we get

$$H(t,\xi)=\frac{1}{(\alpha-\lambda)(n-1)!}\begin{cases}(\alpha-\lambda+\lambda\xi)(1-\xi)^{\alpha-1}t^{n-1}, & 0<t\le\xi<1,\\ (\alpha-\lambda+\lambda\xi)(1-\xi)^{\alpha-1}t^{n-1}-(\alpha-\lambda)(t-\xi)^{n-1}, & 0<\xi\le t<1.\end{cases}\tag{2.5}$$

Thus, by (2.4) and (2.5), we obtain

$$\begin{aligned}(\alpha-\lambda)(n-1)!I^{n-\alpha}w(t)&=\int_0^t\big((1-\xi)^{\alpha-1}(\alpha-\lambda+\lambda\xi)t^{n-1}-(\alpha-\lambda)(t-\xi)^{n-1}\big)q(\xi)\,d\xi\\&\qquad+\int_t^1(1-\xi)^{\alpha-1}(\alpha-\lambda+\lambda\xi)t^{n-1}q(\xi)\,d\xi\\&:=I_1(t)+I_2(t).\end{aligned}$$

From the hypothesis, we deduce that the function $\xi\to q(\xi)$ is continuous and integrable near 0 and the function $\xi\to(1-\xi)^{\alpha-1}q(\xi)$ is continuous and integrable near 1. Hence, I_1 and I_2 are integrable on $(0,1)$. So we get, I_1 and I_2 are differentiable on $(0,1)$ and for each $t\in(0,1)$ we have

$$\begin{aligned}&\frac{d}{dt}((n-1)!(\alpha-\lambda)I^{n-\alpha}w(t))\\&\quad=(n-1)\int_0^t\big((1-\xi)^{\alpha-1}(\alpha-\lambda+\lambda\xi)t^{n-2}-(\alpha-\lambda)(t-\xi)^{n-2}\big)q(\xi)\,d\xi\\&\qquad+(n-1)\int_t^1(1-\xi)^{\alpha-1}(\alpha-\lambda+\lambda\xi)t^{n-2}q(\xi)\,d\xi.\end{aligned}$$

Analogously, using the same arguments as above, we prove that $I^{n-\alpha}w(t)$ is differentiable on $(0,1)$ and for each $t\in(0,1)$ we have

$$\Big(\frac{d}{dt}\Big)^n\big((n-1)!(\alpha-\lambda)I^{n-\alpha}w(t)\big)=-(n-1)!(\alpha-\lambda)q(t).$$

Thus

$$\Big(\frac{d}{dt}\Big)^nI^{n-\alpha}w(t)=-q(t).$$

So, $D^\alpha w(t) + q(t) = 0$ for all $t \in (0,1)$.

Next, let us verify the boundary conditions. Using Proposition 2.5(iii), for each $t \in [0,1]$, we have

$$|w(t)| \leq \eta t^{\alpha-1} \int_0^1 K_1(s)|q(s)|\,ds$$

which implies that $w(0) = 0$.

On the other hand, for each $t \in (0,1)$, we have

$$\begin{aligned}(\alpha-\lambda)\Gamma(\alpha)w(t) &= \int_0^t \big((\alpha-\lambda+\lambda s)(1-s)^{\alpha-1}t^{\alpha-1} - (\alpha-\lambda)(t-s)^{\alpha-1}\big)q(s)\,ds \\ &\quad + \int_t^1 (\alpha-\lambda+\lambda s)t^{\alpha-1}(1-s)^{\alpha-1}q(s)\,ds \\ &:= J_1(t) + J_2(t).\end{aligned} \tag{2.6}$$

It is clear that $\lim\limits_{t\to 0} \frac{|J_1(t)|}{t} = 0$ and $\lim\limits_{t\to 0} \frac{|J_2(t)|}{t} = 0$. Thus $\lim\limits_{t\to 0} \frac{w(t)}{t} = 0$ and hence $w'(0) = 0$. Now, using the fact that J_1 is continuous and integrable near 0 and J_2 is continuous and integrable near 1, we deduce that J_1 and J_2 are differentiable on $(0,1)$ and thus we can take derivatives from both sides of (2.6). So for each $t \in (0,1)$, we have

$$\begin{aligned}(\alpha-\lambda)\Gamma(\alpha)w'(t) &= (\alpha-1)\int_0^t \big((\alpha-\lambda+\lambda s)(1-s)^{\alpha-1}t^{\alpha-2} - (\alpha-\lambda)(t-s)^{\alpha-2}\big)q(s)\,ds \\ &\quad + (\alpha-1)\int_t^1 (\alpha-\lambda+\lambda s)t^{\alpha-2}(1-s)^{\alpha-1}q(s)\,ds \\ &= L_1(t) + L_2(t).\end{aligned}$$

Since $\lim\limits_{t\to 0} \frac{|L_1(t)|}{t} = 0$ and $\lim\limits_{t\to 0} \frac{|L_2(t)|}{t} = 0$, we deduce that $\lim\limits_{t\to 0} \frac{w'(t)}{t} = 0$ and then $w''(0) = 0$.

