Memoirs on Differential Equations and Mathematical Physics
Volume 61, 2014, 37–61

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF n-TH ORDER WITH REGULARLY VARYING NONLINEARITIES

Abstract. The conditions of existence and asymptotic for $t \uparrow \omega$ ($\omega \leq +\infty$) representations of one class of monotonic solutions of n-th order differential equations containing in the right-hand side a sum of terms with regularly varying nonlinearities, are established.

2010 Mathematics Subject Classification. 34D05, 34C11.

Key words and phrases. Ordinary differential equations, regularly varying nonlinearities, asymptotics of solutions.

რეზიუმე. n-ური რიგის ჩვეულებრივი დიფერენციალური განტოლებებისათვის, რომელთა მარჯვენა მხარეები წარმოადგენენ რეგულარულად ცვალებადი არაწრფივობის მქონე წევრთა ჯამს, დადგენილია გარკვეული კლასის მონოტონურ ამონახსნთა არსებობის საკმარისი პირობები და მიღებული მათი ასიმპტოტური წარმოდგენები.

1. Introduction

The theory of regularly varying functions created by J. Karamata in 1930 has been later (see, for example, monographs [1], [2]) extensively developed and widely used in various mathematical researches. Particularly, the last decades of the past century is mentioned by a great interest in studying regularly and slowly varying solutions of various differential equations and in equations of the type

$$y'' = \alpha_0 p(t)\varphi(y),$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function and $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a regularly varying continuous function of order $\sigma \neq 1$ as $y \to Y_0$; here Y_0 equals either zero or $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of Y_0. Among the researches carried out within that period and dedicated to determination of asymptotics as $t \to +\infty$ of monotonic solutions for such equations, of special mention are the works [3], [4] and the monograph [5].

Here, according to the definition of regularly varying function (see E. Seneta [1, Ch. 1, Sect. 1.1, pp. 9–10]),

$$\varphi(y) = |y|^\sigma L(y),$$

where L is slowly varying as $y \to Y_0$ function, i.e., the condition

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{L(\lambda y)}{L(y)} = 1 \ \text{ with any } \ \lambda > 0$$

is satisfied. Considering such representation for φ, such class of equations is a natural extension of the class of generalized second order Emden–Fowler equations

$$y'' = \alpha_0 p(t)|y|^\sigma \operatorname{sign} y.$$

The basic results dealing with asymptotic properties of solutions for the second- and n-th order Emden–Fowler equations, obtained before 1990, can be found in the monograph due to I.T. Kiguradze and T.A. Chanturiya [6, Ch. IV, V, pp. 309–401]. The works [7]–[16], dedicated to the determination of asymptotics of monotonic differential equations of second and higher orders with power nonlinearities are also worth mentioning.

For the last decade, the results obtained in [17]–[22] and also those obtained in [12]–[16] were applied to differential equations

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'), \quad y'' = \sum_{k=1}^{m} \alpha_k p_k(t)\varphi_{k0}(y)\varphi_{k1}(y'),$$
$$y^{(n)} = \alpha_0 p(t)\varphi(y) \ \ (n \geq 2)$$

with nonlinearities, regularly varying as $y \to Y_0$ and $y' \to Y_1$, where $Y_i \in \{0; \pm\infty\}$ $(i = 0, 1)$, and with some additional restrictions to nonlinearity for the first two equations.

In the present paper we consider the following differential equation:

$$y^{(n)} = \sum_{k=1}^{m} \alpha_k p_k(t) \prod_{j=0}^{n-1} \varphi_{kj}(y^{(j)}), \tag{1.1}$$

where $n \geq 2$, $\alpha_k \in \{-1;1\}$ $(k = \overline{1,m})$, $p_k : [a,\omega[\to]0,+\infty[$ $(k = \overline{1,m})$ are continuous functions, $\varphi_{kj} : \triangle_{Y_j} \to]0,+\infty[$ $(k = \overline{1,m};\ j = \overline{0,n-1})$ are continuous and regularly varying as $y^{(j)} \to Y_j$ functions of orders σ_{kj}, $-\infty < a < \omega \leq +\infty$,* $\triangle_{Y_j}$ is one-sided neighborhood of Y_j, Y_j equal either to 0 or to $\pm\infty$. It is assumed that numbers ν_j $(j = \overline{0,n-1})$ determined by

