Mem. Differential Equations Math. Phys. 66 (2015), 141–152

Short Communications

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ANTIPERIODIC BOUNDARY VALUE PROBLEM FOR SYSTEMS OF LINEAR GENERALIZED DIFFERENTIAL EQUATIONS

Abstract. The antiperiodic boundary value problem for systems of linear generalized differential equations is considered. The Green type theorem on the unique solvability of the problem is established and representation of its solution is given. The effective necessary and sufficient (among them spectral) conditions for the unique solvability of the problem are also given.

რეზიუმე. განზოგადებულ წრფივ დიფერენციალურ განტოლებათა სისტემებისთვის განხილულია ანტიპერიოდული სასაზღვრო ამოცანა. დადგენილია გრინის ტიპის თეორემა ამოცანის ცალსახად ამოხსნადობისა და ამონახსნის წარმოდგენის შესახებ. აგრეთვე მოყვანილია ამოხსნადობის ეფექტური აუცილებელი და საკმარისი (მათ შორის, სპექტრალური) პირობები.

2000 Mathematics Subject Classification: 34B37.
Key words and phrases: Nonlocal boundary value problem, antiperiodic problem, linear systems, generalized ordinary differential equations, unique solvability, effective conditions.

In the present paper we study the question of the solvability for the system of linear generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \tag{1}$$

under the $\omega > 0$-antiperiodic condition

$$x(t+\omega) = -x(t) \quad \text{for} \quad t \in \mathbb{R}, \tag{2}$$

where $A = (a_{ik})_{i,k=1}^{n} : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f = (f_i)_{i=1}^{n} : \mathbb{R} \to \mathbb{R}^{n}$ are, respectively, the matrix- and vector-functions with bounded variation components on the every closed interval $[a,b]$ from $\mathbb{R}$, and ω is a fixed positive number.

We establish the Green type theorem on the solvability of the problem (1), (2) and represent the solution of the problem. In addition, we give the effective necessary and sufficient conditions (spectral type) for unique solvability of the problem.

The general linear boundary value problem for the system (1) is investigated sufficiently well (see e.g. [6, 8, 15] and the references therein), and the Green type theorems for the unique solvability are obtained. Certain questions dealt with the periodic problem for the system (1) have been investigated in [2–5, 7, 14] (see also the references therein), but the specifical properties analogous to those established for the ordinary differential case (see e.g. [11]) are not available. As for the antiperiodic problem, it is sufficient far from a full value. Thus the problem under considered in the paper, is very actual.

In the paper we establish some special conditions for the unique solvability of the problem (1), (2).

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1, 8–10, 13, 14] and the references therein).

The theory of generalized ordinary differential equations has been introduced by J. Kurzweil [13] in connection with investigation of the well-possed problem for the Cauchy problem for ordinary differential equations.

Throughout the paper, the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$; $[a,b]$ and $]a,b[$ $(a,b\in\mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}_+^{n\times m} = \big\{(x_{ij})_{i,j=1}^{n,m} : x_{ij}\geq 0 \ \ (i=1,\dots,n;\ \ j=1,\dots,m)\big\}$.

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix.

If $X=(x_{ij})_{i,j=1}^{n,m}\in\mathbb{R}^{n\times m}$, then

$$|X| = (|x_{ij}|)_{i,j=1}^{n,m}\,.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then:

X^{-1} is the matrix inverse to X;
$\det X$ is the determinant of X;
$r(X)$ is spectral radius of X;
X^T is the matrix transposed to X;
$\lambda_0(X)$ and $\lambda^0(X)$ are, respectively, the minimal and maximal eigenvalues of the symmetric X matrix.

I_n is the identity $n\times n$-matrix.

The inequalities between the real matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X : [a,b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\overset{b}{\underset{a}{\mathrm{V}}}(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i=1,\dots,n;\ j=1,\dots,m)$; $V(X)(t) = (V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a) = 0$, $V(x_{ij})(t) = \overset{t}{\underset{a}{\mathrm{V}}}(x_{ij})$ for $a < t \le b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-) = X(a),\ X(b+) = X(b))$.