In a similar way as above, we prove that $w^{(3)}(0) = \cdots = w^{(n-2)}(0) = 0$.

Now, using Fubini's theorem, a simple calculus yields

$$\begin{aligned}(\alpha-\lambda)\Gamma(\alpha)\int_0^1 w(t)\,dt &= \int_0^1 \Big((\alpha-\lambda+\lambda s)(1-s)^{\alpha-1}\int_0^s t^{\alpha-1}\,dt \\ &\quad + \int_s^1 \big((\alpha-\lambda+\lambda s)(1-s)^{\alpha-1}t^{\alpha-1} - (\alpha-\lambda)(t-s)^{\alpha-1}\big)\,dt\Big)q(s)\,ds \\ &= \int_0^1 s(1-s)^{\alpha-1}q(s)\,ds = \frac{(\alpha-\lambda)\Gamma(\alpha)}{\lambda}\int_0^1 G(1,s)q(s)\,ds,\end{aligned}$$

which implies that $w(1) = \lambda \int\limits_0^1 w(t)\,dt$.

Finally, let us prove the uniqueness of the solution. Suppose w_1 and w_2 are two continuous solutions on $[0,1]$ of the boundary value problem (2.3). Then we have $D^\alpha(w_2(t) - w_1(t)) = 0$ on $(0,1)$. Thus, by Lemma 2.3(iii), there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$w_2(t) - w_1(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}.$$

Using the boundary conditions, we find $c_n = \cdots = c_2 = 0$. So we get

$$w_2(t) - w_1(t) = c_1 t^{\alpha-1}. \tag{2.7}$$

On the other hand, using (2.7), we get

$$w_2(1) - w_1(1) = \lambda \int_0^1 w_2(t) - w_1(t)\, dt = \frac{\lambda}{\alpha} c_1.$$

This implies that $c_1 = 0$. Then $w_1 = w_2$. □

In the proofs of our main results we shall use the Guo–Krasnosel'skii fixed point theorem presented below.

Lemma 2.8 ([7])**.** *Let P be the cone of a real Banach space E and let Ω_1, Ω_2 be two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$, *and* $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$,

or

(ii) $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$, *and* $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$,

hold. Then the operator T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $E = \mathcal{C}([0,1])$, the Banach space endowed with the supremum norm $\|u\| = \sup\limits_{t\in[0,1]} |u(t)|$. Let $\theta \in [0, \frac{1}{2})$, and set $J_\theta = [\theta, 1-\theta]$. For a function $b : (0,1) \to (0,+\infty)$, we denote

$$\sigma_b^\theta = \int_\theta^{1-\theta} b(t)K(t)\, dt.$$

Next, define the cone

$$\Omega = \left\{u \in E : \ u(t) \geq 0 \text{ on } [0,1],\ u(t) \geq \lambda^* t^{\alpha-1}\|u\|\right\},$$

and for $r > 0$, let

$$\Omega_r = \left\{u \in \Omega : \ \|x\| < r\right\}.$$

In the rest of the paper, we suppose that the following assumptions hold:

(H_1) $q : (0,1) \to [0,+\infty)$ and $0 < \sigma < \infty$, where $\sigma = \int_0^1 q(t)K_1(t)\, dt$.

(H_2) $a \in \mathcal{C}((0,1), [0+\infty))$ and $0 < \sigma_a^0 < \infty$.

(H_3) $f \in \mathcal{C}([0,1] \times [0,+\infty), [0,+\infty))$.

(H_4) There exists $t_0 \in (0,1)$ such that $f(t_0, u) > 0$ for each $u \in (0,+\infty)$.

Remark. We note that (H_1) implies $0 < \sigma_q^0 < \infty$.

In this work we are concerned with a positive solution of problem (1.1). By a positive solution we mean a function $u \in \mathcal{C}([0,1])$ satisfying (1.1) with $u(t) \geq 0$ for all $t \in [0,1]$ and $u(t) > 0$ for all $t \in (0,1]$.