$$\nu_j = \begin{cases} 1, & \text{if either } Y_j = +\infty, \text{ or } Y_j = 0 \\ & \quad \text{and } \Delta_{Y_j}\text{-right neighborhood of 0,} \\ -1, & \text{if either } Y_j = -\infty, \text{ or } Y_j = 0 \\ & \quad \text{and } \Delta_{Y_j}\text{-left neighborhood of 0,} \end{cases} \tag{1.2}$$

are such that

$$\nu_j\nu_{j+1} > 0 \quad \text{with} \quad Y_j = \pm\infty \text{ and}$$
$$\nu_j\nu_{j+1} < 0 \text{ with } Y_j = 0 \ (j = \overline{0,n-2}). \tag{1.3}$$

Such conditions for ν_j $(j = \overline{0,n-1})$ are necessary for the equation (1.1) to have solutions defined in the left neighborhood of ω, each of which satisfying the conditions

$$y^{(j)}(t) \in \Delta_{Y_j} \text{ with } t \in [t_0,\omega[, \quad \lim_{t\uparrow\omega} y^{(j)}(t) = Y_j \ (j = \overline{0,n-1}). \tag{1.4}$$

Among strictly monotonic, with derivatives up to the $n-1$ order inclusive, in some left neighborhood of ω, solutions of equation (1.1) these ones are of special academic interest, because each of the rest ones admits only one representation of the type

$$y(t) = \pi_\omega^{k-1}(t)[c_{k-1} + o(1)] \ (k = \overline{1,n}),$$

where c_{k-1} $(k = \overline{1,n})$ are the non-zero real constants and

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t-\omega, & \text{if } \omega < +\infty. \end{cases} \tag{1.5}$$

The question on the existence of solutions of (1.1) with similar representations may be solved, in a whole, in a rather simple way by applying, for example, Corollary 8.2 for $\omega = +\infty$ from the monograph of I. T. Kiguradze and T. A. Chanturiya [1, Ch. II, p. 8, p. 207] and the schemes from the works [10], [12] as $\omega \leq +\infty$. As for the solutions with properties (1.4), for lack of particular representations for them, there arises the necessity to single out a class of solutions admitting one to get such representations.

*if $a > 1$, then $\omega = +\infty$, and $\omega - 1 < a < \omega$ if $\omega < +\infty$.

One of such rather wide classes of solutions has been introduced in [14]–[16] dedicated to generalized Emden–Fowler type equations of n-th order,

$$y^{(n)} = \alpha_0 p(t) \prod_{j=0}^{n-1} |y^{(j)}|^{\sigma_j}.$$

For the equation (1.1), this class is determined as follows.

Definition 1.1. A solution y of the equation (1.1) defined on the interval $[t_0, \omega[\subset [a, \omega[$, is called a $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if along with (1.4) the condition

$$\lim_{t\uparrow\omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}} = \lambda_0 \tag{1.6}$$

is satisfied.

If y is a solution of the differential equation (1.1) with properties (1.4) and the functions $\ln |y^{(n-1)}(t)|$ and $\ln |\pi_\omega(t)|$ are comparable with order one (see [23, Ch. 5, Sect. 4,5, pp. 296–301]) as $t \uparrow \omega$, then it is easy to check that this solution is the $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution for some λ_0 depending on the value of $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)}$.

Moreover, using assertions 1, 2, 5 and 9 (on the properties of regularly varying functions) from the monograph [5, Appendix, pp. 115–117], it can be verified that in the case of regularly varying as $t \uparrow \omega$ coefficients p_k ($k = \overline{1,m}$) of the equation (1.1), each of its regularly varying as $t \uparrow \omega$ solutions with properties (1.4) is a $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution for some final or equal to $\pm\infty$ value λ_0.

The aim of this note is to determine the conditions for existence of $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of (1.1) in special cases, where $\lambda_0 = \frac{n-i-1}{n-i}$ as $i \in \{1, \dots, n-1\}$, and also asymptotic representations as $t \uparrow \omega$ for such solutions and their derivatives up to and including $n-1$ order.

