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**BOUNDARY VALUE PROBLEMS  
OF MECHANICS OF CONTINUUM  
MEDIA FOR A SPHERE**

**Abstract.** Boundary value problems of continuum mechanics for the ball and the entire space with a spherical cavity are considered. Solutions are constructed in quadratures using their special representations by harmonic functions. The method enables one to extend easily the results to an arbitrary  $m$ -dimensional ball ( $m = 2, 4, 5, \dots$ ).

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**Key words and Phrases.** Elasticity, thermoelasticity, boundary value problems, Navier–Stokes equations.

**რეზიუმე.** განხილულია უწყვეტ ტანთა მექანიკის სასაზღვრო ამოცანები ბირთვისა და სფერული ღრუს მქონე მთელი სივრცისათვის. ჰარმონიულ ფუნქციათა საშუალებით სპეციალური წარმოდგენის გამოყენებით ამონახსნები აგებულია კვადრატურებში. გამოყენებული მეთოდი საშუალებას იძლევა მიღებული შედეგები ადვილად იქნეს განზოგადებული ნებისმიერი  $m$ -განზომილებიანი ბირთვისათვის ( $m = 2, 4, 5, \dots$ ).

## INTRODUCTION

This memoir deals with constructing in quadratures solutions of the boundary value problems of continuum mechanics for the ball and the entire space with a spherical cavity. Mainly for the sake of definiteness we consider the three-dimensional ball  $B^+$  and  $B^- \equiv \mathbb{R}^3 \setminus B^+$ . The method by which the solutions are constructed enables one to easily extend the results to an arbitrary  $m$ -dimensional ball ( $m = 2, 4, 5, \dots$ ).

Solutions are constructed using their special representations by harmonic functions. In the theory of elasticity for the ball such a representation was used evidently for the first time in the work of R. Marcolongo. It is based on the idea of J. Hadamard and V. Cerruti (also see Refs. J. Boussinesq, E. Almansi, C. Somigliana, O. Tedone, and G. Lauricella in the Bibliography at the end of the book). Later this representation was used in the works of other researchers, in particular, in the work of E. Trefftz, this perhaps being the reason why some authors (A. I. Lurie, W. Nowacki) assign it to E. Trefftz. Besides these representations, we also use the harmonic potential theory, and solutions of ordinary linear differential equations with constant coefficients. With such simple means, we have succeeded in solving explicitly, i.e., in quadratures, quite a number of boundary value problems for the ball  $B^+$  and  $B^-$ .

It is nearly half a century that mathematicians have been maintaining keen interest in these questions. We can cite such names as G. Lamé (1852-1859), Lord Kelvin (1863-1890), H. Lamb (1879-1932), C. W. Borchard (1880), V. Cerruti (1885-1886), C. Somigliana (1887), G. Lauricella (1895-1909), E. Almansi (1897), J. Hadamard (1901), G. Grunberg (1925), R. Sauthwell (1926), A. E. H. Love (1927), N. I. Muskhelishvili (1933-1966), B. G. Galperin (1942), G. Fichera (1949), E. Stenberg (1953-1958), A.I. Lurie (1953-1970), J. Happel (1965), H. Brenner (1965), N. O. Basheleishvili (1970-1981), W. Nowacki (1970), A. F. Ulitko (1985), V. T. Grinchenko (1985), D.G. Natroshvili (1986) and many others (see the Bibliography). And yet despite the fact that about 500 authors can be entered in this list, the monograph like this one is not encountered in the literature. Besides, nearly in all of the works prevalence is given to the method of series (in spherical functions), and the solutions of the problems are also given in the form of series (true, in a few papers the authors have succeeded in summing the series obtained for some problems).

In our memoir series are not used whatsoever. We strictly adhere to the method of representation of solutions by harmonic functions to define which we obtain the Dirichlet or Neumann boundary value problems whose solutions are constructed with the aid of Poisson, Neumann, Bjerknes and Dini formulas (for the sake of integrity of the method it is shown that one can as well derive these formula without resorting to series).

The method in question has enabled us to give a clear and concise account of our results. It takes less than two hundred pages to expose the solutions

of the boundary value problems of elasticity, thermoelasticity, elastic and thermoelastic mixtures, fluid flow, polyharmonic equations, problems with concentrated singularities, and, which is the main point, to give analysis of the solutions obtained.

Furthermore, when potential-type functions are used, this method makes it possible to obtain solutions directly in the form of quadratures. In that case the kernels of such potentials are given mainly in terms of elementary functions, whereas the densities are the known boundary functions. Generally speaking, it is one of the goals of the memoir to derive the most convenient and simplest representations. They may evidently be subjected to a further simplification only by simplifying the Poisson formula.

Potential-type functions representing the solutions of the problems can conveniently be used both for investigating their behaviour and for constructing numerical solutions. In case the boundary functions satisfy the conditions less strict than the existence and uniqueness theorems, it becomes necessary to perform analysis of the constructed formulas. We take particular care of these questions, alongside with the investigation of the differential and boundary properties of the solution. This is yet another feature that makes this memoir differ from the other books available in this field.

Altogether 70 problems are posed. Most of them are difficult to consider, as their solutions require rather serious creative efforts and the overcoming of considerable obstacles of computational nature.

Chapter I is the auxiliary one. It contains the discussion of the basic boundary and differential properties of harmonic potentials and functions represented by the Poisson, Neumann, Bjerknæs and Dini formulas. Here we also introduce the terms and notations to be used throughout the memoir and give the properties of some auxiliary differential operators.

Classical solutions of the boundary value problems and solutions with power singularities at some points are both constructed in quadratures. The construction of such solutions requires knowledge of the asymptotic representation of solutions of the respective equations near singular points. The reader will find the asymptotic formulas of this kind in Subsection 1.7.

In constructing effective solutions of the problems, the existence theorems become meaningless, as the uniqueness theorems come to the foreground. To obtain simple theorems that are easy to prove, one should know the behaviour of solutions at infinity. These topics are treated in Subsection 1.8.

Chapter II is devoted to the solution of the boundary value problems of the classical theory of elasticity. Five problems are solved in quadratures for the ball  $B^+$  and five for the ball  $B^-$ . For all the problems we seek for a continuous solution of the system of basic equations of classical elasticity. The problems have different boundary conditions on the sphere  $S$  ( $S \equiv \partial B^+ = \partial B^-$ ): a displacement vector in Problem I, a stress vector in Problem II, a normal component of the displacement vector and tangent

components of the stress vector in Problem III, a normal component of the stress vector and tangent components of the displacement vector in Problem IV, a linear combination of stress and displacement vectors in Problem V.

Problem III is solved both by the technique proposed here and by the slender method of Hadamard which he applied to this problem in the particular case (when the tangent components of the stress vector on  $S$  are zero). The combination of the two methods simplifies the solution of Problem III.

In Chapter III our consideration involves the polyharmonic equation. The Lauricella, Riquier and mixed-type problems are solved in quadratures. It is a well-known fact that, using the Airy function, the plane problems of elasticity are reduced to the biharmonic problem, such a connection therefore being the key to their solution. The results of this chapter convince that the representations of solutions can as well be successfully applied to higher-order equations.

We have to admit that many authors construct the Green functions for the Lauricella problem and use them to represent the solutions of the respective problems for the polyharmonic equation. But, if the order of the equation is higher than two, the densities of these representations contain not the given functions but rather complex local differential operators of the given functions, which considerably diminishes the practical value of such representations.

Chapter IV deals with the static problems of thermoelasticity which are known to be grouped so that we have the boundary value problems separately for the Laplace equation for temperature and separately for a system of equations of elasticity for the stress vector. Temperature is determined using the Poisson or Neumann formulas, and the problems of thermoelasticity are reduced to the problems of elasticity studied in the second chapter. Such a scheme of reduction is well suitable for theoretical investigations and no new problems arise in this connection.

The situation becomes different when effective solutions are constructed. The found temperature will be contained as volume force in the right-hand side of the system of equations of elasticity and the problems will involve the boundary value of the volume potential with the density of the found temperature, as well as the derivatives of the volume potential. These functions cannot be directly expressed through the boundary data.

The problems of thermoelasticity are solved in this memoir using representations similar to the Marcolongo representation.

Our consideration involves mainly the homogeneous equations whose right-hand sides are zero. It is commonly known that problems for non-homogeneous equations are solved by the Green function which in turn is constructed in quadratures by the same formulas as used to represent solutions of the corresponding problems for homogeneous equations. However, in one case, when the right-hand side of equations of elasticity is a harmonic function gradient, the problems can be solved in quadratures directly, without resorting to the Green function.

Chapter V is concerned with solving the problems for elastic and thermoelastic mixtures. We have omitted the discussion of the mathematical modelling of mixtures and refer the reader to the founders of this theory J. Stepan, C. Truesdell, Kh. A. Rakhmatulin, R. Toupin, P.N. Naghdi, M.A. Biot, A. E. Green, T. R. Steel, A. C. Eringen, R. J. Atkin, P. Chadwick, R. J. Knops, H. Tiersten, M. Jahanmir, P. Villaggio, B. Lompriere and others (see the Bibliography at the end of the memoir).

We consider one variant of two-component elastic isotropic mixtures and solve the boundary value problems for  $B^+$  and  $B^-$ .

In Chapter VI we derive the solutions of the boundary value problems for noncompressible fluid flow. A Stokes-linearized system of Navier-Stokes equations is considered. The solutions of the problems for the ball (mainly of the particular ones having different applications) can be found in H. Lamb, J. Happel and H. Brenner, S. M. Belonosov and K. A. Chernous, and others, as well as in some references cited therein.

In all the known works the problems for the fluid flow sphere are solved using series and leaving out the necessary analysis of the series obtained.

Chapter VII is devoted to the boundary value problems with concentrated singularities. These problems differ from the problems investigated in the preceding chapters only in that their solutions at the ball centre have a singularity of a given power order  $\nu$ . The continuous solutions we have constructed previously will therefore be the solutions of the corresponding problems with concentrated singularities. In the theory of elasticity these problems have no new solutions when  $\nu < 1$ . For the case  $\nu < 2$  the problems with concentrated singularities have – in addition to the solutions constructed in Chapter II – other solutions corresponding to concentrated forces applied to the ball centre; a general solution depends on three arbitrary constants. If  $2 \leq \nu < 3$ , then the class of solutions is even wider, since it additionally includes the solutions corresponding to the so-called “double forces” concentrated at the centre and a general solution depends on twelve arbitrary constants. We point out various possible ways of introducing restrictive assumptions for the unique solvability.

In the same chapter the Navier-Stokes equations are used as an example to illustrate how the two-dimensional problems are directly solved by Marcolongo-type representations.

In Chapter VIII, the last one, an algorithm is developed for numerical realizations of the solutions constructed in the preceding chapters. Though the solutions represented in the form of a potential-type function are continuous in the closed ball  $\bar{B}^+$  as well as in  $\bar{B}^-$ , their kernels tend to infinity as the integration-free point, i.e., the kernel point, approaches the boundary  $S$ . Near the boundary the potential is essentially determined by a neighbourhood of the pole, and therefore the method of integration set division seems important when using cubic formulas. The algorithm of an automatic selection of division steps proposed here increases calculation accuracy and makes the computational facilities more efficient.

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CHAPTER I  
NOTATION AND AUXILIARY PROPOSITIONS

**1.1. Some Notation and Terms.** A Euclidean  $m$ -dimensional space is denoted by  $\mathbb{R}^m$  ( $\mathbb{R}^1 \equiv \mathbb{R}$ ) and the points (vectors) of this space are denoted by  $x, y, z, e, \dots$ . The base vectors  $(1, 0, \dots, 0), \dots, (0, 0, \dots, 1)$  are denoted by  $e^1, \dots, e^m$  and the coordinates of  $x, y, z, e, \dots$  in this base by  $x_1, \dots, x_m; y_1, \dots, y_m; z_1, \dots, z_m; e_1, \dots, e_m; \dots$  respectively. Therefore  $e_j^i = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker symbol.

Let  $\Omega$  be a set from  $\mathbb{R}^m$ . Then  $\partial\Omega$  will denote the boundary of  $\Omega$  and  $\overline{\Omega}$  the closure ( $\overline{\Omega} = \Omega \cup \partial\Omega$ ).

A set of continuous real functions defined on  $\Omega$  is denoted by  $C(\Omega)$ , while  $C^k(\Omega)$ , where  $k$  is a positive integer ( $k \in \mathbb{N}$ ), denotes a set of real functions defined on  $\Omega$  and having in  $\Omega$  all continuous derivatives up to order  $k$  inclusive.

Let  $\Omega$  be a domain from  $\mathbb{R}^m$ ,  $\varphi : \Omega \rightarrow \mathbb{R}$  and  $y \in \partial\Omega$ ;  $\varphi$  is called a continuously extendable function at the point  $y$  if there exists a finite limit

$$\lim_{\Omega \ni x \rightarrow y} \varphi(x) \equiv \varphi_\Omega(y).$$

If  $\varphi \in C(\Omega)$  and  $\varphi$  is a continuously extendable function at each point of the set  $\partial\Omega$ , then  $\varphi$  is said to be a continuously extendable function on  $\partial\Omega$ .  $\varphi$  is called a function of the class  $C(\overline{\Omega})$  if  $\varphi \in C(\Omega)$  and  $\varphi$  is continuously extendable on  $\partial\Omega$ . If  $\varphi \in C(\overline{\Omega})$ , then  $\varphi(y)$  will imply  $\varphi_\Omega(y)$  when  $y \in \partial\Omega$ .

$C^k(\overline{\Omega})$  will denote a set of functions from  $C^k(\Omega)$  which, together with all their derivatives up to order  $k$  inclusive, are continuously extendable on  $\partial\Omega$ . For  $y \in \partial\Omega$  the notation  $\mathcal{D}^\beta \varphi(y)$  will imply the limit

$$\lim_{\Omega \ni x \rightarrow y} \mathcal{D}^\beta \varphi(x) \equiv \mathcal{D}^\beta \varphi(y).$$

Here  $\beta = (\beta_1, \dots, \beta_m)$  is the multiindex,  $|\beta| \equiv \beta_1 + \dots + \beta_m \leq k$  and

$$\mathcal{D}^\beta \varphi(x) \equiv \frac{\partial^{|\beta|} \varphi(x)}{\partial x_1^{\beta_1} \dots \partial x_m^{\beta_m}}.$$

If  $\varphi \in C^k(\Omega)$  or  $\varphi \in C^k(\overline{\Omega})$  for any  $k \in \mathbb{N}$ , then we will write  $\varphi \in C^\infty(\Omega)$  or  $\varphi \in C^\infty(\overline{\Omega})$ .

Let  $\varphi : \Omega \rightarrow \mathbb{R}$ . By definition,  $\varphi \in C^{0,\alpha}(\Omega)$ ,  $0 < \alpha \leq 1$ , if there exists a number  $c > 0$  such that  $\forall x, y \in \Omega$  satisfying the condition  $|x - y| \leq 1$  we have the inequality

$$|\varphi(x) - \varphi(y)| \leq c|x - y|^\alpha \quad \left( |x - y| = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} \right).$$

If  $\varphi \in C^{0,\alpha}(\Omega)$ , then  $\varphi$  is a uniformly continuous function on  $\Omega$ , and if  $\varphi$  is a uniformly continuous function on  $\Omega$ , then  $\varphi \in C(\overline{\Omega})$ .  $\varphi \in C^{k,\alpha}(\Omega)$ , where  $k$  is a nonnegative integer number and  $0 < \alpha \leq 1$ , will denote that



Then  $r$  and  $\omega \equiv (\omega_1, \dots, \omega_m)$  will be the radial and the angular coordinate of the point  $x$  ( $x = r\omega$ ).

We introduce the differential operators

$$\frac{d}{dn(x)} \equiv \sum_{i=1}^m \frac{x_i}{r} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial r}, \quad \mathcal{D}_r \equiv r \frac{\partial}{\partial r} = \sum x_i \frac{\partial}{\partial x_i},$$

$$\mathcal{D}_r^k = \overbrace{\mathcal{D}_r (\mathcal{D}_r (\dots \mathcal{D}_r))}^k, \quad \mathcal{D}_r^0 = I,$$

where  $I$  is the identical operator;

$$\mathcal{D}_k(f)(x) \equiv \frac{\partial f(x)}{\partial x_k} - n_k(x) \frac{df(x)}{dn(x)}.$$

Operator  $\mathcal{D}_k$  is the Gunter operator (see Gunter [1])

$$\begin{aligned} \mathcal{D}(f)(x) &\equiv (\mathcal{D}_1(f)(x), \dots, \mathcal{D}_m(f)(x)), \\ \mathcal{D}_r^{(k)} &\equiv \mathcal{D}_r (\mathcal{D}_r - 1) \dots (\mathcal{D}_r - (k-1)), \quad \mathcal{D}_r^{(0)} \equiv I, \\ \mathcal{D}_r^{[k]} &\equiv (\mathcal{D}_r + 1 + \frac{1}{2})(\mathcal{D}_r + 2 + \frac{1}{2}) \dots (\mathcal{D}_r + k + \frac{1}{2}), \\ \mathcal{D}_r^{[k]-[i]} &\equiv (\mathcal{D}_r + i + 1 + \frac{1}{2})(\mathcal{D}_r + i + 2 + \frac{1}{2}) \dots (\mathcal{D}_r + k + \frac{1}{2}), \quad i < k, \\ \mathcal{D}_r^{[k]-[0]} &= \mathcal{D}_r^{[k]}, \quad \mathcal{D}_r^{[k]-[k]} = I. \end{aligned}$$

**1.3. Dirichlet Problem for the Laplace Equation. The Poisson Formulas.** In solving the boundary value problems of solid mechanics for the ball and the entire space with a spherical cavity essential use is made of the well-known integral representations of the solutions of the Dirichlet and Neumann problems for the Laplace equation. A continuous solution of the Laplace equation will be referred to as a harmonic function. The harmonic function is analytic (see, for example, Petrovski [1]; Bers, John, Schechter [1]). We will obtain here the solutions of the Dirichlet and Neumann problems for the ball and show some of their properties needed for our further purposes.

The solution of the Dirichlet problem for the ball  $B^+ \equiv B(0, \rho)$  - Problem  $D^+ : v \in C(\overline{B^+})$ ;  $\forall x \in B^+ : \Delta v(x) = 0$ ;  $\forall y \in S : v^+(y) = g(y)$ ,  $g \in C(S)$  is given by the Poisson formula (see, for example, Courant [1] or Mikhlin [1])

$$v(x) = \Pi(g)(x) \equiv \frac{1}{\omega_m \rho} \int_S \frac{\rho^2 - |x|^2}{|x-y|^m} g(y) d_y S, \quad (1.1)$$

where  $\omega_m$  is the surface area of the unit sphere in  $\mathbb{R}^m$ .

The Poisson formula gives also the solution of the Dirichlet problem for  $B^- \equiv \mathbb{R}^m \setminus \overline{B(0, \rho)}$  - Problem  $D^-$  :  $v \in C(\overline{B^-})$ ;  $\forall x \in B^- : \Delta v(x) = 0$ ;  $\forall y \in S : v^-(y) = g(y)$ ,  $g \in C(S)$ ,

$$v(x) = \Pi'(g)(x) \equiv -\frac{1}{\omega_m \rho} \int_S \frac{\rho^2 - |x|^2}{|x - y|^m} g(y) d_y S. \quad (1.1')$$

Note that the problem  $D^+$  is uniquely solvable for any function  $g \in C(S)$  by (1.1). Problem  $D^-$  is uniquely solvable for any function  $g \in C(S)$  by (1.1) by imposing additional conditions on the solution  $v$ :

$$\lim_{|x| \rightarrow \infty} v(x) = 0, \quad \text{when } m > 2, \quad (1.2)$$

$$\forall x \in B^- : |v(x)| \leq c = \text{const}, \quad \text{when } m = 2. \quad (1.2')$$

**1.4. Neumann Problem.** The solution of the Neumann problem for the ball  $B^+$  - Problem  $N^+$  :  $v \in C(\overline{B^+})$ ,  $\forall x \in B^+ : \Delta v(x) = 0$ ,  $\forall y \in S : (\frac{\partial v}{\partial n})^+(y) = g(y)$ ,  $g \in C(S)$  - is given by the formula

$$v(x) = \frac{1}{\omega_m} \int_S g(y) d_y S \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^m} - \frac{1}{\rho^{m-2}} \right) \frac{d\tau}{\tau} + c, \quad (1.3)$$

where  $c$  is an arbitrary constant, while that for the spherical cavity  $B^-$  - Problem  $N^-$  :  $v \in C(\overline{B^-})$ ,  $\forall x \in B^- : \Delta v(x) = 0$ ,  $\forall y \in S : (\frac{\partial v}{\partial n})^-(y) = g(y)$ ,  $g \in C(S)$  - is given by the formula

$$v(x) = -\frac{1}{\omega_m} \int_S g(y) d_y S \int_0^1 \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^m} \tau^{m-3} d\tau, \quad m > 2; \quad (1.4)$$

$$v(x) = -\frac{1}{2\pi} \int_S g(y) d_y S \int_0^1 \left( \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^2} - 1 \right) \frac{d\tau}{\tau} + c, \quad m = 2. \quad (1.4')$$

Note that Problem  $N^+$  is solvable only if the given function  $g$  satisfies the condition

$$\int_S g(y) d_y S = 0. \quad (1.5)$$

When (1.5) is fulfilled, all solutions of Problem  $N^+$  are given by (1.3).

When  $m > 2$ , Problem  $N^-$  is uniquely solvable for any function  $g \in C(S)$  by (1.4) if condition (1.2) is imposed on the solution  $v$ . When  $m = 2$ , Problem  $N^-$  is solvable only if condition (1.5) is fulfilled. If the latter condition, as well as condition (1.2') are fulfilled, then the solution is given by (1.4').

**1.5. Remarks. The Neumann and Bjerknes Formulas.** In this book we construct classical smooth solutions and therefore the above-given solutions of Problems  $D^+, D^-, N^+$  and  $N^-$  are sufficient for our purposes. It should be said for the sake of generality that if  $g$  is not a continuous function on  $S$  and only summable -  $g \in L(S)$ , then  $v$  represented by (1.1) satisfies the Laplace equation at each point of the ball  $B^+ - \forall x \in B^+ : \Delta v(x) = 0$  and almost every when the boundary condition  $\forall y \in S \setminus S_g : v^+(y) = g(y)$ , where  $\text{mes } S_g = 0$ . This remark applies to the other problems as well.

In this book we will never use the expansion of a functions in series and, to preserve the integrity of our reasoning, we should note that the Poisson formulas, as well as formulas (1.3), (1.4) and (1.4') can be obtained without taking recourse to series. Indeed, as can be easily verified, the Green function of Problem  $D^+$  for the ball  $B^+$  is given by the formula

$$G(x, y) = \Phi(|x - y|) - \Phi\left(\frac{|y|}{\rho} \left|x - \frac{\rho^2}{|y|^2}y\right|\right),$$

where  $\Phi(|x|)$  is the fundamental solution of the equation  $\Delta v = 0$ :

$$\Phi(t) = \frac{1}{(2 - m)\omega_m t^{m-2}}, \quad m > 2; \quad \Phi(t) = \frac{1}{2\pi} \ln t, \quad m = 2.$$

Hence we immediately obtain the Poisson formulas.

For  $m = 3$  the Green function takes the form

$$G(x, y) = \frac{1}{4\pi} \left( \frac{1}{|x - y|} - \frac{\rho|y|}{||y|^2x - \rho^2y|} \right).$$

Formula (1.3) can be obtained by proceeding as follows: If  $g \in C(S)$  and satisfies condition (1.5), then the harmonic function  $v$  in the domain  $B^+$ , satisfying the equation

$$r \frac{\partial v(x)}{\partial r} = \rho \Pi(g)(x),$$

will be the solution of Problem  $N^+$ . The solution of this equation is given by

$$v(x) = \rho \int_0^{|x|} \Pi(g)\left(\eta \frac{x}{|\eta|}\right) \frac{d\eta}{\eta} + c = \rho \int_0^1 \Pi(g)(\tau x) \frac{d\tau}{\tau} + c,$$

which coincides with (1.3).

To obtain formulas (1.4) and (1.4') note that if  $v$  is a harmonic function in the domain  $B^-$ , satisfying the equation (and also condition (1.5) when  $m = 2$ )

$$r \frac{\partial v(x)}{\partial r} = \rho \Pi'(g)(x),$$

then  $v$  will be the solution of Problem  $N^-$ . If (1.2), (1.2') are fulfilled, the solution of this equation is given by

$$v(x) = -\rho \int_0^1 \Pi'(g)\left(\frac{x}{\tau}\right) \frac{d\tau}{\tau}$$

for  $m > 2$  and by

$$v(x) = -\rho \int_0^1 \Pi'(g)\left(\frac{x}{\tau}\right) \frac{d\tau}{\tau} + c$$

for  $m = 2$  and coincides with (1.4), (1.4').

In the case  $m = 3$  the identities

$$\begin{aligned} \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} &= \tau \frac{\partial}{\partial \tau} \left( \frac{2}{|y - \tau x|} - \frac{1}{\rho} \ln \left( (|y - \tau x| + \rho)^2 - |\tau x|^2 \right) \right), \\ x \in B^+, \quad y \in S, \quad 0 < \tau < 1; \\ \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^3} &= \frac{\partial}{\partial \tau} \left( \frac{2\tau}{|x - \tau y|} - \frac{1}{\rho} \ln \left( |x - y|^2 - |x|^2 - \rho^2 + 2\rho|x - \tau y| + 2\rho^2\tau \right) \right), \\ x \in B^-, \quad y \in S, \quad 0 < \tau < 1, \end{aligned}$$

enable us to easily calculate the one-dimensional integrals in (1.3) and (1.4) and to write the solutions of Problems  $N^+$  and  $N^-$  in the form

$$v(x) = N(g)(x) + c, \quad |x| < \rho; \quad (1.6)$$

$$v(x) = N'(g)(x), \quad |x| > \rho, \quad (1.6')$$

where

$$N(g)(x) \equiv \frac{1}{4\pi\rho} \int_S \left( \frac{2\rho}{|x - y|} - \ln \left( (|x - y| + \rho)^2 - |x|^2 \right) \right) g(y) d_y S, \quad (1.7)$$

$$N'(g)(x) \equiv -\frac{1}{4\pi\rho} \int_S \left( \frac{2\rho}{|x - y|} - \ln \frac{|x - y| + |x| + \rho}{|x - y| + |x| - \rho} \right) g(y) d_y S. \quad (1.7')$$

Formula (1.6) is called the Neumann formula (see Neumann [1], Mikhlín [1]) and (1.6') the Bjerknes formula (Bjerknes [1], Koshliakov, Gliner, Smirnov [1]).

In the case  $m = 2$ , using the identities

$$\begin{aligned} \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^2} - 1 &= 2\tau \frac{\partial}{\partial \tau} \ln \frac{1}{|y - \tau x|}, \quad x \in B^+, \quad y \in S, \quad 0 < \tau < 1; \\ \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^2} - 1 &= 2\tau \frac{\partial}{\partial \tau} \ln \frac{1}{|x - \tau y|}, \quad x \in B^-, \quad y \in S, \quad 0 < \tau < 1 \end{aligned}$$

and calculating the one-dimensional integrals in formulas (1.3) and (1.4'), the solutions of Problems  $N^+$  and  $N^-$  are written in the form

$$v(x) = -\frac{1}{\pi} \int_S \ln|x-y|g(y)d_yS + c, \quad |x| < \rho, \quad (1.8)$$

$$v(x) = \frac{1}{\pi} \int_S \ln|x-y|g(y)d_yS + c, \quad |x| > \rho. \quad (1.8')$$

(1.8) and (1.8') are called the Dini formulas (see Mikhlin [1]).

**1.6. Harmonic Potentials.** Let  $V(g)$  and  $W(g)$  be the simple- and double-layer potentials with density  $g$  ( $m = 3$ ):

$$V(g)(x) \equiv \int_S \frac{g(y)}{|x-y|} d_yS, \quad W(g)(x) \equiv \int_S \frac{\partial}{\partial n(y)} \left( \frac{1}{|x-y|} \right) g(y) d_yS.$$

We have the identities  $\forall x \in B^+$ ,  $\forall y \in S$ :

$$\frac{\rho^2 - |x|^2}{|x-y|^3} = -\left(2\rho \frac{\partial}{\partial n(y)} + 1\right) \frac{1}{|x-y|},$$

$$\frac{\rho^2 - |x|^2}{|x-y|^3} = (2D_r + 1) \frac{1}{|x-y|},$$

$$(D_r + 1) \frac{1}{|x-y|} = -\rho \frac{\partial}{\partial n(y)} \frac{1}{|x-y|},$$

using which one can readily establish that the following formulas are valid  $\forall x \in B^+$ :

$$\Pi(f)(x) = -\frac{1}{2\pi} W(f)(x) - \frac{1}{4\pi\rho} V(f)(x), \quad (1.9)$$

$$\Pi(f)(x) = \frac{1}{4\pi\rho} (2D_r + 1) V(f)(x), \quad (1.10)$$

$$W(f)(x) = -\frac{1}{\rho} (D_r + 1) V(f)(x). \quad (1.11)$$

We would like to point out some differential properties of the simple- and double-layer potentials and also those of the Poisson integral (see Gunter [1]).

**Theorem 1.1.** *If  $g \in C(S)$ , then  $V(g) \in C(\bar{B}^+ \cup \bar{B}^-)$ ,  $W(g) \in C(\bar{B}^\pm)$ ,  $\Pi(g) \in C(\bar{B}^\pm)$ , and if  $\varphi \in C^{k,\alpha}(S)$ ,  $0 < \alpha \leq 1$ ,  $k = 0, 1, \dots$ , then  $V(\varphi) \in C^{k+1,\alpha}(\bar{B}^\pm)$ ,  $W(\varphi) \in C^{k,\alpha}(\bar{B}^\pm)$ ,  $\Pi(\varphi) \in C^{k,\alpha}(\bar{B}^\pm)$ . If  $\varphi \in C^{k,\alpha}(S)$ , then  $\mathcal{D}_k(\varphi) \in C^{k-1,\alpha}(S)$ , where  $\mathcal{D}_k$  is the Gunter operator (see Subsection 1.2).*

Assuming  $\varphi \in C^1(S)$ , we have

$$\int_S \mathcal{D}_k(\varphi)(y) d_yS = \frac{2}{\rho^2} \int_S y_k \varphi(y) d_yS, \quad (1.12)$$

$$V(\mathcal{D}(\varphi)) = \text{grad } V(\varphi) + \frac{2}{\rho}V(n\varphi) + W(n\varphi). \quad (1.13)$$

If  $\varphi \in C^1(S)$ ,  $\psi \in C^2(S)$ , then

$$\frac{\partial V(\varphi)}{\partial x_i} = -W(n_i\varphi) + V\left(\mathcal{D}_i(\varphi) - \frac{2}{\rho}n_i\varphi\right), \quad (1.14)$$

$$\frac{\partial W(\psi)}{\partial x_i} = W(\mathcal{D}_i(\psi)) + V(M_i(\psi)), \quad (1.15)$$

where

$$M_i(\psi) \equiv \sum_{j=1}^m \mathcal{D}_j(n_j\mathcal{D}_i(\psi) - n_i\mathcal{D}_j(\psi)).$$

In the sequel the summation sign  $\sum$  will frequently be omitted and the index repetition in the monomial will imply summation from 1 to  $m$ .

The above formulas readily yield the proof of

**Theorem 1.2.** *If  $\varphi \in C^p(S)$ ,  $p \in \mathbb{N}$ , then the derivatives of the simple-layer potential  $\mathcal{D}_\beta V(\varphi)$ , where  $\beta = (\beta_1, \dots, \beta_m)$  is the multiindex and  $|\beta| \leq p$ , are represented as the sum of the simple- and double-layer potentials with continuous densities.*

**Theorem 1.3.** *If  $\psi \in C^{p+1}(S)$ ,  $p \in \mathbb{N}$ , then the derivatives of the double-layer potential  $\mathcal{D}^\beta W(\psi)$ , where  $\beta = (\beta_1, \dots, \beta_m)$  is the multiindex and  $|\beta| \leq p$ , are represented as the sum of the simple- and double-layer potentials with continuous densities.*

**1.7. Asymptotic Representation of Solutions of the Basic Equations of Elasticity in the Neighbourhood of Singular Points.** Below we will construct not only the classical solutions of the boundary value problems of elasticity but also the solutions having singularities at some isolated points. Such solutions correspond either to forces concentrated at these points or to concentrated sources of various nature. To construct such solutions we will need to establish the behaviour of solutions in the neighbourhood of singular points. It is exactly these aims that will be dealt with in this subsection.

In order to solve more general systems of equations than the system of basic equations of elasticity, we have to establish asymptotic representations (properties) of solutions near singular points. Let us consider a system of differential equations in  $\mathbb{R}^m$

$$A_{ik}(\partial_x)u_k = 0 \quad (A(\partial_x)u = 0), \quad i = 1, \dots, n. \quad (1.16)$$

Here summation is taken over the index  $k$  from 1 to  $n$ ;  $m$  and  $n$  are natural numbers. First assume that  $m > 2$ . Unless stipulated otherwise, the lower indexes  $i$ ,  $k$  and  $s$  will in the sequel vary from 1 to  $n$  and the other lower

indexes from 1 to  $m$ ;  $x \equiv (x_1, \dots, x_m)$ ;  $u \equiv (u_1, \dots, u_n)$  is the unknown vector;

$$A_{ik}(\partial_x) \equiv C_{ijkl} \frac{\partial^2}{\partial x_j \partial x_l}. \quad (1.17)$$

Summation in (1.17) is taken over the indexes  $j$  and  $l$  from 1 to  $m$ ;  $C_{ijkl}$  are the constants which, without loss of generality, can be subjected to the condition  $C_{ijkl} = C_{iklj}$ . Assume further that these coefficients satisfy the restrictions

$$\forall \xi \equiv (\xi_1, \dots, \xi_m) \in \mathbb{R}^m \setminus \{0\}: \det A(\xi) = \det \|A_{ik}(\xi)\|_{n \times n} \neq 0, \quad (1.18)$$

implying the ellipticity of system (1.16).

Setting  $m = n = 3$  and  $C_{ijkl} = C_{klij} = C_{jikl}$  in (1.16), we obtain (see Nowacki [2]; Burchuladze, Gegelia [1]) a system covering the basic static equations of elasticity in terms of displacements for a homogeneous anisotropic medium. If

$$A_{ik}(\partial_x) = \delta_{ik} \mu \Delta + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_k}, \quad (1.19)$$

where  $\Delta$  is the Laplace operator,  $\delta_{ik}$  the Kronecker symbol,  $\lambda$  and  $\mu$  the elastic Lamé constants,  $\mu > 0$ ,  $3\lambda + 2\mu > 0$ ,  $i, k = 1, 2, 3$ , then we obtain the basic static equations of elasticity for a homogeneous isotropic medium.

When  $A_{ik}$  are defined from (1.19), for system (1.16) we construct explicitly, in terms of elementary functions (see Kupradze *et al.* [1]), the matrix of fundamental solutions (Kelvin matrix)

$$\begin{aligned} \Gamma_{ik}(x) &= \frac{\lambda' \delta_{ik}}{|x|} + \frac{\mu' x_i x_k}{|x|^3}, \quad i, k = 1, 2, 3, \\ \lambda' &= \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}, \quad \mu' = \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)}. \end{aligned} \quad (1.20)$$

The matrix  $\Gamma$  possesses the following properties:

- 1)  $\Gamma_{ik} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ ,  $\forall x \in \mathbb{R}^3 \setminus \{0\}: A_{ik}(\partial_x) \Gamma_{ks}(x) = 0$ ;
- 2)  $\forall t \neq 0, \forall x \in \mathbb{R}^3 \setminus \{0\}: (\partial^\alpha \Gamma)(tx) = t^{-|\alpha|-1} (\partial^\alpha \Gamma)(x)$ , where  $\alpha = (\alpha_1, \dots, \alpha_m)$  is an arbitrary multiindex;

$$3) \forall x \in \mathbb{R}^3: \lim_{\delta \rightarrow 0} \int_{\partial B(x, \delta)} T_{ik}(\partial_y, \nu(y)) \Gamma_{ks}(y - x) d_y S = \delta_{is},$$

where  $B(x, \delta)$  is the ball with centre at  $x$  and radius  $\delta$ ,  $\partial B(x, \delta)$  the ball boundary, and  $T(\partial_y, \nu) = \|T_{ik}(\partial_y, \nu)\|$  the stress operator:

$$T_{ik}(\partial_y, \nu(y)) = \lambda \nu_i(y) \frac{\partial}{\partial y_k} + \mu \delta_{ik} \frac{\partial}{\partial \nu(y)},$$

$\nu(y)$  the unit vector of the normal to the sphere  $\partial B(x, \delta)$  at  $y$ , external with respect to  $B(x, \delta)$ .

When (1.16) in a system of the basic equations of a transversally isotropic elastic medium, one can also construct, in terms of elementary functions, the matrix of fundamental solutions (see Kröner [1]).

In the general case, under the assumptions (1.18) there exists (see John [1]) a fundamental matrix  $\Phi = \|\Phi_{ks}\|_{n \times n}$  of the operator  $A_{ik}^*(\partial x)$  conjugate of the operator  $A_{ik}(\partial x)$ :

$$\begin{aligned} A^*(\partial x) &= \|A_{ik}^*(\partial x)\|_{n \times n}, \\ A^*(\partial x) &\equiv C_{klij} \frac{\partial^2}{\partial x_i \partial x_l} = C_{kjil} \frac{\partial^2}{\partial x_i \partial x_l} = A_{ki}(\partial x). \end{aligned} \quad (1.21)$$

The matrix  $\Phi$  possesses the following properties:

- 1)  $\Phi_{ks} \in C^\infty(\mathbb{R}^m \setminus \{0\})$ ,  $\forall x \in \mathbb{R}^m \setminus \{0\}$ :  $A_{ik}^*(\partial x)\Phi_{ks}(x) = 0$ ;
- 2)  $\forall t \neq 0$ ,  $\forall x \in \mathbb{R}^m \setminus \{0\}$ :  $(\partial^\alpha \Phi)(tx) = t^{-|\alpha|+2-m}(\partial^\alpha \Phi)(x)$ , where  $\alpha = (\alpha_1, \dots, \alpha_m)$  is an arbitrary multiindex;

$$3) \forall x \in \mathbb{R}^m: \lim_{\delta \rightarrow 0} \int_{\partial B(x, \delta)} T_{ik}^*(\partial y, \nu(y)) \Phi_{ks}(y-x) d_y S = \delta_{is},$$

where  $T_{ik}^*(\partial y, \nu)$  is the "stress" operator corresponding to the operator  $A^*(\partial x)$ :

$$\begin{aligned} T^*(\partial y, \nu) &\equiv \|T_{ik}^*(\partial y, \nu)\|_{n \times n}, \\ T_{ik}^*(\partial y, \nu) &= C_{kjil} \nu_j \frac{\partial}{\partial y_l} = T_{ki}(\partial y, \nu), \end{aligned} \quad (1.22)$$

$$T(\partial y, \nu) \equiv \|T_{ik}(\partial y, \nu)\|_{n \times n}. \quad (1.23)$$

The main aim of this subsection is to prove

**Theorem 1.4.** *Let  $\Omega$  be a domain from  $\mathbb{R}^m$ ,  $y \in \Omega$ ,  $u \equiv (u_1, \dots, u_m)$  be a solution of the class  $C^2$  of system (1.16) in the domain  $\Omega \setminus \{y\}$  and  $\forall x \in \Omega \setminus \{y\}$ :*

$$|u(x)| \leq \frac{c}{|x-y|^\gamma}, \quad (1.24)$$

where  $c = \text{const}$ ,  $\gamma \geq 0$ . Then  $\forall x \in \Omega \setminus \{y\}$ :

$$u(x) = u^{(0)}(x) + \sum_{|\alpha| \leq [\gamma] + 2 - m} (\partial_x^\alpha \Phi(x-y)) a^{(\alpha)}, \quad (1.25)$$

where  $u^{(0)}$  is a classical solution of system (1.16) in the domain  $\Omega$ ,  $u^{(0)} \in C^2(\Omega)$ ,  $\alpha \equiv (\alpha_1, \dots, \alpha_m)$ , is the multiindex,  $[\gamma]$  is the integer part of the number  $\gamma$ ,  $a^{(\alpha)} \equiv (a_1^{(\alpha)}, \dots, a_n^{(\alpha)})$ ,  $a_i^{(\alpha)} = \text{const}$ . Note that if  $[\gamma] + 2 - m < 0$ , then the second term in (1.25) is missing.

If condition (1.24) is replaced by

$$u(x) = o\left(\frac{1}{|x-y|^q}\right), \quad (1.24')$$

where  $q$  is a natural number, then representation (1.25) holds, where summation is performed up to  $q + 1 - m$ .

Note that the coefficients  $a^{(\alpha)}$  in representation (1.25) are expressed in quadratures in terms of  $u$ .

*Proof.* Since the proof of Theorem 1.4 is available only in the periodic literature in the Russian language (see Buchukuri, Gegelia [1]), we have made up our minds to give it here.

The proof is based on the Green and Somigliana formulas. Let  $\Omega$  be a bounded domain from  $\mathbb{R}^n$  with a smooth boundary  $\partial\Omega$ ,  $u = (u_1, \dots, u_n)$  and  $v = (v_1, \dots, v_n): \Omega \rightarrow \mathbb{R}^n$  belong to the Class  $C^2(\bar{\Omega})$ . Then the Green formula is valid:

$$\begin{aligned} & \int_{\Omega} (v_k(x)A_{ki}(\partial x)u_i(x) - u_i(x)A_{ik}^*(\partial x)v_k(x)) dx = \\ & = \int_{\partial\Omega} v_k(y)T_{ki} \left( (\partial y, \nu(y))u_i(y) - u_i(y)T_{ik}^*(\partial y, \nu(y))v_k(y) \right) d_y S, \end{aligned} \quad (1.26)$$

where  $\nu(y)$  is the unit vector of the normal to the surface  $\partial\Omega$  at the point  $y$ , external with respect to  $\Omega$ ,  $dx$  is a volume element,  $d_y S$  is an element of the area of  $\partial\Omega$ .

Let  $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$  and  $\forall x \in \Omega : A(\partial x)u(x) = 0$ . Then the Somigliana formula is valid  $\forall x \in \Omega$ :

$$\begin{aligned} u_s(x) = & \int_{\partial\Omega} \left( u_i(y)T_{ik}^*(\partial y, \nu(y))\Phi_{ks}(y-x) - \right. \\ & \left. - \Phi_{ks}(y-x)T_{ki}(\partial y, \nu(y))u_i(y) \right) d_y S. \end{aligned} \quad (1.27)$$

The proof of the Green formula (1.26) is immediately obtained from the following equalities which are easy to verify:

$$\begin{aligned} \int_{\Omega} v_k A_{ki}(\partial x)u_i dx &= \int_{\Omega} v_k \frac{\partial}{\partial x_j} (C_{kji}l \frac{\partial u_i}{\partial x_l}) dx = \int_{\partial\Omega} v_k T_{ki}(\partial y, \nu)u_i dS - \\ & - \int_{\Omega} C_{kji}l \frac{\partial v_k}{\partial x_j} \frac{\partial u_i}{\partial x_l} dx, \\ \int_{\Omega} u_i A_{ik}^*(\partial x)v_k dx &= \int_{\partial\Omega} u_i T_{ik}^*(\partial y, \nu)v_k dS - \int_{\Omega} C_{kji}l \frac{\partial v_k}{\partial x_j} \frac{\partial u_i}{\partial x_l} dx. \end{aligned}$$

The Somigliana formula (1.27) is proved, using property 3) of the fundamental matrix  $\Phi$  which can then be written in form

$$\lim_{\delta \rightarrow 0} \frac{C_{klij}}{\delta^{m-1}} \int_{\partial B(x, \delta)} \nu_j(y) \left( \frac{\partial}{\partial y_l} \Phi_{ks} \right) \left( \frac{y-x}{|y-x|} \right) d_y S = \delta_{is}. \quad (1.28)$$

Let  $z \in \Omega$ ,  $\overline{B(z, \delta)} \subset \Omega$ ,  $\delta > 0$ ,  $\forall x \in \Omega : A(\partial_x)u(x) = 0$ ,  $u = (u_1, \dots, u_n)$ ,  $u_k \in C^2(\Omega) \cap C^1(\overline{\Omega})$ ,  $v = (v_1, \dots, v_n)$ ,  $v_k(x) \equiv \Phi_{ks}(x - z)$ . Then  $\forall x \in \Omega \setminus \overline{B(z, \delta)} : A_{ik}^*(\partial_x)v_k(x) = 0$  and, using the Green formula for the domain  $\Omega \setminus \overline{B(z, \delta)}$ , we have

$$\begin{aligned} & \int_{\partial\Omega} (u_i(y)T_{ik}^*(\partial_y, \nu(y))v_k(y) - v_k(y)T_{ki}(\partial_y, \nu(y))u_i(y))d_yS = \\ & = \int_{\partial B(z, \delta)} (v_k(y)T_{ki}(\partial_y, \nu(y))u_i(y) - u_i(y)T_{ik}^*(\partial_y, \nu(y))v_k(y))d_yS. \end{aligned}$$

Moreover,

$$\begin{aligned} & \left| \int_{\partial B(z, \delta)} v_k(y)T_{ki}(\partial_y, \nu(y))u_i(y)d_yS \right| \leq \frac{c}{\delta^{m-2}} \sup_k \int_{\partial B(z, \delta)} \left| \Phi_{ks} \left( \frac{y-z}{|y-z|} \right) \right| d_yS \leq c\delta, \\ & \int_{\partial B(z, \delta)} u_i(y)T_{ik}^*(\partial_y, \nu(y))v_k(y)d_yS = \\ & = \int_{\partial B(z, \delta)} (u_i(y) - u_i(z))T_{ik}^*(\partial_y, \nu(y))\Phi_{ks}(y-z)d_yS + \\ & \quad + u_i(z) \int_{\partial B(z, \delta)} T_{ik}^*(\partial_y, \nu(y))\Phi_{ks}(y-z)d_yS, \\ & \left| \int_{\partial B(z, \delta)} (u_i(y) - u_i(z))T_{ik}^*(\partial_y, \nu(y))\Phi_{ks}(y-z)d_yS \right| \leq c \sup_{B(z, \delta)} |u_i(y) - u_i(z)|, \\ & \lim_{\delta \rightarrow 0} u_i(z) \int_{\partial B(z, \delta)} T_{ik}^*(\partial_y, \nu(y))\Phi_{ks}(y-z)d_yS = u_s(z). \end{aligned}$$

These formulas yield (1.27).

Now we can immediately proceed to proving Theorem 1.4. Let the conditions of this theorem be fulfilled. Choose positive numbers  $r$  and  $r_0$  such that  $3r < r_0$  and  $\overline{B(y, r_0)} \subset \Omega$ . Then  $\forall x \in B(y, r_0) \setminus B(y, 3r)$ , from (1.27) we obtain

$$\begin{aligned} u_s(x) &= \int_{\partial B(y, r_0)} (u_i(z)T_{ik}^*(\partial_z, \nu(z))\Phi_{ks}(z-x) - \\ & \quad - \Phi_{ks}(z-x)T_{ki}(\partial_z, \nu(z))u_i(z))d_zS - \\ & \quad - \int_{\partial B(y, r)} u_i(z)T_{ik}^*(\partial_z, \nu(z))\Phi_{ks}(z-x)d_zS + \end{aligned}$$

$$+ \int_{\partial B(y,r)} \Phi_{ks}(z-x) T_{ki}(\partial_z, \nu(z)) u_i(z) d_z S. \quad (1.29)$$

Denote the first integral on the right-hand side of (1.29) by  $u_s^{(0)}(x)$ . Obviously,  $u^{(0)} \equiv (u_1^{(0)}, \dots, u_n^{(0)})$  belong to the class  $C^2(B(y, r_0))$  and  $\forall x \in B(y, r_0) : A(\partial x)u^{(0)}(x) = 0$ .

Transform the last two integrals in (1.29). Since  $|x - y| \geq 3r$  ( $x \in B(y, r_0) \setminus B(y, 3r)$ ), the function  $z \rightarrow \Phi_{ks}(z - x)$  belongs to the class  $C^\infty(B(y, 3r))$  and can be represented in  $B(y, 3r)$  by means of the Taylor formula

$$\begin{aligned} \Phi_{ks}(z-x) &= \Phi_{ks}(y-x+z-y) = \\ &= \sum_{|\alpha| \leq p} \frac{(z-y)^\alpha}{\alpha!} (\mathcal{D}^\alpha \Phi_{ks})(y-x) + R_{ks}^{(p)}(x, y, z), \\ R_{ks}^{(p)}(x, y, z) &= \sum_{|\alpha|=p+1} \frac{(z-y)^\alpha}{\alpha!} (\mathcal{D}^\alpha \Phi_{ks})(y-x+\theta(z-y)), \end{aligned} \quad (1.30)$$

where  $0 < \theta < 1$ ,  $p$  is an arbitrary nonnegative integer number and  $\alpha$  is the multiindex.

Let us estimate the additional expansion term  $R_{ks}^{(p)}$ . Note that for  $z \in \overline{B(y, 2r)}$  we have

$$\begin{aligned} |y-x+\theta(z-y)| &\geq |y-x| - |z-x| \geq |y-x| - 2r \geq \\ &\geq |y-x| - 2\frac{|y-x|}{3} = \frac{|y-x|}{3}. \end{aligned}$$

Therefore if  $z \in \overline{B(y, 2r)}$ , then  $|(\mathcal{D}^\alpha \Phi_{ks})(y-x+\theta(z-y))| \leq C_x$  and thus we obtain

$$|R_{ks}^{(p)}(x, y, z)| \leq C_x |z-y|^{p+1}. \quad (1.31)$$

In the sequel, in addition to estimate (1.31), we will need the estimates of the derivatives of the additional term. We will prove that  $R_{ks}^{(p)}(x, y, \cdot)$  is analytic in the ball  $B(y, 3r)$  and  $\forall z \in B(y, 2r)$  is written as

$$|\mathcal{D}_z^\beta R_{ks}^{(p)}(x, y, z)| \leq C_{\beta x} |z-y|^{p+1-|\beta|}, \quad (1.32)$$

where  $\beta \equiv (\beta_1, \dots, \beta_n)$  is the multiindex satisfying the inequality  $|\beta| \leq p$ . Estimate (1.32) is obtained from the following simple proposition:

Let  $f \in C^k(\overline{B(y, r_1)})$  and  $\mathcal{D}^\alpha f(y) = 0$  for  $|\alpha| \leq k-1$ , then  $\forall z \in \overline{B(y, r_1)}$  we have

$$|\mathcal{D}^\alpha f(z)| \leq C_\alpha |z-y|^{k-|\alpha|}. \quad (1.33)$$

This will be proved using the induction method. For  $k = 1$  we have  $f \in C^1(\overline{B(y, r_1)})$  and  $f(y) = 0$ . Introducing a function  $g : [0, 1] \rightarrow \mathbb{R}$ ,  $g(t) = f(t(z - y) + y)$ , we can write

$$f(z) = f(z) - f(y) = g(1) - g(0) = g'(\theta) = \sum_{i=1}^m \frac{\partial f(\theta(z - y) + y)}{\partial z_i} (z_i - y_i)$$

and therefore  $\forall z \in \overline{B(y, r_1)}$  we have

$$|f(z)| \leq c_0 |z - y|, \quad c_0 \equiv \sum_{i=1}^m \max_{B(y, r_1)} \left| \frac{\partial f(z)}{\partial z_i} \right|. \quad (1.34)$$

Assume that proposition (1.33) is proved for  $k$  and prove that it holds for  $k + 1$ . In other words, estimate (1.33) is fulfilled and we have to prove that for  $f \in C^{k+1}(\overline{B(y, r_1)})$  and  $\mathcal{D}^\alpha f(y) = 0$ , when  $|\alpha| \leq k$ , we have  $|\mathcal{D}^\alpha f(z)| \leq C_\alpha |z - y|^{k+1-|\alpha|}$ . It is obvious that  $\frac{\partial f}{\partial z_i} \in C^k(\overline{B(y, r_1)})$ ,  $\mathcal{D}^\beta \frac{\partial f(y)}{\partial z_i} = 0$  for  $|\beta| \leq k - 1$ . Applying estimate (1.33) to  $\frac{\partial f}{\partial z_i}$ , we obtain  $\forall z \in \overline{B(y, r_1)}$

$$\left| \mathcal{D}^\beta \frac{\partial f(z)}{\partial z_i} \right| \leq C_\beta |z - y|^{k-|\beta|}, \quad |\beta| \leq k - 1. \quad (1.35)$$

Assume  $\varphi_\beta \equiv \mathcal{D}^\beta f$ . Then  $\varphi_\beta \in C^1(\overline{B(y, r_1)})$  and  $\varphi_\beta(y) = 0$ . Now by virtue of (1.34) we write  $\forall z \in \overline{B(y, r)}$ :

$$\begin{aligned} |\mathcal{D}^\beta f(z)| &= |\varphi_\beta(z)| \leq c_0 |z - y|, \\ C_0 &= \sum_{i=1}^m \max_{B(y, r_1)} \left| \frac{\partial \varphi_\beta(z)}{\partial z_i} \right| = \sum_{i=1}^m \max_{B(y, r_1)} \left| \frac{\partial}{\partial z_i} \mathcal{D}^\beta f(z) \right|. \end{aligned}$$

Hence on account of (1.35) we obtain  $C_0 \leq C_\beta |z - y|^{k-|\beta|}$ ,  $|\mathcal{D}^\beta f(z)| \leq C_\beta |z - y|^{k+1-|\beta|}$ , where  $\beta$  is an arbitrary multiindex whose absolute value is not greater than  $k - 1$ . It remains for us to prove the validity of this inequality in case  $|\beta| = k$  or, which is the same, the validity of the relation  $|\mathcal{D}^\alpha f(z)| \leq C_\alpha |z - y|$ ,  $|\alpha| = k$ . But if  $|\alpha| = k$ , then  $\varphi \equiv \mathcal{D}^\alpha f \in C^1(\overline{B(y, r_1)})$  and  $\varphi(y) = \mathcal{D}^\alpha f(y) = 0$ . Thus we conclude that (1.34) is fulfilled for  $\varphi$ . This completes the proof of the validity of estimate (1.33).

Let us now prove the validity of estimate (1.32). Let  $2r < r_1 < |x - y|$  and (see (1.30))

$$f(z) \equiv \Phi_{ks}(z - x) - \sum_{|\alpha| \leq p} \frac{(z - y)^\alpha}{\alpha!} (\mathcal{D}^\alpha \Phi_{ks})(y - x) = R_{ks}^{(p)}(x, y, z).$$

Then  $f \in C^\infty(\overline{B(y, r_1)})$  and  $\mathcal{D}^\alpha f(y) = 0$  for  $|\alpha| \leq p$ . Now by virtue of estimate (1.33) we have  $\forall z \in \overline{B(y, r_1)}$   $|\mathcal{D}^\alpha f(z)| \leq c_\alpha |z - y|^{p+1-|\alpha|}$ ,  $|\alpha| \leq p$ , which is actually inequality (1.30).

Let us turn our attention to representation (1.29). Using expansion (1.30),  $u_s$  is written as

$$\begin{aligned}
u_s(x) &= u_s^{(0)}(x) + \sum_{|\alpha| \leq p} \frac{(\mathcal{D}^\alpha \Phi_{ks})(y-x)}{\alpha!} \times \\
&\quad \times \left( \int_{\partial B(y,r)} (z-y)^\alpha T_{ki}(z, \nu(z)) u_i(z) d_z S - \right. \\
&\quad \left. - \int_{\partial B(y,r)} u_i(z) T_{ik}^*(z, \nu(z)) (z-y)^\alpha d_z S \right) + I_s^{(p)}(r, x, y), \\
u_s^{(0)}(x) &\equiv \int_{\partial B(y, r_0)} \left( u_i(z) T_{ik}^*(\partial_z, \nu(z)) \Phi_{ks}(z-x) - \right. \\
&\quad \left. - \Phi_{ks}(z-x) T_{ki}(\partial_z, \nu(z)) u_i(z) \right) d_z S, \\
I_s^{(p)}(r, x, y) &\equiv \int_{\partial B(y,r)} \left( R_{ks}^{(p)}(x, y, z) T_{ki}(z, \nu(z)) u_i(z) - \right. \\
&\quad \left. - u_i(z) T_{ik}^*(z, \nu(z)) R_{ks}^{(p)}(x, y, z) \right) d_z S.
\end{aligned} \tag{1.36}$$

We are to prove that if  $p > \gamma - m + 1$ , then

$$\lim_{r \rightarrow 0} I_s^{(p)}(r, x, y) = 0. \tag{1.37}$$

We introduce a function  $\omega^{(0)} : \mathbb{R}^m \rightarrow \mathbb{R}$  possessing the following properties:  $\omega^{(0)} \in C^\infty(\mathbb{R}^m)$ ;  $|z| \leq 1 : \omega^{(0)}(z) = 1$ ;  $|z| \geq 2 : \omega^{(0)}(z) = 0$ . Let  $\omega_r^{(y)}(z) \equiv \omega^{(0)}\left(\frac{z-y}{r}\right)$ . Then it is obvious that

$$\begin{aligned}
&\omega_r^{(y)} \in C^\infty(\mathbb{R}^m); \quad z \in \overline{B(y, r)} : \omega_r^{(y)}(z) = 1; \\
&z \notin B(y, 2r) : \omega_r^{(y)}(z) = 0; \quad |\mathcal{D}^\alpha \omega_r^{(y)}(z)| \leq C_\alpha r^{-|\alpha|},
\end{aligned} \tag{1.38}$$

where  $\alpha$  is an arbitrary multiindex and  $C_\alpha = \max_{\mathbb{R}^m} |\mathcal{D}^\alpha \omega_r^{(0)}(z)|$ .

Introduce a function  $z \rightarrow \tilde{R}_{ks}^{(p)}(x, y, z) \equiv \omega_r^{(y)}(z) R_{ks}^{(p)}(x, y, z)$ . It is obvious that

$$\begin{aligned}
&\tilde{R}_{ks}^{(0)}(x, y, \cdot) \in C^\infty(\mathbb{R}^m); \quad z \in \overline{B(y, r)} : \tilde{R}_{ks}^{(p)}(x, y, z) = R_{ks}^{(p)}(x, y, z); \\
&z \in \partial B(y, r) : T_{ik}^*(\partial z, \nu(z)) \tilde{R}_{ks}^{(p)}(x, y, z) = T_{ik}^*(\partial z, \nu(z)) R_{ks}^{(p)}(x, y, z); \\
&z \notin B(y, 2r) : \tilde{R}_{ks}^{(p)}(x, y, z) = 0; \quad z \in \partial B(y, 2r) : T_{ik}^*(\partial z, \nu(z)) \tilde{R}_{ks}^{(p)}(x, y, z) = 0.
\end{aligned}$$

Applying the Green formula (1.26) to the functions  $u = (u_1, \dots, u_n)$  and  $v = (\tilde{R}_{1s}^{(p)}(x, y, \cdot), \dots, \tilde{R}_{ns}^{(p)}(x, y, \cdot))$  in the domain  $B(y, 2r) \setminus B(y, r)$ , we arrive

at

$$\begin{aligned} & \int_{B(y,2r) \setminus B(y,r)} u_i(z) A_{ik}^*(\partial z) \tilde{R}_{ks}^{(p)}(x, y, z) dz = \\ & = \int_{\partial B(y,r)} \left( u_i(z) T_{ik}^*(\partial z, \nu(z)) R_{ks}^{(p)}(x, y, z) - R_{ks}^{(p)}(x, y, z) T_{ki}(\partial z, \nu(z)) u_i(z) \right) d_z S. \end{aligned}$$

Therefore

$$I_s^{(p)}(r, x, y) = - \int_{B(y,2r) \setminus B(y,r)} u_i(z) A_{ik}^*(\partial z) \tilde{R}_{ks}^{(p)}(x, y, z) dz.$$

Since  $A_{ik}^*(\partial z) \tilde{R}_{ks}^{(p)}(x, y, z) = A_{ik}^*(\partial z) \omega_r^{(y)}(z) R_{ks}^{(p)}(x, y, z)$ , due to (1.32) and (1.38) we obtain  $\forall z \in B(y, 2r)$ :

$$|A_{ik}^*(\partial z) \tilde{R}_{ks}^{(p)}(x, y, z)| \leq c|z - y|^{p-1} + c|z - y|^p r^{-1} + c|z - y|^{p+1} r^{-2},$$

which for  $I_s^{(p)}$  yields the estimate

$$|I_s^{(p)}(r, x, y)| \leq cr^{p-1} \int_{B(y,2r) \setminus B(y,r)} |u(z)| dz \quad (1.39)$$

and by virtue of (1.24) we obtain

$$|I_s^{(p)}(r, x, y)| \leq cr^{p-1} \int_r^{2r} \rho^{-\gamma+m-1} d\rho \leq cr^{p+m-\gamma-1}. \quad (1.40)$$

Inequality (1.40) implies that (1.37) is fulfilled for  $p > \gamma - m + 1$ . In particular, (1.37) is fulfilled for  $p = [\gamma] - m + 2$ .

Now we will consider representation (1.36). The first term on the right-hand side of (1.36) does not depend on  $r$ . The left-hand side of this equality does not depend on  $r$  either. Therefore for  $p = [\gamma] - m + 2$  we have

$$u_s(x) = u_s^{(0)}(x) + \lim_{r \rightarrow 0} \sum_{|\alpha| \leq p} c_k^{(\alpha)}(r) (\mathcal{D}^\alpha \Phi_{ks})(y - x), \quad (1.41)$$

where

$$\begin{aligned} c_k^{(\alpha)}(r) = & \frac{1}{\alpha!} \int_{\partial B(y,r)} (z - y)^\alpha T_{ik}(\partial z, \nu(z)) u_i(z) - \\ & - u_i(z) T_{ik}^*(\partial z, \nu(z)) (z - y)^\alpha d_z S. \end{aligned}$$

Now to complete the proof of Theorem 1.4 it remains for us to prove the existence of constants  $c_k^{(\alpha)}$  such that

$$\lim_{r \rightarrow 0} \sum_{|\alpha| \leq p} c_k^{(\alpha)}(r) (\mathcal{D}^\alpha \Phi_{ks})(y-x) = \sum_{|\alpha| \leq p} c_k^{(\alpha)} (\mathcal{D}^\alpha \Phi_{ks})(y-x).$$

The validity of this equality immediately follows from the following proposition which is easy to prove:

Let  $f^{(k)} \equiv (f_1^{(k)}, \dots, f_n^{(k)}) : \Omega \rightarrow \mathbb{R}^n$ , where  $\Omega \subset \mathbb{R}^m$  and  $k = 1, \dots, q$ ;  $f(x, r) \equiv \sum_{k=1}^q a_k(r) f^{(k)}(x)$ ,  $a_k(r) \in \mathbb{R}$ . If there exists a limit

$$\lim_{r \rightarrow 0} f(x, r) \equiv f^{(0)}(x),$$

then there are constants  $a_k^{(0)}$  such that

$$\forall x \in \Omega : f^{(0)}(x) = \sum_{k=1}^q a_k^{(0)} f^{(k)}(x).$$

Replacing in Theorem 1.4 condition (1.24) by (1.24'), the validity of representation (1.25), where summation is performed up to  $q+1-m$ , can be proved exactly by the same technique as was used for the first part of this theorem. After obtaining formula (1.39), its right-hand side should be estimated in a somewhat different manner. Namely, since  $|z-y|^q u(z) = o(1)$ , (1.39) yields

$$\begin{aligned} |I_s^{(p)}(r, x, y)| &\leq cr^{p-1} \int_{B(y, 2r) \setminus B(y, r)} |z-y|^{-q} |o(1)| dz \leq \\ &\leq cr^{p-1} \sup_{B(y, 2r) \setminus B(y, r)} |o(1)| \int_r^{2r} \rho^{m-1-q} d\rho \leq cr^{p+m-q-1} o(1) \rightarrow 0, \quad r \rightarrow 0, \end{aligned}$$

if  $p = q + 1 - m$ .

Theorem 1.4 is completely proved for  $m > 2$ . For  $m = 2$  the matrix of fundamental solutions  $\Phi = \|\Phi_{ik}\|_{n \times n}$  is written as (see John [1])

$$\begin{aligned} \Phi_{ik}(x) &= a_{ik} \ln |x| + b_{ik}(x), \\ a_{ik} &= -\frac{1}{2\pi} \int_{\partial B(0,1)} A_{ik}^{-1}(\xi) d_\xi S, \quad b_{ik} = -\frac{1}{2\pi} \int_{\partial B(0,1)} A_{ik}^{-1} \ln \left| \frac{x}{|x|} \xi \right| d_\xi S, \end{aligned}$$

where  $\xi = (\xi_1, \xi_2)$ ,  $A^{-1}$  is the inverse of  $A$  matrix,  $A_{ik}^{-1}$  is an element of this matrix,  $x \cdot \xi = x_1 \cdot \xi_1 + x_2 \cdot \xi_2$  is the scalar product of the vectors  $x$  and  $\xi$ .

One can easily verify that  $\det \|a_{ik}\|_{n \times n} \neq 0$ ,  $b_{ik}$  are homogeneous functions of the 0-th order:  $b_{ik}(\lambda x) = b_{ik}(x)$  and  $\mathcal{D}^\alpha \Phi_{ik}(tx) = t^{-|\alpha|} \mathcal{D}^\alpha \Phi(x)$ , where  $\alpha = (\alpha_1, \alpha_2)$  is the multiindex ( $|\alpha| > 0$ ). Moreover, the Green and Somigliana formulas also hold in the case  $m = 2$ . These facts considered,

the proof of Theorem 1.4 for the case  $m = 2$  differs in no way from the case  $m > 2$ . ■

We would like to mention some immediate corollaries of the proved theorem.

The theorem on a removable singularity is applicable to the theory of elasticity as well. Indeed, let  $m > 2$  and the conditions of Theorem 1.4 be fulfilled for  $|\gamma| < m - 2$ . Then from this theorem it follows that  $\forall x \in \Omega \setminus \{y\} : u(x) = u^{(o)}(x)$ , where  $u^{(o)}$  is an infinitely differentiable solution of system (1.16) in the domain  $\Omega$ . In such a situation there exists a finite limit

$$\lim_{x \rightarrow y} u(x) \quad (1.42)$$

and, if we complete the determination of  $u$  at the point  $y$  by a formula  $u(y) = u^{(o)}(y)$ , then  $u$  will be the classical solution of system (1.16) in the entire domain  $\Omega$  including the point  $y$ .

This proposition can be somewhat strengthened taking into consideration condition (1.24'). Finally, as a corollary we arrive at

**Theorem 1.5.** *Let  $\Omega$  be a domain from  $\mathbb{R}^m$  ( $m \geq 2$ ),  $u \equiv (u_1, \dots, u_n) \in C^2(\Omega \setminus \{y\})$ , where  $y \in \Omega$  and  $\forall x \in \Omega \setminus \{y\} : A(\partial x)u(x) = 0$ . Let, furthermore,  $\forall x \in \Omega \setminus \{y\} : u(x) = o(\Phi(x - y))$ . Then  $y$  is a removable singular point for  $u$ , i.e. there exists limit (1.42) and if  $u$  is defined at  $y$  by the formula  $u(y) = u^{(o)}(y)$ , then  $u$  will be the solution of system (1.16) in the entire domain  $\Omega$  including the point  $y$ .*

Theorem 1.5 remains valid if the formula  $u(x) = o(\Phi(x - y))$  is replaced by  $u(x) = o(\ln|x - y|)$  when  $m = 2$ , and by  $u(x) = o(|x - y|^{2-m})$  when  $m > 2$ .

Representation (1.25) immediately implies the validity of

**Theorem 1.6.** *Let the conditions of Theorem 1.4 be fulfilled and in (1.24)  $\gamma > m - 2$  ( $m \geq 2$ ). Then for any multiindex  $\alpha$  we have*

$$|\mathcal{D}^\alpha u(x)| \leq c_\alpha |x - y|^{-[\gamma] - |\alpha|}. \quad (1.43)$$

*In particular, for stress we have the estimate*

$$|T(\partial_x, \nu)u(x)| \leq c |x - y|^{-[\gamma] - 1} \quad (1.44)$$

*uniformly with respect to  $\nu$ .*

Theorem 1.6 remains valid for  $\gamma \geq m - 2 > 0$ .

**1.8. Character of Solutions of the Basic Equations of Elasticity at Infinity.** In constructing effective solutions of the boundary value problems the questions as to the existence of solutions lose their importance and it is the uniqueness theorems that come to the foreground. Indeed, we must make sure that we have constructed all the solutions of the problem posed and the solution we seek for has not been left out of consideration. Such a drawback is encountered in the papers, where effective solutions are constructed in the form of series. Moreover, in verifying whether the constructed effective solution satisfies the uniqueness theorem it is important to have simple conditions to be used for verification. With this end in view, we will somewhat refine and simplify the well-known uniqueness theorems (see Knops, Payne [1]; Kupradze *et al.*[1]).

Our first step will be to investigate the behaviour of solutions near the point at infinity of the basic equations of elasticity in terms of displacements. We will prove

**Theorem 1.7.** *Let  $\Omega$  be a domain from  $\mathbb{R}^m$  containing the point at infinity,  $u$  be the solution of system (1.16) in the domain  $\Omega$  ( $u \in C^2(\Omega)$ ) and anyone of the following conditions be fulfilled:*

$$\lim_{r \rightarrow \infty} \frac{1}{r^{m+p+1}} \int_{B(0,r) \setminus B(0,r/4)} |u(z)| dz = 0, \quad (1.45)$$

$$\lim_{|z| \rightarrow \infty} \frac{|u(z)|}{|z|^{p+1}} = 0, \quad (1.46)$$

$$\int_{\Omega} \frac{|u(z)| dz}{1 + |z|^{m+p+1}} < +\infty, \quad (1.47)$$

where  $p$  is a nonnegative integer number. Then in the neighbourhood of  $|x| = +\infty$

$$u_s(x) = \sum_{|\alpha| \leq p} c_s^{(\alpha)} x^\alpha + \sum_{|\beta| \leq q} d_k^{(\beta)} \mathcal{D}^\beta \Phi_{ks}(x) + \psi_s(x), \quad (1.48)$$

where  $c_s^{(\alpha)} = \text{const}$ ,  $d_s^{(\beta)} = \text{const}$ ,  $\alpha \equiv (\alpha_1, \dots, \alpha_m)$ ,  $\beta \equiv (\beta_1, \dots, \beta_m)$  are the summation multiindexes,  $q$  is an arbitrary nonnegative integer number,

$$\left| |x|^{m+|\gamma|+q-1} \mathcal{D}^\gamma \psi_s(x) \right| \leq c = \text{const}, \quad (1.49)$$

$\gamma \equiv (\gamma_1, \dots, \gamma_m)$  is an arbitrary multiindex. In that case everyone of the three terms on the right-hand side of representation (1.48) is a solution of system (1.16) in the neighbourhood of the point at infinity.