Now, we introduce the following intermediary boundary value problem

$$\begin{cases} D^\alpha x(t) + \mu a(t) f\big(t, [x(t) - w(t)]^*\big) + q(t) = 0 \ \text{ in } \ (0,1), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \lambda \int\limits_0^1 x(s)\, ds, \end{cases} \tag{2.8}$$

$$\begin{cases} D^\alpha w(t) + 2q(t) = 0 \ \text{ in } \ (0,1), \\ w(0) = w'(0) = \cdots = w^{(n-2)}(0) = 0, \quad w(1) = \lambda \int\limits_0^1 w(s)\, ds, \end{cases} \tag{2.9}$$

where $[x(t) - w(t)]^* = \max\{x(t) - w(t), 0\}$ for each $t \in [0,1]$ and w is the unique solution of problem (2.9) given by $w(t) = 2\int\limits_0^1 G(t,s) q(s)\, ds$.

By Lemma 2.7, the solution w of problem (2.9) satisfies

$$w(t) \le 2\eta\sigma t^{\alpha-1} \ \ \forall\, t \in [0,1]. \tag{2.10}$$

We shall prove that there exists a solution $x(t)$ for the boundary value problem (2.8) with $x(t) \ge w(t)$ for any $t \in [0,1]$ and $x(t) > w(t)$ for any $t \in (0,1)$. In this case, $x(t) - w(t)$ represents a positive solution of the boundary value problem (1.1).

Next, we define the operator $T : E \to E$ as follows:

$$Tx(t) = \int\limits_0^1 G(t,s) \Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\, ds \ \ \forall\, t \in [0,1]. \tag{2.11}$$

Lemma 2.9. *Suppose that* (H_1)–(H_4) *hold. Then* $x \in \mathcal{C}([0,1])$ *is a solution of the boundary value problem* (2.8) *if and only if* $x \in \mathcal{C}([0,1])$ *is a solution of the integral equation*

$$x(t) = \int\limits_0^1 G(t,s) \Big(\mu a(s) f(s, [x(s) - w(s)]^*) + q(s)\Big)\, ds.$$

That is, x is a fixed point of the operator T defined by (2.11).

Proof. The proof is immediate from Lemma 2.4, so we omit it here. □

Lemma 2.10. *Suppose that* (H_1)–(H_4) *hold. Then* $T : \Omega \to \Omega$ *is completely continuous.*

Proof. Since G, f are nonnegative continuous functions, using (H_1), (H_2) we conclude that $T : \Omega \to E$ is continuous. Let $x \in \Omega$, then by Proposition 2.5(iv), for all $t \in [0,1]$, it follows that

$$\begin{aligned} Tx(t) &\ge \eta\lambda^* t^{\alpha-1} \int\limits_0^1 K(s) \Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\, ds \\ &\ge \lambda^* t^{\alpha-1} \int\limits_0^1 G(\tau,s) \Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\, ds \ \ \forall\, \tau \in [0,1]. \end{aligned}$$

So, for each $t \in [0,1]$, we have

$$\begin{aligned} Tx(t) &\ge \lambda^* t^{\alpha-1} \max_{\tau \in [0,1]} \Big\{ \int\limits_0^1 G(\tau,s) \Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\, ds \Big\} \\ &= \lambda^* t^{\alpha-1} \|Tx\|. \end{aligned}$$

Then $T(\Omega) \subset \Omega$. Now, let S be a bounded set of Ω, then there exists a positive constant $M > 0$ such that $\|x\| \leq M$ for all $x \in S$. Therefore, $[x(s) - w(s)]^* \leq \|x\| \leq M$.

Let $M_1 := \max\big\{1, \max\limits_{t\in[0,1],\, x\in[0,M]} f(t,x)\big\}$.

From hypotheses (H_1), (H_2) and Proposition 2.5(ii), for all $t \in [0,1]$ and for all $x \in S$, we have

$$Tx(t) \leq \eta \int_0^1 K(s)\Big(\mu a(s) f\big(s, [x(s) - w(s)]^*\big) + q(s)\Big)\, ds \leq M_1 \eta(\mu\sigma_a^0 + \sigma_q^0).$$

So we obtain $\|Tx\| \leq M_1\eta(\mu\sigma_a^0 + \sigma_q^0)$. Hence, $T(S)$ is uniformly bounded.

Now, let us prove that $T(S)$ is equicontinuous on $[0,1]$.

Using Proposition 2.5, we obtain that G is uniformly continuous on $[0,1] \times [0,1]$. Then for $t_1, t_2 \in [0,1]$ and for all $s \in [0,1]$, we get

$$|G(t_2,s) - G(t_1,s)| \to 0 \ \text{ as } \ t_2 \to t_1$$

and

$$|G(t_2,s) - G(t_1,s)| \leq 2\eta M_1(a(s)K(s) + q(s)K(s)).$$

By (H_1) and (H_2), $2\eta M_1(a(s)K(s) + q(s)K(s))$ is a nonnegative integrable function on $(0,1)$. Thus by the Lebesgue control convergence theorem, we obtain

$$|Tx(t_2) - Tx(t_1)| \to 0 \ \text{ as } \ |t_2 - t_2| \to 0,$$

and so $T(S)$ is equicontinuous. Consequently, by Ascoli's theorem, we conclude that $T(S)$ is relatively compact in E. Hence, $T : \Omega \to \Omega$ is completely continuous. This completes the proof. $\square$

3 Main results

We shall give the existence results of positive solutions for the nonlinear boundary value problem (1.1).

