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GLOBAL ATTRACTOR OF A WEAKLY NONLINEAR PARABOLIC SYSTEM WITH DISCONTINUOUS TRAJECTORIES

Abstract. In the paper, we prove the existence of a global attractor for an impulsive dynamical system, which is generated by a weakly nonlinear parabolic system, when its trajectories have jumps at moments of intersection with certain surface of the phase space.

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1 Introduction

One of the possible ways to describe the qualitative behaviour of evolutionary processes with instant impulsive perturbations is the theory of impulsive differential equations [14, 20, 21]. Autonomous equations, which trajectories have impulsive perturbations at moments of intersection with certain subset of the phase space, form an important subclass of impulsive differential equations and called impulsive (or discontinuous) dynamical systems (DS). Some aspects of the long-time behavior of such finite-dimensional systems have been studied in [1, 6, 13, 14, 17, 19–21]. For infinite-dimensional dissipative systems one of the important qualitative characteristics of their behavior is the concept of a global attractor [23]. In [8, 11, 18, 22, 24], the theory of global attractors has been investigated in the case, where the moments of impulsive perturbations are fixed.

First results of applying this theory to impulsive DS with a finite number of discontinuities along the trajectories arose in [4]. In further works [2, 3], using a priori estimates on the behavior of the trajectories in the neighborhood of impulsive set, the authors managed to transfer the basic constructions of the classical DS theory to an impulsive case and obtain abstract theorems on the existence and properties of the global attractor. However, the question of verifying the imposed conditions on the impulsive DS for special infinite-dimensional nonlinear evolution problems remains open. In [12], the authors offered another approach, based on the concept of a uniform attractor and applied it to scalar impulsive parabolic equations with a small nonlinearity. In [7], this approach was generalized to the multi-valued impulsive DS, generated by the solutions of evolution inclusions.

In this paper, using the methods of [12], we investigate the existence of a global attractor of impulsive DS generated by the two-dimensional parabolic system with a small nonlinearity, which solutions have impulsive perturbation at moments of intersection with a certain subset of the phase space. Moreover, the conditions on the parameters of the problem do not guarantee the uniqueness of the solution of the Cauchy problem, which requires to use the theory of global attractors of multi-valued DS [10, 15, 16].

2 Formulation of the problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain. For the unknown functions $u(t,x), v(t,x)$ in $(0,+\infty)\times\Omega$, we consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + \varepsilon f_1(u,v), \\ \dfrac{\partial v}{\partial t} = a\Delta v + 2b\Delta u + \varepsilon f_2(u,v), \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$ is a small parameter,

$$a > 0, \quad |b| < a, \tag{2.2}$$

continuous linear functions $f_i : \mathbb{R}^2 \longmapsto \mathbb{R}$, $i = 1, 2$, satisfy the following condition:

$$\exists\, C > 0 \quad \forall\, u, v \in \mathbb{R} \quad |f_1(u,v)| + |f_2(u,v)| \leq C. \tag{2.3}$$

The space $H = L^2(\Omega) \times L^2(\Omega)$ with the norm $\|z\|_H = \sqrt{\|u\|^2 + \|v\|^2}$ is the phase space of problem (2.1). Here and in the sequel, $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ are the norm and the scalar product in $L^2(\Omega)$, respectively, $\{\lambda_i\}_{i=1}^{\infty} \subset (0,+\infty)$, $\{\psi_i\}_{i=1}^{\infty} \subset H_0^1(\Omega)$ are solutions of the spectral problem $\Delta\psi = -\lambda\psi$, $\psi \in H_0^1(\Omega)$.

Under conditions (2.2), (2.3), for every $\varepsilon > 0$, $z_0 \in H$, there exists at least one solution $z = \begin{pmatrix} u \\ v \end{pmatrix} \in C([0,+\infty); H)$ of problem (2.1), where $z(0) = z_0$ [5].

For the fixed $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\mu > 0$, we consider the following impulsive problem:

When the phase point $z(t)$ meets *the impulsive set*

$$M = \Big\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in H \mid \quad |(u,\psi_1)| \leq \gamma,\ \alpha(u,\psi_1) + \beta(v,\psi_1) = 1 \Big\}, \tag{2.4}$$

then *the impulsive map* $I : M \longmapsto M'$ maps it into a new position $Iz \in M'$, where

$$M' = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in H \mid \ |(u, \psi_1)| \le \gamma, \ \alpha(u, \psi_1) + \beta(v, \psi_1) = 1 + \mu \right\}. \tag{2.5}$$

We choose the set M due to the results from [12], where for a scalar parabolic equation, the impulsive set $M = \{u \in L^2(\Omega) \mid \ (u, \psi_1) = 1\}$.

