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LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS APPROACH FOR ROBIN TYPE PROBLEM OF THE THEORY OF PIEZO-ELASTICITY FOR INHOMOGENEOUS SOLIDS

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. The paper deals with the three-dimensional Robin type boundary value problem (BVP) of piezoelasticity for anisotropic inhomogeneous solids and develops the generalized potential method based on the use of localized parametrix. Using Green's integral representation formula and properties of the localized layer and volume potentials, we reduce the Robin type BVP to the localized boundary-domain integral equations (LBDIE) system. First we establish the equivalence between the original boundary value problem and the corresponding LBDIE system. We establish that the obtained localized boundary-domain integral operator belongs to the Boutet de Monvel algebra and by means of the Vishik-Eskin theory based on the Wiener-Hopf factorization method, we derive explicit conditions under which the localized operator possesses Fredholm properties and prove its invertibility in appropriate Sobolev-Slobodetskii and Bessel potential spaces.

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რეზიუმე. ნაშრომი ეძღვნება ლოკალიზებული პარამეტრიქსის მეთოდის განვითარებას პიეზო-დრეკადობის თეორიის რობენის ტიპის სამგანზომილებიანი ამოცანისთვის არაერთგვაროვანი ანიზოტროპული სხეულების შემთხვევაში. გრინის ინტეგრალური წარმოდგენის ფორმულისა და ლოკალიზებული პოტენციალების თვისებების გამოყენებით რობენის ტიპის ამოცანა დაიყვანება ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემაზე, რომლის შესაბამისი ოპერატორი ეკუთვნის ბუტე დე მონველის ალგებრას. შესწავლილია რობენის ტიპის ამოცანისა და მიღებულ ლოკალიზებულ სასაზღვრო-სივრცულ ინტეგრალურ განტოლებათა სისტემის ექვივალენტობა. შემდეგ, ვიშიკ-ესკინის თეორიის გამოყენებით, რომელიც ეფუძნება ვინერ-ჰოფის ფაქტორიზაციის მეთოდს, დადგენილია პირობები, რომლის დროსაც ლოკალიზებულ სასაზღვრო-სივრცული ინტეგრალური ოპერატორი არის ფრედჰოლმური და ნაჩვენებია მისი შებრუნებადობა შესაბამის სობოლევ-სლობოდეცკისა და ბესელის პოტენციალთა სივრცეებში.

1. Introduction

In the present paper, we consider the three-dimensional Robin type boundary value problem (BVP) of piezoelasticity for anisotropic inhomogeneous solids and develop the generalized potential method based on the use of *localized parametrix*.

Note that the operator, generated by the system of piezoelasticity for inhomogeneous anisotropic solids, is the second order non-self-adjoint strongly elliptic partial differential operator with variable coefficients. In the reference [22] the Dirichlet problem of piezoelasticity theory was analyzed by the LBDIE approach. The same method for the case of scalar elliptic second order partial differential equations with variable coefficients is justified in [13]–[21], [39].

Due to a great theoretical and practical importance, the problems of piezoelasticity became very popular among mathematicians and engineers (for details see, e.g., [51], [43], [27]–[35]). The BVPs and various types of interface problems of piezoelasticity for *homogeneous anisotropic solids*, when the material parameters are constants and the corresponding fundamental solution is available in explicit form, have been investigated in [5], [6], [7], [8], [9], [42], [10] by means of the conventional classical potential methods.

Unfortunately, this classical potential method is not applicable in the case of inhomogeneous solids since for the corresponding system of differential equations with variable coefficients a fundamental solution is not available in explicit form, in general. Therefore, in our analysis we apply the so-called localized parametrix method which leads to the localized boundary-domain integral equations system.

Our main goal here is to show that solutions of the boundary value problem can be represented by *localized potentials* and that the corresponding *localized boundary-domain integral operator* (LBDIO) is invertible, which seems to be very important from the numerical analysis viewpoint, since they lead to very convenient numerical schemes in applications (for details see [38], [46], [47], [49], [50]).

Towards this end, using Green's representation formula and properties of the localized layer and volume potentials, we reduce the Robin type BVP of piezoelasticity to the *localized boundary-domain integral equations (LBDIE) system.* First, we establish the equivalence between the original boundary value problem and the corresponding LBDIE system which proved to be a quite nontrivial problem playing a crucial role in our analysis. Afterwards, we state that the localized boundary-domain integral operator associated with the Robin type BVP belongs to the Boutet de Monvel algebra of pseudo-differential operators. Finally, with the help of the Vishik–Eskin theory based on the factorization Wiener–Hopf method, we investigate the Fredholm properties of the localized boundary-domain integral operator and prove its invertibility in the appropriate Sobolev–Slobodetskii and Bessel potential spaces.

2. Reduction to LBDIE System and the Equivalence Theorems

2.1. **Formulation of the boundary value problem and localized Green's third formula.** Consider the system of statics of piezoelasticity for an inhomogeneous anisotropic medium [43]:

$$A(x,\partial_x)\,U+X=0, \tag{2.1}$$

where $U:=(u_1,u_2,u_3,u_4)^\top$, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $u_4=\varphi$ is the electric potential, $X=(X_1,X_2,X_3,X_4)^\top$, $(X_1,X_2,X_3)^\top$ is a given mass force density, X_4 is a given charge density, $A(x,\partial_x)$ is a formally non-self-adjoint matrix differential operator

$$\begin{aligned}A(x,\partial_x)&=\big[A_{jk}(x,\partial_x)\big]_{4\times4}\\ &:=\begin{bmatrix}[\partial_i\,(c_{ijlk}(x)\,\partial_l)]_{3\times3} & [\partial_i(e_{lij}(x)\partial_l)]_{3\times1}\\ [-\partial_i(e_{ikl}(x)\,\partial_l)]_{1\times3} & \partial_i(\varepsilon_{il}(x)\,\partial_l)\end{bmatrix}_{4\times4},\end{aligned}$$

where $\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial_{x_j}=\partial/\partial x_j$. Here and in what follows, the Einstein summation by repeated indices from 1 to 3 is assumed if not otherwise stated.

The variable coefficients involved in the above equations satisfy the symmetry conditions:

$$c_{ijkl}=c_{jikl}=c_{klij}\in C^\infty,\;\; e_{ijk}=e_{ikj}\in C^\infty,\;\; \varepsilon_{ij}=\varepsilon_{ji}\in C^\infty,\\ i,j,k,l=1,2,3.$$

In view of these symmetry relations, the formally adjoint differential operator $A^*(x,\partial_x)$ reads as

$$\begin{aligned}A^*(x,\partial_x)&=\big[A^*_{jk}(x,\partial_x)\big]_{4\times4}\\ &:=\begin{bmatrix}[\partial_i\,(c_{ijlk}(x)\,\partial_l)]_{3\times3} & [-\partial_i(e_{lij}(x)\partial_l)]_{3\times1}\\ [\partial_i(e_{ikl}(x)\,\partial_l)]_{1\times3} & \partial_i(\varepsilon_{il}(x)\,\partial_l)\end{bmatrix}_{4\times4}.\end{aligned}$$

Moreover, from physical considerations it follows that (see, e.g., [43]):

$$c_{ijkl}(x)\,\xi_{ij}\,\xi_{kl}\geqslant c_0\,\xi_{ij}\,\xi_{ij}\;\text{ for all }\;\xi_{ij}=\xi_{ji}\in\mathbb{R}, \tag{2.2}$$

$$\varepsilon_{ij}(x)\,\eta_i\,\eta_j\geqslant c_1\,\eta_i\,\eta_i\;\text{ for all }\;\eta=(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^3, \tag{2.3}$$

with some positive constants c_0 and c_1.

By virtue of inequalities (2.2) and (2.3) it can easily be shown that the operator $A(x,\partial_x)$ is uniformly strongly elliptic, that is, there is a constant $c>0$ such that

$$\operatorname{Re}A(x,\xi)\,\zeta\cdot\zeta\geqslant c\,|\xi|^2\,|\zeta|^2\;\text{ for all }\;\xi\in\mathbb{R}^3\;\text{ and for all }\;\zeta\in\mathbb{C}^4, \tag{2.4}$$

where $A(x,\xi)$ is the principal homogeneous symbol matrix of the operator $A(x,\partial_x)$ with opposite sign,

$$A(x,\xi) = \big[A_{jk}(x,\xi)\big]_{4\times 4} := \begin{bmatrix} [c_{ijlk}(x)\,\xi_i\,\xi_l]_{3\times 3} & [e_{lij}(x)\,\xi_i\,\xi_l]_{3\times 1} \\ [-e_{ikl}(x)\,\xi_i\,\xi_l]_{1\times 3} & \varepsilon_{il}(x)\,\xi_i\,\xi_l \end{bmatrix}_{4\times 4}. \tag{2.5}$$

Here and in the sequel, the symbol $a\cdot b$ for $a,b\in\mathbb{C}^4$ denotes the scalar product of two vectors, $a\cdot b=\sum\limits_{j=1}^{4} a_j\overline{b_j}$, where the overbar denotes complex conjugation.

In the theory of piezoelasticity, the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n=(n_1,n_2,n_3)$ have the form

$$\sigma_{ij}\,n_i = c_{ijlk}\,n_i\,\partial_l u_k + e_{lij}\,n_i\,\partial_l\varphi \ \text{ for } \ j=1,2,3,$$

while the normal component of the electric displacement vector (with opposite sign) reads as

$$-D_i\,n_i = -e_{ikl}\,n_i\,\partial_l u_k + \varepsilon_{il}\,n_i\,\partial_l\varphi.$$

Let us introduce the following matrix differential operator:

$$\mathcal{T} = \mathcal{T}(x,\partial_x) = \big[\mathcal{T}_{jk}(x,\partial_x)\big]_{4\times 4} := \begin{bmatrix} [c_{ijlk}(x)\,n_i\,\partial_l]_{3\times 3} & [e_{lij}(x)\,n_i\,\partial_l]_{3\times 1} \\ [-e_{ikl}(x)\,n_i\,\partial_l]_{1\times 3} & \varepsilon_{il}(x)\,n_i\,\partial_l \end{bmatrix}_{4\times 4}. \tag{2.6}$$

For a four–vector $U=(u,\varphi)^\top$, we have

$$\mathcal{T}U = \big(\sigma_{i1}\,n_i,\ \sigma_{i2}\,n_i,\ \sigma_{i3}\,n_i,\ -D_i\,n_i\big)^\top. \tag{2.7}$$

Clearly, the components of the vector $\mathcal{T}U$ given by (2.7) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of electro-elasticity and the forth one is the normal component of the electric displacement vector (with opposite sign). In Green's formulae there also appear the following boundary operator associated with the adjoint differential operator $A^*(x,\partial_x)$:

$$\mathcal{M} = \mathcal{M}(x,\partial_x) = \big[\mathcal{M}_{jk}(x,\partial_x)\big]_{4\times 4} := \begin{bmatrix} [c_{ijlk}(x)\,n_i\,\partial_l]_{3\times 3} & [-e_{lij}(x)\,n_i\,\partial_l]_{3\times 1} \\ [e_{ikl}(x)\,n_i\,\partial_l]_{1\times 3} & \varepsilon_{il}(x)\,n_i\,\partial_l \end{bmatrix}_{4\times 4}. \tag{2.8}$$

Introduce the following matrices associated with the boundary operators (2.6) and (2.8)

$$\mathcal{T}(x,\xi) = \big[\mathcal{T}_{jk}(x,\xi)\big]_{4\times 4}$$
$$:= \begin{bmatrix} [c_{ijlk}(x)\, n_i\, \xi_l]_{3\times 3} & [e_{lij}(x)\, n_i\, \xi_l]_{3\times 1} \\ [-e_{ikl}(x)\, n_i\, \xi_l]_{1\times 3} & \varepsilon_{il}(x)\, n_i\, \xi_l \end{bmatrix}_{4\times 4}, \tag{2.9}$$

$$\mathcal{M}(x,\xi) = \big[\mathcal{M}_{jk}(x,\xi)\big]_{4\times 4}$$
$$:= \begin{bmatrix} [c_{ijlk}(x)\, n_i\, \xi_l]_{3\times 3} & [-e_{lij}(x)\, n_i\, \xi_l]_{3\times 1} \\ [e_{ikl}(x)\, n_i\, \xi_l]_{1\times 3} & \varepsilon_{il}(x)\, n_i\, \xi_l \end{bmatrix}_{4\times 4}. \tag{2.10}$$

Further, let $\Omega = \Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega = S \in C^\infty$, $\overline{\Omega} = \Omega \cup S$. Throughout the paper, $n = (n_1, n_2, n_3)$ denotes the unit normal vector to S directed outward with respect to the domain Ω. Set $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$.

By $H^r(\Omega) = H^r_2(\Omega)$ and $H^r(S) = H^r_2(S)$, $r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary, while $\mathcal{D}(\mathbb{R}^3)$ and $\mathcal{D}(\Omega)$ denote classes of infinitely differentiable functions in $\mathbb{R}^3$ with a compact support in $\mathbb{R}^3$ and Ω respectively, and $\mathcal{S}(\mathbb{R}^3)$ stands for the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions in Ω.

For the vector $U = (u_1, u_2, u_3, u_4)^\top$ the inclusion $U = (u_1, u_2, u_3, u_4)^\top \in H^r$ means that all components u_j, $j = \overline{1,4}$, belong to the space H^r.

Let us denote by $U^+ \equiv \{U\}^+$ and $U^- \equiv \{U\}^-$ the traces of U on S from the interior and exterior of Ω, respectively.