Theorem 3.1. *Suppose that conditions* (H_1)–(H_4) *hold. In addition, suppose that there exists* $\theta \in (0, \frac{1}{2})$ *such that*

$$f_\infty := \lim_{x\to\infty} \Big\{ \min_{t\in J_\theta} \frac{f(t,x)}{x} \Big\} = \infty.$$

Then there exists $\mu^* > 0$ *such that for every* $0 < \mu < \mu^*$, *problem* (1.1) *has at least one positive solution.*

Proof. Choose

$$r > \frac{2\eta\sigma}{\lambda^*}\,.$$

Define $\mu^* = \frac{r - 2\eta\sigma_q^0}{M\eta\sigma_a^0}$, where $M = \max\limits_{t\in[0,1],\, x\in[0,r]} f(t,x)$, and let $0 < \mu < \mu^*$.

Then for each $x \in \partial\Omega_r$ and $s \in [0,1]$, we have

$$[x(s) - w(s)]^* \leq x(s) \leq \|x\| = r.$$

Therefore, by Proposition 2.5(ii), for any $x \in \partial\Omega_r$, we have

$$T(x)(t) \leq \eta\mu \int_0^1 K(s)a(s) f\big(s, [x(s) - w(s)]^*\big)\, ds + 2\eta\sigma_q^0 \leq \mu\eta M\sigma_a^0 + 2\eta\sigma_q^0 \leq \mu^*\eta M\sigma_a^0 + 2\eta\sigma_q^0 = r.$$

So we get

$$\|Tx\| \leq \|x\| \ \text{ for } \ x \in \partial\Omega_r. \tag{3.1}$$

Now, if the condition $f_\infty = \infty$ holds, then for $A = \frac{2}{\mu\gamma\lambda^*\sigma_a^\theta\theta^{\alpha-1}}$, there exists $B > 0$ such that $f(t,x) \geq Ax \; \forall\, t \in J_\theta, \forall\, x \geq B$.

Define $R = \max\{2r, \frac{2B}{\lambda^*\theta^{\alpha-1}}\}$. Then, using (2.10), for any $x \in \partial\Omega_R$ and $t \in [0,1]$, we obtain

$$x(t) - w(t) \geq x(t) - 2\eta\sigma t^{\alpha-1} \geq x(t) - 2\eta\sigma \,\frac{x(t)}{\|x\|} \geq x(t)\Big(1 - \frac{2\eta\sigma}{\lambda^* R}\Big) \geq \frac{1}{2}\,x(t) \geq 0.$$

Therefore, we conclude that for all $t \in J_\theta$,

$$[x(t) - w(t)]^* \geq \frac{\lambda^*}{2}\,Rt^{\alpha-1} \geq \frac{\lambda^*}{2}\,R\theta^{\alpha-1} \geq B,$$

and so for any $x \in \partial\Omega_R$ and $t \in J_\theta$, we have

$$f\big(t, [x(t) - w(t)]^*\big) \geq A[x(t) - w(t)]^* \geq \frac{A}{2}\,x(t). \tag{3.2}$$

By (3.2) and Proposition 2.5(v), it follows that for any $x \in \partial\Omega_R$ and $t \in J_\theta$,

$$Tx(t) \geq \mu\gamma \int_\theta^{1-\theta} K(s)a(s)f\big(s, [x(s) - w(s)]^*\big)\,ds \geq \frac{\mu\gamma\lambda^*}{2}\,\sigma_a^\theta\theta^{\alpha-1}AR = R.$$

Then we have

$$\|Tx\| \geq \|x\| \;\; \forall x \in \partial\Omega_R. \tag{3.3}$$

Thus, using (3.1) and (3.3), we deduce by Lemma 2.8 that the operator T has a fixed point in $\overline{\Omega_R}\setminus\Omega_r$. Therefore, by Lemma 2.9, x is a nonnegative continuous solution of problem (2.8) satisfying

$$r < \|x\| \leq R. \tag{3.4}$$

So we deduce that $x - w$ is a nonnegative continuous solution of problem (1.1).