By virtues of the results from [16], these solutions of the equation (1.1) possess the following a priori asymptotic properties.

Lemma 1.1. *Let $y : [t_0, \omega[\to \Delta_{Y_0}$ be an arbitrary $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution of the equation* (1.1)*. Then:*

(1) *if $n > 2$ and $\lambda_0 = \frac{n-i-1}{n-i}$ for some $i \in \{1, \dots, n-2\}$, then for $t \uparrow \omega$,*

$$\begin{gathered} y^{(k-1)}(t) \sim \frac{[\pi_\omega(t)]^{i-k}}{(i-k)!}\, y^{(i-1)}(t) \quad (k = 1, \dots, i-1)^*, \\ y^{(i)}(t) = o\Big(\frac{y^{(i-1)}(t)}{\pi_\omega(t)}\Big), \end{gathered} \tag{1.7}$$

$$y^{(k)}(t) \sim (-1)^{k-i} \frac{(k-i)!}{[\pi_\omega(t)]^{k-i}}\, y^{(i)}(t) \quad (k = i+1, \dots, n); \tag{1.8}$$

*At $i = 1$ these relationships do not exist.

(2) *if $n \ge 2$ and $\lambda_0 = 0$, then for $t \uparrow \omega$,*

$$
\begin{aligned}
&y^{(k-1)}(t) \sim \frac{[\pi_\omega(t)]^{n-k-1}}{(n-k-1)!}\, y^{(n-2)}(t) \ \ (k = 1, \dots, n-2)^*,\\
&y^{(n-1)}(t) = o\Big(\frac{y^{(n-2)}(t)}{\pi_\omega(t)}\Big)
\end{aligned}
\tag{1.9}
$$

and, in the case of existence of (finite or equal to $\pm\infty$) limit $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)}$,

$$
y^{(n)}(t) \sim \frac{-1}{\pi_\omega(t)\, y^{(n-1)}(t)} \ \ \text{with} \ \ t \uparrow \omega. \tag{1.10}
$$

2. Statement of the Main Results

In order to formulate the theorems, we will need some auxiliary notation and one definition.

By virtue of the definition of regularly varying function, the nonlinearity in (1.1) is representable in the form

$$
\varphi_{kj}(y^{(j)}) = |y^{(j)}|^{\sigma_{kj}} L_{kj}(y^{(j)}) \ \ (k = \overline{1,m}; \ \ j = \overline{0,n-1}), \tag{2.1}
$$

where $L_{kj} : \Delta_{Y_j} \to]0, +\infty[$ are continuous and slowly varying as $y^j \to Y_j$ functions, for which with any $\lambda > 0$

$$
\lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta_{Y_j}}} \frac{L_{kj}(\lambda y^{(j)})}{L_{kj}(y^{(j)})} = 1 \ \ (k = \overline{1,m}; \ \ j = \overline{0,n-1}). \tag{2.2}
$$

It is also known (see [1, Ch. 1, Sect. 1.2, pp. 10–15]) that the limits (2.2) are uniformly fulfilled with respect to λ on any interval $[c,d] \subset]0,+\infty[$ (property M_1) and there exist continuously differentiable slowly varying as $y^{(j)} \to Y_j$ functions $L_{0kj} : \Delta_{Y_j} \to]0, +\infty[$ (property M_2) such that

$$
\lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta_{Y_j}}} \frac{L_{kj}(y^{(j)})}{L_{0kj}(y^{(j)})} = 1 \ \ \text{and} \ \ \lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta_{Y_j}}} \frac{y^{(j)} L'_{0kj}(y^{(j)})}{L_{0kj}(y^{(j)})} = 0 \tag{2.3}
$$

$$
(k = \overline{1,m}; \ \ j = \overline{0,n-1}).
$$

Definition 2.1. We say that a slowly varying as $z \to Z_0$ function $L : \Delta_{Z_0} \to]0, +\infty[$, where Z_0 either equals zero, or $\pm\infty$, and Δ_{Z_0} is one-sided neighborhood of Z_0, satisfies condition S_0, if

$$
L\big(\nu e^{[1+o(1)]\ln|z|}\big) = L(z)[1+o(1)] \ \ \text{with} \ \ z \to Z_0 \ \ (z \in \Delta_{Z_0}),
$$

where $\nu = \operatorname{sign} z$.