We also need the following subspace of $H^1(\Omega)$:

$$H^{1,0}(\Omega; A)$$
$$:= \Big\{ U = (u_1, u_2, u_3, u_4)^\top \in H^1(\Omega) : \ A(x, \partial_x) U \in L_2(\Omega) \Big\}. \tag{2.11}$$

For arbitrary complex-valued vector-functions $U = (u_1, u_2, u_3, u_4)^\top$ and $V = (v_1, v_2, v_3, v_4)^\top$ from the space $H^2(\Omega)$, we have the following Green's formulae [9]:

$$\int\limits_{\Omega} \Big[A(x, \partial_x)\, U \cdot V + E(U, V) \Big]\, dx = \int\limits_{S} \big\{ \mathcal{T} U \big\}^+ \cdot \{V\}^+\, dS, \tag{2.12}$$

$$\int\limits_{\Omega} \Big[A(x, \partial_x)\, U \cdot V - U \cdot A^*(x, \partial_x)\, V \Big]\, dx$$
$$= \int\limits_{S} \Big[\big\{ \mathcal{T} U \big\}^+ \cdot \{V\}^+ - \{U\}^+ \cdot \big\{ \mathcal{M} V \big\}^+ \Big]\, dS, \tag{2.13}$$

where

$$E(U, V) = c_{ijlk}\, \partial_i u_j\, \overline{\partial_l v_k} + e_{lij}\, (\partial_l u_4\, \overline{\partial_i v_j} - \partial_i u_j\, \overline{\partial_l v_4}) + \varepsilon_{jl}\, \partial_j u_4\, \overline{\partial_l v_4}. \tag{2.14}$$

Note that by means a standard limiting procedure the above Green's formulae can be generalized to Lipschitz domains and to vector–functions $U \in H^1(\Omega)$ and $V \in H^1(\Omega)$ with $A(x,\partial_x)U \in L_2(\Omega)$ and $A^*(x,\partial_x)V \in L_2(\Omega)$. By virtue of Green's formula (2.12), we can determine a *generalized trace vector* $\mathcal{T}^+U \equiv \{\mathcal{T}U\}^+ \in H^{-1/2}(\partial\Omega)$ for a function $U \in H^{1,0}(\Omega;A)$,

$$\langle \mathcal{T}^+U, V^+\rangle_{\partial\Omega} := \int\limits_{\Omega} A(x,\partial_x)\,U\cdot V\,dx + \int\limits_{\Omega} E(U,V)\,dx, \tag{2.15}$$

where $V \in H^1(\Omega)$ is an arbitrary vector-function.

Here, the symbol $\langle\,\cdot\,,\cdot\,\rangle_S$ denotes the duality between the spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ which extends the usual L_2 inner product

$$\langle f,g\rangle_S = \int\limits_S \sum_{j=1}^N f_j\,\overline{g_j}\,dS \ \text{ for } \ f,g \in L_2(S).$$

Assume that the domain Ω is filled with an anisotropic inhomogeneous piezoelectric material and let us formulate the Robin type boundary value problem:

Find a vector-function $U = (u_1,u_2,u_3,u_4)^\top \in H^{1,0}(\Omega,A)$ satisfying the differential equation

$$A(x,\partial_x)U = f \ \text{ in } \ \Omega \tag{2.16}$$

and the Robin type boundary condition

$$\mathcal{T}^+U + \beta U^+ = \Psi_0 \ \text{ on } \ S, \tag{2.17}$$

where $\Psi_0 = (\Psi_{01},\Psi_{02},\Psi_{03},\Psi_{03})^\top \in H^{-1/2}(S)$, $f = (f_1,f_2,f_3,f_4)^\top \in H^0(\Omega)$ and $\beta = [\beta_{jk}]_{4\times 4}$ is a positive definite constant matrix.

Equation (2.16) is understood in the distributional sense, while the Robin type boundary condition (2.17) is understood in the functional sense defined in (2.15).

Remark 2.1. From the conditions (2.2) and (2.3) it follows that for complex-valued vector-functions the sesquilinear form $E(U,V)$ defined by (2.14) satisfies the inequality

$$\operatorname{Re} E(U,U) \geq c\,(s_{ij}\,\overline{s}_{ij} + \eta_j\overline{\eta}_j) \ \ \forall\, U = (u_1,u_2,u_3,u_4)^\top \in H^1(\Omega)$$

with $s_{ij} = 2^{-1}(\partial_i u_j(x) + \partial_j u_i(x))$ and $\eta_j = \partial_j u_4(x)$, where c is some positive constant. Therefore, the first Green's formula (2.12) along with the Lax–Milgram lemma imply that the above-formulated Robin type BVP is uniquely solvable in the space $H^{1,0}(\Omega;A)$ (see, e.g., [36], [26], [37]).

As it has already been mentioned, our goal here is to develop the LBDIE method for the Robin type boundary value problem.

To this end, we define a *localized matrix parametrix* associated with the fundamental solution $F_1(x) := -[\,4\,\pi\,|x|\,]^{-1}$ of the Laplace operator

$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$,

$$P(x) \equiv P_\chi(x) := F_\chi(x)\, I$$
$$= \chi(x)\, F_1(x)\, I = -\frac{\chi(x)}{4\,\pi\,|x|}\, I \ \text{ with } \ \chi(0) = 1, \tag{2.18}$$

where $F_\chi(x) = \chi(x) F_1(x)$, I is the unit 4×4 matrix and χ is a localizing function (see Appendix A),

$$\chi \in X^k_{1+}, \quad k \geq 4. \tag{2.19}$$

Throughout the paper, we assume that the condition (2.19) is satisfied and χ has a compact support if not otherwise stated.

Denote by $B(y, \varepsilon)$ a ball centered at the point y, of radius $\varepsilon > 0$ and let $\Sigma(y, \varepsilon) := \partial B(y, \varepsilon)$.

In Green's second formula (2.13), let us take in the place of $V(x)$ successively the columns of the matrix $P(x - y)$, where y is an arbitrarily fixed interior point in Ω, and write the identity (2.13) for the region $\Omega_\varepsilon := \Omega \setminus B(y, \varepsilon)$ with $\varepsilon > 0$ such that $\overline{B(y, \varepsilon)} \subset \Omega$. Keeping in mind that $P^\top(x - y) = P(x - y)$, we arrive at the equality

$$\int\limits_{\Omega_\varepsilon} \Big[P(x-y)\, A(x, \partial_x) U(x) - \big[A^*(x, \partial_x) P(x-y)\big]^\top U(x) \Big]\, dx$$
$$= \int\limits_S \Big[P(x-y) \, \big\{\mathcal{T}(x, \partial_x) U(x)\big\}^+ - \big\{\mathcal{M}(x, \partial_x) P(x-y)\big\}^\top \{U(x)\}^+ \Big]\, dS$$
$$- \int\limits_{\Sigma(y,\varepsilon)} \Big[P(x-y)\, \mathcal{T}(x, \partial_x) U(x) - \big\{\mathcal{M}(x, \partial_x) P(x-y)\big\}^\top U(x) \Big]\, d\Sigma(y, \varepsilon). \tag{2.20}$$

The direction of the normal vector on $\Sigma(y, \varepsilon)$ is chosen as outward with respect to $B(y, \varepsilon)$.

It is evident that the operator

$$\mathcal{A}\, U(y) : = \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} \big[A^*(x, \partial_x) P(x-y)\big]^\top U(x)\, dx$$
$$= \text{v.p.} \int\limits_{\Omega} \big[A^*(x, \partial_x) P(x-y)\big]^\top U(x)\, dx \tag{2.21}$$

is a singular integral operator; here and in the sequel, "v.p." denotes the Cauchy principal value integral. If the domain of integration in (2.21) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{A}\, U \equiv \mathbf{A}\, U$, i.e.,

$$\mathbf{A}\, U(y) := \text{v.p.} \int\limits_{\mathbb{R}^3} \big[A^*(x, \partial_x) P(x-y)\big]^\top U(x)\, dx. \tag{2.22}$$

Note that

$$\frac{\partial^2}{\partial x_i\, \partial x_l} \, \frac{1}{|x-y|} = -\frac{4\,\pi\,\delta_{il}}{3}\, \delta(x-y) + \text{v.p.} \, \frac{\partial^2}{\partial x_i\, \partial x_l} \, \frac{1}{|x-y|}\,, \tag{2.23}$$

where δ_{il} is the Kronecker delta, while $\delta(\,\cdot\,)$ is the Dirac distribution. The derivatives in the left-hand side of (2.23) are understood in the distributional sense. In view of (2.18) and taking into account that $\chi(0)=1$, we can write the following equality in the distributional sense:

$$\big[A^*(x,\partial_x)P(x-y)\big]^\top$$
$$=\begin{bmatrix} \Big[\dfrac{\partial}{\partial x_i}\Big(c_{ijlk}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big)\Big]_{3\times3} & \Big[\dfrac{\partial}{\partial x_i}\Big(e_{ikl}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big)\Big]_{3\times1} \\ \Big[-\dfrac{\partial}{\partial x_i}\Big(e_{lij}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big)\Big]_{1\times3} & \dfrac{\partial}{\partial x_i}\Big(\varepsilon_{il}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big) \end{bmatrix}_{4\times4}$$
$$=\begin{bmatrix} \Big[c_{ijlk}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,F_\chi(x-y)\Big]_{3\times3} & \Big[e_{ikl}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,F_\chi(x-y)\Big]_{3\times1} \\ \Big[-e_{lij}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,F_\chi(x-y)\Big]_{1\times3} & \varepsilon_{il}(x)\,\dfrac{\partial^2}{\partial x_i\partial x_l}\,F_\chi(x-y) \end{bmatrix}_{4\times4}$$
$$+\begin{bmatrix} \Big[\dfrac{\partial}{\partial x_i}\,c_{ijlk}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{3\times3} & \Big[\dfrac{\partial}{\partial x_i}e_{ikl}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{3\times1} \\ \Big[-\dfrac{\partial}{\partial x_i}\,e_{lij}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{1\times3} & \dfrac{\partial}{\partial x_i}\varepsilon_{il}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y) \end{bmatrix}_{4\times4}$$
$$=\begin{bmatrix} \big[c_{ijlk}(x)\,k_{il}(x,y)\big]_{3\times3} & \big[e_{ikl}(x)\,k_{il}(x,y)\big]_{3\times1} \\ \big[-e_{lij}(x)\,k_{il}(x,y)\big]_{1\times3} & \varepsilon_{il}(x)\,k_{il}(x,y) \end{bmatrix}_{4\times4}$$
$$+\begin{bmatrix} \Big[\dfrac{\partial}{\partial x_i}\,c_{ijlk}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{3\times3} & \Big[\dfrac{\partial}{\partial x_i}\,e_{ikl}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{3\times1} \\ \Big[-\dfrac{\partial}{\partial x_i}\,e_{lij}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y)\Big]_{1\times3} & \dfrac{\partial}{\partial x_i}\,\varepsilon_{il}(x)\,\dfrac{\partial}{\partial x_l}\,F_\chi(x-y) \end{bmatrix}_{4\times4},$$

where

$$\begin{aligned} k_{il}(x,y) &:= \frac{\delta_{il}}{3}\,\delta(x-y)+\text{v.p.}\,\frac{\partial^2 F_\chi(x-y)}{\partial x_i\partial x_l} \\ &= \frac{\delta_{il}}{3}\,\delta(x-y)-\frac{1}{4\pi}\,\text{v.p.}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{1}{|x-y|}+m_{il}(x,y), \\ m_{il}(x,y) &:= -\frac{1}{4\pi}\,\frac{\partial^2}{\partial x_i\partial x_l}\,\frac{\chi(x-y)-1}{|x-y|}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} &\big[A^*(x,\partial_x)P(x-y)\big]^\top \\ &\quad= \mathbf{b}(x)\,\delta(x-y)+\text{v.p.}\,\big[A^*(x,\partial_x)P(x-y)\big]^\top \\ &\quad= \mathbf{b}(x)\,\delta(x-y)+R(x,y) \\ &\qquad\quad -\text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix} \big[c_{ijlk}(x)\vartheta_{il}(x,y)\big]_{3\times3} & \big[e_{ikl}(x)\vartheta_{il}(x,y)\big]_{3\times1} \\ \big[-e_{lij}(x)\vartheta_{il}(x,y)\big]_{1\times3} & \varepsilon_{il}(x)\vartheta_{il}(x,y) \end{bmatrix}_{4\times4} \end{aligned}$$

$$= \mathbf{b}(x)\,\delta(x-y) + R^{(1)}(x,y)$$
$$- \text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix} \big[c_{ijlk}(y)\,\vartheta_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(y)\vartheta_{il}(x,y)\big]_{3\times 1} \\ \big[-e_{lij}(y)\vartheta_{il}(x,y)\big]_{1\times 3} & \varepsilon_{il}(y)\vartheta_{il}(x,y) \end{bmatrix}_{4\times 4},$$

where

$$\mathbf{b}(x) = \frac{1}{3}\begin{bmatrix} [c_{ljlk}(x)]_{3\times 3} & [e_{lkl}(x)]_{3\times 1} \\ [-e_{llj}(x)]_{1\times 3} & \varepsilon_{ll}(x) \end{bmatrix}_{4\times 4}, \tag{2.24}$$

$$\vartheta_{il}(x,y) = \frac{\partial^2}{\partial x_i \partial x_l}\,\frac{1}{|x-y|}\,, \quad i,l = 1,2,3, \tag{2.25}$$

$$R(x,y) = \begin{bmatrix} \big[c_{ijlk}(x)\,m_{il}(x,y)\big]_{3\times 3} & \big[e_{ikl}(x)\,m_{il}(x,y)\big]_{3\times 1} \\ \big[-e_{lij}(x)\,m_{il}(x,y)\big]_{1\times 3} & \varepsilon_{il}(x)\,m_{il}(x,y) \end{bmatrix}_{4\times 4}$$
$$+ \begin{bmatrix} \Big[\frac{\partial}{\partial x_i}\,c_{ijlk}(x)\,\frac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 3} & \Big[\frac{\partial}{\partial x_i}\,e_{ikl}(x)\,\frac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{3\times 1} \\ \Big[-\frac{\partial}{\partial x_i}\,e_{lij}(x)\,\frac{\partial F_\chi(x-y)}{\partial x_l}\Big]_{1\times 3} & \frac{\partial}{\partial x_i}\,\varepsilon_{il}(x)\,\frac{\partial F_\chi(x-y)}{\partial x_l} \end{bmatrix}_{4\times 4},$$
$$R^{(1)}(x,y) = R(x,y)$$
$$-\frac{1}{4\pi}\begin{bmatrix} \big[c_{ijlk}(x,y))\,\vartheta_{il}(x,y)\big]_{3\times 3} & \big[e_{lij}(x,y)\,\vartheta_{il}(x,y)\big]_{3\times 1} \\ \big[-e_{ikl}(x,y)\,\vartheta_{il}(x,y)\big]_{1\times 3} & \varepsilon_{il}(x,y)\,\vartheta_{il}(x,y) \end{bmatrix}_{4\times 4},$$
$$c_{ijlk}(x,y) := c_{ijlk}(x) - c_{ijlk}(y), \quad e_{lij}(x,y) := e_{lij}(x) - e_{ikl}(y),$$
$$\varepsilon_{il}(x,y) := \varepsilon_{il}(x) - \varepsilon_{il}(y).$$

