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Ivan Kiguradze

PERIODIC TYPE BOUNDARY VALUE PROBLEMS FOR SINGULAR IN PHASE VARIABLES NONLINEAR NONAUTONOMOUS DIFFERENTIAL SYSTEMS

Dedicated to the Blessed Memory of Professor B. Khvedelidze

Abstract. The unimprovable in a certain sense conditions guaranteeing the existence and uniqueness of positive solutions of periodic type boundary value problems for singular in phase variables nonlinear nonautonomous differential systems are established.

რეზიუმე. დადგენილია გარკვეული აზრით არაგაუმჯობესებადი პირობები, რომლებიც უზრუნველყოფენ პერიოდულის ტიპის სასაზღვრო ამოცანების დადებითი ამონახსნების არსებობასა და ერთადერთობას ფაზური ცვლადების მიმართ სინგულარული არაწრფივი არაავტონომიური დიფერენციალური სისტემებისათვის.

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Let $-\infty < a < b < +\infty$, $\mathbb{R}_{0+} =]0, +\infty[$,

$$\mathbb{R}^n_{0+} = \Big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_1 > 0, \dots, x_n > 0\Big\}$$

and $f_i : [a,b] \times \mathbb{R}^n_{0+} \to \mathbb{R}$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodory conditions, i.e. $f_i(\,\cdot\,, x_1, \dots, x_n) : [a,b] \to \mathbb{R}$ $(i = 1, \dots, n)$ are measurable for all $(x_i)_{i=1}^n \in \mathbb{R}^n_{0+}$, $f_i(t, \,\cdot\,, \dots, \,\cdot\,) : \mathbb{R}^n_{0+} \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuous for almost all $t \in [a,b]$ and for any $\rho > 0$ and $\rho_0 \in]0, \rho[$ the function

$$f^*_{\rho_0,\rho}(t) = \max\bigg\{\sum_{i=1}^n \big|f_i(t, x_1, \dots, x_n)\big| : \ \rho_0 \le x_1 \le \rho, \dots, \rho_0 \le x_n \le \rho\bigg\}$$

is integrable on $[a,b]$.

Consider the differential system

$$\frac{du_i}{dt} = f_i(t, u_1, \dots, u_n) \ \ (i = 1, \dots, n) \tag{1}$$

with the boundary conditions

$$u_i(a) = \varphi_i(u_i(b)) \ \ (i = 1, \dots, n), \tag{2}$$

where $\varphi_i : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ are continuous functions.

A particular case of (2) are the periodic boundary conditions

$$u_i(a) = u_i(b) \ \ (i = 1, \dots, n).$$

Thus the conditions (2) we call the periodic type boundary conditions.

A solution $(u_i)_{i=1}^n : [a, b] \to \mathbb{R}^n_{0+}$ of the system (1) satisfying the boundary conditions (2) is called a positive solution of the problem (1), (2).

For singular in phase variables first and second order differential equations, periodic type boundary value problems are studied in detail (see, e.g., [1, 2, 3, 5, 7]). As for the system (1), for it problems of the type (2) are investigated mainly only in the regular case, i.e., in the case where the functions f_i $(i = 1, \dots, n)$ are continuous, or satisfy the local Carathéodory conditions on the set $[a, b] \times \mathbb{R}^n_+$ and $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are continuous functions, where

$$\mathbb{R}_+ = [0, +\infty[\quad \mathbb{R}^n_+ = \Big\{ (x_i)_{i=1}^n : \ x_1 > 0, \dots, x_n > 0 \Big\}$$

(see [1, 2] and the references therein).

Theorems below on the existence of a positive solution of the problem (1), (2) cover the cases in which the system under consideration has singularities in phase variables, in particular, the case where for arbitrary $i, k \in \{1, \dots, n\}$ and $x_j > 0$ $(j = 1, \dots, n;\ j \neq k)$ the equality

$$\lim_{x_k \to 0} \big| f_i(t, x_1, \dots, x_n) \big| = +\infty$$

is fulfilled.

In Theorems 1 and 2 it is assumed, respectively, that the functions f_i $(i = 1, \dots, n)$ and φ_i $(i = 1, \dots, n)$ on the sets $[a, b] \times \mathbb{R}^n_{0+}$ and $\mathbb{R}_{0+}$ satisfy the inequalities

$$\sigma_i\big(f_i(t, x_1, \dots, x_n) - p_i(t)x_i\big) \geq q_i(t, x_i) \ \ (i = 1, \dots, n), \tag{3}$$

$$\sigma_i\big(\varphi_i(x) - \alpha_i x\big) \geq 0 \ \ (i = 1, \dots, n), \tag{4}$$

$$q_i(t, x_i) \leq \sigma_i\big(f_i(t, x_1, \dots, x_n) - p_i(t)x_i\big) \leq$$

$$\leq \sum_{k=1}^n p_{ik}(t, x_1 + \dots + x_n)x_k + q_0(t, x_1, \dots, x_n) \ \ (i = 1, \dots, n), \tag{5}$$

$$\sigma_i(\varphi_i(x) - \alpha_i x) \geq 0, \quad \sigma_i(\varphi_i(x) - \beta_i x) \leq \beta_0 \ \ (i = 1, \dots, n). \tag{6}$$