Note that the coefficients  $c_s^{(\alpha)}$  and  $d_k^{(\alpha)}$  in (1.48) are expressed by quadratures in terms of  $u$ .

*Proof.* It can be assumed without loss of generality that the origin (the point 0) does not belong to the domain  $\Omega$  and  $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ . Let  $m > 2$ ,  $x$  be an arbitrarily chosen point of  $\Omega$  and  $r$  be a large positive integer such that  $x \in B(0, r/8)$  and  $(\mathbb{R}^m \setminus B(0, r/8)) \subset \Omega$ .

Write the Somigliana formula (1.27) for the domain  $\Omega_r \equiv \Omega \cap B(0, r)$ :

$$\begin{aligned} u_s(x) &= \int_{\partial\Omega} \left( u_i(y) T_{ik}^*(\partial y, \nu(y)) \Phi_{ks}(y-x) - \right. \\ &\quad \left. - \Phi_{ks}(y-x) T_{ki}(\partial y, \nu(y)) u_i(y) \right) d_y S + \\ &\quad + \int_{\partial B(0,r)} \left( u_i(y) T_{ik}^*(\partial y, \nu(y)) \Phi_{ks}(y-x) - \right. \\ &\quad \left. - \Phi_{ks}(y-x) T_{ki}(\partial y, \nu(y)) u_i(y) \right) d_y S. \end{aligned} \quad (1.50)$$

Using the Taylor formula, we represent  $\Phi_{ks}(y-x)$  in the neighbourhood of the point  $y$  as

$$\begin{aligned} \Phi_{ks}(y-x) &= \sum_{|\alpha| \leq p} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} (\mathcal{D}^\alpha \Phi_{ks})(y) + R_{ks}(x, y), \\ R_{ks}(x, y) &\equiv \sum_{|\alpha|=p+1} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} (\mathcal{D}^\alpha \Phi_{ks})(y - \theta x), \end{aligned} \quad (1.51)$$

where  $p$  is an arbitrary nonnegative integer number,  $\alpha$  is a multiindex and  $\theta \in ]0; 1[$ .

First we have to prove that  $\forall y \in \partial B(0, r)$

$$|\mathcal{D}_y^\beta R_{ks}(x, y)| \leq c_{ks}^{(p, \beta)}(x) |y|^{-m-p-|\beta|+1}, \quad (1.52)$$

where  $\beta \equiv (\beta_1, \dots, \beta_m)$  is an arbitrary multiindex and  $c_{ks}^{(p, \beta)}(x)$  is a positive integer which is independent of  $y$ .

Indeed, using Taylor formula we can represent  $(\mathcal{D}^\beta \Phi_{ks})(y-x)$  in the neighbourhood of the point  $y$  as

$$\begin{aligned} (\mathcal{D}^\beta \Phi_{ks})(y-x) &= \sum_{|\alpha| \leq p} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} \mathcal{D}^\alpha \mathcal{D}^\beta (\Phi_{ks})(y) + R_{ks}^{(1)}(x, y), \\ R_{ks}^{(1)}(x, y) &\equiv \sum_{|\alpha|=p+1} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} (\mathcal{D}^\alpha \mathcal{D}^\beta \Phi_{ks})(y - \theta_1 x), \quad 0 < \theta_1 < 1. \end{aligned} \quad (1.53)$$

The differentiation of (1.51) gives

$$\mathcal{D}^\beta \Phi_{ks}(y-x) = \sum_{|\alpha| \leq p} \frac{(-1)^{|\alpha|} x^\alpha}{\alpha!} (\mathcal{D}^\alpha \mathcal{D}^\beta \Phi_{ks})(y) + \mathcal{D}_y^\beta R_{ks}(x, y)$$

and therefore  $\mathcal{D}_y^\beta R_{ks}(x, y) = R_{ks}^{(1)}(x, y)$ . Hence by virtue of (1.53) and property 2) of the matrix  $\Phi$  we obtain the estimate

$$|\mathcal{D}_y^\beta R_{ks}(x, y)| \leq \sum_{|\alpha|=p+1} \frac{|x|^{p+1}}{\alpha!} \frac{1}{|y|^{m-|\alpha|-|\beta|-2}} \left| (\mathcal{D}^{\alpha+\beta} \Phi_{ks}) \left( \frac{y}{|y|} - \frac{\theta_1 x}{|y|} \right) \right|.$$

But  $\forall y \in \partial B(0, r)$ :  $|y| = r$  and, besides,  $|x| < \frac{r}{8}$ ,  $0 < \theta_1 < 1$ . Therefore

$$\left| \frac{y}{|y|} \right| - \left| \frac{\theta_1 x}{|y|} \right| \leq \left| \frac{y}{|y|} - \frac{\theta_1 x}{|y|} \right| \leq \left| \frac{y}{|y|} \right| + \left| \frac{\theta_1 x}{|y|} \right|, \quad \frac{7}{8} \leq \left| \frac{y}{|y|} - \frac{\theta_1 x}{|y|} \right| \leq \frac{9}{8}.$$

Since  $\Phi_{ks} \in C^\infty(\mathbb{R}^m \setminus \{0\})$ , we have  $\forall y \in \partial B(0, r)$  and  $|x| < \frac{r}{8}$

$$\left| (\mathcal{D}^{\alpha+\beta} \Phi_{ks}) \left( \frac{y}{|y|} - \frac{\theta_1 x}{|y|} \right) \right| \leq c_{ks}^{(p,\beta)} = \text{const.}$$

We have derived estimate (1.52) which obviously also holds for any  $y \in \overline{B(0, r)} \setminus B(0, r/4)$  and  $|x| < r/8$ .

Estimate (1.52) implies, in particular, that  $\forall y \in \partial B(0, r)$  and  $|x| < r/8$

$$\left| T_{ik}^*(\partial y, \nu(y)) R_{ks}(x, y) \right| \leq c(x) |y|^{-m-p}. \quad (1.54)$$

Let us now turn our attention to representations (1.50) and (1.51). We write

$$\begin{aligned} u_s(x) &= u_s^{(0)}(x) + \sum_{|\alpha| \leq p} \frac{(-1)^\alpha c_s^{(\alpha)}(r)}{\alpha!} x^\alpha + I_s(p, r, x), \\ u_s^{(0)}(x) &\equiv \int_{\partial \Omega} \left( u_i(y) T_{ik}^*(\partial y, \nu(y)) \Phi_{ks}(y-x) - \right. \\ &\quad \left. - \Phi_{ks}(y-x) T_{ki}(\partial y, \nu(y)) u_i(y) \right) d_y S, \\ c_s^{(\alpha)}(r) &\equiv \int_{\partial B(0, r)} \left( u_i(y) T_{ik}^*(\partial y, \nu(y)) \mathcal{D}^{(\alpha)} \Phi_{ks}(y) - \right. \\ &\quad \left. - \mathcal{D}^{(\alpha)} \Phi_{ks}(y) T_{ki}(\partial y, \nu(y)) u_i(y) \right) d_y S, \\ I_s(p, r, x) &\equiv \int_{\partial B(0, r)} \left( u_i(y) T_{ik}^*(\partial y, \nu(y)) R_{ks}(y, x) - \right. \\ &\quad \left. - R_{ks}(y, x) T_{ki}(\partial y, \nu(y)) u_i(y) \right) d_y S. \end{aligned} \quad (1.55)$$

We have to show that in representation (1.55) the coefficients  $c_s^{(\alpha)}(r)$  are independent of  $r$ . Indeed, let  $r_1 > r$  and apply the Green formula (1.26) to the vectors  $u = (u_1, \dots, u_n)$  and  $v = (\mathcal{D}^\alpha \Phi_{1s}, \dots, \mathcal{D}^\alpha \Phi_{ns})$  in the domain  $B(0, r_1) \setminus \overline{B(0, r)}$ . When  $z \in B(0, r_1) \setminus \overline{B(0, r)}$ , we have  $A_{ki}(\partial z) u_i(z) = 0$ ,

and  $A_{ik}^* \mathcal{D}^\alpha \Phi_{ks}(z) = 0$  and thus the volume integrals vanish in (1.26). This results in

$$\begin{aligned} & \int_{\partial B(0,r)} \left( \mathcal{D}^{(\alpha)} \Phi_{ks}(y) T_{ki}(\partial y, \nu(y)) u_i(y) - \right. \\ & \quad \left. - u_i(y) T_{ik}^*(\partial y, \nu(y)) \mathcal{D}^{(\alpha)} \Phi_{ks}(y) \right) d_y S = \\ & = \int_{\partial B(0,r_1)} \left( \mathcal{D}^{(\alpha)} \Phi_{ks}(y) T_{ki}(\partial y, \nu(y)) u_i(y) - \right. \\ & \quad \left. - u_i(y) T_{ik}^*(\partial y, \nu(y)) \mathcal{D}^{(\alpha)} \Phi_{ks}(y) \right) d_y S. \end{aligned}$$

Here the normals to  $\partial B(0, r)$  and  $\partial B(0, r_1)$  are assumed to be external with respect to  $B(0, r)$  and  $B(0, r_1)$ .

The above equality shows that  $c_s^{(\alpha)}(r) = c_s^{(\alpha)}(r_1)$ . We have proved that  $c_s^{(\alpha)}(r)$  is independent of  $I_s(p, r, x)$ . Now (1.55) implies that  $I_s(p, r, x)$  is independent of  $r$ , too.

Below we will subject  $u$  to the condition whose fulfilment leads to

$$\lim_{r \rightarrow +\infty} I_s(p, r, x) = 0. \quad (1.56)$$

Thus, if (1.56) is fulfilled, then we obtain (see (1.55)) the representation

$$u_s(x) = u_s^0(x) + \sum_{|\alpha| \leq p} c_s^{(\alpha)} x^\alpha, \quad (1.57)$$

where  $c_s^{(\alpha)}$  is independent of  $r$  and  $c_s^\alpha \equiv \frac{(-1)^{|\alpha|}}{\alpha!} c_s^{(\alpha)}(r)$ . Now we will establish criteria for the validity of (1.56). To this end we have to transform its representation as follows:

Let  $\omega : \mathbb{R}^m \rightarrow \mathbb{R}$  and possess the properties

$$\begin{aligned} \omega & \in C^\infty(\mathbb{R}^m), \quad \text{supp } \omega \subset B(0, 3) \setminus B(0, 1/3), \\ \forall y & \in B(0, 2) \setminus B(0, 1/2) : \omega(y) = 1. \end{aligned}$$

Then the function  $\omega^{(r)}(y) \equiv \omega\left(\frac{y}{r}\right)$  possesses the properties

$$\begin{aligned} \text{supp } \omega^{(r)} & \subset B(0, 3r) \setminus B(0, r/3), \\ \forall y & \in B(0, 2r) \setminus B(0, r/2) : \omega^{(r)}(y) = 1. \end{aligned}$$

Obviously,  $b_\alpha \equiv \sup_{\mathbb{R}^m} |\mathcal{D}^\alpha \omega(y)| < +\infty$  and therefore

$$|\mathcal{D}^\alpha \omega^{(r)}(y)| = \left| \frac{1}{r^{|\alpha|}} (\mathcal{D}^\alpha \omega)\left(\frac{y}{r}\right) \right| \leq b_\alpha r^{-|\alpha|}.$$

Introduce the notation  $R_{ks}^{(r)}(x, y) \equiv \omega^{(r)}(y)R_{ks}^{(r)}(x, y)$ . Then  $R_{ks}^{(r)}$  will have the properties

$$\begin{aligned} \forall y \in \partial B(0, r/4) : R_{ks}^{(r)}(x, y) &= 0, \quad T_{ik}^*(\partial y, \nu(y))R_{ks}^{(r)}(x, y) = 0; \\ \forall y \in \partial B(0, r) : R_{ks}^{(r)}(x, y) &= R_{ks}(x, y), \quad T_{ik}^*(\partial y, \nu(y))R_{ks}^{(r)}(x, y) = \\ &= T_{ik}^*(\partial y, \nu(y))R_{ks}(x, y). \end{aligned}$$

Applying the Green formula (1.26) to the vectors  $u = (u_1, \dots, u_m)$  and  $v = (R_{1s}^{(r)}(x, \cdot), \dots, R_{ns}^{(r)}(x, \cdot))$  in the domain  $B(0, r) \setminus B(0, r/2)$ , we will have

$$\begin{aligned} &\int_{B(0, r) \setminus B(0, r/4)} \left( R_{ks}^{(r)}(x, z)A_{ki}(\partial z)u_i(z) - u_i(z)A_{ik}^*(\partial z)R_{ks}^{(r)}(x, z) \right) dz = \\ &= \int_{\partial B(0, r) \cup \partial B(0, r/4)} \left( R_{ks}^{(r)}(x, y)T_{ki}(\partial y, \nu(y))u_i(y) - \right. \\ &\quad \left. - u_i(y)T_{ik}^*(\partial y, \nu(y))R_{ks}^{(r)}(x, y) \right) d_y S. \end{aligned}$$

By virtue of the properties of the function  $R_{ks}^{(r)}$  the integral on  $\partial B(0, r/4)$  will vanish. In the integral on  $\partial B(0, r) : R_{ks}^{(r)} = R_{ks}$ . Moreover, if  $z \in B(0, r) \setminus B(0, r/4)$ , then  $A_{ik}(\partial z)u_i(z) = 0$ . The above equality takes the form

$$I_s(p, r, x) = \int_{B(0, r) \setminus B(0, r/4)} u_i(z)A_{ik}^*(\partial z)R_{ks}^{(r)}(x, z) dz. \quad (1.58)$$

Let us estimate this integral. Obviously,

$$\begin{aligned} \frac{\partial^2 R_{ks}^{(r)}(x, y)}{\partial z_i \partial z_j} &= \omega^{(r)}(z) \frac{\partial^2 R_{ks}(x, z)}{\partial z_i \partial z_j} + \frac{\partial \omega^{(r)}(z)}{\partial z_i} \frac{\partial R_{ks}(x, z)}{\partial z_j} + \\ &+ \frac{\partial \omega^{(r)}(z)}{\partial z_j} \frac{\partial R_{ks}(x, z)}{\partial z_i} + \frac{\partial^2 \omega^{(r)}(z)}{\partial z_i \partial z_j} R_{ks}(x, z). \end{aligned}$$

Hence, on account of (1.52), using the above estimate for  $\mathcal{D}^\alpha \omega^{(r)}$ , we obtain

$$\forall z \in B(0, r) \setminus B(0, r/4) : \left| \frac{\partial^2 R_{ks}^{(r)}(x, z)}{\partial z_i \partial z_j} \right| \leq \frac{c_{ks}(x)}{|r|^{m+p+1}}$$

and, finally,

$$|I_s(p, r, x)| \leq \frac{c_{ks}(x)}{|r|^{m+p+1}} \int_{B(0, r) \setminus B(0, r/4)} |u(z)| dz.$$

Thus, if

$$\lim_{r \rightarrow \infty} \frac{1}{r^{m+p+1}} \int_{B(0, r) \setminus B(0, r/4)} |u(z)| dz = 0,$$

then condition (1.56) is fulfilled and therefore representation (1.57) holds.

One may obtain more convenient criteria for the validity of (1.56). Let us prove that if (1.46) is fulfilled, then (1.45) holds and therefore representation (1.57) is valid.

Indeed,

$$\begin{aligned} \frac{1}{r^{m+p+1}} \int_{B(0,r) \setminus B(0,r/4)} |u(z)| dz &\leq \frac{1}{r^m} \int_{B(0,r) \setminus B(0,r/4)} \left| \frac{u(z)}{|z|^{p+1}} \right| dz \leq \\ &\leq c \sup_{r/4 \leq |z| \leq r} \left| \frac{u(z)}{|z|^{p+1}} \right|_{r \rightarrow \infty} \rightarrow 0. \end{aligned}$$

Now we will prove that if (1.47) is fulfilled, then (1.45) holds. Assume the opposite is true. Let (1.47) be fulfilled and (1.45) be not fulfilled. Then there exists a positive integer  $\varepsilon$  and positive integers  $r_k$  ( $k = 1, \dots$ ) such that  $r_{k+1} > 4r_k > 1$ ,  $\mathbb{R}^m \setminus B(0, r/4) \subset \Omega$  and

$$\frac{1}{r_k^{m+p+1}} \int_{B(0,r_k) \setminus B(0,r_k/4)} |u(z)| dz \geq \varepsilon.$$

Therefore

$$\int_{B(0,r_k) \setminus B(0,r_k/4)} \frac{|u(z)|}{|z|^{m+p+1}} dz \geq \frac{1}{r_k^{m+p+1}} \int_{B(0,r_k) \setminus B(0,r_k/4)} |u(z)| dz \geq \varepsilon.$$

Since  $(B(0, r_k) \setminus B(0, r_k/4)) \cap (B(0, r_{k+1}) \setminus B(0, r_{k+1}/4)) = \emptyset$  and  $\Omega \supset \bigcap_{k=1}^{\infty} (B(0, r_k) \setminus B(0, r_k/4))$ , we have

$$\int_{\Omega} \frac{|u(z)|}{|z|^{m+p+1}} dz \geq \sum_{k=1}^{\infty} \int_{B(0,r_k) \setminus B(0,r_k/4)} \frac{|u(z)|}{|z|^{m+p+1}} dz = +\infty,$$

which is the contradiction.

Thus if anyone of conditions (1.45), (1.46) or (1.47) is fulfilled, then  $u$  will take form (1.57).

Write the function  $u_s^{(0)}$  from (1.57) in a more convenient form, or, to be more exact, find an asymptotic representation of  $u^{(0)}(x)$  near  $|x| = +\infty$ .

Let a number  $r_0$  be chosen such that  $(\mathbb{R}^m \setminus B(0, r_0)) \subset \Omega$ ,  $|x| > 2r_0$ . By the Taylor formula  $\Phi_{ks}(y-x)$  is represented in the neighbourhood of  $-x$  as

$$\Phi_{ks}(y-x) = \sum_{|\alpha| \leq q} \frac{y^\alpha}{\alpha!} \mathcal{D}^\alpha \Phi_{ks}(-x) + \sum_{|\alpha|=q+1} \frac{y^\alpha}{\alpha!} (\mathcal{D}^\alpha \Phi_{ks})(\theta y-x), \quad (1.59)$$

where  $\alpha$  is the multiindex,  $q$  an arbitrary nonnegative integer number,  $\theta \in ]0, 1[$ .

We will show that for the function

$$\psi_{ks}(x, y) \equiv \sum_{|\alpha|=q+1} \frac{y^\alpha}{\alpha!} (\mathcal{D}^\alpha \Phi_{ks})(\theta y - x),$$

when  $y \in \partial\Omega$  and  $|x| > 2r_0$ , we have the estimate

$$|\mathcal{D}_x^\beta \psi_{ks}(x, y)| \leq \frac{c_{ks}^{(\beta)}(y)}{|x|^{m+q+|\beta|-1}}, \quad (1.60)$$

where  $c_{ks}^{(\beta)}$  is the bounded nonnegative function defined on  $\partial\Omega$ .

Indeed, due to (1.59) the function  $\mathcal{D}^\beta \Phi_{ks}$  is determined by

$$\begin{aligned} \mathcal{D}^\beta \Phi_{ks}(y - x) &= \sum_{|\alpha| \leq q} \frac{y^\alpha}{\alpha!} \mathcal{D}^\alpha (\mathcal{D}^\beta \Phi_{ks})(-x) + \psi_{ks}^{(1)}(x, y), \\ \psi_{ks}^{(1)}(x, y) &\equiv \sum_{|\alpha|=q+1} \frac{y^\alpha}{\alpha!} \mathcal{D}^\alpha (\mathcal{D}^\beta \Phi_{ks})(\theta_1 y - x). \end{aligned} \quad (1.61)$$

Applying the differentiation operation  $(-1)^{|\beta|} \mathcal{D}_x^\beta$  to both parts of (1.59) and subtracting the result from (1.61), we obtain  $(-1)^{|\beta|} \mathcal{D}_x^\beta \psi_{ks}(x, y) = \psi_{ks}^{(1)}(x, y)$ . But because of the homogeneity of  $\Phi$

$$|\psi_{ks}^{(1)}(x, y)| \leq \sum_{|\alpha|=q+1} \frac{|y|^{|\alpha|}}{\alpha!} \left| (\mathcal{D}^\alpha \mathcal{D}^\beta \Phi_{ks}) \left( \frac{\theta_1 y}{|x|} - \frac{x}{|x|} \right) \right| \frac{1}{|x|^{m-2+|\alpha|+|\beta|}}.$$

These estimates are to be used for the representation of  $u_s^{(0)}(x)$  given by (1.55). Here  $y \in \partial\Omega$  and, moreover,  $(\mathbb{R}^m \setminus B(0, r_0)) \subset \Omega$  and thus  $|y| < r_0$ ,  $|x| > 2r_0$ . That is why  $\forall y \in \partial\Omega$  and  $|x| > 2r_0$

$$\left| \frac{x}{|x|} \right| - \left| \frac{\theta_1 y}{|x|} \right| \leq \left| \frac{x}{|x|} - \frac{\theta_1 y}{|x|} \right| \leq \left| \frac{x}{|x|} \right| + \left| \frac{\theta_1 y}{|x|} \right|, \quad \frac{1}{2} \leq \left| \frac{\theta_1 y}{|x|} - \frac{x}{|x|} \right| \leq \frac{3}{2}.$$

Therefore

$$\begin{aligned} |\mathcal{D}_x^\beta \psi_{ks}(x, y)| &= |\psi_{ks}^{(1)}(x, y)| \leq \frac{c_{ks}^{(\beta)}(y)}{|x|^{m-1+q+|\beta|}}, \\ c_{ks}^{(\beta)} &\equiv \sum_{|\alpha|=q+1} \frac{|y|^{q+1}}{\alpha!} \left| (\mathcal{D}^\alpha \mathcal{D}^\beta \Phi_{ks}) \left( \frac{\theta_1 y}{|x|} - \frac{x}{|x|} \right) \right| \leq c < +\infty \end{aligned}$$

and we have obtained estimate (1.60).

Let us now transform the representation of  $u_s^{(0)}$  (see (1.55)). Replacing here the expression  $T_{ik}^*(\partial y, \nu(y)) \Phi_{ks}(y - x)$  by  $-T_{ik}^*(\partial x, \nu(y)) \Phi_{ks}(y - x)$  and introducing an expression for  $\Phi_{ks}(y - x)$  from (1.59), we obtain

$$u_s^{(0)}(x) = - \sum_{|\alpha| \leq q} \int_{\partial\Omega} \left( u_i(y) \frac{y^\alpha}{\alpha!} T_{ik}(\partial x, \nu(y)) (\mathcal{D}^\alpha \Phi_{ks})(-x) + \right.$$

$$\begin{aligned}
& + \frac{y^\alpha}{\alpha!} \mathcal{D}^\alpha \Phi_{ks}(-x) T_{ki}(\partial y, \nu(y)) u_i(y) \Big) d_y S - \\
& - \int_{\partial\Omega} \left( u_i(y) T_{ik}^*(\partial x, \nu(y)) \psi_{ks}(x, y) + \right. \\
& \left. + \psi_{ks}(x, y) T_{ki}(\partial x, \nu(y)) u_i(y) \right) d_y S.
\end{aligned}$$

Since

$$T_{ik}^*(\partial x, \nu(y)) = C_{klij} \nu_j(y) \frac{\partial}{\partial x_l}$$

we have

$$\begin{aligned}
u_s^{(0)}(x) &= - \sum_{|\alpha| \leq q-1} \frac{(\mathcal{D}_{x_i} \mathcal{D}^\alpha) \Phi_{ks}(-x)}{\alpha!} c_{klij} \int_{\partial\Omega} u_i(y) y^\alpha \nu_j(y) d_y S + \\
& + \sum_{|\alpha| \leq q} \frac{\mathcal{D}^\alpha \Phi_{ks}(-x)}{\alpha!} \int_{\partial\Omega} y^\alpha T_{ki}(\partial y, \nu(y)) u_i(y) d_y S + \psi_s(x), \\
\psi_s(x) &\equiv - \sum_{|\alpha|=q, \partial\Omega} \int_{\partial\Omega} u_i(y) \frac{y^\alpha}{\alpha!} T_{ik}^*(\partial x, \nu(y)) (\mathcal{D}^\alpha \Phi_{ks})(-x) d_y S - \\
& - \int_{\partial\Omega} \left( u_i(y) T_{ik}^*(\partial x, \nu(y)) \psi_{ks}(x, y) + \right. \\
& \left. + \psi_{ks}(x, y) T_{ki}(\partial y, \nu(y)) u_i(y) \right) d_y S.
\end{aligned}$$

Now  $u_s^{(0)}(x)$  can be represented as

$$u_s^{(0)}(x) = \sum_{|\alpha| \leq q} d_k^{(\alpha)} \mathcal{D}^\alpha \Phi_{ks}(x) + \psi_s(x),$$

where  $d_k^{(\alpha)}$  are some constants depending on  $u$  (note that we may write out their explicit expressions), and for  $\psi_s(x)$  we have the estimate (see (1.60))

$$|\mathcal{D}^\gamma \psi_s(x)| \leq \frac{c_s^{(\gamma)}}{|x|^{m+|\gamma|+q-1}},$$

$c_s^{(\gamma)} = \text{const}$ ,  $\gamma = (\gamma_1, \dots, \gamma_m)$  is an arbitrary multiindex.

To summarize the foregoing reasoning, we conclude that if  $\Omega$  is a domain from  $\mathbb{R}^m$  containing the neighbourhood of the point at infinity,  $\partial\Omega$  is a smooth surface,  $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ ,  $\forall x \in \Omega : A(\partial x)u(x) = 0$  and anyone of conditions (1.45), (1.46), (1.47) is fulfilled, then representation (1.48) holds near the point at infinity. The theorem has been proved for  $m > 2$ . If  $m = 2$ , then we have to modify the proof of Theorem 1.7 as we did in proving Theorem 1.6 for  $m = 2$ . In that case, however, we should keep in mind that unlike the case  $m > 2$  the second term on the right-hand side of

(1.48) does not vanish at infinity for  $m = 2$ . That is why it is sometimes convenient to write representation (1.48) in the form

$$u_s(x) = \sum_{|\alpha| \leq p} c_s^{(\alpha)} x^\alpha + \sum_{k=1}^n d_k \Phi_{k_s}(x) + \sum_{1 \leq |\beta| \leq q} d_k^{(\beta)} \mathcal{D}^\beta \Phi_{k_s}(x) + \psi_s(x). \quad (1.48')$$

The proof of Theorem 1.7 is completed. ■

Note that to provide the uniqueness of solutions of the basic static problems of elasticity in the case of infinite domains (domains with a spherical hole), the solutions have to be subjected to additional restrictions. Such restrictions given by the Green formula can be found, for example, in Kupradze *et al.* [1]; Knops, Payne [1]; Fichera [1]; where the restrictions are imposed both on the solution and on its derivatives. Theorem 1.7 enables us, however, to considerably weaken these restrictions. Thus, for example, the requirement for the damping of solution derivatives becomes unnecessary. Indeed, consider, for example, the first basic problem:

Find a vector  $u$  in the domain  $\Omega$  containing the neighbourhood of the point at infinity by the conditions

$$\forall x \in \Omega : A(\partial x)u(x) = 0, \quad (1.62)$$

$$\forall y \in \partial\Omega : u|_{\partial\Omega}(y) = \varphi(y), \quad \lim_{|x| \rightarrow +\infty} u(x) = 0, \quad (1.63)$$

where  $A$  is the differential operator of elasticity (see, Kupradze *et al.* [1]) and  $\varphi$  is the known vector on  $\partial\Omega$ .

Theorem 1.7 implies

**Theorem 1.8.** *Problem (1.62), (1.63) has one solution at most.*

The uniqueness theorems for the rest of the basic problems of elasticity are formulated similarly.

\* \* \*

The material of the initial Subsections 1.1 to 1.6 can be found in many works devoted to the theory of harmonic potentials and equations of mathematical physics. We would like to mention just a few of them: Goursat [1], Kellog [1], Gunter [1], Koshlyakov, Gliner, Smirnov [1], Courant [1], Sobolev [1], Petrovski [1], Miranda [1], Mikhlin [1], Wermer [1], Liapunov [1].

The results of Subsections 1.7 and 1.8 repeat those from Buchukuri, Gegelia [1,2].

CHAPTER II  
BOUNDARY VALUE PROBLEMS OF STATICS  
IN THE CLASSICAL THEORY OF ELASTICITY

**2.1. Formulation of the Problems.** It will be assumed below that a homogeneous isotropic medium occupies either the domain  $B^+$ , i.e. the three-dimensional ball with centre at the origin and radius  $\rho$ , or the domain  $B^-$ , i.e. the space  $\mathbb{R}^3$  with cavity  $\bar{B}^+$ .

For a static state, when there is no mass force, the system of equations of classical elasticity is written in terms of displacement components as (see, for example, Love [1], Sneddon, Berry [1], Kupradze *et al.* [1])

$$\mu\Delta u(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} u(x) = 0, \quad (2.1)$$

where  $x \equiv (x_1, x_2, x_3) \in \mathbb{R}^3$ ,  $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$  is the Laplace operator,  $u \equiv (u_1, u_2, u_3)$  is the displacement vector,  $\lambda$  and  $\mu$  are the Lamé constants:

$$3\lambda + 2\mu > 0, \quad \mu > 0. \quad (2.2)$$

Let  $\tau^{(n)}(x) \equiv (\tau_1^{(n)}(x), \tau_2^{(n)}(x), \tau_3^{(n)}(x))$  be a stress vector at the point  $x$  applied to an area with the normal  $n = (n_1, n_2, n_3)$ . The relation between displacement and stress has the form

$$\tau_i^{(n)} = \lambda n_i \operatorname{div} u + \mu n_j \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i = 1, 2, 3). \quad (2.3)$$

In this chapter, for the domains  $B^+$  and  $B^-$  we will derive solutions in quadratures of the following boundary value problems:

Find in  $B^+$  a continuous solution  $u$  of system (2.1) satisfying one of the boundary conditions below:

$$\forall y \in S : u^+(y) = f(y) \quad (2.4)$$

– Problem (I)<sup>+</sup>;

$$\forall y \in S : (\tau^{(n)})^+(y) = f(y) \quad (2.5)$$

– Problem (II)<sup>+</sup>;

$$\forall y \in S : (n \cdot u)^+(y) = g(y), \quad (\tau^{(n)} - n(n \cdot \tau^{(n)}))^+(y) = l(y) \quad (2.6)$$

– Problem (III)<sup>+</sup>;

$$\forall y \in S : (n \cdot \tau^{(n)})^+(y) = g(y), \quad (u - n(n \cdot u))^+(y) = l(y) \quad (2.7)$$

– Problem (IV)<sup>+</sup>;

$$\forall y \in S : (\tau^{(n)} + \sigma_0 u)^+(y) = f(y) \quad (2.8)$$

– Problem (V)<sup>+</sup>.

Here  $S$  is the boundary of the domain  $B^+$ , i.e. the sphere with centre at the origin and radius  $\rho$ ,  $\sigma_0$  is a positive real number,  $f = (f_1, f_2, f_3)$ ,  $l = (l_1, l_2, l_3)$  and  $g$  the values given on  $S$ . It is assumed that the condition

$$\forall y \in S : (n \cdot l)(y) = 0 \quad (2.9)$$

is fulfilled.

Therefore on the sphere we are given a displacement for Problem (I)<sup>+</sup>, boundary stress for Problem (II)<sup>+</sup>, a normal displacement component and a tangential stress component for Problem (III)<sup>+</sup>, a normal stress component and a tangential displacement component for Problem (IV)<sup>+</sup>, a linear combination of displacement and stress for Problem (V)<sup>+</sup>.

The problems for the domain  $B^-$  are formulated in the same way. They will respectively be denoted by (I)<sup>-</sup>, ..., (V)<sup>-</sup>.

We are interested in deriving classical and regular solutions of the said problems. The function  $u$  is called a regular solution in  $B^+$  if  $u \in C^1(\bar{B}^+) \cap C^2(B^+)$ . The function  $u$  is called a classical solution if  $u \in C(\bar{B}^+) \cap C^2(B^+)$  and, additionally, if  $(\tau^{(n)})^+ \in C(S)$  in the case of Problems (II)<sup>+</sup>, (III)<sup>+</sup>, (IV)<sup>+</sup>, (V)<sup>+</sup>. A regular and classical solution in the domain  $B^-$  is defined similarly.

The boundary value problems of the theory of elasticity for arbitrary domains have been studied with sufficient completeness by the methods of a potential and integral equations in the monograph Kupradze *et al.* [1] (see also Fichera [1]; Knops, Payne [1]; Burchuladze, Gegelia [1]). The existence and uniqueness of regular solutions have been investigated, and the necessary and sufficient conditions have been found for the boundary value problems to be solvable. For our purposes we will need mainly the uniqueness theorems and solvability conditions.

**2.2. Uniqueness Theorems and Solvability Conditions.** In this subsection we are going to present some results from Kupradze *et al.*[1] which will be needed in the sequel.

**Theorem 2.1.** *Problems (I)<sup>+</sup>, (IV)<sup>+</sup>, (V)<sup>+</sup> admit one regular solution at most. The difference between any two regular solutions of Problem (II)<sup>+</sup> can be equal only to a vector of the form*

$$u(x) = [a \times x] + b,$$

where  $a = (a_1, a_2, a_3)$  and  $b = (b_1, b_2, b_3)$  are arbitrary constant vectors. The difference between any two regular solutions of Problem (III)<sup>+</sup> can be equal only to a vector of the form

$$u(x) = [a \times x].$$

Here and in the sequel the symbol  $[u \times v]$  denotes the vector product of the vectors  $u$  and  $v$ .

**Theorem 2.2.** *Problems (I)<sup>-</sup>, (II)<sup>-</sup>, (III)<sup>-</sup>, (IV)<sup>-</sup> and (V)<sup>-</sup> admit one regular solution at most satisfying the condition (see 1.8 of this book)*

$$\lim_{|x| \rightarrow \infty} u(x) = 0.$$

We will give yet another theorem from Kupradze *et al.* [1].

**Theorem 2.3.** *Problems (I)<sup>±</sup>, (II)<sup>-</sup>, (III)<sup>-</sup>, (IV)<sup>±</sup>, (V)<sup>±</sup> have regular solutions for arbitrary sufficiently smooth boundary functions.*

*For Problem (II)<sup>+</sup> to be solvable it is necessary and sufficient that the main vector and the principal moment of external force be equal to zero:*

$$\int_S f(y) d_y S = 0, \quad \int_S [y \times f(y)] d_y S = 0, \quad (2.10)$$

*and for the solvability of Problem (III)<sup>+</sup> it is necessary and sufficient that the principal moment be equal to zero, i.e.*

$$\int_S [y \times l(y)] d_y S = 0. \quad (2.11)$$

**2.3. Special Representations of Solutions of System (2.1).** In constructing solutions of Problems (I)<sup>±</sup>, ..., (V)<sup>±</sup> essential use is made of a special representation of displacements by means of harmonic functions. Namely, the theorem below, which is easy to verify, is valid.

**Theorem 2.4.** *If*

$$u(x) = v(x) + \frac{\rho^2 - r^2}{2} \text{grad } \psi(x), \quad (2.12)$$

*where the vector function  $v \equiv (v_1, v_2, v_3)$  and the scalar function  $\psi$  are defined in the domain  $B^+$  or  $B^-$  and satisfy the conditions*

$$(\mathcal{D}_r + \alpha)\psi = \beta \text{div } v, \quad \Delta v = 0, \quad \Delta \psi = 0, \quad (2.13)$$

$$\alpha = \frac{\mu}{\lambda + 3\mu}, \quad \beta = \frac{\lambda + \mu}{\lambda + 3\mu}, \quad (2.14)$$

*$r \equiv |x|$ , the operator  $\mathcal{D}_r$  is defined in Subsection 1.2, then  $u$  is a solution of equation (2.1) in  $B^+$  or  $B^-$ .*

The converse statement is likewise valid.

**Theorem 2.5.** *If  $u$  is a continuous solution of equation (2.1) in  $B^+(B^-)$ , then there exists a vector function  $v$  and a scalar function  $\psi$  defined and continuous in  $B^+(B^-)$  such that equalities (2.12), (2.13) are fulfilled.*

Indeed, let  $u$  be a continuous solution of equation (2.1) in  $B^+$ , then  $u \in C^\infty(B^+)$ . Applying the operation  $\operatorname{div}$  to equation (2.1), we obtain  $\Delta \operatorname{div} u = 0$ . Let

$$\psi(x) = \frac{\lambda + \mu}{2\mu} \int_0^1 \operatorname{div}_{\tau x} u(\tau x) \frac{d\tau}{\sqrt{\tau}}.$$

It is easy to obtain

$$\Delta \psi = 0, \quad (2\mathcal{D}_r + 1)\psi = \frac{\lambda + \mu}{\mu} \operatorname{div} u. \quad (2.15)$$

Let us set

$$v \equiv u - \frac{\rho^2 - r^2}{2} \operatorname{grad} \psi$$

and verify that equalities (2.12), (2.13) are valid. For this it suffices to show that  $\Delta v = 0$  and  $(\mathcal{D}_r + \alpha)\psi = \beta \operatorname{div} v$ . Taking into account (2.1) and (2.15), we have

$$\begin{aligned} \Delta v &= \Delta u + 3 \operatorname{grad} \psi + 2\mathcal{D}_r \operatorname{grad} \psi = \\ &= -\frac{\lambda + \mu}{\mu} \operatorname{grad} \operatorname{div} u + 2 \operatorname{grad} \mathcal{D}_r \psi + \operatorname{grad} \psi = \\ &= \operatorname{grad} \left( (2\mathcal{D}_r + 1)\psi - \frac{\lambda + \mu}{\mu} \operatorname{div} u \right) = 0. \\ \operatorname{div} v &= \operatorname{div} u + \mathcal{D}_r \psi = \frac{2\mu}{\lambda + \mu} \mathcal{D}_r \psi + \frac{\mu}{\lambda + \mu} \psi + \mathcal{D}_r \psi = \\ &= \frac{\lambda + 3\mu}{\lambda + \mu} \mathcal{D}_r \psi + \frac{\mu}{\lambda + \mu} \psi \end{aligned}$$

i.e.  $(\mathcal{D}_r + \alpha)\psi = \beta \operatorname{div} v$ .

This completes the proof of Theorem 2.5.

We can show the uniqueness of representation (2.12), i.e. if  $u$  is represented by formula (2.12) and by a similar formula

$$u(x) = v_1(x) + \frac{\rho^2 - r^2}{2} \operatorname{grad} \psi_1(x),$$

then  $v_1 = v$  and  $\psi_1 = \psi$ .

By a similar technique we can consider the case when  $u$  is a solution of equation (2.1) in the domain  $B^-$ .

Though we will use Theorem 2.5 nowhere, it guarantees us a possibility of constructing solutions in form (2.12) for all the boundary value problems.

For doing this job Theorem 2.4 is of essential importance. Formula (2.12) is not new. It was given evidently for the first time in 1904 by Marcolongo in [1]. This representation is also encountered in other papers, for example in Trefftz [1], without references to the original sources. In some subsequent papers (for example, Lurie [1], Nowacki [1]) it is called the Trefftz representation. It was the famous Italian mathematician Gaetano Fichera who sent

us Marcolongo's paper which has led us to solving many boundary value problems and to writing this book.

Formula (2.12) will be used to solve Problems (I)<sup>±</sup>, (III)<sup>±</sup> and (V)<sup>±</sup>; Problems (II)<sup>±</sup> and (IV)<sup>±</sup> are solved by other similar representations to be given below.

**Theorem 2.6.** *If*

$$u(x) = v(x) + x(2\mathcal{D}_r + 1)\psi(x) - r^2 \operatorname{grad} \psi(x) + \frac{\rho^2 - r^2}{2} \operatorname{grad} \psi(x), \quad (2.16)$$

where

$$\begin{aligned} \Delta v = 0, \quad \Delta \psi = 0, \\ 2\mathcal{D}_r^2 \psi + 2\frac{2\lambda + \mu}{\lambda + \mu} \mathcal{D}_r \psi + \frac{3\lambda + 2\mu}{\lambda + \mu} \psi = -\operatorname{div} v, \end{aligned} \quad (2.17)$$

then  $u$  is a solution of equation (2.1) in  $\mathbb{R}^3 \setminus S(0, \rho)$ .

**Theorem 2.7.** *If*

$$\begin{aligned} u(x) = v(x) + x(2\mathcal{D}_r + 1)\varphi(x) + (\rho^2 - r^2) \operatorname{grad} \varphi(x) + \\ + \frac{\rho^2 - r^2}{2} \operatorname{grad} \psi(x), \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} \Delta v = 0, \quad \Delta \varphi = 0, \quad \Delta \psi = 0, \\ (\mathcal{D}_r + \alpha)\psi = \beta(\operatorname{div} v + 2\mathcal{D}_r^2 \varphi + 5\mathcal{D}_r \varphi + 3\varphi), \end{aligned} \quad (2.19)$$

then  $u$  is a solution of equation (2.1) in  $\mathbb{R}^3 \setminus S(0, \rho)$ .

It is easy to verify that if  $\varphi$  is a harmonic function, then so are  $x(2\mathcal{D}_r + 1)\varphi(x) - r^2 \operatorname{grad} \varphi(x)$  and  $x(2\mathcal{D}_r + 1)\varphi(x) + (\rho^2 - r^2) \operatorname{grad} \varphi(x)$ , and therefore representations (2.12), (2.16) and (2.18) are equivalent.

**2.4. Solution of Problem (I)<sup>+</sup>.** First we shall derive a formal solution of Problem (I)<sup>+</sup>, assuming that all the operations applied are valid. After constructing the formal solution we shall prove that the obtained formula provides a classical solution of the problem for a continuous boundary function and show that uniqueness theorem holds for classical solutions.

A solution  $u$  of Problem (I)<sup>+</sup> is to be sought for in form (2.12). To this effect in the domain  $B^+$  we find the vector function  $v$  and the scalar function  $\psi$  satisfying conditions (2.13); moreover, on the boundary  $S$  ( $S \equiv \partial B^+$ )  $u$  defined by (2.12) takes the value of  $f$ , i.e.,

$$\begin{aligned} \forall y \in S : u^+(y) \equiv \lim_{B^+ \ni x \rightarrow y} u(x) = \lim_{B^+ \ni x \rightarrow y} v(x) + \\ + \lim_{B^+ \ni x \rightarrow y} \frac{\rho^2 - r^2}{2} \operatorname{grad} \psi(x) = f(y). \end{aligned}$$

When  $x \rightarrow y$ , we have  $r \equiv |x| \rightarrow \rho$  and for  $v$  we obtain the Dirichlet problem

$$\forall x \in B^+ : \Delta v(x) = 0, \quad \forall y \in S : v^+(y) = f(y), \quad (2.20)$$

which is solved by the Poisson formula (1.1)

$$v(x) = \Pi(f)(x) \equiv \frac{1}{4\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|y-x|^3} f(y) d_y S.$$

The substitution of the obtained value of  $v$  in (2.13) gives us the differential equation with respect to  $\psi$

$$r \frac{\partial \psi}{\partial r} + \alpha \psi = \beta \operatorname{div} \Pi(f) \equiv F. \quad (2.21)$$

This is a linear equation with respect to the variable  $r$  whose general solution is given by the formula

$$\psi(x) = e^{-\int_{r_0}^r \frac{\alpha}{\eta} d\eta} \left( c(\vartheta_1, \vartheta_2) + \int_{r_0}^r \frac{1}{t} F\left(\frac{x}{t}\right) e^{\int_{r_0}^t \frac{\alpha}{\eta} d\eta} dt \right), \quad (2.22)$$

where  $c$  is an arbitrary function of its arguments,  $r, \vartheta_1, \vartheta_2$  are the spherical coordinates of the point  $x$ , and  $r_0$  is a constant,  $0 \leq r_0 \leq \rho$ ,  $c$  and  $r_0$  should be defined so that  $\psi$  be a harmonic function in  $B^+$ .

The function  $\psi$  can be represented as the sum  $\psi = \psi_0 + \psi_1$ , where

$$\psi_0(x) = r^{-\alpha} \left( c(\vartheta_1, \vartheta_2) r_0^\alpha - \int_0^{r_0} F\left(\frac{x}{t}\right) t^{\alpha-1} dt \right),$$

$$\psi_1(x) = r^{-\alpha} \int_0^r F\left(\frac{x}{t}\right) t^{\alpha-1} dt.$$

By virtue of (2.2)  $\alpha > 0$  and for the continuity of  $\psi_0$  at the point 0 it is necessary that  $r_0 = 0$ . Then  $\psi_1 = \psi_0$ .

By substituting  $\tau = t/r$ , the function  $\psi = \psi_1$  is reduced to the form

$$\psi(x) = \int_0^1 F(\tau x) \tau^{\alpha-1} d\tau.$$

Hence due to (2.21) and (1.1) we have

$$\psi(x) = \beta \int_0^1 \left( \frac{1}{4\pi\rho} \operatorname{div} \int_S \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} f(y) d_y S \right) \frac{d\tau}{\tau^{2-\alpha}} =$$

$$= \frac{\beta}{4\pi\rho} \operatorname{div} \int_S \left( \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \frac{d\tau}{\tau^{2-\alpha}} \right) f(y) d_y S. \quad (2.23)$$

Substituting the found values of  $\psi$  and  $v = \Pi(f)$  in (2.12), we obtain

$$u(x) = \int_S \mathcal{K}(x, y) f(y) d_y S, \quad (2.24)$$

where  $\mathcal{K} \equiv \|\mathcal{K}_{ij}\|_{3 \times 3}$ ,

$$\begin{aligned} \mathcal{K}_{ij}(x, y) &= \frac{1}{4\pi\rho} \left( \frac{\rho^2 - |x|^2}{|y - x|^3} \delta_{ij} + \frac{\beta(\rho^2 - |x|^2)}{2} \times \right. \\ &\quad \left. \times \frac{\partial^2}{\partial x_i \partial x_j} \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \frac{d\tau}{\tau^{2-\alpha}} \right). \end{aligned} \quad (2.25)$$

The solution of Problem (I)<sup>+</sup> is therefore constructed.

**2.5. Analysis of the Constructed Solution (2.24).** The goal we have set before ourselves in this subsection is to prove that if in (2.24)  $f$  is a continuous function on  $S$ , then (2.24) provides a classical solution of Problem (I)<sup>+</sup>, i.e.  $u$  given by (2.24) satisfies equation (2.1), is continuous in the closed domain  $\bar{B}^+$  and  $\forall y \in S : u^+(y) = f(y)$ . Besides, we shall prove that Problem (I)<sup>+</sup> has the unique classical solution. Note that such conclusions do not follow from the general theory (Kupradze *et al.* [1]).

Let us make a remark on the permutation of the integration order used in formula (2.23). In constructing expressions (2.23) we employed the equality

$$\begin{aligned} &\int_S f(y) d_y S \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \frac{d\tau}{\tau^{2-\alpha}} = \\ &= \int_0^1 \frac{d\tau}{\tau^{2-\alpha}} \int_S \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) f(y) d_y S. \end{aligned}$$

To convince ourselves that this formula is valid it is sufficient to show (see, for example, Saks [1] or Rudin [1]) that

$$\int_S d_y S \int_0^1 \left| \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right| |f(y)| \frac{d\tau}{\tau^{2-\alpha}} < +\infty. \quad (2.26)$$

It is easy to verify that the following equalities hold:

$$\forall x \in B^+, y \in S : \lim_{\tau \rightarrow 0} \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) = 0,$$

$$\lim_{\tau \rightarrow 0} \frac{\partial}{\partial \tau} \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} = \frac{3x_i y_i}{\rho^3}.$$

Therefore  $\forall x \in B^+$ ,  $\forall y \in S$ ,  $\forall \tau \in [0, 1]$ :

$$\left| \frac{1}{\tau} \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \right| \leq c = \text{const.}$$

Now, keeping in mind that  $f \in C(S)$  and  $\alpha > 0$ , we easily find that (2.26) is valid.

Here to derive the estimate we have used the simple proposition which will be useful for our further reasoning as well. If  $f$  is a function defined on  $]0; 1]$ , has on this interval all continuous derivatives up to order  $k$  inclusive,

$$\lim_{t \rightarrow 0} f^{(i)}(t) = 0, \quad i = 0, 1, \dots, k-1, \quad \lim_{t \rightarrow 0} f^{(k)} = c,$$

where  $c$  is a (finite) real number, then

$$f(t) = O(t^k) \quad (2.27)$$

in the neighbourhood of  $t = 0$ .

**Problem 1.** Prove the above-formulated proposition.

We will present some auxiliary estimates and propositions. It is easy to obtain

$$\forall x \in B^+, \forall y \in S, \forall \tau \in [0, 1/2] : |y - \tau x| \geq \rho - |\tau x| \geq \rho/2; \quad (2.28)$$

$$\forall x \in B^+, \forall y \in S, \forall \tau \in [1/2, 1] : |y - \tau x| \geq \frac{|y - x|}{2}. \quad (2.28')$$

Consider the integral

$$\mathcal{I}(x, y) \equiv \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \frac{d\tau}{\tau^{2-\alpha}},$$

which by virtue of (2.28) can be rewritten as

$$\mathcal{I}(x, y) \equiv \int_{1/2}^1 \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} \frac{d\tau}{\tau^{2-\alpha}} + F(x, y),$$

where  $F$  and its derivatives with respect to the Cartesian coordinates of the point  $x$  up to the second order inclusive are bounded functions.

Using the identities

$$\frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} = \frac{1}{|y - \tau x|} + 2r \frac{\partial}{\partial \tau} \frac{1}{|y - \tau x|}, \quad (2.29)$$

$$\frac{1}{|y - \tau x|} = -\frac{1}{\rho} \tau \frac{\partial}{\partial \tau} \ln \frac{(|y - \tau x| + \rho)^2 - |\tau x|^2}{2\tau} \quad (2.29')$$

and inequality (2.28'), we obtain the representation

$$\mathcal{I}(x, y) = \frac{2}{|x - y|} + \mathcal{I}_0(x, y), \quad (2.30)$$

where

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} \mathcal{I}_0(x, y) \right| \leq \frac{c}{|x - y|^2}.$$

Here and in our further estimates  $c$  will denote a positive finite constant which does not always have the same value.

Taking into account representation (2.30) and the equality

$$\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|y - x|} = 3 \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^5} - \frac{\delta_{ij}}{|x - y|^3},$$

we obtain for  $\mathcal{K}$  (see (2.25)) the estimate

$$\forall x \in B^+, \forall y \in S : |\mathcal{K}(x, y)| \leq c \frac{\rho^2 - |x|^2}{|y - x|^3} \quad (2.31)$$

and the representation

$$\mathcal{K} = \mathcal{K}^{(1)} + \mathcal{K}^{(2)} + \mathcal{K}^{(3)}, \quad (2.32)$$

where

$$\begin{aligned} \mathcal{K}^{(1)}(x, y) &= \left\| \frac{1}{4\pi\rho} \frac{\rho^2 - |x|^2}{|y - x|^3} \delta_{ij} \right\|_{3 \times 3}, \\ \mathcal{K}^{(2)}(x, y) &= \left\| \frac{\beta(\rho^2 - |x|^2)}{4\pi\rho} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|x - y|} \right\|_{3 \times 3}, \\ \mathcal{K}^{(3)}(x, y) &= \|(\rho^2 - |x|^2) \chi_{ij}(x, y)\|_{3 \times 3}, \\ |\chi_{ij}(x, y)| &\leq \frac{c}{|x - y|^2}. \end{aligned} \quad (2.33)$$

**Theorem 2.8.** *If  $f \in C^\infty(S)$ , then  $u$  defined by (2.24) is a regular solution of Problem (I)<sup>+</sup>.*

*Proof.* Under the conditions of the theorem the inclusion  $\Pi(f) \in C^{3,\gamma}(\bar{B}^+)$ ,  $0 < \gamma \leq 1$ , holds (Theorem 1.1). Therefore for the integrals

$$\begin{aligned} E_i(x) &\equiv \frac{1}{4\pi\rho} \int_0^1 \left( \frac{1}{\tau^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_S \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} f_i(y) d_y S \right) \tau^\alpha d\tau = \\ &= \int_0^1 \tau^\alpha \frac{\partial^2}{\partial(\tau x_i) \partial(\tau x_j)} \Pi(f_j)(\tau x) d\tau, \end{aligned}$$

$$\begin{aligned}
L_{ik}(x) &\equiv \frac{1}{4\pi\rho} \int_0^1 \left( \frac{1}{\tau^3} \frac{\partial^3}{\partial x_k \partial x_i \partial x_j} \int_S \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} f_j(y) d_y S \right) \tau^{1+\alpha} d\tau = \\
&= \int_0^1 \tau^{1+\alpha} \frac{\partial^3}{\partial(\tau x_k) \partial(\tau x_i) \partial(\tau x_j)} \Pi(f_j)(\tau x) d\tau
\end{aligned}$$

we have the estimates  $\forall x', x'' \in B^+$ :

$$\begin{aligned}
|E_i(x') - E_i(x'')| &\leq c \int_0^1 |\tau x' - \tau x''| \tau^\alpha d\tau = \\
&= c|x' - x''| \int_0^1 \tau^{1+\alpha} d\tau \leq c|x' - x''|, \\
|L_{ik}(x') - L_{ik}(x'')| &\leq c \int_0^1 \tau^{1+\alpha} |\tau x' - \tau x''|^\gamma d\tau \leq c|x' - x''|^\gamma.
\end{aligned}$$

Therefore  $u \in C^1(\bar{B}^+)$  and  $u^+ = f$ . Obviously,  $u \in C^2(B^+)$  and  $u$  is a regular solution of equation (2.1). ■

**Theorem 2.9.**  $\forall x \in B^+$ :

$$\int_S \mathcal{K}_{ij}(x, y) d_y S = \delta_{ij}, \quad (2.34)$$

where  $\mathcal{K}_{ij}$  are defined from (2.25).

Indeed, the vectors  $\overset{1}{u}, \overset{2}{u}, \overset{3}{u}$ , where  $\forall x \in B^+ : \overset{m}{u}_i(x) = \delta_{im}; i, m = 1, 2, 3$ , are regular solutions of the following Problem (I)<sup>+</sup>:

$$\begin{aligned}
\mu \Delta \overset{m}{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \overset{m}{u} &= 0, \\
\overset{m+}{u}_i &= \delta_{im} \quad i = 1, 2, 3.
\end{aligned} \quad (2.35)$$

Since for regular solutions of Problem (I)<sup>+</sup> Uniqueness Theorem 2.1 holds, these solutions will coincide. Theorem 2.9 is proved.

**Theorem 2.10.** *If  $f \in C(S)$ , then  $u$  defined by (2.24) is a classical solution of Problem (I)<sup>+</sup>:  $u \in C(\bar{B}^+) \cap C^2(B^+)$ ,  $u^+ = f$  and  $u$  satisfies equation (2.1) at each point of the domain  $B^+$ .*

*Proof.* If  $f \in C(S)$ , then it is clear that  $u$  defined by (2.24) is continuous and has derivatives of all orders in  $B^+$ . It is likewise clear that  $u$  satisfies

equation (2.1). Let us verify that the boundary condition  $u^+ = f$  is fulfilled. By virtue of (2.24), (2.34),  $\forall x \in B^+$  and  $\forall z \in S$ :

$$u(x) - f(z) = \int_S \mathcal{K}(x, y)(f(y) - f(z))d_y S$$

and on account of (2.31)

$$|u(x) - f(z)| \leq c \int_S \frac{\rho^2 - |x|^2}{|y - x|^3} |f(y) - f(z)| d_y S \rightarrow c|f(z) - f(z)|, \text{ when } x \rightarrow z.$$

Therefore

$$\lim_{B^+ \ni x \rightarrow z \in S} |u(x) - f(z)| = 0.$$

Thus  $u^+ = f$ . Hence, in turn, it follows that  $u \in C(\bar{B}^+)$ . ■

Let us now prove that the classical solution of Problem (I)<sup>+</sup> is unique.

**Theorem 2.11.** *If  $u \in C(\bar{B}^+) \cap C^2(B^+)$ ,  $u$  is a solution of equation (2.1),  $u^+ = 0$ , then  $\forall x \in B^+ : u(x) = 0$ .*

*Proof.* The function  $u$  is uniformly continuous because it is continuous in  $\bar{B}^+$ . Therefore, by virtue of the condition  $u^+ = 0$ , for any  $\varepsilon > 0$  there exists a number  $\delta > 0$  such that  $|u(y)| < \varepsilon$  when  $y \in S(0, \rho_0)$ , where  $\rho_0 = \rho - \delta_0$  and  $0 < \delta_0 \leq \delta$ . Let  $z$  be any fixed point of the domain  $B^+$ . It can be assumed that  $\delta < \rho - |z|$ . Then  $z \in B^+(0, \rho_0)$ .

From the conditions of the theorem it follows that  $u \in C^\infty(B^+(0, \rho))$ . Hence, in particular, we conclude that the inclusions  $u \in C^{3, \gamma}(S(0, \rho_0))$ ,  $0 < \gamma < 1$ ,  $u \in C^1(\bar{B}^+(0, \rho_0)) \cap C^2(B^+(0, \rho_0))$  are valid. By virtue of Theorem 2.8, keeping in mind that the uniqueness theorem holds for regular solutions of Problem (I)<sup>+</sup> (see Theorem 2.1), from (2.24) we obtain the representation

$$u(z) = \int_{S(0, \rho_0)} \mathcal{K}(x, y)u(y)d_y S.$$

Hence due to (2.31) we have

$$\begin{aligned} |u(z)| &\leq \frac{c}{4\pi\rho_0} \int_{S(0, \rho_0)} \frac{\rho_0^2 - |z|^2}{|y - z|^3} |u(y)| d_y S \leq \\ &\leq \frac{c\varepsilon}{4\pi\rho_0} \int_{S(0, \rho_0)} \frac{\rho_0^2 - |z|^2}{|y - z|^3} d_y S = c\varepsilon. \end{aligned}$$

Therefore  $u(z) = 0$ . ■

Combining Theorems 2.10 and 2.11, we obtain

**Theorem 2.12.** *If  $f \in C(S)$ , then  $u$  defined by (2.24) is the unique classical solution of Problem (I)<sup>+</sup>.*

Using the results from Kupradze *et al.* [1], we can establish that solutions obtained by (2.24) are regular under less rigorous restrictions on the boundary function than in Theorem 2.8. The following theorem is valid.

**Theorem 2.13.** *If  $f \in C^{1,\gamma}(S)$ ,  $0 < \gamma \leq 1$ , then  $u$  defined by (2.24) is a regular solution of Problem (I)<sup>+</sup>.*

**2.6. Solution of Problem (I)<sup>-</sup>.** The solution of Problem (I)<sup>-</sup> is similar to that of Problem (I)<sup>+</sup>, and all the properties of the latter apply to Problem (I)<sup>-</sup>. We shall dwell only on the facts characteristic of Problem (I)<sup>-</sup>.

The solution of Problem (I)<sup>-</sup> is written in the form

$$\forall x \in B^- \equiv \mathbb{R}^3 \setminus \bar{B}^+ : u(x) = \int_S \mathcal{K}'(x, y) f(y) d_y S, \quad (2.36)$$

where  $\mathcal{K}'(x, y) \equiv \|\mathcal{K}'_{ij}(x, y)\|_{3 \times 3}$ ,

$$\begin{aligned} \mathcal{K}'_{ij}(x, y) = & \frac{1}{4\pi\rho} \left( \frac{|x|^2 - \rho^2}{|x - y|^3} \delta_{ij} + \right. \\ & \left. + \frac{\beta(|x|^2 - \rho^2)}{2} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^1 \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^3} \tau^{1-\alpha} d\tau \right). \end{aligned} \quad (2.37)$$

Using the estimate  $\forall x \in B^-, \forall y \in S$ :

$$|\mathcal{K}'(x, y)| \leq c \frac{|x|^2 - \rho^2}{|x - y|^3} \quad (2.38)$$

we prove the following auxiliary theorem:

**Theorem 2.14.** *If  $f \in C^\infty(S)$ , then  $u$  defined by (2.36) is a regular solution of Problem (I)<sup>-</sup> and*

$$u_i(x) = o(1), \quad i = 1, 2, 3, \quad (2.39)$$

*in the neighborhood of the point at infinity.*

**Theorem 2.15.**  $\forall x \in B^- :$

$$\int_S \mathcal{K}'(x, y) d_y S = \varkappa(x), \quad (2.40)$$

where  $\varkappa(x) \equiv \|\varkappa_{ij}(x)\|_{3 \times 3}$ ,

$$\varkappa_{ij}(x) = \frac{\rho}{|x|} \delta_{ij} + \frac{\lambda + \mu}{2\lambda + 5\mu} \frac{|x|^2 - \rho^2}{2} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\rho}{|x|}.$$

*Proof.* If we consider the vectors  $\overset{1}{u}, \overset{2}{u}, \overset{3}{u}$ , where  $\overset{k}{u}_i(x) = \varkappa_{ik}(x)$ , then it is easy to note that they satisfy equation (2.1), conditions (2.39) and have the properties  $\overset{k}{u} \in C^1(\bar{B}^-) \cap C^2(B^-)$ ,  $(\overset{k}{u}_i)^- = \delta_{ik} \in C^\infty(S)$ . Then  $\overset{k}{u}$  is the unique regular solution of Problem (I)<sup>-</sup>. Therefore by Theorem 2.14 formula (2.36)

$$\varkappa_{ik}(x) = \int_S \mathcal{K}_{ij}(x, y) \delta_{jk} d_y S,$$

coinciding with (2.40) holds for  $\overset{k}{u}$ . ■

**Theorem 2.16.** *If  $f \in C(S)$ , then  $u$  defined by formula (2.36) is a classical solution of Problem (I)<sup>-</sup>. It is unique in the class of functions satisfying the condition  $u(x) = o(1)$  in the neighbourhood of the point at infinity.*

*Proof.* The proof of this theorem repeats in the main the proof of the corresponding Theorem 2.12. We shall verify only the boundary condition

$$\lim_{B^- \ni x \rightarrow z \in S} u(x) = f(z).$$

Due to (2.36)-(2.40) we have

$$\begin{aligned} |u(x) - \varkappa(x)f(z)| &= \left| \int_S \mathcal{K}'(x, y)(f(y) - f(z)) d_y S \right| \leq \\ &\leq c \int_S \frac{|x|^2 - \rho^2}{|x - y|^3} |f(y) - f(z)| d_y S. \end{aligned}$$

Hence by virtue of the Poisson integral

$$\lim_{B^- \ni x \rightarrow z \in S} (u(x) - \varkappa(x)f(z)) = 0$$

and therefore

$$\lim_{B^- \ni x \rightarrow z \in S} u(x) = \lim_{B^- \ni x \rightarrow z \in S} \varkappa(x)f(z) = f(z). \quad \blacksquare$$

**2.7. Solution of Problem (II)<sup>+</sup>.** A solution  $u$  of Problem (II)<sup>+</sup> is sought for in form (2.16); then, on performing simple calculations, for the stress vector  $\tau^{(n)}$ , defined by formula (2.3), we obtain by virtue of (2.16) and (2.17)

$$\tau^{(n)}(x) = \frac{\mu}{r} h(x) + \frac{\mu(\rho^2 - r^2)}{r} \mathcal{D}_r \text{grad } \psi(x), \quad (2.41)$$

where  $h = (h_1, h_2, h_3)$ ,

$$h_i(x) = x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) - x_i \text{div } v(x) \quad (2.42)$$

(with summation taken over  $j$ ). Thus to find the solution we have to define  $h, v$  and  $\psi$ .

It is easy to verify that  $h$  is a harmonic function in  $B^+$ , and, due to (2.41) and the boundary condition of Problem (II)<sup>+</sup>, we obtain

$$h^+ = \frac{\rho}{\mu} f. \quad (2.42')$$

Thus  $h$  is the solution of the Dirichlet problem for  $B^+$  and it can therefore be represented by the Poisson formula (1.1), which by virtue of (2.10) can be rewritten in the form

$$h(x) = \frac{1}{4\pi\mu} \int_S \left( \Phi_0(x, y) + \frac{3xy}{\rho^3} \right) f(y) d_y S, \quad (2.43)$$

where

$$\Phi_0(x, y) = \frac{\rho^2 - |x|^2}{|y - x|^3} - \frac{1}{\rho} - \frac{3xy}{\rho^3}. \quad (2.44)$$

From (2.42) it follows that

$$\operatorname{div} h = -\operatorname{div} v, \quad (2.45)$$

$$x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) = q_i(x), \quad (2.46)$$

where  $q \equiv (q_1, q_2, q_3)$ ,

$$q_i(x) = h_i(x) - x_i \operatorname{div} h(x). \quad (2.47)$$

Applying the operation  $r \frac{\partial}{\partial r}$  to both parts of (2.46) we have

$$x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) + x_k x_j \left( \frac{\partial^2 v_i(x)}{\partial x_k \partial x_j} + \frac{\partial^2 v_j(x)}{\partial x_k \partial x_i} \right) = r \frac{\partial q_i(x)}{\partial r}.$$

A scalar multiplication of equality (2.46) by  $x$  with a subsequent differentiation gives us

$$x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) + x_k x_j \frac{\partial^2 v_j(x)}{\partial x_k \partial x_i} = \frac{1}{2} \frac{\partial (x \cdot q(x))}{\partial x_i}.$$

Now, taking into account the formula

$$x_k x_j \frac{\partial^2 v_i(x)}{\partial x_k \partial x_j} = r \frac{\partial}{\partial r} r \frac{\partial v_i(x)}{\partial r} - r \frac{\partial v_i(x)}{\partial r} = r^2 \frac{\partial^2 v_i(x)}{\partial r^2},$$

the above equalities yield for  $v$  the equation

$$r^2 \frac{\partial^2 v(x)}{\partial r^2} = r \frac{\partial q(x)}{\partial r} - \frac{1}{2} \operatorname{grad}(x \cdot q(x)), \quad (2.48)$$

and to define  $\psi$  we obtain by virtue of (2.17) and (2.45) the equation

$$2\mathcal{D}_r^2 \psi + 2 \frac{2\lambda + \mu}{\lambda + \mu} \mathcal{D}_r \psi + \frac{3\lambda + 2\mu}{\lambda + \mu} \psi = \operatorname{div} h. \quad (2.49)$$

Our further aim is to solve equations (2.48) and (2.49) and to construct the solution of Problem (II)<sup>+</sup> in form (2.16). Let us begin with equation (2.48).

The second condition of (2.10) implies

$$\int_S y_i f_j(y) d_y S = \int_S y_j f_i(y) d_y S, \quad i, j = 1, 2, 3. \quad (2.50)$$

By virtue of (2.43), (2.47) and (2.50) equation (2.48) can be rewritten as

$$\begin{aligned} r^2 \frac{\partial^2 v_i(x)}{\partial r^2} &= \frac{1}{8\pi\mu} \int_S \left( \left( 2r \frac{\partial \Phi_0(x, y)}{\partial r} - \Phi_0(x, y) \right) \delta_{ik} - x_k \frac{\partial \Phi_0(x, y)}{\partial x_i} - \right. \\ &\quad \left. - 2x_i \frac{\partial}{\partial x_k} \left( r \frac{\partial \Phi_0(x, y)}{\partial r} - \Phi_0(x, y) \right) + r^2 \frac{\partial^2 \Phi_0(x, y)}{\partial x_i \partial x_k} \right) f_k(y) d_y S. \end{aligned} \quad (2.51)$$

Consider a simpler equation

$$r^2 \frac{\partial^2 w}{\partial r^2} = \Phi_0 \quad (2.52)$$

whose particular solution is given by the formula

$$w(x, y) = \int_0^r dt \int_0^r \Phi_0\left(\frac{x}{t}, y\right) \frac{dt}{t^2}.$$

Due to the Dirichlet formula we can write it in the form

$$w = \Phi^{(2)} - \Phi^{(1)}, \quad (2.53)$$

where

$$\Phi^{(m)}(x, y) = \int_0^1 \Phi_0(\tau x, y) \frac{d\tau}{\tau^m}, \quad m = 1, 2.$$

The substitution of (2.53) in (2.52) gives us the identity

$$\Phi_0 = r^2 \frac{\partial^2}{\partial r^2} (\Phi^{(2)} - \Phi^{(1)}). \quad (2.54)$$

Taking into account (2.54) and the relations

$$\mathcal{D}_r \Phi^{(1)} = \Phi_0, \quad \mathcal{D}_r \Phi^{(2)} = \Phi^{(2)} + \Phi_0, \quad (2.55)$$

we obtain the equalities

$$(2\mathcal{D}_r - 1)\Phi_0 = r^2 \frac{\partial^2}{\partial r^2} (\Phi^{(2)} + \Phi^{(1)}), \quad (2.56)$$

$$(\mathcal{D}_r - 1)\Phi_0 = r^2 \frac{\partial^2}{\partial r^2} \Phi^{(1)}. \quad (2.57)$$

Now applying the formulas

$$\begin{aligned} x_k \frac{\partial}{\partial x_i} \left( r^2 \frac{\partial^2}{\partial r^2} \right) &= r^2 \frac{\partial^2}{\partial r^2} \left( x_k \frac{\partial}{\partial x_i} \right), \\ r^2 \frac{\partial^2}{\partial x_i \partial x_k} \left( r^2 \frac{\partial^2}{\partial r^2} \right) &= r^2 \frac{\partial^2}{\partial r^2} \left( r^2 \frac{\partial^2}{\partial x_i \partial x_k} \right) \end{aligned}$$

and keeping in mind (2.54), (2.56) and (2.57), we rewrite equation (2.51) as

$$\begin{aligned} r^2 \frac{\partial^2}{\partial r^2} \left( v_i(x) - \frac{1}{8\pi\mu} \int_S ((\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y)) \delta_{ik} + \right. \\ \left. + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \Phi^{(2)}(x, y)) - 2x_i \frac{\partial \Phi^{(1)}(x, y)}{\partial x_k} + \right. \\ \left. + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y)) \right) f_k(y) d_y S = 0. \end{aligned}$$

Hence for the sought for vector  $v$  we obtain the representation

$$\begin{aligned} v_i(x) &= \frac{1}{8\pi\mu} \int_S \left( (\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y)) \delta_{ik} + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \right. \\ &\quad \left. - \Phi^{(2)}(x, y)) - 2x_i \frac{\partial \Phi^{(1)}(x, y)}{\partial x_k} + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \right. \\ &\quad \left. - \Phi^{(1)}(x, y)) \right) f_k(y) d_y S + c_{ij} x_j + b_i, \end{aligned} \quad (2.58)$$

where  $c_{ij}$ ,  $b_i$  are arbitrary constants.

The additional term  $c_{ij} x_j + b_i$  in (2.58) is due to the fact that the general solution of the equation

$$r^2 \frac{\partial^2 \omega}{\partial r^2} = 0$$

with respect to the harmonic scalar function  $\omega$  is given by the formula  $\omega(x) = a_k x_k + b$ , where  $a_k$  and  $b$  are arbitrary constants.