Evidently, the entries of the matrix-functions $R(x,y)$ and $R^{(1)}(x,y)$ possess weak singularities of type $\mathcal{O}(|x-y|^{-2})$ as $x \to y$. Therefore, we get

$$\text{v.p.}\,[A^*(x,\partial_x)P(x-y)]^\top = R(x,y)$$
$$+ \text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix} -\big[c_{ijlk}(x)\,\vartheta_{il}(x,y)\big]_{3\times 3} & -\big[e_{lij}(x)\,\vartheta_{il}(x,y)\big]_{3\times 1} \\ \big[e_{ikl}(x)\,\vartheta_{il}(x,y)\big]_{1\times 3} & -\varepsilon_{il}(x)\,\vartheta_{il}(x,y) \end{bmatrix}_{4\times 4}, \tag{2.26}$$

$$\text{v.p.}\,[A^*(x,\partial_x)P(x-y)]^\top = R^{(1)}(x,y)$$
$$+ \text{v.p.}\,\frac{1}{4\pi}\begin{bmatrix} -\big[c_{ijlk}(y)\,\vartheta_{il}(x,y)\big]_{3\times 3} & -\big[e_{lij}(y)\,\vartheta_{il}(x,y)\big]_{3\times 1} \\ \big[e_{ikl}(y)\,\vartheta_{il}(x,y)\big]_{1\times 3} & -\varepsilon_{il}(y)\,\vartheta_{il}(x,y) \end{bmatrix}_{4\times 4}. \tag{2.27}$$

Further, by direct calculations one can easily verify that

$$\lim_{\varepsilon\to 0}\int\limits_{\Sigma(y,\varepsilon)} P(x-y)\,\mathcal{T}(x,\partial_x)U(x)\,d\Sigma(y,\varepsilon) = 0, \tag{2.28}$$

$$\lim_{\varepsilon\to 0}\int\limits_{\Sigma(y,\varepsilon)} \big\{\mathcal{M}(x,\partial_x)P(x-y)\big\}^\top U(x)\,d\Sigma(y,\varepsilon)$$
$$=\frac{1}{4\pi}\begin{bmatrix} \Big[c_{ijlk}(y)\int\limits_{\Sigma_1}\eta_i\eta_l\,d\Sigma_1\Big]_{3\times 3} & \Big[e_{ikl}(y)\int\limits_{\Sigma_1}\eta_l\eta_i\,d\Sigma_1\Big]_{3\times 1} \\ \Big[-e_{lij}(y)\int\limits_{\Sigma_1}\eta_i\eta_l\,d\Sigma_1\Big]_{1\times 3} & \varepsilon_{il}(y)\int\limits_{\Sigma_1}\eta_i\eta_l\,d\Sigma_1 \end{bmatrix}_{4\times 4} U(y)$$
$$=\frac{1}{4\pi}\begin{bmatrix} \Big[c_{ijlk}(y)\,\frac{4\,\pi\,\delta_{il}}{3}\Big]_{3\times 3} & \Big[e_{ikl}(y)\,\frac{4\,\pi\,\delta_{li}}{3}\Big]_{3\times 1} \\ \Big[-e_{lij}(y)\,\frac{4\,\pi\,\delta_{il}}{3}\Big]_{1\times 3} & \varepsilon_{il}(y)\,\frac{4\,\pi\,\delta_{il}}{3} \end{bmatrix}_{4\times 4} U(y)$$
$$=\mathbf{b}(y)\,U(y), \tag{2.29}$$

where Σ_1 is a unit sphere, $\eta=(\eta_1,\eta_2,\eta_3)\in\Sigma_1$ and $\mathbf{b}$ is defined by (2.24).

Passing to the limit in (2.20) as $\varepsilon\to 0$ and using the relations (2.21), (2.28) and (2.29), we obtain

$$\mathbf{b}(y)\,U(y)+\mathcal{A}\,U(y)-V(\mathcal{T}^+U)(y)+W(U^+)(y) = \mathcal{P}\big(A(x,\partial_x)U\big)(y),\;\; y\in\Omega, \tag{2.30}$$

where $\mathcal{A}$ is a *localized singular integral operator* given by (2.21), while V, W and $\mathcal{P}$ are the *localized single layer, double layer and Newtonian volume potentials*,

$$V(g)(y):=-\int\limits_S P(x-y)\,g(x)\,dS_x, \tag{2.31}$$
$$W(g)(y):=-\int\limits_S \big[\mathcal{M}(x,\partial_x)\,P(x-y)\big]^\top g(x)\,dS_x, \tag{2.32}$$
$$\mathcal{P}(h)(y):=\int\limits_\Omega P(x-y)\,h(x)\,dx. \tag{2.33}$$

Let us also introduce the scalar volume potential

$$\mathbb{P}(\mu)(y):=\int\limits_\Omega F_\chi(x-y)\,\mu(x)\,dx \tag{2.34}$$

with μ being a scalar density function.

If the domain of integration in the Newtonian volume potential (2.33) is the whole space $\mathbb{R}^3$, we employ the notation $\mathcal{P}\,h\equiv\mathbf{P}\,h$, i.e.,

$$\mathbf{P}(h)(y):=\int\limits_{\mathbb{R}^3} P(x-y)\,h(x)\,dx. \tag{2.35}$$

Mapping properties of the above potentials are investigated in [16].

We refer the relation (2.30) as *Green's third formula*. By a standard limiting procedure we can extend Green's third formula (2.30) to the functions from the space $H^{1,0}(\Omega, A)$. In particular, it holds true for solutions of the above formulated Robin type BVP. In this case, the generalized trace vector $\mathcal{T}^+U$ is understood in the sense of definition (2.15).

For $U = (u_1, \dots, u_4)^\top \in H^1(\Omega)$, one can also derive the following relation:

$$\mathcal{A}\,U(y) = -\mathbf{b}(y)\,U(y) - W(U^+)(y) + \mathcal{Q}\,U(y), \quad \forall\, y \in \Omega, \tag{2.36}$$

where

$$\mathcal{Q}\,U(y) := \begin{bmatrix} \Big[\dfrac{\partial}{\partial y_i}\,\mathbb{P}(c_{ijlk}\,\partial_l u_k)(y) + \dfrac{\partial}{\partial y_i}\,\mathbb{P}(e_{lij}\partial_l u_4)(y)\Big]_{3\times 1} \\ -\dfrac{\partial}{\partial y_i}\,\mathbb{P}(e_{ikl}\partial_l u_k)(y) + \dfrac{\partial}{\partial y_i}\,\mathbb{P}(\varepsilon_{il}\partial_l u_4)(y) \end{bmatrix}_{4\times 4}. \tag{2.37}$$

and $\mathbb{P}$ is defined in (2.34).

In what follows, for our analysis we need the explicit expression of the principal homogeneous symbol matrix $\mathfrak{S}\,(\mathcal{A})(y,\xi)$ of the singular integral operator $\mathcal{A}$. This matrix coincides with the Fourier transform of the singular matrix kernel defined by (2.26). Let $\mathcal{F}$ denote the Fourier transform operator,

$$\mathcal{F}_{z\to\xi}[g] = \int\limits_{\mathbb{R}^3} g(z)\, e^{i\,z\cdot\xi}\, dz,$$

and set

$$\begin{aligned} h_{il}(z) &:= \text{v.p.}\vartheta_{il}(x,t) = \text{v.p.}\,\frac{\partial^2}{\partial z_i \partial z_l}\,\frac{1}{|z|}\,, \\ \widehat{h}_{il}(\xi) &:= \mathcal{F}_{z\to\xi}(h_{il}(z)), \quad i,l = 1,2,3. \end{aligned}$$

In view of (2.23) and taking into account the relations $\mathcal{F}_{z\to\xi}\delta(z) = 1$ and $\mathcal{F}_{z\to\xi}(|z|^{-1}) = 4\pi|\xi|^{-2}$ (see, e.g., [25]), we easily derive

$$\begin{aligned} \widehat{h}_{il}(\xi) := \mathcal{F}_{z\to\xi}(h_{il}(z)) &= \mathcal{F}_{z\to\xi}\Big(\frac{4\pi\delta_{il}}{3}\delta(z) + \frac{\partial^2}{\partial z_i\,\partial z_l}\,\frac{1}{|z|}\Big) \\ &= \frac{4\pi\delta_{il}}{3} + (-i\xi_i)(-i\xi_l)\mathcal{F}_{z\to\xi}\Big(\frac{1}{|z|}\Big) = \frac{4\pi\delta_{il}}{3} - \frac{4\pi\xi_i\xi_l}{|\xi|^2}\,. \end{aligned}$$

Now, for arbitrary $y \in \overline{\Omega}$ and $\xi \in \mathbb{R}^3 \setminus \{0\}$, due to (2.27), we get

$$\begin{aligned} \mathfrak{S}\,(\mathcal{A})(y,\xi) &= -\frac{1}{4\pi}\,\mathcal{F}_{z\to\xi}\begin{bmatrix} \big[c_{ijlk}(y)\,h_{il}(z)\big]_{3\times 3} & \big[e_{ikl}(y)\,h_{il}(z)\big]_{3\times 1} \\ \big[-e_{lij}(y)\,h_{il}(z)\big]_{1\times 3} & \varepsilon_{il}(y)\,h_{il}(z) \end{bmatrix}_{4\times 4} \\ &= -\frac{1}{4\pi}\begin{bmatrix} \big[c_{ijlk}(y)\,\widehat{h}_{il}(z)\big]_{3\times 3} & \big[e_{ikl}(y)\,\widehat{h}_{il}(z)\big]_{3\times 1} \\ \big[-e_{lij}(y)\,\widehat{h}_{il}(z)\big]_{1\times 3} & \varepsilon_{il}(y)\,\widehat{h}_{il}(z) \end{bmatrix}_{4\times 4} \end{aligned}$$

$$= -\mathbf{b}(y) + \frac{1}{|\xi|^2}\begin{bmatrix} \big[c_{ijlk}(y)\,\xi_i\xi_l\big]_{3\times 3} & \big[e_{lij}(y)\,\xi_l\xi_i\big]_{3\times 1} \\ \big[-e_{ikl}(y)\,\xi_i\xi_l\big]_{1\times 3} & \varepsilon_{il}(y)\,\xi_i\xi_l \end{bmatrix}_{4\times 4}$$

$$= \frac{1}{|\xi|^2}\,A(y,\xi) - \mathbf{b}(y). \tag{2.38}$$

As we can see, the entries of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{A})(y,\xi)$ of the operator $\mathcal{A}$ are even rational homogeneous functions in ξ of order 0. It can easily be verified that both the characteristic function of the singular kernel in (2.27) and the Fourier transform (2.38) satisfy the Tricomi condition, i.e., their integral averages over the unit sphere vanish (cf. [40]).

Denote by ℓ_0 the extension operator by zero from $\Omega = \Omega^+$ onto $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$. It is evident that for the function $U \in H^1(\Omega)$ we have

$$(\mathcal{A}\,U)(y) = (\mathbf{A}\ell_0 U)(y) \ \text{ for } \ y \in \Omega.$$

Introduce the notation

$$(\mathbf{K}\ell_0 U)(y) := (\mathbf{b}(y) - \mathbf{I})\,U(y) + (\mathbf{A}\ell_0 U)(y) \ \text{ for } \ y \in \Omega, \tag{2.39}$$

and for our further purposes we rewrite the third Green's formula (2.30) in a more convenient form

$$[\mathbf{I} + \mathbf{K}]\ell_0 U\,(y) - V(\mathcal{T}^+ U)(y) + W(U^+)(y) = \mathcal{P}(A(x,\partial_x)U)(y), \ \ y \in \Omega, \tag{2.40}$$

where $\mathbf{I}$ is the identity operator.

The relation (2.38) implies that the principal homogeneous symbols of the singular integral operators $\mathbf{K}$ and $\mathbf{I} + \mathbf{K}$ read as

$$\mathfrak{S}(\mathbf{K})(y,\xi) = |\xi|^{-2}A(y,\xi) - I \ \ \forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \tag{2.41}$$

$$\mathfrak{S}(\mathbf{I} + \mathbf{K})(y,\xi) = |\xi|^{-2}A(y,\xi) \ \ \forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{2.42}$$

It is evident that the symbol matrix (2.42) is uniformly strongly elliptic due to (2.4)

$$\operatorname{Re}\big(\mathfrak{S}(\mathbf{I} + \mathbf{K})(y,\xi)\,\zeta,\zeta\big) = |\xi|^{-2}\operatorname{Re}\big(A(y,\xi)\,\zeta,\zeta\big) \geq c\,|\zeta|^2 \tag{2.43}$$
$$\forall\, y \in \overline{\Omega}, \ \ \forall\, \xi \in \mathbb{R}^3 \setminus \{0\}, \ \ \forall\, \zeta \in \mathbb{C}^3,$$

where c is the same positive constant as in (2.4).