Now, let us prove that $x - w$ is a positive solution of (1.1), that is, $x(t) - w(t) > 0$ for all $t \in (0,1]$. Since x satisfies (3.4), using (2.10) we obtain

$$x(t) - w(t) \geq t^{\alpha-1}(\lambda^* r - 2\eta\sigma) > 0 \;\; \forall\, t \in (0,1].$$

Hence, $x - w$ is a positive solution of problem (1.1). This completes the proof. □

Theorem 3.2. *Suppose that conditions* (H$_1$)–(H$_4$) *hold. In addition, assume that the following assertions hold:*

(A$_1$) *there exits* $\theta \in (0, \frac{1}{2})$ *such that* $f^*_\infty := \lim\limits_{x\to\infty} \big\{\min\limits_{t\in J_\theta} f(t,x)\big\} = \infty$;

(A$_2$) $f^\infty := \lim\limits_{x\to\infty} \big\{\max\limits_{t\in[0,1]} \frac{f(t,x)}{x}\big\} = 0$.

Then there exists $\mu^* > 0$ *such that problem* (1.1) *has at least one positive solution for every* $\mu > \mu^*$.

Proof. First, suppose that (A$_1$) holds, then there exists $R_0 > 0$ such that

$$f(t,x) \geq \frac{f^*_\infty}{2} \;\; \forall\, t \in J_\theta, \;\; \forall\, x \geq R_0.$$

Now, fix $R_1 > \max\big\{\frac{2R_0}{\lambda^*\theta^{\alpha-1}}, \frac{4\eta\sigma}{\lambda^*}\big\}$. Define $\mu^* = \frac{2R_1}{\gamma\sigma_a^\theta f^*_\infty} > 0$ and let $\mu > \mu^*$. Then, for each $x \in \partial\Omega_{R_1}$ and $t \in [0,1]$, we have

$$x(t) - w(t) \geq x(t) - 2\eta\sigma t^{\alpha-1} \geq x(t) - \frac{2\eta}{\lambda^*}\,\sigma\,\frac{x(t)}{\|x\|} \geq x(t)\Big(1 - \frac{2\eta\sigma}{\lambda^* R_1}\Big) \geq \frac{1}{2}\,x(t) \geq 0.$$

So, for $x \in \partial\Omega_{R_1}$ and $t \in J_\theta$, we get

$$[x(t) - w(t)]^* \geq \frac{1}{2}\, x(t) \geq \frac{1}{2}\, \lambda^* \theta^{\alpha-1} R_1 > R_0.$$

Then for any $x \in \partial\Omega_{R_1}$ and $t \in J_\theta$, we obtain

$$f\big(t, [x(t) - w(t)]^*\big) \geq \frac{f_\infty^*}{2}\,.$$

It follows that for any $x \in \partial\Omega_{R_1}$ and $t \in J_\theta$,

$$Tx(t) \geq \mu\gamma \int\limits_{\theta}^{1-\theta} K(s)a(s)f\big(s, [x(s) - w(s)]^*\big)\, ds \geq \mu\gamma\, \frac{f_\infty^*}{2} \int\limits_{\theta}^{1-\theta} K(s)a(s)\, ds \geq \mu^*\gamma\, \frac{f_\infty^*}{2}\, \sigma_a^\theta = R_1.$$

Thus

$$\|Tx\| \geq \|x\| \;\; \forall\, x \in \partial\Omega_{R_1}.$$

On the other hand, since $f^\infty = 0$, for $\varepsilon = \frac{1}{\mu\eta\sigma_a^0} > 0$, there exists $B > 0$ such that for each $t \in [0,1]$, $x \geq B$, we have $f(t,x) \leq \varepsilon x$. Therefore, we obtain

$$f(t,x) \leq M + \varepsilon x \;\; \forall\, t \in [0,1], \;\; \forall\, x \geq 0,$$

where $M = \max\limits_{t\in[0,1],\; x\in[0,B]} f(t,x)$. Let $M_1 = \max\{1, M\}$ and choose

$$R_2 > \max\left\{2R_1, \mu\eta\sigma_a^0 M_1 \Big(\frac{1}{2} - \mu\sigma_a^0\eta\varepsilon\Big)^{-1}, 2\eta M_1 \sigma_q^0\right\}.$$

It follows that for any $x \in \partial\Omega_{R_2}$ and $t \in [0,1]$,

$$\begin{aligned} Tx(t) &\leq \mu\eta \int\limits_0^1 K(s)a(s)f\big(s, [x(s) - w(s)]^*\big)\, ds + \eta\sigma_q^0 \\ &\leq \mu\eta M\sigma_a^0 + \mu\eta\varepsilon \int\limits_0^1 K(s)a(s)[x(s) - x(s)]^*\, ds + \eta\sigma_q^0 \leq \mu\eta M_1\sigma_a^0 + \mu\eta\sigma_a^0\varepsilon R_2 + \eta M_1 \sigma_q^0 \\ &\leq R_2\Big(\frac{1}{2} - \mu\sigma_a^0\eta\varepsilon\Big) + \mu\eta\sigma_a^0\varepsilon R_2 + \eta M_1\sigma_q^0 = \|x\|. \end{aligned}$$

So, we get

$$\|Tx\| \leq \|x\| \;\; \forall\, x \in \partial\Omega_{R_2}.$$

Thus, by Lemma 2.8, we deduce that the operator T has a fixed point in $\overline{\Omega_{R_2}} \setminus \Omega_{R_1}$. Therefore, by Lemma 2.9, x is a solution of problem (2.8). Thus, we deduce that $x - w$ is a nonnegative solution of problem (1.1).