Here,

$$\sigma_i \in \{-1, 1\}, \ \ \alpha_i > 0, \ \ \beta_i > 0, \ \ \sigma_i(\beta_i - \alpha_i) \geq 0 \ \ (i = 1, \dots, n), \ \ \beta_0 \geq 0,$$

$p_i : [a, b] \to \mathbb{R}$ $(i = 1, \dots, n)$ are integrable functions, $p_{ik} : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are integrable in the first argument and nonincreasing and continuous in the second argument functions, and $q_0 : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is an integrable in the first argument and nonincreasing and continuous in the last n arguments function. Moreover, p_i

and q_i $(i = 1, \dots, n)$ satisfy the conditions

$$\sigma_i \left(\alpha_i \exp\left(\int_a^b p_i(s)\, ds \right) - 1 \right) < 0 \quad (i = 1, \dots, n), \tag{7}$$

$$\sigma_i \left(\beta_i \exp\left(\int_a^b p_i(s)\, ds \right) - 1 \right) < 0 \quad (i = 1, \dots, n), \tag{8}$$

$$\int_a^b q_i(t, x)\, dt > 0 \quad \text{for} \quad x > 0 \quad (i = 1, \dots, n). \tag{9}$$

Along with $(1), (2)$ we consider the auxiliary problem

$$\begin{aligned} \frac{du_i}{dt} = (1-\lambda)\big(p_i(t)u_i + \sigma_i q_i(t, u_i)\big) + \\ + \lambda f_i(t, u_1, \dots, u_n) + \sigma_i \varepsilon \quad (i = 1, \dots, n), \end{aligned} \tag{10}$$

$$u_i(a) = (1-\lambda)\alpha_i u_i(b) + \lambda \varphi_i(u_i(b)) \quad (i = 1, \dots, n), \tag{11}$$

depending on the parameters $\lambda > 0$ and $\varepsilon > 0$.

Theorem 1 (**Principle of a priori boundedness**). *Let the inequalities* (3), (4), (7), *and* (9) *be fulfilled and let there exist positive constants* ε_0 *and* ρ *such that for arbitrary* $\lambda \in [0, 1]$ *and* $\varepsilon \in]0, \varepsilon_0]$ *every positive solution* $(u_i)_{i=1}^n$ *of the problem* $(10), (11)$ *admits the estimates*

$$u_i(t) < \rho \quad (i = 1, \dots, n).$$

Then the problem $(1), (2)$ *has at least one positive solution.*

By $X = (x_{ik})_{i,k=1}^n$ and $r(X)$ we denote the $n \times n$ matrix with components $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the spectral radius of the matrix X, respectively. For any integrable function $p : [a, b] \to \mathbb{R}$ and positive number β satisfying the condition

$$\Delta(p, \beta) = 1 - \beta \exp\left(\int_a^b p(s)\, ds \right) \neq 0,$$

we put

$$g(p, \beta)(t, s) = \\ = \begin{cases} \dfrac{1}{\Delta(p, \beta)} \exp\left(\displaystyle\int_s^t p(\tau)\, d\tau \right) & \text{for} \quad a \le s \le t \le b, \\ \dfrac{\beta}{\Delta(p, \beta)} \exp\left(\displaystyle\int_a^b p(\tau)\, d\tau + \int_s^t p(\tau)\, d\tau \right) & \text{for} \quad a \le t < s \le b \end{cases}$$

and

$$w(p,\beta)(t)=\frac{1}{\Delta(p,\beta)}\left[(1-\beta)\exp\left(\int_a^t p(s)\,ds\right)+\beta\exp\left(\int_a^b p(s)\,ds\right)-1\right].$$

Theorem 2. *Let the inequalities* (5), (6), (8), *and* (9) *be fulfilled and let there exist continuous functions* $\ell_i : [a,b] \to \mathbb{R}_{0+}$ $(i=1,\dots,n)$ *such that*

$$\lim_{x\to+\infty} r(H(x)) < 1, \tag{12}$$

where $H(x) = (h_{ik}(x))_{i,k=1}^n$ *and*

$$h_{ik}(x) =$$

$$= \max\left\{\frac{1}{\ell_i(t)}\int_a^b \big|g(p_i,\beta_i)(t,s)\big| p_{ik}(s,x)\ell_k(s)\,ds : \ a\le t\le b\right\} \ \ (i,k=1,\dots,n).$$

Then the problem (1), (2) *has at least one positive solution.*

This theorem can be proved on the basis of Theorem 1 and Theorems 2.1, 2.2 and 3.1 of [3].

