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Eugene Bravyi

BOUNDARY VALUE PROBLEMS FOR FAMILIES OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. We consider boundary value problems for all equations from a family of linear functional differential equations. The necessary and sufficient conditions for the unique solvability and existence of non-negative (non-positive) solutions are obtained.*

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1 Introduction

In the recent years, the boundary value problems for functional differential equations have been investigated in many works (for example, [1, 6–12]). We offer new conditions for a unique solvability of boundary value problems and the existence of solutions with a given sign. It turns out, these conditions are sharp in some family of equations.

Here we use the following notation: $\mathbf{AC}^{n-1}[0,1]$ is the space of functions $x:[0,1]\to\mathbb{R}$ for which there exist absolutely continuous derivatives of order less than n; $\mathbf{C}[0,1]$ is the space of continuous functions $x:[0;1]\to\mathbb{R}$ with the norm $\|x\|_{\mathbf{C}}=\max\limits_{t\in[0,1]}|x(t)|$; $\mathbf{L}[0,1]$ is the space of integrable functions $z:[0;1]\to\mathbb{R}$ with the norm $\|z\|_{\mathbf{L}}=\int\limits_0^1|z(s)|\,ds$.

We consider general boundary value problems for linear functional differential equations

$$\begin{cases} x^{(n)}(t)=(Tx)(t)+f(t), & t\in[0,1],\\ \ell_i x=\alpha_i, & i=1,\dots,n, \end{cases} \tag{1.1}$$

where $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ is a linear bounded operator; $f\in\mathbf{L}[0,1]$; $\ell_i:\mathbf{AC}^{n-1}[0,1]\to\mathbb{R}$, $i=1,\dots,n$, are linear bounded functionals with the representation

$$\ell_i x=\sum_{j=0}^{n-1}a_{ij}x^{(j)}(0)+\int\limits_0^1\varphi_i(s)x^{(n)}(s)\,ds,\ \ i=1,\dots,n,$$

$\varphi_i:[0,1]\to\mathbb{R}$, $i=1,\dots,n$, are measurable bounded functions, $a_{ij}\in\mathbb{R}$, $i,j=1,\dots,n$; $\alpha_i\in\mathbb{R}$, $i=1,\dots,n$. A solution of (1.1) is a function from the space $\mathbf{AC}^{n-1}[0,1]$ which satisfies for almost all $t\in[0,1]$ the functional differential equation from problem (1.1) and the boundary value conditions from (1.1).

Such problem (1.1) has the Fredholm property (see, for example, [2]), therefore problem (1.1) is uniquely solvable if and only if the homogeneous boundary value problem

$$\begin{cases} x^{(n)}(t)=(Tx)(t), & t\in[0,1],\\ \ell_i x=0, & i=1,\dots,n, \end{cases} \tag{1.2}$$

has only the trivial solution.

We will use the notation $\ell\equiv\{\ell_1,\ell_2,\dots,\ell_n\}$, $\alpha\equiv\{\alpha_1,\alpha_2,\dots,\alpha_n\}$.

An operator $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ is called positive if for every non-negative function $x\in\mathbf{C}[0,1]$ the inequality $(Tx)(t)\ge 0$ holds for a.a. $t\in[0,1]$.

Here we suppose that p^+, $p^-\in\mathbf{L}[0,1]$ are the given non-negative functions.

Definition 1.1. Denote by $\mathbb{S}(p^+,p^-)$ the family of all operators $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ such that

$$T=T^+-T^-,$$

where T^+, $T^-:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ are linear positive operators satisfying the conditions

$$T^+\mathbf{1}=p^+,\quad T^-\mathbf{1}=p^-.$$

Definition 1.2. We say that the pair (p^+,p^-) belongs to the set $\mathbb{A}_{n,\ell}$ if problem (1.1) is uniquely solvable for every operator $T\in\mathbb{S}(p^+,p^-)$.

Definition 1.3. We say that the pair (p^+,p^-) belongs to the set $\mathbb{B}^+_{n,\ell}(\alpha,f)$ if $(p^+,p^-)\in\mathbb{A}_{n,\ell}$ and a unique solution of problem (1.1) is non-negative for every operator $T\in\mathbb{S}(p^+,p^-)$.

Definition 1.4. We say that the pair (p^+,p^-) belongs to the set $\mathbb{B}^-_{n,\ell}(\alpha,f)$ if $(p^+,p^-)\in\mathbb{A}_{n,\ell}$ and a unique solution of problem (1.1) is non-positive for every operator $T\in\mathbb{S}(p^+,p^-)$.