The positivity of the solution is shown as in the proof of the previous theorem. □

Now we state the multiple existence result.

Theorem 3.3. *Assume that $\mu = 1$ and* (H$_1$)–(H$_4$) *hold. In addition, suppose that the following conditions are satisfied:*

(A$_1$) *there exists $R_1 > \frac{4\eta\sigma}{\lambda^*}$ such that $f(t,x) \leq \frac{R_1 - \eta\sigma_q^0}{\eta\sigma_a^0}$ $\forall\, t \in [0,1]$, $x \in [0, R_1]$;*

(A$_2$) *there exists $\theta \in (0, \frac{1}{2})$ such that the following assertion holds: $\exists\, R_2 > 2R_1 :\;\; \gamma\sigma_a^\theta f(t,x) \geq R_2$ $\forall\, t \in J_\theta$, $\forall\, x \in [\frac{3}{4}\lambda^*\theta^{\alpha-1}R_2, R_2]$;*

(A$_3$) $f^\infty = \lim\limits_{x\to\infty} \big\{ \max\limits_{t\in[0,1]} \frac{f(t,x)}{x} \big\} = 0.$

Then problem (1.1) *has two positive solutions.*

Proof. First, suppose that condition (A_1) holds, then for each $x \in \partial\Omega_{R_1}$ and $t \in [0,1]$, we have

$$[x(s)-w(s)]^* \leq x(s) \leq R_1 \ \text{ and } \ [x(s)-w(s)]^* \geq \frac{1}{2}\,x(s) \geq 0.$$

So, for each $x \in \partial\Omega_{R_1}$ and $t \in [0,1]$,

$$f\big(t,[x(t)-w(t)]^*\big) \leq \frac{R_1-\eta\sigma_q^0}{\eta\sigma_a^0}\,.$$

Therefore, for any $x \in \partial\Omega_{R_1}$ and $t \in [0,1]$, we get

$$Tx(t) \leq \eta\int\limits_0^1 K(s)a(s)f\big(s,[x(s)-w(s)]^*\big)\,ds + \eta\sigma_q^0 \leq \eta\sigma_a^0\Big(\frac{R_1-\eta\sigma_q^0}{\eta\sigma_a^0}\Big) + \eta\sigma_q^0 = \|x\|.$$

Thus, we have

$$\|Tx\| \leq \|x\| \ \ \forall\, x \in \partial\Omega_{R_1}. \tag{3.5}$$

On the other hand, if (A_2) holds, it follows that for $R_2 > 2R_1$ and $x \in \partial\Omega_{R_2}$, $t \in [0,1]$,

$$x(t)-w(t) \geq \lambda^* t^{\alpha-1}R_2 - 2\eta\sigma t^{\alpha-1} \geq \lambda^* t^{\alpha-1}R_2 - \frac{1}{2}\,\lambda^* t^{\alpha-1}R_1 \geq \frac{3\lambda^*}{4}\,t^{\alpha-1}R_2.$$

Thus, for all $x \in \partial\Omega_{R_2}$ and $t \in J_\theta$, we have

$$x(t)-w(t) \geq \frac{3}{4}\,\lambda^*\theta^{\alpha-1}R_2.$$

Therefore, for all $x \in \partial\Omega_{R_2}$ and $t \in J_\theta$, we get

$$\gamma\sigma_a^\theta f\big(s,[x(s)-w(s)]\big) \geq R_2.$$

So, for any $x \in \partial\Omega_{R_2}$ and $t \in J_\theta$, we obtain

$$Tx(t) \geq \gamma\int\limits_\theta^{1-\theta} K(s)a(s)f\big(s,[x(s)-w(s)]^*\big)\,ds \geq \gamma\sigma_a^\theta\,\frac{R_2}{\gamma\sigma_a^\theta} = R_2.$$