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\|X\|_s = \sup\{\|X(t)\| : t \in [a,b]\}$, $|X|_s = (\|x_{ij}\|_s)_{i,j=1}^{n,m}$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\mathrm{V}}}(X) < \infty$) with the norm $\|X\|_s$.

$\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

$\mathrm{BV}_\omega^+(\mathbb{R},\mathbb{R}^{n\times m})$ and $\mathrm{BV}_\omega^-(\mathbb{R},\mathbb{R}^{n\times m})$ are the sets of all matrix-functions $G : \mathbb{R} \to \mathbb{R}^{n\times m}$ whose restrictions on $[0,\omega]$ belong to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$ and there exists a constant matrix $C \in \mathbb{R}^{n\times m}$ such that, respectively,

$$G(t+\omega) = G(t) + C \quad \text{for} \quad t \in \mathbb{R}$$

and

$$G(t+\omega) = -G(t) + C \quad \text{for} \quad t \in \mathbb{R}.$$

$\mathrm{BV}([a,b],\mathbb{R}_+^{n\times m}) = \big\{X \in \mathrm{BV}([a,b],\mathbb{R}^{n\times m}) : X(t) \ge O_{n\times m} \text{ for } t \in [a,b]\big\}$.

$s_c, s_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j=1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad \text{for } a < t \le b,$$

and

$$s_c(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \quad \text{for} \quad t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_c(g))$ corresponding to the function $s_c(g)$.

If $a = b$, then we assume

$$\int_a^b x(t)\,dg(t) = 0,$$

and if $a > b$, then we assume

$$\int_a^b x(t)\,dg(t) = -\int_b^a x(t)\,dg(t).$$

Hence $\int_a^b x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [12, 13]).

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \quad \text{for} \quad s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} \in \mathrm{BV}([a,b],\mathbb{R}^{l\times n})$ and $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to \mathbb{R}^{n\times m}$, then

$$S_c(G)(t) \equiv (s_c(g_{ik})(t))_{i,k=1}^{l,n}, \quad S_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j = 1, 2)$$

and

$$\int_a^b dG(\tau)\cdot X(\tau) = \left(\sum_{k=1}^{n} \int_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\right)_{i,j=1}^{l,m}.$$

We introduce the operator. If $X \in \mathrm{BV}_{loc}(\mathbb{R}, ; \mathbb{R}^{n\times n})$,

$$\det(I_n + (-1)^j d_j X(t)) \ne 0 \ \text{ for } t \in \mathbb{R} \ \ (j = 1, 2),$$

and $Y \in \mathrm{BV}_{loc}(\mathbb{R}, ; \mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(0) = O_{n\times m},$$
$$\mathcal{A}(X,Y)(t) = Y(t) - Y(0) + \sum_{0<\tau\le t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1}\,d_1Y(\tau)$$
$$- \sum_{0\le\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\,d_2Y(\tau) \quad \text{for} \quad t > 0,$$
$$\mathcal{A}(X,Y)(t) = -\mathcal{A}(X,Y)(t) \quad \text{for} \quad t < 0.$$

We say that the matrix-function $X \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition if the matrices $S_c(X)(t)$, $S_1(X)(t)$ and $S_2(X)(t)$ are pairwise permutable for every $t \in [a,b]$ and there exists $t_0 \in [a,b]$ such that

$$\int_{t_0}^t S_c(X)(\tau)\,dS_c(X)(\tau) = \int_{t_0}^t dS_c(X)(\tau)\cdot S_c(X)(\tau) \quad \text{for} \ \ t \in [a,b].$$

A vector-function $\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is said to be a solution of the system (1) if

$$x(t)-x(s)=\int\limits_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \text{ for }\ \ s<t;\quad s,t\in\mathbb{R}.$$

Under a solution of the problem (1), (2) we understand a solution x of the system (1), satisfying the condition (2).