From (2.39) it follows that (see, e.g., [3], [26, Theorem 8.6.1]) if $\chi \in X^k$ with integer $k \geqslant r + 2$, then

$$r_{\Omega}\mathbf{K}\,\ell_0 : H^r(\Omega) \longrightarrow H^r(\Omega), \ \ r \geqslant 0, \tag{2.44}$$

since the symbol (2.41) is rational and the operator with the kernel function either $R(x,y)$ or $R^{(1)}(x,y)$ maps $H^r(\Omega)$ into $H^{r+1}(\Omega)$ (cf. [16, Theorem 5.6]). Here and throughout the paper, r_{Ω} denotes the restriction operator to Ω.

Assuming that $U \in H^2(\Omega)$ and applying the differential operator $\mathcal{T}(x,\partial_x)$ to Green's formula (2.40) and using the properties of localized potentials described in Appendix B (see Theorems B.1–B.4) we arrive at the relation:

$$\mathcal{T}^+\mathbf{K}\ell_0 U + (\mathbf{I}-\mathbf{d})(\mathcal{T}^+U) - \mathcal{W}'(\mathcal{T}^+U) + \mathcal{L}(U^+) = \mathcal{T}^+\mathcal{P}(A(x,\partial_x)U) \ \text{ on } \ S, \tag{2.45}$$

where the localized boundary integral operators $\mathcal{W}'$ and $\mathcal{L} := \mathcal{L}^+$ are generated by the localized single- and double-layer potentials and are defined in (B.3) and (B.4), the matrix $\mathbf{d}$ is defined by (B.17), while

$$\mathcal{T}^+\mathbf{K}\ell_0 U \equiv \big\{\mathcal{T}(\mathbf{K}\,\ell_0 U)\big\}^+ \ \text{ on } \ S, \tag{2.46}$$

$$\mathcal{T}^+\mathcal{P}(A(x,\partial_x)U) \equiv \big\{\mathcal{T}\mathcal{P}(A(x,\partial_x)U)\big\}^+ \ \text{ on } \ S. \tag{2.47}$$

2.2. **LBDIE formulation of the Robin type problem and the equivalence theorem.** Let $U \in H^2(\Omega)$ be a solution to the Robin type BVP (2.16), (2.17) with $\psi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega)$. As we have derived above, there hold the relations (2.40) and (2.45), which now can be rewritten in the form

$$[\mathbf{I}+\mathbf{K}]\ell_0 U + W(\Phi) + V(\beta\Phi) = \mathcal{P}(f) + V(\Psi_0) \ \text{ in } \ \Omega, \tag{2.48}$$

$$\mathcal{T}^+\mathbf{K}\ell_0 U + \mathcal{L}(\Phi) + (\mathbf{d}-\mathbf{I})\,\beta\,\Phi + \mathcal{W}'\beta\Phi = \mathcal{T}^+\mathcal{P}(f) + (\mathbf{d}-\mathbf{I})\Psi_0 + \mathcal{W}'(\Psi_0) \ \text{ on } \ S, \tag{2.49}$$

where $\Phi := U^+ \in H^{\frac{3}{2}}(S)$.

One can consider these relations as a LBDIE system with respect to the unknown vector-functions U and Φ. Now we prove the following equivalence theorem.

Theorem 2.2. *Let $\chi \in X^4_{1+}$. The Robin type boundary value problem* (2.16), (2.17) *is equivalent to LBDIE system* (2.48), (2.49) *in the following sense:*

(i) *If a vector-function $U \in H^2(\Omega)$ solves the Robin type BVP* (2.16), (2.17), *then it is unique and the pair $(U,\Phi) \in H^2(\Omega) \times H^{\frac{3}{2}}(S)$ with*

$$\Phi = U^+\,, \tag{2.50}$$

solves the LBDIE system (2.48), (2.49) *and, vice versa;*

(ii) *If a pair $(U,\Phi) \in H^2(\Omega) \times H^{\frac{3}{2}}(S)$ solves the LBDIE system* (2.48), (2.49), *then it is unique and the vector-function U solves the Robin type BVP* (2.16), (2.17), *and relation* (2.50) *holds.*

Proof. (i) The first part of the theorem is trivial and directly follows form the relations (2.40), (2.45), (2.50) and Remark 2.1.

(ii) Now, let a pair $(U,\Phi) \in H^2(\Omega) \times H^{\frac{3}{2}}(S)$ solve the LBDIE system (2.48), (2.49). We apply the differential operator $\mathcal{T}$ to equation (2.48), take its trace on S and compare with (2.49) to obtain

$$\mathcal{T}^+U + \beta\Phi = \Psi_0 \ \text{ on } \ S. \tag{2.51}$$

Further, since $U \in H^2(\Omega)$, we can write the third Green's formula (2.40) which in view of (2.51) can be rewritten as

$$[\mathbf{I}+\mathbf{K}]\,\ell_0 U + V(\beta\Phi) - V(\Psi_0) + W(U^+) = \mathcal{P}(A(x,\partial_x)U) \ \text{ in } \ \Omega. \tag{2.52}$$

From (2.48) and (2.52) it follows that

$$W(U^+ - \Phi) - \mathcal{P}\big(A(x,\partial_x)U - f\big) = 0 \ \text{ in } \ \Omega, \tag{2.53}$$

whence by Lemma 6.4 in [16] we conclude

$$A(x,\partial_x)U = f \ \text{ in } \ \Omega \quad \text{and} \quad U^+ = \Phi \ \text{ on } \ S.$$

Therefore, from (2.51) we get

$$\mathcal{T}^+U + \beta U^+ = \Psi_0 \ \text{ on } \ S. \tag{2.54}$$

Thus U solves the Robin type BVP (2.16), (2.17) and, in addition, equation (2.50) holds.

The uniqueness of a solution to the LBDIE system (2.48), (2.49) in the class $H^2(\Omega) \times H^{\frac{3}{2}}(S)$ directly follows from the above-proven equivalence result and the uniqueness theorem for the Robin type problem (2.16), (2.17) (see Remark 2.1). □

3. Invertibility of the LBDIO Corresponding to the Robin Type BVP

From Theorem 2.2 it follows that the LBDIE system (2.48), (2.49) with a special right-hand side is uniquely solvable in the class $H^2(\Omega, A) \times H^{3/2}(S)$. Here, our main goal is to investigate Fredholm properties of the localized boundary-domain integral operator generated by the left-hand side expressions in (2.48), (2.49) in appropriate functional spaces.

To this end, let us consider the LBDIE system for the unknown pair $(U,\Phi) \in H^2(\Omega) \times H^{3/2}(S)$,

$$(\mathbf{I}+\mathbf{K})\ell_0 U \ + W(\Phi) + V(\beta\Phi) = F_1 \ \text{ in } \ \Omega, \tag{3.1}$$

$$\mathcal{T}^+\mathbf{K}\ell_0 U + \mathcal{L}(\Phi) + (\mathbf{d}-\mathbf{I})\beta\Phi + \mathcal{W}'(\beta\Phi) = F_2 \ \text{ on } \ S, \tag{3.2}$$

where $F_1 \in H^2(\Omega)$ and $F_2 \in H^{1/2}(S)$.

Introduce the notation

$$\mathbf{B} := \mathbf{I} + \mathbf{K}. \tag{3.3}$$

In view of (2.42), the principal homogeneous symbol matrix of the operator $\mathbf{B}$ reads as

$$\mathfrak{S}(\mathbf{B})(y,\xi) = |\xi|^{-2}A(y,\xi) \ \text{ for } \ y \in \overline{\Omega}, \ \ \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{3.4}$$

The entries of the matrix $\mathfrak{S}(\mathbf{B})(y,\xi)$ are even rational homogeneous functions of order 0 in ξ. Moreover, due to (2.4), the matrix $\mathfrak{S}(\mathbf{B})(y,\xi)$ is uniformly strongly elliptic,

$$\operatorname{Re}\big(\mathfrak{S}(\mathbf{B})(y,\xi)\zeta,\zeta\big) \geq c\,|\zeta|^2 \ \text{ for all } \ y \in \overline{\Omega}, \ \ \xi \in \mathbb{R}^3 \setminus \{0\} \ \text{ and } \ \zeta \in \mathbb{C}^3.$$

Consequently, **B** is a uniformly strongly elliptic pseudodifferential operator of zero order (i.e., a singular integral operator) and the partial indices of factorization of the symbol (3.4) are equal to zero (cf. Lemma 1.20 in [12]).

Now we present some auxiliary material needed for our further analysis. Let $\widetilde{y} \in \partial\Omega$ be some fixed point and consider the frozen symbol $\mathfrak{S}(\mathbf{B})(\widetilde{y},\xi) \equiv \mathfrak{S}(\widetilde{\mathbf{B}})(\xi)$, where $\widetilde{\mathbf{B}}$ denotes the operator $\mathbf{B}$ written in a chosen local coordinate system. Further, let $\widehat{\widetilde{\mathbf{B}}}$ denote the pseudodifferential operator with the symbol

$$\widehat{\mathfrak{S}}(\widetilde{\mathbf{B}})(\xi',\xi_3) := \mathfrak{S}(\widetilde{\mathbf{B}})\big((1+|\xi'|)\omega,\xi_3\big)$$
$$\text{with } \omega = \frac{\xi'}{|\xi'|}\,, \ \ \xi=(\xi',\xi_3), \ \ \xi'=(\xi_1,\xi_2).$$

The principal homogeneous symbol matrix $\mathfrak{S}(\widetilde{\mathbf{B}})(\xi)$ of the operator $\widehat{\widetilde{\mathbf{B}}}$ can be factorized with respect to the variable ξ_3,

$$\mathfrak{S}(\widetilde{\mathbf{B}})(\xi) = \mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi)\,\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})(\xi), \tag{3.5}$$

where

$$\mathfrak{S}^{(\pm)}(\widetilde{\mathbf{B}})(\xi) = \frac{1}{\xi_3 \pm i\,|\xi'|}\,\widetilde{A}^{(\pm)}(\xi',\xi_3),$$

$\widetilde{A}^{(\pm)}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors of the first order in ξ_3 of the positive definite polynomial symbol matrix $\widetilde{A}(\xi',\xi_3) \equiv \widetilde{A}(\widetilde{y},\xi',\xi_3)$ (see Theorem 1 in [23], Theorem 1.33 in [45], Theorem 1.4 in [24]), i.e.

$$\widetilde{A}(\xi^{'},\xi_3) = \widetilde{A}^{(-)}(\xi',\xi_3)\,\widetilde{A}^{(+)}(\xi',\xi_3) \tag{3.6}$$

with $\det\widetilde{A}^{(+)}(\xi',\tau) \neq 0$ for $\operatorname{Im}\tau > 0$ and $\det\widetilde{A}^{(-)}(\xi',\tau) \neq 0$ for $\operatorname{Im}\tau < 0$. Moreover, the entries of the matrices $\widetilde{A}^{(\pm)}(\xi',\xi_3)$ are homogeneous functions in $\xi=(\xi',\xi_3)$ of order 1.

Denote by $a^{(\pm)}(\xi')$ the coefficients of ξ_3^4 in the determinants $\det\widetilde{A}^{(\pm)}(\xi',\xi_3)$. Evidently,

$$a^{(-)}(\xi')\,a^{(+)}(\xi') = \det\widetilde{A}(0,0,1) > 0 \ \text{ for } \ \xi' \neq 0. \tag{3.7}$$

It is easy to see that the inverse factor-matrices $[\widetilde{A}^{(\pm)}(\xi',\xi_3)]^{-1}$ have the following structure:

$$\big[\widetilde{A}^{(\pm)}(\xi',\xi_3)\big]^{-1} = \frac{1}{\det\widetilde{A}^{(\pm)}(\xi',\xi_3)}\,\big[p_{ij}^{(\pm)}(\xi',\xi_3)\big]_{4\times 4}, \tag{3.8}$$

where $[p_{ij}^{(\pm)}(\xi',\xi_3)]_{4\times 4}$ is the matrix of co-factors corresponding to the matrix $\widetilde{A}^{(\pm)}(\xi',\xi_3)$. They can be written in the form

$$p_{ij}^{(\pm)}(\xi',\xi_3) = c_{ij}^{(\pm)}(\xi')\,\xi_3^3 + b_{ij}^{(\pm)}(\xi')\,\xi_3^2 + d_{ij}^{(\pm)}(\xi')\,\xi_3 + e_{ij}^{(\pm)}(\xi') \tag{3.9}$$

with $c_{ij}^{(\pm)}$, $b_{ij}^{(\pm)}$, $d_{ij}^{(\pm)}$, and $e_{ij}^{(\pm)}$, $i,j=1,2,3,4$, being homogeneous functions in ξ' of order 0, 1, 2 and 3, respectively.

Denote by Π^+ the Cauchy type integral operator

$$\Pi^+(f)(\xi) = \frac{i}{2\pi} \lim_{t\to 0+} \int\limits_{-\infty}^{+\infty} \frac{h(\xi',\eta_3)\,d\eta_3}{\xi_3 + i\,t - \eta_3}, \quad \xi = (\xi',\xi_3), \quad \xi' = (\xi_1,\xi_2), \tag{3.10}$$

which is well defined for any $\xi \in \mathbb{R}^3$ for a bounded smooth function $h(\xi',\,\cdot\,)$ satisfying the relation $h(\xi',\eta_3) = O(1+|\eta_3|)^{-\nu}$ with some $\nu > 0$.

The following lemma holds (see [22]).

