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THE SHARPNESS OF SOME EXISTENCE STATEMENTS FOR DIFFERENTIAL SYSTEMS WITH NONLOCAL BOUNDARY CONDITIONS

Abstract. Recently, some extensions of results of M. A. Krasnosel'skii and Gustafson–Schmitt for systems of the type $x' = f(t,x)$ with periodic boundary conditions $x(0) = x(1)$ have been obtained for nonlocal boundary conditions of the type $x(1) = \int\limits_0^1 dh(s)\,x(s)$ or $x(0) = \int\limits_0^1 dh(s)\,x(s)$, where h is a real non-decreasing function satisfying some conditions, and containing the periodic boundary conditions as special cases. The situations with periodic and nonlocal boundary conditions are compared through the use of counterexamples, exhibiting the special character of the periodic case. Similar counterexamples also show, in the case of second order systems with some nonlocal boundary conditions, that the sense of some inequalities in the assumptions cannot be reversed.*

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რეზიუმე. პერიოდული ამოცანის $x' = f(t,x)$, $x(0) = x(1)$ ამოხსნადობის კარგად ცნობილი კრასნოსელსკის და გუსტავსონ-შმიტის შედეგები განზოგადებული იქნა $x(1) = \int\limits_0^1 dh(s)\,x(s)$ და $x(0) = \int\limits_0^1 dh(s)\,x(s)$ სახის არალოკალური სასაზღვრო პირობების შემთხვევაში (h ნამდვილი არაკლებადი ფუნქციაა), რომლებიც მოიცავენ პერიოდულ სასაზღვრო პირობებს, როგორც კერძო შემთხვევებს. კონტრმაგალითების საშუალებით შედარებულია პერიოდული და არალოკალური სასაზღვრო პირობების შემთხვევები, რომლებიც ავლენენ პერიოდული შემთხვევის სპეციალურ ხასიათს. ანალოგიური კონტრმაგალითები აგრეთვე გვიჩვენებენ, რომ მეორე რიგის სისტემების შემთხვევაში გარკვეული არალოკალური სასაზღვრო პირობებით ზოგიერთი უტოლობის ნიშნის შებრუნება არ შეიძლება.

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1 Introduction

Let $\langle \cdot \mid \cdot \rangle$ denote the usual inner product in $\mathbb{R}^n$, $|\cdot|$ the corresponding Euclidian norm, and $B_R \subset \mathbb{R}^n$ the open ball of center 0 and radius R. Throughout the paper, *let* $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous.*

Let us first consider the *periodic boundary value problem*

$$x' = f(t,x), \quad x(0) = x(1). \tag{1.1}$$

A classical existence theorem for problem (1.1), more than fifty years old, is the following one.

Existence theorem. *If there exists* $R > 0$ *such that either*

$$\langle u \mid f(t,u)\rangle \geq 0 \ \ \forall\,(t,u) \in [0,1] \times \partial B_R,$$

or

$$\langle u \mid f(t,u)\rangle \leq 0 \ \ \forall\,(t,u) \in [0,1] \times \partial B_R,$$

then problem (1.1) *has at least one solution such that* $x([0,1]) \subset \overline{B}_R$.

The two results are indeed equivalent, the second one being deduced from the first one through the change of variable $\tau = 1 - t$. They are a nonlinear counterpart to the elementary result that, for each $e \in C([0,1], \mathbb{R}^n)$ and each $\lambda \in \mathbb{R} \setminus \{0\}$, the problem

$$x' = \lambda x + e(t), \quad x(0) = x(1)$$

has a solution, a consequence of the fact that 0 is the unique real eigenvalue of the operator $\frac{d}{dt}$ with periodic boundary conditions.

Although the existence theorem above is a special case of a result given by M. A. Krasnosel'skii in 1966 ([6, Theorem 3.2]), and was surely known to him, its explicit statement is not contained in [6], and we did not find an earlier reference. One can just mention that in 1965, F. E. Browder [1] proved the existence of a solution of (1.1) with $\mathbb{R}^n$ replaced by an arbitrary real Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$ when $f : [0,1] \times H \to H$ is continuous, $-f(t,\cdot)$ is monotone for each $t \in [0,1]$ and there exists $R > 0$ such that $\langle u \mid f(t,u)\rangle < 0$ for $(t,u) \in [0,1] \times \partial B_R$.

