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Short Communication

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ON THE ANTIPERIODIC PROBLEM FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. The general theorem (principle of a priori boundedness) on the solvability of the antiperiodic problem for systems of nonlinear generalized ordinary differential equations is given.

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Let n be a natural number, $\omega > 0$ be a real number, $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a matrix-function with bounded total variation components on every closed interval of the real axis, and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector-function belonging to the Carathéodory class corresponding to the matrix-function A on every closed interval of the real axis.

Consider the nonlinear system of generalized ordinary differential equations

$$dx = dA(t) \cdot f(t, x) \tag{1}$$

in the antiperiodic problem

$$x(t + \omega) = -x(t) \ \text{ for } \ t \in \mathbb{R}. \tag{2}$$

We will assume that

$$A(t + \omega) = A(t) + C \ \text{ and } \ f(t + \omega, x) = -f(t, -x) \ \text{ for } \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \tag{3}$$

or

$$A(t + \omega) = -A(t) + C \ \text{ and } \ f(t + \omega, x) = f(t, -x) \ \text{ for } \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \tag{4}$$

where $C \in \mathbb{R}^{n\times n}$ is a constant matrix.

The theorem on the existence of a solution of the problem (1), (2), which will be given below and called the principle of a priori boundedness, generalizes the well known Conti–Opial type theorems (see [6, 7, 12] for the case of ordinary differential equations) and supplements the earlier known criteria of solvability of nonlinear and initial boundary value problems for systems of generalized ordinary differential equations (see [1–5, 11, 13, 14] and the references therein).

Analogous and related questions are investigated in [7–10] (see also the references therein) for the boundary value problems for linear and nonlinear systems of ordinary differential and functional differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1–5, 11, 13, 14] and the references therein).

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$, $[a,b]$ $(a,b\in\mathbb{R})$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{il})_{i,l=1}^{n,m}$ with the norm $\|X\|=\sum\limits_{i,l=1}^{n,m}|x_{il}|$;

$\mathbb{R}_+^{n\times m}=\big\{(x_{il})_{i,l=1}^{n,m}:\ x_{il}\geq 0\ (i=1,\dots,n;\ l=1,\dots,m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix.

If $X=(x_{il})_{i,l=1}^{n,m}\in\mathbb{R}^{n\times m}$, then $|X|=(|x_{il}|)_{i,l=1}^{n,m}$.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then $\det X$ is the determinant of X; I_n is the identity $n\times n$-matrix; $\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1,\dots,\lambda_n$.

$\bigvee\limits_a^b(X)$ is the total variation of the matrix-function $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ on the closed interval $[a,b]$, i.e., the sum of total variations of its components x_{il} $(i=1,\dots,n;\ l=1,\dots,m)$; $V(X)(t)=(v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(0)=0$, $v(x_{il})(t)=\bigvee\limits_0^t(x_{il})$ for $t>0$ and $v(x_{il})(t)=-\bigvee\limits_t^0(x_{il})$ for $t<0$;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X:[a,b]\to\mathbb{R}^{n\times m}$ at the point t (we assume $X(t)=X(a)$ for $t\leq a$ and $X(t)=X(b)$ for $t\geq b$, if necessary); $\triangle^-X(t)=X(t)-X(t-)$, $\triangle^+X(t)=X(t+)-X(t)$;

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<+\infty$);

$\mathrm{BV}_s([a,b],\mathbb{R}^{n\times m})$ is the normed space of all $X\in BV([a,b],\mathbb{R}^{n\times m})$ with the norm $\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$I\subset\mathbb{R}$ is an interval.

$C(I,\mathbb{R}^{n\times m})$ is the set of all continuous matrix-functions $X:I\to\mathbb{R}^{n\times m}$.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in\mathbb{R}_+$ and $x\in B_1$.

An operator $\varphi:\mathrm{BV}([a,b],\mathbb{R}^n)\to\mathbb{R}^n$ is called nondecreasing if for every $x,y\in\mathrm{BV}([a,b],\mathbb{R}^n)$ such that $x(t)\leq y(t)$ for $t\in[a,b]$ the inequality $\varphi(x)(t)\leq\varphi(y)(t)$ holds for $t\in[a,b]$.