We assume that

$$A\in \mathrm{BV}_\omega^+(\mathbb{R},\mathbb{R}^{n\times n})\quad\text{and}\quad f\in \mathrm{BV}_\omega^-(\mathbb{R},\mathbb{R}^n),$$

i.e.,

$$A(t+\omega)=A(t)+C\quad\text{and}\quad f(t+\omega)=-f(t)+c\quad\text{for}\quad t\in\mathbb{R}, \tag{3}$$

where $C\in\mathbb{R}^{n\times n}$ and $c\in\mathbb{R}^n$ are, respectively, some constant matrix and a vector; and

$$\det\big(I_n+(-1)^j d_jA(t)\big)\neq 0\quad\text{for}\quad t\in\mathbb{R}\ \ (j=1,2). \tag{4}$$

If a matrix-function $X\in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is such that $\det(I_n-d_1X(t))\neq 0$ for $t\in[0,\omega]$, then we put

$$\begin{gathered}[X(t)]_0=(I_n-d_1X(t))^{-1},\\ [X(t)]_i=(I_n-d_1X(t))^{-1}\int\limits_0^t dX_-(\tau)\cdot[X(\tau)]_{i-1}\\ \text{for}\quad t\in[0,\omega]\quad(i=1,2,\dots),\end{gathered} \tag{5_1}$$

$$\begin{gathered}(X(t))_0=O_{n\times n},\quad (X(t))_1=X(t),\quad (X(t))_{i+1}=\int\limits_0^t dX_-(\tau)\cdot(X(\tau))_i\\ \text{for}\quad t\in[0,\omega]\quad(i=1,2,\dots),\end{gathered} \tag{6_1}$$

and

$$\begin{gathered}V_1(X)(t)=|(I_n-d_1X(t))^{-1}|V(X_-)(t),\\ V_{i+1}(X)(t)=|(I_n-d_1X(t))^{-1}|\int\limits_0^t dV(X_-)(\tau)\cdot V_i(X)(\tau)\\ \text{for}\ \ t\in[0,\omega]\ \ (i=1,2,\dots),\end{gathered} \tag{7_1}$$

where $X_-(t) \equiv X(t-)$; and if $X \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is such that $\det(I_n + d_2X(t)) \neq 0$ for $t \in [0,\omega]$, then we put

$$[X(t)]_0 = (I_n + d_2X(t))^{-1},$$

$$[X(t)]_i = (I_n + d_2X(t))^{-1} \int_\omega^t dX_+(\tau)\cdot[X(\tau)]_{i-1}$$

$$\text{for} \quad t \in [0,\omega] \quad (i = 1,2,\dots), \tag{5_2}$$

$$(X(t))_0 = O_{n\times n}, \quad (X(t))_1 = X(t), \quad (X(t))_{i+1} = \int_\omega^t dX_+(\tau)\cdot(X(\tau))_i$$

$$\text{for} \quad t \in [0,\omega] \quad (i = 1,2,\dots) \tag{6_2}$$

and

$$V_1(X)(t) = |(I_n + d_2X(t))^{-1}|(V(X_+)(t)(b) - V(X_+)(t)|,$$

$$V_{i+1}(X)(t) = |(I_n + d_2X(t))^{-1}|\left|\int_\omega^t dV(X_+)(\tau)\cdot V_i(X)(\tau)\right|$$

$$\text{for } t \in [0,\omega] \ (i = 1,2,\dots), \tag{7_2}$$

where $X_+(t) \equiv X(t+)$.

Alongside with the system (1), we consider the corresponding homogeneous system

$$dx(t) = dA(t)\cdot x(t). \tag{1_0}$$

Moreover, along with the condition (2), we consider the condition

$$x(0) = -x(\omega). \tag{8}$$

Definition 1. Let the condition (3) hold. A matrix-function $\mathcal{G} : [0,\omega]\times[0,\omega] \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem $(1_0), (8)$ if:

(a) for every $s \in]0,\omega[$, the matrix-function $\mathcal{G}(\cdot,s)$ satisfies the matrix equation

$$dX(t) = dA(t)\cdot X(t)$$

both on $[0,s[$ and $]s,\omega]$;

(b)

$$\mathcal{G}(t,t+) - \mathcal{G}(t,t-) = Y(t)D^{-1}\Big\{Y^{-1}(t)(I_n - d_1A(t))^{-1} + Y(\omega)Y^{-1}(t)(I_n + d_2A(t))^{-1}\Big\} \ \text{ for } \ t \in]a,b[;$$

(c) $\mathcal{G}(t,\cdot) \in BV([0,\omega],\mathbb{R}^{n\times n})$ for every $t \in [0,\omega]$;

(d) the equality

$$\int_0^\omega d_s\,(\mathcal{G}(0,s) + \mathcal{G}(\omega,s))\cdot f(s) = 0$$

holds for every $f \in BV([0,\omega],\mathbb{R}^n)$.