Krasnosel'skii's theorem. *If there exists a bounded open convex set* $C \subset \mathbb{R}^n$*, and functions* $\Phi_i \in C^1(\mathbb{R}^n, \mathbb{R})$ $(i = 1, \dots, r)$ *such that* $\overline{C} = \{u \in \mathbb{R}^n : \ \Phi_i(u) \leq 0 \ (i = 1, \dots, r)\}$, $\nabla\Phi_i(u) \neq 0$ *when* $\Phi_i(u) = 0$ *for some* $u \in \partial C$*, and either*

$$\big\langle \nabla\Phi_i(u) \mid f(t,u)\big\rangle \geq 0 \ \ \forall\,(t,u) \in [0,1] \times \partial C \ \ \textit{and} \ \ \forall\, i \in \alpha(u),$$

or

$$\big\langle \nabla\Phi_i(u) \mid f(t,u)\big\rangle \leq 0 \ \ \forall\,(t,u) \in [0,1] \times \partial C \ \ \textit{and} \ \ \forall\, i \in \alpha(u),$$

where $\alpha(u) := \{i \in \{1, \dots, r\} : \ \Phi_i(u) = 0\}$*, then problem* (1.1) *has at least one solution such that* $x([0,1]) \subset \overline{C}$.

The existence theorem above corresponds to the choice of $C = B_R$, $r = 1$ and $\Phi_1(u) = \frac{1}{2}\,(|u|^2 - R^2)$. A more direct proof of Krasnosel'skii's theorem based upon coincidence degree arguments has been given in 1974 in [7, Corollary 3.2].

Now, if $C \subset \mathbb{R}^n$ is an open convex neighborhood of $0 \in \mathbb{R}^n$, then, for each $u \in \partial C$, there exists some $\nu(u) \in \mathbb{R}^n \setminus \{0\}$ such that $\langle \nu(u) \mid u\rangle > 0$ and $C \subset \{v \in \mathbb{R}^n : \ \langle \nu(u) \mid v - u\rangle < 0\}$. $\nu(u)$ is called an *outer normal* to ∂C at u, and $\nu : \partial C \to \mathbb{R}^n \setminus \{0\}$ an *outer normal field* on ∂C. Notice that ν needs not to be continuous. The second condition means that $\nu(u)$ is orthogonal to a *supporting hyperplane* of C at u [2,5]. In 1974, using arguments similar to those of [7], Gustafson and Schmitt [4] introduced the following elegant existence condition.

Gustafson–Schmitt's theorem. *If there exists a bounded convex open neighborhood* C *of* 0 *in* $\mathbb{R}^n$*, and an outer normal field* ν *on* ∂C *such that either*

$$\big\langle \nu(u) \mid f(t,u)\big\rangle > 0 \ \ \forall\,(t,u) \in [0,1] \times \partial C, \tag{1.2}$$

or

$$\big\langle \nu(u) \mid f(t,u) \big\rangle < 0 \;\; \forall\, (t,u) \in [0,1] \times \partial C, \tag{1.3}$$

then problem (1.1) *has at least one solution such that* $x([0,1]) \in C$.

Notice that the monograph [6] is not quoted in [4], and that the special case where $C = B_R$ is explicitly stated there. The relation between [6] and [4] was explicited in [7,8], where it was also shown that inequalities need not to be strict in Gustafson–Schmitt's assumptions (1.2), (1.3) if one replaces C by $\overline{C}$ in the conclusion. See also [3] for further generalizations. Krasnosel'skii's theorem follows from extended Gustafson–Schmitt's condition because if, without loss of generality, we assume that $0 \in C$ in Krasnosel'skii's statement, then, for $u \in \partial C$ and $i \in \alpha(u)$, $\nabla \Phi_i(u)$ is an outer normal to ∂C at u.

In [10], the following generalizations of problem (1.1)

$$x' = f(t,x), \quad x(1) = \int_0^1 dh(s)\, x(s), \tag{1.4}$$

and

$$x' = f(t,x), \quad x(0) = \int_0^1 dh(s)\, x(s) \tag{1.5}$$

(sometimes called *nonlocal terminal value problem*, and *nonlocal initial value problem*, respectively), have been considered, where

$$h : [0,1] \to \mathbb{R} \;\; \textit{is non-decreasing and} \;\; \int_0^1 dh(s) = 1.$$

Both boundary conditions in (1.4) and (1.5) can be seen as generalizations of the periodic boundary conditions $x(0) = x(1)$, where either $x(0)$ or $x(1)$ is replaced by some average of x over the interval $[0,1]$.

The following theorems are special cases of the results proved in [10] by reduction to a fixed point problem and the use of some version of Leray–Schauder continuation theorem.