Note (see (2.41)) that  $\psi$  exerts no influence on the boundary condition of Problem (II)<sup>+</sup>, and from (2.41), (2.42), (2.42') we obtain

$$\lim_{B^+ \ni x \rightarrow y \in S} \left( x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) - x_i \operatorname{div} v(x) \right) = \frac{\rho}{\mu} f(y). \quad (2.59)$$

Thus the constants  $c_{ij}$  and  $b_i$  are to be chosen such that condition (2.59) be fulfilled. By virtue of (2.55) and (2.58) the bracketed expression in (2.59) takes the form

$$\begin{aligned} x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) - x_i \operatorname{div} v(x) &= \\ = \frac{1}{4\pi\mu} \int_S \Phi_0(x, y) f_i(y) d_y S + (c_{ij} + c_{ji} - c_{kk} \delta_{ij}) x_j. \end{aligned}$$

Therefore for (2.59) to be fulfilled, due to (2.44) it suffices that

$$c_{ij} + c_{ji} - c_{kk}\delta_{ij} = \frac{3}{4\pi\rho^3\mu} \int_S y_j f_i(y) d_y S.$$

Hence

$$c_{ij} + c_{ji} = \frac{3}{4\pi\rho^3\mu} \int_S (y_j f_i(y) - \delta_{ij} y_k f_k(y)) d_y S, \quad i, j = 1, 2, 3.$$

Taking the latter equality into account, we have

$$\begin{aligned} c_{ij}x_j &= \frac{1}{2}(c_{ij} + c_{ji})x_j + \frac{1}{2}(c_{ij} - c_{ji})x_j = \\ &= \frac{1}{8\pi\mu} \int_S \left( x_k \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_k} \right) \frac{3xy}{\rho^3} f_k(y) d_y S + \varepsilon_{ijk} a_j x_k, \\ 2a_1 &= c_{32} - c_{23}, \quad 2a_2 = c_{13} - c_{31}, \quad 2a_3 = c_{21} - c_{12}, \end{aligned}$$

where  $\varepsilon_{ijk}$  is the Levi-Civita symbol.

Finally, for  $v_i$  defined by (2.58) we obtain

$$\begin{aligned} v_i(x) &= \frac{1}{8\pi\mu} \int_S ((\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y))\delta_{ik} + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \\ &\quad - \Phi^{(2)}(x, y) + \frac{3xy}{\rho^3}) - x_i \frac{\partial}{\partial x_k} (2\Phi^{(1)}(x, y) + \frac{3xy}{\rho^3}) + \\ &\quad + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y)) f_k(y) d_y S + \\ &\quad + \varepsilon_{ijk} a_j x_k + b_i. \end{aligned} \quad (2.60)$$

Let us now solve equation (2.49). We rewrite it in the equivalent form

$$2\mathcal{D}_r^2(\psi - c) + 2\frac{2\lambda + \mu}{\lambda + \mu} \mathcal{D}_r(\psi - c) + \frac{3\lambda + 2\mu}{\lambda + \mu} (\psi - c) = F, \quad (2.61)$$

where

$$\begin{aligned} F(x) &= \operatorname{div} h(x) - \operatorname{div} h(0), \quad c = \frac{\lambda + \mu}{3\lambda + 2\mu} \operatorname{div} h(0), \\ \operatorname{div} h(0) &= \frac{3}{4\pi\mu\rho^3} \int_S y \cdot f(y) d_y S = \frac{1}{4\pi\mu} \operatorname{div} \int_S \frac{3x \cdot y}{\rho^3} f(y) d_y S. \end{aligned} \quad (2.62)$$

Equation (2.61) is the Euler equation for the variable  $r$ . Introducing the variable  $t$  by means of the formula  $t = \ln r$ ,  $-\infty < t < \ln \rho$ , we obtain  $x = \frac{x}{r}r = e^t \frac{x}{r}$ ,

$$\left( 2\frac{\partial^2}{\partial t^2} + 2\frac{2\lambda + \mu}{\lambda + \mu} \frac{\partial}{\partial t} + \frac{3\lambda + 2\mu}{\lambda + \mu} \right) \left( \psi\left(\frac{x}{r}e^t\right) - c \right) = F\left(\frac{x}{r}e^t\right), \quad (2.63)$$

whose characteristic equation

$$2k^2 + 2 \frac{2\lambda + \mu}{\lambda + \mu} k + \frac{3\lambda + 2\mu}{\lambda + \mu} = 0 \quad (2.64)$$

has the roots

$$\frac{-(2\lambda + \mu) \pm \sqrt{-2\lambda^2 - 6\lambda\mu - 3\mu^2}}{2(\lambda + \mu)}.$$

We shall apply a separate treatment to each of the three cases

$$-\frac{2}{3}\mu < \lambda < \frac{\sqrt{3}-3}{2}\mu, \quad \lambda = \frac{\sqrt{3}-3}{2}\mu, \quad \lambda > \frac{\sqrt{3}-3}{2}\mu.$$

In these cases the characteristic equation has either two different real roots or a real multiple root, or two complex-conjugate roots. Let  $\lambda > (\sqrt{3}-3)\mu/2$ . This case is related with the situation  $\lambda > 0$  and  $\mu > 0$  which occurs for nearly all real elastic media (Huntington [1]). In the case under consideration equation (2.64) has the complex-conjugate roots  $k_1 + ik_2$  and  $k_1 - ik_2$ , where

$$k_1 = -\frac{2\lambda + \mu}{2(\lambda + \mu)}, \quad k_2 = \frac{\sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}}{2(\lambda + \mu)}, \quad (2.65)$$

$$-1 < k_1 < \frac{1}{2}, \quad k_2 > 0.$$

Now it is not difficult to write out the general solution of the homogeneous equation (2.63) in the form

$$\psi_0\left(\frac{x}{r}e^t\right) - c = e^{k_1 t}(c_1 \cos k_2 t + c_2 \sin k_2 t), \quad (2.66)$$

where  $c_1$  and  $c_2$  are arbitrary functions of the spherical coordinates  $\vartheta_1$  and  $\vartheta_2$  (or, which is the same thing, of the point  $x/r$ ).

The particular solution of (2.63) can be written as

$$\psi\left(\frac{x}{r}e^t\right) - c = \frac{1}{2k_2} \int_{-\infty}^t e^{k_1(t-\xi)} \sin k_2(t-\xi) F\left(e^\xi \frac{x}{r}\right) d\xi. \quad (2.67)$$

This solution can be combined with the general solution (2.66) of the homogeneous equation. But the latter solution must be a harmonic function in  $B^+$  and that is why it is necessary that  $c_1 = 0$  and  $c_2 = 0$ .

Keeping in mind that  $t = \ln r$  in (2.67) and introducing the variable  $\tau \equiv r^{-1}e^\xi$ , we obtain

$$\psi(x) = -\frac{1}{2k_2} \int_0^1 F(\tau x) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{1+k_1}} + c.$$

A similar treatment is applied to the cases  $-2\mu/3 < \lambda < (\sqrt{3}-3)\mu/2$  and  $\lambda = (\sqrt{3}-3)\mu/2$ .

Noting that  $F$  and  $c$  are defined by (2.62), we eventually have

$$\psi(x) = \frac{1}{8\pi\mu} \operatorname{div} \int_S \left( \Psi(x, y) + \frac{\lambda + \mu}{3\lambda + 2\mu} \frac{6xy}{\rho^3} \right) f(y) d_y S, \quad (2.68)$$

where

$$\Psi(x, y) = -\frac{1}{k_2} \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} - \frac{3x \cdot y \tau}{\rho^3} \right) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{2+k_1}}. \quad (2.69)$$

**Problem 2.** Using estimate (2.27), prove the validity of the integration order permutation formula that was used in deriving (2.68).

Now, by virtue of (2.16), (2.60), (2.68) we can proceed to constructing the solution of Problem (II)<sup>+</sup>:

$$\begin{aligned} u_i(x) = & \frac{1}{8\pi\mu} \int_S \left( (\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y)) \delta_{ik} + x_k \frac{\partial}{\partial x_i} \left( \Phi^{(1)}(x, y) - \right. \right. \\ & \left. \left. - \Phi^{(2)}(x, y) + \frac{3xy}{\rho^3} \right) + x_i \frac{\partial}{\partial x_k} \left( (2\mathcal{D}_r - 1) \Psi(x, y) - \right. \right. \\ & \left. \left. - 2\Phi^{(1)}(x, y) - \frac{\lambda}{3\lambda + 2\mu} \frac{3xy}{\rho^3} \right) + \frac{\rho^2 - r^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \Psi(x, y) + \right. \\ & \left. + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y) - \Psi(x, y)) f_k(y) d_y S + \right. \\ & \left. + \varepsilon_{ijk} a_j x_k + b_i. \right. \end{aligned} \quad (2.70)$$

By virtue of (2.41) the stress vector  $\tau^{(n)}$  takes the form

$$\begin{aligned} \tau^{(n)}(x) = & \frac{1}{4\pi r} \int_S \frac{\rho^2 - r^2}{|x - y|^3} f(y) d_y S + \\ & + \frac{\rho^2 - r^2}{8\pi r} \operatorname{grad} \operatorname{div} \int_S (\mathcal{D}_r - 2) \Psi(x, y) f(y) d_y S. \end{aligned} \quad (2.71)$$

**Problem 3.** Construct the solution of Problem (II)<sup>+</sup> when the Lamé constants satisfy the conditions

$$-\frac{2}{3} < \lambda < \frac{\sqrt{3} - 3}{2} \mu, \quad \mu > 0.$$

**Problem 4.** Construct the solution of Problem (II)<sup>+</sup> when the Lamé constants satisfy the conditions

$$\lambda = \frac{\sqrt{3} - 3}{2} \mu, \quad \mu > 0.$$

**2.8. Analysis of the Obtained Solution.** Let us investigate the solution of Problem (II)<sup>+</sup> when the boundary function  $f$  is continuous on  $S$ . It will be proved that the inclusions  $u \in C(\bar{B}^+)$ ,  $(\tau^{(n)})^+ \in C(S)$  are valid for  $u$  and  $\tau^{(n)}$  defined respectively by (2.70) and (2.71).

We begin by transforming  $\Phi^{(1)}$ ,  $\Phi^{(2)}$  and  $\Psi$  defined by (2.53) and (2.69). Applying the formulas

$$\begin{aligned} \int \frac{t^2-1}{t(t^2-2at+1)^{3/2}} dt &= -\frac{2}{\sqrt{t^2-2at+1}} + \ln \frac{\sqrt{t^2-2at+1}+1-at}{t}, \\ \int \left( \frac{t^2-1}{t^2(t^2-2at+1)^{3/2}} + \frac{1}{t^2} + \frac{3a}{t} \right) dt &= -\frac{2}{t\sqrt{t^2-2at+1}} + \\ &+ \frac{3\sqrt{t^2-2at+1}-1}{t} + 3a \ln(\sqrt{t^2-2at+1}+1-at), \quad |a| \leq 1, \end{aligned} \quad (2.72)$$

$\Phi^{(1)}$  and  $\Phi^{(2)}$  can be represented as

$$\begin{aligned} \Phi^{(1)}(x, y) &= \frac{2}{|x-y|} - \frac{1}{\rho} \ln((|x-y|+\rho)^2 - |x|^2) - \\ &\quad - \frac{3xy}{\rho^3} + \frac{1}{\rho} \ln(4\rho^2 - 2), \end{aligned} \quad (2.73)$$

$$\begin{aligned} \Phi^{(2)}(x, y) &= \frac{2}{|x-y|} - \frac{3xy}{\rho^3} \ln((|x-y|+\rho)^2 - |x|^2) - \\ &\quad - \frac{3|x-y|}{\rho^2} + (3 \ln 4\rho^2 - 5) \frac{xy}{\rho^3} + \frac{1}{\rho}. \end{aligned} \quad (2.74)$$

It is easy to verify the validity of the equality

$$\mathcal{D}_r \Psi = (1 + k_1) \Psi + \Psi_1,$$

where

$$\Psi_1(x, y) = \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} - \frac{3xy}{\rho^3} \tau \right) \cos(k_2 \ln \tau) \frac{d\tau}{\tau^{2+k_1}}.$$

Applying identities (2.29), (2.29'), we obtain the following representations for  $\Psi$  and  $\Psi_1$ :

$$\begin{aligned} \Psi(x, y) &= -\frac{2}{\rho} \ln((|x-y|+\rho)^2 - |x|^2) + \\ &\quad + \int_{1/2}^1 \ln((|\tau x - y| + \rho)^2 - |\tau x|^2) \varphi_1(\tau) d\tau + \Psi'(x, y), \end{aligned} \quad (2.75)$$

$$\Psi_1(x, y) = \frac{2}{|x-y|} - \frac{\lambda + 2\mu}{(\lambda + \mu)\rho} \ln((|x-y|+\rho)^2 - |x|^2) +$$

$$+ \int_{1/2}^1 \ln((|\tau x - y| + \rho)^2 - |\tau x|^2) \varphi_2(\tau) d\tau + \Psi'_1(x, y), \quad (2.76)$$

where  $\Psi'$  and  $\Psi'_1$  and their derivatives with respect to the Cartesian coordinates of the point  $x$ , and also  $\varphi_1$  and  $\varphi_2$  are the bounded functions.

Now the solution of Problem (II)<sup>+</sup> represented by (2.70) can be written in the form

$$u = \overset{1}{u} + \overset{2}{u} + \overset{3}{u},$$

where

$$\begin{aligned} \overset{1}{u}(x) &= \int_S L^{(1)}(x, y) f(y) d_y S, \\ \overset{2}{u}(x) &= \int_S (\rho^2 - |x|^2) L^{(2)}(x, y) f(y) d_y S, \\ \overset{3}{u}(x) &= \frac{|x|^2}{8\pi\mu} \int_S L^{(3)}(x, y) f(y) d_y S, \\ L^{(m)} &= \|L_{ij}^{(m)}\|_{3 \times 3}, \quad m = 1, 2, 3, \\ |L^{(1)}(x, y)| &\leq \frac{c}{|x - y|}, \quad |L^{(2)}(x, y)| \leq \frac{c}{|x - y|^2}, \\ L_{ij}^{(3)}(x, y) &= \frac{\partial^2}{\partial x_i \partial x_j} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y) - \Psi(x, y)). \end{aligned}$$

It is easy to show that  $\overset{1}{u} \in C(\bar{B}^+)$ ,  $\overset{2}{u} \in C(\bar{B}^+)$ . We shall prove the inclusion  $\overset{3}{u} \in C(\bar{B}^+)$ . By virtue of equalities (2.73)-(2.75) we have

$$\begin{aligned} \Phi^{(2)}(x, y) - \Phi^{(1)}(x, y) - \Psi(x, y) &= \Phi(x, y) + \\ &+ \int_{1/2}^1 \ln((|\tau x - y| + \rho)^2 - |\tau x|^2) \varphi(\tau) d\tau, \end{aligned}$$

where  $\varphi$ , as well as  $\Phi$  and its derivatives up to the first order are the bounded functions and

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} \Phi(x, y) \right| \leq \frac{c}{|x - y|}, \quad i, j = 1, 2, 3.$$

Thus to prove the inclusion  $\overset{3}{u} \in C(\bar{B}^+)$  it suffices to show that  $\chi \in C(\bar{B}^+)$ , where

$$\chi(x) = \int_S \left( \int_{1/2}^1 \text{grad div} \ln((|\tau x - y| + \rho)^2 - |\tau x|^2) \varphi(\tau) d\tau \right) f(y) d_y S. \quad (2.77)$$

Obviously,  $\chi \in C(B^+)$ ; therefore it is sufficient to prove that  $\chi \in C(\bar{B}^+ \setminus B(0, \rho/2))$ . let  $\rho/2 < x \leq \rho$ . Since for  $z \in B^+$ ,  $y \in S$ :

$$(|z - y| + \rho)^2 - |z|^2 \geq 2\rho|z - y|, \quad \left| \frac{z_i - y_i}{|z - y|} \right| \leq 1, \quad |y_i| \leq \rho,$$

we have

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} \ln((|\tau x - y| + \rho)^2 - |\tau x|^2) \right| \leq \frac{4}{|\tau x - y|^2}.$$

For the expression

$$A_{ij}(x, y) \equiv \int_{1/2}^1 \frac{\partial^2}{\partial x_i \partial x_j} \ln((|\tau x - y| + \rho)^2 - |\tau x|^2) \varphi(\tau) d\tau$$

the above estimate yields

$$\begin{aligned} |A_{ij}(x, y)| &\leq c \int_{1/2}^1 \frac{d\tau}{|\tau x - y|^2} = \\ &= \frac{c}{\rho|x|\sqrt{1-a^2}} \left( \operatorname{arctg} \frac{|x| - a\rho}{\rho\sqrt{1-a^2}} - \operatorname{arctg} \frac{|x| - 2a\rho}{2\rho\sqrt{1-a^2}} \right), \end{aligned}$$

where  $a = \frac{x \cdot y}{\rho|x|} \equiv \cos \gamma$ . Therefore

$$|A_{ij}(x, y)| \leq \frac{c}{\sqrt{1-a^2}}.$$

Let  $x \in B^+$  and  $x_0$  be a point of the sphere  $S$  such that  $|x - x_0| = \min_{z \in S} |x - z|$ . Then

$$|y - x_0| = 2\rho \left| \sin \frac{\gamma}{2} \right|, \quad |y + x_0| = 2\rho \left| \cos \frac{\gamma}{2} \right|,$$

and therefore

$$\sqrt{1-a^2} = \sqrt{1-\cos^2 \gamma} = |\sin \gamma| = \frac{|y - x_0||y + x_0|}{2\rho^2}.$$

Thus

$$|A_{ij}(x, y)| \leq \frac{c}{|y - x_0||y + x_0|}$$

and for  $\chi$  we have the estimate

$$|\chi(x)| \leq c \int_S \frac{|f(y)|}{|y + x_0||y - x_0|} d_y S.$$

Keeping in mind that  $|y - x_0| > \rho$  if  $|y + x_0| \leq \rho$  and, conversely,  $|y + x_0| > \rho$  if  $|y - x_0| \leq \rho$ , and applying the arguments used in proving the continuity of the harmonic simple-layer potential (see Mikhlin [1] or Gunter [1]), it is easy to establish the inclusion  $\chi \in C(\bar{B}^+)$ . Thus  $u \in C(\bar{B}^+)$ .

Now we shall prove the inclusion  $(\tau^{(n)})^+ \in C(S)$ . Using representations (2.75) and (2.76), for  $\tau^{(n)}$  defined by (2.71) we have

$$\tau^{(n)}(x) = \frac{\rho}{|x|} \Pi(f)(x) + \frac{\rho^2 - |x|^2}{4\pi|x|} \int_S \text{grad div} \frac{f(y)}{|x-y|} d_y S + \eta(x), \quad (2.78)$$

where

$$\lim_{B^+ \ni x \rightarrow z \in S} \eta(x) = 0.$$

Now let us prove that if  $F = (F_1, F_2, F_3)$  and  $F_i \in C(S)$ , then

$$\lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2) \text{grad div} \int_S \frac{F(y)}{|x-y|} d_y S = 0. \quad (2.79)$$

Indeed, from the property of the Poisson integral, from formulas (2.31), (2.32), (2.33) and from the estimate

$$\forall x \in \mathbb{R}^3 : \int_S |x-y|^{-3/2} d_y S \leq c, \quad (2.80)$$

where  $c$  depends only on  $\rho$ , it follows that

$$\begin{aligned} \lim_{B^+ \ni x \rightarrow z \in S} \int_S \mathcal{K}^{(1)}(x, y) F(y) d_y S &= F(z), \\ \lim_{B^+ \ni x \rightarrow z \in S} \int_S \mathcal{K}^{(3)}(x, y) F(y) d_y S &= 0. \end{aligned}$$

By virtue of Theorem 2.10

$$\lim_{B^+ \ni x \rightarrow z \in S} \int_S \mathcal{K}(x, y) F(y) d_y S = F(z).$$

These formulas yield (2.79).

Now, using (2.78), (2.79) and the property of the Poisson integral, we easily convince ourselves that

$$\lim_{B^+ \ni x \rightarrow z \in S} \tau^{(n)}(x) = f(z)$$

and therefore the inclusion  $\tau^{(n)} \in C(S)$  is valid.

Thus we have

**Theorem 2.17.** *If  $f \in C(S)$  and conditions (2.10) are satisfied, then  $u$  defined by (2.70) is a classical solution of Problem (II)<sup>+</sup>.*

Note also that for classical solutions of Problem (II)<sup>+</sup> we have a uniqueness theorem similar to Theorem 2.1 which is a corollary of the validity of Green formulas in the theory of elasticity for classical solutions.

**2.9. Solution of Problem (II)<sup>-</sup>.** No new difficulties arise in constructing and solving Problem (II)<sup>-</sup>, since it is solved exactly by the same procedure as above.

The solution of Problem (II)<sup>-</sup> has the form

$$\begin{aligned}
u_i(x) = & \frac{1}{8\pi\mu} \int_S \left( -(\tilde{\Phi}^{(1)}(x, y) + \tilde{\Phi}^{(2)}(x, y))\delta_{ik} + x_k \frac{\partial}{\partial x_k} (\tilde{\Phi}^{(2)}(x, y) - \right. \\
& - \tilde{\Phi}^{(1)}(x, y)) + x_i \frac{\partial}{\partial x_k} ((2\mathcal{D}_r - 1)\tilde{\Psi}(x, y) + 2\tilde{\Phi}^{(1)}(x, y)) + \\
& + \frac{\rho^2 - 3r^2}{2} \frac{\partial^2}{\partial x_i \partial x_j} \tilde{\Psi}(x, y) - r^2 \frac{\partial^2}{\partial x_i \partial x_j} (\tilde{\Phi}^{(2)}(x, y) - \\
& \left. - \tilde{\Phi}^{(1)}(x, y)) \right) f_k(y) d_y S, \tag{2.81}
\end{aligned}$$

while the stress vector has the form

$$\begin{aligned}
\tau^{(n)}(x) = & \frac{1}{4\pi r} \int_S \frac{r^2 - \rho^2}{|x - y|^3} f(y) d_y S + \\
& + \frac{\rho^2 - r^2}{8\pi r} \int_S \text{grad div}(\mathcal{D}_r - 2)\tilde{\Psi}(x, y) f(y) d_y S, \tag{2.82}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Phi}^{(m)}(x, y) & \equiv \int_0^1 \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^3} \tau^{m-1} d\tau, \quad m = 1, 2; \\
\tilde{\Psi}(x, y) & \equiv -\frac{1}{k_2} \int_0^1 \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^3} \tau^{1+k_1} \sin(k_2 \ln \tau) d\tau, \\
k_1 = -\frac{2\lambda + \mu}{2(\lambda + \mu)}, \quad k_2 & = \frac{\sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}}{2(\lambda + \mu)}, \quad -1 < k_1 < \frac{1}{2}, \quad k_2 > 0.
\end{aligned}$$

**Theorem 2.18.** *If  $f \in C(S)$ , then  $u$  defined by formula (2.81) is a classical solution of Problem (II)<sup>-</sup>. It is unique in the class of functions satisfying the condition  $u(x) = o(1)$  in the neighbourhood of the point at infinity.*

**Problem 5.** Prove Theorem 2.18.

**2.10. Solution of Problem (III)<sup>+</sup>.** To solve this problem by the method to be proposed below it is convenient to replace the boundary conditions (see (2.6)) of Problem (III)<sup>+</sup>

$$(n \cdot u)^+ = g, \quad (\tau^{(n)} - n(n \cdot \tau^{(n)}))^+ = l$$

by some equivalent condition as it is done when investigating this problem for an arbitrary domain (Kupradze *et al.* [1]).

Replacing  $\tau^{(n)}$  in (2.6) by its expression from (2.3), we obtain

$$\tau_i^{(n)} - n_i(n \cdot \tau^{(n)}) = \mu n_j \left( \frac{\partial u_i}{\partial y_j} - \frac{\partial u_j}{\partial y_i} \right) + 2\mu n_j \mathcal{D}_i(u_j),$$

where  $\mathcal{D}_i$  is the Gunter operator. On account of the first condition of (2.6) we obtain

$$(n_j \mathcal{D}_i(u_j))^+ = (\mathcal{D}_i(n \cdot u))^+ - (u_j \mathcal{D}_i(n_j))^+ = \mathcal{D}_i(g) - (u_j \mathcal{D}_i(n_j))^+.$$

Since  $n_k(x) = x_k/|x|$ , we have

$$\mathcal{D}_i(n_j)(x) = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}.$$

Hence, taking into account the first condition of (2.6), we obtain

$$(u_j \mathcal{D}_i(n_j))^+(y) = \frac{1}{\rho}(u_i)^+(y) - \frac{y_i}{\rho^2}g(y).$$

Therefore

$$\begin{aligned} (\tau_i^{(n)} - n_i(n \cdot \tau^{(n)}))^+(y) &= \frac{\mu}{\rho} \left( y_j \left( \frac{\partial u_i}{\partial y_j} - \frac{\partial u_j}{\partial y_i} \right) - 2u_i \right)^+(y) + \\ &\quad + 2\mu \mathcal{D}_i(g)(y) + \frac{2\mu}{\rho^2} y_i g(y). \end{aligned}$$

Now the second condition of (2.6) can be replaced by

$$(H(\partial_y)u)^+ = F,$$

where  $F = (F_1, F_2, F_3)$ ,

$$F_i(y) \equiv \frac{\rho}{\mu} l_i(y) - 2\rho \mathcal{D}_i(g)(y) - \frac{2}{\rho} y_i g(y), \quad (2.83)$$

$$H(\partial_x) \equiv \|H_{ij}(\partial_x)\|_{3 \times 3},$$

$$H_{ij}(\partial_x) \equiv \delta_{ij}(\mathcal{D}_r - 2) - x_j \frac{\partial}{\partial x_i}. \quad (2.84)$$

Thus instead of Problem (III)<sup>+</sup> we have an equivalent problem (the equivalency will be proved below) with the boundary conditions

$$(n \cdot u)^+ = g, \quad (2.85)$$

$$(H(\partial_y)u)^+ = F. \quad (2.85')$$

Note that by virtue of (2.9) and the identity  $n_k \mathcal{D}_k = 0$  we have

$$y_k F_k(y) = -2\rho g(y). \quad (2.86)$$

Moreover, by virtue of the Stokes formula

$$\int_S (y_i \mathcal{D}_k - y_k \mathcal{D}_i)(g)(y) d_y S = 0,$$

(2.11) yields

$$\int_S (y_i F_k(y) - y_k F_i(y)) d_y S = 0, \quad i, k = 1, 2, 3. \quad (2.87)$$

A solution of Problem (III)<sup>+</sup> is sought for in the form of (2.12). Then  $H(\partial_x)u$  can be represented as

$$H(\partial_x)u(x) = h(x) + \frac{\mu(\rho^2 - |x|^2)}{\lambda + \mu} \text{grad } \psi(x), \quad (2.88)$$

where  $h = (h_1, h_2, h_3)$ ,

$$\begin{aligned} h_i(x) = & x_j \left( \frac{\partial v_i(x)}{\partial x_j} - \frac{\partial v_j(x)}{\partial x_i} \right) - 2v_i(x) + x_i \mathcal{D}_r \psi(x) + \\ & + \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \psi(x)}{\partial x_i} - \frac{\lambda + 2\mu}{\lambda + \mu} \rho^2 \frac{\partial \psi(x)}{\partial x_i}. \end{aligned} \quad (2.89)$$

From (2.88) and (2.85')  $h^+ = F$ . Using (2.13), it is easy to show that  $\forall x \in B^+ : \Delta h(x) = 0$ . Therefore  $h$  is the solution of the Dirichlet problem and is expressed by the Poisson formula

$$h = \Pi(F). \quad (2.90)$$

Let us calculate  $\text{div } h$  from (2.89); we have

$$\text{div } h = (\mathcal{D}_r - 2) \text{div } v + \mathcal{D}_r^2 \psi + \frac{3\lambda + 5\mu}{\lambda + \mu} \mathcal{D}_r \psi.$$

Hence, excluding  $v$  by means of (2.13), to define  $\psi$  we obtain the equation

$$\mathcal{D}_r^2 \psi + \frac{\lambda}{2(\lambda + 2\mu)} \mathcal{D}_r \psi - \frac{\mu}{\lambda + 2\mu} \psi = \frac{\lambda + \mu}{2(\lambda + 2\mu)} \text{div } h. \quad (2.91)$$

Let us now derive an equation for defining  $v$ . A scalar multiplication of (2.89) by  $x$  gives

$$x \cdot v(x) = -\frac{1}{2} x \cdot h(x) - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - r^2) r \frac{\partial \psi(x)}{\partial r}. \quad (2.92)$$

Rewriting (2.89) in the form

$$\begin{aligned} (\mathcal{D}_r - 1)v_i(x) = & h_i(x) + \frac{\partial(x \cdot v(x))}{\partial x_i} - \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \psi(x)}{\partial x_i} - \\ & - x_i \mathcal{D}_r \psi(x) + \frac{\lambda + 2\mu}{\lambda + \mu} \rho^2 \frac{\partial \psi(x)}{\partial x_i} \end{aligned}$$

and taking into account (2.92), to define  $v$  we obtain the equation

$$(\mathcal{D}_r - 1)v = q, \quad (2.93)$$

where  $q = (q_1, q_2, q_3)$ ,

$$q_i(x) = \frac{1}{2} \left( h_i(x) - x_k \frac{\partial h_k(x)}{\partial x_i} \right) - \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \psi(x)}{\partial x_i} + (\mathcal{D}_r - 1) \left( \frac{\mu}{\lambda + \mu} x_i \psi(x) - \frac{\lambda + 2\mu}{\lambda + \mu} \frac{\rho^2 - r^2}{2} \frac{\partial \psi(x)}{\partial x_i} \right). \quad (2.94)$$

Let us solve these equations.

Consider (2.91) and rewrite it in an equivalent form

$$\mathcal{D}_r^2(\psi - c) + \frac{\lambda}{2(\lambda + 2\mu)} \mathcal{D}_r(\psi - c) - \frac{\mu}{\lambda + 2\mu} (\psi - c) = \eta, \quad (2.95)$$

where

$$c = -\frac{\lambda + \mu}{2\mu} \operatorname{div} h(0),$$

$$\eta(x) = \frac{\lambda + \mu}{2(\lambda + 2\mu)} (\operatorname{div} h(x) - \operatorname{div} h(0)).$$

Note that

$$\operatorname{div} h(0) = \frac{3}{4\pi\rho^4} \int_S y F(y) d_y S = \frac{1}{4\pi\rho} \operatorname{div} \int_S \frac{3(x \cdot y)}{\rho^3} F(y) d_y S.$$

Therefore due to (2.86)

$$c = \frac{3(\lambda + \mu)}{4\pi\mu\rho^3} \int_S g(y) d_y S,$$

$$\eta(x) = \frac{\lambda + \mu}{8\pi\rho(\lambda + 2\mu)} \operatorname{div} \int_S \left( \frac{\rho^2 - |x|^2}{|y - x|^3} - \frac{1}{\rho} - \frac{3xy}{\rho^3} \right) F(y) d_y S.$$

Introducing the variable  $t = \ln r$ , equation (2.95) can be rewritten as

$$\frac{\partial^2(\psi - c)}{\partial t^2} + \frac{\lambda}{2(\lambda + 2\mu)} \frac{\partial(\psi - c)}{\partial t} - \frac{\mu}{\lambda + 2\mu} (\psi - c) = \eta.$$

The characteristic equation

$$k^2 + \frac{\lambda}{2(\lambda + 2\mu)} k - \frac{\mu}{\lambda + 2\mu} = 0$$

has the roots

$$k_1 = \frac{-\lambda + \sqrt{\lambda^2 + 16\lambda\mu + 32\mu^2}}{4(\lambda + \mu)},$$

$$k_2 = \frac{-\lambda - \sqrt{\lambda^2 + 16\lambda\mu + 32\mu^2}}{4(\lambda + \mu)}. \quad (2.96)$$

Note that by virtue of condition (2.2)

$$\begin{aligned} \lambda^2 + 16\lambda\mu + 32\mu^2 &= \lambda^2 + 16\mu(\lambda + 2\mu) > 0, \\ 0 < k_1 < 1, \quad -1 < k_2 < 0. \end{aligned} \quad (2.97)$$

It is not difficult now to write out the solution of equation (2.91) taking into account the fact that  $\psi$  is a harmonic function in  $B^+$ :

$$\begin{aligned} \psi(x) &= \frac{\gamma_0}{4\pi\rho} \operatorname{div} \int_S \left( \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} - \frac{3xy}{\rho^3} \tau \right) (\tau^{-2-k_1} - \right. \\ &\quad \left. - \tau^{-2-k_2}) d\tau \right) F(y) d_y S + \frac{3(\lambda + \mu)}{4\pi\mu\rho^3} \int_S g(y) d_y(S), \end{aligned} \quad (2.98)$$

where

$$\gamma_0 = \frac{\lambda + \mu}{2(k_1 - k_2)(\lambda + 2\mu)}. \quad (2.99)$$

We introduce the notation

$$\Phi^{(m)}(x, y) = \int_0^1 \Phi_0(\tau x, y) \frac{d\tau}{\tau^m}, \quad 0 < m < 3, \quad (2.100)$$

where  $\Phi_0$  is defined by (2.44). Then  $h$  and  $\psi$  defined respectively by (2.90) and (2.98) can be written in the form

$$h(x) = \frac{1}{4\pi\rho} \int_S \left( \Phi_0(x, y) + \frac{1}{\rho} + \frac{3xy}{\rho^3} \right) F(y) d_y S, \quad (2.101)$$

$$\begin{aligned} \psi(x) &= \frac{\gamma_0}{4\pi\rho} \operatorname{div} \int_S \left( \Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y) \right) F(y) d_y S + \\ &\quad + \frac{3(\lambda + \mu)}{4\pi\mu\rho^3} \int_S g(y) d_y S. \end{aligned} \quad (2.102)$$

Now we shall proceed to solving equation (2.93). Let us first rewrite in a convenient form the right-hand side  $q$  defined by (2.94). By virtue of (2.100) it is easy to obtain

$$(\mathcal{D}_r - 1)\Phi^{(m)} = (m - 2)\Phi^{(m)} + \Phi_0.$$

Hence follow the identities

$$\begin{aligned} \Phi_0 &= (\mathcal{D}_r - 1)\Phi^{(2)}, \\ \Phi^{(2+k_1)} - \Phi^{(2+k_2)} &= (\mathcal{D}_r - 1) \left( \frac{1}{k_1} \Phi^{(2+k_1)} - \right. \end{aligned} \quad (2.103)$$

$$-\frac{1}{k_2}\Phi^{(2+k_2)} + \frac{k_1 - k_2}{k_1 k_2}\Phi^{(2)}). \quad (2.103')$$

Taking into account (2.87), (2.101), (2.103) and the equalities

$$x_k \frac{\partial}{\partial x_i} \mathcal{D}_r = \mathcal{D}_r x_k \frac{\partial}{\partial x_i}, \quad (\mathcal{D}_r - 1) \frac{1}{\rho} = -\frac{1}{\rho},$$

we have

$$\begin{aligned} h_i(x) - x_k \frac{\partial h_k(x)}{\partial x_i} &= \frac{1}{4\pi\rho} (\mathcal{D}_r - 1) \int_S \left( \left( \Phi^{(2)}(x, y) - \frac{1}{\rho} \right) \delta_{ik} - \right. \\ &\quad \left. - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) \right) F_k(y) d_y S. \end{aligned} \quad (2.104)$$

Next, calculating  $r^2 \frac{\partial \psi}{\partial x_i}$  from (2.102), taking into account (2.103') and the equality

$$r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\mathcal{D}_r - 1) = (\mathcal{D}_r - 1) r^2 \frac{\partial^2}{\partial x_i \partial x_k},$$

we obtain

$$\begin{aligned} r^2 \frac{\partial \psi(x)}{\partial x_i} &= (\mathcal{D}_r - 1) \frac{\gamma_0}{4\pi\rho} r^2 \frac{\partial^2}{\partial x_i \partial x_k} \int_S \left( \frac{1}{k_1} \Phi^{(2+k_1)}(x, y) - \right. \\ &\quad \left. - \frac{1}{k_2} \Phi^{(2+k_2)}(x, y) + \frac{k_1 - k_2}{k_1 k_2} \Phi^{(2)}(x, y) \right) F_k(y) d_y S. \end{aligned} \quad (2.105)$$

Now by virtue of (2.94), (2.104) and (2.105) we rewrite equation (2.93) as

$$(\mathcal{D}_r - 1)(v_i - p_i) = 0,$$

where

$$\begin{aligned} p_i(x) &= \frac{1}{4\pi\rho} \int_S \left( \frac{1}{2} \left( \left( \Phi^{(2)}(x, y) - \frac{1}{\rho} \right) \delta_{ik} - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) \right) - \right. \\ &\quad - \frac{\mu_0}{\lambda + \mu} r^2 \frac{\partial^2}{\partial x_i \partial x_k} \left( \frac{1}{k_1} \Phi^{(2+k_1)}(x, y) - \frac{1}{k_2} \Phi^{(2+k_2)} + \right. \\ &\quad \left. \left. + \frac{k_1 - k_2}{k_1 k_2} \Phi^{(2)}(x, y) \right) + \gamma_0 \left( \frac{\mu}{\lambda + \mu} x_i - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - \right. \right. \\ &\quad \left. \left. - r^2) \frac{\partial}{\partial x_i} \right) \frac{\partial}{\partial x_k} \left( \Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y) \right) \right) F_k(y) d_y S. \end{aligned} \quad (2.106)$$

Note that the general solution of the homogeneous equation

$$(\mathcal{D}_r - 1)v_i^0 = 0$$

in the class of harmonic functions in  $B^+$  has the form

$$\overset{\circ}{v}_i(x) = c_{ij}x_j,$$

where  $c_{ij}$  are arbitrary constants.

Thus the solution of equation (2.93) will be written in the form

$$v_i(x) = p_i(x) + c_{ij}x_j. \quad (2.107)$$

Therefore the solution of Problem (III)<sup>+</sup> can be constructed by representation (2.12) using (2.102), (2.106), (2.107):

$$\begin{aligned} u_i(x) = & \frac{1}{4\pi\rho} \int_S \left( \frac{1}{2} \left( \left( \Phi^{(2)}(x, y) - \frac{1}{\rho} \right) \delta_{ik} - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) \right) - \right. \\ & - \frac{\gamma_0\mu}{\lambda + \mu} r^2 \frac{\partial^2}{\partial x_i \partial x_k} \left( \frac{1}{k_1} \Phi^{(2+k_1)}(x, y) - \frac{1}{k_2} \Phi^{(2+k_2)}(x, y) + \right. \\ & + \left. \frac{k_1 - k_2}{k_1 k_2} \Phi^{(2)}(x, y) \right) + \frac{\gamma_0\mu}{\lambda + \mu} \left( x_i \frac{\partial}{\partial x_k} - \frac{\rho^2 - r^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \right) \times \\ & \left. \times \left( \Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y) \right) \right) F_k(y) d_y S + c_{ij}. \end{aligned} \quad (2.108)$$

Formula (2.108) contains arbitrary constants  $c_{ij}$  which have to be chosen such that the vector  $u$  represented by this formula satisfy the boundary conditions (2.85) and (2.85'). To this effect we shall establish some properties of the function  $\Phi^{(m)}$  defined by (2.100).

Applying (2.29) and (2.29'), we obtain the representation

$$\begin{aligned} \Phi^{(m)}(x, y) = & \frac{2}{|x - y|} - \frac{2m - 1}{\rho} \ln((|y - x| + \rho)^2 - |x|^2) + \\ & + \int_{1/2}^1 \ln((|\tau x - y| + \rho)^2 - |\tau x|^2) \varphi_1(\tau) d\tau + \Phi_1^{(m)}(x, y), \end{aligned} \quad (2.109)$$

where  $\Phi_1^{(m)}$  and its derivatives with respect to the Cartesian coordinates of the point  $x$ , as well as  $\varphi_1$  are the bounded functions.

In particular, (2.109) yields the equalities

$$\Phi^{(m)}(x, y) = \frac{2}{|x - y|} + \Phi_0^{(m)}(x, y), \quad (2.110)$$

where

$$\begin{aligned} |\Phi_0^{(m)}(x, y)| & \leq \frac{c}{|x - y|}, \quad \left| \frac{\partial}{\partial x_i} \Phi_0^{(m)}(x, y) \right| \leq \frac{c}{|x - y|}, \\ \left| \frac{\partial^2}{\partial x_i \partial x_k} \Phi_0^{(m)}(x, y) \right| & \leq \frac{c}{|x - y|^2}, \end{aligned}$$

$$\begin{aligned} \Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y) &= \frac{2(k_2 - k_1)}{\rho} \ln(|x - y| + \rho)^2 - \\ &- |x|^2 + \int_{1/2}^1 \ln(|\tau x - y| + \rho)^2 - |\tau x|^2 \varphi_2(\tau) d\tau + \Phi_2(x, y), \end{aligned} \quad (2.111)$$

$$\begin{aligned} &k_2 \Phi^{(2+k_1)}(x, y) - k_1 \Phi^{(2+k_2)}(x, y) + (k_1 - k_2) \Phi^{(2)}(x, y) = \\ &= \int_{1/2}^1 \ln(|\tau x - y| + \rho)^2 - |\tau x|^2 \varphi_3(\tau) d\tau + \Phi_3(x, y), \end{aligned} \quad (2.112)$$

where  $\Phi_2$ ,  $\Phi_3$  and their derivatives with respect to the coordinates of the point  $x$ , as well as  $\varphi_2$  and  $\varphi_3$  are the bounded functions.

From (2.108) we have

$$\begin{aligned} x \cdot u(x) &= -\frac{1}{8\pi\rho} \int_S \left( x_k \frac{\rho^2 - |x|^2}{|y - x|^3} - \frac{3xy}{\rho^3} x_k - \right. \\ &- \frac{k_1 k_2 (\rho^2 - |x|^2)}{2(k_1 - k_2)} \frac{\partial}{\partial x_k} (k_1 \Phi^{(2+k_1)}(x, y) - \\ &- \left. k_2 \Phi^{(2+k_2)}(x, y)) \right) F_k(y) d_y S + c_{ik} x_i x_k. \end{aligned} \quad (2.113)$$

But by virtue of (2.110)

$$\left| (\rho^2 - |x|^2) \frac{\partial}{\partial x_k} (k_1 \Phi^{(2+k_1)}(x, y) - k_2 \Phi^{(2+k_2)}(x, y)) \right| \leq c \frac{\rho^2 - |x|^2}{|y - x|^2}.$$

Therefore, if  $F \in C(S)$ , then

$$\lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2) \int_S \frac{\partial}{\partial x_k} (k_1 \Phi^{(2+k_1)}(x, y) - k_2 \Phi^{(2+k_2)}(x, y)) F_k(y) d_y S = 0.$$

Moreover, taking into account the property of the Poisson integral and (2.86), we have

$$\lim_{x \rightarrow z} \frac{-x_k}{8\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|y - x|^3} F_k(y) d_y S = -\frac{1}{2} z_k F_k(z) = \rho g(z).$$

Thus to satisfy condition (2.85) it is necessary that

$$c_{ik} x_i x_k = -\frac{3x_i x_k}{8\pi\rho^4} \int_S y_i F_k(y) d_y S. \quad (2.114)$$

By virtue of (2.87) the symmetrical part  $c_{ik}$  has the form

$$\frac{1}{2}(c_{ik} + c_{ki}) = -\frac{3}{8\pi\rho^4} \int_S y_i F_k(y) d_y S, \quad (2.114')$$

whereas the asymmetrical part  $\frac{1}{2}(c_{ik} - c_{ki})$  remains undefined.

To satisfy (2.85') equality (2.114') is also sufficient. We introduce the notations  $2a_1 \equiv c_{32} - c_{23}$ ,  $2a_2 \equiv c_{13} - c_{31}$ ,  $2a_3 \equiv c_{21} - c_{12}$ . Due to (2.96), (2.99) the function  $u$  defined by (2.108) will take the final form

$$\begin{aligned}
u_i(x) = & \frac{1}{8\pi\rho} \int_S \left( \left( \Phi^{(2)}(x, y) - \frac{1}{\rho} - \frac{3xy}{\rho^3} \right) \delta_{ik} - \right. \\
& - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) + \frac{r^2}{k_1 - k_2} \frac{\partial^2}{\partial x_i \partial x_k} (k_2 \Phi^{(2+k_1)}(x, y) - \\
& - k_1 \Phi^{(2+k_2)}(x, y) + (k_1 - k_2) \Phi^{(2)}(x, y)) - \\
& - \frac{k_1 k_2}{k_1 - k_2} \left( x_i \frac{\partial}{\partial x_k} - \frac{\rho^2 - r^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \right) (\Phi^{(2+k_1)}(x, y) - \\
& \left. - \Phi^{(2+k_2)}(x, y) \right) F_k(y) d_y S + \varepsilon_{ijk} a_j x_k, \tag{2.115}
\end{aligned}$$

where  $a_1, a_2, a_3$  are arbitrary constants.

### 2.11. Properties of the Constructed Solution of Problem (III)<sup>+</sup> in the Closed Domain $\bar{B}^+$ .

**Theorem 2.19.** *If  $l \in C(S)$ ,  $g \in C^1(S)$  (i.e.,  $F \in C(S)$ ), then the inclusion  $u \in C(\bar{B}^+)$  holds for  $u$  defined by (2.115).*

*Proof.* Using (2.110)-(2.112) and the inclusion  $\chi \in C(\bar{B}^+)$  (see (2.77)) it is easy to show that  $\overset{1}{u} \in C(\bar{B}^+)$ , where  $\overset{1}{u} = u - \overset{2}{u}$ ,

$$\overset{2}{u}_i(x) = -\frac{1}{8\pi\rho} \int_S x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) F_k(y) d_y S.$$

Due to (2.83), (2.86) the function  $\overset{2}{u}_i$  can be rewritten as

$$\begin{aligned}
\overset{2}{u}_i(x) = & -\frac{1}{8\pi\rho} \int_S (x_k - y_k) \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) F_k(y) d_y S + \\
& + \frac{1}{4\pi} \int_S \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) g(y) d_y S.
\end{aligned}$$

By virtue of the estimates

$$\begin{aligned}
\left| (x_k - y_k) \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) \right| & \leq \frac{c}{|x - y|}, \\
\left| \frac{\partial}{\partial x_i} \left( \Phi^{(2)}(x, y) - \frac{2}{|x - y|} \right) \right| & \leq \frac{c}{|x - y|}
\end{aligned}$$

and Theorem 1.1 we find that  $\overset{2}{u} \in C(\bar{B}^+)$ . ■

**Theorem 2.20.** *If  $l \in C^{0,\gamma}(S)$ ,  $g \in C^{1,\gamma}(S)$ ,  $0 < \gamma \leq 1$ , then  $u$  defined by (2.115) satisfies the condition  $\mathcal{D}(x \cdot u) \in C(\bar{B}^+ \setminus \{0\})$ , and the stress  $\tau^{(n)}$  corresponding to this displacement  $u$  satisfies the condition  $\tau^{(n)} \in C(\bar{B}^+ \setminus \{0\})$ . Moreover, the second condition of (2.6) and condition (2.85') are equivalent.*

*Proof.* It is easy to verify that the inclusion  $\mathcal{D}(x \cdot u) \in C(\bar{B}^+ \setminus \{0\})$  will be proved if  $\frac{\partial}{\partial x_i}(x \cdot u) \in C(\bar{B}^+)$ , but for this, by virtue of (2.110) and (2.113) it suffices to prove the inclusion  $G_i \in C(\bar{B}^+)$ , where

$$G_i(x) = \int_S \left( x_k \frac{\partial}{\partial x_i} \frac{\rho^2 - |x|^2}{|y-x|^3} + \frac{\mu}{\lambda + 2\mu} (\rho^2 - |x|^2) \frac{\partial^2}{\partial x_i \partial x_k} \frac{1}{|x-y|} \right) F_k(y) d_y S.$$

Taking into account (2.86) and the equality

$$\begin{aligned} (x_k - y_k) \frac{\partial}{\partial x_i} \frac{\rho^2 - |x|^2}{|y-x|^3} &= -(\rho^2 - |x|^2) \frac{\partial^2}{\partial x_i \partial x_k} \frac{1}{|x-y|} - \\ &\quad - \frac{\rho^2 - |x|^2}{|y-x|^3} \delta_{ik} + 2x_i \frac{\partial}{\partial x_k} \frac{1}{|x-y|}, \end{aligned}$$

we have

$$\begin{aligned} G_i(x) &= -\frac{\lambda + \mu}{\lambda + 2\mu} (\rho^2 - |x|^2) \int_S \frac{\partial^2}{\partial x_i \partial x_k} \frac{1}{|x-y|} F_k(y) d_y S - \\ &\quad - \int_S \frac{\rho^2 - |x|^2}{|y-x|^3} F_i(y) d_y S + 2x_i \int_S \frac{\partial}{\partial x_k} \frac{1}{|x-y|} F_k(y) d_y S - \\ &\quad - 2\rho \int_S \frac{\partial}{\partial x_i} \frac{\rho^2 - |x|^2}{|y-x|^3} g(y) d_y S. \end{aligned}$$

Hence due to (2.79), Theorem 1.1 and the property of the Poisson integral we easily conclude that  $G_i \in C(\bar{B}^+)$ .

Now we shall prove the second part of the theorem. By (2.3)

$$\tau^{(n)}(x) = \lambda n(x) \operatorname{div} u(x) + \frac{\mu}{r} (H(\partial_x)u(x) + 2 \operatorname{grad}(x \cdot u(x))),$$

where  $H(\partial_x)$  is defined by (2.84).

On account of (2.115)

$$\begin{aligned} H_{ik}(\partial_x)u_k(x) &= \frac{1}{4\pi\rho} \int_S \left( \frac{\rho^2 - |x|^2}{|y-x|^3} \delta_{ik} + \right. \\ &\quad \left. + \frac{k_1 k_2}{k_1 - k_2} \frac{\rho^2 - |x|^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y)) \right) F_k(y) d_y S, \\ \operatorname{div} u(x) &= -\frac{1}{8\pi\rho} \int_S \left( \frac{3y_k}{\rho^3} + \frac{1}{k_1 - k_2} \frac{\partial}{\partial x_k} ((1 + 2k_1)\Phi^{(2+k_1)}(x, y) - \right. \end{aligned}$$

$$-(1 + 2k_2)\Phi^{(2+k_2)}(x, y)) F_k(y) d_y S.$$

Due to estimate (2.110), Theorem 1.1 and the property of the Poisson integral we conclude that  $H(\partial_x)u \in C(\bar{B}^+)$ ,  $\operatorname{div} u \in C(\bar{B}^+)$ . Hence we easily obtain the proof of the second part of the theorem.

Using the equality

$$\begin{aligned} \tau^{(n)}(x) - n(x)(n(x) \cdot \tau^{(n)}(x)) &= \frac{\mu}{r} H(\partial_x)u(x) + \\ &+ 2\mu \mathcal{D}(n \cdot u)(x) + \frac{2\mu}{r} (n(x) \cdot u(x))n(x), \end{aligned}$$

Theorem 2.19 and the proved parts of Theorem 2.20, it is easy to establish the validity of the second part of the theorem. ■

Thus, using Theorems 2.20 and 2.19 and the fact that a uniqueness theorem similar to Theorem 2.1 holds for a classical solution, we obtain

**Theorem 2.21.** *If  $l \in C^{0,\gamma}(S)$ ,  $g \in C^{1,\gamma}(S)$ ,  $0 < \gamma \leq 1$  and condition (2.9) is satisfied, then  $u$  defined by (2.115) is a classical solution of Problem (III)<sup>+</sup>. The difference between any two classical solutions of this problem can be equal only to the rigid rotation vector  $\mathring{u}(x) \equiv [a \times x]$ .*

Keeping in mind Theorem 2.21 and the investigation of Problem (III)<sup>+</sup> by the methods of a potential and integral equations (Kupradze *et al.* [1]), it is easy to verify that under the conditions of Theorem 2.21 the function  $u$  defined by (2.115) is also a regular solution of Problem (III)<sup>+</sup>.

When solving Problem (III)<sup>+</sup>, we do not manage to weaken the conditions imposed on the boundary functions as it is done during the investigation of this problem for arbitrary domains (Kupradze *et al.* [1]) and as we have succeeded in doing for Problem (I)<sup>+</sup> and (II)<sup>+</sup>.

**Problem 6.** Find out whether the conclusion of Theorem 2.21 is valid or not when  $l \in C(S)$  and  $g \in C^1(S)$ .

**2.12. Two Other Representations of Solutions of Problem (III)<sup>+</sup>.** The disadvantage of the solution of Problem (III)<sup>+</sup> represented by formula (2.115) is that its density contains not the given functions  $g$  and  $l$  (see (2.7)) but the function  $F = (F_1, F_2, F_3)$  which is defined by formula (2.83) and contains a derivative of the given function  $g$ . We must get rid of this drawback, since this lessens the effectiveness of representation (2.115).

It is easy to show that

$$y_m \mathcal{D}_k(g)(y) = \mathcal{D}_k(y_m g)(y) - \left( \delta_{km} - \frac{y_k y_m}{\rho^2} \right) g(y).$$

Hence due to (1.12) we have

$$\int_S y_m \mathcal{D}_k(g)(y) d_y S = \frac{3}{\rho^2} \int_S y_k y_m g(y) d_y S - \delta_{km} \int_S g(y) d_y S.$$

Therefore

$$\int_S (x \cdot y) \mathcal{D}(g)(y) d_y S = \frac{3}{\rho^2} \int_S (x \cdot y) y g(y) d_y S - x \int_S g(y) d_y S, \quad (2.116)$$

$$\int_S \left( \frac{1}{\rho} + \frac{3xy}{\rho^3} \right) \mathcal{D}(g)(y) d_y S = \int_S \left( \frac{9(x \cdot y)}{\rho^5} y + \frac{2y - 3x}{\rho^3} \right) g(y) d_y S. \quad (2.117)$$

By virtue of (1.12), (1.13), (2.116) and the equality

$$\forall x \in B^+, \forall y \in S : \frac{\partial}{\partial n} \frac{1}{|y-x|} = -\frac{1}{2\rho} \frac{\rho^2 - |x|^2}{|y-x|^3} - \frac{1}{2\rho} \frac{1}{|y-x|},$$

we have

$$\begin{aligned} \int_S \left( \frac{1}{|y-x|} - \frac{1}{\rho} - \frac{x \cdot y}{\rho^3} \right) \mathcal{D}_k(g)(y) d_y S &= \int_S \left( \frac{3y_k}{2\rho^2|x-y|} - \right. \\ &\left. - \frac{x_k - y_k}{|x-y|^3} - \frac{y_k}{2\rho^2} \frac{\rho^2 - |x|^2}{|y-x|^3} - \frac{3y_k(x \cdot y)}{\rho^5} + \frac{x_k - 2y_k}{\rho^3} \right) g(y) d_y S. \end{aligned} \quad (2.118)$$

Consider the expression

$$\begin{aligned} &\int_S \Phi^{(m)}(x, y) \mathcal{D}_k(g)(y) d_y S = \\ &= \int_0^1 \left( \int_S \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} - \frac{3x \cdot y \tau}{\rho^3} \right) \mathcal{D}_k(g)(y) d_y S \right) \frac{d\tau}{\tau^m} = \\ &= \int_0^1 (2D_r + 1) \int_S \left( \frac{1}{|y - \tau x|} - \frac{1}{\rho} - \frac{x \cdot y}{\rho} \tau \right) \mathcal{D}_k(g)(y) d_y S \frac{d\tau}{\tau^m}. \end{aligned}$$

Hence, taking into account (2.118) and replacing therein  $x$  by  $\tau x$ , we obtain

$$\int_S \Phi^{(m)}(x, y) \mathcal{D}_k(g)(y) d_y S = \int_S P_k^{(m)}(x, y) g(y) d_y S, \quad (2.119)$$

where

$$P_k^{(m)}(x, y) = (2D_r + 1) \int_0^1 \left( \frac{y_k - \tau x_k}{|y - \tau x|^3} - \frac{y_k}{2\rho^2} \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} + \right.$$

$$+\frac{3y_k}{2\rho^2|y-\tau x|}-\frac{3y_k(x\cdot y)\tau}{\rho^5}+\frac{\tau x_k-2y_k}{\rho^3}\frac{d\tau}{\tau^m}. \quad (2.119')$$

Finally, using (2.83), (2.117), (2.119), for  $u$  defined by (2.115) we have

$$\begin{aligned} u_i(x) &= \frac{1}{8\pi\rho} \int_S \left( \left( \Phi^{(2)}(x, y) - \frac{1}{\rho} - \frac{3x\cdot y}{\rho^3} \right) \delta_{ik} - \right. \\ &\quad - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) + \frac{r^2}{k_1 - k_2} \frac{\partial^2}{\partial x_i \partial x_k} (k_2 \Phi^{(2+k_1)}(x, y) - \\ &\quad - k_1 \Phi^{(2+k_2)}(x, y) + (k_1 - k_2) \Phi^{(2)}(x, y)) - \frac{k_1 k_2}{k_1 - k_2} \left( x_i \frac{\partial}{\partial x_k} - \right. \\ &\quad \left. - \frac{\rho^2 - |x|^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \right) (\Phi^{(2+k_1)}(x, y) - \\ &\quad \left. - \Phi^{(2+k_2)}(x, y) \right) \left( \frac{\rho}{\mu} l_k(y) - \frac{2y_k}{\rho} g(y) \right) d_y S - \\ &\quad - \frac{1}{4\pi} \int_S \left( \left( P_k^{(2)}(x, y) - \frac{2y_k - 3x_k}{\rho^3} - \frac{9(x\cdot y)y_k}{\rho^5} \right) \delta_{ik} - \right. \\ &\quad - x_k \frac{\partial}{\partial x_i} P_k^{(2)}(x, y) + \frac{r^2}{k_1 - k_2} \frac{\partial^2}{\partial x_i \partial x_k} (k_2 P_k^{(2+k_1)}(x, y) - \\ &\quad - k_1 P_k^{(2+k_2)}(x, y) + (k_1 - k_2) P_k^{(2)}(x, y)) - \\ &\quad - \frac{k_1 k_2}{k_1 - k_2} \left( x_i \frac{\partial}{\partial x_k} - \frac{\rho^2 - |x|^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \right) (P_k^{(2+k_1)}(x, y) - \\ &\quad \left. - P_k^{(2+k_2)}(x, y) \right) g(y) d_y S + \varepsilon_{ijk} a_j x_k. \end{aligned} \quad (2.120)$$

This completes the solution of Problem (III)<sup>+</sup>.

Formula (2.120) contains only the given functions  $g$  and  $l$ , but to derive it we had to perform rather lengthy calculations. The same type representation of the solution can be obtained by combining our method with that of Hadamard [1]. In the latter approach we do not have to transfer derivatives from the kernel to the density and, conversely, from the density to the kernel as we did in deriving (2.120). Below we shall discuss the solution of Problem (III)<sup>+</sup> with the aid of Hadamard formulas.

The solution of Problem (III)<sup>+</sup>, i.e. of Problem (2.1), (2.6), can be written as the sum of  $\overset{1}{u}$  and  $\overset{2}{u}$ , where  $\overset{1}{u}$  is the solution of this problem when  $g = 0$  and  $l$  is an arbitrary function;  $\overset{2}{u}$  is the solution when  $l = 0$  and  $g$  is an arbitrary function. Denote the former problem by (III)<sub>0, l</sub><sup>+</sup> and the latter by (III)<sub>g, 0</sub><sup>+</sup>.

The solution of Problem (III)<sub>0, l</sub><sup>+</sup> is evidently given by formula (2.115). However, in the latter formula by virtue of (2.83) the expression  $F$  no longer

contains derivatives of the given functions. In that case  $F_i = l_i \rho / \mu$  and (2.115) takes the form

$$\begin{aligned} \overset{1}{u}_i(x) = & \frac{1}{8\pi\mu} \int_S \left( \left( \Phi^{(2)}(x, y) - \frac{1}{\rho} - \frac{3x \cdot y}{\rho^3} \right) \delta_{ik} - \right. \\ & - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) + \frac{r^2}{k_1 - k_2} \frac{\partial^2}{\partial x_i \partial x_k} (k_2 \Phi^{(2+k_1)}(x, y) - \\ & - k_1 \Phi^{(2+k_2)}(x, y) + (k_1 - k_2) \Phi^{(2)}(x, y)) - \\ & - \frac{k_1 k_2}{k_1 - k_2} \left( x_i \frac{\partial}{\partial x_k} - \frac{\rho^2 - r^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \right) (\Phi^{(2+k_1)}(x, y) - \\ & \left. - \Phi^{(2+k_2)}(x, y) \right) l_k(y) d_y S + \varepsilon_{ijk} a_j x_k. \end{aligned} \quad (2.121)$$

Problem  $(\text{III})_{g,0}^+$  was solved by Hadamard in [1], but the condition  $l = 0$  is rather restrictive. The solution can be expressed by only one scalar function  $\zeta$ :

$$\begin{aligned} \overset{2}{u}(x) = & x \left( \frac{\lambda + 3\mu}{\mu} \mathcal{D}_r + 1 \right) (\mathcal{D}_r - 1) \zeta(x) - r^2 \text{grad} \left( \mathcal{D}_r - \frac{3\lambda + 4\mu}{2\mu} \right) \zeta(x) + \\ & + \frac{\rho^2 - r^2}{2} \text{grad} \left( \frac{\lambda + \mu}{\mu} \mathcal{D}_r^2 + 2 \frac{\lambda + \mu}{\mu} \mathcal{D}_r - 1 \right) \zeta(x), \end{aligned} \quad (2.122)$$

where  $\zeta$  is a harmonic function in  $B^+$  and satisfies the equation

$$\left( \mathcal{D}_r^2 + \frac{\lambda}{2(\lambda + 2\mu)} \mathcal{D}_r - \frac{\mu}{\lambda + 2\mu} \right) \zeta(x) = \frac{\mu}{(\lambda + 2\mu)\rho^2} H(x), \quad (2.123)$$

and  $H$  is a harmonic function in  $B^+$  satisfying the condition  $H^+ = g$ .

Hadamard's consideration is limited to the derivation of formulas (2.122) and (2.123); he notes that Problem  $(\text{III})_{g,0}^+$  is solved by (1.122) and (1.123) to within an additive vector of rigid displacement. There are no other results in Hadamard [1].

To establish the solution uniqueness and to study the differential properties in the closed domain  $\bar{B}^+$  it is necessary to solve equation (2.123) and to investigate the obtained solution. Obviously,  $H$  is given by the Poisson formula  $H = \Pi(g)$ . Taking the latter fact into account, equation (2.123) can be expressed in form (2.98) (see (2.91), (2.95)-(2.98)).

**Problem 7.** Construct the solution of Problem  $(\text{III})^-$  in quadratures.

**2.13. Solution of Problem  $(\text{IV})^+$ .** This problem is solved by the same procedure we used to solve Problem  $(\text{III})^+$ . First, the boundary condition of Problem  $(\text{IV})^+$  (see (2.7))

$$(u - n(n \cdot u))^+ = l, \quad (2.124)$$

$$\left(n \cdot \tau^{(n)}\right)^+ = g \quad (2.125)$$

is replaced by its equivalent form.

In (2.125) instead of  $\tau^{(n)}$  we put its expression from (2.3) and obtain (the summation sign is omitted)

$$\begin{aligned} n \cdot \tau^{(n)} &= \lambda \operatorname{div} u + 2\mu n \frac{\partial u}{\partial n} = (\lambda + 2\mu) \operatorname{div} u - \\ &\quad - 2\mu \mathcal{D}_k(u_k - n_k(n \cdot u)) - 2\mu \mathcal{D}_k(n_k(n \cdot u)), \end{aligned}$$

where  $\mathcal{D}_k$  is the Gunter differential operator which on account of the equalities  $\mathcal{D}_k(n_k) = 2/\rho$ ,  $n_k \mathcal{D}_k = 0$  takes the form

$$n \cdot \tau^{(n)} = (\lambda + 2\mu) \operatorname{div} u - \frac{4\mu}{\rho}(n \cdot u) - 2\mu \mathcal{D}_k(u_k - n_k(n \cdot u)).$$

Therefore

$$\left((\lambda + 2\mu) \operatorname{div} u - \frac{4\mu}{\rho^2}(y \cdot u)\right)^+(y) = h(y), \quad h = g + 2\mu \mathcal{D}_k(l_k). \quad (2.126)$$

Thus instead of the boundary conditions (2.124), (2.125) we have conditions (2.124), (2.126).

The solution of Problem (IV)<sup>+</sup> is sought for in form (2.18):

$$\begin{aligned} u(x) &= v(x) + x(2\mathcal{D}_r + 1)\varphi(x) + (\rho^2 - r^2) \operatorname{grad} \varphi(x) + \\ &\quad + \frac{\rho^2 - r^2}{2} \operatorname{grad} \psi(x), \end{aligned} \quad (2.127)$$

where  $v$ ,  $\varphi$  and  $\psi$  are the wanted continuous functions in the domain  $B^+$ , satisfying the equations (see (2.14))

$$\Delta v = 0, \quad \Delta \varphi = 0, \quad \Delta \psi = 0, \quad (2.128)$$

$$(\mathcal{D}_r + \alpha)\psi = \beta(\operatorname{div} v + 2\mathcal{D}_r^2 \varphi + 5\mathcal{D}_r \varphi + 3\varphi). \quad (2.128')$$

Note that representation (2.127) contains five harmonic functions  $v_1$ ,  $v_2$ ,  $v_3$ ,  $\varphi$  and  $\psi$  interconnected by the same relation (2.128'). Thus the degree of freedom of harmonic functions is equal to four while in other representations (see (2.12) and (2.16)) it is equal to three. Therefore, when using representation (2.127), an additional relation, say,

$$(n \cdot v)^+ = 0 \quad (2.129)$$

has to be introduced to reduce the degree of freedom to three.

By virtue of

$$\begin{aligned} u_i(x) - \frac{x_i}{r^2} x_k u_k(x) &= v_i(x) - \frac{x_i}{r^2} x_k v_k(x) + \\ &\quad + \frac{\rho^2 - r^2}{2} \left( \frac{\partial}{\partial x_i} - \frac{x_i}{r} \frac{\partial}{\partial r} \right) (\psi(x) + 2\varphi(x)), \end{aligned}$$

from (2.129) and (2.124) we have  $v^+ = l$ ,  $\Delta v = 0$ , and  $v$  is therefore given by the Poisson formula  $v = \Pi(l)$ .

Note that condition (2.129) is fulfilled by virtue of (2.9). Thus the vector  $v$  is found.

Let us now find the functions  $\varphi$  and  $\psi$ . By virtue of (2.128')

$$\operatorname{div} u = \frac{\mu}{\lambda + \mu} (2\mathcal{D}_r + 1)\psi.$$

Moreover,

$$x \cdot u(x) = x \cdot v(x) + r^2(2\mathcal{D}_r + 1)\varphi(x) + \frac{\rho^2 - r^2}{2} \mathcal{D}_r(\psi(x) + 2\varphi(x)).$$

Therefore

$$\begin{aligned} (\lambda + 2\mu) \operatorname{div} u(x) - \frac{4\mu}{r^2}(x \cdot u(x)) &= \zeta(x) - \frac{4\mu}{r^2}(x \cdot v(x)) - \\ &- 2\mu \frac{\rho^2 - r^2}{r^2} \mathcal{D}_r(\psi(x) + 2\varphi(x)), \end{aligned} \quad (2.130)$$

where

$$\zeta(x) = (2\mathcal{D}_r + 1) \left( \frac{(\lambda + 2\mu)\mu}{\lambda + \mu} \psi(x) - 4\mu\varphi(x) \right)$$

and from the conditions  $\Delta\varphi = 0$ ,  $\Delta\psi = 0$  we obtain the equality  $\forall x \in B^+$  :  $\Delta\zeta(x) = 0$ , from conditions (2.126), (2.130) the equality  $\zeta^+ = h$ . Therefore  $\zeta = \Pi(h)$ .

Hence

$$(2\mathcal{D}_r + 1)\chi = \gamma\Pi(h), \quad (2.131)$$

$$\begin{aligned} \chi &= \psi - \eta\varphi, \quad \eta = 4(\lambda + \mu)(\lambda + 2\mu)^{-1}, \\ \gamma &= (\lambda + \mu)(\mu(\lambda + 2\mu))^{-1}. \end{aligned} \quad (2.131')$$

Treating (2.131) as a differential equation with respect to the harmonic function  $\chi$ , we obtain its unique solution and write it in the form

$$\chi(x) = \frac{\gamma}{2} \int_0^1 \Pi(h)(\tau x) \frac{d\tau}{\sqrt{\tau}} = \frac{\gamma}{4\pi\rho} V(h)(x),$$

where  $V$  is a simple layer potential (see 1.6). Hence

$$\psi(x) = \eta\varphi(x) + \frac{\gamma}{4\pi\rho} V(h)(x). \quad (2.132)$$

Substituting  $\psi$  and  $v = \Pi(l)$  in (2.128'), we obtain a differential equation

$$2\mathcal{D}_r^2\varphi + \frac{\lambda - 2\mu}{\lambda + 2\mu} \mathcal{D}_r\varphi + \frac{3\lambda + 2\mu}{\lambda + 2\mu} \varphi = F, \quad (2.133)$$

where

$$F = -\operatorname{div} \Pi(l) + \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \Pi(h) - \frac{\lambda + \mu}{8\pi\rho\mu(\lambda + 2\mu)} V(h).$$

Rewrite this equation in an equivalent form

$$2\mathcal{D}_r^2(\varphi - c) + \frac{\lambda - 2\mu}{\lambda + 2\mu} \mathcal{D}_r(\varphi - c) + \frac{3\lambda + 2\mu}{\lambda + 2\mu} (\varphi - c) = F', \quad (2.134)$$

where  $c = F(0)(\lambda + 2\mu)(3\lambda + 2\mu)^{-1}$ ,  $F'(x) = F(x) - F(0)$ ,

$$F(0) = \frac{1}{4\pi\rho^2(\lambda + 2\mu)} \int_S g(y) d_y S.$$

In equation (2.134), replacing the variable  $r$  by  $t$  defined by the formula  $t = \ln r$ , we obtain

$$2\frac{\partial^2(\varphi - c)}{\partial t^2} + \frac{\lambda - 2\mu}{\lambda + 2\mu} \frac{\partial(\varphi - c)}{\partial t} + \frac{3\lambda + 2\mu}{\lambda + 2\mu} (\varphi - c) = F',$$

whose characteristic equation

$$2k^2 + \frac{\lambda - 2\mu}{\lambda + 2\mu} k + \frac{3\lambda + 2\mu}{\lambda + 2\mu} = 0$$

has the roots

$$\frac{-\lambda + 2\mu \pm \sqrt{-23\lambda^2 - 68\lambda\mu - 28\mu^2}}{4(\lambda + 2\mu)}.$$

In common with Problem (II)<sup>+</sup> we shall now consider separately three cases:

$$-\frac{2}{3}\mu < \lambda < \frac{16\sqrt{2} - 34}{23}\mu, \quad \lambda = \frac{16\sqrt{2} - 34}{23}\mu, \quad \lambda > \frac{16\sqrt{2} - 34}{23}\mu.$$

Below a detailed consideration will be given to the case  $\lambda > \frac{16\sqrt{2} - 34}{23}\mu$  and  $\mu > 0$ . A similar treatment can be applied to the other cases.