Lemma 3.1. *Let $\chi \in X^k_{1+}$ with integer $k \geqslant s+2$ and let ℓ_0 be the extension operator by zero from $\mathbb{R}^3_+$ onto the half-space $\mathbb{R}^3_-$. The operator*

$$r_{_{\mathbb{R}^3_+}} \widehat{\widetilde{\mathbf{B}}} \ell_0 : H^s(\mathbb{R}^3_+) \longrightarrow H^s(\mathbb{R}^3_+)$$

is invertible for all $s \geq 0$, where $r_{_{\mathbb{R}^3_+}}$ is the restriction operator to the half-space $\mathbb{R}^3_+$.

Moreover, for $f \in H^s(\mathbb{R}^3_+)$ with $s \geq 0$, the unique solution of the equation

$$r_{_{\mathbb{R}^3_+}} \widehat{\widetilde{\mathbf{B}}} \ell_0 U = f, \tag{3.11}$$

can be represented in the form

$$U_+ := \ell_0 u = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell f)\Big)\Big\}, \tag{3.12}$$

where $\ell f \in H^s(\mathbb{R}^3)$ is an arbitrary extension of f onto the whole space $\mathbb{R}^3$.

Lemma 3.2. *Let the factor matrix $\widetilde{A}^{(+)}(\xi',\tau)$ be as in (3.6), and let $a^{(+)}$ and $c^{(+)}_{ij}$ be as in (3.7) and (3.9), respectively. Then the following equality holds*

$$\frac{1}{2\pi i} \int\limits_{\gamma^-} \big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau = \frac{1}{a^{(+)}(\xi')}\,\big[c^{(+)}_{ij}(\xi')\big]_{4\times 4}, \tag{3.13}$$

and

$$\det\big[c^{(+)}_{ij}(\xi')\big]_{4\times 4} \neq 0 \ \ \textit{for} \ \ \xi' \neq 0. \tag{3.14}$$

Here γ^- is a contour in the lower complex half-plane enclosing all roots of the polynomial $\det \widetilde{A}^{(+)}(\xi',\tau)$ *with respect to τ.*

It is well known that the differential operator $\mathcal{T}(x,\partial_x)$ covers the operator $A(x,\partial_x)$ on the boundary S (see, e.g., [1], [11], [41], [48]), i.e., the problem

$$\widetilde{A}\Big(\xi', i\,\frac{d}{dt}\Big)\,v(\xi',t) = 0, \quad t \in \mathbb{R}_+ = (0,+\infty), \tag{3.15}$$

$$\widetilde{\mathcal{T}}\Big(\xi', i\,\frac{d}{dt}\Big)\,v(\xi',t)\Big|_{t=0} = 0 \tag{3.16}$$

has only the trivial solution in the Schwartz space $\mathcal{S}(\mathbb{R}_+)$ of infinitely smooth, rapidly decreasing vector-functions at infinity. Here, $\widetilde{A}(\xi',\xi_3) := A(\widetilde{y},\xi',\xi_3)$ and $\widetilde{\mathcal{T}}(\xi',\xi_3) := \mathcal{T}(\widetilde{y},\xi',\xi_3)$ correspond, respectively, to the "frozen" differential and co-normal operators at the point $\widetilde{y} \in \partial\Omega$.

The above covering condition implies the following assertion.

Lemma 3.3. *Let* γ^- *be as in Lemma* 3.2. *The matrix*

$$\int\limits_{\gamma^-} \widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau \tag{3.17}$$

is non-singular for all $\xi' \neq 0$.

Proof. Let us consider the following matrix:

$$\int\limits_{\gamma^-} e^{-i\tau t}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau, \quad 0<t<\infty, \tag{3.18}$$

and denote by $v^{(1)}(\xi',t)$, $v^{(2)}(\xi',t)$, $v^{(3)}(\xi',t)$, and $v^{(4)}(\xi',t)$, the columns of the matrix (3.18).

Clearly, $v^{(k)}(\xi',\,\cdot\,) \in \mathcal{S}(\mathbb{R}_+)$, $k=1,2,3,4$.

First we show that $v^{(k)}(\xi',\,\cdot\,)$, $k=\overline{1,4}$, are linearly independent solutions of equation (3.15). Indeed, by direct differentiation it can be easily seen that the vector-functions $v^{(k)}(\xi',t)$, $k=\overline{1,4}$, solve the equation

$$\widetilde{A}^{(+)}\Big(\xi',i\,\frac{d}{dt}\Big)v(\xi',t)=0, \quad 0<t<\infty. \tag{3.19}$$

In view of the decomposition

$$\widetilde{A}\Big(\xi',i\,\frac{d}{dt}\Big)=\widetilde{A}^{(-)}\Big(\xi',i\,\frac{d}{dt}\Big)\widetilde{A}^{(+)}\Big(\xi',i\,\frac{d}{dt}\Big), \tag{3.20}$$

it follows that $v^{(k)}(\xi',t)$, $k=\overline{1,4}$, are solutions of equation (3.15).

Now let us show that the vector-functions $v^{(k)}(\xi',\,\cdot\,)$, $k=\overline{1,4}$, are linearly independent. Assume that for some scalar constants α_k, $k=\overline{1,4}$, the equality

$$\alpha_1\,v^{(1)}(\xi',t)+\alpha_2\,v^{(2)}(\xi',t)+\alpha_3\,v^{(3)}(\xi',t)+\alpha_4\,v^{(4)}(\xi',t)=0 \tag{3.21}$$

holds. Note that the matrix-function (3.18) is continuous at $t=0$. Therefore from (3.21) by passing to the limit, as $t\to 0$, we obtain the following linear algebraic system of equations with respect to $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^\top$,

$$\Big(\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)\alpha=0. \tag{3.22}$$

Due to Lemma 3.2,

$$\det\Big(\int\limits_{\gamma^-}\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)\neq 0 \ \text{ for all } \ \xi'\neq 0,$$

and consequently $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^\top=0$, implying that $v^{(k)}(\xi',\,\cdot\,)$, $k=\overline{1,4}$, are linearly independent solutions of equation (3.15).

Further, let us consider an arbitrary solution of equation (3.15) belonging to the class $\mathcal{S}(\mathbb{R}_+)$,

$$v(\xi',t)=\sum_{k=1}^{4}a_k\,v^{(k)}(\xi',t), \tag{3.23}$$

where a_1, a_2, a_3, a_4 are the scalar constants. If (3.23) satisfies in addition the condition (3.16), then due to the covering condition it should be identical zero. Substituting (3.23) into (3.16), we arrive at the following system of linear algebraic equations with respect to $a=(a_1,a_2,a_3,a_4)^\top$:

$$\Bigg(\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Bigg)a=0. \tag{3.24}$$

Since this system should possess only the trivial solution, we conclude that

$$\det\Bigg(\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Bigg)\neq 0\ \text{ for all }\ \xi'\neq 0,$$

which completes the proof. □

Now, with the above auxiliary results in hand, we can investigate the invertibility of the localized boundary-domain integral operator generated by the left-hand side expressions in the system (3.1), (3.2). We denote this operator by $\mathfrak{R}$,

$$\mathfrak{R}:=\begin{bmatrix} r_{\Omega}\mathbf{B}\ell_0 & -r_{\Omega}W+r_{\Omega}V\beta\\ \mathcal{T}^{+}\mathbf{K}\ell_0 & \mathcal{L}+(\mathbf{d}-\mathbf{I})+\mathcal{W}'\beta\end{bmatrix}_{8\times 8}.$$

Let us introduce the following boundary operators depending on the parameter $t\in[0,1]$,

$$\begin{aligned}\mathcal{T}_t=\mathcal{T}_t(x,\partial_x)&:=(1-t)I\partial_n+t\mathcal{T}(x,\partial_x),\\ \mathcal{M}_t=\mathcal{M}_t(x,\partial_x)&:=(1-t)I\partial_n+t\mathcal{M}(x,\partial_x).\end{aligned} \tag{3.25}$$

Now we can prove the following assertion.

Theorem 3.4. *Let a localizing function* $\chi\in X^{\infty}_{1+}$, $r\geq 1$, *and the conditions*

$$\det\widetilde{\mathcal{T}}_t(\xi',-i\,|\xi'|)\neq 0,\quad \det\widetilde{\mathcal{M}}_t(\,\xi',-i\,|\xi'|)\neq 0, \tag{3.26}$$

be satisfied for all $\xi'\neq 0$ *and for all* $t\in(0,1]$, *where the matrices* $\widetilde{\mathcal{T}}_t(\xi',\xi_3)$ *and* $\widetilde{\mathcal{M}}_t(\xi',\xi_3)$ *are defined as follows:*

$$\begin{aligned}\widetilde{\mathcal{T}}_t(\xi',\xi_3)&:=(1-t)\xi_3I+t\widetilde{\mathcal{T}}(\xi',\xi_3),\\ \widetilde{\mathcal{M}}_t(\xi',\xi_3)&:=(1-t)\xi_3I+t\widetilde{\mathcal{M}}(\xi',\xi_3).\end{aligned} \tag{3.27}$$

Then the operator

$$\mathfrak{R}:H^{r+1}(\Omega)\times H^{r+1/2}(S)\longrightarrow H^{r+1}(\Omega)\times H^{r-1/2}(S) \tag{3.28}$$

is invertible.

Proof. We prove the theorem in four steps, where we show that

Step 1: the operator $r_{_\Omega}\mathbf{B}\ell_0\ :\ H^s(\Omega)\to H^s(\Omega)$ for $s\geq 0$ is Fredholm with zero index;
Step 2: the operator $\mathfrak{R}$ in (3.28) is Fredholm;
Step 3: $\operatorname{Ind}\mathfrak{R}=0$;
Step 4: the operator $\mathfrak{R}$ is invertible.

Step 1. Since (3.4) is a rational function in ξ, we can apply the theory of pseudodifferential operators with the symbol satisfying the transmission conditions (see [25], [3], [44], [45], [4]). With the help of the local principal (see [2] and Lemma 23.9 in [25]) and the above Lemma 3.1 we can deduce that the operator

$$\mathcal{B}:=r_{_\Omega}\mathbf{B}\,\ell_0:H^s(\Omega)\longrightarrow H^s(\Omega)$$

is Fredholm for all $s\geq 0$.

To show that $\operatorname{Ind}\mathcal{B}=0$, we use the fact that the operators $\mathcal{B}$ and $\mathcal{B}_t=r_{_\Omega}(\mathbf{I}+t\,\mathbf{K})\ell_0$, where $t\in[0,1]$, are homotopic. Note that $\mathcal{B}=\mathcal{B}_1$. The principal homogeneous symbol of the operator $\mathcal{B}_t$ has the form

$$\mathfrak{S}(\mathcal{B}_t)(y,\xi)=I+t\,\mathfrak{S}(\mathbf{K})(y,\xi)=(1-t)I+t\mathfrak{S}(\mathbf{B})(y,\xi).$$

It is easy to see that the operator $\mathcal{B}_t$ is uniformly strongly elliptic,

$$\operatorname{Re}\big(\mathfrak{S}(\mathcal{B}_t)(y,\xi)\zeta,\zeta\big)=(1-t)|\zeta|^2+t\operatorname{Re}\big(\mathfrak{S}(\mathbf{B})(y,\xi)\zeta,\zeta\big)\geq c_1\,|\zeta|^2$$

for all $y\in\overline{\Omega}$, $\xi\neq 0$, $\zeta\in\mathbb{C}^3$, and $t\in[0,1]$, $c_1=\min\{1,c\}$, where c is the constant involved in (2.4).

Since $\mathfrak{S}(\mathcal{B}_t)(y,\xi)$ is rational, even and homogeneous of order zero in ξ, as above, we again conclude that the operator

$$\mathcal{B}_t:H^s(\Omega)\longrightarrow H^s(\Omega)$$

is Fredholm for all $s\geq 0$ and for all $t\in[0,1]$. Therefore $\operatorname{Ind}\mathcal{B}_t$ is the same for all $t\in[0,1]$. On the other hand, due to the equality $\mathcal{B}_0=r_{_\Omega}I$, we get

$$\operatorname{Ind}\mathcal{B}=\operatorname{Ind}\mathcal{B}_1=\operatorname{Ind}\mathcal{B}_t=\operatorname{Ind}\mathcal{B}_0=0.$$

Step 2. To investigate Fredholm properties of the operator $\mathfrak{R}$ we apply the local principle (cf. e.g., [25], § 19 and § 22). Due to this principle, we have to show that the operator $\mathfrak{R}$ is locally Fredholm at an arbitrary "frozen" interior point $\widetilde{y}\in S$, and secondly that the so-called generalized *Šapiro–Lopatinskiĭ condition* for the operator $\mathfrak{R}$ holds at an arbitrary "frozen" point $\widetilde{y}\in S$. To obtain the explicit form of this condition we proceed as follows. Let $\mathcal{U}$ be a neighborhood of a fixed point $\widetilde{y}\in\overline{\Omega}$ and let $\widetilde{\psi}_0,\widetilde{\varphi}_0\in\mathcal{D}(\mathcal{U})$ be infinitely differentiable scalar functions such that

$$\operatorname{supp}\widetilde{\psi}_0\cap\operatorname{supp}\widetilde{\varphi}_0\neq\varnothing,\quad \widetilde{y}\in\operatorname{supp}\,\widetilde{\psi}_0\cap\operatorname{supp}\widetilde{\varphi}_0,$$

and consider the operator $\widetilde{\psi}_0\mathfrak{R}\,\widetilde{\varphi}_0$. We consider separately two possible cases: $\widetilde{y}\in\Omega$ and $\widetilde{y}\in S$.

Case 1). Let $\widetilde{y}\in\Omega$. Then we can choose a neighborhood $\mathcal{U}_j$ of the point $\widetilde{y}$ such that $\overline{\mathcal{U}}\subset\Omega$. Therefore the operator $\widetilde{\psi}_0\mathfrak{R}\,\widetilde{\varphi}_0$ has the same Fredholm properties as the operator $\widetilde{\psi}_0 \mathbf{B}\, \widetilde{\varphi}_0$ (see the similar arguments in the proof of Theorem 22.1 in [25]). Then owing to Step 1, we conclude that $\widetilde{\psi}_0 \mathfrak{R}\, \widetilde{\varphi}_0$ is the locally Fredholm operator at interior points of Ω.