If $\alpha:I\to\mathbb{R}$ is a nondecreasing function, then $D_\alpha=\{t\in I:\ \alpha(t+)-\alpha(t-)\neq 0\}$.

$s_1,s_2,s_c:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ are the operators defined by

$$s_1(x)(a)=s_2(x)(a)=0,$$
$$s_1(x)(t)=\sum_{a<\tau\leq t}\triangle^-x(\tau)\ \text{ and }\ s_2(x)(t)=\sum_{a\leq\tau<t}\triangle^+x(\tau)\ \text{ for }\ a<t\leq b,$$

and

$$s_c(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)\ \text{ for }\ t\in[a,b].$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)+\sum_{s<\tau\leq t}x(\tau)\triangle^-g(\tau)+\sum_{s\leq\tau<t}x(\tau)\triangle^+g(\tau)\ \text{ for }\ a\leq s<t\leq b,$$

where $\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$; if $a=b$, then we assume $\int_a^b x(t)\,dg(t)=0$; thus $\int_s^t x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [11,13,14]).

$L([a,b],\mathbb{R};g)$ is the space of all functions $x:[a,b]\to\mathbb{R}$, measurable and integrable with respect to the measure $\mu(g_c(g))$ for which

$$\sum_{a<t\leq b}|x(t)|\triangle^-g(t)+\sum_{a\leq t<b}|x(t)|\triangle^+g(t)<+\infty,$$

with the norm $\|x\|_{L,g} = \int_a^b |x(t)|\,dg(t)$.

If $g_j : [a,b] \to \mathbb{R}$ $(j=1,2)$ are nondecreasing functions, $g(t) \equiv g_1(t) - g_2(t)$, and $x : [a,b] \to \mathbb{R}$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \ \text{ for } \ a \le s \le t \le b.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i=1,\dots,l;\ k=1,\dots,n;\ j=1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int_s^t x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m} \ \text{ for } \ a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j=1,2) \ \text{ and } \ S_c(G)(t) \equiv \big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n}.$$

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^{n\times m}$, then $Car([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

(i) the function $f_{kj}(\,\cdot\,,x) : I \to D_2$ is $\mu(s_c(g_{ik}))$-measurable for every $x \in D_1$;

(ii) the function $f_{kj}(t,\,\cdot\,) : D_1 \to D_2$ is continuous for $\mu(s_c(g_{ik}))$-almost every $t \in I$ and for every $t \in D_{g_{ik}}$, and $\sup\{|f_{kj}(\,\cdot\,,x)| : \ x \in D_0\} \in L([a,b],\mathbb{R};g_{ik})$ for every compact $D_0 \subset D_1$.

If $G_j : [a,b] \to \mathbb{R}^{l\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$, and $X : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \ \text{ for } \ a \le s \le t \le b,$$

$$S_k(G)(t) \equiv S_k(G_1)(t) - S_k(G_2)(t) \ \ (k=1,2), \ \ S_c(G)(t) \equiv S_c(G_1)(t) - S_c(G_2)(t).$$

If $G_1(t) \equiv V(G)(t)$ and $G_2(t) \equiv V(G)(t) - G(t)$, then

$$L([a,b],D;G) = \bigcap_{j=1}^2 L([a,b],D;G_j),$$

$$Car([a,b]\times D_1, D_2; G) = \bigcap_{j=1}^2 Car([a,b]\times D_1, D_2; G_j).$$

If $G(t) \equiv \operatorname{diag}(t,\dots,t)$, then we omit G in the notation containing G.

The inequalities between the vectors and matrices are understood componentwise.

Below we assume that

$$A_1(t) \equiv V(A)(t) \ \text{ and } \ A_2(t) \equiv V(A)(t) - A(t).$$

A vector-function $x : \mathbb{R} \to \mathbb{R}^n$ is said to be a solution of the system (1) if its restriction on every closed interval $[a,b] \subset \mathbb{R}$ belongs to $\mathrm{BV}([a,b],\mathbb{R}^n)$ and

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot f(\tau, x(\tau)) \ \text{ for } \ s \le t.$$

Under a solution of the problem (1), (2) we mean solutions of the system (1) satisfying the condition (2).