Thus in the considered case the characteristic equation has the roots  $k_1 + ik_2$  and  $k_1 - ik_2$ , where

$$k_1 = \frac{2\mu - \lambda}{4(\lambda + 2\mu)}, \quad k_2 = \frac{\sqrt{23\lambda^2 + 68\lambda\mu + 28\mu^2}}{4(\lambda + 2\mu)}, \quad -\frac{1}{2} < k_1 < \frac{1}{2}, \quad k_2 > 0.$$

Now, repeating the arguments used in solving equation (2.61), equation (2.133) can be rewritten as

$$\begin{aligned} \varphi(x) = & \int_S (\alpha_1 \operatorname{div} \Psi(x, y) l(y) + (\alpha_2 Q(x, y) + \\ & + \alpha_3 M(x, y) + \alpha_4) h(y)) d_y S, \end{aligned} \quad (2.135)$$

where  $\Psi$  is given by (2.69),

$$\begin{aligned}
Q(x, y) &= \frac{1}{k_2} \int_0^1 L_0(\tau x, y) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{1+k_1}}, \\
M(x, y) &= \frac{1}{k_2} \int_0^1 L(\tau x, y) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{1+k_1}}, \\
L(x, y) &= \frac{2}{|x-y|} - \frac{2}{\rho}, \\
L_0(x, y) &= \frac{\rho^2 - |x|^2}{|y-x|^3} - \frac{1}{\rho}, \\
\alpha_1 &= -\frac{1}{8\pi\rho}, \quad \alpha_2 = -\frac{\lambda + 3\mu}{16\pi\rho\mu(\lambda + 2\mu)}, \\
\alpha_3 &= \frac{\lambda + \mu}{32\pi\rho\mu(\lambda + 2\mu)}, \quad \alpha_4 = \frac{1}{4\pi\rho^2(3\lambda + 2\mu)}.
\end{aligned}$$

Clearly, to construct the solution of Problem (IV)<sup>+</sup> in form (2.127), besides  $v$  and  $\varphi$ , it suffices to find  $\psi + 2\varphi$ . Due to (2.132) and (2.135) we have

$$\begin{aligned}
\psi(x) + 2\varphi(x) &= \int_S (\beta_1 \operatorname{div} \Psi(x, y) l(y) + (\beta_2 L(x, y) + \\
&\quad + \beta_3 Q(x, y) + \beta_4 M(x, y) + \beta_5 h(y)) d_y S, \tag{2.136}
\end{aligned}$$

where

$$\begin{aligned}
\beta_1 &= -\frac{3\lambda + 4\mu}{4\pi\rho(\lambda + 2\mu)}, \quad \beta_2 = \frac{\lambda + \mu}{8\pi\rho\mu(\lambda + 2\mu)}, \\
\beta_3 &= -\frac{(3\lambda + 4\mu)(\lambda + 3\mu)}{8\pi\rho\mu(\lambda + 2\mu)^2}, \quad \beta_4 = \frac{(\lambda + \mu)(3\lambda + 4\mu)}{16\pi\rho\mu(\lambda + 2\mu)^2}, \\
\beta_5 &= \frac{3\lambda^2 + 11\lambda\mu + 10\mu^2}{4\pi\rho^2\mu(\lambda + 2\mu)(3\lambda + 2\mu)}.
\end{aligned}$$

Thus the solution of Problem (IV)<sup>+</sup> has the form

$$\begin{aligned}
u(x) &= \Pi(l)(x) + x(2\mathcal{D}_r + 1)\varphi(x) + \\
&\quad + \frac{\rho^2 - r^2}{2} \operatorname{grad}(\psi(x) + 2\varphi(x)), \tag{2.137}
\end{aligned}$$

where  $\varphi$  is defined by (2.135) and  $\psi + 2\varphi$  by (2.136).

**Problem 8.** Construct the solution of Problem (IV)<sup>-</sup> in quadratures.

**Problem 9.** Analyze the constructed solution of Problem (IV)<sup>-</sup> and (see (2.137)) prove a theorem similar to Theorem 2.21.

**Problem 10.** Solve Problem (IV)<sup>+</sup> when  $-\frac{2}{3}\mu < \lambda < \frac{16\sqrt{2}-34}{23}\mu$  and  $\lambda = \frac{16\sqrt{2}-34}{23}\mu$ .

**2.14. Analysis of the Solution of Problem (IV)<sup>+</sup>.** Let us discuss some properties of the functions  $Q, L$  and  $M$ . We introduce the notations

$$Q_1(x, y) = - \int_0^1 L_0(\tau x, y) \cos(k_2 \ln \tau) \frac{d\tau}{\tau^{1+k_1}},$$

$$M_1(x, y) = - \int_0^1 L(\tau x, y) \cos(k_2 \ln \tau) \frac{d\tau}{\tau^{1+k_1}}.$$

The following identities

$$(2\mathcal{D}_r + 1)L = 2L_0, \quad (\mathcal{D}_r - k_1)Q = Q_1, \quad (\mathcal{D}_r - k_1)M = M_1$$

are valid.

Using identities (2.29), (2.29'), transform  $Q, Q_1, M$  and  $M_1$  as follows:

$$Q(x, y) = \frac{2\delta(x, y)}{\rho} + \int_{1/2}^1 \delta(\tau x, y) \varphi_1(\tau) d\tau + c_1(x, y),$$

$$Q_1(x, y) = -\frac{2}{|x-y|} + \frac{(\lambda + 6\mu)\delta(x, y)}{2(\lambda + 2\mu)\rho} + \int_{1/2}^1 \delta(\tau x, y) \varphi_2(\tau) d\tau + c_2(x, y),$$

$$M(x, y) = \int_{1/2}^1 \delta(\tau x, y) \varphi_3(\tau) d\tau + c_3(x, y),$$

$$M_1(x, y) = \frac{2\delta(x, y)}{\rho} + \int_{1/2}^1 \delta(\tau x, y) \varphi_4(\tau) d\tau + c_4(x, y),$$

where

$$\delta(x, y) \equiv \ln((|x-y| + \rho)^2 - |x|^2),$$

$c_1, \dots, c_4$  and their derivatives with respect to the coordinates of the point  $x$ , as well as  $\varphi_1, \dots, \varphi_4$  are the bounded functions.

Similarly to Theorems 2.19 and 2.20 we prove

**Theorem 2.22.** *If  $g \in C^{0,\gamma}(S)$ ,  $l \in C^{1,\gamma}(S)$ ,  $0 < \gamma \leq 1$ , then the inclusions  $u \in C(\bar{B}^+)$ ,  $\tau^{(n)} \in C(\bar{B}^+ \setminus \{0\})$ ,  $\mathcal{D}_k(u_k - n_k(n \cdot u)) \in C(\bar{B}^+ \setminus \{0\})$  are valid for  $u$  defined by (2.137).*

Now it is easy to prove

**Theorem 2.23.** *If  $g \in C^{0,\gamma}(S)$ ,  $l \in C^{1,\gamma}(S)$ ,  $0 < \gamma \leq 1$  and condition (2.9) is satisfied, then  $u$  defined by (2.137) is the unique classical solution of Problem (IV)<sup>+</sup>.*

**Problem 11.** Find out whether the conclusion of Theorem 2.23 is valid or not when  $g \in C(S)$  and  $l \in C^1(S)$ .

Finally, note that in common with the case of Problem (III)<sup>+</sup> the solution of Problem (IV)<sup>+</sup> written in form (2.137) can be represented in a form where the density contains no derivative of the boundary function.

**2.15. Solution of Problem (V)<sup>+</sup>.** The solution  $u$  of Problem (V)<sup>+</sup> is to be sought for in form (2.12). By virtue of (2.3) and (2.12)

$$\begin{aligned} \tau^{(n)}(x) + \sigma_0 u(x) &= \frac{\mu}{r} \left( h(x) + \frac{\sigma_0}{\mu} r v(x) \right) + \\ &+ \frac{\mu(\rho^2 - r^2)}{r} \mathcal{D}_r \operatorname{grad} \psi(x) + \frac{\sigma_0(\rho^2 - r^2)}{2} \operatorname{grad} \psi(x), \end{aligned} \quad (2.138)$$

where  $h = (h_1, h_2, h_3)$ ,

$$\begin{aligned} h_i(x) &= \frac{\lambda}{\lambda + \mu} x_i \psi(x) - r^2 \frac{\partial \psi(x)}{\partial x_i} + \frac{\lambda - \mu}{\lambda + \mu} x_i \mathcal{D}_r \psi(x) + \\ &+ x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right). \end{aligned} \quad (2.139)$$

Consider the vector  $\zeta(x) = h(x) + \sigma v(x)$ ,  $\sigma = \sigma_0 \rho / \mu$ . Taking into account the relation (2.13), it is easy to verify that  $\Delta \zeta(x) = 0$ .

Moreover, by virtue of (2.138) and the boundary condition (2.8) we have  $\zeta^+ = \rho f / \mu$ .

Therefore  $\zeta$  is given by the Poisson formula. Thus

$$h + \sigma v = \frac{\rho}{\mu} \Pi(f). \quad (2.140)$$

Hence due to relations (2.13) and (2.139) we obtain

$$\begin{aligned} \operatorname{div} h + \sigma \operatorname{div} v &= \frac{\rho}{\mu} \operatorname{div} \Pi(f), \\ \mathcal{D}_r^2 \psi + \frac{4\lambda + 2\mu + \sigma(\lambda + 3\mu)}{2(\lambda + \mu)} \mathcal{D}_r \psi + \frac{3\lambda + (\sigma + 2)\mu}{2(\lambda + \mu)} \psi &= \frac{\rho}{2\mu} \operatorname{div} \Pi(f). \end{aligned} \quad (2.141)$$

Multiplying scalarly (2.140) by  $x$ , we obtain by virtue of (2.139)

$$\begin{aligned} \mathcal{D}_r(x \cdot v(x)) + \frac{\sigma - 2}{2}(x \cdot v(x)) &= \frac{\rho}{2\mu} x \cdot \Pi(f)(x) + \\ &+ \frac{\mu r^2}{\lambda + \mu} \mathcal{D}_r \psi(x) - \frac{\lambda r^2}{2(\lambda + \mu)} r^2 \psi(x). \end{aligned} \quad (2.142)$$

Keeping in mind (2.139), equality (2.140) can be rewritten as

$$x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) + \sigma v_i(x) = q_i(x), \quad (2.143)$$

where  $q = (q_1, q_2, q_3)$

$$q_i(x) = \frac{\rho}{\mu} \Pi(f_i)(x) - \frac{\lambda}{\lambda + \mu} x_i \psi(x) + r^2 \frac{\partial \psi(x)}{\partial x_i} - \frac{\lambda - \mu}{\lambda + \mu} x_i \mathcal{D}_r \psi(x).$$

Applying the operator  $\mathcal{D}_r$  to (2.143), we obtain

$$\begin{aligned} x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) + x_k x_j \left( \frac{\partial^2 v_i(x)}{\partial x_k \partial x_j} + \frac{\partial^2 v_j(x)}{\partial x_k \partial x_i} \right) + \\ + \sigma \mathcal{D}_r v_i(x) = \mathcal{D}_r q_i(x), \end{aligned} \quad (2.144)$$

and a scalar multiplication by  $x$  and differentiation give us

$$\begin{aligned} x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) + x_k x_j \frac{\partial^2 v_j(x)}{\partial x_k \partial x_i} = \\ = -\frac{1}{2} \frac{\partial}{\partial x_i} (\sigma x \cdot v(x) - x \cdot q(x)). \end{aligned} \quad (2.145)$$

Subtracting (2.145) from (2.144), we have

$$\begin{aligned} x_k x_j \frac{\partial^2 v_i(x)}{\partial x_k \partial x_j} + \sigma \mathcal{D}_r v_i(x) = \\ = \mathcal{D}_r q_i(x) + \frac{1}{2} \frac{\partial}{\partial x_i} (\sigma x \cdot v(x) - x \cdot q(x)). \end{aligned} \quad (2.146)$$

But

$$x_k x_j \frac{\partial^2}{\partial x_k \partial x_j} = x_k \frac{\partial}{\partial x_k} x_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_j} = \mathcal{D}_r^2 - \mathcal{D}_r$$

and therefore

$$\mathcal{D}_r^2 v(x) + (\sigma - 1) \mathcal{D}_r v(x) = \mathcal{D}_r q(x) + \frac{1}{2} \text{grad}(\sigma x \cdot v(x) - x \cdot q(x)). \quad (2.147)$$

Now we can find the harmonic function  $\psi$  from (2.141), then  $x \cdot v(x)$  from (2.142) and, finally,  $v$  from (2.147). Thus we have shown that the solution of Problem (V)<sup>+</sup> can be constructed in form (2.12). We can also prove

**Theorem 2.24.** *If the boundary function  $f \in C(S)$ , then the solution constructed in the above manner is the unique classical solution of Problem (V)<sup>+</sup>.*

**Problem 12.** Find in quadratures a continuous solution  $u = (u_1, u_2, u_3)$  of system (2.1) in the domain  $B^+$ , by the boundary conditions:

$$\forall y \in S : u_1^+(y) = f_1(y), \quad (\tau_2^{(n)})^+(y) = f_2(y), \quad (\tau_3^{(n)})^+(y) = f_3(y),$$

where  $f_1, f_2$  and  $f_3$  are the continuous functions given on  $S$ .

**Problem 13.** Find in quadratures a continuous solution  $u = (u_1, u_2, u_3)$  of system (2.1) in the domain  $B^+$ , by the boundary conditions

$$\forall y \in S : u_1^+(y) = f_1(y), \quad u_2^+(y) = f_2(y), \quad (\tau_3^{(n)})^+(y) = f_3(y),$$

where  $f_1, f_2$  and  $f_3$  are the continuous functions given on  $S$ .

*Indication.* For Problems 12 and 13 see Marcolongo [2].

**Problem 14.** Find in quadratures a continuous solution  $u = (u_1, u_2, u_3)$  of system (2.1) in the domain  $B^-$ , by the boundary conditions

$$\forall y \in S : u_1^-(y) = f_1(y), \quad (\tau_2^{(n)})^-(y) = f_2(y), \quad (\tau_3^{(n)})^-(y) = f_3(y)$$

and by the condition at infinity

$$u(x) = o(1).$$

Here  $f_1, f_2$  and  $f_3$  are the continuous functions given on  $S$ .

**Problem 15.** Find in quadratures a continuous solution  $u = (u_1, u_2, u_3)$  of system (2.1) in the domain  $B^-$ , by the boundary conditions

$$\forall y \in S : u_1^-(y) = f_1(y), \quad u_2^-(y) = f_2(y), \quad (\tau_3^{(n)})^-(y) = f_3(y)$$

and by the condition at infinity

$$u(x) = o(1).$$

Here  $f_1, f_2, f_3$  are the continuous functions given on  $S$ .

*Indication.* Problems 12-15 can be solved by the method described in Subsection 2.15.

**Problem 16.** Write out the final form of the solution of Problem (V)<sup>+</sup> and prove that if the boundary function  $f$  is continuous on  $S$ , then the constructed solution  $u$  is the unique classical solution of this problem.

**Problem 17.** Find in quadratures a continuous solution of system (2.1) in the domain  $B^-$ , by the boundary conditions

$$\forall y \in S : (\tau^{(n)} - \sigma_0 u)^-(y) = f(y)$$

and by the condition at infinity

$$u(x) = o(1).$$

Here  $\sigma_0$  is a positive number and  $f$  is the continuous function given on  $S$ .

In Chapter II we have solved in quadratures all the basic boundary value problems of elasticity, which enables us to construct in quadratures the Green tensors (Kupradze *et al.* [1], Ch. VII, 1.2) of the corresponding problems.

The matrix  $G$  defined on the product  $B^+ \times B^+$  is called the Green tensor of Problem (I)<sup>+</sup> if for any fixed  $y$  from  $B^+$  the function  $G(\cdot, y)$  satisfies system (2.1) in the domain  $B^+ \setminus \{y\}$  and the boundary condition

$$\lim_{B^+ \ni x \rightarrow z \in S} G(x, y) = 0.$$

Moreover, the matrix  $G$  is representable in the form

$$G(x, y) = \Gamma(x - y) + v(x, y),$$

where  $\Gamma$  is a matrix of fundamental solutions (see (7.1)), and  $v(\cdot, y)$  is a regular solution of equation (2.1) in the domain  $B^+$ .

**Problem 18.** Construct in quadratures the Green tensor of Problem (I)<sup>+</sup> for the domain  $B^+$ .

**Problem 19.** Construct the Green tensors for Problems (II)<sup>+</sup>, (III)<sup>+</sup>, (IV)<sup>+</sup>, and (V)<sup>+</sup> in the domain  $B^+$ .

*Indication.* For the Green tensor definition see Kupradze *et al.* [1], Chapter VII, 1.2.

\* \* \*

The statement of the problem solved in this chapter (Problems (I)<sup>±</sup> to (V)<sup>±</sup>), as well as the uniqueness theorems and conditions of solvability are given in the monographs Love [1], Muskhelishvili [2], Kupradze *et al.* [1], Fichera [2], Knops, Payne [1], Burchuladze, Gegelia [1], Buchukuri, Gegelia [1, 2] and others. Problem (I)<sup>+</sup> for  $B^+$  was for the first time solved in the well-known memoirs Lamé [1] (1852) after deriving the basic equations of the theory of elasticity. G. Lamé obtained the solution in the form of a series in spherical functions and in spherical coordinates. Lord Kelvin (1863) gave the solution of Problem (I)<sup>+</sup> in the form of a series in Cartesian coordinates (see Thomson [3]). Subsequently, the results of G. Lamé and Lord Kelvin were repeated and used in the specific problems by various researchers, who sometimes represented them in a different form (for more detailed relevant information see Grinchenko, Ulitko [1]). Evidently the first work where the solution of Problem (I)<sup>+</sup> was obtained in quadratures was that by Borchardt [1] (see also Cerruti [1], Tedone [1], Somigliana [1])). Using representation (2.12), R. Marcolongo gave a simple derivation of formula (2.24) without making a recourse to series (see Marcolongo [1]). The solutions of Problem (I)<sup>±</sup> were constructed also in Love [1], Trefftz [1], Lurie [1], Muskhelishvili [2], Natroshvili [1] and others.

In Thomson [1], Love [1], Trefftz [1] a method is indicated for constructing the solution of Problem (II)<sup>+</sup> with the aid of series. In Lurie [1, 2] the solution is constructed in the form of a series in spherical functions. All the above-mentioned works contain a formal discussion as to the convergence of series without verifying or indicating the necessary conditions for the validity of the results. In Natroshvili [1] the summation of a series representing the solution is performed. D. Natroshvili's formula is somewhat more complicated than representation (2.70).

Problem (III)<sup>+</sup> was solved by J. Hadamard for one particular case when tangential components of the stress vector  $S$  are zero (see Hadamard [1]). This restriction is essential for J. Hadamard's method.

Problems (III)<sup>±</sup> and (VI)<sup>±</sup> are solved by means of series in Natroshvili [2]. By the summation of series D. Natroshvili obtains the solutions in quadratures, but these formulas differ essentially from the ones derived in the this book.

The mentioned problems, as well as particular problems for ball having applications are treated in Bacheleishvili [1-3], Kupradze et al. [1], Love [1], Muskhelishvili [1], Lurie [1], Weber [1], Sternberg, Rosenthal [1], Fichera [1], Timoshenko, Goodier [1], Natroshvili [3,4], Galerkin [1], Giorgashvili [2].

CHAPTER II  
BOUNDARY VALUE PROBLEMS  
FOR A POLYHARMONIC EQUATION

**3.1. Formulation of the Problems. Auxiliary Formulas.** Consider a polyharmonic equation

$$\Delta^{\nu+1}u = 0, \quad \nu = 1, 2, \dots, \quad (3.1)$$

where  $\Delta^{\nu+1} = \Delta(\Delta^\nu)$ ,  $\Delta^1 = \Delta$  is the Laplace operator.

It is well-known (see, for example, Petrovski [1], Courant [1], Miranda [1], Bers, John, Schechter [1]) that any continuous function having in the domain  $\Omega \in \mathbb{R}^m$  all derivatives from the expression  $\Delta^{\nu+1}$  and satisfying equation (3.1) is analytic in  $\Omega$ . We will call it a polyharmonic function of order  $\nu + 1$ .

Note that for our purpose it is quite sufficient to call a function polyharmonic of order  $\nu + 1$ , if in  $\Omega$  it has all continuous partial derivatives up to order  $2(\nu + 1)$  inclusive and satisfies equation (3.1).

The boundary value problems for a polyharmonic equation were investigated by quite a number of researchers (see Nicolesco [1], Vekua [1,3], Miranda [1] and the references cited therein). Here we are going to consider the boundary value problems of three types for the three-dimensional ball. As will be clear from the reasoning below, all our results apply as well to an arbitrary  $m$ -dimensional ball.

**Problem (I)<sup>+</sup>.** Find a polyharmonic function  $u$  of the  $\nu + 1$ -th order in the ball  $B^+$  by the boundary conditions

$$\forall y \in S : \left( \frac{d^k u}{dn^k} \right)^+ (y) = f_k(y), \quad k = 0, 1, \dots, \nu, \quad (3.2)$$

where  $f_0, \dots, f_\nu$  are the functions given on  $S$ .

As previously, it is assumed that  $B^+$  is a three-dimensional ball with centre at the origin and radius  $\rho$ , and  $S$  is its boundary, i.e., the sphere with centre at the origin and radius  $\rho$ .

**Problem (II)<sup>+</sup>.** Find a polyharmonic function  $u$  of the  $\nu + 1$ -th order in the ball  $B^+$  by the boundary conditions

$$\forall y \in S : (\Delta^k u)^+ (y) = f_k(y), \quad k = 0, 1, \dots, \nu. \quad (3.3)$$

**Problem (III)<sup>+</sup>.** Find a polyharmonic function  $u$  of the  $\nu + 1$ -th order in the ball  $B^+$  by the boundary conditions:

$$\begin{aligned} \forall y \in S : \left( \frac{d^h u}{dn^h} \right)^+ (y) &= f_h(y), \quad h = 0, 1, \dots, \varkappa, \quad 1 \leq \varkappa < \nu, \\ \forall y \in S : (\Delta^q u)^+ (y) &= f_q(y), \quad q = \varkappa + 1, \dots, \nu. \end{aligned} \quad (3.4)$$

Problem (I)<sup>+</sup> is also referred to as the Lauricella problem, Problem (II)<sup>+</sup> as the Riquier problem (Vekua [3], Riquier [1]). The problems with zero

data (for  $f_0 = 0, \dots, f_\nu = 0$ ) will be called homogeneous and denoted by the symbols  $(I)_0^+$ ,  $(II)_0^+$ ,  $(III)_0^+$ .

In solving these problems, we will use the representation of a polyharmonic function by harmonic functions. For a polyharmonic function of the  $\nu + 1$ -th order we have the representation (Vekua [3], Nicolesco [1]) known as the Almansi formula

$$u = \sum_{p=0}^{\nu} r^{2p} u_p,$$

where  $u_0, \dots, u_\nu$  are harmonic functions.

Obviously, this representation also yields, in  $B^+$ , the representation

$$u(x) = \sum_{p=0}^{\nu} a_p (r^2 - \rho^2)^p u_p(x), \quad (3.5)$$

where  $a_1, \dots, a_p$  are arbitrary constants,  $r \equiv |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ .

We will give some formulas needed for our further reasoning. It is easy to verify that

$$\Delta \mathcal{D}_r = (\mathcal{D}_r + 2)\Delta,$$

and therefore for the harmonic function  $\psi$  we have the equalities

$$\Delta \mathcal{D}_r^k \psi(x) = \mathcal{D}_r^k \Delta \psi(x) = 0, \quad (3.6)$$

$$\Delta \mathcal{D}_r^{(k)} \psi(x) = \mathcal{D}_r^{(k)} \Delta \psi(x) = 0, \quad (3.7)$$

$$\Delta \mathcal{D}_r^{[k]} \psi(x) = \mathcal{D}_r^{[k]} \Delta \psi(x) = 0, \quad (3.8)$$

where  $\mathcal{D}_r^k, \mathcal{D}_r^{(k)}, \mathcal{D}_r^{[k]}$  are the operators defined in Subsection 1.2.

The method of mathematical induction enables one to easily prove the identities

$$r^k \frac{\partial^k}{\partial r^k} = \mathcal{D}_r^{(k)}, \quad k = 1, 2, \dots; \quad (3.9)$$

$$\frac{d^h r^{2i}}{dr^h} = \frac{(2i)!}{(2i-h)!} r^{2i-h}, \quad i, h = 0, 1, \dots, \quad h \leq 2i; \quad (3.10)$$

$$\Delta^k (r^{2i} \psi(x)) = 4^k i(i-1) \dots (i-k+1) r^{2(i-k)} \mathcal{D}_r^{[i]-[i-k]} \psi(x), \quad (3.11)$$

$$i, k = 1, 2, \dots, \quad k \leq i,$$

where  $\psi$  is a harmonic function.

Using the Leibnitz formulas

$$\frac{d^k (\varphi \psi)}{dr^k} = \sum_{j=0}^k C_k^j \frac{d^{k-j} \varphi}{dr^{k-j}} \frac{d^j \psi}{dr^j} \quad \left( C_k^j = \frac{k!}{j!(k-j)!} \right)$$

and the Newton formula, it is easy to show due to (3.9), (3.10) that

$$\begin{aligned} & r^k \frac{\partial^k}{\partial r^k} (r^2 - \rho^2)^p u_p(x) = \\ & = \sum_{j=0}^k \sum_{i=0}^p C_k^j C_p^i \frac{(-1)^{p-i} (2i)!}{(2i-k+j)!} \rho^{2(p-i)} r^{2i} \mathcal{D}_r^{(j)} u_p(x). \end{aligned} \quad (3.12)$$

Hence, taking into account the equality (Prudnikov, Brichkov, Marichev [1]),

$$\sum_{i=0}^p (-1)^i C_k^j C_p^i \frac{(2i)!}{(2i-k+j)!} = \frac{k!}{j!} (-1)^p 2^{2p-k+j} C_p^{k-j-p},$$

we have

$$\begin{aligned} & \lim_{B^+ \ni x \rightarrow y \in S} \frac{\partial^k}{\partial r^k} (r^2 - \rho^2)^p u_p(x) = \\ & = \left( 2^p \rho^{2p-k} \sum_{i=0}^{\min\{k-p, p\}} \frac{k! C_p^i}{2^i (k-p-i)!} \mathcal{D}_r^{(k-p-i)} u_p \right)^+(y), \quad 0 \leq p \leq k; \end{aligned} \quad (3.13)$$

$$\lim_{B^+ \ni x \rightarrow y \in S} \frac{\partial^k}{\partial r^k} (r^2 - \rho^2)^p u_p(x) = 0, \quad p > k. \quad (3.14)$$

Applying (3.11) and the Newton formula, we obtain

$$\begin{aligned} & \Delta^k (r^2 - \rho^2)^p u_p(x) = \\ & = \sum_{i=k}^p (-1)^{p-i} \frac{4^k p!}{(i-k)! (p-i)!} \rho^{2(p-i)} r^{2(i-k)} \mathcal{D}_r^{[i]-[i-k]} u_p(x), \end{aligned} \quad (3.15)$$

where  $u_p$  is a harmonic function.

It is likewise easy to prove the equality

$$\sum_{i=k}^p (-1)^{p-i} \frac{1}{(i-k)! (p-i)!} \mathcal{D}_r^{[i]-[i-k]} = C_k^{p-k} \mathcal{D}_r^{[k]-[p-k]}, \quad k \leq p \leq 2k,$$

Taking the latter equality into account and setting  $\Delta u_p = 0$ , we obtain from (3.15)

$$\begin{aligned} & \lim_{B^+ \ni x \rightarrow y \in S} \Delta^k (r^2 - \rho^2)^p u_p(x) = \\ & = (4^k p! \rho^{2(p-k)} C_k^{p-k} \mathcal{D}_r^{[k]-[p-k]} u_p)^+(y), \quad k \leq p \leq 2k, \end{aligned} \quad (3.16)$$

$$\lim_{B^+ \ni x \rightarrow y \in S} \Delta^k (r^2 - \rho^2)^p u_p(x) = 0, \quad p < k, \quad p > 2k. \quad (3.17)$$

The latter equalities immediately yield

$$\left( \Delta^k \sum_{p=0}^{\nu} (r^2 - \rho^2)^p u_p \right)^+(y) = \left( \Delta^k \sum_{p=k}^{\min\{\nu, 2k\}} (r^2 - \rho^2)^p u_p \right)^+(y) =$$

$$= \left( \Delta^k \sum_{i=0}^{\min\{\nu-k, k\}} (r^2 - \rho^2)^{k+i} u_{k+i} \right)^+ (y), \quad 1 \leq k \leq \nu. \quad (3.18)$$

**3.2. On the Uniqueness of a Regular Solution.** Below we will construct in quadratures the solutions of Problems (I)<sup>+</sup>, (II)<sup>+</sup>, (III)<sup>+</sup>. Thereby we will surely prove the existence theorems, but it will remain for us to show that we have constructed all solutions and for that we need the uniqueness theorems. We begin by deriving some identities of the Green formula type for regular functions which will underlay the uniqueness theorems.

The function  $u$  defined in the domain  $\Omega$  will be called regular of the  $k$ -th order, if  $u \in C^{2k}(\overline{\Omega})$ ,  $k \in \mathbb{N}$ . It is obvious that if  $u$  is a regular function of the  $k$ -th order, then it will be regular of the  $k-1$ -th order ( $k > 2$ ), and the regular function of the first order will also be regular by the definition from Subsection 2.1.

We write the Green formula in the form

$$\int_{\Omega} (\varphi \Delta \psi - \psi \Delta \varphi) dx = \int_{\partial \Omega} \left( \varphi \frac{d\psi}{dn} - \psi \frac{d\varphi}{dn} \right) dS.$$

It is assumed that  $\Omega$  is the bounded domain with a smooth boundary  $\partial \Omega$ , and  $u$  and  $v$  are the regular functions of the  $\nu+1$ -th order in  $\Omega$ . Applying successively the Green formula to the functions  $\varphi = u$  and  $\psi = \Delta^\nu v$ ,  $\varphi = \Delta u$  and  $\psi = \Delta^{\nu-1} v, \dots, \varphi = \Delta^\nu u$  and  $\psi = v$ , combining the obtained formulas we will have

$$\begin{aligned} & \int_{\Omega} (u \Delta^{\nu+1} v - v \Delta^{\nu+1} u) dx = \\ & = \sum_{k=0}^{\nu} \int_{\partial \Omega} \left( \Delta^k u \frac{d \Delta^{\nu-k} v}{dn} - \frac{d \Delta^k u}{dn} \Delta^{\nu-k} v \right) dS. \end{aligned} \quad (3.19)$$

This identity is called the Gutzmer formula (Gutzmer [1]). It is used instead of the classical Green formula in the theory of polyharmonic functions.

Let  $u$  and  $v$  be any regular functions of the  $\nu+1$ -th order in the domain  $B^+$ . The Gutzmer formula yields the identities (Vekua [3]):

1<sup>0</sup>. For  $\nu+1 = 2k$

$$\begin{aligned} & \int_{B^+} u \Delta^{2k} v dx - \int_{B^+} \Delta^k u \Delta^k v dx = \\ & = \sum_{i=0}^{k-1} \int_S \left( \Delta^i u \frac{d \Delta^{2k-1-i} v}{dn} - \frac{d \Delta^i u}{dn} \Delta^{2k-1-i} v \right)^+ dS; \end{aligned} \quad (3.20)$$

2<sup>0</sup>. For  $\nu + 1 = 2k + 1$

$$\begin{aligned} & \int_{B^+} u \Delta^{2k+1} v \, dx - \int_{B^+} \left( \frac{\partial \Delta^k u}{\partial x_1} \frac{\partial \Delta^k v}{\partial x_1} + \frac{\partial \Delta^k u}{\partial x_2} \frac{\partial \Delta^k v}{\partial x_2} + \right. \\ & \left. + \frac{\partial \Delta^k u}{\partial x_3} \frac{\partial \Delta^k v}{\partial x_3} \right) dx = \sum_{i=0}^{k-1} \int_S \left( \Delta^i u \frac{d \Delta^{2k-i} v}{dn} - \frac{d \Delta^i u}{dn} \Delta^{2k-i} v \right)^+ dS - \\ & \quad - \int_S \left( \Delta^k u \frac{d \Delta^k v}{dn} \right)^+ dS. \end{aligned} \quad (3.21)$$

The function

$$v_0(x-y) = -\frac{1}{4\pi 2^\nu \nu! (2\nu-1)!!} |x-y|^{2\nu-1} \quad (3.22)$$

is the fundamental solution (John [1]) of equation (3.1) in the space  $\mathbb{R}^3$ .

Let  $z \in B^+(0, \rho)$  and  $\varepsilon$  be a number such that  $\bar{B}(z, \varepsilon) \subset B^+(0, \rho)$ . If in the domain  $B^+(0, \rho) \setminus \bar{B}(z, \varepsilon)$  we apply the Gutzmer formula to the regular polyharmonic function  $u$  of the  $\nu + 1$ -th order and to the function  $v : v(x) \equiv v_0(x-z)$  and take into account the equalities

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \sum_{k=1}^{\nu} \int_{S(z, \varepsilon)} \left( \Delta^k u \frac{d \Delta^{\nu-k} v_0}{dn} - \frac{d \Delta^k u}{dn} \Delta^{\nu-k} v_0 \right) dS = 0, \\ & \lim_{\varepsilon \rightarrow 0} \int_{S(z, \varepsilon)} \frac{du}{dn} \Delta^\nu v_0 dS = 0, \\ & \lim_{\varepsilon \rightarrow 0} \int_{S(z, \varepsilon)} \frac{d \Delta^\nu v_0(y-z)}{dn} u(y) d_y S = - \lim_{\varepsilon \rightarrow 0} \int_{S(z, \varepsilon)} \frac{d}{dn} \frac{u(y)}{4\pi|y-z|} d_y S = u(z), \end{aligned}$$

then we will obtain the integral representation of a regular polyharmonic function of the  $\nu + 1$ -th order

$$\begin{aligned} u(x) = & - \sum_{k=0}^{\nu} \int_S \left( (\Delta^k u)^+(y) \frac{d \Delta^{\nu-k} v_0(y-x)}{dn} - \right. \\ & \left. - \left( \frac{d \Delta^k u}{dn} \right)^+(y) \Delta^{\nu-k} v_0(y-x) \right) d_y S. \end{aligned} \quad (3.23)$$

In proving the uniqueness of the regular solutions of Problems (I)<sup>+</sup> and (III)<sup>+</sup> we use

**Lemma 3.1.** *If the function  $u$  defined in  $B^+$  satisfies the boundary conditions*

$$\forall y \in S : \left( \frac{d^k u}{dn^k} \right)^+(y) = 0, \quad k = 0, \dots, \nu, \quad (3.24)$$

then the equalities

$$\forall y \in S : (D^\beta u)^+(y) = 0, \quad |\beta| \leq \nu, \quad (3.25)$$

are fulfilled.

*Proof.* Consider more general conditions than (3.24)

$$\forall y \in S : \left( \frac{d^k u}{dn^k} \right)^+(y) = f_k(y), \quad k = 0, \dots, \nu. \quad (3.26)$$

Let  $f_{\nu-k} \in C^k(S)$ ,  $k = 0, \dots, \nu$ . Then at each point of the surface  $S$  we can calculate the derivatives of  $u$  in the tangential plane along two mutually perpendicular directions  $\tau_1$  and  $\tau_2$  (see Subsection 1.1). Moreover, by (3.26) we have the normal derivatives of the function  $u$ . Therefore at each point of  $S$  the derivatives of  $u$  along any direction are expressed linearly in terms of the derivatives along the directions  $\tau_1, \tau_2$  and  $n$ . Therefore, given conditions (3.24), equalities (3.25) are fulfilled. ■

**Theorem 3.1.** *Problems (I)<sup>+</sup>, (II)<sup>+</sup>, (III)<sup>+</sup> can have one solution at most in the class of regular functions of the  $\nu + 1$ -th order.*

*Proof.* It suffices to show that the homogeneous Problems (I)<sub>0</sub><sup>+</sup>, (II)<sub>0</sub><sup>+</sup>, (III)<sub>0</sub><sup>+</sup> cannot have a solution different from zero. Let  $u_0$  be a regular solution of the  $\nu + 1$ -th order of anyone of Problems (I)<sub>0</sub><sup>+</sup>, (II)<sub>0</sub><sup>+</sup>, (III)<sub>0</sub><sup>+</sup>. Given the boundary conditions for the homogeneous problems (see also Lemma 3.1), from identities (3.20) and (3.21), for  $u = v = u_0$ , we have  $\forall x \in B^+ : \Delta^k u_0(x) = 0$  when  $\nu + 1 = 2k$ , and  $\forall x \in B^+ : \Delta^k u_0(x) = \text{const}$  when  $\nu + 1 = 2k + 1$  (therefore  $\Delta^{k+1} u_0(x) = 0$ ). Repeating the foregoing arguments, we may claim that the equality  $\forall x \in B^+ : \Delta u_0(x) = 0$  is valid and, together with the condition  $\forall y \in S : u^+(y) = 0$ , it yields  $\forall x \in B^+ : u_0(x) = 0$ . ■

**3.3. Solution of Problem (I)<sup>+</sup>.** The solution of this problem is sought for in the form (see (3.5))

$$u(x) = \sum_{p=0}^{\nu} \frac{(|x|^2 - \rho^2)^p}{2^p \rho^p p!} u_p(x), \quad (3.27)$$

where  $u_0, \dots, u_\nu$  are the wanted harmonic functions in the domain  $B^+$ .

Since for the function  $u_0$  we have the Dirichlet problem  $\forall x \in B^+ : \Delta u_0(x) = 0$ ,  $\forall y \in S : u_0^+(y) = f_0(y)$ ,  $u_0$  is given by the Poisson formula (see (1.1))

$$u_0(x) = \Pi(f_0)(x) \equiv \frac{1}{4\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|x-y|^3} f_0(y) d_y S. \quad (3.28)$$

Due to (3.13) and (3.14), for  $u$  represented by equality (3.27) we have

$$\left(\frac{\partial^k u}{\partial r^k}\right)^+(y) = u_k^+(y) + \sum_{p=0}^{k-1} \sum_{i=0}^{\min\{k-p,p\}} \frac{2^{-i} \rho^{p-k} k!}{i!(p-i)!(k-p-i)!} (\mathcal{D}_r^{(k-p-i)} u_p)^+(y),$$

$$k = 1, \dots, \nu.$$

Taking into account the boundary conditions of Problem (I)<sup>+</sup> (see (3.2)), the latter formula yields

$$u_k^+(y) + \sum_{p=0}^{k-1} \sum_{i=0}^{\min\{k-p,p\}} \frac{2^{-i} \rho^{p-k} k!}{i!(p-i)!(k-p-i)!} (\mathcal{D}_r^{(k-p-i)} u_p)^+(y) = f_k(y).$$

The function  $W_k$ , where

$$W_k(x) = u_k(x) + \sum_{p=0}^{k-1} \sum_{i=0}^{\min\{k-p,p\}} \frac{2^{-i} \rho^{p-k} k!}{i!(p-i)!(k-p-i)!} \mathcal{D}_r^{(k-p-i)} u_p(x)$$

is harmonic in  $B^+$  and  $W_k^+(y) = f_k(y)$ , can therefore be written by virtue of the Poisson formula  $\forall x \in B^+$  as

$$W_k(x) = \Pi(f_k)(x).$$

Hence

$$u_k(x) = \Pi(f_k)(x) - \sum_{p=0}^{k-1} \sum_{i=0}^{\min\{k-p,p\}} \frac{2^{-i} \rho^{p-k} k!}{i!(p-i)!(k-p-i)!} \mathcal{D}_r^{(k-p-i)} u_p(x). \quad (3.29)$$

Thus  $u_0$  is expressed by quadratures (formula (3.28)), whereas to calculate  $u_1, \dots, u_\nu$ , we have to additionally use differentiation. The function  $u_k$  is expressed by means of  $u_0, \dots, u_{k-1}$ . Relations (3.28) and (3.29) are the recurrent formulas for defining  $u_0, u_1, \dots, u_\nu$ .

The solution of Problem (I)<sup>+</sup> is provided by (3.27) in which we substitute  $u_0, \dots, u_\nu$  from (3.28) and (3.29). Therefore  $u$  is given by recurrent relations. We can however do without them. Indeed, if we introduce the notation

$$P'_{k-j}(\mathcal{D}_r) \equiv - \sum_{i=0}^{\min\{k-j,j\}} \frac{2^{-i} \rho^{j-k} k!}{i!(j-i)!(k-j-i)!} \mathcal{D}_r^{(k-j-i)},$$

then it becomes obvious (see Subsection 1.2) that  $P'_{k-j}(\xi)$  is a polynomial of the  $k-j$ -th order. Now we can claim that the functions  $u_0, u_1, \dots, u_\nu$  will be represented in the form

$$u_k(x) = \sum_{j=0}^k P''_{k-j}(\mathcal{D}_r) \Pi(f_j)(x), \quad (3.30)$$

where  $P''_{k-j}(\xi)$  is a polynomial of the  $k - j$ -th order.

Substituting the values of  $u_k$  from (3.30) in (3.27), the solution of Problem (I)<sup>+</sup> can be written as

$$u(x) = \sum_{k=0}^{\nu} \sum_{i=0}^k (|x|^2 - \rho^2)^k P_{k-i}(\mathcal{D}_r) \Pi(f_i)(x), \quad (3.31)$$

where  $P_{k-i}(\xi)$  is a polynomial of the  $k - i$ -th order with respect to  $\xi$ .

To write the solution in its final form we need define polynomials  $P_{k-i}$ . For this we are to proceed as follows.

Write the solution  $u$  of Problem (I)<sup>+</sup> as the sum

$$u = \sum_{k=0}^{\nu} {}^k u, \quad (3.32)$$

where  ${}^k u$  is the solution of the problem

$$\begin{aligned} \forall x \in B^+ : \Delta^{\nu+1} {}^k u(x) &= 0, \quad \nu \in \mathbb{N}, \\ \forall y \in S : \left( \frac{d^q {}^k u}{dn^q} \right)^+ (y) &= \delta_{kq} f_k(y), \quad k, q = 0, \dots, \nu, \end{aligned}$$

– Problem (I)<sup>+</sup>, where  $\delta_{kq}$  is the Kronecker symbol.

The solution of Problem (I)<sup>+</sup> is sought for in the form

$${}^k u(x) = \sum_{i=0}^{\nu-k} \frac{1}{(k+i)!} \left( \frac{|x|^2 - \rho^2}{2\rho} \right)^{k+i} {}^k u_i(x), \quad (3.33)$$

where  ${}^k u_0, {}^k u_1, \dots, {}^k u_{\nu-k}$  are the harmonic functions for which on account of (3.28) and (3.29) we obtain

$${}^k u_0 = \Pi(f_k), \quad {}^k u_q = - \sum_{j=0}^{q-1} \sum_{i=0}^{\min\{q-j, k+j\}} \frac{2^{-i} \rho^{j-q} (k+q)!}{i!(q-j-i)!(k+j-i)!} \mathcal{D}_r^{(q-j-i)} {}^k u_j, \\ q = 1, \dots, \nu - k.$$

But by (3.30) we have the representation

$${}^k u_q(x) = P_q(\mathcal{D}_r) \Pi(f_k)(x), \quad q = 0, \dots, \nu - k. \quad (3.34)$$

For these two representations to be identical polynomials  $\overset{k}{P}_q(\xi)$  should be chosen as follows:

$$\begin{aligned} \overset{k}{P}_0(\xi) &= 1, \\ \overset{k}{P}_q(\xi) &= - \sum_{j=0}^{q-1} \sum_{i=0}^{\min\{q-j, k+j\}} \frac{2^{-i} \rho^{j-q} (k+q)!}{i!(q-j-i)!(k+j-i)!} \xi^{(q-j-i)k} \overset{k}{P}_j(\xi), \quad (3.35) \\ k &= 0, 1, \dots, \nu, \quad q = 1, 2, \dots, \nu - k, \end{aligned}$$

where

$$\xi^{(p)} \equiv \xi(\xi - 1) \dots (\xi - (p - 1)). \quad (3.36)$$

Taking (3.33) and (3.34) into account, the solution of Problem (I)<sup>+</sup> takes the form

$$\overset{k}{u}(x) = \sum_{i=0}^{\nu-k} \frac{1}{(k+i)!} \left( \frac{|x|^2 - \rho^2}{2\rho} \right)^{k+i} \overset{k}{P}_i(\mathcal{D}_r) \Pi(f_k)(x) \quad (3.37)$$

and by virtue of (3.32) we finally obtain

$$u(x) = \sum_{k=0}^{\nu} \sum_{i=0}^k \frac{1}{k!} \left( \frac{|x|^2 - \rho^2}{2\rho} \right)^k \overset{k}{P}_{k-i}(\mathcal{D}_r) \Pi(f_i)(x), \quad (3.38)$$

where the polynomials  $\overset{k}{P}_q$  are defined by the recurrent relations (3.35).

Thus we have constructed the formal solution of Problem (I)<sup>+</sup>. We do not know what minimal restrictions have to be imposed on the function  $f_i$  in order that  $u$  defined by (3.38) give the solution of Problem (I)<sup>+</sup>. We do not know either whether the problem has or has not other solutions. It is true that if  $u$  defined by (3.38) is a regular solution of the  $\nu + 1$ -th order, then it is unique (Theorem 3.1), but it still remains for us to investigate under what restrictions imposed on the functions  $f_0, \dots, f_\nu$ ,  $u$  will be a regular solution of the  $(\nu + 1)$ -th order. To elucidate these questions we need one property of the Poisson integral.

**3.4. On One Property of the Poisson Integral.** The main goal of this subsection is to prove

**Theorem 3.2.** *If  $f \in C(S)$ , then*

$$\begin{aligned} \lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2)^k \mathcal{D}_r^p \Pi(f)(x) &= 0, \\ p &= 0, 1, \dots, k; \quad k \in \mathbb{N}. \end{aligned}$$

To this end we will first consider the following auxiliary Problem  $(\overset{0}{I})^+$

$$\begin{aligned} \forall x \in B^+ : \Delta^{p+1} \overset{0}{u}(x) &= 0, \quad p \in \mathbb{N}, \\ \forall y \in S : (\overset{0}{u})^+(y) &= f_0(y), \quad \left( \frac{d^q \overset{0}{u}}{dn^q} \right)^+(y) = 0, \quad q = 1, \dots, p, \end{aligned} \quad (3.39)$$

and write its solution (see (3.37)) in the form

$$\overset{0}{u}(x) = \int_S \overset{p}{K}(x, y) f_0(y) d_y S, \quad (3.40)$$

where

$$\overset{p}{K}(x, y) = \frac{-1}{4\pi\rho} \sum_{i=0}^p \frac{1}{i!} \left( \frac{|x|^2 - \rho^2}{2\rho} \right)^i \overset{0}{P}_i(\mathcal{D}_r) \frac{|x|^2 - \rho^2}{|x - y|^3}. \quad (3.41)$$

The polynomials  $\overset{0}{P}_i$  (see (3.35)) are defined by the relations

$$\begin{aligned} \overset{0}{P}_0(\xi) &= 1, \\ \overset{0}{P}_q(\xi) &= - \sum_{j=0}^{q-1} \sum_{i=0}^{\min\{q-j, j\}} \frac{2^i \rho^{j-q} q!}{i!(q-j-i)!(j-i)!} \xi^{(q-j-i)} \overset{0}{P}_j(\xi), \\ & \quad q = 1, \dots, p. \end{aligned} \quad (3.42)$$

We easily obtain

$$\begin{aligned} \overset{0}{P}_1(\xi) &= -\frac{\xi}{\rho}, \quad \overset{0}{P}_2(\xi) = \frac{\xi}{\rho^2}(\xi + 2), \\ \overset{0}{P}_3(\xi) &= -\frac{\xi}{\rho^3}(\xi + 2)(\xi + 4). \end{aligned} \quad (3.43)$$

**Problem 20.** Is the equality

$$\overset{0}{P}_k(\xi) = \left(-\frac{1}{\rho}\right)^k \xi(\xi + 2) \cdots (\xi + 2(k-1)), \quad k = 1, \dots, p,$$

fulfilled or not?

**Lemma 3.2.** In the polynomial  $\overset{0}{P}_k(\xi)$  the coefficient of  $\xi^k$  is equal to  $\left(-\frac{1}{\rho}\right)^k$ .

*Proof.* The lemma will be proved by the induction method. Assume (see (3.43))

$$\overset{0}{P}_j(\xi) = \left(-\frac{1}{\rho}\right)^j \xi^j + a'_{j-1} \xi^{j-1} + \cdots + a'_0, \quad j = 0, 1, \dots, k-1, \quad (3.44)$$

and prove the equality

$${}^0P_k(\xi) = \left(-\frac{1}{\rho}\right)^k \xi^k + a_{k-1} \xi^{k-1} + \cdots + a_0.$$

The polynomial  ${}^0P_k(\xi)$  defined by (3.42) can be written as

$${}^0P_k(\xi) = -\sum_{j=0}^{k-1} \frac{k!}{\rho^{k-j}(k-j)!j!} \xi^{(k-j)} {}^0P_j(\xi) + a''_{k-1} \xi^{k-1} + \cdots + a''_0$$

or, by virtue of (3.36) and (3.44), as

$${}^0P_k(\xi) = -\sum_{j=0}^{k-1} \frac{k!}{\rho^{k-j}(k-j)!j!} \left(-\frac{1}{\rho}\right)^j \xi^k + a_{k-1} \xi^{k-1} + \cdots + a_0.$$

Therefore

$$\begin{aligned} {}^0P_k(\xi) &= -\frac{1}{\rho^k} \xi^k \sum_{j=0}^{k-1} (-1)^j \frac{k!}{(k-j)!j!} + a_{k-1} \xi^{k-1} + \cdots + a_0 = \\ &= \left(-\frac{1}{\rho}\right)^k \xi^k + a_{k-1} \xi^{k-1} + \cdots + a_0. \quad \blacksquare \end{aligned}$$

**Lemma 3.3.** *The estimate  $\forall x \in B^+, \forall y \in S$  :*

$$\left|(\rho^2 - |x|^2)^k \mathcal{D}_r^p \frac{\rho^2 - |x|^2}{|y-x|^3}\right| \leq c \frac{\rho^2 - |x|^2}{|y-x|^3}, \quad p = 0, \dots, k, \quad k \in \mathbb{N}, \quad (3.45)$$

is valid.

*Proof.* Using the formulas

$$\frac{\rho^2 - |x|^2}{|y-x|^3} = (2\mathcal{D}_r + 1) \frac{1}{|y-x|}, \quad (3.46)$$

$$\left|\mathcal{D}_r^p \frac{1}{|y-x|}\right| \leq \frac{c}{|y-x|^{p+1}}, \quad (3.47)$$

we easily obtain

$$\begin{aligned} \left|(\rho^2 - |x|^2)^k \mathcal{D}_r^p \frac{\rho^2 - |x|^2}{|y-x|^3}\right| &\leq c \frac{(\rho^2 - |x|^2)^k}{|y-x|^{p+2}} \leq \\ &\leq c \frac{(\rho^2 - |x|^2)^{k-p+1}}{|y-x|^3} \leq c \frac{\rho^2 - |x|^2}{|y-x|^3}. \quad \blacksquare \end{aligned}$$

Now we can proceed to proving Theorem 3.2. Let  $p < k$ . Then from (3.46) and (3.47) we obtain

$$\left|(\rho^2 - |x|^2)^k \mathcal{D}_r^p \int_S \frac{\rho^2 - |x|^2}{|y-x|^3} f(y) d_y S\right| \leq$$

$$\leq c(\rho^2 - |x|^2)^{k-p-\alpha} M \int_S \frac{d_y S}{|y-x|^{2-\alpha}}, \quad (3.48)$$

where  $0 < \alpha < 1$ ,  $M \equiv \max_{y \in S} |f(y)|$ . Taking into account the inequality  $k - p - \alpha > 0$  and the estimate

$$\forall x \in \bar{B}^+ : \int_S \frac{d_y S}{|y-x|^{2-\alpha}} \leq c,$$

we readily conclude that Theorem 3.2 holds for  $p < k$ .

Let now  $p = k$ . Then we have to prove the equality

$$\lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2)^k \mathcal{D}_r^k \Pi(f)(x) = 0. \quad (3.49)$$

First we will prove that

$$\lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2) \mathcal{D}_r \Pi(f)(x) = 0. \quad (3.50)$$

To this effect we will consider the problem  $\forall x \in B^+$ :

$$\begin{aligned} \Delta^2 \overset{1}{v}(x) &= 0, \\ \forall y \in S : (\overset{1}{v})^+(y) &= f(y), \quad \left( \frac{d\overset{1}{v}}{dn} \right)^+(y) = 0. \end{aligned} \quad (3.51)$$

Due to (3.40) and (3.43) the solution of this problem can be written in the form

$$\overset{1}{v}(x) = \int_S \overset{1}{K}(x, y) f(y) d_y S, \quad (3.52)$$

where

$$\overset{1}{K}(x, y) = \frac{1}{4\pi\rho} \left( 1 + \frac{\rho^2 - |x|^2}{2\rho^2} \mathcal{D}_r \right) \frac{\rho^2 - |x|^2}{|y-x|^3}.$$

We must show that the equality

$$\lim_{B^+ \ni x \rightarrow z \in S} \overset{1}{v}(x) = f(z) \quad (3.53)$$

is fulfilled for  $f \in C(S)$ . For this we, in turn, have to prove

$$\int_S \overset{1}{K}(x, y) d_y S = 1. \quad (3.54)$$

The function  $\forall x \in B^+ : \overset{1}{v} = 1$  is actually a regular solution of the second order of Problem (3.51) for  $f = 1$ , and, by Theorem 3.1, it is unique. On the other hand, the solution of the same problem can be represented by (3.52), and for  $f = 1$  it is also a regular solution of the second order and hence unique. The coincidence of both solutions gives us equality (3.54).

Using (3.54) and Lemma 3.3, we have

$$\begin{aligned} |\overset{1}{v}(x) - f(z)| &= \left| \int_S \overset{1}{K}(x, y)(f(y) - f(z))d_y S \right| \leq \\ &\leq c \int_S \frac{\rho^2 - |x|^2}{|y - x|^3} |f(y) - f(z)|d_y S, \end{aligned}$$

which yields equality (3.53). From (3.53) we conclude that

$$\lim_{B^+ \ni x \rightarrow z \in S} \left( \Pi(f)(x) + \frac{1}{2\rho^2}(\rho^2 - r^2)\mathcal{D}_r \Pi(f)(x) \right) = f(z). \quad (3.55)$$

By virtue of the property of the Poisson integral the latter equality implies (3.50).

Let us assume that (3.49) is fulfilled for  $k = 1, \dots, p-1$  and prove the equality

$$\lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2)^p \mathcal{D}_r^p \Pi(f)(x) = 0. \quad (3.56)$$

To this end we have to consider the problem

$$\begin{aligned} \forall x \in B^+ : \Delta^{p+1} \overset{p}{v}(x) &= 0, \\ \forall y \in S : v^+(y) = f(y), \quad \left( \frac{d^l \overset{p}{v}}{dn^l} \right)^+(y) &= 0, \quad l = 1, 2, \dots, p, \end{aligned}$$

whose solution due to (3.40) has the form

$$\overset{p}{v}(x) = \int_S \overset{p}{K}(x, y) f(y) d_y S.$$

Repeating the same reasoning as for problem (3.51) in deriving (3.55), we easily find

$$\lim_{B^+ \ni x \rightarrow z \in S} \int_S \overset{p}{K}(x, y) f(y) d_y S = f(z).$$

Hence, remembering the fact that  $\overset{p}{K}$  is given by (3.41) and equalities (3.49) are, by assumption, fulfilled for  $k = 1, 2, \dots, p-1$ , we conclude by virtue of Lemma 3.2 that (3.56) is valid. Theorem 3.2 is proved.

The following similar assertions hold for harmonic simple- and double-layer potentials : if  $f \in C(S)$ , then

$$\begin{aligned} \lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2)^k \mathcal{D}_r^p V(f)(x) &= 0, \\ p = 0, 1, \dots, k+1, \quad k \in \mathbb{N}; \end{aligned} \quad (3.57)$$

$$\begin{aligned} \lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2)^k \mathcal{D}_r^q W(f)(x) &= 0, \\ q = 0, 1, \dots, k, \quad k \in \mathbb{N}. \end{aligned} \quad (3.58)$$

Due to (1.9) and (1.10) these formulas are a corollary of Theorem 3.2.

Using (3.57) and (3.58), we readily convince ourselves by virtue of Theorems 1.2 and 1.3 that the following theorem is valid.

**Theorem 3.3.** *If  $f \in C^{p+1}(S)$ ,  $p = 1, 2, \dots$ , then*

$$\lim_{B^+ \ni x \rightarrow z \in S} (\rho^2 - |x|^2)^k \mathcal{D}_r^{q+p} \Pi(f)(x) = 0, \quad (3.59)$$

$$q = 0, 1, \dots, k, \quad k \in \mathbb{N}.$$

**3.5. Analysis of the Constructed Solution of Problem (I)<sup>+</sup>.** In Subsection 3.2 we introduced the notion of a regular solution of Problem (I)<sup>+</sup> and proved the theorem of the solution uniqueness. To establish the regularity of the solution of Problem (I)<sup>+</sup> expressed by (3.38) we need to impose an increased smoothness on the boundary functions. To weaken this restriction we introduce the notion of a classical solution and prove its uniqueness.

The solution  $u$  of Problem (I)<sup>+</sup> will be called classical (or a classical solution of the  $\nu + 1$ -th order) if

$$\frac{\partial^k u}{\partial r^k} \in C(\bar{B}^+), \quad k = 0, 1, \dots, \nu. \quad (3.60)$$

We begin by proving

**Lemma 3.4.** *If  $f_k \in C^\infty(S)$ ,  $k = 0, 1, \dots, \nu$ , then  $u$  defined by (3.38) is a regular solution of the  $\nu + 1$ -th order in  $B^+$ .*

The proof readily follows from the fact that under the assumptions of the lemma we have the inclusion  $\Pi(f_k) \in C^{p,\gamma}(S)$ , where  $0 < \gamma < 1$ ,  $p = 1, 2, \dots$ .

**Theorem 3.4.** *If  $u$  is a classical solution of Problem (I)<sub>0</sub><sup>+</sup>, then  $\forall x \in B^+ : u(x) = 0$ .*

*Proof.* Let  $z$  be any point from  $B^+$ . Since  $u, \frac{\partial u}{\partial r}, \dots, \frac{\partial^\nu u}{\partial r^\nu}$  are continuous in  $\bar{B}^+$ , for any  $\varepsilon > 0$  there exists, by virtue of the boundary conditions of Problem (I)<sub>0</sub><sup>+</sup>, a number  $\delta > 0$  such that  $\forall y \in S(0, \rho') : \left| \frac{\partial^k u(y)}{\partial r^k} \right| < \varepsilon$  for  $k = 0, 1, \dots, \nu$ , where  $\rho' = \rho - \delta'$ ,  $0 < \delta' < \delta$ . Assuming  $\delta' < \rho - |z|$ , it is obvious that  $z \in B^+(0, \rho')$ .

Under the conditions of the theorem  $u \in C^\infty(B^+(0, \rho'))$ , and therefore, due to Lemma 3.4,  $u$  is a regular solution of the  $\nu + 1$ -th order in  $B^+(0, \rho')$  of Problem (I)<sup>+</sup> with the boundary conditions

$$\left( \frac{\partial^k u}{\partial r^k} \right)^+(y) = \bar{f}_k(y),$$

where  $\bar{f}_k(y) \equiv \left(\frac{\partial^k u}{\partial r^k}\right)(y)$ , when  $y \in S(0, \rho')$ . Thus (3.38) is fulfilled in the domain  $B^+(0, \rho')$  :

$$u(z) = \sum_{k=0}^{\nu} \sum_{i=0}^k \frac{1}{k!} \left(\frac{|z|^2 - \rho'^2}{2\rho'}\right)^k P_{k-i}^i(\mathcal{D}_r) \Pi(f_i)(z).$$

Now by virtue of Lemma 3.3 we have

$$|u(z)| \leq c \int_S \frac{\rho'^2 - |z|^2}{|y - z|^3} \sum_{k=0}^{\nu} |\bar{f}_k(y)| d_y S \leq c\varepsilon.$$

Thus,  $\forall z \in B^+ : u(z) = 0$ . ■

**Theorem 3.5.** *If  $f_i \in C^{\nu+1-i}(S)$ ,  $i = 0, 1, \dots, \nu - 1$ ,  $f_\nu \in C(S)$ , then  $u$  defined by (3.38) is the classical solution of Problem (I)<sup>+</sup>.*

*Proof.* The function  $u$  defined by (3.38) has form (3.5) and it is therefore a polyharmonic function of the  $\nu + 1$ -th order. As for condition (3.60), it is obtained from Theorems 3.2 and 3.3 and from the fact that if  $f \in C^{k+1}(S)$ , then  $\Pi(f) \in C^k(\bar{B}^+)$ ,  $k \in \mathbb{N}$ , and if  $f \in C(S)$ , then  $\Pi(f) \in C(\bar{B}^+)$ . One can also see immediately that condition (3.2) is satisfied. ■

Combining Theorems 3.4 and 3.5, we arrive at

**Theorem 3.6.** *If  $f \in C^{\nu+1-i}(S)$ ,  $i = 0, 1, \dots, \nu - 1$ ,  $f_\nu \in C(S)$ , then  $u$  defined by (3.38) is the unique classical solution of Problem (I)<sup>+</sup>.*

We have completed the investigation of the Lauricella problem (Problem (I)<sup>+</sup>).

**3.6. Solution of Problem (II)<sup>+</sup>.** In the first place we have to solve an auxiliary differential equation

$$\forall x \in B^+ : \mathcal{D}_r^{[k]} \psi(x) = F(x), \quad k \in \mathbb{N}, \quad (3.61)$$

where the operator  $\mathcal{D}_r^{[k]}$  is defined (see Subsection 1.2) as follows:

$$\mathcal{D}_r^{[k]} = \left(\mathcal{D}_r + \frac{3}{2}\right) \left(\mathcal{D}_r + \frac{5}{2}\right) \cdots \left(\mathcal{D}_r + \frac{2k+1}{2}\right).$$

We introduce the notation

$$J^k(F)(x) \equiv \int_0^1 \frac{(1-\tau)^{k-1}}{(k-1)!} F(\tau x) \sqrt{\tau} d\tau, \quad k \in \mathbb{N}, \quad (3.62)$$

and prove

**Lemma 3.5.** *If  $F$  is a harmonic function in  $B^+$ , then in the class of harmonic functions equation (3.61) has the unique solution defined by the formula*

$${}^k\psi(x) = J^k(F)(x). \quad (3.63)$$

*Proof.* Consider the equation

$$r \frac{\partial \psi(x)}{\partial r} + \alpha \psi(x) = F(x), \quad \alpha = \frac{2k+1}{2}, \quad k \in \mathbb{N}. \quad (3.64)$$

It is easy to find that in the class of harmonic functions it has the unique solution (see the solution of equation (2.21))

$$\psi(x) = \int_0^1 F(\tau x) \tau^{\alpha-1} d\tau. \quad (3.65)$$

Hence it follows that in the class of harmonic functions equation (3.61) has, for  $k = 1$ , the unique solution

$${}^1\psi(x) = J^1(F)(x).$$

Thus Lemma 3.5 holds for  $k = 1$ . Assume now that (3.63) is fulfilled and prove that in the class of harmonic functions the equation

$$\mathcal{D}_r^{[k+1]}\psi(x) = F(x) \quad (3.66)$$

has the unique solution

$${}^{k+1}\psi(x) = J^{k+1}(F)(x). \quad (3.67)$$

Indeed, since

$$\mathcal{D}_r^{[k+1]} = \mathcal{D}_r^{[k]} \left( \mathcal{D}_r + \frac{2k+3}{2} \right)$$

and, by assumption, the solution of (3.61) is uniquely defined by (3.63), from (3.66) we obtain

$$\left( \mathcal{D}_r + \frac{2k+3}{2} \right)^{k+1} \psi(x) = \frac{1}{(k-1)!} \int_0^1 F(\tau x) (1-\tau)^{k-1} \sqrt{\tau} d\tau$$

or

$$\left( \mathcal{D}_r + \frac{2k+3}{2} \right)^{k+1} \psi(x) = \frac{1}{(k-1)!} \int_0^r F\left(\frac{t}{r}x\right) \left(1 - \frac{t}{r}\right)^{k-1} r^{-\frac{3}{2}} \sqrt{t} dt.$$

Hence, due to the fact that the solution of (3.64) is given by (3.65), it is easy to establish that the representation

$$\psi^{k+1}(x) = \frac{1}{(k-1)!} r^{-\frac{2k+3}{2}} \int_0^r dt_1 \int_0^{t_1} F\left(\frac{t}{r}\right) (t_1 - t)^{k-1} \sqrt{t} dt$$

is unique.

Applying here

$$\int_0^r dt_1 \int_0^{t_1} (t_1 - t)^{n-1} f(t) dt = \frac{1}{n} \int_0^r (r - t)^n f(t) dt,$$

we readily convince ourselves that representation (3.67) is valid. ■

We will mention some of the properties of the operator  $J^k$  defined by (3.62). If  $F$  is a harmonic function in  $B^+$ , then the following equalities are valid  $\forall x \in B^+$ :

$$\mathcal{D}_r^{[k]} J^k(F)(x) = J^k(\mathcal{D}_r^{[k]} F)(x) = F(x), \quad (3.68)$$

$$\begin{aligned} J^k(\mathcal{D}_r^{[k]-[p]} F)(x) &= \mathcal{D}_r^{[k]-[p]} J^k(F)(x) = J^p(F)(x), \\ p &\leq k, \quad k, p \in \mathbb{N}. \end{aligned} \quad (3.69)$$

Now we proceed to solving Problem (II)<sup>+</sup>. Its solution is sought for in the form (see (3.5))

$$u(x) = \sum_{p=0}^{\nu} \frac{1}{4^p p!} (|x|^2 - \rho^2)^p u_p(x), \quad (3.70)$$

where  $u_\nu, \dots, u_0$  are the desired harmonic functions in  $B^+$ .

By virtue of (3.16) and (3.17), from (3.70) we obtain

$$(\Delta^\nu u)^+(y) = (\mathcal{D}_r^{[\nu]} u_\nu)^+(y).$$

Hence, taking into account the boundary condition of Problem (II)<sup>+</sup> (see (3.3)) we have

$$\forall y \in S : (\mathcal{D}_r^{[\nu]} u_\nu)^+(y) = f_\nu(y).$$

Since  $u_\nu$  is sought for in the class of harmonic functions,  $\mathcal{D}_r^{[\nu]} u_\nu$  is harmonic in  $B^+$  (see (3.8)) and therefore for  $u_\nu$  we have

$$\mathcal{D}_r^{[\nu]} u_\nu(x) = \Pi(f_\nu)(x). \quad (3.71)$$

Let  $k$  be a natural number  $1 \leq k < \nu$ . Then, by virtue of (3.16) and (3.18) we have from (3.70)

$$\begin{aligned} (\Delta^k u)^+ &= \left( \sum_{i=0}^{\min\{\nu-k, k\}} \frac{1}{4^{k+i}(k+i)!} \Delta^k (r^2 - \rho^2)^{k+i} u_{k+i} \right)^+ = \\ &= \left( \sum_{i=0}^{\min\{\nu-k, k\}} \left( \frac{\rho^2}{4} \right)^i C_k^i \mathcal{D}_r^{[k]-[i]} u_{k+i} \right)^+ = \\ &= \left( \mathcal{D}_r^{[k]} u_k + \sum_{i=1}^{\min\{\nu-k, k\}} \left( \frac{\rho^2}{4} \right)^i C_k^i \mathcal{D}_r^{[k]-[i]} u_{k+i} \right)^+. \end{aligned}$$

By the boundary condition of Problem (II)<sup>+</sup> the latter equality yields

$$\left( \mathcal{D}_r^{[k]} u_k + \sum_{i=1}^{\min\{\nu-k, k\}} \left( \frac{\rho^2}{4} \right)^i C_k^i \mathcal{D}_r^{[k]-[i]} u_{k+i} \right)^+ = f_k.$$

Hence, by virtue of the fact that the function

$$\mathcal{D}_r^{[k]} u_k + \sum_{i=1}^{\min\{\nu-k, k\}} \left( \frac{\rho^2}{4} \right)^i C_k^i \mathcal{D}_r^{[k]-[i]} u_{k+i}$$

is harmonic in  $B^+$ , we obtain

$$\mathcal{D}_r^{[k]} u_k = \Pi(f_k) - \sum_{i=1}^{\min\{\nu-k, k\}} \left( \frac{\rho^2}{4} \right)^i C_k^i \mathcal{D}_r^{[k]-[i]} u_{k+i}, \quad 1 \leq k < \nu. \quad (3.72)$$

From (3.71) and (3.72) we now obtain, due to Lemma 3.5 and formula (3.69), the representations  $\forall x \in B^+$ :

$$u_\nu(x) = J(\Pi(f_\nu))(x), \quad (3.73)$$

$$\begin{aligned} u_p(x) &= J(\Pi(f_p))(x) - \\ &= \sum_{k=1}^{\min\{\nu-p, p\}} \left( \frac{\rho^2}{4} \right)^k \frac{p!}{k!(p-k)!} J(u_{p+k})(x), \quad 1 \leq p < \nu, \end{aligned} \quad (3.74)$$

which are recurrent relations for defining the functions  $u_\nu, \dots, u_1$ , contained in (3.70). It remains for us to define  $u_0$ . This can obviously be done using the Dirichlet problem  $\forall x \in B^+ : \Delta u_0(x) = 0, \forall y \in S : (u_0)^+(y) = f_0(y)$ . Therefore  $u_0$  can be represented by the Poisson integral

$$u_0 = \Pi(f_0). \quad (3.75)$$

Thus the solution of Problem  $(\text{II})^+$  is given by (3.70), where the functions  $u_\nu, \dots, u_0$  are defined by (3.73)–(3.75).

**3.7. Alternative Representation of Solutions of Problem  $(\text{II})^+$ .** To investigate solutions of Problem  $(\text{II})^+$  it is of importance to represent solutions in the form showing explicitly the role of each of the boundary functions  $f_0, \dots, f_\nu$ . To this end we will consider the special boundary value problems

$$\begin{aligned} \forall x \in B^+ : \Delta^{k+1} u^k(x) &= 0, \quad k = 0, 1, \dots, \nu; \\ \forall y \in S : (\Delta^p u^k)^+(y) &= \delta_{kp} f_p(y), \quad p = 0, 1, \dots, k, \end{aligned}$$

and denote them by  $(\text{II})^+$ .

