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BOUNDED SOLUTIONS OF NONLINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

Abstract. For systems of nonlinear differential equations with deviating arguments, sufficient conditions for the existence and uniqueness of bounded on $(-\infty, +\infty)$ solutions are established.

რეზიუმე. გადახრილ არგუმენტებიან არაწრფივ დიფერენციალურ განტოლებათა სისტემებისთვის დადგენილია $(-\infty, +\infty)$ შუალედში შემოსაზღვრული ამონახსნების არსებობისა და ერთადერთობის საკმარისი პირობები.

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Consider the system of nonlinear differential equations with deviating arguments

$$x_i'(t) = g_i(t)x_i(t) + f_i(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))) \quad (i = 1, \dots, n), \tag{1}$$

where $\tau_{ij} : \mathbb{R} \to \mathbb{R}$ $(i, j = 1, \dots, n)$ are measurable in any finite interval functions, $g_i \in L_{loc}(\mathbb{R}, \mathbb{R})$ $(i = 1, \dots, n)$ and $f_i : \mathbb{R}^{n+1} \to \mathbb{R}$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodory conditions.

A vector function $(x_i)_{i=1}^n : \mathbb{R} \to \mathbb{R}^n$ is said to be a **bounded solution of the system** (1) if it is absolutely continuous in any finite interval, satisfies the system (1) almost everywhere on $\mathbb{R}$ and

$$\sup\bigg\{\sum_{i=1}^n |x_i(t)| : \ t \in \mathbb{R}\bigg\} < +\infty.$$

For systems of ordinary differential equations, the problem on the existence of bounded solutions is investigated in detail (see, [4–7] and the references therein). In particular, for both linear [5] and essentially nonlinear differential systems [4, 6], I. Kiguradze has established unimprovable in a certain sense conditions guaranteeing, respectively, the existence and uniqueness of a bounded solution.

By R. Hakl [1, 2] effective sufficient conditions are established for the existence of a unique solution of a linear differential system with deviating arguments

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^n p_{ij}(t)x_j(\tau_{ij}(t)) + q_i(t) \quad (i = 1, \dots, n).$$

In the present paper, based on the method of a priori estimates elaborated in [3, 4, 8–10], the Kiguradze type theorems on the existence and uniqueness of a bounded solution of the system (1) are established.

Throughout the paper the following notation is used.

$\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, \infty)$.

$\mathbb{R}^n$ is the space of n-dimensional vectors $x = (x_i)_{i=1}^n$ with the components $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$.

$\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices $X = (x_{ij})_{i,j=1}^n$ with the components $x_{ij} \in \mathbb{R}$ $(i, j = 1, \dots, n)$.

$\mathbb{R}_+^{n\times n} = \{X = (x_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n} : x_{ij} \in \mathbb{R}_+ \ (i, j = 1, \dots, n)\}$.

$r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{n\times n}$.

$L_{loc}(\mathbb{R}, \mathbb{R})$ is the space of summable in any finite interval functions $u : \mathbb{R} \to \mathbb{R}$.

Theorem 1. *Let there exist a constant matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$, a nonnegative number b, and nonnegative functions p_{ij}, $q_i \in L_{loc}(\mathbb{R},\mathbb{R})$ $(i,j = 1,\dots,n)$ such that*

$$r(A) < 1, \tag{2}$$

$$\big|f_i(t,x_1,\dots,x_n)\big| \le \sum_{j=1}^n p_{ij}(t)|x_j| + q_i(t) \quad \text{for } t\in\mathbb{R},\ (x_j)_{j=1}^n \in \mathbb{R}^n \ (i=1,\dots,n),$$

$$\bigg|\int_{t_i}^t \exp\bigg(\int_s^t g_i(\xi)\,d\xi\bigg) p_{ij}(s)\,ds\bigg| \le a_{ij} \quad \text{for } t\in\mathbb{R} \ (i,j=1,\dots,n), \tag{3}$$

$$\sum_{i=1}^n \bigg|\int_{t_i}^t \exp\bigg(\int_s^t g_i(\xi)\,d\xi\bigg) q_i(s)\,ds\bigg| \le b \quad \text{for } t\in\mathbb{R}, \tag{4}$$

where $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$. Then the system (1) *has at least one bounded solution.*