The main purpose of this work is to establish the existence and investigate the properties of the global attractor of impulsive DS, generated by the solution of problem (2.1)–(2.5), for some class of compact-valued impulsive maps I, which have the following form:

$$\text{for } \ z = \sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M$$

$$Iz \subseteq I_0 z = \left\{ \begin{pmatrix} c_1' \\ d_1' \end{pmatrix} \psi_1 + \sum_{i=2}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \mid \ |c_1'| \le \gamma, \ \alpha c_1' + \beta d_1' = 1 + \mu \right\}. \tag{2.6}$$

In a particular case, the single-valued map $I : M \longmapsto M'$

$$I\left(\sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \right) = \begin{pmatrix} c_1 \\ d_1 + \dfrac{\mu}{\beta} \end{pmatrix} \psi_1 + \sum_{i=2}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i,$$

and the compact-valued map $I \equiv I_0$ are the partial cases of formula (2.6).

The main result of the work is to prove the fact that for an arbitrary compact-valued map I, which satisfies (2.6), and for a sufficiently small $\varepsilon > 0$, in the phase space H the impulsive problem (2.1), (2.4), (2.6) generates impulsive (multi-valued) DS $\widetilde{G}_\varepsilon$, which has the global attractor Θ_ε and

$$\operatorname{dist}(\Theta_\varepsilon, \Theta) \longrightarrow 0, \ \ \varepsilon \to 0, \tag{2.7}$$

where

$$\Theta = \bigcup_{t \in [0,\tau], \, |c_1| \le \gamma} \left\{ \begin{pmatrix} c_1 \\ \frac{1+\mu-\alpha c_1}{\beta} - 2bc_1 t \end{pmatrix} e^{-a\lambda_1 t} \psi_1 \mid \ (1 + \mu - 2b\beta c_1 \tau) e^{-a\lambda_1 \tau} = 1 \right\} \cup \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3 Construction of impulsive MDS

Let $P(H)$ ($\beta(H)$) be a set of all non-empty (non-empty bounded) subsets of H.

Definition 3.1 ([15])**.** A multi-valued map $G : \mathbb{R}_+ \times H \to P(H)$ is called a multi-valued dynamical system (MDS), if the following conditions are satisfied:

$$\forall\, x \in H \ \ G(0, x) = x \quad \text{and} \quad \forall\, t, s \ge 0 \ \ G(t + s, x) \subseteq G(t, G(s, x)).$$

MDS G is called strict if $\forall\, x \in H \ \forall\, t, s \ge 0 \ G(t + s, x) = G(t, G(s, x))$.

Remark 3.1. If G is a single-valued map, then we obtain the definition of a semigroup. However, we do not impose any continuity conditions on it, which is important when we consider impulsive systems.

Definition 3.2 ([18])**.** A subset $\Theta \subset H$ is called a global attractor of MDS G, if

(1) Θ is a compact set;

(2) Θ is a uniformly attracting set, i.e.,

$$\forall\, B \in \beta(H) \ \ \operatorname{dist}(G(t, B), \Theta) \longrightarrow 0, \ \ t \to \infty;$$

(3) Θ is minimal among all closed uniformly attracting sets.

The next result follows from [18] and provides a criterion of the existence of a global attractor for dissipative MDS.

Lemma 3.1. *Assume that* MDS G *satisfies the dissipativity condition*

$$\exists\, B_0 \in \beta(H) \;\; \forall\, B \in \beta(H) \;\; \exists\, T = T(B) > 0 \;\; \forall\, t \geq T \;\; G(t, B) \subset B_0.$$

Then the following conditions are equivalent:

(1) MDS G *has global attractor* Θ*;*

(2) MDS G *is asymtotically compact, i.e.,* $\forall\, t_n \nearrow \infty \; \forall\, B \in \beta(H)$

$$\forall\, \xi_n \in G(t_n, B) \;\; \textit{the sequence } \{\xi_n\} \textit{ is precompact in } H.$$

Moreover,

$$\Theta = \omega(B_0) := \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} G(t, B_0)}.$$

Now let us consider a special subclass of MDS called impulsive MDS. Impulsive MDS $\widetilde{G}$ consists of non-empty closed set $M \subset H$ (impulsive set), compact-valued map $I : M \to P(H)$ (impulsive map) and some set K of continuous maps $\varphi : [0, +\infty) \to H$, which satisfy the following assumptions:

(K1) $\forall\, x \in H \; \exists\, \varphi \in K$: $\varphi(0) = x$;

(K2) $\forall\, \varphi \in K \; \forall\, s \geq 0 \; \varphi(\,\cdot\, + s) \in K$.

We denote $K_x = \{\varphi \in K \mid \; \varphi(0) = x\}$.