Thus,

$$\|Tx\| \geq \|x\| \ \ \forall\, x \in \partial\Omega_{R_2}. \tag{3.6}$$

Now, hypothesis (A_3) implies that for $\varepsilon = \frac{1}{\eta\sigma_a^0}$, there exists $B > 0$ such that $f(t,x) \leq \varepsilon x \ \forall\, x \geq B$. Therefore, we obtain

$$f(t,x) \leq M + \varepsilon x \ \ \forall\, t \in [0,1], \ \ x \geq 0,$$

where $M = \max\limits_{t\in[0,1],\ x\in[0,B]} f(t,x)$. Put $M_1 = \max\{1, M\}$ and choose

$$R_3 > \max\Big\{2R_2, \eta\sigma_a^0 M_1\Big(\frac{1}{2}-\sigma_a^0\eta\varepsilon\Big)^{-1}, 2\eta M_1\sigma_q^0\Big\}.$$

Then for any $x \in \partial\Omega_{R_3}$ and $t \in [0,1]$, we have

$$\begin{aligned}
Tx(t) &\leq \eta\int\limits_0^1 K(s)a(s)f\big(s,[x(s)-w(s)]^*\big)\,ds + \eta\sigma_q^0 \\
&\leq \eta M\sigma_a^0 + \mu\eta\varepsilon\int\limits_0^1 K(s)a(s)[x(s)-x(s)]^*\,ds + \eta\sigma_q^0 \leq \eta M_1\sigma_a^0 + \eta\sigma_a^0\varepsilon R_3 + \eta M_1\sigma_q^0 \\
&\leq R_3\Big(\frac{1}{2}-\sigma_a^0\eta\varepsilon\Big) + \eta\sigma_a^0\varepsilon R_3 + \eta M_1\sigma_q^0 = \|x\|.
\end{aligned}$$

So, we get

$$\|Tx\| \le \|x\| \ \ \forall x \in \partial\Omega_{R_3}. \tag{3.7}$$

Therefore, due to Lemma 2.8 and using (3.5), (3.6) and (3.7), we deduce that the operator T has two fixed points x_1 and x_2, respectively, in $\overline{\Omega_{R_2}} \setminus \Omega_{R_1}$ and $\overline{\Omega_{R_3}} \setminus \Omega_{R_2}$. Therefore, by Lemma 2.9, problem (2.8) admits two nonnegative solutions $R_1 < \|x_1\| < R_2 < \|x_2\| < R_3$. Thus, problem (1.1) has two nonnegative solutions. The positivity of the solutions is shown in the same manner as in proving Theorem 3.1. □

4 Examples

In this section, we present some examples illustrating our results. We remark that by the following examples it can immediately be verified that conditions (H_1)–(H_4) hold.

Example 4.1. We consider the following nonlinear fractional differential equations

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu\,\dfrac{1}{t}\,(u(t))^2 - \dfrac{1}{1-t} = 0 \ \text{ in } \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)\,ds. \end{cases} \tag{4.1}$$

Let $f(t,u) = u^2$, $a(t) = \frac{1}{t}$, $\lambda = 1$ and $q(t) = \frac{1}{1-t}$. By a direct calculation, we obtain $f_\infty = \infty$ for any $\theta \in (0, \frac{1}{2})$. We also get $\sigma_a^0 \approx 0.3009$, $\sigma_q^0 \approx 0.2006$ and $\sigma = 0.5015$. Choose $r = 5$, then by a simple calculation we get $\mu^* = 0.34547$. Then by Theorem 3.1, problem (4.1) has at least one positive solution for every $0 < \mu < 0.34547$.

Example 4.2. Consider the following boundary value problem

$$\begin{cases} D^{\frac{7}{3}}u(t) + \mu\,\dfrac{1}{t}\left(100 + \dfrac{1}{1+\sqrt{u}}\right) - \dfrac{1}{1-t} = 0 \ \text{ in } \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{4.2}$$

Let $f(t,u) = 100 + \frac{1}{\sqrt{u}+1}$, $a(t) = \frac{1}{t}$ and $q(t) = \frac{1}{1-t}$. By a direct calculation, we obtain $f^\infty = 0$ and for $\theta = \frac{1}{4}$ we have $f^*_\infty = 100$. We also obtain $\sigma_a^0 \approx 0.35995$, $\sigma_q^0 \approx 0.26996$, $\sigma \approx 0.62991$ and $\sigma_a^\theta \approx 0.16979$. Choose $R_1 = 50$ and $R_2 = 102$. A simple calculation yields $\mu^* = 39.889$. So Theorem 3.2 ensures the existence of a solution of problem (4.2) such that $50 < \|u + w\| < 102$ for every $\mu > 39.889$.