The Green matrix of the problem (1_0) exists and it is unique in the following sense. If $\mathcal{G}(t,s)$ and $\mathcal{G}_1(t,s)$ are two matrix-functions satisfying the conditions (a)–(d) of Definition 1, then

$$\mathcal{G}(t,s) - \mathcal{G}_1(t,s) \equiv Y(t)H_*(s),$$

where $H_* \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ is a matrix-function such that

$$H_*(s+) = H_*(s-) = C = const \quad \text{for} \quad s \in [0,\omega],$$

and $C \in \mathbb{R}^{n\times n}$ is a constant matrix.

In particular,

$$\mathcal{G}(t,s) = \begin{cases} -Y(t)\,(I_n + Y(\omega))^{-1}\,Y^{-1}(s) & \text{for } \ 0 \le s < t \le \omega, \\ Y(t)\,(I_n + Y(\omega))^{-1}\,Y(\omega)Y^{-1}(s) & \text{for } \ 0 \le t < s \le \omega, \\ \text{an arbitrary} & \text{for } \ \ t = s. \end{cases}$$

Theorem 1. *Let the conditions* (3) *and* (4) *hold. Then the problem* (1)*,* (2) *has the unique solution x if and only if the corresponding homogeneous system* (1_0) *has only the trivial solution satisfying the condition* (8)*, i.e., when*

$$\det(Y(0) + Y(\omega)) \neq 0, \tag{9}$$

where Y is a fundamental matrix of the system (1_0)*. If the last condition holds, then the solution x admits the notation*

$$x(t) = \int\limits_0^\omega d_s\mathcal{G}(t,s)\cdot f(s) \quad \textit{for} \quad t \in [0,\omega], \tag{10}$$

where $\mathcal{G} : [a,b]\times[a,b] \to \mathbb{R}^{n\times n}$ is the Green matrix $\mathcal{G}$ of the problem (1_0)*,* (8)*.*

Corollary 1. *Let the conditions* (3) *and* (4) *hold, and the matrix-function A satisfy the Lappo–Danilevskiĭ condition. Then the problem* (1)*,* (8) *has the unique solution if and only if*

$$\det\Big(I_n + \exp(S_0(A)(\omega)) \prod_{0\le\tau<\omega}(I_n + d_2A(\tau)) \prod_{a<\tau\le\omega}(I_n - d_1A(\tau))^{-1}\Big) \neq 0.$$

Note that if the matrix-function A satisfies the Lappo–Danilevskiĭ condition, then the matrix-function Y is defined by $Y(a) = I_n$ and

$$Y(t) \equiv \exp(S_0(A)(t)) \prod_{0\le\tau<t}(I_n + d_2A(\tau)) \prod_{0<\tau\le t}(I_n - d_1A(\tau))^{-1}$$

is the fundamental matrix of the system (1_0).

Remark 1. Let the system (1_0) have a nontrivial ω-antiperiodic solution. Then there exist $f \in \mathrm{BV}^-_\omega(\mathbb{R},\mathbb{R}^n)$ such that the system (1) has no ω-antiperiodic solution.

In general, it is quite difficult to verify the condition (9) directly even in the case where one is able to write out the fundamental matrix of the system (1_0) explicitly. Therefore it is important to seek for effective conditions which would guarantee the absence of nontrivial ω-antiperiodic solutions of the homogeneous system (1_0). Below we present the results concerning our question. Analogous results have been obtained by T. Kiguradze for the ordinary differential equations (see [11, 12]).