Let $B \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, $\eta : [a,b] \to \mathbb{R}^n$ and $q : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ be a matrix-function, a vector-function and an operator, respectively. Then by a solution of the system of generalized ordinary differential inequalities

$$dx - dB(t)\cdot x \le d\eta(t) + dq(x) \ \ (\ge) \ \text{ for } \ t \in [a,b]$$

we mean a vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ such that

$$x(t) - x(s) - \int_s^t dB(\tau)\cdot x(\tau) \le \eta(t) - \eta(s) + q(x)(t) - q(x)(s) \ \ (\ge) \quad \text{for } \ a \le s \le t \le b.$$

In addition, if the vector-function $\eta : [a,b] \to \mathbb{R}^n$ is nondecreasing and $g : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n_+)$ is a positive homogeneous nondecreasing operator, then by $\Omega_{B,\eta,g}$ we denote a set of all solutions of the system

$$|dx - dB(t)\cdot x| \le d\eta(t) + dg(|x|).$$

If $\eta(t) \equiv 0$ and q is a trivial operator, then we omit η and q in the notations containing ones. Thus Ω_B is the set of all solutions of the homogeneous system of generalized differential equations

$$dx = dB(t)\cdot x.$$

We define

$$\alpha_l(t) = \sum_{i=1}^n v(a_{il})(t) \ \ (l = 1,\dots,n) \ \text{ and } \ \alpha(t) = \sum_{i=1}^n \alpha_i(t) \ \text{ for } \ t \in \mathbb{R}.$$

Under the conditions (3) or (4), it is not difficult to verify that if a vector-function x is a solution of the system (1), then the vector-function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ will be a solution of the system (1), as well. Indeed, by the definition of a solution of the system, using (3) or (4), we have

$$\begin{aligned} y(t) - y(s) = -\big(x(t+\omega) - x(s+\omega)\big) &= -\int_{s+\omega}^{t+\omega} dA(\tau)\cdot f(\tau, x(\tau)) \\ &= -\int_s^t dA(\tau+\omega)\cdot f\big(\tau+\omega, x(\tau+\omega)\big) = \int_s^t dA(\tau)\cdot f(\tau, y(\tau)) \ \text{ for } \ s < t. \end{aligned}$$

Therefore, if $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is a solution of the system (1) on the closed interval $[0,\omega]$ satisfying the condition

$$x(\omega) = -x(0), \tag{5}$$

then its ω-antiperiodic continuation, i.e. the vector-function $y(t) = (-1)^k x(t - k\omega)$ for $k\omega \le t < (k+1)\omega$ $(k = 0, \pm1, \pm2, \dots)$, will be a solution of the ω-antiperiodic problem (1), (2).

In connection with this fact we consider the boundary value problem (1), (5) on the closed interval $[0,\omega]$. Below, we will give the sufficient conditions guaranteeing the solvability of the later and hence of the problem (1), (2), as well.

Definition 1. The pair (P,l) of the matrix-function $P \in Car([0,\omega]\times\mathbb{R}^n, \mathbb{R}^{n\times n}; A)$ and the continuous operator $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \times \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ is said to be consistent if:

(i) for any fixed $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, the operator $l(x,\cdot\,) : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z \in \mathbb{R}^n$, x and $y \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, the inequalities

$$\|P(t,z)\| \le \xi(t, \|z\|), \ \ \|l(x,y)\| \le \xi_0(\|x\|_s)\cdot\|y\|_s$$

are fulfilled for $\mu(g_c(\alpha))$-almost all $t \in [0,\omega]$ and for $t \in D_\alpha$, where $\xi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function, and $\xi : [0,\omega]\times\mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing in the second variable function such that $\xi(\cdot, s) \in L([0,\omega],\mathbb{R}_+;\alpha)$ for every $s \in \mathbb{R}_+$;