Remark 3.2. If in assumption (K1), for every $x \in H$, there exists a unique $\varphi \in K$ such that $\varphi(0) = x$, then K_x consists of a single trajectory φ, and the equality $V(t, x) = \varphi(t)$ defines a classical semigroup $V : \mathbb{R}_+ \times H \longmapsto H$.

A phase point of impulsive MDS moves along the trajectories of K, and at the moment of meeting the set M, it immediately jumps onto a new position from the set IM. For the "well-posedness" of the impulsive problem we assume the following conditions [4]:

$$M \cap IM = \varnothing, \tag{3.1}$$

$$\forall\, x \in M \;\; \forall\, \varphi \in K_x \;\; \exists\, \tau = \tau(\varphi) > 0 \;\; \forall\, t \in (0, \tau) \;\; \varphi(t) \notin M. \tag{3.2}$$

We denote

$$\forall\, \varphi \in K \;\; M^+(\varphi) = \Big(\bigcup_{t>0} \varphi(t)\Big) \cap M.$$

If $M^+(x) \neq \varnothing$, then there exists a moment of time $s := s(\varphi) > 0$ such that $\forall\, t \in (0, s)$ $\varphi(t) \notin M$, $\varphi(s) \in M$. Therefore, we can define the following function $s : K \to (0, +\infty]$:

$$s(\varphi) = \begin{cases} s & \text{if } M^+(\varphi) \neq \varnothing, \\ +\infty & \text{if } M^+(\varphi) = \varnothing. \end{cases}$$

Let us construct impulsive trajectory $\widetilde{\varphi}$, which starts from the point $x_0 \in H$. Let $\varphi_0 \in K_{x_0}$.

If $M^+(\varphi_0) = \varnothing$, then define $\widetilde{\varphi}$ on $[0, +\infty)$ as

$$\widetilde{\varphi}(t) = \varphi_0(t) \;\; \forall\, t \geq 0.$$

If $M^+(\varphi_0) \neq \varnothing$, then for $s_0 = s(\varphi_0) > 0$, $x_1 = \varphi_0(s_0) \in M$ and $x_1^+ \in Ix_1$ define $\widetilde{\varphi}$ on $[0, s_0]$ as

$$\widetilde{\varphi}(t) = \begin{cases} \varphi_0(t), & t \in [0, s_0), \\ x_1^+, & t = s_0. \end{cases}$$

Let $\varphi_1 \in K_{x_1^+}$. If $M^+(\varphi_1) = \varnothing$, then define $\widetilde{\varphi}$ on $[0, +\infty)$ as

$$\widetilde{\varphi}(t) = \varphi_1(t - s_0) \ \ \forall\, t \geq s_0.$$

If $M^+(\varphi_1) \neq \varnothing$, then for $s_1 = s(\varphi_1) > 0$, $x_2 = \varphi_1(s_1) \in M$ and $x_2^+ \in Ix_2$, we define $\widetilde{\varphi}$ on $[s_0, s_0+s_1]$ as

$$\widetilde{\varphi}(t) = \begin{cases} \varphi_1(t - s_0), & t \in [s_0, s_0 + s_1), \\ x_2^+, & t = s_0 + s_1. \end{cases}$$

Continuing this procedure, we obtain the impulsive trajectory $\widetilde{\varphi}$ with a finite or infinite number of impulsive points $\{x_n^+\}_{n\geq 1} \subset IM$, corresponding moments of time $\{s_n\}_{n\geq 0} \subset (0, \infty)$ and the functions $\{\varphi_n\}_{n\geq 0} \subset K$.

Let

$$t_0 = 0, \quad t_{n+1} := \sum_{k=0}^{n} s_k.$$

If $\widetilde{\varphi}$ has an infinite number of jumps, then it is defined by the formula

$$\forall\, n \geq 0 \ \ \forall\, t \geq 0 \ \ \widetilde{\varphi}(t) = \begin{cases} \varphi_n(t - t_n), & t \in [t_n, t_{n+1}), \\ x_{n+1}^+, & t = t_{n+1}. \end{cases} \tag{3.3}$$

By $\widetilde{K_x}$ we denote the set of all impulsive trajectories, which start from the point x.

Let us assume that

$$\forall\, x \in H \ \text{ every } \widetilde{\varphi} \in \widetilde{K}_x \text{ is defined on } [0, +\infty). \tag{3.4}$$

Remark 3.3. Due to the construction, every impulsive trajectory is right continuous. Moreover, from (3.1) and (3.3) we obtain: $\forall\, x \in H \ \forall\, \widetilde{\varphi} \in \widetilde{K}_x$, $\forall\, t > 0 \ \widetilde{\varphi}(t) \notin M$.