It is obvious that if  $u^k$  is the solution of Problem  $(\text{II})^+$ , then

$$u(x) = \sum_{k=0}^{\nu} u^k(x) \quad (3.76)$$

is the solution of Problem  $(\text{II})^+$ . It is likewise obvious that

$$u^0(x) = \Pi(f_0)(x). \quad (3.77)$$

Write the solution of Problem  $(\text{II})^+$  in the form (see (3.5), (3.70))

$$u^k(x) = \sum_{j=1}^k \frac{1}{j!} \left( \frac{|x|^2 - \rho^2}{4} \right)^j u_j^k(x), \quad (3.78)$$

where  $u_1^k, \dots, u_k^k$  are harmonic functions. By virtue of (3.73), (3.74), the latter functions are defined by

$$\begin{aligned} u_k^k(x) &= J(\Pi(f_k))(x), \\ u_j^k(x) &= - \sum_{i=1}^{\min\{k-j, j\}} \frac{j!}{i!(j-i)!} \left( \frac{\rho^2}{4} \right)^i J(u_{i+j}^k)(x), \\ & \quad j = 1, 2, \dots, k-1. \end{aligned} \quad (3.79)$$

Hence

$$u_j^k(x) = \sum_{|\alpha|=k-j} a_\alpha J^{\alpha} J(\Pi(f_k))(x), \quad j = 1, 2, \dots, k, \quad (3.80)$$

where  $\alpha = (\alpha_1, \dots, \alpha_{k-1})$  is the multiindex,  $0 \leq \alpha_i \leq k-j$ ,  $a_\alpha$  are the constants,

$$J^{\alpha}(F) = J^{\alpha_1} (J^{\alpha_2} \dots (J^{\alpha_{k-1}}(F))),$$

$\overset{0}{J}$  is the identical operator, and

$$\begin{aligned} \overset{p}{J}\overset{k}{J}(F)(x) &= \overset{p}{J}(\overset{k}{J}(F))(x) = \\ &= \int_0^1 \int_0^1 \frac{(1-\tau_1)^p}{(p-1)!} \frac{(1-\tau_2)^k}{(k-1)!} F(\tau_1 \tau_2 x) \sqrt{\tau_1 \tau_2} d\tau_1 d\tau_2. \end{aligned}$$

Now the solution of Problem (II)<sup>+</sup> takes the form

$$\overset{k}{u}(x) = \sum_{j=0}^k \frac{1}{j!} \left( \frac{|x|^2 - \rho^2}{4} \right)^j \sum_{|\alpha|=k-j} a_\alpha \overset{\alpha}{J}\overset{k}{J}(\Pi(f_k))(x). \quad (3.81)$$

Thus (see (3.76)) to solve Problem (II)<sup>+</sup> we have

$$u(x) = \sum_{k=0}^{\nu} \sum_{j=0}^k \frac{1}{j!} \left( \frac{|x|^2 - \rho^2}{4} \right)^j \sum_{|\alpha|=k-j} a_\alpha \overset{\alpha}{J}\overset{k}{J}(\Pi(f_k))(x). \quad (3.82)$$

**3.8. Uniqueness Theorem for Problem (II)<sup>+</sup>.** First we should indicate some properties of the functions  $\overset{k}{u}_j$  and  $\overset{k}{u}$  represented by (3.80) and (3.81).

**Lemma 3.6.** *If  $f_k \in C(S)$ , then the inclusion*

$$\begin{aligned} \mathcal{D}_r^h \overset{k}{u}_j &\in C(\bar{B}^+), \\ k = 1, 2, \dots, \nu; \quad j = 1, 2, \dots, k; \quad h = 0, 1, \dots, 2k - j, \end{aligned} \quad (3.83)$$

is valid for the function  $\overset{k}{u}_j$  defined by (3.80).

*Proof.* It is easy to verify the validity of the equality

$$\mathcal{D}_r \overset{k}{J}(\Pi(f)) = \overset{k-1}{J}(\Pi(f)) - \frac{2k+1}{2} \overset{k}{J}(\Pi(f)), \quad k \in \mathbb{N},$$

which, in turn, yields

$$\begin{aligned} \mathcal{D}_r \overset{p}{J}\overset{k}{J}(\Pi(f)) &= \overset{p-1}{J}\overset{k}{J}(\Pi(f)) - \frac{2p+1}{2} \overset{p}{J}\overset{k}{J}(\Pi(f)), \\ p = 1, 2, \dots, \quad k = 0, 1, \dots \end{aligned} \quad (3.84)$$

But if  $f \in C(S)$ , then  $\Pi(f) \in C(\bar{B}^+)$  and therefore  $v \in C(\bar{B}^+)$ , where

$$v(x) = \int_0^1 \frac{(1-\tau)^{k-1}}{(k-1)!} \Pi(\tau x) \sqrt{\tau} d\tau.$$

It is likewise easy to establish that

$$\overset{p}{J}\overset{k}{J}(\Pi(f)) \in C(\bar{B}^+), \quad p, k = 0, 1, \dots$$

Hence, using the induction method, we conclude by virtue of (3.81) and (3.84) that inclusion (3.83) is valid. ■

**Lemma 3.7.** *If  $f_k \in C(S)$ , then the inclusion*

$$\Delta^m \overset{k}{u} \in C(\bar{B}^+), \quad m, k = 0, 1, \dots, \nu, \quad (3.85)$$

is valid for the function  $\overset{k}{u}$  defined by (3.81).

*Proof.* By Lemma 3.6

$$\mathcal{D}_r^m \overset{k}{u}_j \in C(\bar{B}^+), \quad k = 1, \dots, \nu; \quad m, j = 0, \dots, k.$$

Hence on account of (3.15) we have

$$\Delta^m (r^2 - \rho^2)^j \overset{k}{u}_j \in C(\bar{B}^+), \quad k = 1, 2, \dots, \nu; \quad m, j = 0, \dots, k.$$

Now (3.78) yields

$$\Delta^m \overset{k}{u} \in C(\bar{B}^+), \quad k = 1, 2, \dots, \nu; \quad m = 0, 1, \dots, k. \quad (3.86)$$

Moreover,

$$\forall x \in B^+ : \Delta^m \overset{k}{u}(x) = 0, \quad k = 0, 1, \dots, \nu; \quad m > k. \quad (3.87)$$

Therefore inclusion (3.85) follows from (3.86). ■

**Lemma 3.8.** *If  $f_k \in C(S)$ , then we have the estimate*

$$\forall x \in B^+ : |\Delta^m \overset{k}{u}(x)| \leq c \max_{y \in S} |f_k(y)|, \quad m, k = 0, 1, \dots, \nu, \quad (3.88)$$

for  $\overset{k}{u}$  defined by (3.81).

*Proof.* On account of formulas (3.15), (3.78)

$$\Delta^m \overset{k}{u} = \sum_{j=m}^k \frac{1}{4^j j!} \sum_{i=m}^j \frac{(-1)^{j-i} 4^m j!}{(i-m)!(j-i)!} \rho^{2(j-i)} r^{2(i-m)} \mathcal{D}_r^{[i]-[i-m]} \overset{k}{u}_j.$$

Hence by virtue of (3.80)

$$\begin{aligned} |\Delta^m \overset{k}{u}(x)| &\leq c \sum_{j=m}^k \sum_{i=m}^j |\mathcal{D}_r^{[i]-[i-m]} \overset{k}{u}_j(x)| \leq \\ &\leq c \sum_{j=m}^k \sum_{i=m}^j \sum_{|\alpha|=k-j} |\mathcal{D}_r^{[i]-[i-m]} J J^{\alpha k}(\Pi(f_k))(x)|. \end{aligned}$$

Note that  $\mathcal{D}_r^{[j]-[i-m]}$  is a polynomial of order  $m$  with respect to  $\mathcal{D}_r$ . Therefore taking into account (3.84) and the estimate

$$\begin{aligned} & \left| \int_0^1 \int_0^1 \frac{(1-\tau_1)^p}{p!} \frac{(1-\tau_2)^q}{q!} \sqrt{\tau_1 \tau_2} \Pi(f_k)(\tau_1 \tau_2 x) d\tau_1 d\tau_2 \right| \leq \\ & \leq \int_0^1 \int_0^1 \frac{(1-\tau_1)^p}{p!} \frac{(1-\tau_2)^q}{q!} \sqrt{\tau_1 \tau_2} \max_{y \in S} |f_k(y)| \Pi(1)(x) d\tau_1 d\tau_2 \leq \\ & \leq \max_{y \in S} |f_k(y)| \int_0^1 \int_0^1 \frac{(1-\tau_1)^p}{p!} \frac{(1-\tau_2)^q}{q!} \sqrt{\tau_1 \tau_2} d\tau_1 d\tau_2 \leq c \max_{y \in S} |f_k(y)|, \end{aligned}$$

one can readily establish that

$$\forall x \in B^+ : |\Delta^m u^k(x)| \leq c \max_{y \in S} |f_k(y)|, \quad k=1, \dots, \nu; \quad m=0, \dots, k, \quad (3.89)$$

which by virtue of (3.87) yields (3.88). ■

Let us introduce the notion of a classical solution of Problem (II)<sup>+</sup>.

The solution  $u$  of Problem (II)<sup>+</sup> will be called classical (or a classical solution of the  $\nu + 1$ -th order) if  $\Delta^k u \in C(\bar{B}^+)$ ,  $k=0, 1, \dots, \nu$ .

For the classical solution we have

**Theorem 3.7.** *If  $u$  is a classical solution of Problem (II)<sub>0</sub><sup>+</sup>, then  $\forall x \in B^+ : u(x) = 0$ .*

*Proof.* Assume  $z$  to be arbitrarily chosen in  $B^+$ . Since  $u, \Delta u, \dots, \Delta^\nu u$  are continuous in  $\bar{B}^+$ , for any  $\varepsilon > 0$  there exists a number  $\delta > 0$  such that  $|\Delta^k u(y)| < \varepsilon$ ,  $k=0, 1, \dots, \nu$  for  $y \in S(0, \rho')$ , where  $\rho' = \rho - \delta'$ ,  $0 < \delta' \leq \delta$ . Besides, if the condition  $\delta' < \rho - |z|$  is imposed on  $\delta'$ , then  $z \in B^+(0, \rho')$

The problem

$$\begin{aligned} \forall x \in B(0, \rho') : \Delta^{\nu+1} u(x) &= 0, \\ \forall y \in S(0, \rho') : (\Delta^k u)^+(y) &= \bar{f}_k(y), \quad k=0, 1, \dots, \nu, \end{aligned}$$

where  $\bar{f}_k(y) \equiv \Delta^k u(y)$ , for  $y \in S(0, \rho')$ , has, due to (3.82), the solution

$$\begin{aligned} u(x) &= \sum_{k=0}^{\nu} u^k(x) = \\ &= \sum_{k=0}^{\nu} \sum_{j=0}^k \frac{1}{j!} \left( \frac{|x|^2 - \rho'^2}{4} \right)^j \sum_{|\alpha|=k-j} a_\alpha J^\alpha (\Pi(\bar{f}_k))(x). \end{aligned} \quad (3.90)$$

Under the conditions of the theorem  $\bar{f} \in C^\infty(S(0, \rho'))$ . Now it is easy to find that this is the regular solution of the  $\nu + 1$ -th order in  $B^+(0, \rho')$  and therefore it is unique (see Theorem 3.1).

Now by virtue of Lemma 3.8 (3.90) yields

$$|u(z)| \leq c\varepsilon.$$

Thus  $\forall z \in B^+ : u(z) = 0$ . ■

Lemma 3.7 and Theorem 3.7 provide the proof of

**Theorem 3.8.** *If  $f_k \in C(S)$ ,  $k = 0, 1, \dots, \nu$ , then  $u$  defined by (3.76)-(3.79) is the unique classical solution of Problem (II)<sup>+</sup>.*

**3.9. On One Algorithm of the Solution of Problem (II)<sup>+</sup>.** This subsection will deal with an algorithm of constructing solutions of Problem (II)<sup>+</sup> that differs from those discussed above. We begin by considering an auxiliary problem

$$\begin{aligned} \forall x \in B^+ : \Delta \overset{p}{v}(x) &= (|x|^2 - \rho^2)^p F_p(x), \\ \forall y \in S : \overset{p}{v}^+(y) &= 0, \quad p = 0, 1, \dots, \end{aligned} \quad (3.91)$$

where  $F_p$  is a harmonic function in  $B^+$ , for the nonhomogeneous Laplace equation.

The solution of this problem is sought for in the form

$$\overset{p}{v}(x) = \sum_{l=1}^{p+1} \frac{(|x|^2 - \rho^2)^l}{4l} \psi_l(x), \quad (3.92)$$

where  $\psi_l$  are the desired harmonic functions. It is obvious that  $\overset{p}{v}$  satisfies the boundary condition of problem (3.91).

The equality

$$\begin{aligned} \Delta((r^2 - \rho^2)^l \psi_l) &= 4l(r^2 - \rho^2)^{l-1} \left( \mathcal{D}_r + \frac{2l+1}{2} \right) \psi_l + \\ &+ 4l(l-1)\rho^2(r^2 - \rho^2)^{l-2} \psi_l \end{aligned}$$

enables us to readily conclude that  $\overset{p}{v}$  defined by (3.92) is the solution of equation (3.91) if the functions  $\psi_1, \dots, \psi_{p+1}$  satisfy the conditions

$$\begin{aligned} \left( \mathcal{D}_r + \frac{2p+3}{2} \right) \psi_{p+1} &= F_p, \\ \left( \mathcal{D}_r + \frac{2m+1}{2} \right) \psi_m &= -m\rho^2 \psi_{m+1}, \quad m = 1, 2, \dots, p. \end{aligned}$$

But, as shown above, in the class of harmonic functions equation (3.64) has the unique solution  $u$  represented in form (3.65). Therefore we write

$$\begin{aligned}\psi_{p+1}(x) &= \int_0^1 F_p(\tau x) \tau^{\alpha_{p+1}-1} d\tau, \\ \psi_m(x) &= -m\rho^2 \int_0^1 \psi_{m+1}(\tau x) \tau^{\alpha_m-1} d\tau, \quad m=1, 2, \dots, p,\end{aligned}\tag{3.93}$$

where

$$\alpha_i = \frac{2i+1}{2}, \quad i=1, 2, \dots, p+1.$$

Thus the solution of problem (3.91) is given in form (3.92), where the functions  $\psi_{p+1}, \dots, \psi_1$  are defined by the recurrent relations (3.93).

Let us now turn our attention to Problem (II)<sup>+</sup>:

$$\begin{aligned}\forall x \in B^+ : \Delta^{\nu+1} u(x) &= 0, \quad \nu \in \mathbb{N}, \\ \forall y \in S : (\Delta^k u)^+(y) &= f_k(y), \quad k=0, 1, \dots, \nu.\end{aligned}$$

Introducing the notation,  $v_i = \Delta^i u$ ,  $i=0, 1, \dots, \nu$ , it is obvious that  $v_0 = u$ ,  $\Delta v_{i-1} = v_i$ ,  $i=1, 2, \dots, \nu-1$ ,  $\Delta v_\nu = 0$  and the condition

$$\forall y \in S : v_k^+(y) = f_k(y), \quad k=0, 1, \dots, \nu,$$

is fulfilled. Therefore for  $v_\nu$  we have the problem

$$\begin{aligned}\forall x \in B^+ : \Delta v_\nu(x) &= 0, \\ \forall y \in S : v_\nu^+(y) &= f_\nu(y),\end{aligned}$$

whence we obtain

$$v_\nu(x) = \Pi(f_\nu)(x).\tag{3.94}$$

Now for the functions  $v_{\nu-1}, \dots, v_0$  we have the following recurrent problems:

$$\begin{aligned}\forall x \in B^+ : \Delta v_k(x) &= v_{k+1}(x), \\ \forall y \in S : v_k^+(y) &= f_k(y), \quad k=\nu-1, \dots, 1, 0.\end{aligned}\tag{3.95}$$

Considering these problems successively and remembering the fact that the solution of equations (3.91) is represented in form (3.92), problems (3.95) can be replaced by the problems

$$\begin{aligned}\forall x \in B^+ : \Delta v_k(x) &= \sum_{p=0}^{\nu-k-1} a_{kp} (|x|^2 - \rho^2)^p F_p(x), \\ \forall y \in S : v_k^+(y) &= f_k(y), \quad k=\nu-1, \dots, 0,\end{aligned}\tag{3.96}$$

where  $F_p$  are the known harmonic functions and  $a_{kp}$  the defined constants.

To solve problems (3.96) it suffices to solve problem (3.91) and to write the solution of (3.96) as

$$v_k = \sum_{p=0}^{\nu-k-1} a_{kp} \overset{p}{v} + \Pi(f_k), \quad k = 0, 1, \dots, \nu - 1,$$

where  $\overset{p}{v}$  is defined by (3.92).

Thus the proposed algorithm actually reduces the construction of solutions of Problem (II)<sup>+</sup> to problem (3.91), i.e. to the Dirichlet problem for the nonhomogeneous Laplace equation whose special right-hand side contains the zero boundary condition and is, in turn, reduced to finding the functions  $\psi_1, \dots, \psi_{p+1}$  from the recurrent relations (3.93).

It should be said that the solution of Problem (II)<sup>+</sup> by the algorithm in question is effective for small  $\nu$ . For an arbitrary  $\nu$  the problem of constructing solutions of Problem (II)<sup>+</sup> by this algorithm in the closed form and of finding the differential properties of solutions is evidently a difficult one.

**3.10. Solution of Problem (III)<sup>+</sup>.** Let us consider the following two problems:

**Problem (III)<sub>1</sub><sup>+</sup>.** Find a polyharmonic function  $u'$  of the  $\nu + 1$ -th order in  $B^+$  by the boundary conditions

$$\left(\frac{d^h u'}{dn^h}\right)^+(y) = f_h(y), \quad h = 0, 1, \dots, \varkappa, \quad 1 \leq \varkappa < \nu, \quad (3.97)$$

$$(\Delta^q u')^+(y) = 0, \quad q = \varkappa + 1, \dots, \nu. \quad (3.98)$$

**Problem (III)<sub>2</sub><sup>+</sup>.** Find a polyharmonic function  $u''$  of the  $\nu + 1$ -th order in  $B^+$  by the boundary conditions

$$\left(\frac{d^h u''}{dn^h}\right)^+(y) = 0, \quad h = 0, 1, \dots, \varkappa, \quad 1 \leq \varkappa < \nu, \quad (3.99)$$

$$(\Delta^q u'')^+(y) = f_q(y), \quad q = \varkappa + 1, \dots, \nu. \quad (3.100)$$

Obviously, these problems are the particular cases of Problem (III)<sup>+</sup> whose solution  $u$  is representable as the sum

$$u = u' + u''. \quad (3.101)$$

Thus the solution of Problem (III)<sup>+</sup> reduces to the solution of Problems (III)<sub>1</sub><sup>+</sup> and (III)<sub>2</sub><sup>+</sup>.

The solution of Problem (III)<sub>1</sub><sup>+</sup> is sought for in the form

$$u'(x) = \sum_{p=0}^{\varkappa} \frac{|x|^2 - \rho^2}{2^p \rho^p p!} u'_p(x),$$

where  $u'_0, \dots, u'_\varkappa$  are the desired harmonic functions. It can be easily verified that  $\forall x \in B^+ : \Delta^q u'(x) = 0$ , for  $q = \varkappa + 1, \dots, \nu$  and therefore conditions (3.98) are fulfilled automatically. Just like in the case of deriving the solution of Problem (I)<sup>+</sup> (see (3.35), (3.38)), conditions (3.97) enable us to construct solutions of Problem (III)<sub>1</sub><sup>+</sup> in the form

$$u'(x) = \sum_{k=0}^{\varkappa} \sum_{i=0}^k \frac{1}{k!} \left( \frac{|x|^2 - \rho^2}{2\rho} \right)^k P_{k-i}^i(D_r) \Pi(f_i)(x), \quad (3.102)$$

where

$$\begin{aligned} P_0^k(\xi) &= 1, \quad P_l^k(\xi) = \\ &= - \sum_{j=0}^{l-1} \sum_{i=0}^{\min\{l-j, k+j\}} \frac{2^{-i} \rho^{j-l} (k+l)!}{i! (l-j-i)! (k+j-i)!} \xi^{(l-j-i)k} P_j^k(\xi), \quad (3.103) \\ k &= 0, 1, \dots, \varkappa; \quad l = 1, 2, \dots, \varkappa - k; \\ \xi^{(p)} &\equiv \xi(\xi-1) \cdots (\xi-p+1). \end{aligned}$$

Now consider Problem (III)<sub>2</sub><sup>+</sup>. Its solution is represented by

$$u''(x) = \sum_{p=\varkappa+1}^{\nu} \frac{(|x|^2 - \rho^2)^p}{4^p p!} u_p''(x), \quad (3.104)$$

where  $u''_{\varkappa+1}, \dots, u''_{\nu}$  are the desired harmonic functions.

We observe that by virtue of (3.14) conditions (3.99) are automatically fulfilled for  $u''$  if

$$\mathcal{D}_r^k u_p'' \in C(\bar{B}^+), \quad k = 0, 1, \dots, \varkappa; \quad p = \varkappa + 1, \dots, \nu. \quad (3.105)$$

Like in the case of Problem (II)<sup>+</sup> (see (3.73), (3.74)), to satisfy conditions (3.99) we have the following recurrent relations for  $u_{\nu}, \dots, u_{\varkappa+2}, u_{\varkappa+1}$ :

$$\begin{aligned} u_{\nu}''(x) &= J^{\nu}(\Pi(f_{\nu}))(x), \quad u_j''(x) = J^j(\Pi(f_j))(x) - \\ &- \sum_{i=1}^{\min\{\nu-j, j\}} \left( \frac{\rho^2}{4} \right)^i c_j^i J^i(u_{i+j}'')(x), \quad j = \varkappa + 1, \dots, \nu - 1. \quad (3.106) \end{aligned}$$

Note that if  $f_i \in C(S)$ ,  $i = \varkappa + 1, \dots, \nu$ , then inclusion (3.105) holds (see the proof of Lemma 3.6) for the functions  $u_{\varkappa+1}, \dots, u_{\nu}$ .

Thus the solution of Problem (III)<sup>+</sup> has form (3.101), where  $u'$  is defined by (3.102), (3.103) and  $u''$  by (3.104), (3.106).

The regular solutions of Problem (III)<sup>+</sup> were defined in Subsection 3.2, wherein their uniqueness was also proved. But the uniqueness theorem applies to a wider class of functions than the class of regular functions.

The solution  $u$  will be called the classical solution (or the classical solution of the  $\nu + 1$ -th order) of Problem (III)<sup>+</sup> if

$$\begin{aligned} \frac{d^h u}{dr^h} &\in C(\bar{B}^+), \quad h = 0, 1, \dots, \varkappa, \quad 1 \leq \varkappa < \nu, \\ \Delta^q u &\in C(\bar{B}^+), \quad q = \varkappa + 1, \dots, \nu. \end{aligned}$$

Like for the classical solutions of Problems (I)<sup>+</sup> (II)<sup>+</sup> (see Theorems 3.6 and 3.8), one can prove

**Theorem 3.9.** *If  $f_i \in C^{\varkappa+1-i}(S)$ ,  $i = 0, 1, \dots, \varkappa - 1$ ,  $f_j \in C(S)$ ,  $j = \varkappa, \dots, \nu$ , then  $u = u' + u''$ , where  $u'$  and  $u''$  are respectively defined from (3.102), (3.103) and (3.104), (3.106), is the unique classical solution of Problem (III)<sup>+</sup>.*

**Problem 21.** Give a detailed proof of Theorem 3.9.

**3.11. On the Green Functions.** The solution of Problem (I)<sup>+</sup> (Lauricella problem, see (3.2)) for a polyharmonic equation can be represented by means of the Green function. The Green function of Problem (I)<sup>+</sup> for equation (3.1) is denoted by  $G_{\nu+1}$  and described as the function defined on the product  $B^+ \times B^+$ . Note that for any fixed  $y \in B^+$  the function  $G_{\nu+1}(\cdot, y)$  is a continuous solution of (3.1) in the domain  $B^+ \setminus \{y\}$ . Thus  $\forall x \in B^+ \setminus \{y\}$ :

$$\Delta^{\nu+1} G_{\nu+1}(x, y) = 0.$$

Moreover,  $\forall z \in S$ :

$$\begin{aligned} (G_{\nu+1}(\cdot, y))^+(z) &= 0, \\ \left(\frac{dG_{\nu+1}(\cdot, y)}{dn}\right)^+(z) &= 0, \dots, \left(\frac{d^\nu G_{\nu+1}(\cdot, y)}{dn^\nu}\right)^+(z) = 0, \end{aligned}$$

and hence we have the representation

$$G_{\nu+1}(x, y) = v_0(|x - y|) + g_{\nu+1}(x, y),$$

where  $v_0$  is the fundamental solution (see (3.22)) of (3.1) and  $g_{\nu+1}(\cdot, y)$  the continuous solution of (3.1) in  $B^+$ .

**Problem 22.** Construct the Green function of Problem (II)<sup>+</sup> (Riquier problem, see (3.3)) for the domain  $B^+$ .

*Indication.* See Markhasev [1] and Nicolesco [1].

**Problem 23.** Construct the Green function of Problem (III)<sup>+</sup> (see (3.4)).

Using the Green function  $G_{\nu+1}$ , the solution of Problem (I)<sup>+</sup> is written in the form

$$u(x) = - \sum_{k=0}^{[\frac{\nu}{2}]} \int_S \left( (\Delta^k u)^+(y) \frac{d\Delta_y^{\nu-k} G_{\nu+1}(y, x)}{dn} - \left( \frac{d\Delta^k u}{dn} \right)^+(y) \Delta_y^{\nu-k} G_{\nu+1}(y, x) \right) d_y S, \quad (3.107)$$

where  $[\frac{\nu}{2}]$  is the integer part of the number  $\frac{\nu}{2}$ .

This formula is given for the case  $m = 2$  in Nicolesco [1], Vekua [3], and for any  $m$  in Nicolesco [1]; Bremerkamp, Bottema [1], Boggio [1,2]. It can be obtained from (3.23) if  $v_0$  is replaced by  $G_{\nu+1}$ .

In addition to  $u$  and  $\frac{du}{dn}$  given on the boundary (see (3.2)), this formula also contains  $\Delta u, \frac{d\Delta u}{dn}, \dots$ ; which are not directly given on the boundary. It is true that all these functions can be expressed by means of the given functions  $f_k$ , but such a connection is of local character, decreasing essentially the effectiveness of representation (3.107). It was Bremerkamp and Bottema who paid attention to this fact. In some particular cases ( $\nu = 2$  and  $m = 2$ ,  $\nu = 1$  and  $m$  is arbitrary) these authors succeeded in expressing the integrand in (3.107) through the known functions. Note that our solution representation formulas are free from this drawback.

**Problem 24.** Find a polyharmonic function  $u$  of the  $\nu + 1$ -th order in the domain  $B^-$  by the boundary conditions

$$\forall y \in S : \left( \frac{d^k u}{dn^k} \right)^-(y) = f_k(y), \quad k = 0, 1, \dots, \nu,$$

where  $f_0, \dots, f_\nu$  are the known functions on  $S$  (Problem (I)<sup>-</sup>).

**Problem 25.** Find a polyharmonic function  $u$  of the  $\nu + 1$ -th order in  $B^-$  by the boundary conditions

$$\forall y \in S : (\Delta^k u)^-(y) = f_k(y), \quad k = 0, 1, \dots, \nu,$$

where  $f_0, \dots, f_\nu$  are the known functions on  $S$  (Problem (II)<sup>-</sup>).

**Problem 26.** Find a polyharmonic function  $u$  of the  $\nu + 1$ -th order in  $B^-$  by the boundary conditions

$$\begin{aligned} \forall y \in S : \left( \frac{d^h u}{dn^h} \right)^-(y) &= f_h(y), \quad h = 0, 1, \dots, \varkappa, \\ (\Delta^q u)^-(y) &= f_q(y), \quad q = \varkappa + 1, \dots, \nu, \end{aligned}$$

where  $f_0, \dots, f_\nu$  are the known functions (Problem (III)<sup>-</sup>).

**Problem 27.** Find a condition (at infinity) that provides the uniqueness of the solutions of Problems (I)<sup>-</sup>, (II)<sup>-</sup>, (III)<sup>-</sup>.

\* \* \*

In conclusion, we observe that the polyharmonic equation (3.1) becomes, for  $\nu = 0$ , the Laplace equation which was dealt with in Chapter I. In Chapter III consideration was given to the case  $\nu = 1, 2, \dots$ . When  $\nu = 1$ , equation (3.1) is called biharmonic. Using the Airy stress function, one can reduce systems of equations of plane elasticity to the biharmonic equation. When  $\nu > 1$ , the polyharmonic equation is the simplest differential equation of a higher order. Evidently this and other reasons explain special interest of the researchers in this equation. Among numerous works devoted to the polyharmonic equation we would like to mention Mathieu [1], Gutzmer [1], Venske [1], Almansi [1-3], Lauricella [1-3], Volterra [1], Levi-Civita [1], Goursat [2], Marcolongo [3], Boggio [1], Quintilli [1], Pizzeti [1], Riquier [1], Riesz [1], Nicolesco [1,2], Saks [1], Muskhelishvili [3], Picone [1], Privalov [1], Vekua [1-4], Schröder [1,2], Tolotti [1], Miranda [1,2], Fichera [1], Bremerkamp [1-3], Bremerkamp and Bottama [1], Markhasev [1], Chicinadze [7,8] and others.

CHAPTER IV  
BOUNDARY VALUE PROBLEMS OF THERMOELASTICITY

**4.1. Formulation of Problems and Auxiliary Propositions.** Consider a system of static equations of thermoelasticity (Nowacki [1], Kovalenko [1], Kupradze *et al.* [1])

$$\begin{aligned} \mu \Delta u(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} u(x) - \gamma \operatorname{grad} \theta(x) &= 0, \\ \Delta \theta(x) &= 0, \end{aligned} \quad (4.1)$$

where  $x = (x_1, x_2, x_3)$  is a point of the Euclidean space  $\mathbb{R}^3$ ,  $\Delta$  the Laplace operator,  $u = (u_1, u_2, u_3)$  the displacement vector,  $\theta$  the temperature,  $\lambda$  and  $\mu$  the Lamé constants,  $\gamma = \alpha(3\lambda + 2\mu)$ ,  $\alpha$  the linear thermal expansion coefficient. The constants  $\lambda$ ,  $\mu$  and  $\gamma$  satisfy the conditions

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad \gamma \neq 0. \quad (4.2)$$

System (4.1) contains the unknown displacement  $u$  which is a vector value and the unknown temperature  $\theta$  which is a scalar value, i.e. we are to find the pair  $(u, \theta)$  or the four of scalar values  $(u_1, u_2, u_3, \theta)$ .

The vector determined by

$$P^{(n)} \equiv \tau^{(n)} - \gamma \theta n, \quad (4.3)$$

where  $\tau^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)}, \tau_3^{(n)})$ ,

$$\tau_i^{(n)} = \lambda n_i \operatorname{div} u + \mu n_j \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (4.3')$$

is the stress vector in the theory of elasticity, is called the stress vector in the theory of thermoelasticity.

As previously,  $B^+$  denotes the ball with centre at the origin and radius  $\rho$ ,  $B^-$  denotes the entire space with a spherical cavity  $B^+$  ( $B^- \equiv \mathbb{R}^3 \setminus \bar{B}^+$ ),  $S \equiv \partial B^+ = \partial B^- = \{x \in \mathbb{R}^3 \mid |x| = \rho\}$ .

This chapter is totally devoted to solving in quadratures the following boundary value problems:

Find in  $B^+$  and in  $B^-$  a continuous vector  $u$  and a continuous function  $\theta$  such that the pair  $(u, \theta)$  satisfy in  $B^+$  and accordingly in  $B^-$  system (4.1), using the boundary condition

$$(u)^\pm = f, \quad (\theta)^\pm = \eta \quad (4.4)$$

– Problem (I.I) $^\pm$ ; or

$$(u)^\pm = f, \quad \left( \frac{\partial \theta}{\partial n} \right)^\pm = \eta \quad (4.5)$$

– Problem (I.II) $^\pm$ ; or

$$(\tau^{(n)} - \gamma \theta n)^\pm = f, \quad (\theta)^\pm = \eta \quad (4.6)$$

– Problem (II.I) $^\pm$ ; or

$$(\tau^{(n)} - \gamma\theta n)^\pm = f, \quad \left(\frac{\partial\theta}{\partial n}\right)^\pm = \eta \quad (4.7)$$

– Problem (II.II) $^\pm$ ; or

$$(n \cdot u)^\pm = g, \quad (\tau^{(n)} - n(n \cdot \tau^{(n)}))^\pm = l, \quad (\theta)^\pm = \eta \quad (4.8)$$

– Problem (III.I) $^\pm$ ; or

$$(n \cdot u)^\pm = g, \quad (\tau^{(n)} - n(n \cdot \tau^{(n)}))^\pm = l, \quad \left(\frac{\partial\theta}{\partial n}\right)^\pm = \eta \quad (4.9)$$

– Problem (III.II) $^\pm$ ; or

$$(n \cdot \tau^{(n)} - \gamma\theta)^\pm = g, \quad (u - n(n \cdot u))^\pm = l, \quad (\theta)^\pm = \eta \quad (4.10)$$

– Problem (IV.I) $^\pm$ ; or

$$(n \cdot \tau^{(n)} - \gamma\theta)^\pm = g, \quad (u - n(n \cdot u))^\pm = l, \quad \left(\frac{\partial\theta}{\partial n}\right)^\pm = \eta \quad (4.11)$$

– Problem (IV.II) $^\pm$ .

Here  $f = f = (f_1, f_2, f_3)$ ,  $l = l = (l_1, l_2, l_3)$ ,  $g$  and  $\eta$  are functions given on  $S$ .

If  $f = 0$ ,  $l = 0$ ,  $g = 0$ ,  $\eta = 0$ , then the corresponding problems will be called homogeneous and denoted by the same symbols adding to them the subscript “zero”: (I.I) $_0^\pm, \dots, (IV.II)_0^\pm$ .

A solution  $(u, \theta)$  of system (4.1) will be called regular if  $u, \theta \in C^1(\bar{B}^\pm) \cap C^2(B^\pm)$ . A solution  $(u, \theta)$  of system (4.1) will however be called classical if  $u, \theta \in C(\bar{B}^\pm) \cap C^2(B^\pm)$  and, additionally, if  $(\frac{\partial\theta}{\partial n})^\pm \in C(S)$  in the case of Problem (I.II) $^\pm$ , if  $(\tau^{(n)})^\pm \in C(S)$  in the case of Problems (II.I) $^\pm, (III.I)^\pm, (IV.I)^\pm$ , and if  $(\tau^{(n)})^\pm, (\frac{\partial\theta}{\partial n})^\pm \in C(S)$  in the case of Problems (II.II) $^\pm, (III.II)^\pm, (IV.II)^\pm$ .

The uniqueness theorems are proved in Kupradze *et al.* [1] and Nowacki [1]. Moreover, regular solutions of these problems exist if the boundary data satisfy – in addition to certain conditions of smoothness – conditions of solvability as well.

The results obtained in Kupradze *et al.* [1]; Buchukuri, Gegelia [1,2] readily imply the following theorems needed for our purposes.

**Theorem 4.1.** *If regular solutions  $(u, \theta)$  of the homogeneous Problems (I.I) $_0^\pm, \dots, (IV.II)_0^\pm$  satisfy, near the point at infinity, the conditions*

$$u(x) = o(1), \quad \theta(x) = o(1),$$

*then these problems have only trivial solutions*

$$\forall x \in B^- : u(x) = 0, \quad \theta(x) = 0.$$

**Theorem 4.2.** *The homogeneous Problems (I.I)<sub>0</sub><sup>+</sup> and (IV.I)<sub>0</sub><sup>+</sup> have only trivial regular solutions  $\forall x \in B^+ : u(x) = 0, \theta(x) = 0$ .*

**Theorem 4.3.** *All regular solutions of Problem (I.II)<sub>0</sub><sup>+</sup> are given in the form  $\forall x \in B^+ : u(x) = 0, \theta(x) = \theta_0$ , where  $\theta_0$  is an arbitrary constant.*

**Theorem 4.4.** *All regular solutions of Problem (II.I)<sub>0</sub><sup>+</sup> are given in form  $\forall x \in B^+ : u(x) = [a \times x] + b, \theta(x) = 0$ , where  $a$  and  $b$  are arbitrary three-dimensional constant vectors.*

**Theorem 4.5.** *All regular solutions of Problem (II.II)<sub>0</sub><sup>+</sup> are given in form  $\forall x \in B^+ : u(x) = \gamma\theta_0 \overset{\circ}{u}(x) + [a \times x] + b$ , where  $\theta_0$  is an arbitrary constant;  $a$  and  $b$  are arbitrary three-dimensional constant vectors, and  $\overset{\circ}{u}$  is a solution of the problem*

$$\begin{aligned} \mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u &= 0, \\ (\tau^{(n)})^+ &= n. \end{aligned} \quad (4.12)$$

Note that the latter problem is solvable and, as one can easily verify, its solution is given by

$$\overset{\circ}{u}(x) = \frac{1}{3\lambda + 2\mu} x + [a' \times x] + b', \quad (4.13)$$

where  $a'$  and  $b'$  are arbitrary three-dimensional constant vectors.

**Theorem 4.6.** *All regular solutions of Problem (III.I)<sub>0</sub><sup>+</sup> are given in the form  $\forall x \in B^+ : u(x) = [a \times x], \theta(x) = 0$ , where  $a$  is an arbitrary three-dimensional constant vector.*

**Theorem 4.7.** *All regular solutions of Problem (III.II)<sub>0</sub><sup>+</sup> are given in the form  $\forall x \in B^+ : u(x) = [a \times x], \theta(x) = \theta_0$ , where  $a$  is an arbitrary three-dimensional constant vector and  $\theta_0$  an arbitrary constant.*

**Theorem 4.8.** *All regular solutions of Problem (IV.II)<sub>0</sub><sup>+</sup> are given in the form  $\forall x \in B^+ : u(x) = \gamma\theta_0 \overset{1}{u}(x), \theta(x) = \theta_0$ , where  $\theta_0$  is an arbitrary constant and  $\overset{1}{u}$  is a solution of the problem*

$$\begin{aligned} \mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u &= 0, \\ (n\tau^{(n)})^+(y) = 1, \quad (u - n(n \cdot u))^+(y) &= 0, \end{aligned} \quad (4.14)$$

given by the formula

$$\overset{1}{u}(x) = \frac{1}{3\lambda + 2\mu} x. \quad (4.15)$$

Theorems 4.1 to 4.8 are easily proved if one takes into account the fact that the static problems of thermoelasticity are divided into problems for  $\theta$  (boundary value problems for the Laplace equation) and problems for  $u$  (boundary value problems of classical elasticity). For example, if  $(u, \theta)$  is a regular solution of Problem (II.II) $_0^+$ , then

$$\Delta\theta = 0, \quad \left(\frac{\partial\theta}{\partial n}\right)^+ = 0,$$

$$\mu\Delta u + (\lambda + \mu)\text{grad div } u - \gamma\text{grad } \theta = 0, \quad (\tau^{(n)} - \gamma\theta n)^+ = 0.$$

The former of these problems is the Neumann problem for the Laplace equation and has the solution  $\theta = \theta_0$ , where  $\theta_0$  is an arbitrary constant. The latter problem, upon substitution  $\theta = \theta_0$ , takes the form

$$\mu\Delta u + (\lambda + \mu)\text{grad div } u = 0, \quad (\tau^{(n)})^+ = \gamma\theta_0 n.$$

This actually is the second problem of classical elasticity, and keeping in mind the uniqueness of its solution, as well as the fulfilment of the solvability conditions

$$\int_S n(y) d_y S = 0, \quad \int_S [y \times n(y)] d_y S = 0,$$

one can easily make sure that Theorem 4.5 is valid.

Let us discuss in a more elaborate way the conditions of solvability of the boundary value problems.

Problems (I.I) $^\pm$ , (I.II) $^-$ , (II.I) $^-$ , (III.I) $^\pm$ , (III.II) $^-$ , (IV.I) $^\pm$  and (IV.II) $^-$  are solvable for any sufficiently smooth (see Kupradze *et al.*[1]) boundary data (i.e. for any smooth  $f, l, g$  and  $\eta$ ).

Consider Problem (II.I) $^+$ :

$$\mu\Delta u + (\lambda + \mu)\text{grad div } u - \gamma\text{grad } \theta = 0, \tag{4.16}$$

$$(\tau^{(n)} - \gamma\theta n)^+ = f,$$

$$\Delta\theta = 0, \quad (\theta)^+ = \eta. \tag{4.17}$$

Problem (4.17) is the Dirichlet problem for the Laplace equation and it is solvable for any continuous  $\eta$ . Substituting  $\theta$  from (4.17) in (4.16), we obtain the second boundary value problem of classical elasticity

$$\mu\Delta u + (\lambda + \mu)\text{grad div } u = \Phi, \quad (\tau^{(n)})^+ = F, \tag{4.18}$$

where  $\Phi = \gamma\text{grad } \theta$ ,  $F = f + \gamma n(\theta)^+$ .

For this problem to be solvable (see Kupradze *et al.* [1]) it is necessary that the main vector and the principal moment of external force be both equal to zero:

$$\int_{B^+} \Phi(x) dx - \int_S F(y) d_y S = 0,$$

$$\int_{B^+} [x \times \Phi(x)] dx - \int_S [y \times F(y)] d_y S = 0$$

or, taking into account the expressions for  $\Phi$  and  $F$ , that

$$\int_S f(y) d_y S = 0, \quad \int_S [y \times f(y)] d_y S = 0. \quad (4.19)$$

These conditions will be regarded as fulfilled for Problem (II.I)<sup>+</sup>.

The conditions of solvability are

$$\int_S f(y) d_y S = 0, \quad \int_S [y \times f(y)] d_y S = 0, \quad \int_S \eta(y) d_y S = 0 \quad (4.20)$$

for Problem (II.II)<sup>+</sup>;

$$\int_S [y \times l(y)] d_y S = 0 \quad (4.21)$$

for Problem (III.I)<sup>+</sup>;

$$\int_S [y \times l(y)] d_y S = 0, \quad \int_S \eta(y) d_y S = 0 \quad (4.22)$$

for Problem (III.II)<sup>+</sup>, and

$$\int_S \eta(y) d_y S = 0 \quad (4.23)$$

for Problems (I.II)<sup>+</sup> and (IV.II)<sup>+</sup>.

**4.2. Special Representations of Solutions.** In constructing in quadratures the solutions of the problems stated in the preceding subsection, we make use of special representations of displacements by harmonic functions that are similar to (2.12), (2.16), (2.18). The theorems below are easy to prove.

**Theorem 4.9.** *If*

$$u = v + \frac{\rho^2 - r^2}{2} \text{grad } \psi, \quad (4.24)$$

where  $\Delta v = 0$ ,  $\Delta \psi = 0$ ,

$$(\mathcal{D}_r + \alpha)\psi = \beta(\text{div } v - \delta\theta), \quad (4.25)$$

$\rho$  is a positive integer constant,  $\theta$  any solution of the equations  $\Delta\theta = 0$ ,  $r \equiv |x|$ ,  $\mathcal{D}_r \equiv r \frac{\partial}{\partial r} = x_k \frac{\partial}{\partial x_k}$ ,

$$\alpha \equiv \frac{\mu}{\lambda + 3\mu}, \quad \beta \equiv \frac{\lambda + \mu}{\lambda + 3\mu}, \quad \delta \equiv \frac{\gamma}{\lambda + \mu}, \quad (4.26)$$

then the pair  $(u, \theta)$  is a solution of system (4.1) in  $\mathbb{R}^3 \setminus S(0, \rho)$ .

**Theorem 4.10.** *If*

$$u(x) = v(x) + x(2\mathcal{D}_r + 1)\psi(x) + \frac{\rho^2 - 3r^2}{2} \text{grad } \psi(x), \quad (4.27)$$

where  $\Delta v = 0$ ,  $\Delta\psi = 0$ ,

$$\begin{aligned} 2\mathcal{D}_r^2\psi + 2\frac{2\lambda + \mu}{\lambda + \mu}\mathcal{D}_r\psi + \frac{3\lambda + 2\mu}{\lambda + \mu}\psi &= \delta\theta - \text{div } v, \\ \Delta\theta &= 0, \end{aligned} \quad (4.28)$$

then  $(u, \theta)$  is a solution of system (4.1) in  $\mathbb{R}^3 \setminus S(0, \rho)$  and

$$\frac{r}{\mu}(\tau^{(n)} - \gamma n\theta) = h + (\rho^2 - r^2)\mathcal{D}_r \text{grad } \psi, \quad (4.29)$$

where  $n(x) \equiv \frac{x}{|x|}$ , and  $h = h = (h_1, h_2, h_3)$ ,

$$h_i(x) = x_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right) - x_i \text{div } v(x). \quad (4.30)$$

**Theorem 4.11.** *If*

$$u(x) = v(x) + x(2\mathcal{D}_r + 1)\varphi(x) + \frac{\rho^2 - r^2}{2} \text{grad}(\psi(x) + 2\varphi(x)), \quad (4.31)$$

where  $\Delta v = 0$ ,  $\Delta\varphi = 0$ ,  $\Delta\psi = 0$ ,

$$\begin{aligned} (\mathcal{D}_r + \alpha)\psi &= \beta(\text{div } v - \delta\theta + 2\mathcal{D}_r^2\varphi + 5\mathcal{D}_r\varphi + 3\varphi), \\ \Delta\theta &= 0, \end{aligned} \quad (4.32)$$

then the pair  $(u, \theta)$  is a solution of system (4.1) in  $\mathbb{R}^3 \setminus S(0, \rho)$ .

**Problem 28.** Prove Theorems 4.9 to 4.11.

Representation (4.24) will be used to prove Problems (I.I) $^\pm$ , (I.II) $^\pm$ , (III.I) $^\pm$ , (III.II) $^\pm$ . Representation (4.27) is convenient to prove Problems (II.I) $^\pm$  and (II.II) $^\pm$ , whereas Problems (IV.I) $^\pm$  and (IV.II) $^\pm$  are solved by (4.31).

Note that in dealing with the problems of thermoelasticity for  $\theta$  we have the Dirichlet or Neumann problem and therefore  $\theta$  can be regarded as determined. Our task is to determine  $u$ .

In the rest of the subsections of this chapter we will concern ourselves mainly with the construction of solutions of the boundary value problems of thermoelasticity and formulate the main properties of solutions without proving. These properties (as will be clear from the form of solutions) readily follow from the corresponding properties of solutions of the boundary value problems of classical elasticity and from the properties of solutions of the Dirichlet or Neumann problems for the Laplace equation.

**4.3. Solution of Problems (I.I)<sup>±</sup>, (I.II)<sup>±</sup>.** We are to find the displacement vector  $u$  in form (4.24). Then in the case of Problem (I.I)<sup>+</sup>, we have the Dirichlet problem for  $\theta$  and  $v$  in  $B^+$ .  $\theta$  and  $v$  are determined by the Poisson formulas

$$\theta = \Pi(\eta), \quad v = \Pi(f). \quad (4.33)$$

Substituting the found values of  $\theta$  and  $v$  in (4.25) and treating the latter as a differential equation for  $\psi$ , we obtain

$$(\mathcal{D}_r + \alpha)\psi = F, \quad (4.34)$$

where

$$F = \beta(\operatorname{div} \Pi(f) - \delta \Pi(\eta)).$$

The solution of (4.34) in the class of harmonic functions has the form

$$\begin{aligned} \psi(x) = & \frac{\beta}{4\pi\rho} \operatorname{div} \int_S \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \frac{d\tau}{\tau^{2-\alpha}} f(y) d_y S - \\ & - \frac{\beta\delta}{4\pi\rho} \int_S \int_0^1 \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} \frac{d\tau}{\tau^{1-\alpha}} \eta(y) d_y S. \end{aligned} \quad (4.35)$$

Using (4.24), (4.33) and (4.35), the solution of Problem (I.I)<sup>+</sup> can be written as

$$\begin{aligned} u(x) &= \int_S K(x, y) f(y) d_y S + \gamma \int_S \Theta(x, y) \eta(y) d_y S, \\ \theta(x) &= \Pi(\eta)(x), \end{aligned} \quad (4.36)$$

where  $K$  is determined by (2.25) and

$$\begin{aligned} \Theta &= \Theta = (\Theta_1, \Theta_2, \Theta_3), \\ \Theta_k(x, y) &= -\frac{\rho^2 - |x|^2}{8\pi(\lambda + 3\mu)\rho} \frac{\partial}{\partial x_k} \int_0^1 \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} \frac{d\tau}{\tau^{1-\alpha}}. \end{aligned}$$

Problem (I.II)<sup>+</sup> is solved similarly by the formula

$$\begin{aligned} u(x) &= \int_S K(x, y) f(y) d_y S + \gamma \int_S \tilde{\Theta}(x, y) \eta(y) d_y S, \\ \theta(x) &= N(\eta), \end{aligned} \quad (4.37)$$

where  $K$  is determined by (2.25) and  $N$  by (1.7),  $\tilde{\Theta} = \tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3)$ ,

$$\tilde{\Theta}_k(x, y) = -\frac{\rho^2 - |x|^2}{8\pi(\lambda + 3\mu)\rho} \frac{\partial}{\partial x_k} \int_0^1 \left( \frac{2\rho}{|y - \tau x|} - \right.$$

$$-\ln(|\tau x - y| + \rho)^2 - |\tau x|^2) \frac{d\tau}{\tau^{1-\alpha}}.$$

**Theorem 4.12.** *If  $f, \eta \in C(S)$ , then the pair  $(u, \theta)$  determined by (4.36) is the unique classical solution of Problem (I.I)<sup>+</sup>.*

**Theorem 4.13.** *If  $f, \eta \in C(S)$  and  $\eta$  satisfies condition (4.23), then the pair  $(u, \theta)$  determined by (4.37) is the classical solution of Problem (I.II)<sup>+</sup>. In this case  $u$  is determined uniquely and  $\theta$  to within an arbitrary constant.*

**Problem 29.** Prove Theorems 4.12 and 4.13.

In a manner similar to the above one the solution of Problem (I.I)<sup>-</sup> is given by

$$\begin{aligned} u(x) &= \int_S K'(x, y) f(y) d_y S + \gamma \int_S \Theta'(x, y) \eta(y) d_y S, \\ \theta(x) &= \Pi'(\eta)(x) \end{aligned} \quad (4.38)$$

and the solution of Problem (I.II)<sup>-</sup> by

$$\begin{aligned} u(x) &= \int_S K'(x, y) f(y) d_y S + \gamma \int_S \tilde{\Theta}'(x, y) \eta(y) d_y S, \\ \theta(x) &= N'(\eta)(x), \end{aligned} \quad (4.39)$$

where  $K'$  is determined by (2.37),  $\Pi'$  by (1.1') and  $N'$  by (1.7'),  $\Theta' = \Theta' = (\Theta'_1, \Theta'_2, \Theta'_3)$ ,  $\tilde{\Theta}' = \tilde{\Theta}' = (\tilde{\Theta}'_1, \tilde{\Theta}'_2, \tilde{\Theta}'_3)$ ,

$$\begin{aligned} \Theta'_k(x, y) &= -\frac{|x|^2 - \rho^2}{8\pi(\lambda + 3\mu)\rho} \frac{\partial}{\partial x_k} \int_0^1 \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^3} \tau^{-\alpha} d\tau, \\ \tilde{\Theta}'_k(x, y) &= \frac{|x|^2 - \rho^2}{8\pi(\lambda + 3\mu)\rho} \frac{\partial}{\partial x_k} \int_0^1 \left( \frac{2\rho\tau}{|x - \tau y|} - \ln \frac{|x - \tau y| + |x| + \tau\rho}{|x - \tau y| + |x| - \tau\rho} \right) \frac{d\tau}{\tau^{1+\alpha}}. \end{aligned}$$

**Theorem 4.14.** *If  $f, \eta \in C(S)$ , then the pair  $(u, \theta)$  defined by (4.38) [(4.39)] is the unique classical solution of Problem (I.I)<sup>-</sup> [(I.II)<sup>-</sup>].*

**Problem 30.** Prove Theorem 4.14.

**4.4. Solution of Problems (II.I)<sup>+</sup>, (II.II)<sup>+</sup>.** The solution is to be found in form (4.27). One can easily verify that  $h$  determined by (4.30) is a harmonic function in  $B^+$ , whereas from (4.29), (4.6), (4.7) we obtain  $(h)^+ = \frac{\rho}{\mu} f$ .

Thus  $h$  is the solution of the Dirichlet problem and is written as (2.43).

To find  $v$  we obtain equation (2.48) whose solution is given by (2.60), *i.e.*

$$\begin{aligned} v_i(x) = & \frac{1}{8\pi\mu} \int_S \left( (\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y)) \delta_{ik} + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \right. \\ & \left. - \Phi^{(2)}(x, y) + \frac{3x \cdot y}{\rho^3}) - x_i \frac{\partial}{\partial x_k} (2\Phi^{(1)}(x, y) + \frac{3xy}{\rho^3}) + \right. \\ & \left. + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y)) \right) f_k(y) d_y S + \\ & + \varepsilon_{ijk} a_j x_k + b_i, \end{aligned} \quad (4.40)$$

where  $a_1, a_2, a_3, b_1, b_2, b_3$  are arbitrary constants,  $\Phi^{(1)}$  and  $\Phi^{(2)}$  are determined by (2.53).

Formula (4.30) implies  $\operatorname{div} v = -\operatorname{div} h$  and to determine  $\psi$  we obtain (see also (4.28)) the equation

$$2\mathcal{D}_r^2 \psi + 2 \frac{2\lambda + \mu}{\lambda + \mu} \mathcal{D}_r \psi + \frac{3\lambda + 2\mu}{\lambda + \mu} \psi = \delta\theta + \operatorname{div} h, \quad (4.41)$$

which can be rewritten as

$$\begin{aligned} 2\mathcal{D}_r^2(\psi - c) + 2 \frac{2\lambda + \mu}{\lambda + \mu} \mathcal{D}_r(\psi - c) + \frac{3\lambda + 2\mu}{\lambda + \mu}(\psi - c) &= F, \\ F(x) = \delta(\theta(x) - \theta(0)) + \operatorname{div} h(x) - \operatorname{div} h(0), & \quad (4.41') \\ c = \frac{\lambda + \mu}{3\lambda + 2\mu} (\delta\theta(0) + \operatorname{div} h(0)). & \end{aligned}$$

The latter equation actually coincides with (2.61), both equations differing only in their right-hand sides. Therefore the solution of equation (4.41') can be immediately written when  $\mu > 0$ ,  $\lambda > \frac{\sqrt{3}-3}{2}\mu$ , and if, besides, we take into account the fact that  $\theta$  is expressed by the Poisson formula in the case of Problem (II.I)<sup>+</sup> or by the Neumann formula in the case of Problem (II.II)<sup>+</sup>, we will have

$$\psi(x) = \frac{\gamma}{8\pi\rho(\lambda + \mu)} \int_S \mathcal{P}_1(x, y) \eta(y) d_y S + \frac{1}{8\pi\mu} \operatorname{div} \int_S \Psi(x, y) f(y) d_y S + c_1$$

for Problem (II.I)<sup>+</sup>, or

$$\psi(x) = \frac{\gamma}{8\pi\rho(\lambda + \mu)} \int_S \mathcal{P}_2(x, y) \eta(y) d_y S + \frac{1}{8\pi\mu} \operatorname{div} \int_S \Psi(x, y) f(y) d_y S + c_2$$

for Problem (II.II)<sup>+</sup>.

Here  $\Psi$  is determined by (2.69),

$$\begin{aligned}\mathcal{P}_1(x, y) &= -\frac{1}{k_2} \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{1+k_1}}, \\ \mathcal{P}_2(x, y) &= -\frac{1}{k_2} \int_0^1 \left( \frac{2\rho}{|y - \tau x|} - 2 - \ln \frac{(|\tau x - y| + \rho)^2 - |\tau x|^2}{4\rho^2} \right) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{1+k_1}}, \\ c_1 &= \frac{\gamma}{4\pi\rho^2(3\lambda + 2\mu)} \int_S \eta(y) d_y S + \frac{3(\lambda + \mu)}{4\pi\rho^3\mu(3\lambda + 2\mu)} \int_S y f(y) d_y S, \\ c_2 &= \frac{3(\lambda + \mu)}{4\pi\rho^3\mu(3\lambda + 2\mu)} \int_S y f(y) d_y S, \\ k_1 &= -\frac{2\lambda + \mu}{2(\lambda + \mu)}, \quad k_2 = \frac{\sqrt{2\lambda^2 + 6\lambda\mu + 3\mu^2}}{2(\lambda + \mu)}, \quad -1 < k_1 < \frac{1}{2}, \quad k_2 > 0.\end{aligned}$$

Hence for the solution of Problem (II.I)<sup>+</sup> we finally obtain

$$\begin{aligned}\theta &= \Pi(\eta), \\ u_i(x) &= \frac{1}{8\pi\mu} \int_S \left( (\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y)) \delta_{ik} + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \right. \\ &\quad \left. - \Phi^{(2)}(x, y) + \frac{3xy}{\rho^3}) + x_i \frac{\partial}{\partial x_k} ((2\mathcal{D}_r - 1)\Psi(x, y) - 2\Phi^{(1)}(x, y) - \right. \\ &\quad \left. - \frac{\lambda}{3\lambda + 2\mu} \frac{3xy}{\rho^3}) + \frac{\rho^2 - r^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \Psi(x, y) + \right. \\ &\quad \left. + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y) - \Psi(x, y)) \right) f_k(y) d_y S + \\ &\quad + \frac{\gamma}{8\pi\rho(\lambda + \mu)} \int_S \left( x_i \left( \frac{2(\lambda + \mu)}{(3\lambda + 2\mu)\rho} + \mathcal{P}_1(x, y) + 2\mathcal{D}_r \mathcal{P}_1(x, y) \right) + \right. \\ &\quad \left. + \frac{\rho^2 - 3r^2}{2} \frac{\partial}{\partial x_i} \mathcal{P}_1(x, y) \right) \eta(y) d_y S + \varepsilon_{ijk} a_j x_k + b_i \quad (4.42)\end{aligned}$$

and for the stress vector (see (4.29), (4.3))

$$\begin{aligned}\tau^{(n)}(x) - \gamma\theta(x)n(x) &= \frac{1}{4\pi|x|} \int_S \frac{\rho^2 - |x|^2}{|x - y|^3} f(y) d_y S + \\ &\quad + \frac{\rho^2 - |x|^2}{8\pi|x|} \text{grad div} \int_S (\mathcal{D}_r - 2)\Psi(x, y) f(y) d_y S + \\ &\quad + \frac{\gamma\mu(\rho^2 - |x|^2)}{8\pi\rho(\lambda + \mu)|x|} \text{grad} \int_S (\mathcal{D}_r - 1)\mathcal{P}_1(x, y) \eta(y) d_y S.\end{aligned}$$

The solution of Problem (II.II)<sup>+</sup> is written as

$$\begin{aligned}
\theta(x) &= N(\eta)(x) + \theta_0, \\
u_i(x) &= \frac{1}{8\pi\mu} \int_S \left( (\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y)) \delta_{ik} + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \right. \\
&\quad \left. - \Phi^{(2)}(x, y) + \frac{3xy}{\rho^3}) + x_i \frac{\partial}{\partial x_k} ((2\mathcal{D}_r - 1)\Psi(x, y) - 2\Phi^{(1)}(x, y) - \right. \\
&\quad \left. - \frac{\lambda}{3\lambda + 2\mu} \frac{3xy}{\rho^3}) + \frac{\rho^2 - |x|^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \Psi(x, y) + \right. \\
&\quad \left. + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y) - \Psi(x, y)) \right) f_k(y) d_y S + \\
&\quad + \frac{\gamma}{8\pi\rho(\lambda + \mu)} \int_S (\mathcal{P}_2(x, y) + 2\mathcal{D}_r \mathcal{P}_2(x, y) + \\
&\quad + \frac{\rho^2 - 3r^2}{2} \frac{\partial}{\partial x_i} \mathcal{P}_2(x, y)) \eta(y) d_y S + \\
&\quad + \frac{\gamma\theta_0}{3\lambda + 2\mu} x_i + \varepsilon_{ijk} a_j x_k + b_i
\end{aligned} \tag{4.43}$$

and the stress vector as

$$\begin{aligned}
\tau^{(n)}(x) - \gamma\theta(x)n(x) &= \frac{1}{4\pi|x|} \int_S \frac{\rho^2 - |x|^2}{|x - y|^3} f(y) d_y S + \\
&\quad + \frac{\rho^2 - |x|^2}{8\pi|x|} \text{grad div} \int_S (\mathcal{D}_r - 2)\Psi(x, y) f(y) d_y S + \\
&\quad + \frac{\gamma\mu(\rho^2 - |x|^2)}{8\pi\rho(\lambda + \mu)|x|} \text{grad} \int_S (\mathcal{D}_r - 1)\mathcal{P}_2(x, y) \eta(y) d_y S.
\end{aligned} \tag{4.44}$$

**Theorem 4.15.** *If  $f, \eta \in C(S)$  and conditions (4.19) [(4.20)] are fulfilled, then  $(u, \theta)$  determined by (4.42) [(4.43)] is the classical solution of Problem (II.I)<sup>+</sup> [(II.II)<sup>+</sup>].*

Problems (II.I)<sup>-</sup> and (II.II)<sup>-</sup> are solved quite similarly.

**Problem 31.** Prove Theorem 4.15.

**Problem 32.** Construct the solutions of Problems (II.I)<sup>-</sup> and (II.II)<sup>-</sup>.

Note that for the classical solutions of Problems (II.I)<sup>+</sup> and (II.II)<sup>+</sup> we have uniqueness theorems similar to Theorems 4.1, 4.4 and 4.5. They are corollaries of the Green formulas for classical solutions.

*Remark.* Consider the problem

$$\mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u = 0, \quad (\tau^{(n)} - \gamma\theta_0 n)^+ = 0, \quad (4.45)$$

where  $\theta_0$  is an arbitrary constant.

Solve this problem by the method proposed above. The solution  $u$  should have form (4.27); then for  $v$  and  $\psi$  we obtain the equations

$$\begin{aligned} r^2 \frac{\partial^2 v}{\partial r^2} &= 0, \\ 2\mathcal{D}_r^2 \psi + 2 \frac{2\lambda + \mu}{\lambda + \mu} \mathcal{D}_r \psi + \frac{3\lambda + 2\mu}{\lambda + \mu} \psi &= \frac{\gamma}{\lambda + \mu} \theta_0, \end{aligned}$$

whose solutions in the class of harmonic functions have the form

$$v(x) = a' \times x + b', \quad \psi(x) = \frac{\gamma\theta_0}{3\lambda + 2\mu},$$

where  $a'$  and  $b'$  are any three-dimensional constant vectors.

Now, by virtue of (4.27), the solution of problem (4.45) takes the form

$$u(x) = \frac{\gamma\theta_0}{3\lambda + 2\mu} x + a' \times x + b'.$$

The latter circumstance explains why the term  $\frac{\gamma\theta_0}{3\lambda + 2\mu} x_i$  has appeared in (4.43), since in the case of Problem (II.II)<sup>+</sup>  $\theta$  is determined to within an arbitrary constant term  $\theta_0$ .

Also note that we have constructed here the solution of problem (4.12) in form (4.13).

**4.5. Solution of Problems (III.I)<sup>±</sup>, (III.II)<sup>±</sup>.** In common with the case of Problem (III)<sup>±</sup> of classical elasticity (see Subsection 2.10), to solve Problems (III.I)<sup>±</sup>, (III.II)<sup>±</sup> it is convenient to replace the boundary condition

$$(\tau^{(n)} - n(n \cdot \tau^{(n)}))^+ = l$$

by

$$(H(\partial_y)u)^+ = F, \quad (4.46)$$

where  $F = F = (F_1, F_2, F_3)$ ,

$$\begin{aligned} F_i(y) &= \frac{\rho}{\mu} l_i(y) - 2\rho \mathcal{D}_i(g)(y) - \frac{2}{\rho} y_i g(y), \\ H(\partial_x) &= \|H_{ij}(\partial_x)\|_{3 \times 3}, \quad H_{ij}(\partial_x) = \delta_{ij}(\mathcal{D}_r - 2) - x_j \frac{\partial}{\partial x_i}. \end{aligned}$$

Let us consider Problem (III.I)<sup>+</sup>. Remember that in that case  $\theta$  is the harmonic function satisfying the boundary condition  $(\theta)^+ = \eta$  and therefore given by the Poisson formula

$$\theta(x) = \frac{1}{4\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|y-x|^3} \eta(y) d_y S. \quad (4.47)$$

Finding  $u$  in form (4.24), we write  $H(\partial_x)u$  as

$$H(\partial_x)u(x) = h'(x) + \frac{\mu}{\lambda + \mu}(\rho^2 - r^2) \text{grad } \psi(x) - \delta x \theta(x),$$

where  $h' = h' = (h'_1, h'_2, h'_3)$ ,

$$\begin{aligned} h'_i(x) &= x_j \left( \frac{\partial v_i(x)}{\partial x_j} - \frac{\partial v_j(x)}{\partial x_i} \right) - 2v_i(x) + x_i \mathcal{D}_r \psi(x) + \\ &+ \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \psi(x)}{\partial x_i} - \frac{\lambda + 2\mu}{\lambda + \mu} \rho^2 \frac{\partial \psi(x)}{\partial x_i} + \delta x_i \theta(x). \end{aligned} \quad (4.48)$$

Applying relation (4.25), it is easy to verify that  $\forall x \in B^+ : \Delta h'(x) = 0$ . Moreover, by virtue of (4.46) and the condition  $(\theta)^+ = \eta$ , we have  $(h')^+ = F'$ , where  $F'(y) = F(y) + \delta \eta(y)y$ . Therefore  $h$  is expressed by the Poisson formula  $h' = \Pi(F')$ .

Let us now derive equations to determine  $\psi$  and  $v$ , assuming  $h'$  to be known.

Applying the operator  $\text{div}$  to (4.48), we obtain due to (4.25)

$$A_r \psi = \frac{\lambda + \mu}{2(\lambda + 2\mu)} \text{div } h' - \frac{\gamma}{2(\lambda + 2\mu)} (2\mathcal{D}_r + 1)\theta, \quad (4.49)$$

where

$$A_r \equiv \mathcal{D}_r^2 + \frac{\lambda}{2(\lambda + 2\mu)} \mathcal{D}_r - \frac{\mu}{\lambda + 2\mu}. \quad (4.50)$$

Using the same procedure as in solving Problem (III)<sup>±</sup> of classical elasticity (see Subsection 2.10), we obtain for  $v$  the equation

$$(\mathcal{D}_r - 1)v_i = q'_i, \quad (4.51)$$

where

$$\begin{aligned} q'_i(x) &= \frac{1}{2} \left( h'_i(x) - x_k \frac{\partial h'_k(x)}{\partial x_i} \right) - \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \psi(x)}{\partial x_i} + \\ &+ \frac{\delta}{2} r^2 \frac{\partial \theta(x)}{\partial x_i} + (\mathcal{D}_r - 1) \left( \frac{\mu}{\lambda + \mu} x_i \psi(x) - \right. \\ &\left. - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - r^2) \frac{\partial \psi(x)}{\partial x_i} \right). \end{aligned} \quad (4.52)$$

Write the solution of (4.49) as the sum

$$\psi = \psi_1 + \psi_2, \quad (4.53)$$

where

$$A_r \psi_1 = \frac{\lambda + \mu}{2(\lambda + 2\mu)} \operatorname{div} h', \quad (4.54)$$

$$A_r \psi_2 = -\frac{\gamma}{2(\lambda + 2\mu)} (2\mathcal{D}_r + 1)\theta. \quad (4.55)$$

A solution of (4.54) in the class of harmonic functions is given to within a constant term by (see (2.98))

$$\psi_1(x) = \frac{\gamma_0}{4\pi\rho} \operatorname{div} \int_S (\Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y)) F'(y) d_y S, \quad (4.56)$$

where  $\gamma_0$  is the constant determined by (2.99) and  $\Phi^{(m)}$  is obtained from (2.100).

Finding the solution of (4.55) in the form

$$\psi_2 = \tilde{\psi} + \frac{\gamma}{2\mu}\theta, \quad (4.57)$$

we obtain for  $\tilde{\psi}$  the equation

$$A_r \tilde{\psi} = (\alpha_1 \mathcal{D}_r^2 + \alpha_2 \mathcal{D}_r)\theta,$$

where  $\alpha_1 = -\gamma(2\mu)^{-1}$ ,  $\alpha_2 = -\gamma(\lambda + 4\mu)(4\mu(\lambda + 2\mu))^{-1}$ .

The solution of this equation in the class of harmonic functions is given by the formula

$$\begin{aligned} \tilde{\psi}(x) = & \frac{1}{4\pi\rho(k_1 - k_2)} (\alpha_1 \mathcal{D}_r^2 + \alpha_2 \mathcal{D}_r) \int_S (\tilde{\Phi}^{(1+k_1)}(x, y) - \\ & - \tilde{\Phi}^{(1+k_2)}(x, y)) \eta(y) d_y S, \end{aligned} \quad (4.58)$$

where

$$\tilde{\Phi}^{(m)}(x, y) = \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \frac{d\tau}{\tau^m},$$

$k_1$  and  $k_2$  are expressed by formulas (2.96).

By virtue of (4.57) and (4.58) we have

$$\begin{aligned} \psi_2(x) = & \frac{\gamma}{8\pi\rho\mu} \int_S \frac{\rho^2 - |x|^2}{|y - x|^3} \eta(y) d_y S + \frac{1}{4\pi\rho(k_1 - k_2)} (\alpha_1 \mathcal{D}_r^2 + \\ & + \alpha_2 \mathcal{D}_r) \int_S (\tilde{\Phi}^{(1+k_1)}(x, y) - \tilde{\Phi}^{(1+k_2)}(x, y)) \eta(y) d_y S. \end{aligned} \quad (4.59)$$

Thus the solution of (4.49) has form (4.53), where  $\psi_1$  and  $\psi_2$  are determined by (4.56) and (4.59).