Theorem 1.1. *If* $h(0) < h(\alpha)$ *for some* $\alpha \in (0,1)$ *and if there exists an open, bounded, convex neighborhood* C *of* 0 *in* $\mathbb{R}^n$ *and an outer normal field* ν *on* ∂C *such that*

$$\big\langle \nu(u) \mid f(t,u) \big\rangle \geq 0 \;\; \forall\, (t,u) \in [0,1] \times \partial C, \tag{1.6}$$

then problem (1.4) *has at least one solution* x *such that* $x([0,1]) \in \overline{C}$.

Theorem 1.2. *If* $h(\alpha) < h(1)$ *for some* $\alpha \in (0,1)$ *and if there exists an open, bounded, convex neighborhood* C *of* 0 *in* $\mathbb{R}^n$ *and an outer normal field* ν *on* ∂C *such that*

$$\big\langle \nu(u) \mid f(t,u) \big\rangle \leq 0 \;\; \forall\, (t,u) \in [0,1] \times \partial C, \tag{1.7}$$

then problem (1.5) *has at least one solution* x *such that* $x([0,1]) \subset \overline{C}$.

The following consequences of Theorems 1.1 and 1.2, corresponding to $C = B_R$, are also given in [10].

Corollary 1.1. *If* $h(0) < h(\alpha)$ *for some* $\alpha \in (0,1)$ *and if there exists* $R > 0$ *such that*

$$\langle u \mid f(t,u) \rangle \geq 0 \;\; \forall\, (t,u) \in [0,1] \times \partial B_R, \tag{1.8}$$

Then problem (1.4) *has at least one solution* x *such that* $x([0,1]) \in \overline{B}_R$.

Corollary 1.2. *If $h(\alpha) < h(1)$ for some $\alpha \in (0,1)$ and if there exists $R > 0$ such that*

$$\langle u \mid f(t,u)\rangle \leq 0 \ \ \forall\, (t,u) \in [0,1] \times \partial B_R, \tag{1.9}$$

then problem (1.5) *has at least one solution x such that $x([0,1]) \in \overline{B}_R$.*

Comparing those statements with our first existence theorem for the periodic problem, we see that the sense of the inequality in conditions (1.6) or (1.8) and (1.7) or (1.9) depends upon the boundary condition. On the other hand, as it is easily verified by direct computation, the system

$$x' = \lambda x + e(t),$$

with each of the three-point boundary conditions

$$x(1) = \frac{1}{2}\left[x\Big(\frac{1}{2}\Big) + x(0)\right], \ \ x'(0) = \frac{1}{2}\left[x\Big(\frac{1}{2}\Big) + x(1)\right],$$

has a solution for each $e \in C([0,1],\mathbb{R}^n)$ and each $\lambda \in \mathbb{R} \setminus \{0\}$. This is again a consequence of the fact that the only real eigenvalue of $\frac{d}{dt}$ with each boundary condition is 0. Hence a natural question is to know if the conclusion of the above corollaries still holds when the sense of the corresponding inequality upon f is reversed.

The aim of this paper is to show by some counterexamples that the answer is negative in general, which of course implies that the same negative answer holds for Theorems 1.1 and 1.2. In this sense, the existence conditions given in [10] are sharp.

The construction of our counterexamples in Section 4 depends upon the study of the associated complex eigenvalue problem in Section 2 and of the corresponding Fredholm alternative in Section 3 for some special three-point boundary conditions.

In Section 4, we exhibit a (complex) eigenvalue λ and show the existence of a function $e \in C([0,1],\mathbb{C})$ such that the equation

$$z' = \lambda z + e(t),$$

with the corresponding multipoint boundary conditions, has no solution $z : [0,1] \to \mathbb{C}$ and such that, for the equivalent 2-dimensional system obtained by letting

$$x_1 = \Re z, \ \ x_2 = \Im z, \ \ f_1(t,x) = \Re(\lambda z + e(t)), \ \ f_2(t,x) = \Im(\lambda z + e(t)),$$

$\langle u \mid f(t,u)\rangle$ has the opposite sign to the one in the corresponding corollary, for all $t \in [0,1]$ and all sufficiently large $|u|$. We complete this 2-dimensional counterexample by a 3-dimensional one, from which counterexamples can easily be obtained in all dimensions $n \geq 2$.

In Section 4, we also give an example of periodic problem (1.1) having no solution and such that $\langle x, f(t,x)\rangle$ changes sign when $|x| = R$ and $R > 0$ is sufficiently large. Hence, the assumptions of the existence theorem for periodic problems are sharp as well.