Lemma 3.2 ([7]). *Assume that the conditions* (K1), (K2), (3.1), (3.2), (3.4) *are satisfied. Then the formula* $\widetilde{G}(t, x) = \{\widetilde{\varphi}(t) \mid \ \widetilde{\varphi} \in \widetilde{K}_x\}$ *defines MDS* $\widetilde{G} : \mathbb{R}_+ \times H \to P(H)$*, which we call impulsive MDS.*

4 Existence of global attractor of impulsive MDS, generated by problem (2.1), (2.4), (2.6)

Problem (2.1) generates a family of continuous maps:

$$K^\varepsilon = \big\{ z : [0, +\infty) \to H \mid \ z \text{ is a solution of } (2.1) \big\},$$

which due to the autonomy of the problem (2.1) satisfies the conditions (K1), (K2).

Lemma 4.1. *Under conditions* (2.2), (2.3) *and the inequality*

$$2\beta\gamma \leq 1 \tag{4.1}$$

for a sufficiently small ε*, problem* (2.1), (2.4), (2.6) *generates impulsive MDS, and every impulsive trajectory, which starts from the set* M'*, has an infinite number of impulsive perturbations.*

Remark 4.1. Here and in the sequel, under the expression *for a sufficiently small* ε we mean that some property holds for all $\varepsilon \in (0, \varepsilon_0)$, where ε_0 depends only of the parameters of problem (2.1).

Proof of Lemma 4.1. Let us verify conditions (3.1), (3.2) and (3.4). Condition (3.1) follows from the definition of the sets M and M'. Due to conditions (2.2), (2.3) and Poincaré inequality, there exists $\delta > 0$ such that for every solution z of the problem (2.1) and for almost all $t > 0$ we get the inequality

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|_H^2 + \delta \|z(t)\|_H^2 \leq \varepsilon\sqrt{2}\, C \|z(t)\|_H. \tag{4.2}$$

Then, for a sufficiently small ε, we obtain

$$\forall z \in K^\varepsilon \ \ \forall t \geq 0 \ \ \|z(t)\|_H^2 \leq \|z(0)\|_H^2 e^{-\delta t} + 1. \tag{4.3}$$

Moreover, for every $z = \begin{pmatrix} u \\ v \end{pmatrix} \in K^\varepsilon$ and for every $i \geq 1$ we get the following equalities:

$$(u(t), \psi_i) = (u(0), \psi_i) e^{-a\lambda_i t} + \varepsilon \int_0^t e^{-a\lambda_i (t-s)} (f_1(u(s), v(s)), \psi_i)\, ds, \tag{4.4}$$

$$\begin{aligned}(v(t), \psi_i) = \big((v(0), \psi_i) - 2b\lambda_i (u(0), \psi_i) t\big) e^{-a\lambda_i t} + \varepsilon \int_0^t e^{-a\lambda_i (t-s)} (f_2(u(s), v(s)), \psi_i)\, ds \\ - \varepsilon 2b\lambda_i \int_0^t (t-s) e^{-a\lambda_i (t-s)} (f_1(u(s), v(s)), \psi_i)\, ds.\end{aligned} \tag{4.5}$$

Further, for the sake of simplicity, we denote $\psi := \psi_1$, $\lambda := \lambda_1$ and for $z \in K^\varepsilon$ consider the function

$$g_\varepsilon(t) = \alpha(u(t), \psi) + \beta(v(t), \psi).$$

From (4.4), (4.5), for $z(0) \in M$, we deduce

$$g_\varepsilon(t) = e^{-a\lambda t}(1 - 2\beta b\lambda t(u(0), \psi)) + \varepsilon F_\varepsilon(t),$$

where the function $F_\varepsilon \in C^1([0, \infty))$, $F_\varepsilon(0) = 0$, depends on $z \in K^\varepsilon$, however

$$\exists\, C_1 > 0 \ \ \forall \varepsilon \in (0, 1) \ \ \sup_{t \geq 0} \big(|F_\varepsilon(t)| + |F_\varepsilon'(t)|\big) \leq C_1. \tag{4.6}$$

From (2.2) and (2.3) we get

$$g_\varepsilon'(0) = -a\lambda - 2\beta b\lambda (u(0), \psi) + \varepsilon F_\varepsilon'(0).$$

Since $|(u(0), \psi)| \leq \gamma$, from (4.1) and (4.6) for a sufficiently small ε there exists $\tau = \tau(z(0), \varepsilon) > 0$ such that $\forall t \in (0, \tau)$ $g_\varepsilon(t) < 1$. Thus we get property (3.2).