Now proceed to solving equation (4.51). Write its right-hand part determined by (4.52) as the sum  $q'_i = q_i^{(1)} + q_i^{(2)}$ , where

$$\begin{aligned} q_i^{(1)}(x) &= \frac{1}{2} \left( h'_i(x) - x_k \frac{\partial h'_k(x)}{\partial x_i} \right) - \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \psi_1(x)}{\partial x_i} + \\ &+ (\mathcal{D}_r - 1) \left( \frac{\mu}{\lambda + \mu} x_i \psi_1(x) - \frac{\lambda + 2\mu}{\lambda + \mu} \frac{\rho^2 - r^2}{2} \frac{\partial \psi_1(x)}{\partial x_i} \right), \end{aligned} \quad (4.60)$$

$$\begin{aligned} q_i^{(2)}(x) &= -\frac{\mu}{\lambda + \mu} r^2 \frac{\partial}{\partial x_i} \left( \psi_2(x) - \frac{\gamma}{2\mu} \theta(x) \right) + \\ &+ (\mathcal{D}_r - 1) \left( \frac{\mu}{\lambda + \mu} x_i \psi_2(x) - \frac{\lambda + 2\mu}{\lambda + \mu} \frac{\rho^2 - r^2}{2} \frac{\partial \psi_2(x)}{\partial x_i} \right). \end{aligned} \quad (4.61)$$

Like in solving Problem (III)<sup>+</sup> of classical elasticity (see (2.106)),  $q_i^{(1)}$  can be represented as

$$q_i^{(1)}(x) = (\mathcal{D}_r - 1) P_i^{(1)}(x), \quad (4.62)$$

where

$$\begin{aligned} P_i^{(1)}(x) &= \frac{1}{4\pi\rho} \int_S \left( \frac{1}{2} (\Phi^{(2)}(x, y) - \frac{1}{\rho}) \delta_{ik} - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) \right) - \\ &- \frac{\gamma_0 \mu}{\lambda + \mu} r^2 \frac{\partial^2}{\partial x_i \partial x_k} \left( \frac{1}{k_1} \Phi^{(2+k_1)}(x, y) - \frac{1}{k_2} \Phi^{(2+k_2)}(x, y) \right) + \\ &+ \frac{k_1 - k_2}{k_1 k_2} \Phi^2(x, y) + \gamma_0 \left( \frac{\mu}{\lambda + \mu} x_i - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - r^2) \frac{\partial}{\partial x_i} \right) \times \\ &\times \frac{\partial}{\partial x_k} \left( \Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y) \right) F'_k(y) d_y S. \end{aligned} \quad (4.63)$$

Substituting  $\psi_2$  from (4.59) in (4.61) and using

$$r^2 \frac{\partial}{\partial x_i} \mathcal{D}_r \psi = (\mathcal{D}_r - 1) r^2 \frac{\partial \psi}{\partial x_i}, \quad (4.64)$$

we obtain

$$q_i^{(2)}(x) = (\mathcal{D}_r - 1) P_i^{(2)}(x), \quad (4.65)$$

where

$$\begin{aligned} P_i^{(2)}(x) &= \frac{-\mu}{4\pi\rho(k_1 - k_2)(\lambda + \mu)} r^2 \frac{\partial}{\partial x_i} \int_S (\alpha_1 \mathcal{D}_r + \alpha_2) (\tilde{\Phi}^{(1+k_1)}(x, y) - \\ &- \tilde{\Phi}^{(1+k_2)}(x, y)) \eta(y) d_y S + \left( \frac{\mu}{\lambda + \mu} x_i - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - \right. \end{aligned}$$

$$\begin{aligned}
& -r^2) \frac{\partial}{\partial x_i} \left( \frac{\gamma}{8\pi\rho\mu} \int_S \frac{\rho^2 - r^2}{|y-x|^3} \eta(y) d_y S + \frac{1}{4\pi\rho(k_1 - k_2)} (\alpha_1 \mathcal{D}_r^2 + \right. \\
& \left. + 2\mathcal{D}_r) \int_S (\tilde{\Phi}^{(1+k_1)}(x, y) - \tilde{\Phi}^{(1+k_2)}(x, y)) \eta(y) d_y S \right).
\end{aligned}$$

Now it is obvious that the solution of (4.51) has the form

$$v_i(x) = P_i^{(1)}(x) + P_i^{(2)}(x) + c_{ij}x_j,$$

where  $c_{ij}$  ( $i, j = 1, 2, 3$ ) are arbitrary constants.

Substituting the found values of  $v$  and  $\psi$  in (4.24), we obtain the classical solution of Problem (III.I)<sup>+</sup> when  $\eta, l \in C^{0,\alpha}(S)$ ,  $g \in C^{1,\alpha}(S)$ ,  $0 < \alpha \leq 1$  and condition (4.21) is fulfilled. In that case, too, like in the theory of elasticity (see (2.114) and (2.114')), the symmetrical part of the matrix  $\|c_{ij}\|$  is determined uniquely, whereas the nonsymmetrical part

$$\frac{1}{2}(c_{ij} - c_{ji}) \equiv \varepsilon_{ijk} a_k$$

remains undetermined.

And now consider Problem (III.II)<sup>+</sup>. Using representation (4.24) write  $H(\partial_x)u$  as

$$\begin{aligned}
H(\partial_x)u(x) &= h''(x) + \frac{\mu}{\lambda + \mu}(\rho^2 - r^2) \text{grad } \psi(x) + \\
&+ \delta \left( (\rho^2 - r^2) \frac{\partial \theta(x)}{\partial x_i} + 2x_i \mathcal{D}_r \theta(x) \right), \quad (4.66)
\end{aligned}$$

where  $h'' = (h'_1, h'_2, h'_3)$ ,

$$\begin{aligned}
h''_i(x) &= x_j \left( \frac{\partial v_i(x)}{\partial x_j} - \frac{\partial v_j(x)}{\partial x_i} \right) - 2v_i(x) + x_i \mathcal{D}_r \psi(x) + \\
&+ \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \psi(x)}{\partial x_i} - \frac{\lambda + 2\mu}{\lambda + \mu} \rho^2 \frac{\partial \psi(x)}{\partial x_i} - \delta \left( (\rho^2 - \right. \\
&\left. - r^2) \frac{\partial \theta(x)}{\partial x_i} + 2x_i \mathcal{D}_r \theta(x) \right). \quad (4.67)
\end{aligned}$$

Due to (4.25) it is obvious that  $h''$  is harmonic in  $B^+$  and, moreover, from conditions (4.1) and  $(\frac{\partial \theta}{\partial n})^+ = \eta$  we have  $(h'')^+ = F''$ , where  $F''(y) = F(y) - 2\rho\delta y\eta(y)$ . Therefore  $h'' = \Pi(F'')$ .

Proceeding as above, we obtain for  $\psi$  and  $v$  the equations

$$A_r \psi = \frac{\lambda + \mu}{2(\lambda + 2\mu)} \text{div } h'' + \frac{\gamma}{2(\lambda + 2\mu)} (2\mathcal{D}_r^2 + 3\mathcal{D}_r + 2)\theta, \quad (4.68)$$

$$(\mathcal{D}_r - 1)v_i = q''_i, \quad (4.69)$$

where  $A_r$  is determined by equality (4.50),

$$\begin{aligned}
q_i''(x) &= \frac{1}{2} \left( h_i''(x) - x_k \frac{\partial h_k''(x)}{\partial x_i} \right) - \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \psi(x)}{\partial x_i} - \\
&\quad - \delta r^2 \frac{\partial \theta(x)}{\partial x_i} + (\mathcal{D}_r - 1) \left( \frac{\mu}{\lambda + \mu} x_i \psi(x) - \right. \\
&\quad \left. - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - r^2) \frac{\partial \psi(x)}{\partial x_i} - \right. \\
&\quad \left. \frac{\delta}{2} (\rho^2 + r^2) \frac{\partial \theta(x)}{\partial x_i} + \delta x_i \theta(x) \right). \tag{4.70}
\end{aligned}$$

The solution of (4.68) is written as the sum  $\psi = \psi^1 + \psi^2$ , where

$$A_r \psi^1 = \frac{\lambda + \mu}{2(\lambda + 2\mu)} \operatorname{div} h'', \tag{4.71}$$

$$A_r \psi^2 = \frac{\gamma}{2(\lambda + 2\mu)} (2\mathcal{D}_r^2 + 3\mathcal{D}_r + 2)\theta. \tag{4.72}$$

The solution of (4.71) is given by

$$\psi^1(x) = \frac{\gamma_0}{4\pi\rho} \operatorname{div} \int_S (\Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y)) F''(y) d_y S. \tag{4.73}$$

If  $\psi^2$  is represented as

$$\psi^2 = \tilde{\psi} - \frac{\gamma}{\mu} \theta, \tag{4.74}$$

then we obtain for  $\tilde{\psi}$  the equation

$$A_r \tilde{\psi} = \beta_0 (2\mathcal{D}_r^2 + \mathcal{D}_r) \theta, \tag{4.75}$$

where

$$\beta_0 \equiv \frac{\gamma(\lambda + 3\mu)}{2\mu(\lambda + 2\mu)}. \tag{4.76}$$

Keeping in mind that in the case of Problem (III.II)<sup>+</sup>  $\theta$  is expressed by the Neumann formula

$$\theta(x) = \frac{1}{4\pi\rho} \int_S \left( \frac{2\rho}{|y-x|} - \ln((|y-x| + \rho)^2 - |x|^2) \right) \eta(y) d_y S, \tag{4.77}$$

the solution of equation (4.75) is written in the form

$$\tilde{\psi}(x) = \frac{\beta_0}{4\pi\rho(k_1 - k_2)} (2\mathcal{D}_r^2 + \mathcal{D}_r) \times$$

$$\times \int_S (\mathcal{N}^{(1+k_1)}(x, y) - \mathcal{N}^{(1+k_2)}(x, y)) \eta(y) d_y S, \quad (4.78)$$

where  $k_1$  and  $k_2$  are determined by (2.96) and

$$\mathcal{N}^{(m)}(x, y) \equiv \int_0^1 \left( \frac{2\rho}{|y - \tau x|} - 2 - \ln \frac{(|y - \tau x| + \rho)^2 - |\tau x|^2}{4\rho^2} \right) \frac{d\tau}{\tau^m}.$$

Thus the solution of equation (4.72) is given in the form

$$\begin{aligned} \overset{2}{\psi} = & -\frac{\gamma}{4\pi\rho\mu} \int_S \left( \frac{2\rho}{|y - x|} - \ln(|y - x| + \rho)^2 - |x|^2 \right) \eta(y) d_y S + \\ & + \frac{\beta_0}{4\pi\rho(k_1 - k_2)} (2\mathcal{D}_r^2 + \mathcal{D}_r) \times \\ & \times \int_S ((\mathcal{N}^{(1+k_1)}(x, y) - \mathcal{N}^{(1+k_2)}(x, y)) \eta(y) d_y S \end{aligned} \quad (4.79)$$

and that of equation (4.68) in the form  $\psi = \overset{1}{\psi} + \overset{2}{\psi}$ , where  $\overset{1}{\psi}$  and  $\overset{2}{\psi}$  are determined by (4.73) and (4.79).

Let us consider equation (4.69) whose right-hand part can be represented as  $q_i'' = \overset{(1)}{q}_i + \overset{(2)}{q}_i$ , where

$$\begin{aligned} \overset{(1)}{q}_i(x) = & \frac{1}{2} \left( h_i''(x) - x_k \frac{\partial h_k''(x)}{\partial x_i} \right) - \frac{\mu}{\lambda + \mu} r^2 \frac{\partial \overset{1}{\psi}(x)}{\partial x_i} + \\ & + (\mathcal{D}_r - 1) \left( \frac{\mu}{\lambda + \mu} x_i \overset{1}{\psi}(x) - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - r^2) \frac{\partial \overset{1}{\psi}(x)}{\partial x_i} \right), \end{aligned} \quad (4.80)$$

$$\begin{aligned} \overset{(2)}{q}_i(x) = & -\frac{\mu}{\lambda + \mu} r^2 \frac{\partial}{\partial x_i} \left( \overset{2}{\psi}(x) + \frac{\gamma}{\mu} \theta(x) \right) + \\ & + (\mathcal{D}_r - 1) \left( \frac{\mu}{\lambda + \mu} x_i \overset{2}{\psi}(x) - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - r^2) \frac{\partial \overset{2}{\psi}(x)}{\partial x_i} - \right. \\ & \left. - \frac{\delta}{2} (\rho^2 + r^2) \frac{\partial \theta(x)}{\partial x_i} + \delta x_i \theta(x) \right). \end{aligned} \quad (4.80')$$

Similarly to (4.62) we write

$$\overset{(1)}{q}_i = (\mathcal{D}_r - 1) \overset{(1)}{r}_i,$$

where

$$\overset{(1)}{r}_i(x) = \frac{1}{4\pi\rho} \int_S \left( \frac{1}{2} ((\Phi^{(2)}(x, y) - \frac{1}{\rho}) \delta_{ik} - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y)) - \right.$$

$$\begin{aligned}
& - \frac{\gamma_0 \mu}{\lambda + \mu} r^2 \frac{\partial^2}{\partial x_i \partial x_k} \left( \frac{1}{k_1} \Phi^{(2+k_1)}(x, y) - \frac{1}{k_2} \Phi^{(2+k_2)}(x, y) + \right. \\
& + \frac{k_1 - k_2}{k_1 k_2} \Phi^{(2)}(x, y) \left. + \gamma_0 \left( \frac{\mu}{\lambda + \mu} x_i - \frac{\lambda + 2\mu}{2(\lambda + \mu)} (\rho^2 - r^2) \frac{\partial}{\partial x_i} \right) \times \right. \\
& \left. \times \frac{\partial}{\partial x_k} \left( \Phi^{(2+k_1)}(x, y) - \Phi^{(2+k_2)}(x, y) \right) \right) F_k''(y) d_y S.
\end{aligned}$$

Using (4.77), (7.79) and identity (4.64), we obtain for  $\overset{(2)}{q}_i$  determined by (4.80)

$$\overset{(2)}{q}_i = (\mathcal{D}_r - 1) \overset{(2)}{r}_i,$$

where

$$\begin{aligned}
\overset{(2)}{r}_i(x) = & - \frac{\mu \beta_0}{4\pi \rho (k_1 - k_2) (\lambda + \mu)} r^2 \frac{\partial}{\partial x_i} \int_S (2\mathcal{D}_r + 1) (\mathcal{N}^{(1+k_1)}(x, y) - \\
& - \mathcal{N}^{(1+k_2)}(x, y)) \eta(y) d_y S + \frac{\beta_0}{4\pi \rho (k_1 - k_2)} \left( \frac{\mu x_i}{\lambda + \mu} - \right. \\
& - \frac{\lambda + 2\mu}{\lambda + \mu} \frac{\rho^2 - r^2}{2} \frac{\partial}{\partial x_i} \left. \right) \int_S (2\mathcal{D}_r^2 + \mathcal{D}_r) (\mathcal{N}^{(1+k_1)}(x, y) - \\
& - \mathcal{N}^{(1+k_2)}(x, y)) \eta(y) d_y S + \\
& + \frac{\gamma}{8\pi \rho \mu (\lambda + \mu)} ((\lambda + \mu) \rho^2 - (\lambda + 3\mu) r^2) \frac{\partial}{\partial x_i} \int_S \left( \frac{2\rho}{|x - y|} - \right. \\
& \left. - \ln(|y - x| + \rho)^2 - |x|^2 \right) \eta(y) d_y S.
\end{aligned}$$

Thus the solution of equation (4.69) has the form

$$v_i = \overset{(1)}{r}_i + \overset{(2)}{r}_i + c_{ij} x_j,$$

where  $c_{ij}$  ( $i, j = 1, 2, 3$ ) are arbitrary constants.

To construct the solution of Problem (III.II)<sup>+</sup> in form (4.24) is now quite a simple matter. It is likewise easy to verify that the solution obtained is the classical one if the boundary data satisfy the conditions (4.22) and  $\eta, l \in C^{0,\alpha}(S)$ ,  $g \in C^{1,\alpha}(S)$ ,  $0 < \alpha \leq 1$ .

**Problem 33.** Prove the proposition that we have just formulated.

**Problem 34.** Construct the solutions of Problems (III.I)<sup>-</sup> and (III.II)<sup>-</sup>.

**4.6. Solution of Problems (IV.I)<sup>±</sup>, (IV.II)<sup>±</sup>.** Like in the case of Problem (IV)<sup>±</sup> of classical elasticity, the boundary condition

$$(n \cdot \tau^{(n)} - \gamma\theta)^+ = g$$

should be replaced by an equivalent one (*cf.* (2.126))

$$((\lambda + 2\mu) \operatorname{div} u - \frac{4\mu}{\rho^2}(y \cdot u) - \gamma\theta)^+(y) = g'(y), \quad (4.81)$$

where

$$g' \equiv g + 2\mu\mathcal{D}_k(l_k).$$

Assuming additionally that the condition  $(n \cdot v)^+ = 0$  is fulfilled, we are to find  $u$  in form (4.31). Since

$$\begin{aligned} u_i(x) - \frac{x_i}{r^2} x_k u_k(x) &= v_i(x) - \frac{x_i}{r^2} x_k v_k(x) + \\ &\frac{\rho^2 - r^2}{2} \left( \frac{\partial}{\partial x_i} - \frac{x_i}{r} \frac{\partial}{\partial r} \right) (\psi(x) + 2\varphi(x)), \end{aligned}$$

then from the boundary conditions  $(n \cdot v)^+ = 0$  and  $(u - n(n \cdot u))^+ = l$  it follows that  $v^+ = l$ . Thus  $v = \Pi(l)$ .

Further, on account of (4.31) and (4.32) we have

$$\begin{aligned} (\lambda + 2\mu) \operatorname{div} u(x) - \frac{4\mu}{r^2}(x \cdot u(x)) - \gamma\theta(x) &= \\ = \zeta(x) - 4\mu \frac{x_i v_i(x)}{r^2} - 2\mu \frac{\rho^2 - r^2}{2} \frac{\partial}{\partial r} (\psi(x) + 2\varphi(x)), \end{aligned} \quad (4.82)$$

$$\zeta = \frac{\mu(\lambda + 2\mu)}{\lambda + \mu} (2\mathcal{D}_r + 1)\psi - 4\mu(2\mathcal{D}_r + 1)\varphi + \frac{\gamma\mu}{\lambda + \mu}\theta. \quad (4.83)$$

Since  $\varphi$ ,  $\psi$  and  $\theta$  are harmonic, it is easy to note that  $\zeta$ , too, is harmonic in  $B^+$ . Moreover, by virtue of (4.81), (4.82) and the condition  $(n \cdot v)^+ = 0$  we have for  $\xi$  the condition  $\zeta^+ = g'$  and therefore  $\zeta = \Pi(g')$ .

Now equality (4.83) can be treated as the equation

$$(2\mathcal{D}_r + 1)\chi = \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\Pi(g') - \frac{\gamma}{\lambda + 2\mu}\theta \quad (4.84)$$

for  $\chi \equiv \psi - 4(\lambda + \mu)(\lambda + 2\mu)^{-1}\varphi$ .

To solve this equation we have to consider Problems (IV.I)<sup>+</sup> and (IV.II)<sup>+</sup> separately.

For Problem (IV.I)<sup>+</sup> equation (4.84) can be rewritten as

$$(2\mathcal{D}_r + 1)\chi = \Pi(\xi), \quad \xi = \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}g' - \frac{\gamma}{\lambda + 2\mu}\eta. \quad (4.85)$$

Identity (2.29) enables us to write the harmonic function  $\chi$  satisfying (4.85) as

$$\chi(x) = \frac{1}{4\pi\rho} \int_S \frac{1}{|y-x|} \xi(y) d_y S. \quad (4.86)$$

For Problem (IV.II)<sup>+</sup> the function  $\theta$  is given by the Neumann formula (see (1.6), (1.7)) and therefore (4.84) takes the form

$$(2\mathcal{D}_r + 1)\chi = \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \Pi(g') - \frac{\gamma}{\lambda + 2\mu} \mathcal{N}(\eta). \quad (4.87)$$

Write the solution of this equation in the form

$$\chi = \chi' - \frac{\gamma}{\lambda + 2\mu} \chi'', \quad (4.88)$$

where

$$(2\mathcal{D}_r + 1)\chi' = \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \Pi(g'), \quad (4.89)$$

$$(2\mathcal{D}_r + 1)\chi'' = \mathcal{N}(\eta). \quad (4.90)$$

Equation (4.89) actually coincides with (4.85) and thus its solution can be written as

$$\chi'(x) = \frac{\lambda + \mu}{4\pi\rho\mu(\lambda + 2\mu)} \int_S \frac{1}{|x-y|} g'(y) d_y S. \quad (4.91)$$

As to (4.90), its solution in the class of harmonic functions is

$$\chi'' = \frac{1}{2} \int_0^1 \mathcal{N}(\eta)(\tau x) \frac{d\tau}{\sqrt{\tau}}. \quad (4.92)$$

Using the identity

$$\frac{\partial}{\partial \tau} \ln((|y - \tau x| + \rho)^2 - |\tau x|^2) = \frac{1}{\tau} - \frac{2}{\tau|y - \tau x|},$$

we obtain

$$\begin{aligned} \int_0^1 \ln((|y - \tau x| + \rho)^2 - |\tau x|^2) \frac{d\tau}{\sqrt{\tau}} &= 2 \ln((|y - x| + \rho)^2 - |x|^2) + \\ &+ \int_0^1 \left( \frac{2\rho}{|y - \tau x|} - 2 \right) \frac{d\tau}{\sqrt{\tau}}. \end{aligned}$$

By virtue of the latter equality and the condition

$$\int_S \eta(y) d_y S = 0,$$

(4.92) yields

$$\chi''(x) = -\frac{1}{4\pi\rho} \int_S \ln((|y-x|+\rho)^2 - |x|^2) \eta(y) d_y S.$$

Thus the solution of equation (4.87) is given by the formula

$$\begin{aligned} \chi(x) &= \frac{\lambda + \mu}{4\pi\rho(\lambda + 2\mu)} \int_S \frac{1}{|y-x|} g'(y) d_y S + \\ &+ \frac{\gamma}{4\pi\rho(\lambda + 2\mu)} \int_S \ln((|y-x|+\rho)^2 - |x|^2) \eta(y) d_y S. \end{aligned}$$

From now on constructing the solutions in form (4.32) for Problems (IV.I)<sup>±</sup> and (IV.II)<sup>±</sup> and establishing their boundary properties (theorems similar to Theorems 2.22, 2.23) can be conducted as in the case of Problem (IV)<sup>±</sup> of classical elasticity.

**Problem 35.** Derive the final form of the solutions of Problems (IV.I)<sup>±</sup> and (IV.II)<sup>±</sup>.

**Problem 36.** For Problems (IV.I)<sup>±</sup> and (IV.II)<sup>±</sup> prove theorems similar to Theorems 2.22 and 2.23.

**4.7. A Remark on Solutions of Boundary Value Problems of Thermoelasticity and on Solutions of Boundary Value Problems for Nonhomogeneous Equations of Classical Elasticity.** As said above, the boundary value static problems of thermoelasticity are divided so that we have boundary value problems separately for the Laplace equation for the temperature  $\theta$  and separately for the system of equations of classical elasticity for the displacement vector  $u$ . To clarify our reasoning we will consider, as an example, Problem (IV.II)<sup>+</sup> solved in the preceding subsection. It is formulated as follows:

Given the necessary condition for the problem to be solvable

$$\int_S \eta(y) d_y S = 0, \tag{4.23}$$

find a pair  $(u, \theta)$  satisfying in  $B^+$  the system

$$\begin{aligned} \mu\Delta u(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} u(x) - \gamma \operatorname{grad} \theta(x) &= 0, \\ \Delta \theta(x) &= 0 \end{aligned} \tag{4.1}$$

and the boundary condition

$$(n \cdot \tau^{(n)} - \gamma\theta)^+ = g, \quad (u - n(n \cdot u))^+ = l, \quad \left(\frac{\partial\theta}{\partial n}\right)^+ = \eta, \quad (4.11)$$

where  $l = l = (l_1, l_2, l_3)$ ,  $\eta$ , and  $g$  are functions given on  $S = \partial B^+$ .

Thus to determine  $\theta$  we have the Neumann problem

$$\Delta\theta = 0, \quad \left(\frac{\partial\theta}{\partial n}\right)^+ = \eta,$$

whose solution is given in quadratures by the Neumann formula (see (1.7))  $\theta = N(\eta)$ .

Substituting the found value  $\theta$  in (4.1) and the boundary condition (4.11), we obtain Problem (IV)<sup>+</sup> of classical elasticity

$$\begin{aligned} \mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u &= F, \\ (n \cdot \tau^{(n)})^+(y) &= g_1, \quad (u - n(n \cdot u))^+ = l, \end{aligned} \quad (4.93)$$

where  $F \equiv \gamma \operatorname{grad} \mathcal{N}(\eta)$ ,  $g_1 = g + \gamma(N(\eta))^+$ .

Thus the problems of thermoelasticity are reduced to the problems of classical elasticity and no new difficulties arise here from the viewpoint of theoretical research. That is why the static problems of thermoelasticity have no theoretical interest of their own.

We have quite a different situation in constructing effective solutions. The above-proposed methods are not adequate for this purpose. The reason is that in problem (4.93) the system of equations is nonhomogeneous, whereas we can solve immediately only a homogeneous system; moreover, the right-hand part  $F$  and the boundary condition (the function  $g_1$ ) depend on  $\theta$ . To have a clear idea of the difficulty arisen let us discuss this question in greater detail.

Write the solution  $u$  of problem (4.93) as  $u = v + \overset{\circ}{u}$ , where  $\overset{\circ}{u}$  is the particular solution of system (4.93). Then for  $v$  we obtain the problem

$$\begin{aligned} \mu\Delta v + (\lambda + \mu) \operatorname{grad} \operatorname{div} v &= 0, \\ (n \cdot \tau_v^{(n)})^+ &= f, \quad (v - n(n \cdot v))^+ = \chi, \end{aligned} \quad (4.94)$$

where  $\tau_v^{(n)}$  is the stress corresponding to the displacement  $v$  (see (2.3))

$$f \equiv g_1 - (n \cdot \tau_u^{(n)})^+, \quad \chi \equiv l - (\overset{\circ}{u} - n(n \cdot \overset{\circ}{u}))^+$$

$(\tau_u^{(n)})^+$  is the stress corresponding to the displacement  $\overset{\circ}{u}$  (see (2.3)).

We know how to solve problem (4.94) (see (2.137)). We can also construct  $\overset{\circ}{u}$  explicitly, *i.e.* in quadratures:

$$\overset{\circ}{u}(x) = -\frac{1}{2} \int_{B^+} \Gamma(x-y) F(y) dy = -\frac{\gamma}{2} \int_{B^+} \Gamma(x-y) \operatorname{grad} \mathcal{N}(\eta)(y) dy,$$

where  $\Gamma$  is the Kelvin matrix (Kupradze *et al.* [1]), but we do not know how to express explicitly  $f$  and  $\chi$  in terms of the known functions  $g$  and  $l$ .

One may invent other ways of dividing the problems, but they will make the question even more difficult.

It should be noted that the method used to solve the problems of thermoelasticity enables one to solve the problems of classical elasticity for a nonhomogeneous system

$$\mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u = F \quad (4.95)$$

in the particular case when the right-hand part  $F$  is given in the form

$$F = \operatorname{grad} \omega, \quad (4.96)$$

where  $\omega$  is a harmonic function.

Let us investigate this question more thoroughly. Consider, for example, Problem (I)<sup>+</sup> for system (4.95): Find  $u$  in  $B^+$ , satisfying (4.95) and the boundary condition  $u^+ = f$ . (4.95) can be rewritten as

$$\mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u - \gamma \operatorname{grad} \theta = 0, \quad (4.97)$$

where  $\theta = \frac{\omega}{\gamma}$  ( $\gamma > 0$ ). Then (4.96) implies  $\Delta\theta = 0$ . Moreover,  $u^+ = f$  and  $\theta^+ = \frac{\omega^+}{\gamma}$  are the known functions. Thus we have obtained Problem (I.I)<sup>+</sup> of thermoelasticity whose solution has been derived above.

Other problems for a nonhomogeneous system like (4.95), (4.96) are solved in a similar manner.

The above reasoning enables us to conclude that, in addition to their independent interest, the solutions of the boundary value problems of thermoelasticity yield the solutions of the boundary value problems of classical elasticity in the presence of mass force if the latter is representable as (4.96).

We would also like to note that the discussed effective solutions of the problems of elasticity make it possible to construct effectively the Green functions of the corresponding problems and to use the Green functions to construct effective solutions of the problems for nonhomogeneous equations with an arbitrary right-hand side.

**Problem 37.** Find in quadratures a solution  $(u, \theta)$ , continuous in the domain  $B^+$ , of system (4.1) by the boundary condition

$$(\tau^{(n)} + \sigma_0 u - \gamma \theta n)^+ = f, \quad (\theta)^+ = \eta$$

– Problem (V.I)<sup>+</sup>.

**Problem 38.** Formulate Problems (V.II)<sup>+</sup>, (V.I)<sup>-</sup>, (V.II)<sup>-</sup> and find their solutions in quadratures.

**Problem 39.** Find in quadratures a solution  $(u, \theta)$ , continuous in the domain  $B^+$ , of system (4.1) by the boundary conditions

$$(u_1)^+ = f_1, \quad (u_2)^+ = f_2, \quad (\tau_3^{(n)} - \gamma\theta n_3)^+ = f_3, \quad (\theta)^+ = \eta.$$

**Problem 40.** Find in quadratures a solution  $(u, \theta)$ , continuous in the domain  $B^+$ , of system (4.1) by the boundary conditions

$$(\tau_1^{(n)} - \gamma\theta n_1)^+ = f_1, \quad (\tau_2^{(n)} - \gamma\theta n_2)^+ = f_2, \quad (u_3)^+ = f_3, \quad (\theta)^+ = \eta.$$

**Problem 41.** Solve Problems 39 and 40, where the boundary condition  $(\theta)^+ = \eta$  is replaced by the condition  $(\frac{d\theta}{dn})^+ = \eta$ .

**Problem 42.** Solve Problems 39 to 42 for the domain  $B^-$ .

\* \* \*

General questions of the theory of thermoelasticity, as well as boundary value problems (formulation of the problems, uniqueness and existence theorems, solvability conditions) are investigated in the monographs Lebedev [1], Melan, Parkus [1], Chadwick [1], Nowacki [1-3], Kovalenko [1,2], Timoshenko, Goodier [1], Ilyushin, Pobedrya [1], Boley, Weiner [1], Kupradze *et al.* [1] and others.

Particular boundary value problems for the ball are considered in Grünberg [1], Kowalenko [2], Lurie [2], Timoshenko, Goodier [2], Mahalanabis [1], Natroshvili [3] and others. The results of Chapter IV are published in Chichinadze, Gegelia [1].

CHAPTER V  
PROBLEMS OF THE THEORY OF ELASTIC MIXTURES

**5.1. Statement of the Question.** Our discussion here will involve one of the versions of the basic static equations of the linear theory of elastic mixtures for two isotropic materials (Khoroshun, Soltanov [1]; Natroshvili, Jagmaidze, Svanadze [1]; Truesdell, Toupin [1]; Green, Naghdi [1]; Steel [1]; Green, Steel [1]; Atkin, Chadwick, Steel [1]; Knops, Steel [1]; Tiersten, Jahanmir [1]; Villaggio [1]). In this theory two displacement vectors  $\overset{1}{u} = (\overset{1}{u}_1, \overset{1}{u}_2, \overset{1}{u}_3)$  and  $\overset{2}{u} = (\overset{2}{u}_1, \overset{2}{u}_2, \overset{2}{u}_3)$  are introduced at every point of the domain occupied by the mixture, and the static equations are written in the form

$$\begin{aligned} a_1 \Delta \overset{1}{u} + b_1 \operatorname{grad} \operatorname{div} \overset{1}{u} + c \Delta \overset{2}{u} + d \operatorname{grad} \operatorname{div} \overset{2}{u} &= 0, \\ c \Delta \overset{1}{u} + d \operatorname{grad} \operatorname{div} \overset{1}{u} + a_2 \Delta \overset{2}{u} + b_2 \operatorname{grad} \operatorname{div} \overset{2}{u} &= 0, \end{aligned} \quad (5.1)$$

where  $a_1, a_2, b_1, b_2, c, d$  are the elastic constants of the mixture.

It will be assumed that the elastic mixture of two isotropic materials occupies the domain  $B^+$  or  $B^-$ . For these domains we will solve Problem (I) $^\pm$ :

Find vectors  $\overset{1}{u}$  and  $\overset{2}{u}$  which are a solution of system (5.1) in  $B^+$  or  $B^-$  if  $\forall y \in S$ :

$$(\overset{m}{u})^+(y) = \overset{m}{f}(y) \quad \text{or} \quad (\overset{m}{u})^-(y) = \overset{m}{f}(y), \quad m = 1, 2, \quad (5.2)$$

where  $\overset{m}{f} = (\overset{m}{f}_1, \overset{m}{f}_2, \overset{m}{f}_3)$  are the vectors given on  $S$ .

The solution  $\overset{1}{u}, \overset{2}{u}$  of Problem (I) $^+$  [(I) $^-$ ] is called regular if  $\overset{m}{u} \in \mathbb{C}^1(\bar{B}^+) \cap \mathbb{C}^2(B^+)$  [ $\overset{m}{u} \in \mathbb{C}^1(\bar{B}^-) \cap \mathbb{C}^2(B^-)$ ],  $m = 1, 2$ . On the other hand, the solution  $\overset{1}{u}, \overset{2}{u}$  of problem (I) $^+$  [(I) $^-$ ] will be called classical if  $\overset{m}{u} \in \mathbb{C}(\bar{B}^+) \cap \mathbb{C}^2(B^+)$  [ $\overset{m}{u} \in \mathbb{C}(\bar{B}^-) \cap \mathbb{C}^2(B^-)$ ],  $m = 1, 2$ .

As is well-known (Natroshvili, Jagmaidze, Svanadze [1]; Knops, Steel [1]; Atkin, Chadwick, Steel [1]), if the constants  $a_1, a_2, b_1, b_2, c$  and  $d$  satisfy some conditions (necessary and sufficient conditions for potential energy to be positive-definite), then the uniqueness theorem of a regular solution of the first boundary value problem is valid. As to the question of the uniqueness of a classical solution of this problem, it will be considered further on.

In deriving solutions of Problems (I) $^+$  and (I) $^-$  our main task is to represent the vectors  $\overset{1}{u}$  and  $\overset{2}{u}$  in terms of harmonic functions. We have

**Theorem 5.1.** *If*

$$\overset{m}{u} = \overset{m}{v} + \frac{\rho^2 - r^2}{2} \operatorname{grad} \overset{m}{\psi}, \quad m = 1, 2 \quad (5.3)$$

and there are

$$((2a_1 + b_1)\mathcal{D}_r + a_1)\overset{1}{\psi} + ((2c + d)\mathcal{D}_r + c)\overset{2}{\psi} = \operatorname{div}(b_1\overset{1}{v} + d\overset{2}{v}), \quad (5.4)$$

$$((2c + d)\mathcal{D}_r + c)\overset{1}{\psi} + ((2a_2 + b_2)\mathcal{D}_r + a_2)\overset{2}{\psi} = \operatorname{div}(d\overset{1}{v} + b_2\overset{2}{v}), \quad (5.5)$$

where  $\Delta\overset{m}{v} = 0$ ,  $\Delta\overset{m}{\psi} = 0$ ,  $m = 1, 2$ ;  $\mathcal{D}_r \equiv r\frac{\partial}{\partial r} = x_k\frac{\partial}{\partial x_k}$ ;  $r \equiv |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ ,  $\rho$  is an arbitrary nonnegative constant, then  $\overset{1}{u}$ ,  $\overset{2}{u}$  is the solution of system (5.1) in  $\mathbb{R}^3 \setminus S(0, \rho)$ .

**Problem 43.** Prove Theorem 5.1.

**5.2. Solution of Problem (I)<sup>+</sup>.** We seek  $\overset{1}{u}$  and  $\overset{2}{u}$  in the form of (5.3). Then, by virtue of the boundary conditions (5.2), for  $\overset{1}{v}$  and  $\overset{2}{v}$  we obtain the Dirichlet problem in  $B^+$  and hence they can be expressed by the Poisson formula

$$\overset{m}{v}(x) = \Pi(\overset{m}{f})(x) \equiv \frac{1}{4\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|y - x|^3} \overset{m}{f}(y) dy, \quad m = 1, 2. \quad (5.6)$$

Substituting the obtained values of  $\overset{1}{v}$  and  $\overset{2}{v}$  in (5.4), (5.5), to define  $\overset{1}{\psi}$  and  $\overset{2}{\psi}$  we obtain a system of differential equations

$$((2a_1 + b_1)\mathcal{D}_r + a_1)\overset{1}{\psi} + ((2c + d)\mathcal{D}_r + c)\overset{2}{\psi} = \operatorname{div}\Pi(b_1\overset{1}{f} + d\overset{2}{f}), \quad (5.7)$$

$$((2c + d)\mathcal{D}_r + c)\overset{1}{\psi} + ((2a_2 + b_2)\mathcal{D}_r + a_2)\overset{2}{\psi} = \operatorname{div}\Pi(d\overset{1}{f} + b_2\overset{2}{f}). \quad (5.8)$$

Applying the operator  $(2a_2 + b_2)\mathcal{D}_r + a_2$  to (5.7) and the operator  $(2c + d)\mathcal{D}_r + c$  to (5.8) and performing subtraction, we obtain for  $\overset{1}{\psi}$  the equation

$$\begin{aligned} & \left( ((2a_1 + b_1)\mathcal{D}_r + a_1)((2a_2 + b_2)\mathcal{D}_r + a_2) - \right. \\ & \left. - ((2c + d)\mathcal{D}_r + c)((2c + d)\mathcal{D}_r + c) \right) \overset{1}{\psi} = F', \end{aligned} \quad (5.9)$$

where

$$\begin{aligned} F' \equiv & ((2a_2 + b_2)\mathcal{D}_r + a_2) \operatorname{div}\Pi(b_1\overset{1}{f} + d\overset{2}{f}) - \\ & - ((2c + d)\mathcal{D}_r + c) \operatorname{div}\Pi(d\overset{1}{f} + b_2\overset{2}{f}). \end{aligned} \quad (5.10)$$

Equation (5.9) can be written in the form

$$\mathcal{D}_r^2\overset{1}{\psi} + 2\alpha\mathcal{D}_r\overset{1}{\psi} + \beta\overset{1}{\psi} = \gamma F', \quad (5.11)$$

where

$$\begin{aligned}\alpha &\equiv \frac{1}{2} \frac{4a_1a_2 + a_1b_2 + a_2b_1 - 4c^2 - 2cd}{(2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2}, \\ \beta &\equiv \frac{a_1a_2 - c^2}{(2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2}, \\ \gamma &\equiv \frac{1}{(2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2}.\end{aligned}\quad (5.12)$$

Similarly, from systems (5.7) and (5.8) we obtain for  $\psi^2$  the equation

$$\mathcal{D}_r^2 \psi^2 + 2\alpha \mathcal{D}_r \psi^2 + \beta \psi^2 = \gamma F'', \quad (5.13)$$

where

$$\begin{aligned}F'' &\equiv ((2a_1 + b_2)\mathcal{D}_r + a_1) \operatorname{div} \Pi(df^1 + b_2 f^2) - \\ &- ((2c + d)\mathcal{D}_r + c) \operatorname{div} \Pi(b_1 f^1 + df^2).\end{aligned}\quad (5.14)$$

Thus the left-hand sides of equations (5.11) and (5.13) coincide. Note that due to representation (5.3) it is sufficient for  $\psi^1$  and  $\psi^2$  to be defined to within an arbitrary constant term. Therefore instead of equation (5.11) or (5.13) it is sufficient to solve their equivalent equation

$$\mathcal{D}_r^2 \psi(x) + 2\alpha \mathcal{D}_r \psi(x) + \beta \psi(x) = \gamma(F(x) - F(0)).$$

Introducing the variable  $t$  instead of the variable  $r$  by the formula  $t = \ln r$ , we obtain the linear equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\alpha \frac{\partial}{\partial t} + \beta\right) \psi\left(\frac{x}{r} e^t\right) = \gamma\left(F\left(\frac{x}{r} e^t\right) - F(0)\right). \quad (5.15)$$

Let us consider the corresponding characteristic equation

$$k^2 + 2\alpha k + \beta = 0 \quad (5.16)$$

and assume that  $\alpha^2 > \beta$  (the case  $\alpha^2 \leq \beta$  is treated quite similarly); then equation (5.16) has two different real roots

$$k_1 = -\alpha + \sqrt{\alpha^2 - \beta}, \quad k_2 = -\alpha - \sqrt{\alpha^2 - \beta} \quad (5.17)$$

and the solution of equation (5.15) in the class of harmonic functions is given by the formula (see the solution of equation (2.91))

$$\psi(x) = \frac{\gamma}{k_1 - k_2} \int_0^1 (\xi^{-1-k_1} - \xi^{-1-k_2}) (F(\xi x) - F(0)) d\xi. \quad (5.18)$$

By virtue of the identity  $\mathcal{D}_r \operatorname{div} = \operatorname{div}(\mathcal{D}_r - 1)$ , for  $F'$  and  $F''$  defined from (5.10) and (5.14) we have

$$\begin{aligned} F' &= \operatorname{div}(\mathcal{D}_r \Pi(\overset{1}{\mathcal{F}'}) + \Pi(\overset{1}{\mathcal{F}''})), \\ F'' &= \operatorname{div}(\mathcal{D}_r \Pi(\overset{2}{\mathcal{F}'}) + \Pi(\overset{2}{\mathcal{F}''})), \end{aligned}$$

where

$$\begin{aligned} \overset{1}{\mathcal{F}'} &\equiv (2a_2 + b_2)(b_1 \overset{1}{f} + \overset{2}{df}) - (2c + d)(\overset{1}{df} + b_2 \overset{2}{f}), \\ \overset{1}{\mathcal{F}''} &\equiv -(a_2 + b_2)(b_1 \overset{1}{f} + \overset{2}{df}) + (c + d)(\overset{1}{df} + b_2 \overset{2}{f}), \\ \overset{2}{\mathcal{F}'} &\equiv (2a_1 + b_1)(\overset{1}{df} + b_2 \overset{2}{f}) - (2c + d)(b_1 \overset{1}{f} + \overset{2}{df}), \\ \overset{2}{\mathcal{F}''} &\equiv -(a_1 + b_1)(\overset{1}{df} + b_2 \overset{2}{f}) + (c + d)(b_1 \overset{1}{f} + \overset{2}{df}). \end{aligned}$$

Replacing  $F$  in (5.18) by  $F'$  and  $F''$ , we obtain for  $\overset{1}{\psi}$  and  $\overset{2}{\psi}$

$$\begin{aligned} \overset{m}{\psi}(x) &= \frac{\gamma}{4\pi\rho(k_1 - k_2)} \operatorname{div} \left( \mathcal{D}_r \int_S (\Phi^{(2+k_1)}(x, y) - \right. \\ &\quad \left. - \Phi^{(2+k_2)}(x, y)) \overset{m}{\mathcal{F}'}(y) d_y S + \int_S (\Phi^{(2+k_1)}(x, y) - \right. \\ &\quad \left. - \Phi^{(2+k_2)}(x, y)) \overset{m}{\mathcal{F}''}(y) d_y S \right), \quad m = 1, 2, \end{aligned} \quad (5.19)$$

where

$$\Phi^{(\delta)}(x, y) = \int_0^1 \Phi_0(\tau x, y) \frac{d\tau}{\tau^\delta}; \quad \delta = k_1 + 2, k_2 + 2, \quad (5.20)$$

$$\Phi_0(x, y) \equiv \frac{\rho^2 - |x|^2}{|y - x|^3} - \frac{1}{\rho} - \frac{3xy}{\rho^3}. \quad (5.21)$$

Taking into account the identity

$$\mathcal{D}_r \Phi^{(\delta)}(x, y) = (\delta - 1) \Phi^{(\delta)}(x, y) + \Phi_0(x, y),$$

we finally obtain from (5.19)

$$\overset{m}{\psi}(x) = \frac{\gamma}{4\pi\rho(k_1 - k_2)} \operatorname{div} \int_S \sum_{p=1}^2 \Phi^{(2+k_p)}(x, y) \overset{m}{F}^{(k_p)}(y) d_y S, \quad (5.22)$$

where  $\overset{m}{F}^{(k_p)}$  is a linear combination of the vectors  $\overset{1}{f}$  and  $\overset{2}{f}$ :

$$\overset{1}{F}^{(k_p)} \equiv (a_2 + (2a_2 + b_2)k_p)(b_1 \overset{1}{f} + \overset{2}{df}) - (c + (2c + d)k_p)(\overset{1}{df} + b_2 \overset{2}{f}),$$

$$\overset{2}{F}^{(k_p)} \equiv (c + (2c + d)k_p)(b_1 \overset{1}{f} + \overset{2}{df}) - (a_1 + (2a_1 + b_1)k_p)(\overset{1}{df} + b_2 \overset{2}{f}).$$

Now, from formulas (5.3), (5.6), (5.22) we obtain the solution of Problem (I)<sup>+</sup>:

$$\begin{aligned} \overset{m}{u}(x) &= \frac{1}{4\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|y-x|^3} \overset{m}{f}(y) d_y S + \\ &+ \frac{\gamma(\rho^2 - |x|^2)}{8\pi\rho(k_1 - k_2)} \text{grad div} \int_S \sum_{p=1}^2 \Phi^{(2+k_p)}(x, y) \overset{m}{F}^{(k_p)}(y) d_y S, \quad m=1, 2. \end{aligned} \quad (5.23)$$

We have

**Theorem 5.2.** *If  $\overset{1}{f} \in C(S)$ ,  $\overset{2}{f} \in C(S)$ , then  $\overset{1}{u}, \overset{2}{u}$  defined by (5.23) are the unique classical solution of Problem (I)<sup>+</sup>.*

*If  $\overset{1}{f} \in \mathbb{C}^{1,h}(S)$ ,  $\overset{2}{f} \in \mathbb{C}^{1,h}(S)$ ,  $0 < h \leq 1$ , then  $\overset{1}{u}, \overset{2}{u}$  defined by (5.23) are the unique regular solutions of Problem (I)<sup>+</sup>.*

**Problem 44.** Prove Theorem 5.2.

**Problem 45.** Construct the solution of Problem (I)<sup>+</sup> when  $\alpha^2 \leq \beta$ .

**5.3. Solution of Problem (I)<sup>-</sup>.** The procedure of solving Problem (I)<sup>-</sup> is the same as for Problem (I)<sup>+</sup>. We seek for the solution in the form of (5.3). Then, with the boundary conditions (5.2) taken into account, we have for  $\overset{1}{v}$  and  $\overset{2}{v}$  the Dirichlet problem for B<sup>-</sup> and therefore they are given by the Poisson formula

$$\overset{m}{v}(x) = \Pi'(\overset{m}{f})(x) \equiv \frac{1}{4\pi\rho} \int_S \frac{|x|^2 - \rho^2}{|y-x|^3} \overset{m}{f}(y) d_y S, \quad m=1, 2. \quad (5.24)$$

After substituting the obtained values of  $\overset{1}{v}$  and  $\overset{2}{v}$  in (5.4), (5.5), to define  $\overset{1}{\psi}$  and  $\overset{2}{\psi}$  we have the equations

$$\mathcal{D}_r^2 \overset{m}{\psi} + 2\alpha \mathcal{D}_r \overset{m}{\psi} + \beta \overset{m}{\psi} = \gamma \overset{m}{\mathcal{F}}, \quad m=1, 2,$$

where  $\alpha, \beta$  and  $\gamma$  are defined by (5.12) and

$$\begin{aligned} \overset{m}{\mathcal{F}} &= \text{div } \mathcal{D}_r \Pi'(\overset{m}{\mathcal{F}}') + \text{div } \Pi'(\overset{m}{\mathcal{F}}''), \quad m=1, 2, \\ \overset{1}{\mathcal{F}}' &= (2a_2 + b_2)(b_1 \overset{1}{f} + \overset{2}{df}) - (2c + d)(\overset{1}{df} + b_2 \overset{2}{f}), \\ \overset{1}{\mathcal{F}}'' &= -(a_2 + b_2)(b_1 \overset{1}{f} + \overset{2}{df}) + (c + d)(\overset{1}{df} + b_2 \overset{2}{f}), \\ \overset{2}{\mathcal{F}}' &= (2a_1 + b_1)(\overset{1}{df} + b_2 \overset{2}{f}) - (2c + d)(b_1 \overset{1}{f} + \overset{2}{df}), \\ \overset{2}{\mathcal{F}}'' &= -(a_1 + b_1)(\overset{1}{df} + b_2 \overset{2}{f}) + (c + d)(b_1 \overset{1}{f} + \overset{2}{df}). \end{aligned} \quad (5.25)$$

Consider the equation

$$\mathcal{D}_r^2 \psi + 2\alpha \mathcal{D}_r \psi + \beta \psi = \gamma \mathcal{F} \quad (5.26)$$

and replace the variable  $r$  by  $t$ :

$$t = -\ln r, \quad -\infty < t < -\ln \rho, \quad \rho \leq r < \infty.$$

Then (5.26) transforms to the equation

$$\left( \frac{\partial^2}{\partial t^2} - 2\alpha \frac{\partial}{\partial t} + \beta \right) \psi \left( \frac{x}{r} e^{-t} \right) = \gamma \mathcal{F} \left( \frac{x}{r} e^{-t} \right),$$

whose solution assuming that  $\alpha^2 > \beta$  has, in the class of harmonic functions, the form

$$\psi(x) = \frac{\gamma}{k_1 - k_2} \int_0^1 (\tau^{-1-k_1} - \tau^{-1-k_2}) \mathcal{F} \left( \frac{x}{r} \right) d\tau,$$

where

$$k_1 = \alpha + \sqrt{\alpha^2 - \beta}, \quad k_2 = \alpha - \sqrt{\alpha^2 - \beta}.$$

If here we replace  $\mathcal{F}$  by  $\overset{1}{\mathcal{F}}$  and  $\overset{2}{\mathcal{F}}$  defined by (5.25), we obtain for  $\overset{1}{\psi}$  and  $\overset{2}{\psi}$  the equation

$$\overset{m}{\psi}(x) = \frac{\gamma}{4\pi\rho(k_1-2)} \operatorname{div} \int_S \sum_{p=1}^2 \tilde{\Phi}^{(k_p)}(x, y) \overset{m}{\mathcal{F}}^{(k_p)}(y) d_y S, \quad m=1, 2, \quad (5.27)$$

where

$$\begin{aligned} \overset{1}{\mathcal{F}}^{(k_p)} &= (c + d + (2c + d)k_p) (df + b_2 f^2) - \\ &\quad - (a_2 + b_2 + (2a_2 + b_2)k_p) (b_1 f + df^2), \\ \overset{2}{\mathcal{F}}^{(k_p)} &= (a_1 + b_1 + (2a_1 + b_1)k_p) (df + b_2 f^2) - \\ &\quad - (c + d + (2c + d)k_p) (b_1 f + df^2), \\ \tilde{\Phi}^{(\delta)}(x, y) &= \int_0^1 \tilde{\Phi}_0 \left( \frac{x}{\tau}, y \right) \frac{d\tau}{\tau^{1+\delta}}, \quad \tilde{\Phi}_0(x, y) = \frac{|x|^2 - \rho^2}{|x - y|^3}, \\ \mathcal{D}_r \tilde{\Phi}^{(\delta)}(x, y) &= -\delta \tilde{\Phi}^{(\delta)}(x, y) - \tilde{\Phi}_0(x, y). \end{aligned}$$

Now due to formulas (5.3), (5.24), (5.27) the solution of Problem (I)<sup>-</sup> can be written as

$$\overset{m}{u}(x) = \frac{1}{4\pi\rho} \int_S \frac{|x|^2 - \rho^2}{|x - y|^3} \overset{m}{f}(y) d_y S +$$

$$+ \frac{\gamma(|x|^2 - \rho^2)}{8\pi\rho(k_1 - k_2)} \operatorname{grad} \operatorname{div} \int_S \sum_{p=1}^2 \tilde{\Phi}^{k_p}(x, y) \mathcal{F}^{(k_p)}(y) d_y S, \quad m = 1, 2. \quad (5.28)$$

Taking into account the identity

$$\begin{aligned} \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^3} &= \frac{1}{|x - \tau y|} + 2\tau \frac{\partial}{\partial \tau} \frac{1}{|x - \tau y|}, \\ \frac{1}{x - \tau y} &= \frac{1}{\rho} \frac{\partial}{\partial \tau} \ln(-|x - y|^2 + |x|^2 + \rho^2 + 2\rho|x - \tau y| + 2\tau\rho^2), \end{aligned}$$

we obtain the representation

$$\tilde{\Phi}^{(\delta)}(x, y) = \frac{2}{|x - y|} + \Phi^{(\delta)}(x, y), \quad (5.29)$$

where

$$\left| \frac{\partial}{\partial x_i} \Phi^{(\delta)}(x, y) \right| \leq \frac{c}{|x - y|}, \quad \left| \frac{\partial^2}{\partial x_i \partial x_k} \Phi^{(\delta)}(x, y) \right| \leq \frac{c}{|x - y|^2}.$$

We have

**Theorem 5.3.** *If  $f, \bar{f} \in C(S)$ , then  $\overset{1}{u}, \overset{2}{u}$  defined by (5.28) are the unique classical solution of Problem (I)<sup>-</sup> provided that*

$$\bar{u}(x) = O\left(\frac{1}{|x|}\right), \quad \frac{\partial \bar{u}(x)}{\partial x_i} = o\left(\frac{1}{|x|}\right), \quad k = 1, 2, \quad i = 1, 2, 3.$$

**Problem 46.** Prove Theorem 5.3.

*Indication.* Make use of representation (5.29) and the formula

$$\lim_{B \ni x \rightarrow y \in S} (|x|^2 - \rho^2) \operatorname{grad} \operatorname{div} \int_S \frac{F(y)}{|y - x|} d_y S = 0,$$

where  $F = (F_1, F_2, F_3)$ ,  $F_i \in C(S)$ . By virtue of the results of Subsection 2.6 the last formula is proved similarly to (2.79).

Theorem 5.3 can also be proved similarly to Theorem 2.16 (see Theorems 2.14 and 2.15). One should bear in mind that the pair  $\overset{1}{u}^{(k)}, \overset{2}{u}^{(k)}$ , where

$$\begin{aligned} \overset{m}{u}_i^{(k)}(x) &= \overset{m}{\kappa}_{ik}(x), \quad i, k = 1, 2, 3; \quad m = 1, 2, \\ \overset{m}{\kappa}_{ik} &= \frac{\rho}{|x|} \delta_{ik} + A_m \frac{\rho^2 - |x|^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \frac{\rho}{|x|}, \\ A_1 &= \frac{\gamma}{(k_1 - 2)(k_2 - 2)} ((b_2 + d)(3c + 2d) - (b_1 + d)(3a_2 + 2b_2)), \\ A_2 &= \frac{\gamma}{(k_1 - 2)(k_2 - 2)} ((b_1 + d)(3c + 2d) - (b_2 + d)(3a_1 + 2b_1)), \end{aligned}$$

satisfies system (5.1) in  $B^-$ , the conditions at infinity

$$u^{(k)}(x) = O(|x|^{-1}), \quad \frac{\partial}{\partial x_i} u^{(k)}(x) = O(|x|^{-2})$$

and the boundary conditions

$$(u_i^{(k)})^-(y) = \delta_{ik}.$$

**Problem 47.** Construct in quadratures the solution of Problem (II)<sup>+</sup> (see Natroshvili, Jagmaidze, Svanadze [1]).

**5.4. Boundary Value Problems of Thermoelasticity of Elastic Mixtures.** Consider a simplified system of thermoelastic equations of the linear theory of elastic mixtures for two isotropic materials (Khoroshun, Soltanov [1])

$$\begin{aligned} a_1 \Delta \overset{1}{u} + b_1 \operatorname{grad} \operatorname{div} \overset{1}{u} + c \Delta \overset{2}{u} + d \operatorname{grad} \operatorname{div} \overset{2}{u} - \gamma_1 \operatorname{grad} \overset{1}{\theta} &= 0, \\ c \Delta \overset{1}{u} + d \operatorname{grad} \operatorname{div} \overset{1}{u} + a_2 \Delta \overset{2}{u} + b_2 \operatorname{grad} \operatorname{div} \overset{2}{u} - \gamma_2 \operatorname{grad} \overset{2}{\theta} &= 0, \\ \Delta \overset{1}{\theta} &= 0, \\ \Delta \overset{2}{\theta} &= 0, \end{aligned} \quad (5.30)$$

where  $\overset{1}{u} = (\overset{1}{u}_1, \overset{1}{u}_2, \overset{1}{u}_3)$  and  $\overset{2}{u} = (\overset{2}{u}_1, \overset{2}{u}_2, \overset{2}{u}_3)$  and are the displacement vectors,  $\overset{1}{\theta}$  and  $\overset{2}{\theta}$  the temperature values,  $a_1, a_2, b_1, b_2, c, d, \gamma_1$  and  $\gamma_2$  the constants.

For this system we will formulate the following problems:

Find a quadruplet  $(\overset{1}{u}, \overset{2}{u}, \overset{1}{\theta}, \overset{2}{\theta})$  which is a solution of system (5.30) in  $B^+$  or in  $B^-$  by the boundary conditions

$$(\overset{m}{u})^\pm = \overset{m}{f}, \quad (\overset{m}{\theta})^\pm = \overset{m}{\eta}, \quad m = 1, 2$$

– Problem (I.I)<sup>±</sup>, or

$$(\overset{m}{u})^\pm = \overset{m}{f}, \quad \left(\frac{\partial \overset{m}{\theta}}{\partial n}\right)^\pm = \overset{m}{\eta}, \quad m = 1, 2$$

– Problem (I.II)<sup>±</sup>, where  $\overset{m}{f} = (f_1^m, f_2^m, f_3^m)$  and  $\overset{m}{\eta}$ ,  $m = 1, 2$ , are the given functions on  $S$ .

It is easy to prove

**Theorem 5.4.** *If*

$$\overset{m}{u}(x) = \overset{m}{v}(x) + \frac{\rho^2 - |x|^2}{2} \operatorname{grad} \overset{m}{\psi}(x), \quad m = 1, 2,$$

where  $\Delta \overset{m}{v} = 0$ ,  $\Delta \overset{m}{\psi} = 0$ ,

$$((2a_1 + b_1)\mathcal{D}_r + a_1)\overset{1}{\psi} + ((2c + d)\mathcal{D}_r + c)\overset{2}{\psi} = \operatorname{div}(b_1 \overset{1}{v} + d \overset{2}{v}) - \gamma_1 \overset{1}{\theta},$$

$$((2c + d)\mathcal{D}_r + c)\psi^1 + ((2a_2 + b_2)\mathcal{D}_r + a_2)\psi^2 = \operatorname{div}(d\dot{v}^1 + b_2\dot{v}^2) - \gamma_2\theta^2,$$

$\rho$  is an arbitrary positive constant,  $\theta^1$  and  $\theta^2$  are arbitrary solutions of the equation  $\Delta\theta = 0$ , then the quadruplet  $(\dot{u}^1, \dot{u}^2, \theta^1, \theta^2)$  is a solution of system (5.30) in  $\mathbb{R}^3 \setminus S(0, \rho)$ .

Having the solutions of the thermostatic boundary value problems of classical elasticity and Theorem 5.4, one can easily construct the solutions of Problems (I.I) $^\pm$  and (I.II) $^\pm$ .

**Problem 48.** Solve in quadratures Problems (I.I) $^\pm$  and (I.II) $^\pm$ .

It has to be noted that when solving these problems, we take it for granted that we have the solutions of the static problems of two isotropic elastic mixtures (I) $^+$  and (I) $^-$  for the nonhomogeneous system (5.1) in the particular case (see 4.7) when the right-hand sides of (5.1) are represented in the form  $\operatorname{grad} F$ , where  $F$  is a harmonic function.

\* \* \*

Similar investigations are carried out in a sufficiently great number of works by the well-known scientists J. Stefan, C. Truesdell, X.A. Rakhmatulin, R. Toupin, P.N. Naghdi, M.A. Biot, A.E. Green, T.R. Steel, A.C. Eringen, R.J. Atkin, P. Chadwick, R.J. Knops, H. Tiersten, M. Jahanmir, P. Villaggio, B. Lampriere and others. A relevant survey and bibliography can be found in Rakhmatulin [1], Atkin, Crain [1], Cryer [1], Khoroshun, Soltanov [1], Filippov [1], Natroshvili, Jagmaidze, Svanadze [1], Rushitsky [1] (see also Giorgashvili [1], Chichinadze [4]).

CHAPTER VI  
**PROBLEMS FOR LINEARIZED STATIONARY  
 NAVIER-STOKES EQUATIONS**

**6.1. Formulation of the Problems and the Uniqueness Theorems.** A homogeneous system of Stokes-linearized equations for a stationary flow of noncompressible viscous fluid has the form (Lamb[1])

$$\begin{aligned}\mu\Delta v(x) - \text{grad } P(x) &= 0, \\ \text{div } v(x) &= 0,\end{aligned}\tag{6.1}$$

where  $x \equiv (x_1, x_2, x_3)$  is a point from the three-dimensional Euclidean space  $\mathbb{R}^3$ ,  $\Delta$  the three-dimensional Laplace operator,  $v \equiv (v_1, v_2, v_3)$  the velocity vector,  $P$  the pressure,  $\mu$  the viscosity coefficient.

Let  $\sigma^{(n)}(x) \equiv (\sigma_1^{(n)}(x), \sigma_2^{(n)}(x), \sigma_3^{(n)}(x))$  be the stress vector at the point  $x$  directed towards the (unit) vector  $n \equiv (n_1, n_2, n_3)$ . The relationship between the pressure, velocity vector and stress is expressed by

$$\sigma_i^{(n)}(x) = -P(x)n_i + \mu n_j \left( \frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i} \right), \quad i = 1, 2, 3.\tag{6.2}$$

It is assumed that some viscous fluid fills up the domain  $B^+$  or  $B^-$ .

Let us formulate the boundary value problems:

In  $B^+$  or  $B^-$  find a solution  $(v, P)$  of system (6.1) by one of the following conditions on the boundary:

$$\forall y \in S : (v)^\pm(y) = f(y)\tag{6.3}$$

– the first problem (Problem (I) $^\pm$ ),

$$\forall y \in S : (\sigma^{(n)})^\pm(y) = f(y)\tag{6.4}$$

– the second problem (Problem (II) $^\pm$ ),

$$\forall y \in S : (n \cdot v)^\pm(y) = g(y), \quad (\sigma^{(n)} - n(n \cdot \sigma^{(n)}))^\pm(y) = l(y)\tag{6.5}$$

– the third problem (Problem (III) $^\pm$ ),

$$\forall y \in S : (n \cdot \sigma^{(n)})^\pm(y) = g(y), \quad (v - n(n \cdot v))^\pm(y) = l(y)\tag{6.6}$$

– the fourth problem (Problem (IV) $^\pm$ ). Here  $f = (f_1, f_2, f_3)$ ,  $l = (l_1, l_2, l_3)$  and  $g$  are the functions given on  $S$ ,  $n \equiv n(y)$  is the unit normal vector to the surface  $S$  at the point  $y$ , which is external with respect to  $B^+$  and

$$\forall y \in S : (n(y) \cdot l(y)) = 0.\tag{6.7}$$

The pair  $(v, P)$  is called a regular solution of system (6.1) if  $v \in C^1(\bar{B}^\pm) \cap C^2(B^\pm)$ ,  $P \in C(\bar{B}^\pm) \cap C^2(B^\pm)$ . The solution  $(v, P)$  is classical if  $v \in C(\bar{B}^\pm) \cap C^2(B^\pm)$ ,  $P \in C^1(B^\pm)$  and, moreover,  $(\sigma^{(n)})^\pm \in C(S)$  in Problems (II) $^\pm$ , (III) $^\pm$  and (IV) $^\pm$ .

Problems (I)<sup>±</sup> and (II)<sup>±</sup> were investigated by the methods of a potential and integral equations in Lichtenstein [1], Odqvist [1], Ladizhenskaya [1]. The necessary and sufficient conditions for the solvability of boundary value problems are also obtained there. For Problem (I)<sup>+</sup> and these condition are written as

$$\int_S y f(y) d_y S = 0, \quad (6.8)$$

and for Problem (II)<sup>+</sup> as

$$\int_S f(y) d_y S = 0, \quad \int_S [y \times f(y)] d_y S = 0. \quad (6.9)$$

In the case of Problem (III)<sup>+</sup>, in addition to condition (6.7), we have the conditions

$$\int_S g(y) d_y S = 0, \quad (6.10)$$

$$\int_S [y \times l(y)] d_y S = 0. \quad (6.11)$$

The sufficiency of (6.8) and (6.10) follows from the second equation of system (6.1). Conditions (6.9) show that the main vector and the principal moment of external force are zero, (6.11) is the consequence of the fact that the main moment of external force is zero.

Problems (I)<sup>±</sup>, ..., (IV)<sup>±</sup> with zero boundary conditions will be denoted by the previous symbols with the subscript "zero" added: (I)<sub>0</sub><sup>±</sup>, ..., (IV)<sub>0</sub><sup>±</sup>.

For regular solutions of these problems we have the following uniqueness theorems:

**Theorem 6.1.** *If (v, P) is a regular solution of Problem (I)<sub>0</sub><sup>+</sup>, then v = 0, P = p<sub>0</sub>, where p<sub>0</sub> is an arbitrary constant.*

**Theorem 6.2.** *If (v, P) is a regular solution of Problem (II)<sub>0</sub><sup>+</sup>, then v(x) = [a × x] + b, P = 0, where a and b are arbitrary constant three-dimensional vectors.*

**Theorem 6.3.** *If (v, P) is a regular solution of Problem (III)<sub>0</sub><sup>+</sup>, then v(x) = [a × x], P = p<sub>0</sub>, where a is an arbitrary constant vector, p<sub>0</sub> is any constant number.*

**Theorem 6.4.** *Problem (IV)<sub>0</sub><sup>+</sup> may have only the trivial solution v = 0, P = 0.*

**Theorem 6.5.** *If in the neighbourhood of the point at infinity the regular solutions of Problems (I) $^-$ , (II) $^-$ , (III) $^-$ , (IV) $^-$  satisfy the conditions*

$$P(x) = o(|x|^{-1}), \quad v_i(x) = O(|x|^{-1}), \quad \frac{\partial v_i(x)}{\partial x_k} = o(|x|^{-1}), \quad (6.12)$$

*then these problems may have only the trivial solution  $v = 0$ ,  $P = 0$ .*

One can easily prove Theorems 6.1–6.5 using the Green formulas. For the regular solution of system (6.1) in  $B^+$  the Green formula is written in the form

$$\int_S v \cdot \sigma^{(n)} dS = \frac{\mu}{2} \int_{B^+} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) dx, \quad (6.13)$$

and for the regular solution of system (6.1) in  $B^-$  that satisfies conditions (6.12) it has the form

$$\int_S v \cdot \sigma^{(n)} dS = -\frac{\mu}{2} \int_{B^-} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) dx. \quad (6.14)$$

Note that for the classical solutions of Problems (II) $^\pm$ , (III) $^\pm$  and (IV) $^\pm$  (when these problems are considered for  $B^-$  and satisfy conditions (6.12)) the Green formulas (6.13) and (6.14) are valid and hence so are uniqueness theorems 6.2 - 6.5. The uniqueness of the classical solutions of Problems (I) $^+$  and (I) $^-$  will be discussed below when deriving their solutions.

To solve these problems by the method we propose it is essential to represent the solution  $(v, P)$  of system (6.1) by means of harmonic functions. Such a representation can be obtained by the theorem to be given below. It is assumed as above that  $r \equiv |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$  and  $\rho$  is any positive number,  $\mathcal{D}_r \equiv r \frac{\partial}{\partial r}$ .

**Theorem 6.6.** *If*

$$v = u + \frac{\rho^2 - r^2}{2} \text{grad } \psi, \quad P = -\mu(2\mathcal{D}_r + 1)\psi, \quad (6.15)$$

*where  $\Delta u = 0$ ,  $\Delta \psi = 0$  and  $u$  and  $\psi$  are interconnected by the relation*

$$\mathcal{D}_r \psi = \text{div } u, \quad (6.16)$$

*then the pair  $(v, P)$  is the solution of system (6.1) in  $\mathbb{R}^3 \setminus S(0, \rho)$ .*

Representation (6.15) will be used to solve Problems (I) $^\pm$  and (III) $^\pm$ . In solving Problems (II) $^\pm$  and (IV) $^\pm$ , it is convenient to use other equivalent representations.

**Theorem 6.7.** *If*

$$\begin{aligned} v(x) &= u(x) + x(2\mathcal{D}_r + 1)\psi(x) - r^2 \operatorname{grad} \psi(x) + \\ &\quad + \frac{\rho^2 - r^2}{2} \operatorname{grad} \psi(x), \\ P(x) &= -\mu(2\mathcal{D}_r + 1)\psi(x), \end{aligned} \quad (6.17)$$

where

$$\begin{aligned} \Delta u &= 0, \quad \Delta \psi = 0, \\ 2\mathcal{D}_r^2 \psi + 4\mathcal{D}_r \psi + 3\psi &= -\operatorname{div} u, \end{aligned} \quad (6.18)$$

then the pair  $(v, P)$  is the solution of system (6.1) in  $\mathbb{R}^3 \setminus S$ .

**Theorem 6.8.** *If*

$$\begin{aligned} v(x) &= u(x) + x(2\mathcal{D}_r + 1)\varphi(x) + (\rho^2 - r^2) \operatorname{grad} \varphi(x) + \\ &\quad + \frac{\rho^2 - r^2}{2} \operatorname{grad} \psi(x), \\ P(x) &= -\mu(2\mathcal{D}_r + 1)\psi(x), \end{aligned} \quad (6.19)$$

where

$$\begin{aligned} \Delta u &= 0, \quad \Delta \psi = 0, \quad \Delta \varphi = 0 \\ \mathcal{D}_r \psi &= \operatorname{div} u + 2\mathcal{D}_r^2 \varphi + 5\mathcal{D}_r \varphi + 3\varphi, \end{aligned} \quad (6.20)$$

then the pair  $(v, P)$  is the solution of system (6.1) in  $\mathbb{R}^3 \setminus S$ .

The procedure of solving the boundary value problems for linearized stationary Navier-Stokes equations is similar to the method used for the corresponding boundary value problems of classical elasticity. That is why to avoid repetition in constructing solutions and establishing their properties we will note without proofs some principal facts as regards solutions of the boundary value problems.

In the sequel we will deal mainly with solutions of the problems for the domain  $B^+$ . Problems for the domain  $B^-$  are solved in a similar manner.