Finally, in Section 5, we construct in a similar way a counterexample related to the following *nonlocal boundary value problem for a second order system* considered in [9]

$$x'' = g(t,x,x'), \ \ x(0) = 0, \ \ x'(1) = \int_0^1 dh(s)\, x'(s), \tag{1.10}$$

where $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and* $h : [0,1] \to \mathbb{R}$ *is non-decreasing and* $\int_0^1 dh(s) = 1.$ The following existence result is proved in [9].

Theorem 1.3. *If $h(0) < h(\alpha)$ for some $\alpha \in (0,1)$ and if there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν on ∂C such that*

$$\big\langle \nu(v) \mid g(t,u,v)\big\rangle \geq 0 \ \ \forall\, (t,u,v) \in [0,1] \times \overline{C} \times \partial C,$$

then problem (1.10) *has at least one solution x such that $x([0,1]) \subset \overline{C}$ and $x'([0,1]) \subset \overline{C}$.*

Its special case where $C = B_R$ goes as follows.

Corollary 1.3. *If $h(0) < h(\alpha)$ for some $\alpha \in (0,1)$ and if there exists $R > 0$ such that*

$$\langle v \mid g(t,u,v)\rangle \geq 0 \;\; \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R, \tag{1.11}$$

then problem (1.10) *has at least one solution x such that $x([0,1]) \subset \overline{B}_R$ and $x'([0,1]) \subset \overline{B}_R$.*

In Section 5, we exhibit a counterexample showing that a statement like Corollary 1.3 does not hold if the sense of inequality (1.11) is reversed.

Notice that one could consider as well the problem

$$x'' = g(t,x,x'), \;\; x(0) = 0, \;\; x'(0) = \int_0^1 dh(s)\, x'(s), \tag{1.12}$$

when $h(\alpha) < h(1)$ for some $\alpha \in (0,1)$ and the assumptions on g, and prove, mimicking the approach of [9], the following existence result.

Theorem 1.4. *If $h(\alpha) < h(1)$ for some $\alpha \in (0,1)$ and if there exists an open, bounded, convex neighborhood C of 0 in $\mathbb{R}^n$ and an outer normal field ν on ∂C such that*

$$\big\langle \nu(v) \mid g(t,u,v)\big\rangle \leq 0 \;\; \forall\, (t,u,v) \in [0,1] \times \overline{C} \times \partial C,$$

then problem (1.12) *has at least one solution x such that $x([0,1]) \subset \overline{C}$ and $x'([0,1]) \subset \overline{C}$.*

We leave to the reader the task of stating the corresponding corollary analog to Corollary 1.3 and of constructing a counterexample to an existence statement with reversed inequalities.

In analogy with the periodic case for first order differential systems, the two-point boundary value problem

$$x'' = g(t,x,x'), \;\; x(0) = 0, \;\; x'(0) = x'(1) \tag{1.13}$$

is a special case of both problems (1.10) and (1.12). Hence, the existence of a solution to problem (1.13) is insured if there exists $R > 0$ such that either

$$\langle v \mid g(t,u,v)\rangle \geq 0 \;\; \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R,$$

or

$$\langle v \mid g(t,u,v)\rangle \leq 0 \;\; \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R.$$

2 First order eigenvalue problems

We consider the eigenvalue problem

$$z'(t) = \lambda z(t), \;\; z(1) = \frac{1}{2}\left[z(0) + z\Big(\frac{1}{2}\Big)\right], \tag{2.1}$$

where $\lambda \in \mathbb{C}$, $z : [0,1] \to \mathbb{C}$. Its three-point boundary condition is a special case of the one in Corollary 1.1 with

$$h(s) = \begin{cases} 0 & \text{if } \; s = 0, \\ 1/2 & \text{if } \; s \in (0,1/2] \\ 1 & \text{if } \; s \in (1/2,1]. \end{cases}$$

Proposition 2.1. *The eigenvalues of problem* (2.1) *are $\lambda_{tc,1,k} = 2k(2\pi i)$ and $\lambda_{tc,2,k} = -\log 4 + (2k+1)(2\pi i)$ $(k \in \mathbb{Z})$. They are located in the left part of the complex plane.*