Theorem 2. *Let there exist a constant matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$, a nonnegative number b, and nonnegative functions $p_{ij} \in L_{loc}(\mathbb{R},\mathbb{R})$ $(i,j=1,\dots,n)$ such that along with* (2), (3) *the conditions*

$$\begin{aligned}&\big|f_i(t,x_1,\dots,x_n) - f_i(t,y_1,\dots,y_n)\big| \\ &\quad\le \sum_{j=1}^n p_{ij}(t)|x_j - y_j| \quad \text{for } t\in\mathbb{R},\ (x_j)_{j=1}^n \in \mathbb{R}^n,\ (y_j)_{j=1}^n \in \mathbb{R}^n \ (i=1,\dots,n),\end{aligned} \tag{5}$$

$$\sum_{i=1}^n \bigg|\int_{t_i}^t \exp\bigg(\int_s^t g_i(\xi)\,d\xi\bigg) |f_i(s,0\dots,0)|\,ds\bigg| \le b \quad \text{for } t\in\mathbb{R} \tag{6}$$

and

$$\limsup_{t\to t_i} \int_0^t g_i(s)\,ds = +\infty \ \ (i=1,\dots,n) \tag{7}$$

be fulfilled, where $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$. Then the system (1) *has one and only one bounded solution.*

Let us describe a scheme of proving the above-formulated theorems.

For an arbitrary natural number m, we consider the system of differential equations

$$x_i'(t) = g_i(t)x_i(t) + \lambda f_i\big(t, x_1(\tau_{i\,1m}(t)),\dots,x_n(\tau_{i\,nm}(t))\big) \ \ (i=1,\dots,n) \tag{8}$$

and the system of differential equations

$$\big|x_i'(t) - g_i(t)x_i(t)\big| \le \sum_{j=1}^n p_{ij}(t)\big|x_j(\tau_{i\,jm}(t))\big| + q_i(t) \ \ (i=1,\dots,n) \tag{9}$$

with the boundary conditions

$$x_i(\sigma_i m) = 0 \ \ (i=1,\dots,n). \tag{10}$$

Here $\lambda\in[0,1]$, $\sigma_i \in \{-1,1\}$ $(i=1,\dots,n)$,

$$\tau_{i\,jm}(t) = \begin{cases} \tau_{ij}(t) & \text{for } |\tau_{ij}(t)| \le m, \\ m & \text{for } \tau_{ij}(t) > m, \\ -m & \text{for } \tau_{ij}(t) < -m \end{cases}$$

and $p_{ij} \in L_{loc}(\mathbb{R},\mathbb{R})$, $q_i \in L_{loc}(\mathbb{R},\mathbb{R})$ $(i,j=1,\dots,n)$ are nonnegative functions.

An absolutely continuous vector function $(x_i)_{i=1}^n : [-m,m] \to \mathbb{R}^n$ is said to be a **solution of the system** (8) (of **the system** (9)) if it almost everywhere on $[-m,m]$ satisfies this system. A solution of the system (8) (of the system (9)), satisfying the boundary conditions (10), is called a **solution of the problem** (8)**,** (10) (of **the problem** (9)**,** (10)).

The following lemmas are valid.

Lemma 1. *Let there exist a positive constant ρ such that for an arbitrary natural number m and arbitrary $\lambda \in [0,1]$ every solution of the problem* (8), (10) *admits the estimate*

$$\max\Big\{\sum_{i=1}^{n}|x_i(t)| : \ -m \le t \le m\Big\} \le \rho. \tag{11}$$

Then the system (1) *has at least one bounded solution.*

Lemma 2. *Let inequalities* (2)–(4), *where $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$, $A=(a_{ij})_{i,j=1}^{n} \in \mathbb{R}_+^{n\times n}$ and $b \in \mathbb{R}_+$, be fulfilled. Moreover, let the condition*

$$\sigma_i = \begin{cases} 1 & \text{if } t_i = +\infty, \\ -1 & \text{if } t_i = -\infty \end{cases}$$

for any $i \in \{1,\dots,n\}$ be fulfilled. Then there exists a positive constant ρ such that for an arbitrary natural m every solution of the problem (9), (10) *admits the estimate* (11).