Now we pass to the case, where

$$\sigma_i p_i(t) \le 0 \ \text{ for } \ a\le t\le b, \quad \sigma_i\int_a^b p_i(t)\,dt < 0 \ \ (i=1,\dots,n)$$

and the inequalities (5) have the form

$$q_i(t,x_i) \le \sigma_i\big(f_i(t,x_1,\dots,x_n) - p_i(t)x_i\big) \le$$

$$\le |p_i(t)|\sum_{k=1}^n \frac{h_{ik}(x_1+\cdots+x_n)}{|w(p_k,\beta_k)(t)|}\,x_k + q_0(t,x_1,\dots,x_n) \ \ (i=1,\dots,n), \tag{13}$$

where $h_{ik} : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ $(i,k=1,\dots,n)$ are continuous nonincreasing functions, and σ_i, q_i $(i=1,\dots,n)$ and q_0 are the numbers and functions satisfying the above conditions.

>From Theorem 2 it follows the following corollary.

Corollary 1. *If along with* (6), (8) *and* (13) *the inequality* (12) *is fulfilled, where* $H(x) = (h_{ik}(x))_{i,k=1}^n$, *then the problem* (1), (2) *has at least one positive solution.*

As an example, we consider the problems

$$\frac{du_i}{dt} = \sigma_i\Big(\sum_{k=1}^n p_{ik}u_k + f_{0i}(t,u_1,\dots,u_n)\Big) \ \ (i=1,\dots,n), \tag{14}$$

$$u_i(a) = u_i(b) \ \ (i=1,\dots,n), \tag{15}$$

and

$$\frac{du_i}{dt} = \sigma_i \sum_{k=1}^{n} \frac{|1-\beta_k| h_{ik}}{(1-\beta_k)(t-a)+\beta_k(b-a)}\, u_k +$$
$$+ \sigma_i f_{0i}(t, u_1, \dots, u_n) \ \ (i = 1, \dots, n), \tag{16}$$
$$u_i(a) = \beta_i u_i(b) \ \ (i = 1, \dots, n), \tag{17}$$

where $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, p_{ik} $(i, k = 1, \dots, n)$ and β_i $(i = 1, \dots, n)$ are the constants satisfying the inequalities

$$p_{ii} < 0, \ \ p_{ik} \geq 0 \ \ (i \neq k; \ \ i, k = 1, \dots, n), \tag{18}$$
$$\beta_i > 0, \ \ \sigma_i(\beta_i - 1) < 0 \ \ (i = 1, \dots, n), \tag{19}$$

h_{ik} $(i, k = 1, \dots, n)$ are nonnegative constants and $f_{0i} : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are functions satisfying the local Carathédoty conditions. Moreover, on the set $[a, b] \times \mathbb{R}^n_{0+}$ the inequalities

$$q_i(t, x_i) \leq f_{0i}(t, x_1, \dots, x_n) \leq q_0(t, x_1, \dots, x_n) \ \ (i = 1, \dots, n)$$

are fulfilled, where $q_0 : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is an integrable in the first argument and nonincreasing and continuous in the last n arguments function, and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are integrable in the first argument and nonincreasing in the second argument functions satisfying the conditions (9).

Corollary 2. *For the existence of at least one positive solution of the problem* (14), (15) *it is necessary and sufficient that real parts of the eigenvalues of the matrix* $(p_{ik})^n_{i,k=1}$ *be negative.*

Corollary 3. *For the existence of at least one positive solution of the problem* (16), (17) *it is necessary and sufficient that the matrix* $H = (h_{ik})^n_{i,k=1}$ *satisfy the inequality*

$$r(H) < 1. \tag{20}$$

Remark 1. In the conditions of Corollaries 2 and 3 the functions f_{0i} $(i = 1, \dots, n)$ may have singularities of arbitrary order in the least n arguments. For example, in (14) and (16) we may assume that

$$f_{0i}(t, x_1, \dots, x_n) = \sum_{k=1}^{n} \big(q_{1ik}(t) x_k^{-\mu_{1ik}} + q_{2ik} \exp(x_k^{-\mu_{2ik}})\big) \ \ (i = 1, \dots, n),$$

where μ_{1ik}, μ_{2ik} $(i, k = 1, \dots, n)$ are positive constants and $q_{1ik} : [a, b] \to \mathbb{R}_+$, $q_{2ik} : [a, b] \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are integrable functions such that

$$\int_a^b \big(q_{1ii}(t) + q_{2ii}(t)\big)\, dt > 0 \ \ (i = 1, \dots, n).$$

The uniqueness of a positive solution of the problem (1), (2) can be proved only in the case where each function f_i has the singularity in the i-th phase variable only. More precisely, we consider the case when the system (1) has the following form

$$\frac{du_i}{dt} = p_i(t)u_i + \sigma_i\big(f_{0i}(t, u_1, \dots, u_n) + q_i(t, u_i)\big) \quad (i = 1, \dots, n). \tag{21}$$