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*At $n = 2$ these relationships do not exist.

Remark 2.1. If the slowly varying as $z \to Z_0$ function $L : \Delta_{Z_0} \to]0, +\infty[$ satisfies the condition S_0, then for every slowly varying as $z \to Z_0$ function $l : \Delta_{Z_0} \to]0, +\infty[$,

$$L(zl(z)) = L(z)[1 + o(1)] \text{ when } z \to Z_0 \ \ (z \in \Delta_{Z_0}).$$

The validity of this statement follows from the theorem of representation (see [1, Ch. 1, Sect. 1.2, p. 10]) of slowly varying function l and property M_1 of function L.

Remark 2.2 (see [22]). If slowly varying as $z \to Z_0$ function $L : \Delta_{Z_0} \to]0, +\infty[$ satisfies condition S_0, then the function $y : [t_0, \omega[\to \Delta_{Y_0}$ is continuously differentiable and such that

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \frac{y'(t)}{y(t)} = \frac{\xi'(t)}{\xi(t)} \, [r + o(1)] \text{ when } t \uparrow \omega,$$

where r is the non-zero real constant, ξ is continuously differentiable in some left neighborhood of ω real function, for which $\xi'(t) \neq 0$, then

$$L(y(t)) = L\big(\nu|\xi(t)|^r\big)[1 + o(1)] \text{ when } t \uparrow \omega,$$

where $\nu = \operatorname{sign} y(t)$ in the left neighborhood of ω.

Remark 2.3. If slowly varying as $z \to Z_0$ function $L : \Delta_{Z_0} \to]0, +\infty[$ satisfies condition S_0 and the function $r : \Delta_{Z_0} \times K \to \mathbb{R}$, where K is compact in $\mathbb{R}^m$, is such that

$$\lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} r(z, v) = 0 \text{ unifornly with respect to } v \in K,$$

then

$$\lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} \frac{L(\nu e^{[1 + r(z,v)] \ln |z|})}{L(z)} = 1$$

$$\text{uniformly with respect to } v \in K, \text{ where } \nu = \operatorname{sign} z.$$

Indeed, if it shouldn't be true, then there would exist a sequence $\{v_n\} \in K$ and a sequence $\{z_n\} \in \Delta_{Z_0}$ converging to Z_0 such that the inequality

$$\liminf_{n \to +\infty} \Big| \frac{L(\nu e^{[1 + r(z_n, v_n)] \ln |z_n|})}{L(z_n)} - 1 \Big| > 0 \tag{2.5}$$

is fulfilled.

Thus it is clear that there is the function $v : \Delta_{Z_0} \to K$ such that $v(z_n) = v_n$. For this function it is obvious that $\lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} r(z, v(z)) = 0$ and hence

$$\lim_{\substack{z \to Z_0 \\ z \in \Delta_{Z_0}}} \frac{L(\nu e^{[1 + r(z, v(z))] \ln |z|})}{L(z)} = 1,$$

which contradicts the inequality (2.5).

Finally, let us introduce auxiliary definitions assuming

$$\mu_{ki}=n-i-1+\sum_{j=0}^{i-2}\sigma_{kj}(i-j-1)-\sum_{j=i+1}^{n-1}\sigma_{kj}(j-i)\ \ (k=\overline{1,m};\ \ i=\overline{1,n}),$$

$$\gamma_k=1-\sum_{j=0}^{n-1}\sigma_{kj},\quad \gamma_{ki}=1-\sum_{j=i}^{n-1}\sigma_{kj}\ \ (k=\overline{1,m};\ \ i=\overline{1,n-1}),$$

$$C_{ki}=\frac{1}{(n-i)!}\prod_{j=0}^{i-1}[(i-j-1)!]^{-\sigma_{kj}}\prod_{j=i+1}^{n-1}[(j-i)!]^{\sigma_{kj}}\ \ (k=\overline{1,m};\ i=\overline{1,n-1}),$$

$$J_{ki}(t)=\int_{A_{ki}}^{t}p_k(s)|\pi_\omega(s)|^{\mu_{ki}}\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}L_{kj}\big(\nu_j|\pi_\omega(s)|^{i-j-1}\big)\,ds\ \ (k=\overline{1,m};\ i=\overline{1,n}),$$