Case 2). Now let $\widetilde{y} \in S$. Then at this point we have to "froze" the operator $\widetilde{\psi}_0\, \mathfrak{R}\, \widetilde{\varphi}_0$, which means that we can choose a neighborhood $\mathcal{U}$ of the point $\widetilde{y}$ sufficiently small such that at the local co-ordinate system with the origin at the point $\widetilde{y}$ and the third axis coinciding with the normal vector at the point $\widetilde{y} \in S$, the following decomposition

$$\widetilde{\psi}_0 \mathfrak{R}\, \widetilde{\varphi}_0 = \widetilde{\psi}_0 \Big(\widehat{\widetilde{\mathfrak{R}}} + \widetilde{\mathbf{N}} + \widetilde{\mathbf{M}} \Big)\, \widetilde{\varphi}_0, \tag{3.29}$$

holds, where $\widetilde{\mathbf{N}}$ is a bounded operator with a small norm

$$\widetilde{\mathbf{N}}\ : H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2) \longrightarrow H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2),$$

while $\widetilde{\mathbf{M}}$ is a bounded operator

$$\widetilde{\mathbf{M}}\ : H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2) \longrightarrow H^{r+2}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2);$$

the operator $\widehat{\widetilde{\mathfrak{R}}}$ is defined in the upper half-space $\mathbb{R}^3_+$ as follows

$$\widehat{\widetilde{\mathfrak{R}}} := \begin{bmatrix} r_+ \widehat{\widetilde{\mathbf{B}}} \ell_0 & r_+ \widehat{\widetilde{W}} \\ (\widehat{\widetilde{\mathcal{T}}^+ \widetilde{\mathbf{K}}}) \ell_0 & \widehat{\widetilde{\mathcal{L}}} \end{bmatrix} \ \text{ with } \ r_+ = r_{_{\mathbb{R}^3_+}}$$

and possesses the following mapping property

$$\widehat{\widetilde{\mathfrak{R}}}\ : H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2) \longrightarrow H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2). \tag{3.30}$$

The operators with "hat" involved in the expression of $\widehat{\widetilde{\mathfrak{R}}}$, are defined as follows: for the operator $\widetilde{G}$, the operator $\widehat{\widetilde{G}}$ denotes that in $\mathbb{R}^n$ $(n = 2, 3)$ constructed by the symbol

$$\widehat{\mathfrak{S}}(\widetilde{G})(\xi) := \mathfrak{S}(\widetilde{G})\big((1 + |\xi'|)\omega, \xi_3\big) \ \text{ if } \ n = 3$$

and

$$\widehat{\mathfrak{S}}(\widetilde{G})(\xi) := \mathfrak{S}(\widetilde{G})\big((1 + |\xi'|)\omega\big) \ \text{ if } \ n = 2,$$

where $\omega = \frac{\xi'}{|\xi'|}$, $\xi = (\xi', \xi_n)$, $\xi' = (\xi_1, \dots, \xi_{n-1})$.

The generalized Šapiro–Lopatinskiĭ condition is related to the invertibility of the operator (3.30). Indeed, let us write the system corresponding to the operator $\widehat{\widetilde{\mathfrak{R}}}$:

$$r_+ \widehat{\widetilde{\mathbf{B}}} \ell_0 \widetilde{U} + r_+ \widehat{\widetilde{W}} \widetilde{\Phi} = \widetilde{F}_1 \ \text{ in } \ \mathbb{R}^3_+\,, \tag{3.31}$$

$$(\widehat{\widetilde{\mathcal{T}}^+ \widetilde{\mathbf{K}}}) \ell_0 \widetilde{U} + \widehat{\widetilde{\mathcal{L}}}\, \widetilde{\Phi} = \widetilde{F}_2 \ \text{ on } \ \mathbb{R}^2\,, \tag{3.32}$$

where $\widetilde{F}_1 \in H^2(\mathbb{R}^3_+)$, $\widetilde{F}_2 \in H^{1/2}(\mathbb{R}^2)$.

Note that the operator $r_+\widehat{\widetilde{\mathbf{B}}}\ell_0$ is a singular integral operator with even rational elliptic principal homogeneous symbol. Then due to Lemma 3.1, the operator

$$r_+\widehat{\widetilde{\mathbf{B}}}\ell_0\ : H^{r+1}(\mathbb{R}^3_+) \longrightarrow H^{r+1}(\mathbb{R}^3_+)$$

is invertible. Therefore from equation (3.31) we can define $\widetilde{U}$. (3.31)

$$\ell_0\widetilde{U} = \ell_0\big[r_+\widehat{\widetilde{\mathbf{B}}}\ell_0\big]^{-1}\widetilde{f} = \\ = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\ell\widetilde{f})\Big)\Big\}, \quad (3.33)$$

where $\widetilde{f} = \widetilde{F}_1 - r_+\widehat{\widetilde{W}}\widetilde{\Phi}$, ℓ is an extension operator from $\mathbb{R}^3_+$ to $\mathbb{R}^3$ preserving the function space, while ℓ_0 is an extension operator $\mathbb{R}^3_+$ to $\mathbb{R}^3_-$ by zero, the operator Π^+ involved in (3.33) is defined in (3.10); here $\widehat{\mathfrak{S}}^{(\pm)}(\cdot)$ denote the so-called "plus" and "minus" factors in the factorization of the corresponding symbol $\widehat{\mathfrak{S}}(\cdot)$ with respect to the variable ξ_3. Note that the function $\ell_0\widetilde{U}$ in (3.33) does not depend on the extension operator ℓ.

Substituting (3.33) into (3.32), we get the following pseudodifferential equation with respect to the unknown function $\widetilde{\Phi}$:

$$-(\widehat{\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}})\mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\widehat{\widetilde{W}}\widetilde{\Phi})\Big)\Big\} + \widehat{\widetilde{\mathcal{L}}}\,\widetilde{\Phi} \\ = \widetilde{F} \ \text{ on } \ \mathbb{R}^2, \quad (3.34)$$

where

$$\widetilde{F} = \widetilde{F}_2 - \widehat{\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}}\ell_0\,\big[r_+\widehat{\widetilde{\mathbf{B}}}\ell_0\big]^{-1}\widetilde{F}_1.$$

It can be shown that

$$\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}\,v\,(y') = \Big[\mathcal{F}^{-1}_{\xi\to y}\big[\widetilde{\mathcal{T}}(-i\xi)\mathfrak{S}(\widetilde{\mathbf{K}})(\xi)\mathcal{F}(v)(\xi)\big]\Big]_{y_3=0+} \\ = \mathcal{F}^{-1}_{\xi'\to y'}\Big[\Pi'\big[\widetilde{\mathcal{T}}(-i\xi)\mathfrak{S}(\widetilde{\mathbf{K}})(\xi)\mathcal{F}(v)(\xi)\big]\Big], \quad (3.35)$$

where the operator Π' is defined as follows:

$$\Pi'(g)(\xi') = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} g(\xi',\xi_3)\,d\xi_3 \ \text{ for } \ g\in L_1(\mathbb{R}^3)$$

while (for details see [21], Appendix C)

$$\Pi'(g)(\xi') = \lim_{x_3\to 0+} r_+\mathcal{F}^{-1}_{\xi_3\to x_3}\,[g(\xi',\xi_3)] = -\frac{1}{2\pi}\int\limits_{\gamma^-} g(\xi',\zeta)\,d\zeta,$$

if the following conditions hold:

(i) $g(\xi',\xi_3)$ is rational in ξ_3 and the denominator does not vanish for nonzero real $\xi = (\xi',\xi_3) \in \mathbb{R}^3 \setminus \{0\}$,

(ii) $g(\xi',\xi_3)$ is homogeneous of order $m \in \mathbb{Z} := \{0,\pm1,\pm2,\dots\}$ in $\xi = (\xi',\xi_3)$, and

(iii) $g(\xi', \xi_3)$ is infinitely differentiable with respect to real $\xi = (\xi', \xi_3)$ for $\xi' \neq 0$,

and γ^- is a contour in the lower complex half-plane orientated counter-clockwise and enclosing all the poles of the rational function g.

It is clear that if $g(\xi', \zeta)$ is analytic with respect to ζ in the lower half-plane ($\operatorname{Im}\zeta < 0$), then

$$\Pi'(g)(\xi') = 0 \ \text{ for all } \ \xi'.$$

Further, we can represent the double-layer potential as

$$W(\varphi) = \mathbf{P}\big(\mathcal{M}^\top(\Phi \otimes \delta_S)\big), \tag{3.36}$$

where the distribution $\mathcal{M}^\top(\Phi \otimes \delta_S)$ is supported on the boundary S and is defined by the relation

$$\big\langle \mathcal{M}^\top(\Phi \otimes \delta_S), \psi)\big\rangle_{\mathbb{R}^3} := \langle \Phi, \mathcal{M}\psi\rangle_S \ \ \forall\, \psi \in \mathcal{D}(\mathbb{R}^3).$$

In the case if $S = \mathbb{R}^2$ is the boundary of the half-space, the distribution $\widetilde{\Phi} \otimes \delta_S$ is the direct product $\widetilde{\Phi} \otimes \delta_S = \widetilde{\Phi}(x_1, x_2) \times \delta(x_3)$ and in view of (3.35), we can write

$$\begin{aligned}
&(\widehat{\widetilde{\mathcal{T}}^+\widetilde{\mathbf{K}}})\, \mathcal{F}^{-1}_{\xi\to\widetilde{x}}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})(\xi)\big]^{-1}\Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathcal{F}(\widehat{\widetilde{W}}\widetilde{\Phi})\Big)(\xi)\Big\}\,(\widetilde{y}')\\
&\qquad = \mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\Big\{\Pi'\Big[\widehat{\widetilde{\mathcal{T}}}\,\widehat{\mathfrak{S}}(\widetilde{\mathbf{K}})\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\\
&\qquad\qquad \times \Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\widehat{\mathfrak{S}}(\widetilde{\mathbf{P}})\widehat{\widetilde{\mathcal{M}}}^\top\Big)\Big](\xi')\mathcal{F}_{\widetilde{x}'\to\xi'}\widetilde{\Phi}\Big\}.
\end{aligned} \tag{3.37}$$

By virtue of the above relations, equation (3.34) can be rewritten in the form

$$\mathcal{F}^{-1}_{\xi'\to y'}\,\big[\,\widehat{e}\,(\xi')\,\mathcal{F}(\widetilde{\Phi})(\xi')\big] = \widetilde{F}(y') \ \text{ on } \ \mathbb{R}^2, \tag{3.38}$$

where

$$\widehat{e}(\xi') = e\Big((1 + |\xi'|)\,\omega\Big), \ \ \omega = \frac{\xi'}{|\xi'|} \tag{3.39}$$

with $e(\,\cdot\,)$ being a homogeneous matrix function of order 1 given by the equality

$$\begin{aligned}
e(\xi') = &-\Pi'\Big\{\widetilde{\mathcal{T}}\,\mathfrak{S}(\widetilde{\mathbf{K}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^+\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\widetilde{\mathcal{M}^\top}\Big)\Big\}(\xi')\\
&+ \mathfrak{S}(\widetilde{\mathcal{L}})(\xi') \ \ \forall\, \xi' \neq 0.
\end{aligned} \tag{3.40}$$

If $\det e(\xi')$ is different from zero for all $\xi' \neq 0$, then $\det \widehat{e}(\xi') \neq 0$ for all $\xi' \in \mathbb{R}^2$, and the corresponding pseudodifferential operator

$$\widehat{\mathbf{E}} : H^s(\mathbb{R}^2) \longrightarrow H^{s-1}(\mathbb{R}^2),$$

generated by the left hand-side expression in (3.38), is invertible for all $s \in \mathbb{R}$. In particular, it follows that the system of equations (3.31), (3.32) is uniquely solvable with respect to $(\widetilde{U}, \widetilde{\Phi})$ in the space $H^2(\mathbb{R}^3_+) \times H^{3/2}(\mathbb{R}^2)$ for arbitrary right-hand sides $(\widetilde{F}_1, \widetilde{F}_2) \in H^2(\mathbb{R}^3_+) \times H^{1/2}(\mathbb{R}^2)$. Consequently, the operator $\widehat{\widehat{\mathfrak{R}}}$ in (3.30) is invertible, which implies that the operator (3.29) possesses left and right regularizers. In turn, this yields that the operator (3.28) possesses left and right regularizers, as well. Thus the operator (3.28) is Fredholm if the matrix

$$e(\xi') = -\Pi'\Big\{\widetilde{\mathcal{T}}\,\mathfrak{S}(\widetilde{\mathbf{K}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1} \times \Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^{\top}\Big)\Big\}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}})(\xi') \tag{3.41}$$

is non-singular for all $\xi' \neq 0$. This condition is called the *Šapiro–Lopatinskiĭ condition* (cf. [25], Theorems 12.2 and 23.1, and also formulas (12.27), (12.25)). Let us show that in our case the Šapiro–Lopatinskiĭ condition holds. To this end, let us note that the principal homogeneous symbols $\mathfrak{S}(\widetilde{\mathbf{K}})$, $\mathfrak{S}(\widetilde{\mathbf{B}})$, $\mathfrak{S}(\widetilde{\mathbf{P}})$, and $\mathfrak{S}(\widetilde{\mathcal{L}})$ of the operators $\mathbf{K}$, $\mathbf{B}$, $\mathbf{P}$, and $\mathcal{L}$ in the chosen local co-ordinate system involved in formula (3.41) read as:

$$\begin{gathered}
\mathfrak{S}(\widetilde{\mathbf{K}})(\xi) = |\xi|^{-2}\widetilde{A}(\xi) - I,\\
\mathfrak{S}(\widetilde{\mathbf{B}})(\xi) = |\xi|^{-2}\widetilde{A}(\xi), \quad \mathfrak{S}(\widetilde{\mathbf{P}})(\xi) = -|\xi|^{-2}I,\\
\mathfrak{S}(\widetilde{\mathcal{L}})(\xi') = \frac{1}{2|\xi'|}\,\widetilde{\mathcal{T}}(\xi', -i|\xi'|)\,\widetilde{\mathcal{M}}^{\top}(\xi', -i|\xi'|),\\
\xi = (\xi', \xi_3), \ \ \xi' = (\xi_1, \xi_2).
\end{gathered} \tag{3.42}$$

Recall that the matrices $\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})$ and ${}^{(-)}(\widetilde{\mathbf{B}})$ are the so-called "plus" and "minus" factors in the factorization of the symbol $\mathfrak{S}(\widetilde{\mathbf{B}})$ with respect to the variable ξ_3.