The particular cases of (21) are the differential systems

$$\frac{du_i}{dt} = \sigma_i\bigg(\sum_{k=1}^{n} p_{ik}u_k + q_i(t, u_i)\bigg) \quad (i = 1, \dots, n) \tag{22}$$

and

$$\frac{du_i}{dt} = \sigma_i\bigg(\sum_{k=1}^{n} \frac{|1-\beta_k|h_{ik}}{(1-\beta_k)(t-a)+\beta_k(b-a)}\, u_k + q_i(t, u_i)\bigg) \quad (i=1, \dots, n). \tag{23}$$

Here $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, $p_i : [a, b] \to \mathbb{R}$ $(i = 1, \dots, n)$ are integrable functions, $f_{0i} : [a, b] \times \mathbb{R}_+^n \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodoty conditions, and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are integrable in the first argument and nonincreasing and continuous in the second argument functions. Moreover, p_i and q_i $(i = 1, \dots, n)$ satisfy the conditions (8) and (9). As for p_{ik} and β_i $(i, k = 1, \dots, n)$, they are the constants satisfying the inequalities (18) and (19), and h_{ik} $(i, k = 1, \dots, n)$ are nonnegative constants.

Theorem 3. *Let on the sets $[a, b] \times \mathbb{R}_+^n$ and $\mathbb{R}_+$ the conditions*

$$\sigma_i\big(f_{0i}(t, x_1, \dots, x_n) - f_{0i}(t, y_1, \dots, y_n)\big)\operatorname{sgn}(x_i - y_i) \le$$
$$\le \sum_{k=1}^{n} p_{ik}(t)|x_k - y_k| \quad (i = 1, \dots, n)$$

and

$$\sigma_i(\varphi_i(x) - \alpha_i x) \ge 0, \quad \sigma_i\Big[(\varphi_i(x) - \varphi_i(y))\operatorname{sgn}(x - y) - \beta_i|x - y|\Big] \le 0$$
$$(i = 1, \dots, n)$$

hold, where $p_{ik} : [a, b] \to \mathbb{R}_+$ $(i, k = 1, \cdots, n)$ are integrable functions. Let, moreover, there exist continuous functions $\ell_i : [a, b] \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ such that the matrix $H = (h_{ik})_{i,k=1}^n$ with the components

$$h_{ik} = \max\left\{\frac{1}{\ell_i(t)}\int_a^b \big|g(p_i, \beta_i)(t, s)\big|p_{ik}(s)\ell_k(s)\,ds : \ a \le t \le b\right\} \ (i, k = 1, \dots, n)$$

satisfies the inequality (20)*. Then the problem* (21), (2) *has a unique positive solution.*

Theorem 3 results in the following corollaries.

Corollary 4. *For the existence of a unique positive solution of the problem* (22), (15) *it is necessary and sufficient that real parts of eigenvalues of the matrix $(p_{ik})_{i,k=1}^n$ be negative.*

Corollary 5. *For the existence of a unique positive solution of the problem* (23), (17) *it is necessary and sufficient that the matrix* $H = (h_{ik})_{i,k=1}^{n}$ *satisfy the inequality* (20).

Remark 2. In the conditions of Theorem 3 and its corollaries, the functions q_i $(i = 1, \dots, n)$ may have singularities of arbitrary order in the second argument. For example, in (21), (22) and (23) we may assume that

$$q_i(t,x) = q_{i1}(t)x^{-\mu_{i1}} + q_{i2}(t)\exp(x^{-\mu_{i2}}) \ \ (i = 1, \dots, n),$$

where $\mu_{i1} > 0$, $\mu_{i2} > 0$ $(i = 1, \dots, n)$, and $q_{ik} : [a,b] \to \mathbb{R}_+$ $(i = 1, \dots, n;$ $k = 1, 2)$ are integrable functions such that

$$\int_a^b \big(q_{i1}(t) + q_{i2}(t)\big)\, dt > 0 \ \ (i = 1, \dots, n).$$

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Author's address:

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia.

E-mail: `kig@rmi.ge`