Theorem 2. *Let the conditions* (3) *and* (4) *hold. Then the system* (1) *has the unique ω-antiperiodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} \big([A(0)]_i + [A(\omega)]_i\big)$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{11}$$

where

$$M_{k,m} = V_m(A)(c) + \left(\sum_{i=0}^{m-1} |[A(\cdot)]_i|_s\right) \cdot |M_k^{-1}|[V_k(A)(0) + V_k(A)(\omega)],$$

$[A(t)]_i$ $(i = 0, \dots, m-1)$ and $V_i(A)(t)$ $(i = 0, \dots, m-1)$ are defined, respectively, by (5_l) and (7_l) for some $l \in \{1, 2\}$, and $c = (2-l)\omega$.

Corollary 2. *Let the conditions* (3) *and* (4) *hold. Then the system* (1) *has the unique ω-antiperiodic solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} [(A(0))_i + (A(\omega))_i]$$

is nonsingular and the inequality (11) *holds, where*

$$M_{k,m} = (V(A)(c))_m$$
$$+\left(I_n + \sum_{i=0}^{m-1} |(A(\cdot))_i|_s\right) \cdot |M_k^{-1}| \left[(V(A)(0))_k + (V(A)(\omega))_k\right],$$

$(A(t))_i$ $(i = 0, \dots, m-1)$ and $(V(A)(t))_i$ $(i = 0, \dots, m-1)$ are defined by (6_l) for some $l \in \{1, 2\}$, and $c = (2-l)\omega$.

Corollary 3. *Let the conditions* (3) *and* (4) *hold. Let, moreover, there exist a natural j such that*

$$(A(0))_i = -(A(\omega))_i \quad (i = 1, \dots, j-1)$$

and

$$\det\big((A(0))_j + (A(\omega))_j\big) \neq 0,$$

where $(A(t))_i$ $(i = 0, \dots, j)$ *are defined by* (6_l) *for some* $l \in \{1, 2\}$*. Then there exists* $\varepsilon_0 > 0$ *such that the system*

$$dx(t) = \varepsilon dA(t) \cdot x(t) + df(t)$$

has one and only one ω*-antiperiodic solution for every* $\varepsilon \in]0, \varepsilon_0[$.

Theorem 3. *Let the conditions* (3) *and* (4) *hold, and let a matrix-function* $A_0 \in \mathrm{BV}^+_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ *be such that*

$$\det\big(I_n + (-1)^j d_j A_0(t)\big) \neq 0 \quad for \quad t \in [0, \omega] \quad (j = 1, 2)$$

and the homogeneous system

$$dx(t) = dA_0(t) \cdot x(t)$$

has only the trivial ω*-antiperiodic solution. Let, moreover, the matrix-function* $A \in \mathrm{BV}^+_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$ *admit the estimate*

$$\int\limits_0^\omega |\mathcal{G}_0(t,\tau)| dV(S_0(A - A_0))(\tau) + \sum_{0<\tau\leq\omega} |\mathcal{G}_0(t, \tau-) \cdot d_1(A(\tau) - A_0(\tau))| \\ + \sum_{0\leq\tau<\omega} |\mathcal{G}_0(t, \tau+) \cdot d_2(A(\tau) - A_0(\tau))| \leq M,$$

where $\mathcal{G}_0(t,\tau)$ *is the Green matrix of the problem* (1_0)*,* (8)*, and* $M \in \mathbb{R}^{n\times n}_+$ *is a constant matrix such that*

$$r(M) < 1.$$

Then the system (1) *has one and only one* ω*-antiperiodic solution.*

The presentation (10) can be replaced by a more simple and suitable form if we introduce the concept of the Green matrix for the problem $(1_0), (2)$.

Definition 2. The matrix function $\mathcal{G}_\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem $(1_0), (2)$ if:

(a) $\mathcal{G}_\omega(t+\omega, \tau+\omega) = \mathcal{G}_\omega(t, \tau)$, $\mathcal{G}_\omega(t, t+\omega) + \mathcal{G}_\omega(t, \tau) = -I_n$ for $t, \tau \in \mathbb{R}$;

(b) the matrix function $\mathcal{G}_\omega(., \tau) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a fundamental matrix of the system (1_0) for every $\tau \in \mathbb{R}$.