$$J_{kii}(t)=\int_{A_{kii}}^{t}|J_{ki}(s)|^{\frac{1}{\gamma_{ki}}}\,ds\ \ (k=\overline{1,m};\ \ i=\overline{1,n}),$$

where each of the limits of integration A_{km}, A_{kmm} $(m\in\{0,1\})$ is chosen equal to the point $a_0\in[a,\omega[$ (on the right of which, i.e., as $t\in[a_0,\omega[$, the integrand function is continuous) if under this value of limits of integration the corresponding integral tends to $\pm\infty$ as $t\uparrow\omega$, and equal to ω if at such value of limits of integration it tends to zero as $t\uparrow\omega$.

Theorem 2.1. *Let $n>2$, $i\in\{1,\dots,n-2\}$ and for some $s\in\{1,\dots,m\}$ the inequalities*

$$\limsup_{t\uparrow\omega}\frac{\ln p_k(t)-\ln p_s(t)}{\beta\ln|\pi_\omega(t)|}<$$
$$<\beta\sum_{\substack{j=0\\ j\neq i-1}}^{n-1}(\sigma_{sj}-\sigma_{kj})(i-j-1)\ \ \text{at all}\ \ k\in\{1,\dots,m\}\setminus\{s\},\qquad(2.6_i)$$

be fulfilled, where $\beta=\operatorname{sign}\pi_\omega(t)$ for $t\in[a,\omega[$. Moreover, let $\gamma_s\gamma_{si}\neq0$ and the functions L_{sj} for all $j\in\{0,\dots,n-1\}\setminus\{i-1\}$ satisfy condition S_0. Then for the existence of $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i})$-solutions of the equation (1.1) *it is necessary, and if algebraic equation*

$$\sum_{j=i+1}^{n-1}\frac{\sigma_{sj}}{(j-i)!}\prod_{m=1}^{j-i}(m-\rho)+\sigma_{si}=\frac{1}{(n-i)!}\prod_{m=1}^{n-i}(m-\rho)\qquad(2.7)$$

has no roots with zero real part it is sufficient that (*along with* (1.3)) *the inequalities*

$$\begin{gathered}\nu_j\nu_{j-1}(i-j)\pi_\omega(t)>0\ \ \text{at all}\ \ j\in\{1,\dots,n-1\}\setminus\{i\},\\ \nu_i\nu_{i-1}\gamma_s\gamma_{si}J_{sii}(t)>0,\end{gathered}\qquad(2.8_i)$$

$$\nu_i\alpha_s(-1)^{n-i-1}\pi_\omega^{n-i-1}(t)\gamma_{si}J_{si}(t)>0\qquad(2.9_i)$$

be fulfilled in some left neighborhood of ω, *as well as the conditions*

$$\begin{gathered}\nu_{j-1}\lim_{t\uparrow\omega}|\pi_\omega(t)|^{i-j}=Y_{j-1}\ \ \text{at all}\ \ j\in\{1,\dots,n\}\setminus\{i\},\\ \nu_{i-1}\lim_{t\uparrow\omega}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}=Y_{i-1},\end{gathered}\tag{2.10_i}$$

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{si}(t)}{J_{si}(t)}=-\gamma_{si},\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}=0.\tag{2.11_i}$$

Moreover, each solution of that kind admits as $t\uparrow\omega$ *the asymptotic representations*

$$y^{(j-1)}(t)=\frac{[\pi_\omega(t)]^{i-j}}{(i-j)!}\,y^{(i-1)}(t)[1+o(1)]\ \ (j=1,\dots,i-1),\tag{2.12_i}$$

$$y^{(j)}(t)=(-1)^{j-i}\frac{(j-i)!}{[\pi_\omega(t)]^{j-i}}\cdot\frac{\gamma_{si}J'_{sii}(t)}{\gamma_sJ_{sii}(t)}\,y^{(i-1)}(t)[1+o(1)]\qquad (j=i,\dots,n-1),\tag{2.13_i}$$