In this paper, we give an effective description of the sets $\mathbb{A}_{n,\ell}$, $\mathbb{B}^{+}_{n,\ell}(\alpha, f)$, $\mathbb{B}^{-}_{n,\ell}(\alpha, f)$ under the following condition. We suppose that the boundary value problem

$$\begin{cases} x^{(n)}(t) = f(t), \quad t \in [0,1], \\ \ell_i x = \alpha_i, \quad i = 1, \dots, n, \end{cases} \tag{1.3}$$

is uniquely solvable. Then its solution w has a representation

$$w(t) \equiv \sum_{i=1}^{n} \alpha_i x_i(t) + (Gf)(t), \ \ t \in [0,1],$$

where the functions x_1, $x_2, \dots, x_n$ form a fundamental system of solutions to the equation $x^{(n)} = \mathbf{0}$; $G : \mathbf{L}[0,1] \to \mathbf{AC}^{n-1}[0,1]$ is the Green operator defined by the equality

$$(Gf)(t) = \int_0^1 G(t,s) f(s)\, ds, \ \ t \in [0,1];$$

$G(t,s)$ is the Green function of problem (1.3). Note, that the Green function $G(t,s)$ has a representation

$$G(t,s) = C(t,s) + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i(t) \varphi_j(s), \ \ t, s \in [0,1],$$

where

$$C(t,s) = \begin{cases} \dfrac{(t-s)^{n-1}}{(n-1)!}, & 0 \le s \le t \le 1, \\ 0, & 0 \le t < s \le 1, \end{cases}$$

$c_{ij} \in \mathbb{R}$, $i, j \in \{1, 2, \dots, n\}$.

2 The unique solvability for all equations with operators from the family $\mathbb{S}(p^+, p^-)$

Denote

$$p(t) \equiv p^+(t) - p^-(t), \quad v(t) \equiv 1 - (Gp)(t), \ \ t \in [0,1],$$
$$g_{t_2,t_1,v}(s) \equiv G(t_2,s)v(t_1) - G(t_1,s)v(t_2), \quad s \in [0,1], \ \ 0 \le t_1 \le t_2 \le 1,$$
$$[a]^+ \equiv \frac{|a| + a}{2}, \quad [a]^- \equiv \frac{|a| - a}{2} \ \text{ for any } a \in \mathbb{R}.$$

Theorem 2.1. *The pair (p^+, p^-) belongs to the set $\mathbb{A}_{n,\ell}$ if and only if one of the following conditions holds:*

(1) $v(t) > 0$ *for all* $t \in [0,1]$ *and*

$$\int_0^1 \left(p^+(s)[g_{t_2,t_1,v}(s)]^- + p^-(s)[g_{t_2,t_1,v}(s)]^+\right) ds < v(t_2) \ \text{ for all } 0 \le t_1 \le t_2 \le 1;$$

(2) $v(t) < 0$ *for all* $t \in [0,1]$ *and*

$$\int_0^1 \left(p^+(s)[g_{t_2,t_1,v}(s)]^+ + p^-(s)[g_{t_2,t_1,v}(s)]^-\right) ds < -v(t_2) \ \text{ for all } 0 \le t_1 \le t_2 \le 1.$$

For proving Theorem 2.1, we need the following lemma (see [3,4]).

Lemma 2.1. *Boundary value problem* (1.2) *has only the trivial solution for every operators* $T \in \mathbb{S}(p^+, p^-)$ *if and only if the boundary value problem*

$$\begin{cases} x^{(n)}(t) = p_1(t)x(t_1) + p_2(t)x(t_2), & t \in [0,1], \\ \ell_i x = 0, \quad i = 1, \dots, n, \end{cases} \tag{2.1}$$

has only the trivial solution for every functions p_1, p_2 *and points* t_1, t_2 *such that*

$$p_1, p_2 \in \mathbf{L}[0,1], \tag{2.2}$$

$$p_1 + p_2 = p^+ - p^-, \tag{2.3}$$

$$-p^-(t) \le p_i(t) \le p^+(t), \ \ t \in [0,1], \ \ i = 1, 2, \tag{2.4}$$

$$0 \le t_1 \le t_2 \le 1. \tag{2.5}$$

Proof of Theorem 2.1. Boundary value problem (2.1) is equivalent to the equation

$$x(t) = (Gp_1)(t)x(t_1) + (Gp_2)(t)x(t_2), \ \ t \in [0,1].$$

This equation has only the trivial solution if and only if the algebraic system

$$x(t_1) = (Gp_1)(t_1)x(t_1) + (Gp_2)(t_1)x(t_2), \quad x(t_2) = (Gp_1)(t_2)x(t_1) + (Gp_2)(t_2)x(t_2)$$