Theorem 4. *Let the conditions* (3) *and*

$$\det\big(I_n \pm d_j A(t)\big) \neq 0 \quad for \quad t \in \mathbb{R}$$

hold and the boundary value problem (1_0)*,* (2) *have only the trivial solution. Then the system* (1) *has the unique* ω*-antiperiodic solution* x *and it admits the notation*

$$x(t) = \int\limits_t^{t+\omega} \mathcal{G}_\omega(t,\tau) d\mathcal{A}\left(A, \mathcal{A}(-A, f)\right)(\tau) \quad for \quad t \in \mathbb{R},$$

where $\mathcal{G}_\omega$ *is the Green matrix of the problem* $(1_0), (2)$.

If $s \in \mathbb{R}$ and $\beta \in \mathrm{BV}[0,\omega],\mathbb{R})$ are such that

$$1 + (-1)^j d_j\beta(t) \neq 0 \quad \text{for} \quad (-1)^j(t-s) < 0 \quad (j = 1,2),$$

then by $\gamma_s(\beta)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t)\,d\beta(t), \quad \gamma(s) = 1.$$

It is known (see [9,10]) that

$$\gamma_s(\beta)(t) = \begin{cases} \exp(s_0(\beta)(t) - s_0(\beta)(s)) \prod\limits_{s<\tau\le t}(1 - d_1\beta(\tau))^{-1} \\ \quad \times \prod\limits_{s\le\tau<t}(1 + d_2\beta(\tau)) \ \text{ for } \ s < t \le \omega, \\ \exp(s_0(\beta)(t) - s_0(\beta)(s)) \prod\limits_{t<\tau\le s}(1 - d_1\beta(\tau)) \\ \quad \times \prod\limits_{t\le\tau<s}(1 + d_2\beta(\tau))^{-1} \ \text{ for } \ 0 \le t < s, \\ \qquad 1 \qquad\qquad \text{for } \ t = s. \end{cases} \tag{12}$$

Let $g : [0,\omega] \to \mathbb{R}$ be a nondecreasing function, and $P = (p_{ik})_{i,k=1}^n$, where $p_{ik} \in L([0,\omega],\mathbb{R};g)$ $(i,k = 1,\dots,n)$. Then by $Q_\omega(P;g)$ we denote the set of all matrix-functions $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ such that

$$b_{ik}(t) = \int\limits_0^t p_{ik}(\tau)dg(\tau) \quad \text{for} \quad t \in [0,\omega] \quad (i,k = 1,\dots,n),$$

where

$$b_{ik}(t) \equiv a_{ik}(t) - \frac{1}{2}\bigg(\sum_{l=1}^n \sum_{0<\tau\le t} d_1a_{li}(\tau)\cdot d_1a_{lk}(\tau) - \sum_{0\le\tau<t} d_2a_{li}(\tau)\cdot d_2a_{lk}(\tau)\bigg) \quad (i,k = 1,\dots,n).$$

Theorem 5. *Let the conditions* (3) *and* $A \in Q_\omega(P;g)$ *hold. Let, moreover, either*

$$\begin{gathered} \sum_{i,k=1}^n p_{ik}(t)x_ix_k \ge \alpha(t)\sum_{i=1}^n x_i^2 \ \textit{ for } \ \mu(g)\text{-a.a. } t \in [0,\omega], \ (x_i)_{i=1}^n \in \mathbb{R}^n, \\ 1 - 2\alpha(t)d_1g(t) > 0, \quad 1 + 2\alpha(t)d_2g(t) \neq 0 \ \textit{ for } \ 0 \le t < \omega \end{gathered} \tag{13}$$

and

$$\gamma_\omega(g_\alpha)(0) > 1 \tag{14}$$

or

$$\begin{gathered} \sum_{i,k=1}^n p_{ik}(t)x_ix_k \le \beta(t)\sum_{i=1}^n x_i^2 \ \textit{ for } \ \mu(g)\text{-a.a. } t \in [0,\omega], \ (x_i)_{i=1}^n \in \mathbb{R}^n, \\ 1 + 2\beta(t)d_2g(t) > 0, \quad 1 - 2\beta(t)d_1g(t) \neq 0 \ \textit{ for } \ 0 < t \le \omega \end{gathered} \tag{15}$$