(iii) there exists a positive number β such that for any $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, $q \in L([0,\omega],\mathbb{R}^n;A)$ and $c_0 \in \mathbb{R}^n$, for which the conditions

$$\det\big(I_n - \triangle^- A(t)\cdot P(t,x(t))\big) \neq 0 \ \text{ for } \ t \in [0,\omega],$$
$$\det\big(I_n + \triangle^+ A(t)\cdot P(t,x(t))\big) \neq 0 \ \text{ for } \ t \in [0,\omega]$$

hold, an arbitrary solution x of the boundary value problem

$$dy = dA(t)\cdot\big(P(t,x(t))y + q(t)\big), \ \ l(x,y) = c_0$$

admits the estimate

$$\|y\|_s \le \beta\big(\|c_0\| + \|q\|_{L,\alpha}\big).$$

Theorem 1. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times R^n, R^n; A)$ and let there exist a positive number ρ and a consistent pair (P,l) of the matrix-function $P \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and the continuous operator $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\times\mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ such that an arbitrary solution of the problem*

$$dx = dA(t)\cdot\big(P(t,x)x + \lambda[f(t,x) - P(t,x)]x\big), \tag{6}$$
$$\lambda(x(0)+x(\omega)) + (1-\lambda)l(x,x) = 0 \tag{7}$$

admits the estimate

$$\|x\|_s \le \rho \tag{8}$$

for any $\lambda \in]0,1[$. Then the problem (1), (2) *is solvable.*

Definition 2. Let $\mathcal{S} \subset \mathrm{BV}_s([0,\omega],\mathbb{R}^{n\times n})$, $\mathcal{L}$ be a subset of the set of all bounded vector-functionals $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$, and $y \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$. We say that:

(i) a matrix-function $B_0 \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{E}^n_{\mathcal{S}}$ if the condition

$$\det\big(I_n - \triangle^- B_0(t)\big) \neq 0 \ \text{ and } \ \det\big(I_n + \triangle^+ B_0(t)\big) \neq 0 \ \text{ for } \ t \in [0,\omega] \tag{9}$$

holds and there exists a sequence $B_k \in \mathcal{S}$ $(k=1,2,\dots)$ such that $\lim\limits_{k\to+\infty}\|B_k - B_0\|_s = 0$;

(ii) a vector-functional $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ belongs to the set $\mathcal{E}^n_{\mathcal{L}}(y)$ if there exists a sequence $l_k \in \mathcal{L}$ $(k=1,2,\dots)$ such that $\lim\limits_{k\to+\infty} l_k(y) = l_0(y)$.

Definition 3. Let $g_0 : \mathrm{BV}([0,\omega],\mathbb{R}^n_+)\to\mathrm{BV}([0,\omega],\mathbb{R}^n)$ be a positive homogeneous nondecreasing operator, and $h_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}^n_+)\to\mathbb{R}^n_+$ be a positive homogeneous operator. We say that the pair $(\mathcal{S},\mathcal{L})$ of a set $\mathcal{S}\subset\mathrm{BV}_s([0,\omega],\mathbb{R}^{n\times n})$ and a set $\mathcal{L}$ of some vector-functionals $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ belongs to the Opial class $\mathcal{O}^n_{g_0,h_0}$ if:

(i) every operator $l \in \mathcal{L}$ is linear and continuous with respect to the norm $\|\cdot\|_s$;

(ii) there exist the numbers $r_0, \xi_0 \in \mathbb{R}_+$ and a nondecreasing function $\varphi : [0,\omega]\to\mathbb{R}$ such that the inequalities

$$\|B(0)\| \le r_0, \ \ \|B(t) - B(s)\| \le \varphi(t) - \varphi(s) \ \text{ for } \ 0 \le s < t \le \omega,$$
$$\|l(y)\| \le \xi_0\|y\|_s$$

are fulfilled for any $B \in \mathcal{S}$, $l \in \mathcal{L}$ and $y \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$;

(iii) for $B_0 \in \mathcal{E}^n_{\mathcal{S}}$, if the function $y \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ is a solution of the system

$$|dy - dB_0(t)\cdot y| \le dg_0(|y|)$$

under the condition

$$|l_0(y)| \le h_0(|y|),$$

where $l_0 \in \mathcal{E}^n_{\mathcal{L}}(y)$, then $y(t) \equiv 0$.