with respect to $x(t_1)$, $x(t_2)$ has only the trivial solution, that is, when

$$\Delta(t_1, t_2, p_1, p_2) \equiv \begin{vmatrix} 1 - (Gp_1)(t_1) & -(Gp_2)(t_1) \\ -(Gp_1)(t_2) & 1 - (Gp_2)(t_2) \end{vmatrix}$$

$$= \begin{vmatrix} 1 - (Gp_1)(t_1) & v(t_1) \\ -(Gp_1)(t_2) & v(t_2) \end{vmatrix} = v(t_2) + \int_0^1 p_1(s) g_{t_2,t_1,v}(s)\, ds \ne 0, \tag{2.6}$$

We use Lemma 2.1. From the form of the set of admissible function p_i (2.4), it follows that $\Delta(t_1, t_2, p_1, p_2)$ does not equal to zero for every t_i, p_i, $i = 1, 2$, if and only if the conditions of Theorem 2.1 are fulfilled. It guarantees the unique solvability of all problems (2.1) under the conditions (2.2)–(2.5). □

3 Examples

Consider the Cauchy problem

$$\begin{cases} \dot{x}(t) = (Tx)(t) + f(t), & t \in [0,1], \\ x(0) = \alpha_1. \end{cases}$$

As an immediate result from Theorem 2.1, we have

Corollary 3.1. *The pair* (p^+, p^-) *belongs to the set* $\mathbb{A}_{1,\{x(0)\}}$ *if and only if the inequality*

$$1 + \int_0^{t_1} p^-(s)\, ds \Big(1 - \int_{t_1}^{t_2} p^-(s)\, ds \Big) - \int_0^{t_2} p^+(s)\, ds + \int_0^{t_1} p^+(s)\, ds \int_{t_1}^{t_2} p^+(s)\, ds > 0$$

holds for all $0 \le t_1 \le t_2 \le 1$.

Now we can easily get the following known assertion.

Corollary 3.2 ([5]).

$(p^+, \mathbf{0}) \in \mathbb{A}_{1,\{x(0)\}}$ *if and only if* $\int\limits_0^1 p^+(s)\,ds < 1$;

$(\mathbf{0}, p^-) \in \mathbb{A}_{1,\{x(0)\}}$ *if and only if* $\int\limits_0^1 p^-(s)\,ds < 3$.

Set $p^+(t) \equiv \mathcal{T}^+ t$, $p^-(t) \equiv \mathcal{T}^- t$, $t \in [0,1]$, where $\mathcal{T}^+ \geq 0$, $\mathcal{T}^- \geq 0$.

Corollary 3.3. *The pair* (p^+, p^-) *belongs to the set* $\mathbb{A}_{1,\{x(0)\}}$ *if and only if*

$$0 \leq \mathcal{T}^+ < 2, \ \ 0 \leq \mathcal{T}^- < 1 + \sqrt{5}$$

or

$$0 \leq \mathcal{T}^+ < 2, \ \ \mathcal{T}^- > 1 + \sqrt{5},$$
$$(\mathcal{T}^-)^2\,(6 - \mathcal{T}^-)(\mathcal{T}^- + 2) - (\mathcal{T}^+)^2\,(4 - \mathcal{T}^+)^2 + 2\mathcal{T}^+\mathcal{T}^-(\mathcal{T}^+\mathcal{T}^- - 2\mathcal{T}^+ - 4\mathcal{T}^-) > 0.$$

Consider the Cauchy problem for the second order functional differential equation

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), & t \in [0,1], \\ x(a) = \alpha_1, & \dot{x}(a) = \alpha_2, \end{cases}$$

From Theorem 2.1, we have

Corollary 3.4.

$(\mathbf{0}, \mathcal{T}^-) \in \mathbb{A}_{2,\{x(0),\dot{x}(0)\}}$ *if and only if* $\mathcal{T}^- < 16$;

$(\mathbf{0}, p^-) \in \mathbb{A}_{2,\{x(0),\dot{x}(0)\}}$ *if* $p^-(t) \leq 16$ *for all* $t \in [0,1]$, $p^- \not\equiv 16$.

Consider the Dirichlet boundary value problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t), & t \in [0,1], \\ x(0) = \alpha_1, & x(1) = \alpha_2, \end{cases}$$

Corollary 3.5.