Example 4.3. Consider the following boundary value problem:

$$\begin{cases} D^{\frac{7}{3}}u(t) + \mu\,\dfrac{1}{t}\,f(t,u) - \dfrac{1}{1-t} = 0 \ \text{ in } \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{4.3}$$

where

$$f(t,u) = \begin{cases} \dfrac{1}{3}\,u, & 0 \le u \le 12, \\ 10000u - 119996, & 12 < u \le 13.78, \\ u + 17790.3, & 13.78 < u \le 50, \\ 2523u^{\frac{1}{2}}, & u > 50. \end{cases}$$

Then problem (4.3) admits two positive solutions. In fact, let $a(t) = \frac{1}{t}$ and $q(t) = \frac{1}{1-t}$. By a direct calculation, we get $\sigma_a^0 \approx 0.35995$, $\sigma_q^0 \approx 0.26996$ and $\sigma \approx 0.62991$. Choose $R_1 = 12 > \frac{4\eta\sigma}{\lambda^*}$, then for any $t \in [0,1]$, $u \in [0,12]$, $f(t,u) \leq \frac{R_1 - \eta\sigma_q^0}{\eta\sigma_a^0} \approx 18.3$. Thus condition ($A_1$) is satisfied. On the other hand, for $\theta = \frac{1}{4}$, we have $\sigma_a^\theta \approx 0.16979$. Take $R_2 = 50$, then $R_2 > 2R_1$ and for any $t \in J_\theta$ and for all $u \in [\frac{3}{4}\,\lambda^*\theta^{\alpha-1}R_2, R_2]$, we have $f(t,u) \geq \frac{R_2}{\gamma\sigma_a^\theta} \approx 1994.5$ which implies that condition (A_2) is satisfied.

Finally, since $f^\infty = 0$, the assertion (A_3) is satisfied. Consequently, by Theorem 3.3, problem (4.3) admits two positive solutions u_1 and u_2 satisfying

$$R_1 \leq \|u_1 + w\| \leq R_2 \leq \|u_2 + w\| \leq R_3.$$

Acknowledgement

The authors want to thank the referees for their careful reading of the paper and constructive suggestions and comments.

References

[1] Zh. Bai, Existence of solutions for some third-order boundary-value problems. *Electron. J. Differential Equations* **2008**, No. 25, 6 pp.

[2] R. Bourguiba, F. Toumi and O. K. Wannassi, Existence and nonexistence results for a system of integral boundary value problems with parametric dependence. *Journal of Dynamics and Differential Equations* (submitted).

[3] A. Cabada and Z. Hamdi, Multiplicity results for integral boundary value problems of fractional order with parametric dependence. *Fract. Calc. Appl. Anal.* **18** (2015), no. 1, 223–237.

[4] J. Henderson and R. Luca, Positive solutions for a system of nonlocal fractional boundary value problems. *Fract. Calc. Appl. Anal.* **16** (2013), no. 4, 985–1008.

[5] J. Henderson and R. Luca, Positive solutions for a system of semipositone coupled fractional boundary value problems. *Bound. Value Probl.* **2016**, Paper No. 61, 23 pp.

[6] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[7] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*. Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron P. Noordhoff Ltd. Groningen, 1964.

[8] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*. Edited and with a foreword by S. M. Nikol'skiĭ. Translated from the 1987 Russian original. Revised by the authors. Gordon and Breach Science Publishers, Yverdon, 1993.

[9] W. R. Schneider, Fractional diffusion. In: *Dynamics and Stochastic Processes (Lisbon, 1988)*, 276–286, Lecture Notes in Phys., 355, Springer, New York, 1990.

[10] W. R. Schneider and W. Wyss, Fractional diffusion and wave equations. *J. Math. Phys.* **30** (1989), no. 1, 134–144.

[11] F. Toumi, Existence and nonexistence of positive solutions for a system of nonlinear singular fractional differential equations. *Fract. Differ. Calc.* **5** (2015), no. 2, 183–198.

[12] G. Wang, B. Ahmad and L. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order. *Nonlinear Anal.* **74** (2011), no. 3, 792–804.

[13] X. Zhang, L. Liu and Y. Wu, Multiple positive solutions of a singular fractional differential equation with negatively perturbed term. *Math. Comput. Modelling* **55** (2012), no. 3-4, 1263–1274.

[14] W.-X. Zhou, J.-G. Peng and Y.-D. Chu, Multiple positive solutions for nonlinear semipositone fractional differential equations. *Discrete Dyn. Nat. Soc.* **2012**, Art. ID 850871, 10 pp.

(Received 29.09.2017)

Authors' addresses:

Rim Bourguiba
University of Monastir, Faculty of Sciences of Monastir, UR11ES53, 5019 Monastir, Tunisia.
E-mail: Rym.Bourguiba@fsm.rnu.tn

Faten Toumi
1. University of Monastir, Faculty of Sciences of Monastir, UR11ES53, 5019 Monastir, Tunisia.
2. University of Kairouan, Preparatory Institute for Engineering Studies of Kairouan, 3100 Kairouan, Tunisia.
E-mail: Faten.Toumi@fsb.rnu.tn