Let us prove (3.4). Due to estimation (4.3), condition (3.4) is satisfied if z do not intersect the set M. Thus, let us take arbitrarily $z \in K^\varepsilon$ from $z(0) = z_0 \in M'$ and consider the function $g_\varepsilon(t)$, which has the form

$$g_\varepsilon(t) = e^{-a\lambda t}(1 + \mu - 2\beta b\lambda t(u_0, \psi)) + \varepsilon F_\varepsilon(t).$$

Since $g_\varepsilon(0) = 1 + \mu$, $\limsup_{t \to \infty} g_\varepsilon(t) \leq \varepsilon C_1$, for a sufficient small $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that

$$\forall t \in (0, s_\varepsilon) \ \ g_\varepsilon(t) > 1, \ \ g_\varepsilon(s_\varepsilon) = 1. \tag{4.7}$$

Let us show that for a sufficiently small $\varepsilon > 0$ the inequality

$$|(u(s_\varepsilon), \psi)| \leq \gamma \tag{4.8}$$

is fulfilled, i.e., $z(s_\varepsilon) \in M$. Indeed, for a sufficiently small ε, from (4.7) we have the next inequality

$$\Big(1 + \frac{\mu}{2}\Big) e^{a\lambda s_\varepsilon} \geq 1 + \mu - 2\beta b\lambda s_\varepsilon (u_0, \psi). \tag{4.9}$$

As $|(u_0, \psi)| \leq \gamma$, from (4.9) we obtain

$$s_\varepsilon \geq \widetilde{s}, \tag{4.10}$$

where $\widetilde{s} > 0$ does not depend on ε, z_0 and is the root of the equation

$$\Big(1 + \frac{\mu}{2}\Big) e^{a\lambda \widetilde{s}} = 1 + \mu - 2\beta |b| \lambda \widetilde{s} \gamma.$$

Then from (4.4) we deduce the estimation

$$|(u(s_\varepsilon), \psi)| \le \gamma e^{-a\lambda\widetilde{s}} + \varepsilon C_1,$$

from which for a sufficiently small $\varepsilon > 0$ we obtain (4.8). Thus, every impulsive trajectory, which starts from the set M', has an infinite number of impulsive points and, due to estimation (4.10), we have (3.4). □

Therefore, for a sufficiently small ε, the impulsive multi-valued dynamical system $\widetilde{G}_\varepsilon : R_+ \times H \to P(H)$,

$$\forall\, t \ge 0 \;\; \forall\, z_0 \in H \;\; \widetilde{G}_\varepsilon(t, z_0) = \big\{ z(t) \mid \;\; z(\,\cdot\,) \in \widetilde{K}^\varepsilon_{z_0} \big\}, \tag{4.11}$$

is correctly defined, where $\widetilde{K}^\varepsilon_{z_0}$ is the set of all impulsive trajectories of problem (2.1), (2.4), (2.6), which start from the point z_0.

The main result of this paper is the following

Theorem. *For a sufficiently small* $\varepsilon > 0$, *under conditions* (2.2), (2.3), (4.1), *the impulsive* MDS (4.11) *has a global attractor* Θ_ε. *Moreover, the limit equality* (2.7) *is fulfilled.*

Proof. Let us verify the dissipativity property. If for $\|z_0\| \le R$ the impulsive trajectory $z \in \widetilde{K}^\varepsilon_{z_0}$ does not have impulsive points, then from (4.3) it follows that

$$\|z(t)\| \le \sqrt{2} \;\; \forall\, t \ge T = \frac{1}{\delta} \ln R^2.$$

Otherwise, for a sufficiently small ε, using the function g_ε, for the moment $s_\varepsilon = s(z) > 0$, we obtain the inequality

$$e^{-a\lambda s_\varepsilon}\big(\alpha(u_0, \psi) + \beta(v_0, \psi) - 2\beta b\lambda s_\varepsilon(u_0, \psi)\big) \ge \frac{1}{2}\,,$$

thus, we deduce $s_\varepsilon \le s(R)$, where $s(R) > 0$ is a solution of the equation

$$\frac{1}{2}\, e^{a\lambda s_\varepsilon} = \sqrt{\alpha^2 + \beta^2}\, R + 2\beta|b|\lambda s_\varepsilon R.$$

After this, the phase point jumps into the point $z_1^+ = z(s_\varepsilon) \in I(z(s_\varepsilon - 0))$. Due to the form of the impulsive map (2.6), we deduce the estimation

$$\forall\, z \in H \;\forall\, z^+ \in I(z) \;\; \|z^+\|^2_H \le \kappa^2 + \|z\|^2_H, \tag{4.12}$$

where $\kappa^2 := \gamma^2 + (\frac{1+\mu+\alpha\gamma}{\beta})^2$. In particular,

$$\|z(s_\varepsilon)\|^2_H \le \kappa^2 + R^2 + 1.$$

Therefore, it suffices to prove the dissipativity condition only for those impulsive trajectories, which start from the set IM, i.e., for a sufficiently small ε, it suffices to prove that

$$\begin{gathered} \exists\, R_0 > 0 \;\; \forall\, R > 0 \;\; \exists\, T = T(R) > 0 \;\; \forall\, z_0 \in IM, \;\; \|z_0\|_H \le R, \\ \forall\, z \in \widetilde{K}^\varepsilon_{z_0} \;\; \forall\, t \ge T \;\; \|z(t)\|_H \le R_0. \end{gathered} \tag{4.13}$$