If $g_0(y)(t) \equiv \int\limits_0^t dG_0(\tau)\cdot q_0(y)(\tau)$ for $y \in \mathrm{BV}([0,\omega],\mathbb{R}_+^n)$, where $G_0 : [0,\omega] \to \mathbb{R}^n$ is a nondecreasing matrix-function, and $q_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}_+^n) \to \mathrm{BV}_s([0,\omega],\mathbb{R}_+^n)$ is a positive homogeneous operator, then we write $\mathcal{O}^n_{G_0,q_0,h_0}$ instead of $\mathcal{O}^n_{g_0,h_0}$.

Definition 4. Let $P \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and let $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\times\mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous vector-functional. We say that the pair (B_0,l_0) of the matrix-function $B_0 \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ and the vector-functional $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the set $\mathcal{E}^n_{A,P,l}$ if there exists a sequence $x_k \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ $(k = 1,2,\dots)$ such that the conditions

$$\lim_{k\to+\infty}\int\limits_a^t dA(\tau)\cdot P(\tau,x_k(\tau)) = B_0(t) \ \text{ uniformly on } \ [0,\omega] \tag{10}$$

$$\lim_{k\to+\infty} l(x_k,y) = l_0(y) \ \text{ for } \ y\in\Omega_{B_0}$$

are valid.

Definition 5. We say that the pair (P,l) of the matrix-function $P \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and the continuous operator $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\times\mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ belongs to the Opial class $\mathcal{O}^n_A$ with respect to the matrix-function A if:

(i) for any fixed $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, the operator $l(x,\cdot\,) : \mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ is linear;

(ii) for any $z\in\mathbb{R}^n$, x and $y \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$, the inequalities

$$\|P(t,z)\| \le \xi(t), \quad \|l(x,y)\| \le \xi_0\|y\|_s \tag{11}$$

are fulfilled for $\mu(g_c(\alpha))$-almost all $t\in[0,\omega]$ and for $t \in D_\alpha$, where $\xi_0 \in R_+$, and $\xi \in L([0,\omega],\mathbb{R}_+;\alpha)$;

(iii) the problem

$$dy = dB_0(t)\cdot y, \quad l_0(y) = 0$$

has only a trivial solution for every pair $(B_0,l_0)\in\mathcal{E}^n_{A,P,l}$.

Remark 1. By (10) and (11), the condition $\|\triangle^- A(t)\|\cdot\xi(t) < 1$ and $\|\triangle^+ A(t)\|\cdot\xi(t) < 1$ for $t\in[0,\omega]$ guarantees the condition (9).

Corollary 1. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n;A)$ and let there exist a positive number ρ and a pair $(P,l)\in\mathcal{O}^n_A$ such that an arbitrary solution of problem* (6), (7) *admits the estimate* (8) *for any $\lambda\in]0,1[$. Then the problem* (1), (2) *is solvable.*

Corollary 2. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n;A)$, $P\in L([0,\omega],\mathbb{R}^{n\times n};A)$, and let $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\to\mathbb{R}^n$ be a bounded linear operator such that*

$$\det\big(I_n - \triangle^- A(t)\cdot P(t)\big)\neq 0 \ \text{ and } \ \det\big(I_n + \triangle^+ A(t)\cdot P(t)\big)\neq 0 \ \text{ for } \ t\in[0,\omega]$$

and the problem

$$dy = dA(t)\cdot P(t)y, \ \ l(y) = 0$$

has only a trivial solution. Let, moreover, there exist a positive number ρ such that an arbitrary solution of the problem

$$dx = dA(t)\cdot\big(P(t)x + \lambda[f(t,x) - P(t)x]\big),$$
$$\lambda(x(0)+x(\omega)) + (1-\lambda)l(x) = 0$$

admits the estimate (8) *for any $\lambda\in]0,1[$. Then the problem* (1), (2) *is solvable.*

The following result is analogous of the well-known one belonging to R. Conti and Z. Opial for boundary value problems for ordinary nonlinear differential equations (see [6,7,12]).