We rewrite (3.40) in the form

$$\begin{aligned}
e(\xi') &= -\Pi'\Big\{\widetilde{\mathcal{T}}\,\Big(\mathfrak{S}(\widetilde{\mathbf{B}}) - I\Big)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\\
&\qquad \times \Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^{\top}\Big)\Big\}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}})(\xi')\\
&= e_1(\xi') + e_2(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}})(\xi'),
\end{aligned} \tag{3.43}$$

where $\mathfrak{S}(\widetilde{\mathcal{L}})(\xi')$ is defined in (3.42) and

$$e_1(\xi') = -\Pi'\Big\{\widetilde{\mathcal{T}}\,\mathfrak{S}(\widetilde{\mathbf{B}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1} \times \Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^{\top}\Big)\Big\}(\xi'), \tag{3.44}$$

$$e_2(\xi') = \Pi'\Big\{\widetilde{\mathcal{T}}\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^{\top}\Big)\Big\}(\xi'). \tag{3.45}$$

By direct calculations we get

$$\begin{aligned}
&\Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^{\top}\Big)(\xi')\\
&= \frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{-\infty}^{+\infty}\frac{\big([\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\big)(\xi', \eta_3)\widetilde{\mathcal{M}}^{\top}(-i\xi', -i\eta_3)}{\xi_3 + i\,t - \eta_3}\,d\eta_3
\end{aligned}$$

$$= -\frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{-\infty}^{+\infty}\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\eta_3)\widetilde{\mathcal{M}}^\top(-i\xi',-i\eta_3)}{(\xi_3+i\,t-\eta_3)(|\xi'|^2+\eta_3^2)}\,d\eta_3$$

$$= \frac{i}{2\pi}\lim_{t\to 0+}\int\limits_{\gamma^-}\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\tau)\widetilde{\mathcal{M}}^\top(-i\xi',-i\tau)}{(\xi_3+i\,t-\tau)(|\xi'|^2+\tau^2)}\,d\tau$$

$$= \frac{1}{2\pi}\lim_{t\to 0+}\frac{2\pi i[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{(\xi_3+i\,t+i|\xi'|)\,2\,(-i|\xi'|)}$$

$$= -\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|\,(\xi_3+i|\,\xi'|)}\,. \tag{3.46}$$

Now, from (3.44) by virtue of (3.46), we derive

$$\begin{aligned}
e_1(\xi') &= -\Pi'\Big\{\widetilde{\mathcal{T}}\,\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\\
&\qquad\times\Pi^+\Big(\Big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\Big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^\top\Big)\Big\}(\xi')\\
&= -\Pi'\Big\{\widetilde{\mathcal{T}}\,\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\Pi^+\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^\top\Big)\Big\}(\xi')\\
&= \Pi'\Big\{\widetilde{\mathcal{T}}(-i\xi',-i\xi_3)\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',\xi_3)\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|\,(\xi_3+i|\,\xi'|)}\Big)\Big\}(\xi')\\
&= -i\,\Pi'\Big\{\frac{\widetilde{\mathcal{T}}(\xi',\xi_3)\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',\xi_3)}{\xi_3+i|\xi'|}\Big\}(\xi')\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\Big)\\
&= \frac{i}{2\,\pi}\int\limits_{\gamma^-}\frac{\widetilde{\mathcal{T}}(\xi',\tau)\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',\tau)}{\tau+i|\,\xi'|}\,d\tau\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\Big)\\
&= -\mathcal{T}(\xi',-i|\xi'|)\,\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})(\xi',-i\,|\xi'|)\\
&\qquad\times\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\\
&= -\frac{1}{2\,|\xi'|}\,\widetilde{\mathcal{T}}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|).
\end{aligned} \tag{3.47}$$

Quite similarly, from (3.45), with the help of (3.46) and Lemma 3.2, we find

$$
\begin{aligned}
e_2(\xi') &= \Pi'\Big\{\widetilde{\mathcal{T}}\,[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}\Pi^+\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}^\top\Big)\Big\}(\xi')\\
&= -\Pi'\Big\{\widetilde{\mathcal{T}}(-i\xi',-i\xi_3)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}(\xi',\xi_3)\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2|\xi'|\,(\xi_3+i|\xi'|)}\Big)\Big\}(\xi')\\
&= i\,\Pi'\Big\{\widetilde{\mathcal{T}}(\xi',\xi_3)\,\frac{[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\xi_3)}{\xi_3+i|\,\xi'|}\Big\}(\xi')\\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})]^{-1}(\xi',-i|\xi'|)\,\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\Big)\\
&= \frac{i}{2|\xi'|}\Big(-\frac{1}{2\pi}\int\limits_{\gamma^-}\frac{\widetilde{\mathcal{T}}(\xi',\tau)\,[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}})]^{-1}(\xi',\tau)}{\tau+i|\,\xi'|}\,d\tau\Big)\\
&\qquad\times\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)\\
&= -\frac{i}{4\,\pi\,|\xi'|}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\\
&\qquad\times(-2\,i\,|\xi'|)\big[A^-(\xi',-i\,|\xi'|)\big]^{-1}\,\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|)\\
&= -\Big(\frac{1}{2\,\pi}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)\\
&\qquad\times\big[\widetilde{A}^{(-)}(\xi',-i\,|\xi'|)\big]^{-1}\,\widetilde{\mathcal{M}}^\top(\xi',-i|\xi'|).
\end{aligned}
\tag{3.48}
$$

Therefore, in view of relations (3.43), (3.42), (3.47), and (3.48) we finally obtain

$$
\begin{aligned}
e(\xi') = -\Big(\frac{1}{2\,\pi}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\,\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)&\\
\times\big[\widetilde{A}^{(-)}\big(\xi',-i\,|\xi'|\big)\big]^{-1}\,\widetilde{\mathcal{M}}^\top\big(\xi',-i|\xi'|\big)&.
\end{aligned}
$$

Since

$$
\det\Big(\int\limits_{\gamma^-}\widetilde{\mathcal{T}}(\xi',\tau)\big[\widetilde{A}^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big)\neq 0\ \text{ for all }\ \xi'\neq 0
$$

due to Lemma 3.3, and $\det\widetilde{A}^{(-)}(\xi',-i\,|\xi'|)\neq 0$ and $\det\widetilde{\mathcal{M}}(\xi',-i\,|\xi'|)\neq 0$ for all $\xi'\neq 0$ in accordance with (3.6) and (3.26), respectively, we deduce that

$$
\det e(\xi')\neq 0\ \text{ for all }\ \xi'\neq 0.
$$

Therefore for the operator $\mathfrak{R}$ the Šapiro–Lopatinskiĭ condition holds and the operator

$$
\mathfrak{R}: H^{r+1}(\Omega)\times H^{r+1/2}(S)\longrightarrow H^{r+1}(\Omega)\times H^{r-1/2}(S)
$$

is Fredholm for $r \geq 1$.

Step 3. We can now show that $\operatorname{Ind} \mathfrak{R} = 0$. To this end, for $t \in [0,1]$ let us consider the operator

$$\mathfrak{R}_t := \begin{bmatrix} r_{\Omega} \mathbf{B}_t \ell_0 & r_{\Omega} W_t + r_{\Omega} V \beta \\ t\, (\mathcal{T}^{+} \mathbf{K}) \ell_0 & \mathcal{L}_t + (\mathbf{d} - \mathbf{I}) + \mathcal{W}' \beta \end{bmatrix} \tag{3.49}$$

with $\mathbf{B}_t = \mathbf{I} + t\,\mathbf{K}$ and prove that it is homotopic to the operator $\mathfrak{R} = \mathfrak{R}_1$, where

$$W_t(g)(y) := -\int\limits_S \big[\mathcal{M}_t(x, \partial_x)\, P(x-y)\big]^{\top} g(x)\, dS_x, \quad y \in S, \quad t \in [0,1], \tag{3.50}$$

and

$$\mathcal{L}_t g(y) := \big[\mathcal{T}_t(y, \partial_y) W_t\, g(y)\big]^{+}, \quad y \in S, \quad t \in [0,\, 1], \tag{3.51}$$

with $\mathcal{T}_t(y, \partial_y)$ and $\mathcal{M}_t(y, \partial_y)$ defined in (3.25). Clearly, $\mathcal{L} = \mathcal{L}_1$.

We have to check that for the operator $\mathfrak{R}_t$ the Šapiro–Lopatinskiĭ condition is satisfied for all $t \in [0,1]$. Indeed, in this case the matrix associated with the Šapiro–Lopatinskiĭ condition reads as (cf. (3.40))

$$\begin{aligned} e_t(\xi') &= -\Pi' \Big\{ \widetilde{\mathcal{T}}_t \Big(\mathfrak{S}(\widetilde{\mathbf{B}}_t) - I \Big) \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t) \big]^{-1} \\ &\qquad \times \Pi^{+} \Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t) \big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}_t^{\top} \Big) \Big\}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}}_t)(\xi') \\ &= e_t^{(1)}(\xi') + e_t^{(2)}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}}_t)(\xi'), \end{aligned} \tag{3.52}$$

where

$$\begin{aligned} e_t^{(1)}(\xi') &= -\Pi' \Big\{ \widetilde{\mathcal{T}}_t \mathfrak{S}(\widetilde{\mathbf{B}}_t) \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t) \big]^{-1} \\ &\qquad \times \Pi^{+} \Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t) \big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}})\, \widetilde{\mathcal{M}}_t^{\top} \Big) \Big\}(\xi') \\ &= -\frac{1}{2\,|\xi'|}\, \widetilde{\mathcal{T}}_t(\xi', -i|\xi'|)\, \widetilde{\mathcal{M}}_t^{\top} \big(\xi', -i|\xi'|\big), \\ e_t^{(2)}(\xi') &= \Pi' \Big\{ \widetilde{\mathcal{T}}_t \big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t) \big]^{-1} \Pi^{+} \Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t) \big]^{-1} \mathfrak{S}(\widetilde{\mathbf{P}}) \widetilde{\mathcal{M}}_t \Big) \Big\}(\xi'), \\ \mathfrak{S}(\widetilde{\mathcal{L}}_t)(\xi') &= \frac{1}{2\,|\xi'|}\, \widetilde{\mathcal{T}}_t(\xi', -i|\xi'|)\, \widetilde{\mathcal{M}}_t^{\top} \big(\xi', -i|\xi'|\big). \end{aligned} \tag{3.53}$$

We have to show that $e_t(\xi')$ is non-singular for all $\xi' \neq 0$ and $t \in [0,1]$.

By direct calculations, we get

$$
\begin{aligned}
e_t^{(2)}(\xi') &= \Pi'\Big\{\widetilde{\mathcal{T}}_t\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\Pi^+\Big(\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}\mathfrak{S}(\widetilde{\mathbf{P}})\,\widetilde{\mathcal{M}}_t^\top\Big)\Big\}(\xi') \\
&= -\Pi'\bigg\{\widetilde{\mathcal{T}}_t(-i\xi',-i\xi_3)\big[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)\big]^{-1}(\xi',\xi_3) \\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',-i|\xi'|)\,\widetilde{\mathcal{M}}_t^\top(\xi',-i|\xi'|)}{2\,|\xi'|(\xi_3+i|\,\xi'|)}\Big)\bigg\}(\xi') \\
&= i\Pi'\Big\{\frac{\widetilde{\mathcal{T}}_t(\xi',\xi_3)[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',\xi_3)}{\xi_3+i|\,\xi'|}\Big\}(\xi') \\
&\qquad\times\Big(\frac{[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}_t^\top(\xi',-i|\xi'|)}{2\,|\xi'|}\Big) \\
&= \frac{i}{2\,|\xi'|}\Big(-\frac{1}{2\,\pi}\int\limits_{\gamma^-}\frac{\widetilde{\mathcal{T}}_t(\xi',\tau)[\mathfrak{S}^{(+)}(\widetilde{\mathbf{B}}_t)]^{-1}(\xi',\tau)}{\tau+i|\,\xi'|}\,d\tau\Big) \\
&\qquad\times\big[\mathfrak{S}^{(-)}(\widetilde{\mathbf{B}}_t)\big]^{-1}(\xi',-i|\xi'|)\widetilde{\mathcal{M}}_t^\top\big(\xi',-i|\xi'|\big) \\
&= -\frac{i}{4\,\pi\,|\xi'|}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}_t(\xi',\tau)\big[\widetilde{A}_t^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\big(-2\,i\,|\xi'|\big) \\
&\qquad\times\big[\widetilde{A}_t^{(-)}(\xi',-i|\xi'|)\big]^{-1}\widetilde{\mathcal{M}}_t^\top\big(\xi',-i|\xi'|\big) \\
&= -\Big(\frac{1}{2\,\pi}\int\limits_{\gamma^-}\widetilde{\mathcal{T}}_t(\xi',\tau)\big[\widetilde{A}_t^{(+)}(\xi',\tau)\big]^{-1}\,d\tau\Big) \\
&\qquad\times\big[\widetilde{A}_t^{(-)}(\xi',-i|\xi'|)\big]^{-1}\widetilde{\mathcal{M}}_t^\top\big(\xi',-i|\xi'|\big),
\end{aligned}
\tag{3.54}
$$

where $\widetilde{A}_t(\xi)=(1-t)|\xi|^2I+t\widetilde{A}(\xi)$ and $\widetilde{A}_t(\xi',\xi_3)=\widetilde{A}_t^{(-)}(\xi',\xi_3)\widetilde{A}_t^{(+)}(\xi',\xi_3)$, $\widetilde{A}_t^{(\pm)}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors in ξ_3 of the polynomial symbol matrix $\widetilde{A}_t(\xi',\xi_3)$.