$(\mathcal{T}^+, \mathbf{0}) \in \mathbb{A}_{2,\{x(0),x(1)\}}$ *if and only if* $\mathcal{T}^+ < 32$;

$(p^+, \mathbf{0}) \in \mathbb{A}_{2,\{x(0),x(1)\}}$ *if* $p^+(t) \leq 32$ *for all* $t \in [0,1]$, $p^+ \not\equiv 32$.

4 Non-negative (non-positive) solutions for all equations with operators from the family $\mathbb{S}(p^+, p^-)$

Suppose $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$, $f \in \mathbf{L}$ and

$$\sum_{i=1}^n |\alpha_i| + \int\limits_0^1 |f(s)|\,ds > 0.$$

For every $0 \leq t_1 \leq t_2 \leq 1$, define

$$g_{t_2,t_1,w}(s) \equiv G(t_2, s)w(t_1) - G(t_1, s)w(t_2), \ \ s \in [0,1],$$

$$R_1(t_1,t_2) \equiv w(t_1) + \int\limits_0^1 \left(p^+(s)[g_{t_2,t_1,w}(s)]^- + p^-(s)[g_{t_2,t_1,w}(s)]^+\right) ds,$$

$$R_2(t_1,t_2) \equiv w(t_2) + \int\limits_0^1 \left(p^+(s)[g_{t_2,t_1,w}(s)]^+ + p^-(s)[g_{t_2,t_1,w}(s)]^-\right) ds,$$

$$R_3(t_1,t_2) \equiv w(t_1) - \int\limits_0^1 \left(p^+(s)[g_{t_2,t_1,w}(s)]^+ + p^-(s)[g_{t_2,t_1,w}(s)]^-\right) ds,$$

$$R_4(t_1,t_2) \equiv w(t_2) - \int\limits_0^1 \left(p^+(s)[g_{t_2,t_1,w}(s)]^- + p^-(s)[g_{t_2,t_1,w}(s)]^+\right) ds.$$

Theorem 4.1. *Suppose* $(p^+,p^-) \in \mathbb{A}_{n,\ell}$.

The pair (p^+,p^-) *belongs to the set* $\mathbb{B}^+_{n,\ell}(\alpha,f)$ *if and only if one of the following conditions holds:*

(1) $v(t) > 0$, $w(t) \ge 0$ *for all* $t \in [0,1]$ *and* $R_3(t_1,t_2) \ge 0$, $R_4(t_1,t_2) \ge 0$ *for all* $0 \le t_1 \le t_2 \le 1$*;*

(2) $v(t) < 0$, $w(t) \le 0$ *for all* $t \in [0,1]$ *and* $R_1(t_1,t_2) \le 0$, $R_2(t_1,t_2) \le 0$ *for all* $0 \le t_1 \le t_2 \le 1$.

The pair (p^+,p^-) *belongs to the set* $\mathbb{B}^-_{n,\ell}(\alpha,f)$ *if and only if one of the following conditions holds:*

(1) $v(t) < 0$, $w(t) \ge 0$ *for all* $t \in [0,1]$ *and* $R_3(t_1,t_2) \ge 0$, $R_4(t_1,t_2) \ge 0$ *for all* $0 \le t_1 \le t_2 \le 1$*;*

(2) $v(t) > 0$, $w(t) \le 0$ *for all* $t \in [0,1]$ *and* $R_1(t_1,t_2) \le 0$, $R_2(t_1,t_2) \le 0$ *for all* $0 \le t_1 \le t_2 \le 1$.

Lemma 4.1. *Let* $(p^+,p^-) \in \mathbb{A}_{n,\ell}$. *Then the set of all solutions of problems* (1.1) *for all operators* $T \in \mathbb{S}(p^+,p^-)$ *coincides with the set of solutions of the boundary value problem*

$$\begin{cases} x^{(n)}(t) = p_1(t)x(t_1) + p_2(t)x(t_2) + f(t), & t \in [0,1], \\ \ell_i x = \alpha_i, \quad i = 1,\dots,n, \end{cases} \tag{4.1}$$

for all functions p_1, p_2 *and points* t_1, t_2 *satisfying conditions* (2.2)–(2.5).