Corollary 3. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n;A)$ and let the pair $(P,l) \in \mathcal{O}_A^n$ be such that*

$$|f(t,x) - P(t,x)x| \le \beta(t,\|x\|) \ \text{ for } \ t \in [0,\omega], \ \ x \in \mathbb{R}^n, \tag{12}$$

$$|x(0) + x(\omega) - l(x,x)| \le l_0(|x|) + l_1(\|x\|_s) \ \text{ for } \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n), \tag{13}$$

where $\beta \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n;A)$ is a nondecreasing in the second variable vector-function, $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is a positive homogeneous continuous operator, and $l_1 \in C(\mathbb{R}_+,\mathbb{R}_+^n)$. Let, moreover,

$$\lim_{k\to+\infty} \frac{1}{\rho} \int\limits_a^b dV(A)(\tau)\cdot\beta(\tau,\rho) = 0_n, \qquad \lim_{\rho\to+\infty} \frac{l_1(\rho)}{\rho} = 0_n. \tag{14}$$

Then the problem (1), (2) *is solvable.*

By $Y_P(x)$ we denote the fundamental matrix of the system

$$dy = dA(t)\cdot P(t,x(t))\,y$$

for every $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ satisfying the condition $Y_P(x)(a) = I_n$.

Corollary 4. *Let $A \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n;A)$, $P \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n};A)$ and a continuous operator $l : \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\times\mathrm{BV}_s([0,\omega],\mathbb{R}^n) \to \mathbb{R}^n$, satisfying conditions* (i) *and* (ii) *of Definition 5, be such that the conditions* (12)–(14) *hold, where $\beta \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n;A)$ is a nondecreasing in the second variable vector-function, $l_0 : \mathrm{BV}_s([0,\omega],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is a positive homogeneous continuous operator, and $l_1 \in C(\mathbb{R}_+,\mathbb{R}_+^n)$. Let, moreover,*

$$\inf\big\{|\det(l(x,Y_P(x)))| : \ \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)\big\} > 0. \tag{15}$$

Then the problem (1), (2) *is solvable.*

Remark 2. In Corollary 4, the condition (15) cannot be replaced by the condition

$$\det\big(l(x,Y_P(x))\big) \neq 0 \ \text{ for } \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n). \tag{16}$$

The corresponding example for ordinary differential systems, i.e., for the case $A(t) \equiv \mathrm{diag}(t,\dots,t)$ has been constructed in [8]. Basing on this example, it is not difficult to construct analogous examples for the case $A(t) \not\equiv \mathrm{diag}(t,\dots,t)$. Consider the scalar boundary value problem

$$dx = \Big(\frac{|x|x}{1+|x|} + 1\Big)\, d\alpha(t), \ \ x(0) = -x(\omega),$$

where $\alpha(t) = 0$ for $0 \le t \le c$ and $\alpha(t) = -2$ for $c < t \le \omega$, and $c = \omega/2$. Every solution of the system has the form

$$x(t) = \begin{cases} x(0) & \text{for } \ 0 \le t \le c, \\ x(0) - 2\Big(\dfrac{|x(0)|x(0)}{1+|x(0)|} + 1\Big) & \text{for } \ c < t \le \omega. \end{cases}$$

This problem cannot be solvable because the equation $x(0) + x(\omega) = 0$ is unsolvable with respect to $x(0)$. On the other hand, if we assume $P(t,x) = \frac{|x|}{1+|x|}$ and $l(x,y) = y(0) + y(\omega)$, then

$$Y(t) = \begin{cases} 1 & \text{for } \ 0 \le t \le c, \\ 1 - \dfrac{2|x(c)|}{1+|x(c)|} & \text{for } \ c < t \le \omega \end{cases}$$

for $x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n)$ and, therefore,

$$\det\big(l(x,Y_P(x))\big) = \frac{2}{1+|x(c)|} \ \text{ for } \ x \in \mathrm{BV}_s([0,\omega],\mathbb{R}^n).$$

Consequently, all the conditions of Corollary 4 are fulfilled except of the condition (15), instead of which we have the condition (16).

Remark 3. In particular, in the results obtained above we can assume $l(x,y) \equiv x(0) + x(\omega)$ and $l(x) = l(x,x) \equiv x(0) + x(\omega)$. Thus, for instance, the second estimate in condition (ii) of Definition 1 is fulfilled. The condition (7) in Theorem 1 and Corollary 1 as well as the analogous condition in Corollary 2 coincide with the condition (2). The condition (13) is valid for $l_0 \equiv 0$ and $l_1 \equiv 0$.

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