Proof. The eigenvalue problem (2.1) has a nontrivial solution if and only if $\lambda \in \mathbb{C}$ is such that

$$e^{\lambda} = \frac{1}{2} + \frac{1}{2}\, e^{\lambda/2}. \tag{2.2}$$

Set $\mu := e^{\lambda/2}$, so that equation (2.2) becomes the equation in μ

$$\mu^2 - \frac{1}{2}\,\mu - \frac{1}{2} = 0,$$

whose solutions are $\mu_{tc,1} = 1$, $\mu_{tc,2} = -\frac{1}{2}$. The equation $e^{\lambda/2} = \mu_{tc,1} = 1$ is satisfied for $\frac{\lambda}{2} = 2k\pi i$ $(k \in \mathbb{Z})$ which gives the eigenvalues

$$\lambda_{tc,1,k} = 2k(2\pi i) \ \ (k \in \mathbb{Z}).$$

The equation $e^{\lambda/2} = \mu_{tc,2} = -\frac{1}{2}$ is satisfied for $\frac{\lambda}{2} = -\log 2 + \pi i + 2k\pi i = -\log 2 + (2k+1)\pi i$ $(k \in \mathbb{Z})$, which gives the eigenvalues

$$\lambda_{tc,2,k} = -\log 4 + (2k+1)(2\pi i) \ \ (k \in \mathbb{Z}).$$ □

Similarly, we consider the eigenvalue problem

$$z'(t) = \lambda z(t), \ \ z(0) = \frac{1}{2}\left[z\Big(\frac{1}{2}\Big) + z(1)\right], \tag{2.3}$$

where $\lambda \in \mathbb{C}$, $z : [0,1] \to \mathbb{C}$. Its multi-point boundary condition is a special case of the one in Corollary 1.2 with

$$h(s) = \begin{cases} 0 & \text{if } s \in [0, 1/2), \\ 1/2 & \text{if } s \in [1/2, 1), \\ 1 & \text{if } s = 1. \end{cases}$$

Proposition 2.2. *The eigenvalues of problem* (2.3) *are* $\lambda_{ic,1,k} = 2k(2\pi i)$ *and* $\lambda_{ic,2,k} = \log 4 + (2k+1)(2\pi i)$ $(k \in \mathbb{Z})$. *They are located in the left right part of the complex plane.*

Proof. The eigenvalue problem (2.3) has a nontrivial solution if and only if $\lambda \in \mathbb{C}$ is such that

$$1 = \frac{1}{2}\, e^{\lambda/2} + \frac{1}{2}\, e^{\lambda}. \tag{2.4}$$

Set $\mu := e^{\lambda/2}$, so that equation (2.4) becomes the equation in μ

$$\frac{1}{2}\,\mu^2 + \frac{1}{2}\,\mu - 1 = 0$$

whose solutions are $\mu_{ic,1} = 1$ and $\mu_{ic,2} = -2$. Consequently, we obtain, as above,

$$\lambda_{ic,1,k} = 2k(2\pi i) \ \ (k \in \mathbb{Z})$$

and

$$\lambda_{ic,2,k} = \log 4 + (2k+1)(2\pi i) \ \ (k \in \mathbb{Z}).$$ □

Remark 2.1. The situation can be compared with the spectrum for the periodic boundary conditions

$$z' = \lambda z, \ \ z(0) = z(1)$$

which, as easily seen, is made of the eigenvalues $\lambda_{p,k} = k(2\pi i)$ $(k \in \mathbb{Z})$. One can see that, in the case of (2.1), half of the eigenvalues of the periodic problem move to the line $\Re z = -\log 4$, and, in the case of (2.3), the same half moves to the line $\Re z = \log 4$. The spectra have lost their symmetry with respect to the imaginary axis.

3 Fredholm alternative

The construction of our counterexamples requires the use of the Fredholm alternative for the corresponding forced boundary value problems.

Proposition 3.1. *λ is an eigenvalue of* (2.1) (*resp.* (2.3)) *if and only if there exists a continuous function e such that the nonhomogeneous problem* (3.1) (*resp.* (3.2)) *has no solution.*