Theorem 1 follows from Lemmas 1 and 2.

Assume now that the conditions of Theorem 2 are fulfilled. Then by Theorem 1, the system (1) has at least one bounded solution $(x_i)_{i=1}^n$. Our aim is to show that an arbitrary bounded solution $(\overline{x}_i)_{i=1}^n$ of that system coincides with $(x_i)_{i=1}^n$. Assume

$$u_i(t) = \overline{x}_i(t) - x_i(t) \ \ (i=1,\dots,n)$$

and

$$\rho_i = \sup\big\{|u_i(t)| : \ t \in \mathbb{R}\big\} \ \ (i=1,\dots,n).$$

Then according to the condition (5), the vector function $(u_i)_{i=1}^n$ is a bounded solution of the system of differential inequalities

$$\big|u_i'(t) - g_i(t)u_i(t)\big| \le \sum_{j=1}^{n} p_{ij}(t)\rho_j \ \ (i=1,\dots,n).$$

If we now take the conditions (3) and (7) into account, then it becomes clear that

$$|u_i(t)| \le \sum_{j=1}^{n}\bigg|\int_{t_i}^{t}\exp\bigg(\int_{s}^{t} g_i(\xi)\,d\xi\bigg)p_{ij}(s)\,ds\bigg|\rho_j \le \sum_{j=1}^{n} a_{ij}\rho_j \ \text{ for } \ t \in \mathbb{R} \ \ (i=1,\dots,n)$$

and

$$\rho_i \le \sum_{j=1}^{n} a_{ij}\rho_j \ \ (i=1,\dots,n).$$

Hence, in view of (2), it follows that

$$\rho_i = 0 \ \ (i=1,\dots,n),$$

and, consequently,

$$\overline{x}_i(t) \equiv x_i(t) \ \ (i=1,\dots,n).$$

Example. Consider the differential equation

$$x'(t) = g(t)\big[x(t) + a|x(\tau(t))| + b\big], \tag{12}$$

where $a \in \mathbb{R}_+$, $b > 0$, $\tau : \mathbb{R} \to \mathbb{R}$ is a measurable in any infinite interval function and $g \in L_{loc}(\mathbb{R},\mathbb{R})$ is a nonnegative function such that

$$\int_{0}^{+\infty} g(s)\,ds = +\infty. \tag{13}$$

The equation (12) follows from the system (1) in case

$$n=1, \ \ \tau_1(t)=\tau(t), \ \ g_1(t)=g(t), \ \ f_1(t,x_1) = g_1(t)(a|x_1|+b). \tag{14}$$

On the other hand, the equalities (13) and (14) guarantee the fulfilment of the conditions (3), (5)–(7), where

$$t_1 = +\infty, \ \ a_{11} = a, \ \ p_{11}(t) = a_{11}g_1(t),$$

whence by Theorem 2, it follows that if

$$a < 1, \tag{15}$$

then the equation (12) has a unique bounded solution.

Let us now show that the condition (15) is also necessary for the existence of a bounded solution of the equation (1). Indeed, let the equation (12) have a bounded solution x. If we put

$$\delta = \inf\big\{|x(t)| : \ t \in \mathbb{R}\big\},$$

then with regard for (13), we find

$$\begin{aligned} -x(t) &= \int\limits_t^{+\infty} \exp\bigg(\int\limits_s^t g(\xi)\,d\xi\bigg) g(s)\big[a|x(\tau(s))| + b\big]\,ds \\ &\geq (a\delta + b) \int\limits_t^{+\infty} \exp\bigg(\int\limits_s^t g(\xi)\,d\xi\bigg) g(s)\,ds = a\delta + b > 0 \ \text{ for } \ t \in \mathbb{R} \end{aligned}$$

and

$$\delta \geq a\delta + b.$$

Consequently, the inequality (15) is fulfilled.

The above-constructed example shows that the condition (2) in Theorems 1 and 2 is unimprovable and it cannot be replaced by the condition

$$r(A) \leq 1.$$

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