Analogously to Lemma 3.3, we can prove that the matrix

$$
\int\limits_{\gamma^-}\widetilde{\mathcal{T}}_t(\xi',\tau)\big[\widetilde{A}_t^{(+)}(\xi',\tau)\big]^{-1}\,d\tau
$$

is non-singular for all $\xi'\neq 0$ and for all $t\in[0,1]$.

Therefore, by (3.52), (3.54) and (3.26) we have

$$
\det e_t(\xi')=\det e_t^{(2)}(\xi')\neq 0 \ \text{ for all } \ \xi'\neq 0 \ \text{ and for all } \ t\in[0,1], \tag{3.55}
$$

which implies that for the operator $\mathfrak{R}_t$ the Šapiro–Lopatinskiĭ condition is satisfied.

Hence the operator

$$
\mathfrak{R}_t\ : H^{r+1}(\Omega)\times H^{r+1/2}(S)\longrightarrow H^{r+1}(\Omega)\times H^{r-1/2}(S)
$$

is Fredholm for all $r\geq 1$ and for all $t\in[0,1]$.

Further, we prove that the index of the operator

$$\mathfrak{R}_0 = \begin{bmatrix} r_{\Omega}\mathbf{I}\,\ell_0 & r_{\Omega}W_0 + r_{\Omega}V\beta \\ 0 & \mathcal{L}_0 + (\mathbf{d}-\mathbf{I}) + \mathcal{W}'\beta \end{bmatrix} : H^{r+1}(\Omega)\times H^{r+1/2}(S) \longrightarrow H^{r+1}(\Omega)\times H^{r-1/2}(S)$$

is zero. Towards this end, first we show that the index of the operator $\mathcal{L}_t$ equals zero for all $t\in[0,\,1]$.

The principal homogeneous symbol matrix of the operator $\mathcal{L}_t$ reads as

$$\mathfrak{S}(\widetilde{\mathcal{L}}_t)(\xi') = \frac{1}{2\,|\xi'|}\,\widetilde{\mathcal{T}}_t\big(\xi',-i|\xi'|\big)\widetilde{\mathcal{M}}_t^{\top}\big(\xi',-i|\xi'|\big)$$

and is elliptic due to (3.26). Consequently, the operator $\mathcal{L}_t : H^{s+1/2}(S)\to H^{s-1/2}(S)$ with $s\in\mathbb{R}$ is Fredholm for all $t\in[0,1]$. Moreover, the principal part of the operator $\mathcal{L}_0 : H^{1/2}(S)\to H^{-1/2}(S)$ is selfadjoint due to the equality

$$\mathcal{L}_0 g = \mathcal{L}_{\Delta} g,$$

where $\mathcal{L}_{\Delta}$ stands for the trace of the normal derivative of the localized harmonic double-layer potential,

$$\mathcal{L}_{\Delta} g(y) = -\bigg\{\frac{\partial}{\partial n(y)}\int\limits_S \frac{\partial P(x-y)}{\partial n(x)}\,g(x)\,dS_x\bigg\}^{+}.$$

Therefore,

$$\operatorname{Ind}\mathcal{L} = \operatorname{Ind}\mathcal{L}_1 = \operatorname{Ind}\mathcal{L}_t = \operatorname{Ind}\mathcal{L}_0 = 0 \ \text{ for all } \ t\in[0,\,1] \ \text{ and for all } \ s\in\mathbb{R},$$

implying that the index of the operator $\mathfrak{R}_0$ equals zero. Since the family of the operators $\mathfrak{R}_t$ for $t\in[0,\,1]$ are homotopic, we conclude that

$$\operatorname{Ind}\mathfrak{R} = \operatorname{Ind}\mathfrak{R}_1 = \operatorname{Ind}\mathfrak{R}_t = \operatorname{Ind}\mathfrak{R}_0 = 0 \ \text{ for all } \ t\in[0,\,1] \ \text{ and for all } \ r\geq 1.$$

Step 4. From the equivalence Theorem 2.2 it follows that $\operatorname{Ker}\mathfrak{R} = \{0\}$ in the space $H^{r+1}(\Omega)\times H^{r+1/2}(S)$ for all $r\geq 1$ and, consequently, the operator

$$\mathfrak{R} : H^{r+1}(\Omega)\times H^{r+1/2}(S) \longrightarrow H^{r+1}(\Omega)\times H^{r-1/2}(S)$$

is invertible for all $r\geq 1$. □

Corollary 3.5. *Let a localizing function $\chi\in X^4_{1+}$ and the condition* (3.26) *be fulfilled. Then the operator*

$$\mathfrak{R} \,:\, H^2(\Omega)\times H^{3/2}(S) \longrightarrow H^2(\Omega)\times H^{1/2}(S)$$

is invertible.

Proof. It is word for word repeats the above proof with $r=1$. □

4. Appendix A: Classes of Localizing Functions

Here we introduce the classes of localizing functions used in the main text of the paper (for details see the reference [16]).

Definition A.1. We say $\chi \in X^k$ for integer $k \geq 0$ if $\chi(x) = \breve{\chi}(|x|)$, $\breve{\chi} \in W_1^k(0,\infty)$ and $\varrho\breve{\chi}(\varrho) \in L_1(0,\infty)$. We say $\chi \in X_+^k$ for integer $k \geq 1$ if $\chi \in X^k$, $\chi(0) = 1$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\widehat{\chi}_s(\omega)}{\omega} > 0 & \text{for } \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int_0^\infty \varrho\breve{\chi}(\varrho)\, d\varrho & \text{for } \omega = 0, \end{cases} \tag{A.1}$$

and $\widehat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$

$$\widehat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}(\varrho) \sin(\varrho\omega)\, d\varrho. \tag{A.2}$$

We say $\chi \in X_{1+}^k$ for integer $k \geq 1$ if $\chi \in X_+^k$ and

$$\omega\widehat{\chi}_s(\omega) \leq 1, \ \ \forall\, \omega \in \mathbb{R}. \tag{A.3}$$

Evidently, we have the following embeddings: $X^{k_1} \subset X^{k_2}$ and $X_+^{k_1} \subset X_+^{k_2}$, $X_{1+}^{k_1} \subset X_{1+}^{k_2}$ for $k_1 > k_2$. The class X_+^k is defined in terms of the sine-transform. The following lemma provides us with an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class (for details see [16]).

Lemma A.2. *Let $k \geq 1$. If $\chi \in X^k$, $\breve{\chi}(0) = 1$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0,\infty)$, and $\breve{\chi}$ is a non-increasing function on $[0,+\infty)$, then $\chi \in X_+^k$.*

The following examples for χ are presented in [16],

$$\chi_1(x) = \begin{cases} \left[1 - \dfrac{|x|}{\varepsilon}\right]^k & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A.4}$$

$$\chi_2(x) = \begin{cases} \exp\left[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\right] & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A.5}$$

$$\chi_3(x) = \begin{cases} \left(1 - \dfrac{|x|}{\varepsilon}\right)^2 \left(1 - 2\dfrac{|x|}{\varepsilon}\right) & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon. \end{cases} \tag{A.6}$$

One can notice that $\chi_1 \in X_+^k$, while $\chi_2 \in X_+^\infty$ due to Lemma A.2, and $\chi_3 \in X_+^2$. Moreover, $\chi_1 \in X_{1+}^k$ for $k = 2$ and $k = 3$, and $\chi_3 \in X_{1+}^2$, while $\chi_1 \notin X_{1+}^1$ and $\chi_2 \notin X_{1+}^\infty$ (for details see [16]).

5. Appendix B: Properties of Localized Potentials

Here we collect some theorems describing mapping properties of the localized potentials. The proofs can be found in [16] (see also [26], Chapter 8 and the references therein).

Here we employ the notation V, W, and $\mathcal{P}$ introduced in the main text for the localized layer and volume potentials, see (2.31)–(2.33). Further, let us introduce the boundary operators generated by the localized layer potentials associated with the localized parametrix $P(x-y) \equiv P_\chi(x-y)$,

$$\mathcal{V}g(y) := -\int_S P(x-y)\, g(x)\, dS_x, \quad y \in S, \tag{B.1}$$

$$\mathcal{W}g(y) := -\int_S \big[\mathcal{M}(x,\partial_x)P(x-y)\big]^\top g(x)\, dS_x, \;\; y \in S, \tag{B.2}$$

$$\mathcal{W}'g(y) := -\int_S \big[\mathcal{T}(y,\partial_y)P(x-y)\big] g(x)\, dS_x, \;\; y \in S, \tag{B.3}$$

$$\mathcal{L}^\pm g(y) := \big[\mathcal{T}(y,\partial_y)Wg(y)\big]^\pm, \;\; y \in S, \tag{B.4}$$

where $\mathcal{T}(x,\partial_x)$ and $\mathcal{M}(x,\partial_x)$ are defined in (2.6) and (2.8).

Theorem B.1. *The following operators are continuous:*

$$\mathcal{P} : \widetilde{H}^s(\Omega) \longrightarrow H^{s+2,s}(\Omega;\Delta), \;\; -\frac{1}{2} < s < \frac{1}{2}\,, \;\; \chi \in X^1, \tag{B.5}$$

$$: H^s(\Omega) \longrightarrow H^{s+2,s}(\Omega;\Delta), \;\; -\frac{1}{2} < s < \frac{1}{2}\,, \;\; \chi \in X^1, \tag{B.6}$$

$$: H^s(\Omega) \longrightarrow H^{\frac{5}{2}-\varepsilon,\frac{1}{2}-\varepsilon}(\Omega;\Delta), \;\; \frac{1}{2} \le s < \frac{3}{2}\,, \;\; \forall\, \varepsilon \in (0,1), \;\; \chi \in X^2, \tag{B.7}$$

where Δ is the Laplace operator.

Theorem B.2. *The following operators are continuous:*

$$V : H^{s-\frac{3}{2}}(S) \longrightarrow H^s(\mathbb{R}^3), \;\; s < \frac{3}{2}\,, \;\; \textit{if } \chi \in X^1, \tag{B.8}$$

$$: H^{s-\frac{3}{2}}(S) \longrightarrow H^{s,s-1}(\Omega^\pm;\Delta), \;\; \frac{1}{2} < s < \frac{3}{2}\,, \;\; \textit{if } \chi \in X^2, \tag{B.9}$$

$$W : H^{s-\frac{1}{2}}(S) \longrightarrow H^s(\Omega^\pm), \;\; s < \frac{3}{2}\,, \;\; \textit{if } \chi \in X^2, \tag{B.10}$$

$$: H^{s-\frac{1}{2}}(S) \longrightarrow H^{s,s-1}(\Omega^\pm;\Delta), \;\; \frac{1}{2} < s < \frac{3}{2}\,, \;\; \textit{if } \chi \in X^3. \tag{B.11}$$

Theorem B.3. *If $\chi \in X^k$ has a compact support and $-\frac{1}{2} \le s \le \frac{1}{2}$, then the following localized operators are continuous:*

$$V : H^s(S) \longrightarrow H^{s+\frac{3}{2}}(\Omega^\pm) \;\; \textit{for } \; k = 2, \tag{B.12}$$

$$W : H^{s+1}(S) \longrightarrow H^{s+\frac{3}{2}}(\Omega^\pm) \;\; \textit{for } \; k = 3. \tag{B.13}$$

Theorem B.4. *Let $\psi \in H^{-\frac{1}{2}}(S)$ and $\varphi \in H^{\frac{1}{2}}(S)$. Then the following jump relations hold on S:*

$$V^{+}\psi = V^{-}\psi = \mathcal{V}\psi, \quad \chi \in X^{1}, \tag{B.14}$$

$$W^{\pm}\varphi = \mp\,\mathbf{d}\varphi + \mathcal{W}\varphi, \quad \chi \in X^{2}, \tag{B.15}$$

$$\mathcal{T}^{\pm}V\psi = \pm\,\mathbf{d}\psi + \mathcal{W}'\psi, \quad \chi \in X^{2}, \tag{B.16}$$

where

$$\mathbf{d}(y) := \frac{1}{2}\begin{bmatrix} [c_{ijlk}(y)\,n_i\,n_l]_{3\times 3} & [e_{lij}(y)\,n_i\,n_l]_{3\times 1} \\ [-e_{ikl}(y)\,n_i\,n_l]_{1\times 3} & \varepsilon_{il}(y)\,n_i\,n_l \end{bmatrix}_{4\times 4}, \quad y \in S, \tag{B.17}$$

and $\mathbf{d}(y)$ is strongly elliptic due to (2.4) *.*

Theorem B.5. *Let $-\frac{1}{2} \leq s \leq \frac{1}{2}$. The following operators*

$$\mathcal{V} : H^{s}(S) \longrightarrow H^{s+1}(S), \quad \chi \in X^{2}, \tag{B.18}$$

$$\mathcal{W} : H^{s+1}(S) \longrightarrow H^{s+1}(S), \quad \chi \in X^{3}, \tag{B.19}$$

$$\mathcal{W}' : H^{s}(S) \longrightarrow H^{s}(S), \quad \chi \in X^{3}, \tag{B.20}$$

$$\mathcal{L}^{\pm} : H^{s+1}(S) \longrightarrow H^{s}(S), \quad \chi \in X^{3}, \tag{B.21}$$

are continuous.

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