**6.2. Solution of Problem (I)<sup>+</sup>.** The solution of this problem is to be found in form (6.15). Then, as can be easily verified, for  $u$  we have the Dirichlet problem for the ball whose solution is given by the Poisson formula

$$\forall x \in B^+ : u(x) = \Pi(f)(x) \equiv \frac{1}{4\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|y - x|^3} f(y) d_y S. \quad (6.21)$$

After substituting the found value of  $u$  in (6.16) and treating it as a differential equation for the harmonic function  $\psi$  with the right-hand side  $F(x) = \operatorname{div} \Pi(f)(x)$ , we will have

$$\psi(x) = \int_0^r F\left(\frac{x}{r}\eta\right) \frac{d\eta}{\eta} + c, \quad c = \text{const.}$$

The substitution of  $\tau = \eta/r$  brings it to the form

$$\psi(x) = \int_0^1 \operatorname{div} \Pi(f)(\tau x) \frac{d\tau}{\tau^2} + c. \quad (6.22)$$

By virtue of condition (6.8)  $\psi$  can also be written as

$$\psi(x) = \frac{1}{4\pi\rho} \operatorname{div} \int_S \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} - \frac{3\tau}{\rho^3} x \cdot y \right) \frac{d\tau}{\tau^2} f(y) d_y S + c.$$

Taking into account the notation of (2.44), (2.53) and representation (2.74), we have

$$\psi(x) = \frac{1}{4\pi\rho} \operatorname{div} \int_S \left( \frac{2}{|x - y|} - \frac{3}{\rho^2} \chi(x, y) \right) f(y) d_y S + c, \quad (6.23)$$

where

$$\chi(x, y) = |x - y| + \frac{x \cdot y}{\rho} \ln \left( (|x - y| + \rho)^2 - |x|^2 \right). \quad (6.24)$$

Now from (6.15), (6.21) and (6.22) it is easy to write the solution of Problem (I)<sup>+</sup>

$$v(x) = \Pi(f)(x) + \frac{\rho^2 - |x|^2}{2} \operatorname{grad} \int_0^1 \operatorname{div} \Pi(f)(\tau x) \frac{d\tau}{\tau^2}, \quad (6.25)$$

$$P(x) = -\mu(2\mathcal{D}_r + 1) \int_0^1 \operatorname{div} \Pi(f)(\tau x) \frac{d\tau}{\tau^2} + p_0, \quad p_0 = \text{const.},$$

or, taking into account (6.21) and (6.23), its equivalent form

$$\begin{aligned} v(x) &= \frac{1}{4\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|x - y|^3} f(y) d_y S + \\ &+ \frac{\rho^2 - |x|^2}{4\pi\rho} \operatorname{grad} \operatorname{div} \int_S \left( \frac{1}{|x - y|} + \frac{3\chi(x, y)}{2\rho^2} \right) f(y) d_y S, \end{aligned} \quad (6.26)$$

$$P(x) = -\frac{\mu}{2\pi\rho} \operatorname{div} \int_S \left( \frac{\rho^2 - |x|^2}{|x - y|^3} + \frac{1}{|x - y|} - \frac{3\chi(x, y)}{2\rho^2} \right) f(y) d_y S + p_0.$$

**6.3. Analysis of the Solution.** First we have to derive some estimates needed for our further reasoning.

The identity

$$\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|x-y|} = 3 \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} - \frac{\delta_{ij}}{|x-y|^3}, \quad (6.27)$$

allows us to represent the vector  $v$  determined by (6.26) in the form

$$\begin{aligned} v(x) = & \frac{3(\rho^2 - |x|^2)}{4\pi\rho} \int_S \left\| \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} - \right. \\ & \left. - \frac{1}{2\rho^2} \frac{\partial^2 \chi(x, y)}{\partial x_i \partial x_j} \right\|_{3 \times 3} f(y) d_y S. \end{aligned} \quad (6.28)$$

Calculating the second derivatives of the function  $\chi$  and taking into account the estimates

$$\begin{aligned} \forall x \in B^+, \quad \forall y \in S : \\ |x \cdot y| \leq \rho^2, \quad |x - y| \leq 2\rho, \quad |x_i - y_i| \leq |x - y|, \quad \rho - |x| \leq |x - y|, \\ 2\rho|x - y| \leq (|x - y| + \rho)^2 - |x|^2 \leq 8\rho|x - y|, \end{aligned}$$

we obtain

$$\left| \frac{\partial^2 \chi(x, y)}{\partial x_i \partial x_j} \right| \leq \frac{32\rho}{|x-y|^2} \quad (6.29)$$

$$\left| \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} - \frac{1}{2\rho^2} \frac{\partial^2 \chi(x, y)}{\partial x_i \partial x_j} \right| \leq \frac{33}{|y-x|^3}, \quad i, j = 1, 2, 3. \quad (6.30)$$

Now one can readily prove

**Theorem 6.9.** *If  $f \in C^{3,\gamma}(S)$ ,  $0 < \gamma \leq 1$  and satisfies condition (6.8), then the pair  $(v, P)$  determined by (6.25) is a regular solution of Problem (I)<sup>+</sup>.*

**Theorem 6.10.** *If  $x \in B^+$ , then*

$$\frac{3(\rho^2 - |x|^2)}{4\pi\rho} \int_S \left( \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} - \frac{1}{2\rho^2} \frac{\partial^2 \chi(x, y)}{\partial x_i \partial x_j} \right) d_y S = \delta_{ij}, \quad (6.31)$$

where  $\chi$  is determined by (6.24).

*Proof.* Consider the pair  $(\overset{k}{v}, \overset{k}{P})$ , where  $\overset{k}{v}_i(x) = \delta_{ik}$ ,  $\overset{k}{P}(x) = 0$ ;  $i, k = 1, 2, 3$ ;  $x \in B^+$ . Obviously, this pair is a regular solution of Problem (I)<sup>+</sup> (condition (6.8) is fulfilled). Then by Theorem 6.9 formula (6.28) is valid and in the case under consideration it coincides with (6.31). ■

Next we are going to consider the uniqueness of the classical solution of Problem (I)<sup>+</sup>.

**Theorem 6.11.** *If the pair  $(v, P)$ , where  $P \in C^1(B^+)$ ,  $v \in C(\bar{B}^+) \cap C^2(B^+)$ , is a solution of system (6.1) and  $v^+ = 0$ , then  $\forall x \in B^+ : v(x) = 0$ ,  $P(x) = p_0$ , where  $p_0$  is an arbitrary constant.*

*Proof.* Let  $z$  be any fixed point in the domain  $B^+$ . Since  $v$  is continuous in  $\bar{B}^+$ , it is uniformly continuous. Therefore by virtue of the boundary condition  $v^+ = 0$ , for any number  $\varepsilon > 0$  there exists a number  $\delta > 0$  such that  $|v(y)| < \varepsilon$  for  $y \in S(0, \rho_0)$ , where  $\rho_0 = \rho - \delta_0$ ,  $0 < \delta_0 \leq \delta$ . One may assume that  $\delta_0 < \rho - |z|$  and therefore  $z \in B^+(0, \rho_0)$ .

From the condition of the theorem it follows that  $v \in C^2(\bar{B}^+(0, \rho_1))$ ,  $P \in C^1(\bar{B}^+(0, \rho_1))$ , where  $0 < \rho_1 \leq \rho_0$ , and thus one easily obtains  $v, P \in C^\infty(\bar{B}^+(0, \rho_0))$ . In that case from Theorem 6.9 we obtain the following (unique) representation:

$$|v(z)| = \frac{3(\rho_0^2 - |z|^2)}{4\pi\rho_0} \int_{S(0, \rho_0)} \left\| \frac{(z_i - y_i)(z_j - y_j)}{|z - y|^5} - \frac{1}{2\rho_0^2} \frac{\partial^2 \chi(z, y)}{\partial z_i \partial z_j} \right\|_{3 \times 3} v(y) d_y S.$$

On account of (6.30) we have

$$\begin{aligned} v(z) &\leq \frac{3(\rho_0^2 - |z|^2)}{4\pi\rho_0} \sum_{i,j=1}^3 \int_{S(0, \rho_0)} \left| \frac{(z_i - y_i)(z_j - y_j)}{|z - y|^5} - \right. \\ &\quad \left. - \frac{1}{2\rho_0^2} \frac{\partial^2 \chi(z, y)}{\partial z_i \partial z_j} \right| |v(y)| d_y S \leq \\ &\leq \frac{891\varepsilon}{4\pi\rho_0} \int_{S(0, \rho_0)} \frac{\rho_0 - |z|^2}{|z - y|^3} d_y S = 891\varepsilon. \end{aligned} \quad (6.32)$$

Hence, remembering that  $\varepsilon$  is arbitrary, we have  $v(z) = 0$ . Therefore  $\forall x \in B^+ : v(x) = 0$ . System (6.1) now yields  $\text{grad } P = 0$  and hence  $P = p_0$ . ■

**Theorem 6.12.** *If  $f \in C(S)$  and  $f$  satisfies condition (6.8), then the pair  $(v, P)$  determined by (6.26) is a classical solution of Problem (I)<sup>+</sup>. Moreover,  $v$  is unique and  $P$  is determined to within an arbitrary constant term.*

*Proof.* One may easily verify that when  $f \in C(S)$ ,  $v$  and  $P$  determined by (6.26) have derivatives of all orders in  $B^+$  and satisfy system (6.1).

By virtue of (6.28) and (6.31) we have

$$\begin{aligned} v(x) - f(z) &= \frac{3(\rho^2 - |x|^2)}{4\pi\rho} \int_S \left\| \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^5} - \right. \\ &\quad \left. - \frac{1}{2\rho^2} \frac{\partial^2 \chi(x, y)}{\partial x_i \partial x_j} \right\|_{3 \times 3} (f(y) - f(z)) d_y S. \end{aligned}$$

Now (6.30) yields

$$|v(x) - f(z)| \leq \frac{981}{4\pi\rho} \int_S \frac{\rho^2 - |x|^2}{|y - x|^3} |f(y) - f(z)| d_y S.$$

On account of the property of the Poisson integral  $v^+ = f$ . Hence in turn we obtain the inclusion  $v \in C(\bar{B}^+)$ . As to the second part of the theorem, its proof follows from the preceding theorem. ■

**Problem 49.** Construct the Green tensor for Problem (I)<sup>+</sup>.

**6.4. Solution of Problem (I)<sup>-</sup>.** The solution will be sought for in form (6.15). Then for  $u$  we have the Dirichlet problem in  $B^-$  and therefore  $u$  is given by the Poisson formula

$$\forall x \in B^- : u(x) = \Pi'(f)(x) \equiv \frac{1}{4\pi\rho} \int_S \frac{|x|^2 - \rho^2}{|x - y|^3} f(y) d_y S. \quad (6.33)$$

Taking the latter formula into account, from relation (6.16) we obtain for the harmonic function  $\psi$

$$\psi(x) = - \int_0^1 \operatorname{div} \Pi'(f)\left(\frac{x}{\tau}\right) d\tau \quad (6.34)$$

or

$$\psi(x) = - \frac{1}{4\pi\rho} \operatorname{div} \int_S \int_0^1 \frac{|x|^2 - \tau^2 \rho^2}{|x - \tau y|^3} \tau d\tau f(y) d_y S. \quad (6.35)$$

On account of (2.72)  $\psi$  can be written as

$$\psi(x) = - \frac{1}{4\pi\rho} \operatorname{div} \int_S \left( \frac{2}{|x - y|} + \frac{3}{\rho^2} \chi'(x, y) \right) f(y) d_y S, \quad (6.36)$$

where

$$\chi'(x, y) = |x| - |x - y| + \frac{x \cdot y}{\rho} \ln \frac{|x - y| + |x| - \rho}{|x - y| + |x| + \rho}. \quad (6.37)$$

Now taking into account (6.15), (6.33) and (6.34), we obtain the solution of Problem (I)<sup>-</sup> by

$$\begin{aligned} v(x) &= \Pi'(f)(x) + \frac{|x|^2 - \rho^2}{2} \operatorname{grad} \operatorname{div} \int_0^1 \Pi'(f)\left(\frac{x}{\tau}\right) d\tau, \\ P(x) &= \mu(2\mathcal{D}_r + 1) \operatorname{div} \int_0^1 \Pi'(f)\left(\frac{x}{\tau}\right) d\tau \end{aligned} \quad (6.38)$$

or, taking into account (6.33) and (6.36), by

$$\begin{aligned}
v(x) &= \frac{1}{4\pi\rho} \int_S \frac{|x|^2 - \rho^2}{|x-y|^3} f(y) d_y S + \\
&\quad + \frac{|x|^2 - \rho^2}{4\pi\rho} \operatorname{grad} \operatorname{div} \int_S \left( \frac{1}{|x-y|} + \frac{3\chi'(x,y)}{2\rho^2} \right) f(y) d_y S, \\
P(x) &= -\frac{\mu}{2\pi\rho} \operatorname{div} \int_S \left( \frac{|x|^2 - \rho^2}{|x-y|^3} - \frac{1}{|x-y|} - \frac{3\chi'(x,y)}{2\rho^2} \right) f(y) d_y S.
\end{aligned} \tag{6.39}$$

By virtue of identity (6.27) the vector  $v$  determined by (6.39) can be written as

$$\begin{aligned}
v(x) &= \frac{3(|x|^2 - \rho^2)}{4\pi\rho} \int_S \left\| \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} \right\| + \\
&\quad + \frac{1}{2\rho^2} \left\| \frac{\partial^2 \chi'(x,y)}{\partial x_i \partial x_j} \right\|_{3 \times 3} f(y) d_y S.
\end{aligned} \tag{6.40}$$

**6.5. Analysis of the Solution.** We begin by giving some necessary estimates. For  $\chi'$  the following estimates  $\forall x \in B^-(0, \rho) \cap B^+(0, 2\rho)$ ,  $\forall y \in S$  hold:

$$\left| \frac{\partial^2 \chi'(x,y)}{\partial x_i \partial x_j} \right| \leq \frac{48\rho}{|x-y|^2}, \tag{6.41}$$

$$\left| \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} + \frac{1}{2\rho^2} \frac{\partial^2 \chi'(x,y)}{\partial x_i \partial x_j} \right| \leq \frac{73}{|x-y|^3}, \quad i, j = 1, 2, 3. \tag{6.42}$$

Note that the pair  $(v, P)$  determined by (6.38) satisfies conditions (6.12) and, as a result, we have

**Theorem 6.13.** *If  $f \in C^{3,\gamma}(S)$ ,  $0 < \gamma \leq 1$ , then the pair  $(v, P)$  determined by (6.38) is the unique regular solution of Problem (I)<sup>-</sup> satisfying conditions (6.12).*

**Theorem 6.14.** *If  $x \in B^-$ , then*

$$\frac{3(|x|^2 - \rho^2)}{4\pi\rho} \int_S \left( \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} + \frac{1}{2\rho^2} \frac{\partial^2 \chi'(x,y)}{\partial x_i \partial x_j} \right) d_y S = \chi_{ij}(x), \tag{6.43}$$

where  $\chi(x) = \|\chi_{ij}(x)\|_{3 \times 3}$ ,

$$\chi_{ij}(x) = \frac{\rho}{|x|} \delta_{ij} + \frac{|x|^2 - \rho^2}{4} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\rho}{|x|}.$$

*Proof.* Consider the pairs  $(\overset{k}{v}, \overset{k}{P})$ ,  $k = 1, 2, 3$ , where

$$\begin{aligned}\overset{k}{v}_i(x) &= \frac{\rho}{|x|} \delta_{ik} + \frac{|x|^2 - \rho^2}{4} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\rho}{|x|}, \\ \overset{k}{P}(x) &= -\frac{3}{2} \mu \frac{\partial}{\partial x_k} \frac{\rho}{|x|}.\end{aligned}$$

One can easily observe that the pairs  $(\overset{k}{v}, \overset{k}{P})$  are the regular solutions of system (6.1), satisfy conditions (6.12) and  $(\overset{k}{v}_i)^+ = \delta_{ik}$ . But in that case by Theorem 6.5 the pair  $(\overset{k}{v}, \overset{k}{P})$  is the unique regular solution of Problem (I)<sup>-</sup>. Therefore formula (6.40) satisfied for  $\overset{k}{v}$  coincides with (6.43). ■

**Theorem 6.15.** *If  $f \in C(S)$ , then the pair  $(v, P)$  determined by (6.39) is the unique classical solution of Problem (I)<sup>-</sup> satisfying conditions (6.12).*

*Proof.* Let us verify whether the boundary condition  $(v)^- = f$  is fulfilled. Due to formulas (6.40), (6.42) and (6.43) we have

$$\begin{aligned}|v(x) - \varkappa(x)f(z)| &= \left| \frac{3(|x|^2 - \rho^2)}{4\pi\rho} \int_S \left\| \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^5} + \right. \right. \\ &\quad \left. \left. + \frac{1}{2\rho^2} \frac{\partial^2 \chi'(x, y)}{\partial x_i \partial x_j} \right\|_{3 \times 3} (f(y) - f(z)) d_y S \right| \leq \\ &\leq \frac{1971}{4\pi\rho} \int_S \frac{|x|^2 - \rho^2}{|x - y|^3} |f(y) - f(z)| d_y S.\end{aligned}$$

Hence by virtue of the property of the Poisson integral

$$\forall z \in S : \lim_{B^- \ni x \rightarrow z} (v(x) - \varkappa(x)f(z)) = 0$$

and therefore

$$\forall z \in S : \lim_{B^- \ni x \rightarrow z} v(x) = \lim_{B^- \ni x \rightarrow z} \varkappa(x)f(z) = f(z).$$

The remainder part of the theorem is likewise easily proved. ■

*Remark.* Due to (2.79) and the property of the Poisson integral Theorems 6.12 and 6.15 can be proved in a simpler manner if the solution of Problem (I)<sup>+</sup> is represented in form (6.26) and that of Problem (I)<sup>-</sup> in form (6.39).

**6.6. Solution of Problem (II)<sup>+</sup>.** Using representation (6.17) for the solution of this problem, we will construct the corresponding stress vector  $\sigma^{(n)}$  by formula (6.2). Taking into account relation (6.18), we can write

$$\sigma^{(n)}(x) = \frac{\mu}{r}h(x) + \frac{\mu(\rho^2 - r^2)}{r} \mathcal{D}_r \text{grad } \psi(x), \quad (6.44)$$

where  $h = (h_1, h_2, h_3)$ ,

$$h_i(x) = x_j \left( \frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i} \right) - x_i \text{div } u(x).$$

The form of stress (6.44) formally coincides with the form of stress of classical elasticity (2.41). Therefore, taking into account the boundary condition (6.4) of Problem (II)<sup>+</sup> and repeating the arguments used in solving Problem (II)<sup>+</sup> of classical elasticity, we can write the sought for vector in the form

$$\begin{aligned} u_i(x) = & \frac{1}{8\pi\mu} \int_S ((\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y))\delta_{ik} + \\ & + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \Phi^{(2)}(x, y)) - 2x_i \frac{\partial}{\partial x_k} \Phi^{(1)}(x, y) - \\ & - r^2 \frac{\partial^2}{\partial x_i \partial x_j} (\Phi^{(1)}(x, y) - \Phi^{(2)}(x, y)) f_k(y) d_y S + \\ & + c_{ik} x_k + b_i, \end{aligned} \quad (6.45)$$

where  $c_{ik}$  and  $b_i$  are arbitrary constants and  $\Phi^{(1)}$  and  $\Phi^{(2)}$  are determined by (2.53) as follows:

$$\Phi^{(m)}(x, y) = \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} - \frac{3x \cdot y}{\rho^3} \tau \right) \frac{d\tau}{\tau^m}, \quad m = 1, 2. \quad (6.46)$$

Having  $u$  in form (6.45), for the harmonic function  $\psi$  we obtain, by virtue of (6.18), the equation

$$\mathcal{D}_r^2 \psi + 2\mathcal{D}_r \psi + \frac{3}{2} \psi = F, \quad (6.47)$$

where

$$F(x) = \frac{1}{8\pi\mu} \text{div} \int_S \frac{\rho^2 - |x|^2}{|y - x|^3} f(y) d_y S.$$

The latter equation is the particular case of equation (2.61) whose solution can be written as

$$\psi(x) = \frac{1}{8\pi\mu} \text{div} \int_S \Psi_0(x, y) f(y) d_y S, \quad (6.48)$$

where

$$\Psi_0(x, y) = -\sqrt{2} \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \sin \frac{\ln \tau}{\sqrt{2}} \frac{d\tau}{\tau}. \quad (6.49)$$

Formulas (6.45) and (6.48) will now be used to construct  $v$  and  $P$ . We have

$$\begin{aligned} v_i(x) &= \frac{1}{8\pi\mu} \int_S \left( (\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y)) \delta_{ik} + \right. \\ &\quad + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \Phi^{(2)}(x, y)) + \\ &\quad + x_i \frac{\partial}{\partial x_k} ((2D_r - 1)\Psi_0(x, y) - 2\Phi^{(1)}(x, y)) + \\ &\quad + \frac{\rho^2 - r^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \Psi_0(x, y) + \\ &\quad + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y) - \Psi_0(x, y)) \Big) f_k(y) d_y S + \\ &\quad + c_{ik} x_k + b_i, \\ P(x) &= -\frac{1}{8\pi} \frac{\partial}{\partial x_k} \int_S (2D_r - 1) \Psi_0(x, y) f_k(y) d_y S. \end{aligned}$$

The expression for  $v$  contains arbitrary constants  $c_{ik}$  and  $b_i$  which should be chosen such that the boundary condition (6.4) of Problem (II)<sup>+</sup> be fulfilled. The final solution of Problem (II)<sup>+</sup> can be written in the form

$$\begin{aligned} v_i(x) &= \frac{1}{8\pi\mu} \int_S \left( (\Phi^{(1)}(x, y) + \Phi^{(2)}(x, y)) \delta_{ik} + x_k \frac{\partial}{\partial x_i} (\Phi^{(1)}(x, y) - \right. \\ &\quad - \Phi^{(2)}(x, y) + \frac{3x \cdot y}{\rho^3}) + x_i \frac{\partial}{\partial x_k} ((2D_r - 1)\Psi_0(x, y) - \\ &\quad - 2\Phi^{(1)}(x, y) - \frac{3x \cdot y}{\rho^3}) + \frac{\rho^2 - r^2}{2} \frac{\partial^2}{\partial x_i \partial x_k} \Psi_0(x, y) + \\ &\quad + r^2 \frac{\partial^2}{\partial x_i \partial x_k} (\Phi^{(2)}(x, y) - \Phi^{(1)}(x, y) - \Psi_0(x, y)) \Big) f_k(y) d_y S + \\ &\quad + \varepsilon_{ijk} a_j x_k + b_i, \quad i = 1, 2, 3, \\ P(x) &= -\frac{1}{8\pi} \operatorname{div} \int_S (2D_r - 1) \Psi_0(x, y) f(y) d_y S, \end{aligned} \quad (6.50)$$

where  $a_1, a_2, a_3, b_1, b_2, b_3$  are arbitrary constants,  $\varepsilon_{ijk}$  is the Levy-Civita

symbol. The stress vector  $\sigma^{(n)}$  has the form

$$\begin{aligned}\sigma^{(n)}(x) &= \frac{1}{4\pi r} \int_S \frac{\rho^2 - |x|^2}{|y-x|} f(y) d_y S + \\ &+ \frac{\rho^2 - r^2}{8\pi r} \text{grad div} \int_S (\mathcal{D}_r - 2) \Psi_0(x, y) f(y) d_y S.\end{aligned}$$

**Theorem 6.16.** *If  $f \in C(S)$  and conditions (6.9) are fulfilled, then  $(v, P)$  determined by (6.50) is the classical solution of Problem (II)<sup>+</sup>.*

**Problem 50.** Prove Theorem 6.16.

**Problem 51.** Construct the Green tensor for Problem (II)<sup>+</sup>.

**6.7. Solution of Problem (III)<sup>+</sup>.** First of all we replace condition (6.5) by its equivalent form. Like in the case of Problem (III)<sup>+</sup> of classical elasticity, using expression (6.2) for the stress  $\sigma^{(n)}$  and the first condition of (6.5), we obtain  $\forall y \in S$ :

$$\begin{aligned}(\sigma^{(n)} - n(n \cdot \sigma^{(n)}))^+(y) &= \frac{\mu}{\rho} (H(\partial y)v)^+(y) + \\ &+ 2\mu \mathcal{D}(g)(y) + \frac{2\mu}{\rho^2} y g(y),\end{aligned}\tag{6.51}$$

where  $H(\partial x) \equiv \|H_{ij}(\partial x)\|_{3 \times 3}$ ,

$$H_{ij}(\partial x) \equiv \delta_{ij}(\mathcal{D}_r - 2) - x_j \frac{\partial}{\partial x_i}.\tag{6.52}$$

Now by virtue of representation (6.51) replace the second boundary condition of (6.5) by its equivalent form and write the boundary conditions of Problem (III)<sup>+</sup> in the final form

$$(n \cdot v)^+ = g,\tag{6.53}$$

$$(H(\partial y)v)^+ = F,\tag{6.54}$$

where  $F = (F_1, F_2, F_3)$ ,

$$F(y) = \frac{\rho}{\mu} l(y) - 2\rho \mathcal{D}(g)(y) - \frac{2}{\rho} y g(y).\tag{6.55}$$

Note that

$$y F(y) = -2\rho g(y),\tag{6.56}$$

$$\int_S (y_i F_k(y) - y_k F_i(y)) d_y S = 0.\tag{6.57}$$

Formula (6.56) is the consequence of condition (6.7) and of the identity  $n_i D_i = 0$ , whereas (6.57) follows from condition (6.11) and the formula

$$\int_S (y_i \mathcal{D}_k - y_k \mathcal{D}_i)(g)(y) d_y S = 0.$$

The solution of Problem (III)<sup>+</sup> will be sought for in form (6.15). Calculate  $H(\partial x)v$  by

$$H(\partial x)v(x) = h(x), \quad (6.58)$$

where  $h = (h_1, h_2, h_3)$ ,

$$\begin{aligned} h_i(x) = & x_j \left( \frac{\partial u_i(x)}{\partial x_j} - \frac{\partial u_j(x)}{\partial x_i} \right) - \\ & - 2u_i(x) + x_i \mathcal{D}_r \psi(x) - \rho^2 \frac{\partial \psi(x)}{\partial x_i}. \end{aligned} \quad (6.59)$$

Due to relation (6.16) it is easy verify that  $h$  is harmonic in  $B^+$  and, using conditions (6.54) and (6.58), we have  $h^+ = F$ . Therefore  $h$  is given by the Poisson formula  $h = \Pi(F)$ . On the other hand,

$$\operatorname{div} h = (\mathcal{D}_r - 2) \operatorname{div} u + \mathcal{D}_r^2 \psi + 3\mathcal{D}_r \psi.$$

Hence, taking into account (6.16), for  $\psi$  we obtain the equation

$$\mathcal{D}_r^2 \psi + \frac{1}{2} \mathcal{D}_r \psi = \frac{1}{2} \operatorname{div} h. \quad (6.60)$$

To derive an equation for  $u$  equality (6.59) should be rewritten as

$$(\mathcal{D}_r - 1)u_i(x) = \frac{\partial(x \cdot u(x))}{\partial x_i} + h_i(x) - x_i \mathcal{D}_r \psi(x) + \rho^2 \frac{\partial \psi(x)}{\partial x_i}. \quad (6.61)$$

But by virtue of the same equality (6.59) we have

$$x \cdot u(x) = -\frac{x \cdot h(x)}{2} - \frac{\rho^2 - r^2}{2} \mathcal{D}_r \psi(x).$$

The substitution of the value  $x \cdot u(x)$  from the latter equality in (6.61) gives for  $u$  the equation

$$(\mathcal{D}_r - 1)u = q, \quad (6.62)$$

where  $q = (q_1, q_2, q_3)$ ,

$$q_i(x) = \frac{1}{2} \left( h_i(x) - x_k \frac{\partial h_k(x)}{\partial x_i} \right) - (\mathcal{D}_r - 1) \frac{\rho^2 - r^2}{2} \frac{\partial \psi(x)}{\partial x_i}. \quad (6.63)$$

To construct the solution of Problem (III)<sup>+</sup> in form (6.15) we will first solve (6.60) and then (6.62).

Rewrite (6.60) in its equivalent form

$$\mathcal{D}_r^2 \psi + \frac{1}{2} \mathcal{D}_r \psi = \eta, \quad (6.64)$$

where

$$\eta(x) = \frac{1}{8\pi\rho} \operatorname{div} \int_S \left( \frac{\rho^2 - |x|^2}{|y-x|^3} - \frac{1}{\rho} - \frac{3x \cdot y}{\rho^3} \right) F(y) d_y S.$$

To ascertain that equations (6.60) and (6.64) are equivalent, note that

$$\operatorname{div} \int_S (x \cdot y) F(y) d_y S = 0.$$

Indeed, by virtue of (6.10) and (6.56) we write

$$\operatorname{div} \int_S (x \cdot y) F(y) d_y S = \int_S y \cdot F(y) d_y S = -2\rho \int_S g(y) d_y S = 0.$$

Seeking for the solution of (6.64) (which is the particular case of (2.95)) in the class of harmonic functions in  $B^+$ , we obtain

$$\psi(x) = \frac{1}{4\pi\rho} \operatorname{div} \int_S \left( \Phi^{(2)}(x, y) - \Phi^{(1,5)}(x, y) \right) F(y) d_y S + c, \quad (6.65)$$

where  $c = \text{const}$ ,

$$\Phi^{(m)}(x, y) = \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} - \frac{3x \cdot y}{\rho^3} \tau \right) \frac{d\tau}{\tau^m}, \quad m=1, 5; 2. \quad (6.66)$$

As to equation (6.62), following the procedure of solving Problem (III)<sup>+</sup>, we can write its solution in the form

$$\begin{aligned} u_i(x) = & \frac{1}{8\pi\rho} \int_S \left( \left( \Phi^{(2)}(x, y) - \frac{1}{\rho} \right) \delta_{ik} - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) - \right. \\ & \left. - (\rho^2 - r^2) \frac{\partial^2}{\partial x_i \partial x_k} \left( \Phi^{(2)}(x, y) - \Phi^{(1,5)}(x, y) \right) \right) F_k(y) d_y S + c_{ij} x_j, \end{aligned} \quad (6.67)$$

where  $c_{ij}$  ( $i, j = 1, 2, 3$ ) are arbitrary constants.

Substituting the function  $\psi$  from (6.65) and the function  $u$  from (6.67) in (6.15) and choosing  $c_{ij}$  such that the boundary conditions (6.5) be fulfilled,

the solution  $(v, P)$  of Problem (III)<sup>+</sup> takes the form

$$\begin{aligned} v_i(x) &= \frac{1}{8\pi\rho} \int_S \left( (\Phi^{(2)}(x, y) - \frac{1}{\rho} - \frac{3x \cdot y}{\rho^3}) \delta_{ik} - \right. \\ &\quad \left. - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) \right) F_k(y) d_y S + \varepsilon_{ijk} a_j x_k, \\ P(x) &= -\frac{\mu}{4\pi\rho} \operatorname{div} \int_S \phi^{(2)}(x, y) F(y) d_y S + p_0, \end{aligned} \quad (6.68)$$

where  $\varepsilon_{ijk}$  is the Levy-Civita symbol,  $a = (a_1, a_2, a_3)$  an arbitrary constant vector,  $p_0$  an arbitrary constant number.

Note that the function  $\Phi^{(2)}$  determined by (6.66) can be represented as (see (2.74))

$$\begin{aligned} \Phi^{(2)}(x, y) &= \frac{2}{|x - y|} - \frac{3(x \cdot y)}{\rho^3} \ln ( (|x - y| + \rho)^2 - |x|^2 ) - \\ &\quad - \frac{3|x - y|}{\rho^2} + (6 \ln 2\rho - 5) \frac{x \cdot y}{\rho^3} + \frac{1}{\rho}. \end{aligned}$$

Similarly to Theorem 2.21 one can prove

**Theorem 6.17.** *If  $l \in C^{0,\gamma}(S)$ ,  $g \in C^{1,\gamma}(S)$ ,  $0 < \gamma \leq 1$ , and  $l$  and  $g$  satisfy conditions (6.7), (6.10), (6.11), then the pair  $(v, P)$  determined by (6.68) is the classical solution of Problem (III)<sup>+</sup>.*

**Problem 52.** Prove that the difference between any two classical solutions of Problem (III)<sup>+</sup> has the form  $(\overset{0}{v}, \overset{0}{P})$ , where  $\overset{0}{v}(x) \equiv [a \times x]$ ,  $\overset{0}{P} \equiv p_0$ ,  $p_0$  is a constant number, and  $a$  a constant three-dimensional vector.

Note that in common with representation (2.68) the density of the solution of Problem (III)<sup>+</sup> contains a derivative of the function  $g$  given on  $S$  (compare with (2.120)). After discarding derivatives from the density, the solution takes the form

$$\begin{aligned} v_i(x) &= \frac{1}{8\pi\rho} \int_S \left( (\Phi^{(2)}(x, y) - \frac{1}{\rho} - \frac{3x \cdot y}{\rho^3}) \delta_{ik} - x_k \frac{\partial}{\partial x_i} \Phi^{(2)}(x, y) \right) \times \\ &\quad \times \left( \frac{\rho}{\mu} l_k(y) - \frac{2}{\rho} y_k g(y) \right) d_y S - \frac{1}{4\pi} \int_S \left( (P_k^{(2)}(x, y) - \frac{2y_k - 3x_k}{\rho^3} - \right. \\ &\quad \left. - \frac{9y_k}{\rho^5} x \cdot y) \delta_{ik} - x_k \frac{\partial}{\partial x_i} P_k^{(2)}(x, y) \right) g(y) d_y S + \varepsilon_{ijk} a_j x_k, \\ P(x) &= -\frac{\mu}{4\pi\rho} \frac{\partial}{\partial x_k} \int_S \Phi^{(2)}(x, y) \left( \frac{\rho}{\mu} l_k(y) - \frac{2}{\rho} y_k g(y) \right) d_y S + \\ &\quad + \frac{\mu}{2\pi} \frac{\partial}{\partial x_k} \int_S P_k^{(2)}(x, y) g(y) d_y S + p_0, \end{aligned} \quad (6.69)$$

where

$$P_k^{(2)}(x, y) = (2D_r + 1) \int_0^1 \left( \frac{y_k - \tau x_k}{|y - \tau x|^3} - \frac{y_k}{2\rho^2} \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} + \right. \\ \left. + \frac{3y_k}{2\rho^2} \frac{1}{|y - \tau x|} - \frac{3y_k(x \cdot y)}{\rho^3} \tau + \frac{\tau x_k - 2y_k}{\rho^3} \right) \frac{d\tau}{\tau^2}.$$

**Problem 53.** Construct the solution of Problem (III)<sup>-</sup>.

**6.8. Solution of Problem (IV)<sup>+</sup>.** Replace the first of the boundary conditions (6.6) by its equivalent form. By virtue of (6.2)

$$n \cdot \sigma^{(n)} = -P + 2\mu n \cdot \frac{\partial v}{\partial n}.$$

Using the identity

$$n \cdot \frac{\partial v}{\partial n} = \operatorname{div} v - D_k(v_k - n_k(n \cdot v)) - \frac{2(n \cdot v)}{\rho},$$

and taking into account the second equation of system (6.1), we obtain

$$n \cdot \sigma^{(n)} = -\frac{4\mu}{\rho}(n \cdot v) - P - 2\mu D_k(v_k - n_k(n \cdot v)).$$

The boundary conditions of Problem (IV)<sup>+</sup> can now be rewritten as

$$\left( v - \frac{y(y \cdot v)}{\rho^2} \right)^+(y) = l(y), \quad (6.70)$$

$$\left( -\frac{4y \cdot v}{\rho^2} - \frac{P}{\mu} \right)^+(y) = h(y), \quad (6.71)$$

where

$$h = \frac{g}{\mu} + 2D_k(l_k). \quad (6.72)$$

The solution of Problem (IV)<sup>+</sup> is to be sought for in form (6.19) and, additionally, it is to be required that the condition  $(n \cdot u)^+ = 0$  be fulfilled.

It is easy to verify that

$$v_i(x) - \frac{x_i}{r^2} x_k v_k(x) = u_i(x) - \frac{x_i}{r^2} x_k u_k(x) + \\ + \frac{\rho^2 - r^2}{2} \left( \frac{\partial}{\partial x_i} - \frac{x_i}{r} \frac{\partial}{\partial r} \right) (\psi(x) + 2\varphi(x)).$$

Hence and from (6.70) it follows that  $u^+ = l$ . Moreover, since  $u$  is harmonic in  $B^+$ , it can be represented by the Poisson formula  $u = \Pi(l)$ .

Let us now find  $\psi$  and  $\varphi$ . By virtue of (6.19)

$$-\frac{4}{r^2} x_i \cdot v(x) - \frac{P(x)}{\mu} = \zeta(x) - \frac{4}{r^2} x \cdot u(x) -$$

$$-\frac{2}{r^2}(\rho^2 - r^2)\mathcal{D}_r(\psi(x) + 2\varphi(x)), \quad (6.73)$$

where  $\zeta = (2\mathcal{D}_r + 1)(\psi - 4\varphi)$ . Since  $\Delta\varphi = 0$ ,  $\Delta\psi = 0$ , it readily follows that  $\Delta\zeta = 0$ . Moreover, from (6.71), (6.73) and the condition  $(nu)^+ = 0$  we obtain  $\zeta^+ = h$ . Therefore  $\zeta = \Pi(h)$ . Hence we have

$$(2\mathcal{D}_r + 1)(\psi - 4\varphi) = \Pi(h).$$

Treating the latter relation as a diffeential equation for  $\psi - 4\varphi$ , we can write its solution in the class of harmonic functions as

$$\psi(x) - 4\varphi(x) = \frac{1}{2} \int_0^1 \Pi(h)(\tau x) \frac{d\tau}{\sqrt{\tau}} = \frac{1}{4\pi\rho} V(h)(x),$$

where

$$V(h)(x) \equiv \int_S \frac{h(y)}{|x-y|} d_y S.$$

Thus

$$\psi(x) = 4\varphi(x) + \frac{1}{4\pi\rho} V(h)(x). \quad (6.74)$$

If in (6.20) we replace  $\psi$  by its expression from (6.74) and set  $u = \Pi(l)$ , then for  $\varphi$  we obtain the equation

$$\mathcal{D}_r^2 + \frac{1}{2}\mathcal{D}_r\varphi + \frac{3}{2}\varphi = F, \quad (6.75)$$

where

$$F(x) = -\frac{1}{2} \operatorname{div} \Pi(l)(x) + \frac{1}{8\pi\rho} \mathcal{D}_r V(h)(x). \quad (6.76)$$

In the class of harmonic functions the solution of (6.75) has the form

$$\varphi(x) = -\frac{1}{k_2} \int_0^1 F(\tau x) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{1+k_1}}, \quad (6.77)$$

where  $k_1 = -\frac{1}{4}$ ,  $k_2 = \frac{\sqrt{23}}{4}$ .

Substituting the value  $F$  from (6.76) in (6.77), we finally obtain for  $\varphi$

$$\begin{aligned} \varphi(x) = & \frac{1}{8\pi\rho k_2} \operatorname{div} \int_S \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{2+k_1}} l(y) d_y S - \\ & - \frac{1}{8\pi k_2} \mathcal{D}_r \int_S \int_0^1 \frac{\sin(k_2 \ln \tau)}{|y - \tau x|} \frac{d\tau}{\tau^{1+k_1}} h(y) d_y S, \end{aligned} \quad (6.78)$$

which due to (6.74) gives for  $\psi$

$$\begin{aligned} \psi(x) = & \frac{1}{2\pi\rho k_2} \operatorname{div} \int_S \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^3} - \frac{1}{\rho} \right) \sin(k_2 \ln \tau) \frac{d\tau}{\tau^{2+k_1}} l(y) d_y S - \\ & - \frac{1}{2\pi\rho k_2} \mathcal{D}_r \int_S \int_0^1 \frac{\sin(k_2 \ln \tau)}{|y - \tau x|} \frac{d\tau}{\tau^{1+k_1}} h(y) d_y S + \\ & + \frac{1}{4\pi\rho} \int_S \frac{h(y)}{|y - x|} d_y S. \end{aligned} \quad (6.79)$$

Thus the solution of Problem (IV)<sup>+</sup> has the form

$$\begin{aligned} v(x) = & \Pi(l)(x) + x(2\mathcal{D}_r + 1)\varphi(x) + \frac{\rho^2 - r^2}{2} \operatorname{grad} (\psi(x) + 2\varphi(x)), \\ P(x) = & -\mu(2\mathcal{D}_r + 1)\psi(x), \end{aligned} \quad (6.80)$$

where  $\varphi$  and  $\psi$  are determined by (6.78) and (6.79).

**Theorem 6.18.** *If  $g \in C^{0,\gamma}(S)$ ,  $l \in C^{1,\gamma}(S)$ ,  $0 < \gamma \leq 1$ , and condition (6.7) is satisfied, then the pair  $(v, P)$  determined by (6.80) is the unique classical solution of Problem (IV)<sup>+</sup>.*

**Problem 54.** Prove Theorem 6.18.

\* \* \*

Solutions of the problems for the sphere, mainly of particular ones, having applications are derived in the monographs Lamb [1], Happel, Brenner [1], Belonosov, Chernous [1] (see the bibliography given therein, also Chichinadze [1, 2, 5]).

In all the works of which we know the problems are solved by means of series and the solutions are also given in the form of series without a necessary analysis of the representations obtained.

CHAPTER VII  
**PROBLEMS WITH CONCENTRATED SINGULARITIES  
 AND TWO-DIMENSIONAL PROBLEMS**

**7.1. Problems with Concentrated Singularities.** The class of solutions considered in the preceding chapters can be essentially widened if we take into consideration pointwise singularities. But first we have to make a remark. Let an elastic homogeneous isotropic medium with the Lamé constants  $\lambda$  and  $\mu$  fill up the entire space  $\mathbb{R}^3$  and the force equal to two unities and directed along the  $x_j$ -axis be applied to the origin (point  $0 = (0, 0, 0)$ ). Then (see, for example, Love [1] or Kupradze *et al.* [1]) the displacement of the point  $x = (x_1, x_2, x_3)$  produced by this force is calculated by the formula  $\Gamma^j(x) = (\Gamma_{1j}(x), \Gamma_{2j}(x), \Gamma_{3j}(x))$ , where

$$\begin{aligned}\Gamma_{kj}(x) &= \frac{\lambda' \delta_{kj}}{|x|} + \mu' \frac{x_k x_j}{|x|^3}, \quad k, j = 1, 2, 3, \\ \lambda' &\equiv \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}, \quad \mu' \equiv \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)}.\end{aligned}\tag{7.1}$$

Thus each column (as well as each row) of the matrix

$$\Gamma(x) \equiv \|\Gamma_{ij}(x)\|_{3 \times 3}\tag{7.2}$$

considered as a vector satisfies the system of basic equations of elasticity

$$\mu \Delta u(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} u(x) = 0\tag{7.3}$$

at each point  $x$  of the entire space except the origin.

$\Gamma$  is called the matrix of fundamental solutions of equations (7.3) or the Kelvin matrix (see Love [1], Kupradze *et al.* [1]).

Consider the vector

$${}^0 u(x) = c_1 \Gamma^1(x) + c_2 \Gamma^2(x) + c_3 \Gamma^3(x),$$

where  $c_1, c_2$ , and  $c_3$  are some constants. The restriction of this vector on the sphere  $S(0, \rho)$  will be denoted by  ${}^0 f$ . Thus

$$\forall y \in S : {}^0 f(y) \equiv c_1 \Gamma^1(y) + c_2 \Gamma^2(y) + c_3 \Gamma^3(y).$$

Now let us solve the following problem:

In the ball  $B^+ \equiv B(0, \rho)$  find a continuous vector  $u$  which in this domain satisfies system (7.3) and on the boundary  $S$  takes the value  $f - {}^0 f$ , where  $f$  is the known vector on  $S$ .

This problem was solved in quadratures in Subsection 2.4. In the latter formula replace  $f$  by  $f - {}^0 f$ . Denote the obtained solution by  $u$  and consider the vector

$$V(x) = u(x) + {}^0 u(x).$$

Obviously,  $V$  is a solution of the following problem:

In  $B^+$  find a vector  $V$  which is continuous in  $B^+ \setminus \{0\}$ , satisfies system (7.3) in  $B^+ \setminus \{0\}$ , takes the value  $f$  on  $S$  and satisfies the condition

$$|V(x)| \leq \frac{c}{|x|} \quad (7.4)$$

in the neighbourhood of the origin.

Therefore the solution  $V$  of this problem contains the displacement produced by forces concentrated at the point  $0 = (0, 0, 0)$  and three arbitrary constants. From the theorem proved in Subsection 1.7 it follows that there are no other solutions of the problem.

We can make the class of solutions even wider. For this we have to elucidate the meaning of derivatives of the Kelvin matrix. Apply force equal to  $2/h$  of the unity and directed along the  $x_1$ -axis to the origin and force equal to  $2/h$  and directed in the opposite to the  $x_1$ -axis direction to the point  $\overset{h}{x} \equiv (-h, 0, 0)$ . Then the displacement produced by the action of these two forces can be calculated by the formula

$$\frac{1}{h}\Gamma^1(x) - \frac{1}{h}\Gamma^1(x - \overset{h}{x}) = \frac{\Gamma^1(x_1, x_2, x_3) - \Gamma^1(x_1 + h, x_2, x_3)}{h}.$$

Passing to the limit as  $h \rightarrow 0$ , we will obtain an expression for the displacement of  $x$  produced by the action of the said "double force". It will be equal to a derivative of  $\Gamma^1$  with respect to  $x_1$ . Derivatives of  $\frac{\partial \Gamma^j}{\partial x_i}$  have a similar meaning.

Obviously, everyone of the nine vectors  $\frac{\partial \Gamma^j}{\partial x_i}$  satisfies system (7.3) at any point  $x$  of the space  $\mathbb{R}^3$  except the origin and at this point there is a singularity of the type  $O(\frac{1}{|x|^2})$ .

Consider the vector

$$\overset{0}{u}(x) = \sum_{j=1}^3 c_j \Gamma^j(x) + \sum_{i,j=1}^3 c_{ij} \frac{\partial \Gamma^j(x)}{\partial x_i}, \quad (7.5)$$

where  $c_i$  and  $c_{ij}$  are some real constants. Denoting the restriction of this vector on  $S$  by  $\overset{0}{f}$ , we have

$$\forall y \in S : \overset{0}{f}(y) \equiv \overset{0}{u}(y) = \sum_{j=1}^3 c_j \Gamma^j(y) + \sum_{i,j=1}^3 c_{ij} \frac{\partial \Gamma^j(y)}{\partial y_i}.$$

Let the continuous vector  $u$  be the solution of system (7.3) in  $B^+$  and on  $S$  take the value  $f - \overset{0}{f}$ , where  $f$  is the known vector on  $S$ . The vector  $u$  is given by (2.24), where  $f$  is replaced by  $f - \overset{0}{f}$ . Now  $V(x) = u(x) + \overset{0}{u}(x)$  will satisfy system (7.3) at each point  $x$  of the domain  $B^+$  except the origin and take the value  $f$  on  $S$ , whereas at the point  $0 = (0, 0, 0)$  it will satisfy the estimate  $|V(x)| \leq c/|x|^2$ . The solution  $V$  contains displacements caused

by the concentrated force and “double force” acting at the point 0; it also contains twelve arbitrary constants.

We can now formulate the general problem with concentrated singularities:

Find a continuous solution  $V$  of system (7.3) in the domain  $B^+ \setminus \{0\}$  (the ball with a hole at the centre) which satisfies the condition  $V^+ = f$  and the estimate

$$|V(x)| \leq \frac{c}{|x|^\nu} \quad (7.6)$$

where  $f = (f_1, f_2, f_3)$  is a given continuous vector on  $S$ ,  $c = \text{const}$ , and  $\nu$  is a nonnegative number. The problem will be called Problem  $(I)_{cs}^+$ .

The results of 1.7 imply the validity of

**Theorem 7.2.** *Any solution of system (7.3) which satisfies condition (7.6) is given by the formula*

$$V(x) = u(x) + \sum_{|\alpha| \leq [\nu]-1} D^\alpha \Gamma(x) a^{(\alpha)}, \quad (7.7)$$

where  $\alpha \equiv (\alpha_1, \alpha_2, \alpha_3)$  is a multiindex,  $\Gamma$  the Kelvin matrix,  $[\nu]$  the integer part of the number  $\nu$ ,  $a^{(\alpha)} \equiv (a_1^{(\alpha)}, a_2^{(\alpha)}, a_3^{(\alpha)})$ ,  $a_i^{(\alpha)} = \text{const}$ ,  $u$  the continuous solution of system (7.3) in  $B^+$ . Note the absence of the second term in (7.7) when  $[\nu] = 0$ .

From Theorem 7.1 it follows that the solution of Problem  $(I)_{cs}^+$  is written in form (7.7). Fix the constants  $a^{(\alpha)}$  in any order you like and consider the vector

$$\overset{0}{u}(x) = \sum_{|\alpha| \leq [\nu]-1} D^\alpha \Gamma(x) a^{(\alpha)}. \quad (7.8)$$

Obviously,  $\overset{0}{u}$  is the solution of system (7.3) in the domain  $\mathbb{R}^3 \setminus \{0\}$ , satisfying estimate (7.6). Denote the restriction of  $\overset{0}{f}$  on  $S$  by  $\overset{0}{u}$ :

$$\forall y \in S : \overset{0}{f}(y) = \sum_{|\alpha| \leq [\nu]-1} \partial^\alpha \Gamma(y) a^{(\alpha)}. \quad (7.9)$$

Let  $u$  be the solution of Problem  $(I)^+$  for the boundary function  $f - \overset{0}{f}$ . Clearly,  $u$  is given by (2.24), where  $f$  is replaced by  $f - \overset{0}{f}$ . In that case the solution of Problem  $(I)_{cs}^+$  is

$$V(x) = u(x) + \sum_{|\alpha| \leq [\nu]-1} D^\alpha \Gamma(x) a^{(\alpha)}. \quad (7.10)$$

This formula contains arbitrary constants  $a^{(\alpha)}$  and various-type forces concentrated at the point  $0 = (0, 0, 0)$ . From Theorem 7.1 it follows that

there are no other solutions of Problem  $(I)_{cs}^+$ . Thus the solution of Problem  $(I)_{cs}^+$  is nonunique. The arbitrary constants contained in the solution can be determined by imposing various additional restrictions on the solutions. One may give, for example, the intensity of force applied to the point 0 or displacements at some points of the domain. But another approach to this problem seems to us more interesting.

Note that on  $S$ :  $|x| = \rho = \text{const}$ ,  $x_1 = \rho \cos \varrho_1$ ,  $x_2 = \rho \cos \varrho_1 \cos \varrho_2$ ,  $x_3 = \rho \sin \varrho_1 \sin \varrho_2$  and therefore  $\overset{0}{f}$  determined by (7.9) is a trigonometric polynomial. Determine the coefficient of this polynomial in such a manner that  $\overset{0}{f}$  be the least deviated from  $f$ . Then in the particular case when  $f$  is represented in form (7.9) the solution of Problem  $(I)_{cs}^+$  will be  $\overset{0}{u}$  determined by (7.8) and, clearly, it is given explicitly in terms of elementary functions, without quadratures. However, in the general case the solution is given in form (7.10), where the estimate

$$|u(x)| \leq c \max_{y \in S} |f(y) - \overset{0}{f}(y)|$$

holds for  $u$  (see (2.31) and the property of the Poisson integral).

We will give some other problems which can be solved by the same method:

Find in quadratures a solution  $u$  of system (7.3) which is continuous in the domain  $B^+ \setminus \{0\}$  and satisfies the estimate

$$|u(x)| \leq \frac{c}{|x|^\nu} \quad (7.11)$$

and anyone of the following boundary conditions with concentrated singularities:

PROBLEM 55.

$$\forall y \in S : (\tau^{(n)})^+(y) = f(y)$$

-Problem  $(II)_{cs}^+$ ;

PROBLEM 56.

$$\forall y \in S : (n \cdot u)^+(y) = g(y), \quad (\tau^{(n)} - n(n \cdot \tau^{(n)}))^+(y) = l(y)$$

-Problem  $(III)_{cs}^+$ ;

PROBLEM 57.

$$\forall y \in S : (n \cdot \tau^{(n)})^+(y) = g(y), \quad (u - n(n \cdot u))^+(y) = l(y)$$

- Problem  $(IV)_{cs}^+$ ;

PROBLEM 58.

$$\forall y \in S : (\tau^{(n)} + \sigma_0 u)^+(y) = f(y)$$

-Problem  $(V)_{cs}^+$ .

**Problem 55.** Find in quadratures a continuous solution of system (7.3) in the domain  $B^+ \setminus \{\overset{1}{x}, \dots, \overset{n}{x}\}$  by the boundary condition  $u^+ = f$ , for which we have the estimate

$$|u(x)| \leq \sum_{i=1}^n \frac{c}{|x - \overset{i}{x}|^{\nu_i}}. \quad (7.12)$$

Here  $\overset{1}{x}, \dots, \overset{n}{x} \in B^+$ ;  $\nu_1, \dots, \nu_n$  are nonnegative constants,  $c = \text{const}$ ,  $f = (f_1, f_2, f_3)$  is a given function on  $S$ .

**Problem 56.** Formulate and solve Problems  $(\text{II})_{cs}^+$ ,  $(\text{III})_{cs}^+$ ,  $(\text{IV})_{cs}^+$ ,  $(\text{V})_{cs}^+$  with concentrated singularities when condition (7.11) is replaced by condition (7.12).

**Problem 57.** Formulate and solve the boundary value problems with concentrated singularities for a polyharmonic equation.

**Problem 58.** Formulate and solve the boundary value problems with concentrated singularities of the thermoelasticity theory.

**Problem 59.** Formulate and solve boundary value problems with concentrated singularities for elastic mixtures and a fluid flow.

**7.2. Two-Dimensional Problems.** An elastic body undergoes plane deformation parallel to the  $x_1x_2$ -plane if the third displacement component  $u = (u_1, u_2, u_3)$  is equal to zero:  $u_3 = 0$ , and the first two components  $u_1$  and  $u_2$  depend only on  $x_1$  and  $x_2$ . Then Hook's law gives us the following dependence of the stress tensor  $\|\tau_{ij}\|$  on the displacement vector (see Muskhelishvili [1], Kupradze *et al* [1]):

$$\begin{aligned} \tau_{11} &= \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2\mu \frac{\partial u_1}{\partial x_1}, \quad \tau_{13} = 0, \\ \tau_{12} &= \mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right), \quad \tau_{23} = 0, \\ \tau_{22} &= \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2\mu \frac{\partial u_2}{\partial x_2}, \quad \tau_{33} = \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right). \end{aligned} \quad (7.13)$$

The equilibrium equation rewritten in terms of stress components takes the form

$$\begin{aligned} \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} &= 0, \\ \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} &= 0, \end{aligned} \quad (7.14)$$

and the compatibility equation (see Muskhelishvili [1]) in the plane theory will be written as (there is no mass force)

$$\Delta(\tau_{11} + \tau_{22}) = 0. \quad (7.15)$$

From relations (7.14) it follows (see, for example, Love [1], Muskhelishvili [1]) that there exists a function  $A$  such that

$$\tau_{11} = \frac{\partial^2 A}{\partial x_2^2}, \quad \tau_{12} = -\frac{\partial^2 A}{\partial x_1 \partial x_2}, \quad \tau_{22} = \frac{\partial^2 A}{\partial x_1^2}. \quad (7.16)$$

Equations (7.15) and (7.16) enable us to conclude that  $A$  satisfies the biharmonic equation.

$A$  is called the stress function or the Airy function (G.B. Airy, 1862). After the Airy function is found, the stress tensor can be found by simple differentiation. Moreover, the basic problems of plane elasticity are immediately reduced to the boundary value problems investigated in Chapter III. Let us, for example, consider the second basic problem of plane elasticity (or the first problem following Muskhelishvili's notation). It is required here to determine a plane deformed state of an elastic medium when the stress tensor is given on the boundary. That is, on the boundary we have

$$\begin{aligned} \tau_1^{(n)} &= \tau_{11}n_1 + \tau_{12}n_2 = \frac{\partial^2 A}{\partial x_2^2}n_1 - \frac{\partial^2 A}{\partial x_1 \partial x_2}n_2 = f_1, \\ \tau_2^{(n)} &= \tau_{21}n_1 + \tau_{22}n_2 = -\frac{\partial^2 A}{\partial x_1 \partial x_2}n_1 + \frac{\partial^2 A}{\partial x_1^2}n_2 = f_2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial^2 A}{\partial x_1 \partial x_2} \frac{dx_1}{ds} + \frac{\partial^2 A}{\partial x_2^2} \frac{dx_2}{ds} &= -f_1, \\ \frac{\partial^2 A}{\partial x_1^2} \frac{dx_1}{ds} + \frac{\partial^2 A}{\partial x_1 \partial x_2} \frac{dx_2}{ds} &= f_2 \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\partial A}{\partial x_1} &= \int f_2 ds, \quad \frac{\partial A}{\partial x_2} = -\int f_1 ds, \\ \frac{dA}{dn} &= -\frac{\partial A}{\partial x_1} \frac{dx_2}{ds} + \frac{\partial A}{\partial x_2} \frac{dx_1}{ds}, \\ A &= \int \frac{\partial A}{\partial x_1} dx_1 + \frac{\partial A}{\partial x_2} dx_2. \end{aligned}$$

Thus on the boundary  $\frac{dA}{dn}$  and  $A$  will be expressed as functions of the arc coordinate  $s$  by means of the known functions  $f_1$  and  $f_2$ .

To find the Airy function we have obtained here the Lauricella problem for the biharmonic equation, which was solved in Chapter III.

All plane problems can be solved in a similar manner. But these problems can also be solved by the method proposed in Chapter II. Special representations of solutions, Poisson-type formulas and formulas providing solutions of ordinary linear differential equations are the main tool used in solving three-dimensional problems. All these means apply as well to the plane case. Moreover, the resulting ordinary differential equations do not

actually differ from the corresponding equations of the three-dimensional problems.

In this book we do not pursue the aim to carry out a thorough investigation of two-dimensional problems, as they have been solved in quite a number of works whose survey is beyond the framework of this book. For sufficiently full information the interested reader is referred to Love [1] and Muskhelishvili [1]. To illustrate our reasoning, below we will give the solution of the first problem for the linearized Navier-Stokes system.

The linearized Navier-Stokes system is written in the form

$$\begin{aligned}\mu\Delta v(x) - \text{grad } p(x) &= 0, \\ \text{div } v(x) &= 0,\end{aligned}\tag{7.17}$$

where  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $v = (v_1, v_2)$  is the velocity vector,  $p$  the pressure,  $\mu$  the viscosity coefficient,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad \text{grad} = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right).$$

**Theorem 7.3.** *If*

$$\begin{aligned}v &= u + \frac{\rho^2 - r^2}{2} \text{grad } \psi, \\ p &= -2\mu D_r \psi\end{aligned}\tag{7.18}$$

where  $\Delta u = 0$ ,  $\Delta \psi = 0$ ,  $r \equiv |x| = \sqrt{x_1^2 + x_2^2}$ ,  $D_r \equiv r \frac{\partial}{\partial r} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ ,  $u$  and  $\psi$  are interconnected by the relation

$$D_r \psi = \text{div } u,\tag{7.19}$$

then the pair  $(v, p)$  is a solution of system (7.15).

**Problem 60.** Prove Theorem 7.2.

Let  $K^+$  be the circle in a two-dimensional Euclidean space with centre at the origin and radius  $\rho$ :  $K^+ \equiv \{x \in \mathbb{R}^2 \mid |x| < \rho\}$  and  $S$  be the circumference  $S = \partial K^+ \equiv \{x \in \mathbb{R}^2 \mid |x| = \rho\}$ .

For system (7.17) we will consider the first boundary value problem:

Find the pair  $(v, p)$  which is a solution of system (7.17) in  $K^+$  if

$$\lim_{K^+ \ni x \rightarrow y \in S} v(x) = f(y),\tag{7.20}$$

where  $f = (f_1, f_2)$  is a function given on  $S$ .

Note that for this problem to be solvable it is necessary and sufficient that

$$\int_S f(y) n(y) d_y S = 0,\tag{7.21}$$

where  $n = (n_1, n_2)$  is the external normal to  $S$ .

The solution of the problem is to be sought for in form (7.18). Then, as can be easily verified, for  $u$  we have the Dirichlet problem

$$\begin{aligned} \forall x \in K^+ : \Delta u(x) &= 0, \\ \forall y \in S : \lim_{K^+ \ni x \rightarrow y} u(x) &= f(y), \end{aligned}$$

whose solution is given in the Poisson integral form

$$u(x) = \Pi(f)(x) = \frac{1}{2\pi\rho} \int_S \frac{\rho^2 - r^2}{|x - y|^2} f(y) d_y S. \quad (7.22)$$

Introducing the polar coordinates  $y = (\rho \cos \omega, \rho \sin \omega)$  and  $x = (r \cos \theta, r \sin \theta)$ , for the points  $y \in S$  and  $x \in K^+$  the Poisson integral can be rewritten as

$$\Pi(f)(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2r\rho \cos(\omega - \theta) + r^2} f^*(\omega) d\omega, \quad (7.23)$$

where  $f^*(\omega) \equiv f(y) = f(\rho \cos \omega, \rho \sin \omega)$ .

The substitution of (7.22) in relation (7.19) enables us to treat the latter as a differential equation for  $\psi$ :

$$r \frac{\partial \psi}{\partial r} = \operatorname{div} \Pi(f).$$

Solving this equation in the class of harmonic functions, we have

$$\psi(x) = \int_0^1 \operatorname{div} \Pi(f)(\tau x) \frac{d\tau}{\tau^2}, \quad (7.24)$$

and, due to (7.21),

$$\psi(x) = \frac{1}{2\pi\rho} \operatorname{div} \int_S \int_0^1 \left( \frac{\rho^2 - |\tau x|^2}{|y - \tau x|^2} - 1 - \frac{2\tau}{\rho^2} \sum_{k=1}^2 x_k y_k \right) \frac{d\tau}{\tau^2} f(y) d_y S.$$

Now using (7.22), (7.24) and (7.18), the solution of the first problem for the circle can be constructed in the form

$$v(x) = \Pi(f)(x) + \frac{\rho^2 - |x|^2}{2} \operatorname{grad} \int_0^1 \operatorname{div} \Pi(f)(\tau x) \frac{d\tau}{\tau^2}, \quad (7.25)$$

$$p(x) = -2\mu \operatorname{div} \Pi(f)(x).$$

**Theorem 7.4.** *If  $f \in \mathbb{C}(S)$  and  $f$  satisfies condition (7.21), then the pair  $(v, p)$  determined by (7.25) is a solution of system (7.17),  $v$  satisfies condition (7.20) and  $p \in \mathbb{C}^1(K^+)$ ,  $V \in \mathbb{C}(\bar{K}^+) \cap \mathbb{C}^2(K^+)$ . Moreover,  $v$  is determined uniquely, whereas  $p$  to within an arbitrary constant term.*

**Problem 61.** Prove Theorem 7.3.

**Problem 62.** Solve the boundary value problems of the classical theory of elasticity for the circle and the entire plane with a circular hole.

**Problem 63.** Solve the Lauricella, Riquier and mixed-type problems for a polyharmonic equation for the circle.

**Problem 64.** Solve the plane boundary value problems of thermoelasticity for the circle and the entire plane with a circular hole.

**Problem 65.** Solve the boundary value problems of fluid flow for the circle and for the circle exterior.

\* \* \*

Singular solutions of various equations of continuum mechanics were constructed as early as the 19th century and were widely used to construct special potentials giving particular solutions of the considered equations (see Kelvin [1], Love [1], Kupradze et al. [1] and others). Singular solutions of the boundary value problems of continuum mechanics are constructed here for the first time (see Buchukuri, Gegelia [1,2]).

Vast literature is devoted to problems for the circle. An interested reader is referred to Muskhelishvili [1], Love [1], Basheleishvili [1, 2, 3]).

CHAPTER VIII  
**CALCULATION OF SINGULAR INTEGRALS  
 OF THE POISSON INTEGRAL TYPE**

**8.1. Preliminary Remarks.** In the foregoing chapters we have constructed in quadratures the solutions of many boundary value problems of continuum mechanics for the ball and the entire space with a spherical cavity. These quadratures are integrals on the sphere which is the boundary of the considered ball. They are expressed mainly by simple combinations of Poisson integrals or by integrals of the Poisson integral type. Such representations prove convenient in constructing numerical algorithms of solutions. Our proposed scheme of constructing numerical solutions of the problems possesses all the advantages of the method of boundary integral equations such as: it decreases the problem dimension by unity, makes it possible to calculate the unknown values at any point without obtaining them at other points of the considered domain and so on (see Goldstein [1], Rizzo [1], Lachat, Watson [1], Wendland [1] and others). To apply the method of boundary integral equations one needs to solve the obtained integral equations. To substantiate the methods of solving such equations we have to perform their rather sophisticated investigation; solutions are represented as integrals of the Poisson integral type whose kernels are given in terms of elementary functions and densities are the given boundary conditions. Despite the fact that the methods of calculating such integrals do not actually differ from those commonly used for calculating multidimensional (in particular, two-dimensional) integrals, still the former methods need all the same a certain modification.

**8.2. Poisson Integral.** In calculating the integrals which are the solutions of the considered boundary value problems one encounters problems of the same nature as those for the Poisson integral regarded as the simplest one among such-type integrals.

The Poisson integral

$$\begin{aligned} u(\rho_0, \vartheta_0, \varphi_0) &= \Pi(f)(x^0) \equiv \\ &= \frac{\rho}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{(\rho^2 - \rho_0^2) \sin \vartheta \tilde{f}(\vartheta, \varphi)}{(\rho^2 - 2\rho\rho_0 \cos \gamma + \rho_0^2)^{3/2}} d\varphi d\vartheta \end{aligned} \quad (8.1)$$

is the solution of the Dirichlet problem for the ball  $B(0, \rho) \equiv \{x \in \mathbb{R}^3 \mid |x| < \rho\}$ :

$$\Delta u(x) = 0, \quad x \in B(0, \rho); \quad u|_{\partial B(0, \rho)}(y) = f(y).$$

Here  $f$  is a given continuous function given on  $\partial B(0, \rho)$ ;

$$\tilde{f}(\vartheta, \varphi) \equiv f(\rho \cos \varphi_0 \sin \vartheta, \rho \sin \varphi \sin \vartheta, \rho \cos \vartheta); \quad (8.2)$$

$(\rho_0, \vartheta_0, \varphi_0)$  are the spherical coordinates of the point  $x^0 = (x_1^0, x_2^0, x_3^0)$ ;

$$\begin{aligned}x_1^0 &= \rho_0 \cos \varphi_0 \sin \vartheta_0, \\x_2^0 &= \rho_0 \sin \varphi_0 \sin \vartheta_0, \\x_3^0 &= \rho_0 \cos \vartheta_0;\end{aligned}$$

$(\vartheta, \varphi)$  are the angular coordinates of the point  $y$  on the  $\partial B(0, \rho)$ :

$$\begin{aligned}y_1 &= \rho \cos \varphi \sin \vartheta, \\y_2 &= \rho \sin \varphi \sin \vartheta, \\y_3 &= \rho \cos \vartheta;\end{aligned}$$

$\gamma$  is the angle formed by the vectors  $x^0$  and  $y$ .

Introducing the notation  $\tau \equiv \rho_0/\rho$ , we rewrite (8.1) in a form better suitable for calculation

$$u(\rho_0, \vartheta_0, \varphi_0) = \frac{1 - \tau^2}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\sin \vartheta}{(1 - 2\tau \cos \gamma + \tau^2)^{3/2}} \tilde{f}(\vartheta, \varphi) d\varphi d\vartheta. \quad (8.3)$$

For our purpose it is convenient to rewrite integral (8.3) in terms of the iterated integral and to use Simpson's quadrature formula (see, for example, Hamming [1] or Collatz [1]) for one-dimensional integrals.

**8.3. Simpson's Method and Runge's Principle.** In calculating the integral

$$\mathcal{I} \equiv \int_a^b f(t) dt \quad (8.4)$$

its value is approximately replaced by the sum

$$\begin{aligned}S(f, a, b, m) &\equiv \frac{h}{3} \left( f(a) + f(b) + 2 \sum_{k=1}^{m-1} f(a + 2kh) + \right. \\ &\quad \left. + 4 \sum_{k=0}^{m-1} f(a + (2k + 1)h) \right),\end{aligned} \quad (8.5)$$

where  $h \equiv \frac{b-a}{2m}$ .

Note that (8.5) contains the value of  $f$  at  $2m - 1$  points of the segment  $[a, b]$ .

Denote the error of Simpson's formula by  $R(h)$ :

$$R(h) \equiv \int_a^b f(t) dt - S(f, a, b, m). \quad (8.6)$$

If  $f$  has, on  $[a, b]$ , continuous derivatives of fourth order  $f \in \mathbb{C}^4([a, b])$ , then

$$R(h) = -\frac{(b-a)h^4}{180} f^{(4)}(\xi)$$

for some  $\xi \in ]a, b[$ . Therefore to estimate the error we obtain the formula

$$|R(h)| \leq \frac{(b-a)h^4}{180} \max_{a \leq \xi \leq b} |f^{(4)}(\xi)|. \quad (8.7)$$

Estimate (8.7) is rather rough and its application frequently leads to a substantial increase of  $m$  in sum (8.5). This particularly refers to those integrals for which  $f^{(4)}$  sharply increases in the neighbourhood of some point of  $[a, b]$ . We are going to show below that such a situation occurs in calculating integrals of the Poisson-integral type.

In practice, the error is frequently estimated using Runge's principle consisting in the following: if the condition

$$\sigma \equiv |S(f, a, b, 2m) - S(f, a, b, m)| \leq \varepsilon. \quad (8.8)$$

is fulfilled for some  $m$ , then  $S(f, a, b, 2m)$  is taken as an approximate value of integral (8.4) and the number  $\varepsilon$  for the error. As has been established, for Simpson's formula the error can be estimated by

$$R(h) \approx \frac{\sigma}{15}. \quad (8.9)$$

**8.4. Integral Calculation Algorithm.** Program No.1 realizes the aforementioned method of calculating integral (8.4) as a function *integral1()* in the language *C*.

We wish to note rightaway that the programs given in this chapter are written in the language *C* (the TURBO C version) and realized on the computer IBM AT with the processor model 80286/80287.

To calculate integral (8.3) it suffices to apply twice the function of calculating a one-dimensional integral. We will estimate the error for each case such that the calculation error of (8.3) be not greater than  $\varepsilon$ .