Proof. It is shown in [10] (or by direct verification) that the non-homogeneous problems

$$Lz := z' - z = e(t), \;\; z(0) = \frac{1}{2}\, z\Big(\frac{1}{2}\Big) + \frac{1}{2}\, z(1)$$

and

$$Mz := z' + z = e(t), \;\; z(0) = \frac{1}{2}\, z\Big(\frac{1}{2}\Big) + \frac{1}{2}\, z(1)$$

have a unique solution $z = L^{-1}e$ and $z = M^{-1}e$ for every $e \in C([0,1],\mathbb{C})$, and that the linear mappings L^{-1} and M^{-1} are compact in the space $C([0,1],\mathbb{C})$. As a consequence, each problem

$$z' - \lambda z = e(t), \;\; z(1) = \frac{1}{2}\, z(0) + \frac{1}{2}\, z\Big(\frac{1}{2}\Big) \tag{3.1}$$

and

$$z' + \lambda z = e(t), \;\; z(0) = \frac{1}{2}\, z\Big(\frac{1}{2}\Big) + \frac{1}{2}\, z(1) \tag{3.2}$$

can be written equivalently

$$z = (\lambda - 1)L^{-1}z + L^{-1}e, \;\; z = (\lambda + 1)M^{-1}z + M^{-1}e,$$

so that the Fredholm alternative follows from Riesz theory of linear compact operators. □

4 Counterexamples to Corollaries 1.1 and 1.2 with opposite vector fields sign conditions

We now finalize the construction of our counterexamples.

We first consider the case of a three-point boundary condition of terminal type, and apply Proposition 3.1 to the case of the eigenvalue $\lambda_{tc,2,0} = -\log 4 + (4k+2)\pi i$ of (2.1). Let $e : [0,1] \to \mathbb{C}$ be a continuous function such that the problem

$$z'(t) = (-\log 4 + 2\pi i)z(t) + e(t), \;\; z(1) = \frac{1}{2}\, z(0) + \frac{1}{2}\, z\Big(\frac{1}{2}\Big) \tag{4.1}$$

has no solution. Setting $z(t) = x_1(t) + ix_2(t)$, $e(t) = e_1(t) + ie_2(t)$, problem (4.1) is equivalent to the planar real system

$$\begin{cases} x_1'(t) = -(\log 4)x_1(t) - 2\pi x_2(t) + e_1(t), \\ x_2'(t) = 2\pi x_1(t) - (\log 4)x_2(t) + e_2(t), \\ x_1(1) = \dfrac{1}{2}\, x_1(0) + \dfrac{1}{2}\, x_1\Big(\dfrac{1}{2}\Big), \\ x_2(1) = \dfrac{1}{2}\, x_2(0) + \dfrac{1}{2}\, x_2\Big(\dfrac{1}{2}\Big). \end{cases} \tag{4.2}$$

Let

$$f(t,u) := \big(-(\log 4)u_1 - 2\pi u_2 + e_1(t), 2\pi u_1 - (\log 4)u_2 + e_2(t)\big).$$

For (4.2), we have

$$\begin{aligned} \langle u \mid f(t,u)\rangle &= u_1\big[-(\log 4)u_1 - 2\pi u_2 + e_1(t)\big] + u_2\big[2\pi u_1 - (\log 4)u_2 + e_2(t)\big] \\ &= -(\log 4)(u_1^2 + u_2^2) + u_1 e_1(t) + u_2 e_2(t) \\ &\le -(\log 4)|u|^2 + |e(t)|\,|u| < 0, \end{aligned} \tag{4.3}$$

when $|u| \geq R$ for some sufficiently large R.

Conclusion. *For problem* (1.4) *with the conditions of Corollary* 1.1 *on f and the existence of some $R > 0$ such that*

$$\langle u \mid f(t,u)\rangle \leq 0 \ \ \forall\,(t,u) \in [0,1] \times \partial B_R,$$

there is no existence theorem similar to Corollary 1.1.

In the case of the three-point conditions of initial type, we similarly apply Proposition 3.1 to the case of the eigenvalue $\lambda_{ic,2,0} = \log 4 + 2\pi i$ of (2.3). Let $e : [0,1] \to \mathbb{C}$ be a continuous function such that the problem

$$z'(t) = (\log 4 + 2\pi i)z(t) + e(t), \ \ z(1) = \frac{1}{2}\, z(0) + \frac{1}{2}\, z\Big(\frac{1}{2}\Big) \tag{4.4}$$

has no solution. Setting $z(t) = x_1(t) + ix_2(t)$, $e(t) = e_1(t) + ie_2(t)$, problem (4.4) is equivalent to the planar real system

$$\begin{cases} x_1'(t) = (\log 4)x_1(t) - 2\pi x_2(t) + e_1(t), \\ x_2'(t) = 2\pi x_1(t) + (\log 4)x_2(t) + e_2(t), \\ x_1(0) = \dfrac{1}{2}\, x_1\Big(\dfrac{1}{2}\Big) + \dfrac{1}{2}\, x_1(1), \\ x_2(0) = \dfrac{1}{2}\, x_2\Big(\dfrac{1}{2}\Big) + \dfrac{1}{2}\, x_2(1). \end{cases} \tag{4.5}$$