and

$$\gamma_0(g_\beta)(\omega) < 1, \tag{16}$$

where $\alpha, \beta \in L([0,\omega], \mathbb{R}; g)$*, the functions* $\gamma_\omega(g_\alpha), \gamma_0(g_\beta)$ *are defined by* (12)*, and*

$$g_\alpha(t) \equiv 2\int_0^t \alpha(\tau)dg(\tau) \quad \text{and} \quad g_\beta(t) \equiv 2\int_0^t \beta(\tau)dg(\tau) \tag{17}$$

Then the system (1) *has the unique* ω*-antiperiodic solution.*

Corollary 4. *Let the conditions* (3) *and* $A \in Q_\omega(P; g)$ *hold. Let, moreover, either the conditions* (13) *and* (14) *or the conditions* (15) *and* (16) *hold, where* $\alpha(t) \equiv \lambda_0(P^*(t))$*,* $\beta(t) \equiv \lambda^0(P^*(t))$*,* $P^*(t) \equiv P(t) + P^T(t)$*, the functions* $\gamma_\omega(g_\alpha), \gamma_0(g_\beta)$ *are defined by* (12)*, and the functions* g_α *and* g_β *are defined by* (17)*. Then the system* (1) *has the unique* ω*-antiperiodic solution.*

Acknowledgement

The present paper was supported by the Shota Rustaveli National Science Foundation (Grant No. FR/182/5-101/11).

References

1. M. Ashordia, On the stability of solutions of the multipoint boundary value problem for the system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **6** (1995), 1–57.
2. M. Ashordia, On the method of the construction of a solution of the periodic boundary value problem for a system of linear generalized ordinary differential equations. *Reports Enlarged Session Sem. I. N. Vekua Inst. Appl. Math.* **11** (1996), No. 1-3 , 14-17.
3. M. Ashordia, On the existence of nonnegative solutions of the periodic boundary value problem for a system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **10** (1997), 153–156.
4. M. Ashordia, On the question of solvability of the periodic boundary value problem for a system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **11** (1997), 159–162.
5. M. Ashordia, On existence of solutions of the periodic boundary-value problem for nonlinear system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **16** (1999), 150–153.
6. M. Ashordia, On the solvability of linear boundary value problems for systems of generalized ordinary differential equations. *Funct. Differ. Equ.* **7** (2000), No. 1-2, 39–64 (2001).
7. M. Ashordia, On the solvability of the periodic type boundary value problem for linear systems of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **44** (2008), 133–142.
8. M. Ashordia, On the general and multipoint boundary value problems for linear systems of generalized ordinary differential equations, linear impulse and linear difference systems. *Mem. Differential Equations Math. Phys.* **36** (2005), 1–80.
9. J. Groh, A nonlinear Volterra–Stieltjes integral equation and Gronwall inequality in one dimension, *Illinois J. Math.* **2(24)** (1980), 244–263.

10. T. H. Hildebrandt, On systems of linear differentio–Stieltjes integral equations, *Illinois J. Math.* **3** (1959), 352–373.
11. I. T. Kiguradze, The Initial Value Problem and Boundary Value Problems for Systems of Ordinary Differential Equations. Vol. I. Linear Theory. (Russian) *Metsniereba, Tbilisi*, 1997.
12. T. Kiguradze, Some boundary value problems for systems of linear differential equations of hyperblic type. *Mem. Differential Equations Math. Phys.* **1** (1994), 1–144.
13. J. Kurzweil, Generalized ordinary differential equations. *Czechoslovak Math. J.* **8(83)** (1958), 360–388.
14. Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations. boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht–Boston, Mass.–London,* 1979.
15. Š. Schwabik and M. Tvrdý, Boundary value problems for generalized linear differential equations. *Czechoslovak Math. J.* **29(104)** (1979), No. 3, 451–477.

(Received 20.05.2014)

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