Proof. Let y be a solution of problem (4.1) for some functions p_1, p_2 and for some points t_1, t_2 satisfying conditions (2.2)–(2.5). Then y is a solution of problem (1.1), where $T = T^+ - T^-$ and the positive operators T^+, T^- are defined by the equalities

$$(T^+x)(t) = p^+(t)\zeta(t)x(t_1) + p^+(t)(1-\zeta(t))x(t_2), \ \ t \in [0,1],$$
$$(T^-x)(t) = p^-(t)(1-\zeta(t))x(t_1) + p^-(t)\zeta(t)x(t_2), \ \ t \in [0,1],$$

$\zeta : [0,1] \to [0,1]$ is a measurable function such that

$$p_1(t) = p^+(t)\zeta(t) - p^-(t)(1-\zeta(t)), \ \ t \in [0,1].$$

Therefore, $T \in \mathbb{S}(p^+,p^-)$.

Conversely, let y be a solution of problem (1.1) with $T \in \mathbb{S}(p^+,p^-)$. Let

$$\min_{t\in[0,1]} y(t) = y(t_1), \qquad \max_{t\in[0,1]} y(t) = y(t_2).$$

Then for positive operators T^+, T^- such that $T^+\mathbf{1} = p^+$, $T^-\mathbf{1} = p^-$ the following inequalities hold:

$$p^+(t)y(t_1) \le (T^+y)(t) \le p^+(t)y(t_2), \ \ t \in [0,1],$$
$$p^-(t)y(t_1) \le (T^-y)(t) \le p^-(t)y(t_2), \ \ t \in [0,1].$$

Therefore, there exist measurable functions $\zeta, \xi : [0,1] \to [0,1]$ such that

$$(T^+y)(t) = p^+(t)(1-\zeta(t))y(t_1) + p^+(t)\zeta(t)y(t_2), \ \ t \in [0,1],$$

$$(T^-y)(t) = p^-(t)(1-\xi(t))y(t_1) + p^-(t)\xi(t)y(t_2), \quad t \in [0,1].$$

So, the function y satisfies problem (4.1) for the functions

$$p_1(t) = (T^+\mathbf{1})(t)(1-\zeta(t)) - (T^-\mathbf{1})(t)(1-\xi(t)), \quad t \in [0,1],$$
$$p_2(t) = (T^+\mathbf{1})(t)\zeta(t) - (T^-\mathbf{1})(t)\xi(t), \quad t \in [0,1].$$

It is clear that equality (2.3) and inequalities (2.4) hold. If $t_1 > t_2$, then by renumbering p_1, p_2, t_1, t_2, condition (2.5) will be valid. □

Proof of Theorem 4.1. Find when solutions of (1.1) retain their sign for all $T \in \mathbb{S}(p^+, p^-)$. Use Lemma 4.1. The maximal and minimal values $x_1 \equiv x(t_1)$, $x_2 \equiv x(t_2)$ of a unique solution of problem (1.1) satisfy the system

$$\begin{cases} x_1 = w(t_1) + (Gp_1)(t_1)x_1 + (Gp_2)(t_1)x_2, \\ x_2 = w(t_2) + (Gp_1)(t_2)x_1 + (Gp_2)(t_2)x_2 \end{cases} \tag{4.2}$$

for some p_1, $p_2 \in \mathbf{L}[0,1]$ such that conditions (2.3), (2.4) are fulfilled.

Note that $w \not\equiv \mathbf{0}$.

From (4.2), we obtain

$$x_1 = \frac{\Delta_1(t_1,t_2,p_1,p_2)}{\Delta(t_1,t_2,p_1,p_2)}, \quad x_2 = \frac{\Delta_2(t_1,t_2,p_1,p_2)}{\Delta(t_1,t_2,p_1,p_2)},$$

where the functional $\Delta(t_1,t_2,p_1,p_2)$ is defined by equality (2.6) and retains its sign (the conditions of Theorem 2.1 are fulfilled, therefore $\operatorname{sgn}(\Delta(t_1,t_2,p_1,p_2)) = \operatorname{sgn}(1 - Gp)$); the functionals $\Delta_1(t_1,t_2,p_1,p_2)$ and $\Delta_2(t_1,t_2,p_1,p_2)$ are defined by the equalities

$$\begin{aligned} \Delta_1(t_1,t_2,p_1,p_2) &\equiv \begin{vmatrix} w(t_1) & -(Gp_2)(t_1) \\ w(t_2) & 1-(Gp_2)(t_2) \end{vmatrix} = w(t_1) - \int_0^1 p_2(s) g_{t_2,t_1,w}(s)\,ds, \\ \Delta_2(t_1,t_2,p_1,p_2) &\equiv \begin{vmatrix} 1-(Gp_1)(t_1) & w(t_1) \\ -(Gp_1)(t_2) & w(t_2) \end{vmatrix} = w(t_2) + \int_0^1 p_1(s) g_{t_2,t_1,w}(s)\,ds. \end{aligned} \tag{4.3}$$