Let

$$f(t,u) := \big((\log 4)u_1 - 2\pi u_2 + e_1(t), 2\pi u_1 + (\log 4)u_2 + e_2(t)\big).$$

For (4.5), we have

$$\begin{aligned} \langle u \mid f(t,u)\rangle &= u_1[(\log 4)u_1 - 2\pi u_2 + e_1(t)] + u_2\big[2\pi u_1 + (\log 4)u_2 + e_2(t)\big] \\ &= (\log 4)(u_1^2 + u_2^2) + u_1 e_1(t) + u_2 e_2(t) \\ &\geq (\log 4)|u|^2 - |e(t)|\,|u| > 0, \end{aligned}$$

when $|u| \geq R$ for some sufficiently large R.

Conclusion. *For problem* (1.5) *with the conditions of Corollary* 1.2 *on f and the existence of some $R > 0$ such that*

$$\langle u \mid f(t,u)\rangle \geq 0 \ \ \forall\,(t,u) \in [0,1] \times \partial B_R,$$

there is no existence theorem similar to Corollary 1.2.

Remark 4.1. The symmetry-breaking for the spectra of the three-point boundary value problems of terminal or initial type explains the difference in the existence conditions for the nonlinear problems with the three-point boundary conditions and with the periodic conditions. The presence of the complex spectrum in the left or the right half-plane influences like a ghost the existence of solutions of the real nonlinear systems. Maybe extra conditions upon f could provide existence results with the sign conditions of the counterexamples.

Remark 4.2. Strictly speaking, our counterexamples do not cover the case of $n = 1$ or of n odd. For $n = 3$, if one adds the equations

$$x_3' = -(\log 4)x_3 + \frac{\log 4}{4}\,(x_1 + x_2), \ \ x_3(1) = \frac{1}{2}\,\Big[x_3(0) + x_3\Big(\frac{1}{2}\Big)\Big]$$

or

$$x_3' = (\log 4)x_3 + \frac{\log 4}{4}\,(x_1 + x_2), \ \ x_3(0) = \frac{1}{2}\,\Big[x_3\Big(\frac{1}{2}\Big) + x_3(1)\Big]$$

to (4.2) or to (4.5), respectively, the corresponding boundary value problems have no solutions and the nonlinear parts verify the opposite sign conditions to Corollaries 1.1 and 1.2, respectively. Of course, the counterexamples for $n = 2$ and $n = 3$ easily provide counterexamples in any dimension $n \geq 2$. The case $n = 1$ remains open.

Remark 4.3. The periodic problem

$$z' = 2\pi i z + e^{2\pi i t}, \quad z(0) = z(1) \tag{4.6}$$

has no solution. Indeed, if z is a possible solution, then

$$(e^{-2\pi i t} z)' = 1,$$

which gives a contradiction, by integration over $[0,1]$ and use of the boundary conditions.

Letting $z = x_1 + i x_2$, the following problem

$$x_1' = -2\pi x_2 + \cos(2\pi t), \quad x_2' = 2\pi x_1 + \sin(2\pi t), \quad x_1(0) = x_1(1), \quad x_2(0) = x_2(1),$$

equivalent to (4.6), has no solution. On the other hand, letting

$$\begin{gathered} f_1(t, x_1, x_2) = -2\pi x_2 + \cos(2\pi t), \quad f_2(t, x_1, x_2) = 2\pi x_1 + \sin(2\pi t), \\ x = (x_1, x_2), \quad f(t,x) = \big(f_1(t, x_1, x_2), f_2(t, x_1, x_2)\big), \end{gathered}$$

we have

$$\begin{aligned} \langle x, f(t,x) \rangle &= -2\pi x_2 x_1 + \cos(2\pi t) x_1 + 2\pi x_1 x_2 + \sin(2\pi t) x_2 \\ &= \cos(2\pi t) x_1 + \sin(2\pi t) x_2. \end{aligned}$$

For $x = R[\cos(2\pi\theta), \sin(2\pi\theta)] \in \partial B_R$ $(\theta \in [0,1])$, we have

$$\begin{aligned} \langle x, f(t,x) \rangle &= R\big[\cos(2\pi t)\cos(2\pi\theta) + \sin(2\pi t)\sin(2\pi\theta)\big] \\ &= R\cos[2\pi(t-\theta)] \quad (t, \theta \in [0,1]), \end{aligned}$$

which implies that, for each $t \in [0,1]$, $\langle x, f(t,x)\rangle$ takes both positive and negative values on ∂B_R, and shows that, for n even, the assumptions of the existence theorems for periodic problems given at the beginning of the Introduction are sharp.