But if $\{s^i_\varepsilon\}_{i=0}^\infty$ are the moments of the impulsive perturbation for $z \in \widetilde{K}^\varepsilon_{z_0}$, then from (4.3), (4.12) and inequality (4.10) we find that for $k \ge 0$,

$$\Big\| z\Big(\sum_{i=0}^{k} s^i_\varepsilon\Big)\Big\|^2_H \le e^{-\delta(k+1)\overline{s}} R^2 + \frac{\kappa^2}{1 - e^{-\delta\overline{s}}}\,. \tag{4.14}$$

Thus, from the last inequality and formula (4.3) follows (4.13), where $R_0 = 2 + \frac{\kappa^2}{1-e^{-\delta\overline{s}}}$.

Let us prove that $\widetilde{G}_\varepsilon$ is asymptotically compact. Towards this end, we fix an arbitrary solution $z=\begin{pmatrix}u\\ v\end{pmatrix}$ of problem (2.1). Considering every equation in (2.1) as a linear equation with right-hand side $h_1(t)=\varepsilon f_1(u(t),v(t))$, $h_2(t)=2b\Delta u(t)+\varepsilon f_2(u(t),v(t))$, from the regularity lemma [23] we deduce that there exists a constant $C_2>0$, which depends only on the parameters of problem (2.1) and does not depend on ε, such that for almost all $t>0$,

$$\frac{d}{dt}\,\|u(t)\|^2_{H^1_0}+a\|\Delta u(t)\|^2\le C_2, \tag{4.15}$$

$$\frac{d}{dt}\,\|u(t)\|^2+a\|u(t)\|^2_{H^1_0}\le C_2, \tag{4.16}$$

$$\frac{d}{dt}\,\|v(t)\|^2_{H^1_0}+a\|\Delta v(t)\|^2\le \frac{4b^2}{a}\,\|\Delta u(t)\|^2+C_2, \tag{4.17}$$

$$\frac{d}{dt}\,\|v(t)\|^2+a\|v(t)\|^2_{H^1_0}\le \frac{4b^2}{a}\,\|u(t)\|^2_{H^1_0}+C_2. \tag{4.18}$$

From (4.15), (4.16) and the Uniform Gronwall Lemma [23] we obtain

$$\forall\, t>0\ \ \|u(t)\|^2_{H^1_0}\le C_2t+\frac{\|u(0)\|^2}{at}+\frac{2C_2}{a}\,. \tag{4.19}$$

Then from (4.17)–(4.19) and the Uniform Gronwall Lemma we have

$$\forall\, t>0\ \forall\, r\in(0,t)\ \|v(t)\|^2_{H^1_0}\le\Big(\frac{4b^2}{a^2}+1\Big)\Big(\frac{\|u(0)\|^2+\|v(0)\|^2}{ar}+\frac{2C_2}{a}\Big)+\frac{C_2r+\|u(t-r)\|^2_{H^1_0}}{a}\,. \tag{4.20}$$

Assume that $r=\frac{t}{2}$ and from (4.19), (4.20) we get the following estimation:

$$\begin{aligned}\forall\, t>0\ \ \|v(t)\|^2_{H^1_0}\le\Big(\frac{4b^2}{a^2}+1\Big)&\Big(\frac{2(\|u(0)\|^2+\|v(0)\|^2)}{at}+\frac{2C_2}{a}\Big)\\ &+\frac{C_2t}{2a}\Big(1+\frac{1}{2a}\Big)+\frac{2\|u(0)\|^2}{ta^2}+\frac{2C_2}{a^2}\,.\end{aligned} \tag{4.21}$$

Now, let $z_0^{(n)}=\sum\limits_{i=1}^{\infty}\begin{pmatrix}c_i^{(n)}\\ d_i^{(n)}\end{pmatrix}\cdot\psi_i$, $\|z_0^{(n)}\|_H\le R$, be an arbitrary bounded sequence of initial data, $\xi_n\in\widetilde{G}_\varepsilon(t_n,z_0^{(n)})$, $t_n\nearrow+\infty$. Then $\xi_n=z_n(t_n)$, where $z_n\in\widetilde{K}^\varepsilon_{z_0^{(n)}}$. If z_n does not have impulsive points, then for the function $y_n(t)=z_n(t+t_n-1)$, $t\ge0$ we obtain

$$y_n\in\widetilde{K}^\varepsilon_{z_n(t_n-1)},\ \ \xi_n=z_n(t_n)=y_n(1).$$

From (4.3) we find that $\|z_n(t_n-1)\|\le\sqrt{2}\ \forall\, n\ge N(R)$. Therefore, from estimates (4.19), (4.21), the sequence $\{y_n(1)=\xi_n\}$ is bounded in $H^1_0(\Omega)\times H^1_0(\Omega)$ and, hence, is precompact in H.