Denote by  $\tilde{S}(f, a, b, \delta)$  sum (8.5) for  $m$  such that

$$\left| \int_a^b f(t) dt - S(f, a, b, m) \right| \leq \delta. \quad (8.10)$$

By Runge's principle  $\tilde{S}(f, a, b, \delta)$  can be replaced by  $S(f, a, b, 2n)$ , where

$$|S(f, a, b, 2n) - S(f, a, b, n)| \leq \delta.$$

Denote by  $F$  the function

$$F(\vartheta, \varphi) \equiv \frac{\sin \vartheta \tilde{f}(\vartheta, \varphi)}{(1 - 2\tau \cos \gamma + \tau^2)^{3/2}},$$

and by  $\mathcal{I}_1$  the integral

$$\mathcal{I}_1(\vartheta) \equiv \int_0^{2\pi} F(\vartheta, \varphi) d\varphi.$$

On account of (8.10), for  $\mathcal{I}_1(\rho)$  we have

$$|\mathcal{I}_1(\vartheta) - \tilde{\mathcal{I}}_1(\vartheta)| \leq \delta_1, \quad (8.11)$$

where

$$\tilde{\mathcal{I}}_1(\vartheta) \equiv \tilde{S}(F(\vartheta, \cdot), 0, 2\pi, \delta_1).$$

Let  $\tilde{u}(\rho_0, \vartheta_0, \varphi_0)$  be an approximate value of integral (8.3):

$$\tilde{u}(\rho_0, \vartheta_0, \varphi_0) \equiv \frac{1 - \tau^2}{4\pi} \tilde{S}(\tilde{\mathcal{I}}_1, 0, \pi, \delta_2). \quad (8.12)$$

Then

$$\begin{aligned} & |u(\rho_0, \vartheta_0, \varphi_0) - \tilde{u}(\rho_0, \vartheta_0, \varphi_0)| \leq \\ & \leq \frac{1 - \tau^2}{4\pi} \left| \int_0^\pi \mathcal{I}_1(\vartheta) d\vartheta - \tilde{S}(\mathcal{I}_1, 0, \pi, \delta_2) \right| d\vartheta + \\ & + \frac{1 - \tau^2}{4\pi} |\tilde{S}(\mathcal{I}_1, 0, \pi, \delta_2) - \tilde{S}(\tilde{\mathcal{I}}_1, 0, \pi, \delta_2)|. \end{aligned} \quad (8.13)$$

By virtue of (8.11) the first term on the right-hand side of this inequality is less than  $\delta_2$ . Let us estimate the second term. We observe that

$$|S(f, a, b, m)| \leq (b - a) \max_{a \leq x \leq b} |f(x)|.$$

Therefore

$$|\tilde{S}(\mathcal{I}_1, 0, \pi, \delta_2) - \tilde{S}(\tilde{\mathcal{I}}_1, 0, \pi, \delta_2)| \leq \pi \max_{0 \leq \vartheta \leq \pi} |\mathcal{I}_1(\vartheta) - \tilde{\mathcal{I}}_1(\vartheta)| \leq \pi \delta_1.$$

(8.13) now yields

$$|u(\rho_0, \vartheta_0, \varphi_0) - \tilde{u}(\rho_0, \vartheta_0, \varphi_0)| \leq \frac{1 - \tau^2}{4\pi} (\delta_2 + \pi \delta_1).$$

If

$$\delta_1 = \delta_2 = \frac{4\pi\varepsilon}{(1 - \tau^2)(\pi + 1)}, \quad (8.14)$$

then we will finally obtain

$$|u(\rho_0, \vartheta_0, \varphi_0) - \tilde{u}(\rho_0, \vartheta_0, \varphi_0)| \leq \varepsilon.$$

Program No.2 employs the above-described method to calculate integral (8.3). An approximate value is calculated by means of (8.12) and (8.11), while the values  $\delta_1$  and  $\delta_2$  by (8.14). The program also counts up the number

of nodes ( $n$ ) at which the values of  $\tilde{f}$  are calculated and computes the time ( $t$ ) spent on calculation. In this program  $f$  is the boundary function given by the user.  $f$  must be a function of the Cartesian coordinates of a point of the sphere and introduced in the program between the lines /\*begin of boundary function\*/ and /\*end of boundary function\*/. In our case the function  $f$  is given by the formula  $f(x_1, x_2, x_3) = 3x_1^2 - x_2^2 + 3x_3 - 4$ . If, for example,  $f(x_1, x_2, x_3) = x_1 - x_2 - 1$ , then this part of the program has to be changed as follows:

```
/*begin of boundary function*/,  
return (3 * x1 - x2 + 4);,  
/*end of boundary function*/.
```

**8.5. A Remark on the Calculation of the Poisson Integral Near the Boundary.** In practice it is important to know the desired values at points close to the boundary, but when the point  $x^0 \equiv (\rho_0, \vartheta_0, \varphi_0)$  approaches the boundary  $\partial B(0, \rho)$ , in calculating the Poisson integral at  $x^0$ , we observe an sharp increase in the number of nodes ( $n$ ) at which the values of  $f$  are calculated, which accordingly brings about the same increase in the time ( $t$ ). Let us find out what reason underlies this phenomenon.

If we assume

$$A \equiv (1 - 2\tau \cos \gamma + \tau^2)^{1/2},$$

then

$$A^2 = (1 - \tau)^2 + 4\tau \sin^2 \frac{\gamma}{2} > (1 - \tau)^2$$

and  $1/A < 1/d$ , where  $d \equiv 1 - \tau$ . Moreover, if

$$\sin \frac{\gamma}{2} < \frac{d}{2\sqrt{1-d}},$$

then  $A^2 < 2(1 - \tau)^2$  and

$$\frac{1}{A} > \frac{1}{\sqrt{2}d}.$$

Hence, using the estimate  $1/A < 1/d$ , we conclude that  $1/A$  has order  $1/d$  for small  $\gamma$ .

(8.7) implies that to estimate the error of Simpson's formula it is necessary to estimate derivatives of the integrand function of fourth order. In our case this function is

$$F(\vartheta, \varphi) = \mathcal{K}(\tau, \vartheta, \varphi) \tilde{f}(\vartheta, \varphi),$$

$$\mathcal{K}(\tau, \vartheta, \varphi) \equiv \frac{\sin \vartheta}{(1 - 2\tau \cos \gamma + \tau^2)^{3/2}} = \frac{\sin \vartheta}{A^3}. \quad (8.15)$$

(8.3) and (8.7) imply that it suffices to estimate the fourth-order derivatives of  $F$  with respect to  $\varphi$  and  $\vartheta$ . We assume

$$\forall(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi] : \left| \frac{\partial^{\alpha+\beta} \tilde{f}(\vartheta, \varphi)}{\partial \vartheta^\alpha \partial \varphi^\beta} \right| \leq c, \quad \alpha + \beta \leq 4. \quad (8.16)$$

Let us estimate derivatives of the kernel  $\mathcal{K}$ . We have

$$\begin{aligned}
\frac{\partial \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi} &= \frac{3\tau \sin \vartheta}{A^5} \cdot \frac{\partial(\cos \gamma)}{\partial \varphi}, \\
\frac{\partial^2 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi^2} &= \frac{3\tau \sin \vartheta}{A^5} \cdot \frac{\partial^2(\cos \gamma)}{\partial \varphi^2} + \frac{15\tau^2 \sin \vartheta}{A^7} \left( \frac{\partial(\cos \gamma)}{\partial \varphi} \right)^2, \\
\frac{\partial^3 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi^3} &= \frac{3\tau \sin \vartheta}{A^5} \cdot \frac{\partial^3(\cos \gamma)}{\partial \varphi^3} + \frac{45\tau^2 \sin \vartheta}{A^7} \cdot \frac{\partial(\cos \gamma)}{\partial \varphi} \cdot \frac{\partial^2(\cos \gamma)}{\partial \varphi^2} + \\
&\quad + \frac{105\tau^3 \sin \vartheta}{A^9} \left( \frac{\partial(\cos \gamma)}{\partial \varphi} \right)^3, \\
\frac{\partial^4 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi^4} &= \frac{3\tau \sin \vartheta}{A^5} \cdot \frac{\partial^4(\cos \gamma)}{\partial \varphi^4} + \frac{15\tau^2 \sin \vartheta}{A^7} \left( 3 \left( \frac{\partial^2(\cos \gamma)}{\partial \varphi^2} \right)^2 + \right. \\
&\quad \left. + 4 \frac{\partial(\cos \gamma)}{\partial \varphi} \cdot \frac{\partial^3(\cos \gamma)}{\partial \varphi^3} \right) + \frac{630\tau^3 \sin \vartheta}{A^9} \left( \frac{\partial(\cos \gamma)}{\partial \varphi} \right)^2 \frac{\partial^2(\cos \gamma)}{\partial \varphi^2} + \\
&\quad + \frac{945\tau^4 \sin \vartheta}{A^{11}} \left( \frac{\partial(\cos \gamma)}{\partial \varphi} \right)^4, \\
\frac{\partial \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta} &= \frac{\cos \vartheta}{A^3} + \frac{3\tau \sin \vartheta}{A^5} \cdot \frac{\partial(\cos \gamma)}{\partial \vartheta}, \\
\frac{\partial^2 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta^2} &= -\frac{\sin \vartheta}{A^3} + \frac{3\tau}{A^5} \left( 2 \cos \vartheta \frac{\partial(\cos \gamma)}{\partial \vartheta} + \sin \vartheta \frac{\partial^2(\cos \gamma)}{\partial \vartheta^2} \right) + \\
&\quad + \frac{15\tau^2 \sin \vartheta}{A^7} \left( \frac{\partial(\cos \gamma)}{\partial \vartheta} \right)^2, \\
\frac{\partial^3 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta^3} &= -\frac{\cos \vartheta}{A^3} + \frac{3\tau}{A^5} \left( \sin \vartheta \frac{\partial^3(\cos \gamma)}{\partial \vartheta^3} + 3 \cos \vartheta \frac{\partial^2(\cos \gamma)}{\partial \vartheta^2} - \right. \\
&\quad \left. - 3 \sin \vartheta \frac{\partial(\cos \gamma)}{\partial \vartheta} \right) + \frac{45\tau^2}{A^7} \left( \cos \vartheta \left( \frac{\partial(\cos \gamma)}{\partial \vartheta} \right)^2 + \right. \\
&\quad \left. + \sin \vartheta \frac{\partial(\cos \gamma)}{\partial \vartheta} \cdot \frac{\partial^2(\cos \gamma)}{\partial \vartheta^2} \right) + \frac{105\tau^3 \sin \vartheta}{A^9} \left( \frac{\partial(\cos \gamma)}{\partial \vartheta} \right)^3, \\
\frac{\partial^4 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta^4} &= \frac{\sin \vartheta}{A^3} + \frac{3\tau}{A^5} \left( \sin \vartheta \frac{\partial^4(\cos \gamma)}{\partial \vartheta^4} - 6 \sin \vartheta \frac{\partial^2(\cos \gamma)}{\partial \vartheta^2} + \right. \\
&\quad \left. + 4 \cos \vartheta \frac{\partial^3(\cos \gamma)}{\partial \vartheta^3} - 5 \cos \vartheta \frac{\partial(\cos \gamma)}{\partial \vartheta} \right) + \\
&\quad + \frac{15\tau^2}{A^7} \left( 4 \sin \vartheta \frac{\partial(\cos \gamma)}{\partial \vartheta} \cdot \frac{\partial^3(\cos \gamma)}{\partial \vartheta^3} + 12 \cos \vartheta \frac{\partial(\cos \gamma)}{\partial \vartheta} \times \right. \\
&\quad \left. \times \frac{\partial^2(\cos \gamma)}{\partial \vartheta^2} + 3 \sin \vartheta \left( \frac{\partial^2(\cos \gamma)}{\partial \vartheta^2} \right)^2 \right) + \\
&\quad + \frac{10\tau^3}{A^9} \left( 42 \cos \vartheta \left( \frac{\partial(\cos \gamma)}{\partial \vartheta} \right)^3 + 73 \sin \vartheta \left( \frac{\partial(\cos \gamma)}{\partial \vartheta} \right)^2 \times \right. \\
&\quad \left. \times \frac{\partial^2(\cos \gamma)}{\partial \vartheta^2} \right) + \frac{945\tau^4 \sin \vartheta}{A^{11}} \left( \frac{\partial(\cos \gamma)}{\partial \vartheta} \right)^4,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial(\cos \gamma)}{\partial \varphi} &= -\sin \vartheta \sin \vartheta_0 \sin(\varphi - \varphi_0), \\
\frac{\partial^2(\cos \gamma)}{\partial \varphi^2} &= -\sin \vartheta \sin \vartheta_0 \cos(\varphi - \varphi_0), \\
\frac{\partial^3(\cos \gamma)}{\partial \varphi^3} &= \sin \vartheta \sin \vartheta_0 \sin(\varphi - \varphi_0), \\
\frac{\partial^4(\cos \gamma)}{\partial \varphi^4} &= \sin \vartheta \sin \vartheta_0 \cos(\varphi - \varphi_0), \\
\frac{\partial(\cos \gamma)}{\partial \vartheta} &= \cos \vartheta \sin \vartheta_0 \cos(\varphi - \varphi_0) - \sin \vartheta \cos \vartheta_0, \\
\frac{\partial^2(\cos \gamma)}{\partial \vartheta^2} &= -\sin \vartheta \sin \vartheta_0 \cos(\varphi - \varphi_0) - \cos \vartheta \cos \vartheta_0, \\
\frac{\partial^3(\cos \gamma)}{\partial \vartheta^3} &= -\cos \vartheta \sin \vartheta_0 \cos(\varphi - \varphi_0) + \sin \vartheta \cos \vartheta_0, \\
\frac{\partial^4(\cos \gamma)}{\partial \vartheta^4} &= \sin \vartheta \sin \vartheta_0 \cos(\varphi - \varphi_0) + \cos \vartheta \cos \vartheta_0.
\end{aligned}$$

Since  $\gamma$  is the angle formed by the vectors  $x^0$  and  $y$ , we have either  $|\vartheta - \vartheta_0| \leq \gamma$  and  $|\varphi - \varphi_0| \leq \gamma$  or  $2\pi - \gamma < |\varphi - \varphi_0| < 2\pi$ . Therefore

$$|\sin(\varphi - \varphi_0)| \leq \frac{d}{2\sqrt{1-d}}, \quad |\sin(\vartheta - \vartheta_0)| \leq \frac{d}{2\sqrt{1-d}}.$$

Hence we conclude that

$$\begin{aligned}
\frac{\partial(\cos \gamma)}{\partial \varphi} &\sim d, \quad \frac{\partial^3(\cos \gamma)}{\partial \varphi^3} \sim d, \\
\frac{\partial(\cos \gamma)}{\partial \vartheta} &= \sin(\vartheta - \vartheta_0) - 2 \sin \vartheta_0 \cos \vartheta \sin^2 \frac{\varphi - \varphi_0}{2} \sim d, \\
\frac{\partial^3(\cos \gamma)}{\partial \vartheta^3} &= -\frac{\partial(\cos \gamma)}{\partial \vartheta} \sim d.
\end{aligned}$$

Applying these estimates, we arrive at

$$\begin{aligned}
\frac{\partial^\alpha \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi^\alpha} &\sim \frac{1}{d^{\alpha+3}}, \quad \alpha = 0, 1, 2, 3, 4; \\
\frac{\partial^\alpha \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta^\alpha} &\sim \frac{1}{d^{\alpha+3}}, \quad \alpha = 0, 1, 2, 3, 4.
\end{aligned} \tag{8.17}$$

Since

$$\begin{aligned}
\frac{\partial^4 F(\vartheta, \varphi)}{\partial \varphi^4} &= \mathcal{K}(\tau, \vartheta, \varphi) \frac{\partial^4 \tilde{f}(\vartheta, \varphi)}{\partial \varphi^4} + 4 \frac{\partial \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi} \frac{\partial^3 \tilde{f}(\vartheta, \varphi)}{\partial \varphi^3} + \\
&+ 6 \frac{\partial^2 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi^2} \cdot \frac{\partial^2 \tilde{f}(\vartheta, \varphi)}{\partial \varphi^2} + 4 \frac{\partial^3 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi^3} \frac{\partial \tilde{f}(\vartheta, \varphi)}{\partial \varphi} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{\partial^4 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \varphi^4} \tilde{f}(\vartheta, \varphi), \\
\frac{\partial^4 F(\vartheta, \varphi)}{\partial \vartheta^4} & = \mathcal{K}(\tau, \vartheta, \varphi) \frac{\partial^4 \tilde{f}(\vartheta, \varphi)}{\partial \vartheta^4} + 4 \frac{\partial \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta} \frac{\partial^3 \tilde{f}(\vartheta, \varphi)}{\partial \vartheta^3} + \\
& + 6 \frac{\partial^2 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta^2} \cdot \frac{\partial^2 \tilde{f}(\vartheta, \varphi)}{\partial \vartheta^2} + 4 \frac{\partial^3 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta^3} \frac{\partial \tilde{f}(\vartheta, \varphi)}{\partial \vartheta} + \\
& + \frac{\partial^4 \mathcal{K}(\tau, \vartheta, \varphi)}{\partial \vartheta^4} \tilde{f}(\vartheta, \varphi),
\end{aligned}$$

by virtue of (8.16) and (8.17) we obtain

$$\left| \frac{\partial^4 F(\vartheta, \varphi)}{\partial \varphi^4} \right| \sim \frac{1}{d^7}, \quad \left| \frac{\partial^4 F(\vartheta, \varphi)}{\partial \vartheta^4} \right| \sim \frac{1}{d^7} \quad (8.18)$$

for

$$\sin \frac{\gamma}{2} \leq \frac{d}{2\sqrt{1-d}}.$$

(8.7) and (8.18) imply that the lesser  $d \equiv 1 - \tau = 1 - \vartheta_0/\vartheta$ , i.e. the closer the point  $x^0$  to the boundary  $\partial B(0, \rho)$ , the greater the number of nodes at which the values of  $\tilde{f}$  are to be calculated in order to achieve a given precision and, accordingly, the greater the time required for calculation. The reason of this phenomenon is evident from estimates (8.18) and formula (8.7): fourth-order derivatives of the integrand function sharply increase as the point approaches the boundary (the approximation order is equal to  $1/(\rho - \rho_0)^7$ ).

**8.6. Calculation of the Integral Near the Boundary.** Here we shall discuss the technique of improving the computation algorithm. Our aim is for a given error, find ways of decreasing the number of nodes and, accordingly, the computation time when the point  $x^0$  is near the boundary. This can be done in various ways at different levels.

**8.6.1. Method of Separation Singularities.** If we consider the Dirichlet problem for the ball  $B(0, \rho)$  with the boundary function  $f = 1$ , then its solution  $u$  is given by the Poisson formula, where we set  $\tilde{f} = 1$ . On the other hand, it is clear that  $u = 1$  and therefore we obtain (see (8.3))

$$\frac{1 - \tau^2}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\sin \vartheta}{(1 - 2\tau \cos \gamma + \tau^2)^{3/2}} d\varphi d\vartheta = 1;$$

(8.3) can be rewritten as

$$\begin{aligned}
u(\rho_0, \vartheta_0, \varphi_0) & = \tilde{f}(\vartheta_0, \varphi_0) + \frac{1 - \tau^2}{4\pi} \times \\
& \times \int_0^\pi \int_0^{2\pi} \frac{\sin \vartheta}{(1 - 2\tau \cos \gamma + \tau^2)^{3/2}} (\tilde{f}(\vartheta, \varphi) - \tilde{f}(\vartheta_0, \varphi_0)) d\varphi d\vartheta, \quad (8.19)
\end{aligned}$$

where the integrand function is

$$F_1(\vartheta, \varphi) = \frac{\sin \vartheta}{(1 - 2\tau \cos \gamma + \tau^2)^{3/2}} (\tilde{f}(\vartheta, \varphi) - \tilde{f}(\vartheta_0, \varphi_0)).$$

Lagrange's theorem implies

$$|\tilde{f}(\vartheta, \varphi) - \tilde{f}(\vartheta_0, \varphi_0)| \leq c\gamma,$$

where

$$c \equiv \max \left( \left| \frac{\partial \tilde{f}}{\partial \varphi} \right|, \left| \frac{\partial \tilde{f}}{\partial \vartheta} \right| \right).$$

This makes it possible to somewhat improve the estimate of the fourth-order derivative of the integrand function. We now have

$$\left| \frac{\partial^4 F_1(\vartheta, \varphi)}{\partial \varphi^4} \right| \sim \frac{1}{d^6}, \quad \left| \frac{\partial^4 F_1(\vartheta, \varphi)}{\partial \vartheta^4} \right| \sim \frac{1}{d^6} \quad (8.20)$$

for

$$\sin \frac{\gamma}{2} < \frac{d}{2\sqrt{1-d}}.$$

Thus, if the value of  $u(\rho_0, \vartheta_0, \varphi_0)$  is calculated by means of (8.19) instead of (8.3), then with the same number of nodes the precision will be somewhat improved (compare (8.20) with (8.18)).

**8.6.2. Transition to a New Coordinate System.** Note that if  $x^0$  is on the  $Ox_3$ -axis, i.e.  $\varphi_0 = 0$  and  $\vartheta_0 = 0$ , then estimate (8.20) will be improved. Indeed, in that case  $\gamma = \vartheta$  and the kernel  $\mathcal{K}$  does not depend on  $\varphi$ . Therefore

$$\begin{aligned} \left| \frac{\partial^\alpha \mathcal{K}(\rho, \vartheta, \varphi)}{\partial \vartheta^\alpha} \right| &\sim \frac{1}{d^{\alpha+2}}, \\ \left| \frac{\partial^\alpha \mathcal{K}(\rho, \vartheta, \varphi)}{\partial \varphi^\alpha} \right| &= 0, \quad \alpha = 0, 1, 2, 3, 4. \end{aligned}$$

With these estimates taken into account, we obtain

$$\left| \frac{\partial^4 F(\vartheta, \varphi)}{\partial \varphi^4} \right| \leq c, \quad \left| \frac{\partial^4 F_1(\vartheta, \varphi)}{\partial \vartheta^4} \right| \sim \frac{1}{d^5} \quad (8.21)$$

for

$$\sin \frac{\gamma}{2} < \frac{d}{2\sqrt{1-d}}.$$

Also note that since  $\mathcal{K}$  does not contain the variable  $\varphi$ , (8.19) can be rewritten as

$$u(\rho_0, \vartheta_0, \varphi_0) = \tilde{f}(\vartheta_0, \varphi_0) + \frac{1 - \tau^2}{4\pi} \int_0^\pi \frac{\sin \vartheta}{(1 - 2\tau \cos \gamma + \tau^2)^{3/2}} \times$$

$$\times \left( \int_0^{2\pi} \tilde{f}(\vartheta, \varphi) d\varphi - 2\pi \tilde{f}(\vartheta_0, \varphi_0) \right) d\vartheta, \quad (8.19')$$

where  $\vartheta_0 = 0$  and  $\varphi_0 = 0$ .

The equality (8.19') is somewhat more efficient than (8.19) because of the fact that the kernel is calculated in the external integral.

If  $x^0$  is not on the  $Ox_3$ -axis, this can be achieved by rotating the coordinate system. In that case the transformation of coordinates looks like

$$\begin{aligned} y_1 &= y'_1 + \frac{x_3^0 y'_3 - x_1^0 y'_1 - x_2^0 y'_2 + y'_3}{1 + x_3^0} x_1^0, \\ y_2 &= y'_2 + \frac{x_3^0 y'_3 - x_1^0 y'_1 - x_2^0 y'_2 + y'_3}{1 + x_3^0} x_2^0, \\ y_3 &= x_3^0 y'_3 - x_1^0 y'_1 - x_2^0 y'_2. \end{aligned}$$

Here  $(y_1, y_2, y_3)$  are the previous and  $(y'_1, y'_2, y'_3)$  the new coordinates of the point  $y$ .

If we introduce the notations  $g(y'_1, y'_2, y'_3) \equiv f(y_1, y_2, y_3)$  and  $\tilde{g}(\vartheta', \varphi') \equiv \tilde{f}(\vartheta, \varphi)$ , then in the new system  $\gamma = \vartheta'$ ,

$$\mathcal{K}(\tau, \vartheta', \varphi') = \frac{\sin \vartheta'}{(1 - 2\tau \cos \vartheta' + \tau^2)^{3/2}}$$

and (8.19) will take the form

$$\begin{aligned} u(\rho_0, \vartheta_0, \varphi_0) &= \tilde{f}(\vartheta_0, \varphi_0) + \frac{1 - \tau^2}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\sin \vartheta'}{(1 - 2\tau \cos \vartheta' + \tau^2)^{3/2}} \times \\ &\times (\tilde{g}(\vartheta', \varphi') - \tilde{g}(\vartheta'_0, \varphi'_0)) d\varphi' d\vartheta'. \end{aligned}$$

Like above, the kernel can be moved outside the internal integral, which results in

$$\begin{aligned} u(\rho_0, \vartheta_0, \varphi_0) &= \tilde{f}(\vartheta_0, \varphi_0) + \frac{1 - \tau^2}{4\pi} \int_0^\pi \frac{\sin \vartheta' d\vartheta'}{(1 - 2\tau \cos \vartheta' + \tau^2)^{3/2}} \times \\ &\times \left( \int_0^{2\pi} \tilde{g}(\vartheta', \varphi') d\varphi' - 2\pi \tilde{f}(\vartheta_0, \varphi_0) \right) d\vartheta'. \quad (8.19'') \end{aligned}$$

Let us determine how accurate the computation of the internal and external integrals in (8.19'') should be for the error of computation of  $u(\rho_0, \vartheta_0, \varphi_0)$  not to exceed  $\varepsilon$ .

We introduce the notations

$$\begin{aligned}\mathcal{I}_1(\vartheta') &\equiv \int_0^{2\pi} \tilde{g}(\vartheta', \varphi') d\varphi' - 2\pi \tilde{f}(\vartheta_0, \varphi_0), \\ \tilde{S}(\tilde{g}, 0, 2\pi, \delta_1) &\equiv S(\tilde{g}, 0, 2\pi, m)\end{aligned}$$

for  $m$  such that (see (8.5))

$$\begin{aligned}\left| \int_0^{2\pi} \tilde{g}(\vartheta', \varphi') d\varphi' - S(\tilde{g}, 0, 2\pi, m) \right| &\leq \delta_1, \\ \tilde{\mathcal{I}}_1(\vartheta') &\equiv \tilde{S}(\tilde{g}, 0, 2\pi, \delta_1) - 2\pi \tilde{f}(\vartheta_0, \varphi_0).\end{aligned}$$

Then

$$|\mathcal{I}_1(\vartheta') - \tilde{\mathcal{I}}_1(\vartheta')| \leq \delta_1.$$

After an approximate calculation of the internal integral, the integrand function in the external integral takes the form  $\mathcal{K}(\tau, \vartheta') \tilde{\mathcal{I}}_1(\vartheta')$ , where

$$\mathcal{K}(\tau, \vartheta') \equiv \frac{\sin \vartheta'}{(1 - 2\tau \cos \vartheta' + \tau^2)^{3/2}}.$$

Therefore

$$\left| \int_0^\pi \mathcal{K}(\tau, \vartheta') \tilde{\mathcal{I}}_1(\vartheta') d\vartheta' - \tilde{S}(\mathcal{K}(\tau, \vartheta') \tilde{\mathcal{I}}_1(\vartheta'), 0, \pi, \delta_2) \right| \leq \delta_2.$$

Denoting by  $\tilde{u}(\rho_0, \vartheta_0, \varphi_0)$  an approximate value of  $u(\rho_0, \vartheta_0, \varphi_0)$ ,

$$\tilde{u}(\rho_0, \vartheta_0, \varphi_0) = \tilde{f}(\vartheta_0, \varphi_0) + \frac{1 - \tau^2}{4\pi} \tilde{S}(\mathcal{K}(\tau, \vartheta') \tilde{\mathcal{I}}_1(\vartheta'), 0, \pi, \delta_2),$$

we obtain

$$\begin{aligned}|u(\rho_0, \vartheta_0, \varphi_0) - \tilde{u}(\rho_0, \vartheta_0, \varphi_0)| &\leq \frac{1 - \tau^2}{4\pi} \int_0^\pi \mathcal{K}(\tau, \vartheta') |\mathcal{I}_1(\vartheta') - \tilde{\mathcal{I}}_1(\vartheta')| d\vartheta' + \\ &+ \frac{1 - \tau^2}{4\pi} \left| \int_0^\pi \mathcal{K}(\tau, \vartheta') \tilde{\mathcal{I}}_1(\vartheta') d\vartheta' - \tilde{S}(\mathcal{K}(\tau, \vartheta') \tilde{\mathcal{I}}_1(\vartheta'), 0, \pi, \delta_2) \right|.\end{aligned}$$

Since

$$\frac{1 - \tau^2}{4\pi} \int_0^\pi \frac{\sin \vartheta' d\vartheta'}{(1 - 2\tau \cos \vartheta' + \tau^2)^{3/2}} = 1,$$

we have

$$|u(\rho_0, \vartheta_0, \varphi_0) - \tilde{u}(\rho_0, \vartheta_0, \varphi_0)| \leq \frac{\delta_1}{2\pi} + \frac{1 - \tau^2}{4\pi} \delta_2.$$

Setting

$$\delta_1 = \delta_2 = \frac{4\pi\varepsilon}{3 - \tau^2},$$

we obtain the estimate

$$|u(\rho_0, \vartheta_0, \varphi_0) - \tilde{u}(\rho_0, \vartheta_0, \varphi_0)| \leq \varepsilon.$$

Program No.3 computes the Poisson integral by means of representation (8.19'). Note that the time of computation at points close to the boundary is less than that in Program No.2, though we admit that the proposed technique does not lead to great success as regards this problem.

**8.6.3. Technique of Choosing a Variable Step.** One may accomplish a further improvement of the computational technique for the Poisson integral by giving up the uniform distribution of nodes on the integration interval. From the analysis of (8.19'') it is obvious that when condition (8.16) is fulfilled, the approachment of the point  $x^0$  to the boundary does not much affect the computation of the internal integral. The computation of the external integral however becomes more difficult because the kernel

$$\mathcal{K}(\tau, \vartheta') = \frac{\sin \vartheta'}{(1 - 2\tau \cos \vartheta' + \tau^2)^{3/2}}$$

increases and tends to infinity as  $x^0$  approaches the boundary. When computation is performed with a constant step, a loss of accuracy occurs (see estimates (8.7) and (8.17)) on a part of the integration interval, where derivatives of  $\mathcal{K}$  sharply increase. Therefore one may improve the effectiveness as follows: the greater the values of  $\mathcal{K}$  and their derivatives on this part of the integration interval, the lesser the division step should be. We will give a description of the technique realizing this idea for a one-dimensional integral.

Suppose we have to calculate the integral

$$\int_a^b f(t) dt \tag{8.22}$$

to within  $\varepsilon$  when  $f$  sharply increases near one of the integration limits. Let, for example,  $f$  tend to infinity at zero,  $a$  be a small positive number and  $b > a$ . One may assume, say, that

$$f(t) \sim \frac{1}{t^p}$$

in the neighbourhood  $t = 0$ .

A positive function  $\delta$  determined on the segment  $[a, b]$  so that

$$\int_a^b \delta(t) dt = 1 \tag{8.23}$$

will be called a node's distribution function.

Let, besides,

$$\Delta(x) \equiv \int_a^x \delta(t) dt. \quad (8.23')$$

Divide  $[a, b]$  by the points  $a = a_0 < a_1 < a_2 < \dots < a_n = b$  into  $n$  parts such that the condition

$$\left| \int_{a_k}^{a_{k+1}} f(t) dt - S(f, a_k, a_{k+1}, 2) \right| \leq \varepsilon (\Delta(a_{k+1}) - \Delta(a_k)), \quad (8.24)$$

where  $S$  is the sum determined by (8.5), be fulfilled on each part  $[a_k, a_{k+1}]$ . By Runge's principle for this it is sufficient that

$$\left| S(f, a_k, a_{k+1}, 1) - S(f, a_k, a_{k+1}, 2) \right| \leq \varepsilon (\Delta(a_{k+1}) - \Delta(a_k)). \quad (8.25)$$

Then, denoting by  $\overset{*}{S}(f, a, b, \varepsilon)$  the sum

$$\overset{*}{S}(f, a, b, \varepsilon) \equiv \sum_{k=0}^{n-1} S(f, a_k, a_{k+1}, 2), \quad (8.26)$$

we will have

$$\begin{aligned} \left| \int_a^b f(t) dt - \overset{*}{S}(f, a, b, \varepsilon) \right| &\leq \sum_{k=0}^{n-1} \left| \int_{a_k}^{a_{k+1}} f(t) dt - S(f, a_k, a_{k+1}, 2) \right| \leq \\ &\leq \varepsilon \sum_{k=0}^{n-1} (\Delta(a_{k+1}) - \Delta(a_k)) = \varepsilon \int_a^b \delta(t) dt = \varepsilon. \end{aligned}$$

Therefore  $\overset{*}{S}(f, a, b, \varepsilon)$  is the desired approximate value of integral (8.22). Let the points  $a_0, \dots, a_n$  be chosen such that for any  $k = 0, \dots, n$ :

$$\left| S(f, a_k, a_{k+1}, 1) - S(f, a_k, a_{k+1}, 2) \right| \sim \varepsilon (\Delta(a_{k+1}) - \Delta(a_k)).$$

The equalities (8.7) and

$$\Delta(a_{k+1}) - \Delta(a_k) = (a_{k+1} - a_k) \delta(\xi_k), \quad a_k < \xi_k < a_{k+1},$$

imply

$$\frac{1}{15} \frac{(a_{k+1} - a_k)^5}{180 \cdot 4^4} |f^{(4)}(\eta_k)| \sim \varepsilon (a_{k+1} - a_k) \delta(\xi_k),$$

or, which is the same,

$$\begin{aligned} (a_{k+1} - a_k)^4 &\sim \frac{180 \cdot 15 \cdot 4^4 \varepsilon \delta(\xi_k)}{|f^{(4)}(\eta_k)|}, \\ a_k &< \xi_k, \eta_k < a_{k+1}. \end{aligned} \quad (8.27)$$

Relation (8.27) gives us the criterion for choosing points  $a_k$ . The integration step

$$h_k = \frac{a_{k+1} - a_k}{4}$$

depends on values of  $\delta$  and  $f^{(4)}$  on a given part of the integration interval. By an appropriate choice of  $\delta$  one can obtain a various extent of dependence of the step on  $f$ . In particular, if

$$\delta(x) \equiv \frac{f^{(4)}(x)}{\int_a^b f^{(4)}(t) dt},$$

then  $h_k$  will not almost depend on the behaviour of  $f$  on  $[a, b]$ , i.e. we will have integration "with a nearly constant step".

Consider a simplest case when  $\delta$  is constant on  $[a, b]$ . Then by virtue of (8.23), (8.23') we have

$$\delta(x) = \frac{1}{b-a}, \quad \Delta(x) = \frac{x-a}{b-a} \quad (8.28)$$

and condition (8.25) becomes

$$|S(f, a_k, a_{k+1}, 1) - S(f, a_k, a_{k+1}, 2)| \leq \frac{\varepsilon(a_{k+1} - a_k)}{b-a}. \quad (8.29)$$

Now (see (8.27))

$$h_k = O\left(\frac{1}{\sqrt[4]{|f^{(4)}(a_k)|}}\right).$$

We will give an algorithm realizing the discussed method of computing integral (8.22). For the function *integral2()* calculating a one-dimensional integral by this algorithm see Program No.4.

To provide a given precision the integration interval is divided into a certain number of subsegments (whose number in the program is determined by the parameter *segnumber*).

The procedure of an approximate calculation of the integral under condition (8.29) is performed for each subsegment in turn. For this the next-in-turn subsegment  $[a_k, a_{k+1}]$  is divided by the points  $x_i$  ( $i = 0, 1, 2, 3, 4$ ) into four equal parts ( $x_0 = a_k, x_4 = a_{k+1}$ ) and therefore  $[a_k, a_{k+1}] = [x_0, x_4]$ . The sum  $S(f, a_k, a_{k+1}, 2)$  is calculated at  $x_0, x_1, x_2, x_3, x_4$  and the sum  $S(f, a_k, a_{k+1}, 1)$  at  $x_0, x_2, x_4$ . If condition (8.29) is fulfilled, then the obtained sum  $S(f, a_k, a_{k+1}, 2)$  is added to the sum already evaluated and we proceed to considering the next subsegment  $[a_{k+1}, a_{k+2}]$ . If condition (8.29) does not hold for the subsegment  $[a_k, a_{k+1}]$ , we take  $[a_k, x_2]$  as a new segment and enter the calculated values of the pairs  $(x_3, f(x_3))$  and  $(x_4, f(x_4))$  in the stack where they are stored.

The subsegment  $[a_k, x_2]$  is divided by the points  $a_k = y_0, y_1, y_2, y_3, y_4 = x_2$  into four equal parts, the entire procedure is repeated and will go on until condition (8.29) becomes fulfilled on some subsegment  $[a_k, t_2]$ . The

obtained value of  $S(f, a_k, t_2, 2)$  is then added to the sum already calculated. The next subsegment is chosen as follows: the point  $t_2$  playing now the part of  $x_0$  is chosen as the subsegment's left-hand limit; the coordinates of two points that have been stored in the stack last are retrieved from there and treated as the points  $x_2$  and  $x_4$ , whereas

$$x_1 = \frac{x_0 + x_2}{2}, \quad x_3 = \frac{x_2 + x_4}{2}.$$

The previous treatment is applied to the subsegment  $[x_0, x_4]$  and the procedure is repeated.

Consider two cases: either the stack becomes empty and we proceed to the next subsegment  $[a_k, a_{k+1}]$  or the stack becomes overflowed. The latter case will mean that the algorithm is unable to compute the integral to within a given accuracy.

Let us determine a maximum possible size of the stack. It is understood that each entry of the coordinates of two points in the stack brings about a twofold decrease of the integration step. Let  $m$  be a number such that  $2^{-m}$  is equal to machine zero (it is assumed that the initial value of the step  $h$  is 1). Now the entry of the coordinates of  $2m$  points in the stack means that the integration step has become equal to machine zero. This is the maximum size of the stack.

In the program in question  $h$  is described by the type *double* and therefore its minimum positive value is  $1.7 \cdot 10^{-308} \sim 2^{-1024}$ . Thus the maximum size of the stack is that having enough storage room for the coordinates of 2048 points or 4096 numbers of the type *double*.

In practice, for a given accuracy the stack size can be taken considerably lesser than the above-indicated one. In particular, due to (8.27) we see that even for the minimum value  $\varepsilon = 1.7 \cdot 10^{-308}$  and the maximum value  $f^{(4)}(\eta_k) = 1.7 \cdot 10^{308}$  the integration step  $h$  will be about  $10^{-153}$  and therefore the stack size can be decreased twofold.

The algorithm automatically provides the fulfilment of condition (8.29) in each integration stage. Among the advantages of the algorithm is that the calculated value of the integrand function which is not needed at the moment is saved in the stack to be used subsequently. Since the computer executes the operations of entering the numbers in the stack and retrieving them from there much more quickly than the operation of evaluating the integrand function, one achieves economy of computation time.

The algorithm has certain disadvantages. If  $f^{(4)}$  changes but little on some integration interval, then the algorithm somewhat slows down its work as compared with the algorithm with an automatic choice of a constant step (the function *integral1()*, since a certain time is spent on checking the fulfilment of condition (8.29), as well as on entering the calculated values of the integrand function in the stack and retrieving them from there.

The algorithm works well only within a certain accuracy range. Thus for a relative accuracy about  $10^{-9}$  the stack may get overflowed and, as a result,

the program may come to an abnormal termination. Such an accuracy is worse than the theoretical value  $10^{-308}$  obtained above. The reason lies in the peculiarity of machine arithmetics. If  $x^0$  is a sufficiently large and  $h$  a sufficiently small number, then after their addition by the computer an equality

$$x^0 + h = x^0.$$

is obtained. This happens if

$$\left| \frac{h}{x^0} \right| < d,$$

where  $d = 2^{-52} \approx 10^{-16}$  for numbers  $x_0$  and  $h$  of type *double*. Thus if  $|f^{(4)}|$  achieves its maximum value at the point  $x_0$ , then in the neighbourhood of  $x^0$  the integration step cannot be lower than  $d|x^0|$  and there is no sense in further decreasing the step  $h$ . When the step approaches this boundary the accuracy may worsen due to roundoff errors. As a result, if the accuracy achieved is not sufficient for the fulfilment of condition (8.29), the stack will get overflowed.

The situation as described above will not occur if  $x^0 = 0$  or  $x^0$  is a sufficiently small number. In that case the accuracy can be increased if by the transformation of coordinates we will make the function  $f^{(4)}$  reach its maximum value at  $x^0 = 0$ . From this standpoint the application of formula (8.19'') possesses an additional advantage as the kernel  $K(\tau', \vartheta')$  reaches its maximum value near the point  $\vartheta' = 0$ .

The accuracy can be increased by taking another nodes distribution function. Substituting then condition (8.29) for (8.25), one may choose a function  $\delta$  such that the required integration step increase. For example, if

$$\delta(x) = \frac{|f^{(4)}(x)|^\alpha}{\int_a^b |f^{(4)}(t)|^\alpha dt}, \quad 1 > \alpha > 0,$$

the integration step increases in the neighbourhood of the point  $x^0$  and the algorithm will work in a more stable mode.

The analysis we did above makes it possible to construct a very convenient algorithm for calculating the Poisson integral. Program No.5 is obtained from Program No.3 by replacing in the external integral the function *integral1()* by the function *integral2()*. Both in Program No.5 and in Program No.3 the internal integral is calculated by the function *integral1()*. This technique saves computation time if derivatives of the boundary function do not sharply change on the sphere  $B(0, \rho)$ .

Observe that in case the boundary function varies considerably on the sphere, the function *integral2()* can be used to calculate the internal integral as well.

As compared with Program No.2, when the point approaches the boundary the computation time in Program No.5 not only increases, but, on the contrary, decreases. A check for various boundary functions showed that

Program No.5 calculates very effectively the Poisson integral with a relative accuracy up to  $10^{-10}$ . When the point approaches the boundary the computation time as compared with Program No.2 decreases several dozen times.

/\*PROGRAM No.1\*/

**8.7. Programs for Numerical Realizations of Solutions.** */\* Function calculating one-dimensional integral by Simpson formula with automatic selection of constant integration step\*/*

```

#define absval(x) ((x)>=0?(x):- (x))  /*Absolute value of x*/
double integrall(leftend, rightend, delta, segnumber, func)
double leftend, rightend;  /*Left and right limits of integration*/
double delta;  /*Precision*/
unsigned segnumber;  /*Number of preliminary subsegments*/
double (*func)(double x);  /*Pointer to integrand function*/
{double step, h;  /*Integration steps*/
double x;  /*Argument of integrand function*/
double s0, s1, s2;  /*Parts of integral sum*/
double s_old, s_new;  /*Old and new values of integral sum*/
unsigned long i, k;  /*Counters*/
k=segnumber;
delta*=3;  /*Take into account coefficient 1/3 in Simpson formula*/
h=(rightend-leftend)/(2*k);  /*Initial value of integration step*/
s2=0;
s0=(*func)(leftend)+(*func)(rightend);
/*Sum of function values at ends of segment*/
s1=(*func)(x=leftend+h);
for(i=1; i<k; i++)
s2+=(*func)(x+=h);  /*Sum at even*/
{s1+=(*func)(x+=h);  /*and odd nodes*/
}
snew=h*(s0+4*s1+2*s2);
do
{step=h*0.5;  /*New value of integration step*/
k*=2;
s2+=s1;  /*New value of sum at even nodes*/
sold=snew;
s1=(*func)(x=leftend+step);
for(i=1; i<k; i++)
s1+=(*func)(x+=h);  /*Calculate new sum at odd nodes*/
h=step;
snew=h*(s0+4*s1+2*s2);
} while(absval(s_new-s_old)>delta);
/*Compare new and old values of sum.

```

```

        Terminate loop if required precision is reached*/
    return(s_new/3); /*Take into account coefficient 1/3*/
}

```

```

        /*PROGRAM No.2*/

```

```

/* Demonstrating program of calculating Poisson integral by algorithm
with automatic selection of constant integration step (function integral1())*/

```

```

#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <conio.h>
#include <dos.h>
#define absval(x) ((x)>=0?(x):-x) /*Absolute value of x*/

```

```

double r; /*Radius of sphere centered at origin*/
double r0, phi0, theta0; /*Spherical coordinates of point*/
double kernel; /*value of Poisson kernel*/
double th; /*Current value of angle theta*/
double delta, epsilon; /*Required precision*/
double ratio;
unsigned segnumber=4; /*Number of preliminary subsegments*/
unsigned long n; /*Number of nodes*/
unsigned long n1;
struct coords{double x1, x2, x3;} current, pos; /*Cartesian coordinates*/

```

```

double f(double x1, double x2, double x3);
/*Boundary value function*/

```

```

double fix_time(void);
/*Function fix_time() fixes system time in seconds*/

```

```

double integral1(double leftend, double rightend, double delta,
                unsigned segnumber, unsigned long *nodes,
                double (*func)(double x));
/* Function calculating one-dimensional integral by Simpson formula.
It differs from function given in Program No.1 only in that it counts
number of nodes at which function values are calculated*/

```

```

double f1(double phi);
/*Boundary value function in spherical coordinates*/

```

```

double int1(double theta);
/*Function int1() calculates value of internal integral*/

```

```

double int2(double,double,double,double,double);
/*Function int2() calculates value of external integral*/

main()
{double t1, t2;
char ch;
clrscr(); /*Clear screen*/
do
{printf("R=");
scanf("%lf", &r); /*Enter radius*/
printf("R0=");
scanf("%lf", &r0); /*Enter spherical coordinates of point*/
printf("phi0=");
scanf("%lf", &phi0);
printf("theta0=");
scanf("%lf", &theta0);
printf("delta="); /*Enter value of required precision*/
scanf("%lf", &epsilon);
t1=fix_time(); /*Fix time of beginning of calculation*/
printf("I1=%0.15lf", int2(r, r0, phi0, theta0, epsilon));
/*Calculate Poisson integral and display its value*/
t2=fix_time(); /*Fix time of end of calculation*/
printf("\nt=%0.2lf", t2-t1); /*Display time spent on calculation*/
printf("\nn=%ld", n); /*Display counted number of nodes*/
printf("\nI0=%0.15lf", f(r0*pos.x1, r0*pos.x2, r0*pos.x3));
/*Calculate precise value of Poisson integral. Since in our
example f is harmonic function, it coincides with value
of f at point (r0, phi, theta)*/
printf("\nmore?[Y/N]\n");
ch=getch();
}
while((ch=='y')||(ch=='Y')); /*Repeat calculation if 'Y'*/
}

double f(double x1, double x2, double x3)
/*Begin of boundary function*/
return(x1*x1-x2*x2+3*x3-4);
/*End of boundary function*/

double fix_time(void)
/*Function fix_time() fixes system time of computer in seconds*/

{struct time t;
gettime(&t);

```

```

return(3600*t.ti_hour+60*t.ti_min+
      t.ti_sec+0.01*t.ti_hund);
}

double integral1(leftend, rightend, delta, segnumber, nodes, func)
/*Function calculating one-dimensional integral by Simpson formula.
It differs from function given in Program No.1 only in that it counts
number of nodes at which function values are calculated*/
double leftend, rightend; /*Left and right limits of integration*/
double delta; /*Precision*/
unsigned segnumber; /*Number of preliminary subsegments*/
unsigned long *nodes; /*Number of nodes*/
double (*func)(double x); /*Pointer to integrand function*/
{double step, h; /*Integration steps*/
double x; /*Argument of integrand function*/
double s0, s1, s2; /*Parts of integral sum*/
double s_old, s_new; /*Old and new values of integral sum*/
unsigned long i, k; /*Counters*/
k=segnumber;
delta*=3; /*Take into account coefficient 1/3 in Simpson formula*/
h=(rightend-leftend)/(2*k); /*Initial value of integration step*/
s2=0;
s0=(*func)(leftend)+(*func)(rightend);
/*Sum of function values at ends of segment*/
s1=(*func)(x=leftend+h);
for(i=1;i<k;i++)
  {s2+=(*func)(x+=h); /*Sum at even*/
  s1+=(*func)(x+=h); /*and odd nodes*/
  }
s_new=h*(s0+4*s1+2*s2);
do
  {step=h*0.5; /*New value of integration step*/
  k*=2;
  s2+=s1; /*New value of sum at even nodes*/
  s_old=s_new;
  s1=(*func)(x=leftend+step);
  for(i=1; i<k; i++)
    s1+=(*func)(x+=h); /*Calculate new sum at odd nodes*/
  h=step;
  s_new=h*(s0+4*s1+2*s2);
  } while(absval(s_new-s_old)>delta);
/*Compare new and old values of sum.
Leave cycle if required precision is reached*/
*nodes+=2*k+1; /*Count number of nodes on given subsegment*/
}

```

```

return(s_new/3);  /*Take into account coefficient 1/3*/
}

double f1(double phi)
/*Boundary function in spherical coordinates*/

{double a, cos_gamma;
current.x2=r*sin(th);
current.x1=current.x2*cos(phi);
current.x2*=sin(phi);
current.x3=r*cos(th);
cos_gamma=cos(th)*cos(theta0)+sin(th)*sin(theta0)*cos(phi-phi0);
a=1-2*ratio*cos_gamma+ratio*ratio;
kernel=sin(th)/sqrt(a*a*a);  /*Value of Poisson kernel*/
return kernel*f(current.x1, current.x2, current.x3);
}

double int1(double theta)
/*Function int1() evaluates internal integral*/

{th=theta;
return integrall(0, 2*M_PI, delta, segnumber, &n, f1);
/*M_PI is value of pi defined in file math.h*/
}

double int2(radius, r0, phi0, theta0, epsilon)
/*Function int2() calculates value of external integral*/
double radius;  /*Radius of sphere*/
double r0, phi0, theta0;  /*Spherical coordinates*/
double epsilon;  /*Required precision*/
{r=radius;
ratio=r0/r;
n=0;  /*Number of nodes at which function values are calculated*/
delta=epsilon*4*M_PI/((1-ratio*ratio)*(M_PI+1));
pos.x2=sin(theta0);
pos.x1=pos.x2*cos(phi0);
pos.x2*=sin(phi0);
pos.x3=cos(theta0);
return (1-ratio*ratio)*integrall(0,M_PI,delta,segnumber,&n1,int1)/
(4*M_PI);
}

```

/\*PROGRAM No.3\*/

```

/*Demonstrating program of calculating Poisson integral by
representation (8.19)*/
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <conio.h>
#include <dos.h>
#define RANGE -0.8
#define absval(x) ((x)>0?(x):- (x)) /*Absolute value of x*/

double r; /*Radius of sphere centered at origin*/
double r0, phi0, theta0; /*Spherical coordinates of point*/
double f0; /*Value of function f at point (r, phi0, theta0)*/
double fr0; /*Value of function f at point (r0, phi0, theta0)*/
double kernel; /*Value of Poisson kernel*/
double th; /*Current value of angle theta*/
double delta, epsilon; /*Required precision*/
double ratio;
int sign;
unsigned segnumber=4; /*Number of preliminary subsegments*/
unsigned long n; /*Number of nodes*/
unsigned long n1;
struct coords{double x1, x2, x3;} new, old, pos;
/*Cartesian coordinates*/

double f(double x1, double x2, double x3);
/*Boundary function*/

double fix_time(void);
/*Function fix_time() fixes system time in seconds*/

struct coords newcoords(struct coords old);
/*Transformation of Cartesian coordinates*/

double integral1(double leftend, double rightend,
                 double delta, unsigned segnumber, unsigned long *nodes,
                 double (*func)(double x));
/*Function calculates one-dimensional integral by Simpson formula.
It coincides with function from PROGRAM No.2*/

double f1(double phi);
/*Calculates function f() by means of spherical coordinates of x*/

double int1(double theta);

```

```

/*Function int1() calculates value of internal integral*/

double int2(double radius, double r0, double phi0,
            double theta0, double epsilon);
/*Function int2() calculates value of external integral*/

main()
{double t1, t2;
 char ch;
 clrscr(); /*Clear screen*/
 do
 {printf("R=");
 scanf("%lf", &r); /*Enter radius*/
 printf("R0=");
 scanf("%lf", &r0); /*Enter spherical coordinates of point*/
 printf("phi0=");
 scanf("%lf", &phi0);
 printf("theta0=");
 scanf("%lf", &theta0);
 printf("delta="); /*Enter value of required precision*/
 scanf("%lf", &epsilon);
 t1=fix_time(); /*Fix time of beginning of calculation*/
 printf("I1=%0.15lf", int2(r, r0, phi0, theta0, epsilon));
 /*Calculate Poisson integral and display its value*/
 t2=fix_time(); /*Fix time of end of calculation*/
 printf("\nt=%0.2lf", t2-t1); /*Display time spent on calculation*/
 printf("\nn=%ld", n); /*Display counted number of nodes*/
 printf("\nI0=%0.15lf", fr0);
 /*Calculate precise value of Poisson integral. Since in our example
 f is harmonic function, it coincides with value of f
 at point (r0, phi, theta) i.e fr0*/
 printf("\nmore?[Y/N]\n");
 ch=getch();
 }
 while((ch=='y')||(ch=='Y')); /*Repeat calculation if 'Y'*/
 }

double f(double x1, double x2, double x3)
/*Begin of boundary function*/
{return(x1*x1-x2*x2+3*x3-4);}
/*End of boundary function*/

double fix_time(void)
/*Function fix_time() fixes system time in seconds*/

```

```

{struct time t;
  gettimeofday(&t);
  return(3600*t.ti_hour+60*t.ti_min+
         t.ti_sec+0.01*t.ti_hund);
}

struct coords newcoords(struct coords old)
/*Transformation of Cartesian coordinates*/
{struct coords current;
  double a;
  (sign) /*If point x is close to pole (-1,0,0), perform transformation if
    {old.x1=-old.x1; old.x2=-old.x2; old.x3=-old.x3;
    }
  current.x3=old.x3*pos.x3-old.x1*pos.x1-old.x2*pos.x2;
  a=(current.x3+old.x3)/(1+pos.x3);
  current.x1=old.x1+a*pos.x1;
  current.x2=old.x2+a*pos.x2;
  return current;
}

double integral1(leftend, rightend, delta, segnumber, nodes, func)
double leftend, rightend; /*Left and right limits of integration*/
double delta; /*Precision*/
unsigned segnumber; /*Number of preliminary subsegments*/
unsigned long *nodes;
double (*func)(double x); /*Pointer to integrand function*/
{double step, h; /*Integration steps*/
  double x; /*Argument of integrand function*/
  double s0, s1, s2; /*Parts of integral sum*/
  double s_old, s_new; /*Old and new values of integral sum*/
  unsigned long i, k; /*Counters*/
  k=segnumber;
  delta*=3; /*Take into account coefficient 1/3 in Simpson formula*/
  h=(rightend-leftend)/(2*k); /*Initial value of integration step*/
  s2=0;
  s0=(*func)(leftend)+(*func)(rightend);
  /*Sum of function values at ends of segment*/
  s1=(*func)(x=leftend+h);
  for(i=1;i<k;i++)
    {s2+=(*func)(x+=h); /*Sum at even*/
      s1+=(*func)(x+=h); /*and odd nodes*/
    }
  s_new=h*(s0+4*s1+2*s2);
do

```

```

{step=h*0.5; /*New value of integration step*/
k*=2;
s2+=s1; /*New value of sum at even nodes*/
s_old=s_new;
s1=(*func)(x=leftend+step);
for(i=1; i<k; i++)
s1+=(*func)(x+=h); /*Calculate new sum at odd nodes*/
h=step;
s_new=h*(s0+4*s1+2*s2);
} while(absval(s_new-s_old)>delta);
/* Compare new and old values of sum.
Terminate loop if required precision is reached*/
*nodes+=2*k+1; /* Count number of nodes on given subsegment*/
return(s_new/3); /* Take into account coefficient 1/3*/
}

```

```

double f1(double phi)
/*Calculates function f() by means of spherical coordinates of x*/
{old.x2=sin(th);
old.x1=old.x2*cos(phi);
old.x2*=sin(phi);
old.x3=cos(th);
new=newcoords(old);
return f(r*new.x1, r*new.x2, r*new.x3);
}

```

```

double int1(double theta)
/*Function int1() evaluates internal integral*/
{double a;
th=theta;
a=1-2*ratio*cos(th)+ratio*ratio;
kernel=sin(th)/sqrt(a*a*a);
return kernel*(integral1(0, 2*M_PI, delta, segnumber, &n, f1)-2*M_PI*f0);
/*M_PI is value of pi defined in file math.h*/
}

```

```

double int2(radius, r0, phi0, theta0, epsilon)
/*Function int2() evaluates external integral*/
double radius; /*Radius of sphere*/
double r0, phi0, theta0; /*Spherical coordinates*/
double epsilon; /*Required precision*/
{r=radius;
ratio=r0/r;
n=0; /*Number of nodes at which function values are calculated*/
}

```

```

delta=epsilon*4*M_PI/((1-ratio*ratio)*(3-ratio*ratio));
pos.x2=sin(theta0);
pos.x1=pos.x2*cos(phi0);
pos.x2*=sin(phi0);
pos.x3=cos(theta0);
f0=f(r*pos.x1, r*pos.x2, r*pos.x3);
fr0=f(r0*pos.x1, r0*pos.x2, r0*pos.x3);
if(pos.x3<RANGE) /*If point is close to (-1,0,0)*/
  {sign=1; /*Perform transformation x'=-x*/
    pos.x1=-pos.x1;
    pos.x2=-pos.x2;
    pos.x3=-pos.x3;
  }
else sign=0;
return (f0+(1-ratio*ratio)*integral1(0,M_PI,delta,segnumber,&n1,int1)
        /(4*M_PI));
}

```

/\*PROGRAM No.4\*/

*/\*Function for calculation of one-dimensional integral by Simpson formula*

*with automatic selection of variable integration step\*/*

```
#define STACKSIZE 800 /*Define maximum size of stack*/
```

```
#define absval(x) ((x)>0?(x):-x) /*Absolute value of x*/
```

```

double integral(double leftend, /*Left and */
                double rightend, /*right limits of integral*/
                double delta, /*Required precison*/
                int segnumber, /*Number of preliminary subsegments*/
                double (*func)(double x), /*Pointer to integrand function*/
                double *stack) /*Pointer to stack*/

```

```

{double step, h; /*Integration steps*/
  double s0, s1, s2; /*Parts of integral sum*/
  double sum; /*Integral sum*/
  double knotsx[5]; /*Coordinates of nodes*/
  double knotsy[5]; /*Values of function at nodes*/
  double * stackptr; /*Pointer to stack top*/
  double * topptr; /*Pointer to end of stack domain*/
  register int k, i; /*Counters*/
  if(rightend==leftend) return 0;
  delta*=3/absval(rightend-leftend); /*Correct value of delta*/
  h=(rightend-leftend)/(4*segnumber); /*Initial value of inegration step*/
  sum=0;
  knotsx[0]=leftend; /*Define initial value of left node*/

```

```

knotsy[0]=(*func)(knotsx[0]);
for(k=0; k<segnumber; k++ )
/*Perform integral computation process
on each initial subsegment separately*/
{step=h;
for(i=1; i<5;i++) /*Arrange five nodes on each initial subsegment*/
{knotsx[i]=knotsx[i-1]+step;
knotsy[i]=(*func)(knotsx[i]);
}
stackptr=stack; /*Set stack pointer to stack bottom*/
topptr=stack+STACKSIZE; /*Calculate position of end of stack domain*/
while(1) /*Infinite loop*/
{s0=knotsy[0]+knotsy[4]; /*Calculate values of sums on subsegment*/
s1=2*(s0+4*knotsy[2]);
s2=s0+4*(knotsy[1]+knotsy[3])+2*knotsy[2];
if(absval(s2-s1)<=delta) /*If required precision is reached*/
{sum+=step*s2; /*finish to calculate sum on given subsegment*/
if(stackptr==stack) break; /*If stack is empty, break loop*/
/*Otherwise retrieve from stack coordinates of two nodes and
arrange nodes on new subsegment*/
knotsx[0]=knotsx[4];
knotsy[0]=knotsy[4];
knotsy[2]=*--stackptr;
knotsx[2]=*--stackptr;
knotsy[4]=*--stackptr;
knotsx[4]=*--stackptr;
}
else /*If error is greater than required one*/
{if (stackptr>=topptr) /*If stack overflowed,
terminate program*/
{printf("\n Stack is overflowed\n");
exit(1);
}
*stackptr++=knotsx[4]; /*Store in stack values of two*/
*stackptr++=knotsy[4]; /*right nodes*/
*stackptr++=knotsx[3];
*stackptr++=knotsy[3];
knotsx[4]=knotsx[2]; /*Arrange nodes on new subsegment*/
knotsy[4]=knotsy[2];
knotsx[2]=knotsx[1];
knotsy[2]=knotsy[1];
}
step=0.5*(knotsx[2]-knotsx[0]); /*Calculate new value of step*/
knotsy[1]=(*func)(knotsx[1]=knotsx[0]+step); /*and values of */
knotsy[3]=(*func)(knotsx[3]=knotsx[4]-step); /*missing nodes*/

```

```

    } /*End of loop*/
    knotsx[0]=knotsx[4]; /*Value of left node on new segment*/
    knotsy[0]=knotsy[4];
}
return(sum/3); /*Take into account coefficient 1/3 in Simpson's formula*/
}

```

/\*PROGRAM No.5\*/

/\*Demonstrating program of calculating Poisson integral by  
representation (8.19)\*/

```

#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <conio.h>
#include <dos.h>
#define RANGE -0.8
#define STACKSIZE 800
#define absval(x) ((x)>0?(x):- (x)) /*Absolute value of x*/

double stack[STACKSIZE];
double r; /*Radius of sphere centered at origin*/
double r0, phi0, theta0; /*Spherical coordinates of point*/
double f0; /*Value of function f at point (r, phi0, theta0)*/
double fr0; /*Value of function f at point (r0, phi0, theta0)*/
double kernel; /*Value of Poisson kernel*/
double th; /*Current value of angle theta*/
double delta, epsilon; /*Required precision*/
double ratio;
int sign;
unsigned segnumber=4; /*Number of preliminary subsegments*/
unsigned long n; /*Number of nodes*/
struct coords{double x1, x2, x3;} new, old, pos;
/*Cartesian coordinates*/

double f(double x1, double x2, double x3);
/*Boundary value function*/

double fix_time(void);
/*Function fix_time() fixes system time in seconds*/

struct coords newcoords(struct coords old);
/*Transformation of Cartesian coordinates*/

```

```

double integral1(double leftend, double rightend,
                 double delta, unsigned segnumber, unsigned long *nodes,
                 double (*func)(double x));
/*Function calculates one-dimensional integral by Simpson formula.
   It coincides with function from PROGRAM No.2*/

double integral2(double leftend, double rightend, double delta,
                 int segnumber, double (*func)(double x), double *stack);
/*Coincides with function from PROGRAM No.4*/

double f1(double phi);
/*Calculates function f() by means of spherical coordinates of x*/

double int1(double theta);
/*Function int1() calculates value of internal integral*/

double int2(double radius, double r0, double phi0,
            double theta0, double epsilon);
/*Function int2() calculates value of external integral*/

main()
{double t1, t2;
 char ch;
 clrscr(); /*Clear screen*/
 do
 {printf("R=");
 scanf("%lf", &r); /*Enter radius*/
 printf("R0=");
 scanf("%lf", &r0); /*Enter spherical coordinates of point*/
 printf("phi0=");
 scanf("%lf", &phi0);
 printf("theta0=");
 scanf("%lf", &theta0);
 printf("delta="); /*Enter value of required precision*/
 scanf("%lf", &epsilon);
 t1=fix_time(); /*Fix time of beginning of calculation*/
 printf("I1=%0.15lf", int2(r, r0, phi0, theta0, epsilon));
 /*Calculate Poisson integral and display its value*/
 t2=fix_time(); /*Fix time of end of calculation*/
 printf("\nt=%0.2lf",t2-t1); /*Display time spent on calculation*/
 printf("\nn=%ld",n); /*Display counted number of nodes*/
 printf("\nI0=%0.15lf", fr0);
 /*Calculate precise value of Poisson integral. Since in our example
   f is harmonic function, it coincides with value of f

```

```

        at point (r0, phi, theta) i.e fr0*/
        printf("\nmore?[Y/N]\n");
        ch=getch();
    }
    while((ch=='y')||(ch=='Y'));    /*Repeat calculation if 'Y'*/
}

double f(double x1, double x2, double x3)
/*Begin of boundary value function*/
{return(x1*x1-x2*x2+3*x3-4);}
/*End of boundary value function*/

double fix_time(void)
/*Function fix_time() fixes system time in seconds*/
{struct time t;
 gettime(&t);
 return(3600*t.ti_hour+60*t.ti_min+
        t.ti_sec+0.01*t.ti_hund);
}

struct coords newcoords(struct coords old)
/*Transformation of Cartesian coordinates*/
{struct coords current;
 double a;
 if(sign) /*If point x is close to pole (-1,0,0),
 perform transformation x'=-x*/
 {old.x1=-old.x1; old.x2=-old.x2; old.x3=-old.x3;
 }
 current.x3=old.x3*pos.x3-old.x1*pos.x1-old.x2*pos.x2;
 a=(current.x3+old.x3)/(1+pos.x3);
 current.x1=old.x1+a*pos.x1;
 current.x2=old.x2+a*pos.x2;
 return current;
}

double integrall1(leftend, rightend, delta, segnumber, nodes, func)
double leftend, rightend; /*Left and right limits of integration*/
double delta; /*Precision*/
unsigned segnumber; /*Number of preliminary subsegments*/
unsigned long *nodes; /*Number of nodes*/
double (*func)(double x); /*Pointer to integrand function*/
{double step, h; /*Integration steps*/
 double x; /*Argument of integrand function*/

```

```

double s0, s1, s2; /*Parts of integral sum*/
double s_old, s_new; /*Old and new values of integral sum*/
unsigned long i, k; /*Counters*/
k=segnumber;
delta*=3; /*Take into account coefficient 1/3 in Simpson formula*/
h=(rightend-leftend)/(2*k); /*Initial value of integration step*/
s2=0;
s0=(*func)(leftend)+(*func)(rightend);
/*Sum of function values at ends of segment*/
s1=(*func)(x=leftend+h);
for(i=1;i<k;i++)
    {s2+=(*func)(x+=h); /*Sum at even*/
      s1+=(*func)(x+=h); /*and odd nodes*/
    }
s_new=h*(s0+4*s1+2*s2);
do
    {step=h*0.5; /*New value of integration step*/
      k*=2;
      s2+=s1; /*New value of sum at even nodes*/
      s_old=s_new;
      s1=(*func)(x=leftend+step);
      for(i=1; i<k; i++)
          s1+=(*func)(x+=h); /*Calculate new sum at odd nodes*/
      h=step;
      s_new=h*(s0+4*s1+2*s2);
    } while(absval(s_new-s_old)>delta);
/*Compare new and old values of sum.
  Terminate loop if required precision is reached*/
*nodes+=2*k+1; /*Count number of nodes on given subsegment*/
return(s_new/3); /*Take into account coefficient 1/3*/
}

double integral2(double leftend, /*Left and */
                double rightend, /*right limits of integral*/
                double delta, /*Required precision*/
                int segnumber, /*Number of preliminary subsegments*/
                double (*func)(double x), /*Pointer to integrand function*/
                double *stack) /*Pointer to stack*/

{double step, h; /*Integration steps*/
  double s0, s1, s2; /*Parts of integral sum*/
  double sum; /*Integral sum*/
  double knotsx[5]; /*Coordinates of nodes*/
  double knotsy[5]; /*Values of function at nodes*/
  double * stackptr; /*Pointer to stack top*/

```

```

double * topptr;  /*Pointer to end of stack domain*/
register int k, i; /*Counters*/
if(rightend==leftend) return 0;
delta*=3/absval(rightend-leftend); /*Correct value of delta*/
h=(rightend-leftend)/(4*segnumber); /*Initial value of inegration step*/
sum=0;
knotsx[0]=leftend; /*Define initial value of left node*/
knotsy[0]=(*func)(knotsx[0]);
for(k=0; k<segnumber; k++ )
/*Perform integral computation process
on each initial subsegment separately*/
{step=h;
for(i=1; i<5;i++) /*Arrange five nodes on each initial subsegment*/
{knotsx[i]=knotsx[i-1]+step;
knotsy[i]=(*func)(knotsx[i]);
}
stackptr=stack; /*Set stack pointer to stack bottom*/
topptr=stack+STACKSIZE; /*Calculate position of end of stack domain*/
while(1) /*Infinite loop*/
{s0=knotsy[0]+knotsy[4]; /*Calculate values of sums on subsegment*/
s1=2*(s0+4*knotsy[2]);
s2=s0+4*(knotsy[1]+knotsy[3])+2*knotsy[2];
if(absval(s2-s1)<=delta) /*If required precision is reached*/
{sum+=step*s2; /*finish to calculate sum on given subsegment*/
if(stackptr==stack) break; /*If stack is empty, break loop*/
/*Otherwise retrieve from stack coordinates of two nodes and
knotsx[0]=knotsx[4];
knotsy[0]=knotsy[4];
knotsy[2]= *--stackptr;
knotsx[2]= *--stackptr;
knotsy[4]= *--stackptr;
knotsx[4]= *--stackptr;
}
else /*If error is greater than required one*/
{if(stackptr<=topptr) /*If stack overflowed, terminate program*/
{printf("\n Stack is overflowed\n");
exit(1);
}
*stackptr++=knotsx[4]; /*Store in stack values of two*/
*stackptr++=knotsy[4]; /*right nodes*/
*stackptr++=knotsx[3];
*stackptr++=knotsy[3];
knotsx[4]=knotsx[2]; /*Arrange nodes on new subsegment*/
knotsy[4]=knotsy[2];
knotsx[2]=knotsx[1];
}
}
}

```

```

        knotsy[2]=knotsy[1];
    }
    step=0.5*(knotsx[2]-knotsx[0]); /*Calculate new value of step*/
    knotsy[1]=(*func)(knotsx[1]=knotsx[0]+step); /*and values of */
    knotsy[3]=(*func)(knotsx[3]=knotsx[4]-step); /*missing nodes*/
} /*End of loop*/
knotsx[0]=knotsx[4]; /*Value of left node on new segment*/
knotsy[0]=knotsy[4];
}
return(sum/3); /*Take into account coefficient 1/3 in Simpson's formula*/
}

```

```

double f1(double phi)
/*Calculates function f() by means of spherical coordinates of x*/
{old.x2=sin(th);
old.x1=old.x2*cos(phi);
old.x2*=sin(phi);
old.x3=cos(th);
new=newcoords(old);
return f(r*new.x1, r*new.x2, r*new.x3);
}

```

```

double int1(double theta)
/*Function int1() calculates value of internal integral*/
{double a;
th=theta;
a=1-2*ratio*cos(th)+ratio*ratio;
kernel=sin(th)/sqrt(a*a*a);
return kernel*(integral1(0, 2*M_PI, delta, segnumber, &n, f1)-2*M_PI*f0);
/*M_PI is value of pi defined in file math.h*/
}

```

```

double int2(radius, r0, phi0, theta0, epsilon)
/*Function int2() calculates value of external integral*/
double radius; /*Radius of sphere*/
double r0, phi0, theta0; /*Spherical coordinates*/
double epsilon; /*Required precision*/
{r=radius;
ratio=r0/r;
n=0; /*Number of nodes at which function values are calculated*/
delta=epsilon*4*M_PI/((1-ratio*ratio)*(3-ratio*ratio));
pos.x2=sin(theta0);
pos.x1=pos.x2*cos(phi0);
pos.x2*=sin(phi0);
}

```

```
pos.x3=cos(theta0);
f0=f(r*pos.x1, r*pos.x2, r*pos.x3);
fr0=f(r0*pos.x1, r0*pos.x2, r0*pos.x3);
if(pos.x3<RANGE) /*If point is close to (-1,0,0)*/
  {sign=1; /*Perform transformation x'=-x*/
    pos.x1=-pos.x1;
    pos.x2=-pos.x2;
    pos.x3=-pos.x3;
  }
else sign=0;
return (f0+(1-ratio*ratio)*integral2(0,M_PI,delta,segnumber,int 1,stack)
      /(4*M_PI));
}
```

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