5 Second order differential systems

As in Section 2, we start with the following "eigenvalue problem"

$$z''(t) = \lambda z'(t), \quad z(0) = 0, \quad z'(1) = \frac{1}{2}\, z'(0) + \frac{1}{2}\, z'\Big(\frac{1}{2}\Big), \tag{5.1}$$

where $\lambda \in \mathbb{C}$, $x : [0,1] \to \mathbb{C}$. Notice that it is not the classical eigenvalue associated to z'' in which $\lambda z'$ must be replaced by λz.

Proposition 5.1. *All the "eigenvalues" $\lambda_{bc,j,k}$ $(j = 1,2;\ k \in \mathbb{Z})$ of the multipoint boundary value problem* (5.1) *have real part equal to* 0 *or* $-\log 4$*, and hence are located in the left part of the complex plane.*

Proof. Setting $w(t) = z'(t)$, so that, using $z(0) = 0$, $z(t) = \int\limits_0^t w(s)\,ds$, problem (5.1) is equivalent to the eigenvalue problem

$$w'(t) = \lambda w(t), \quad w(1) = \frac{1}{2}\, w(0) + \frac{1}{2}\, w\Big(\frac{1}{2}\Big),$$

i.e., to the eigenvalue problem (2.3). Hence the result follows from Proposition 2.1. □

We now deduce, from the first order case, the Fredholm alternative.

Proposition 5.2. *λ is an "eigenvalue" of* (5.1) *if and only if there exists a continuous function e such that the nonhomogeneous problem* (5.2) *has no solution.*

Proof. In a similar way as in Proposition 5.1, the non-homogeneous problem

$$z'' - \lambda z' = e(t), \ \ z(0) = 0, \ \ z'(1) = \frac{1}{2}\, z'(0) + \frac{1}{2}\, z'\Big(\frac{1}{2}\Big) \tag{5.2}$$

is equivalent, with $w = z'$, to the non-homogeneous problem

$$w' - \lambda w = e(t), \ \ w(1) = \frac{1}{2}\,\Big[w(0) + w\Big(\frac{1}{2}\Big)\Big],$$

and then the conclusion follows from Proposition 3.1. □

To construct the counterexample, we apply Proposition 5.2 to the case of the "eigenvalue" $\lambda_{bc,2,0} = -\log 4 + 2\pi i$ of (5.1). Let $e : [0,1] \to \mathbb{C}$ be a continuous function such that the problem

$$z''(t) = (-\log 4 + 2\pi i) z'(t) + e(t), \ \ z(0) = 0, \ \ z'(1) = \frac{1}{2}\, z'(0) + \frac{1}{2}\, z'\Big(\frac{1}{2}\Big) \tag{5.3}$$

has no solution. Setting $z(t) = x_1(t) + i x_2(t)$, $e(t) = e_1(t) + i e_2(t)$, problem (5.3) is equivalent to the planar real system

$$\begin{cases} x_1''(t) = -(\log 4) x_1'(t) - 2\pi x_2'(t) + e_1(t), \\ x_2''(t) = 2\pi x_1'(t) - (\log 4) x_2'(t) + e_2(t), \\ x_1(0) = 0, \ \ x_1'(1) = \dfrac{1}{2}\, x_1'(0) + \dfrac{1}{2}\, x_1'\Big(\dfrac{1}{2}\Big), \\ x_2(0) = 0, \ \ x_2'(1) = \dfrac{1}{2}\, x_2'(0) + \dfrac{1}{2}\, x_2'\Big(\dfrac{1}{2}\Big). \end{cases}$$

Let

$$g(t,v) := \big(-(\log 4) v_1(t) - 2\pi v_2(t) + e_1(t), 2\pi v_1(t) - (\log 4) v_2(t) + e_2(t) \big).$$

By (4.3), we obtain $\langle v, g(t,v)\rangle < 0$, when $|v| \geq R$ for some sufficiently large R.

Conclusion. *For problem* (1.10) *with the conditions of Corollary* 1.3 *on g and the existence of some $R > 0$ such that*

$$\langle v, g(t,u,v)\rangle \leq 0 \ \ \forall\, (t,u,v) \in [0,1] \times \overline{B}_R \times \partial B_R,$$

there is no existence theorem similar to Corollary 1.3.

Similar conclusions hold for problem (1.12).

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