Otherwise, without loss of generality, from the previous arguments we can assume that $z_0^{(n)}\in IM$, $\|z_0^{(n)}\|_H\le R$. Let $\{T_{i+1}^{(n)}=\sum\limits_{k=0}^{i}s_k^{(n)}\}_{i=0}^{\infty}$ be the moments of impulsive perturbation for $z_n(\,\cdot\,)=\begin{pmatrix}u_n(\,\cdot\,)\\ v_n(\,\cdot\,)\end{pmatrix}$, $\{\eta_i^{(n)+}=z_n(T_i^{(n)})\}_{i=1}^{\infty}\subset IM$ be the corresponding impulsive points. Let us prove the precompactness of the sequence $\{\eta_i^{(n)+}\}$. From the dissipativity condition (4.13), the estimation

$$\overline{s}\le s_k^{(n)}\le\widehat{s}, \tag{4.22}$$

and the estimates (4.19), (4.21), we get the existence of the constant $C(R)$, independent of ε, such that

$$\forall\, i\ge1\ \ \forall\, n\ge1\ \ \big\|u_n(T_i^{(n)}-0)\big\|^2_{H^1_0}+\big\|v_n(T_i^{(n)}-0)\big\|^2_{H^1_0}\le C(R). \tag{4.23}$$

Then from (2.6) and (4.23) for all $i \geq 1$, $n \geq 1$ we deduce the estimation

$$\left\|u_n(T_i^{(n)})\right\|^2_{H_0^1} + \left\|v_n(T_i^{(n)})\right\|^2_{H_0^1} \leq C(R) + 2\lambda\gamma^2. \tag{4.24}$$

Therefore, due to (4.24) and the compactness of the embedding $H_0^1(\Omega) \subset L^2(\Omega)$, there follows the required precompactness of the set $\{\eta_i^{(n)+} \mid i \geq 1, n \geq 1\}$ in H. Then for the sequence $\xi_n \in \widetilde{G}_\varepsilon(t_n, z_0^{(n)})$, for every $n \geq 1$, there exists a number $i = i(n)$, $i(n) \to \infty$, $n \to \infty$, such that $t_n \in [T_{i(n)}^{(n)}, T_{i(n)+1}^{(n)})$. Thus, from the inclusion

$$\xi_n = z_n(t_n) \in \widetilde{G}_\varepsilon\big(t_n - T_{i(n)}^{(n)}, \eta_{i(n)}^{(n)+}\big) \tag{4.25}$$

it follows that $\xi_n = y_n(\tau_n)$, where $\tau_n := t_n - T_{i(n)}^{(n)}$, $y_n \in K^\varepsilon$ is a sequence of solutions of the (non-perturbed) problem (2.1), where $y_n(0) = \eta_{i(n)}^{(n)+}$. Since from the previous arguments on some subsequence we have $\eta_{i(n)}^{(n)+} \to \eta$ in H, and from the inclusion $\tau_n \in [0, \widehat{s}]$ on some subsequence we have $\tau_n \to \tau \in [0, \widehat{s}]$, from the regularity results [9] of the solutions of the problem (2.1) we deduce the following result:

$$y_n(\tau_n) \longrightarrow y(\tau) \text{ in } H, \text{ where } y \in K^\varepsilon,\ y(0) = \eta. \tag{4.26}$$

Thus, the sequence $\{\xi_n\}$ is precompact in H, and from Lemma 3.1 we deduce the existence of the global attractor

$$\Theta_\varepsilon = \bigcap_{s>0} \overline{\bigcup_{t\geq s} \widetilde{G}_\varepsilon(t, B_0)}, \tag{4.27}$$

where the dissipative set B_0 is defined from (4.14) and does not depend on ε.

Let us prove convergence (2.7). It suffices to show that for $\varepsilon_k \to 0$, $\xi^{(k)} \in \Theta_{\varepsilon_k}$, on the subsequence

$$\xi^{(k)} \longrightarrow \xi \in \Theta \text{ in } H, \quad k \to \infty.$$

From (4.27), there exist the sequences $\{t_k \nearrow \infty\}$, $\{z_k^0\} \subset B_0$, $z_k \in \widetilde{K}^{\varepsilon_k}_{z_k^0}$, such that $\forall\, k \geq 1$ $\|\xi^{(k)} - z_k(t_k)\| \leq 1/k$. If z_k do not have impulsive perturbations, then using (4.2) we obtain the estimation

$$\forall\, t \geq 0 \ \ \|z_k(t)\|^2_H \leq \|z_k^0\|^2_H e^{-\delta t} + \frac{2\varepsilon_k^2 C^2}{\delta^2}\,,$$

from which it follows that $\xi^{(k)} \to 0$ in H.