Find the maximum and the minimum of $\Delta_1(t_1,t_2,p_1,p_2)$, $\Delta_2(t_1,t_2,p_1,p_2)$ with respect to p_1, p_2 at the fixed rest arguments. From representations (4.3) we have

$$R_1(t_1,t_2) = \max_{-p^- \le p_2 \le p^+} \Delta_1(t_1,t_2,p_1,p_2), \quad R_2(t_1,t_2) = \max_{-p^- \le p_1 \le p^+} \Delta_2(t_1,t_2,p_1,p_2),$$
$$R_3(t_1,t_2) = \min_{-p^- \le p_2 \le p^+} \Delta_1(t_1,t_2,p_1,p_2), \quad R_4(t_1,t_2) = \min_{-p^- \le p_1 \le p^+} \Delta_2(t_1,t_2,p_1,p_2),$$

that proves the theorem. □

5 Example

As an illustrative example, consider the Dirichlet problem

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + 1, \quad t \in [0,1], \\ x(0) = 0, \quad x(1) = 0. \end{cases} \tag{5.1}$$

From Theorem 4.1 we immediately obtain a sharp condition for the existence of non-positive solutions of (5.1).

Corollary 5.1. *If $p^+(t) \le 11 + 5\sqrt{5}$ for all $t \in [0,1]$, then $(p^+, \mathbf{0}) \in \mathbb{B}^-_{2,\{x(0),x(1)\}}((0,0),\mathbf{1})$. The constant $11 + 5\sqrt{5}$ is sharp.*

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References

[1] A. M. Bica, M. Curila and S. Curila, Two-point boundary value problems associated to functional differential equations of even order solved by iterated splines. *Appl. Numer. Math.* **110** (2016), 128–147.

[2] E. Bravyi, A note on the Fredholm property of boundary value problems for linear functional differential equations. *Mem. Differential Equations Math. Phys.* **20** (2000), 133–135.

[3] E. I. Bravyi, Solvability of boundary value problems for linear functional differential equations. (Russian) *R&C Dynamics, Izhevsk, Russia*, 2011.

[4] E. I. Bravyi, On the best constants in the solvability conditions for the periodic boundary value problem for higher-order functional differential equations. (Russian) *Differ. Uravn.* **48** (2012), no. 6, 773-780; translation in *Differ. Equ.* **48** (2012), no. 6, 779–786.

[5] E. Bravyi, R. Hakl and A. Lomtatidze, Optimal conditions for unique solvability of the Cauchy problem for first order linear functional differential equations. *Czechoslovak Math. J.* **52(127)** (2002), no. 3, 513–530.

[6] A. Calamai and G. Infante, Nontrivial solutions of boundary value problems for second-order functional differential equations. *Ann. Mat. Pura Appl. (4)* **195** (2016), no. 3, 741–756.

[7] A. Domoshnitsky, R. Hakl and B. Půža, Multi-point boundary value problems for linear functional-differential equations. *Georgian Math. J.* **24** (2017), no. 2, 193–206.

[8] P. Drygaś, Functional-differential equations in a class of analytic functions and its application to elastic composites. *Complex Var. Elliptic Equ.* **61** (2016), no. 8, 1145–1156.

[9] I. Kiguradze and Z. Sokhadze, On nonlinear boundary value problems for higher order functional differential equations. *Georgian Math. J.* **23** (2016), no. 4, 537–550.

[10] S. Mukhigulashvili, The nonlocal boundary value problems for strongly singular higher-order nonlinear functional-differential equations. *Ital. J. Pure Appl. Math.* No. 35 (2015), 23–50.

[11] N. Partsvania, On some nonlinear boundary value problems on a finite and an infinite intervals for systems of functional differential equations. *Mem. Differ. Equ. Math. Phys.* **67** (2016), 137–140.

[12] A. L. Skubachevskiĭ, Boundary value problems for elliptic functional-differential equations and their applications. (Russian) *Uspekhi Mat. Nauk* **71** (2016), no. 5(431), 3–112; translation in *Russian Math. Surveys* **71** (2016), no. 5, 801–906.

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Author's address:

Perm National Research Polytechnic University, 29 Komsomolsky pr., Perm 614990, Russia.
E-mail: bravyi@perm.ru