Otherwise, if z_k have impulsive perturbations, then under conditions (4.25) for $\xi_k = z_k(t_k)$, we obtain the equality

$$\xi_k = y_k(\tau_k), \quad y_k \in K^{\varepsilon_k}_{\eta_k^+},$$

whence, using the notation from the previous part of the proving, it follows that

$$\tau_k := t_k - T_{i(k)}^{(k)} \longrightarrow \tau, \ \ \eta_k^+ := \eta_{i(k)}^{(k)+} \longrightarrow \eta, \ \ i(k) \to \infty, \ \ k \to \infty.$$

Since $\tau_k \in [0, s_{i(k)}^{(k)}]$ and the point $s_k := s_{i(k)}^{(k)}$ satisfies inequality (4.22) and is a solution of the equation

$$e^{-a\lambda s_k}(1 + \mu - 2\beta b\lambda s_k(u_0^k, \psi)) + \varepsilon_k F_{\varepsilon_k}(s_k) = 1,$$

where u_0^k, v_0^k are the components of the vector $\eta_k^+ \in IM$, for $k \to \infty$ we obtain that on the subsequence $s_k \to s$, where $\tau \in [0, s]$ and s is a solution of the equation

$$e^{-a\lambda s}(1 + \mu - 2\beta b\lambda s(u_0, \psi)) = 1, \tag{4.28}$$

$$|(u_0, \psi_1)| \leq \gamma, \quad \alpha(u_0, \psi_1) + \beta(v_0, \psi_1) = 1 + \mu. \tag{4.29}$$

From (4.4), (4.5) we deduce that $\forall\, i \geq 1$

$$(u_k(\tau_k), \psi_i) = (u_0^k, \psi_i)e^{-a\lambda_i\tau_k} + \varepsilon_k \int_0^{\tau_k} e^{-a\lambda_i(\tau_k - s)}(f_1(u_k(s), v_k(s)), \psi_i)\, ds,$$

$$(v_k(\tau_k), \psi_i) = \big((v_0^k, \psi_i) - 2b\lambda_i (u_0^k, \psi_i)\tau_k\big)e^{-a\lambda_i\tau_k} + \varepsilon_k \int_0^{\tau_k} e^{-a\lambda_i(\tau_k - s)}(f_2(u_k(s), v_k(s)), \psi_i)\,ds$$
$$- \varepsilon_k 2b\lambda_i \int_0^{\tau_k} (\tau_k - s)e^{-a\lambda_i(\tau_k - s)}(f_1(u_k(s), v_k(s)), \psi_i)\,ds.$$

Analogously to (4.26), we can assume that

$$\xi_k = \begin{pmatrix} u_k(\tau_k) \\ v_k(\tau_k) \end{pmatrix} \longrightarrow \xi = y(\tau) = \begin{pmatrix} u(\tau) \\ v(\tau) \end{pmatrix} \text{ in } H, \text{ where } y \in K^\varepsilon,\ y(0) = \eta.$$

Then, as $k \to \infty$, we obtain

$$(u(\tau), \psi_1) = (u_0, \psi_1)e^{-a\lambda_i\tau}, \tag{4.30}$$
$$(v(\tau), \psi_1) = \big((v_0, \psi_1) - 2b\lambda_1(u_0, \psi_1)\tau\big)e^{-a\lambda_1\tau}, \tag{4.31}$$

where $\tau \in [0, s]$, s is a unique root of equation (4.28) under fixed u_0, v_0 from (4.29).

Taking into account a "non-impulsive" character of the coordinates $j \geq 2$ along each impulsive trajectory, from (2.2) we get

$$\forall\, j \geq 2 \ \ |(u_0^k, \psi_j)| + |(v_0^k, \psi_j)| \longrightarrow 0, \ \ k \to \infty. \tag{4.32}$$

Then from (4.30)–(4.32) we obtain that $\xi \in \Theta$ and (2.7) takes place. □

Remark 4.2. As is shown in [12], for the impulsive DS the global attractor Θ is, generally speaking, not invariant set of the semiflow $\widetilde{G}$. However, the set $\Theta \setminus M$ [4] may have such a property. The invariance property can be obtained from the explicit formula of Θ, when $\varepsilon = 0$. It turns out that if, additionally, the map I is upper-semicontinuous, this fact is valid for a sufficiently small $\varepsilon > 0$, i.e., the equality

$$\forall\, t \geq 0 \ \ \widetilde{G}_\varepsilon(t, \Theta_\varepsilon \setminus M) = \Theta_\varepsilon \setminus M$$

is satisfied. It will be done in the forthcoming papers.

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