$$\frac{|y^{(i-1)}(t)|^{\gamma_s}}{L_{si-1}(y^{(i-1)}(t))}=|\gamma_{si}C_{si}|\left|\frac{\gamma_s}{\gamma_{si}}J_{sii}(t)\right|^{\gamma_{si}}[1+o(1)]\ \ \text{with}\ \ t\uparrow\omega,\tag{2.14_i}$$

and in case $\omega=+\infty$ *there is* $i+1$*-parameter family of solutions if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si}>0$ *is valid, and* $i-1+l$*-parameter family if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si}<0$ *is valid, in case* $\omega<+\infty$, *there is* $r+1$*-parameter family if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si}>0$ *is valid, and* r*-parameter family if the inequality* $\nu_i\nu_{i-1}\gamma_s\gamma_{si}<0$ *is valid, where* l *is a number of roots of the equation* (2.7) *with negative real part and* r *is a number of its roots with positive real part.*

Remark 2.4. Algebraic equation (2.7) has a fortiori no roots with zero real part, if $\sum\limits_{j=i}^{n-2}|\sigma_{sj}|<|1-\sigma_{sn-1}|$.

In Theorem 2.1, asymptotic representation for $y^{(i-1)}$ is written implicitly. The following theorem shows an additional restriction under which this representation may be presented explicitly.

Theorem 2.2. *If the conditions of Theorem* 2.1 *are fulfilled and a slowly varying at* $y^{(i-1)}\to Y_{i-1}$ *function* L_{si-1} *satisfies condition* S_0, *then for each* $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i})$*-solution of the equation* (1.1), *asymptotic representations* (2.12_i), (2.13_i) *and*

$$y^{(i-1)}(t)=\nu_{i-1}\left|\gamma_{si}C_{si}L_{si-1}\Big(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}\Big)\right|^{\frac{1}{\gamma_s}}\times \\ \times\left|\frac{\gamma_s}{\gamma_{si}}J_{sii}(t)\right|^{\frac{\gamma_{si}}{\gamma_s}}[1+o(1)]\tag{2.15_i}$$

hold when $t\uparrow\omega$.

3. Proof of Theorems

Proof of Theorem 2.1. *Necessity.* Let $y : [t_0,\omega[\rightarrow \Delta_{Y_0}$ be an arbitrary $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i}$-solution of the equation (1.1). Then the conditions (1.4) are satisfied, there is $t_1\in[a,\omega[$ such that $\nu_j y^{(j)}(t)>0$ $(j=\overline{0,n-1})$ for $t\in[t_1,\omega[$ and by Lemma 1.1, the asymptotic relations (1.7), (1.8) hold. From (1.7) and (1.8) we obtain the relations

$$\frac{y^{(j)}(t)}{y^{(j-1)}(t)}=\frac{i-j+o(1)}{\pi_\omega(t)}\quad (j=\overline{1,n})\ \text{ when }\ t\uparrow\omega \tag{3.1_i}$$

and therefore

$$\ln|y^{(j-1)}(t)|=\big[i-j+o(1)\big]\ln|\pi_\omega(t)|\quad (j=\overline{1,n})\ \text{ when }\ t\uparrow\omega. \tag{3.2_i}$$

By virtue of (3.1_i), the first of inequalities (2.8_i) are fulfilled, and by virtue of (3.2_i), the first of conditions (2.10_i) are satisfied.

Taking into account (3.2_i), the representations (2.1) and the conditions

$$\lim_{\substack{y^{(j)}\to Y_j\\ y^{(j)}\in\Delta_{y_j}}}\frac{\ln L_{kj}(y^{(j)})}{\ln|y^{(j)}|}=0\quad (k=\overline{1,m},\ \ j=\overline{0,n-1}), \tag{3.3}$$

which are satisfied due to the properties of slowly varying functions (see [1, Ch. 1, p. 1.5, p. 24]), we find that

$$\begin{gathered}\ln\varphi_{kj}(y^{(j)}(t))=\sigma_{kj}\ln|y^{(j)}(t)|+\ln L_{kj}(y^{(j)}(t))=\\ =\big[\sigma_{kj}+o(1)\big]\ln|y^{(j)}(t)|=\big[\sigma_{kj}(i-j-1)+o(1)\big]\ln|\pi_\omega(t)|\\ (k=\overline{1,m},\ \ j=\overline{0,n-1})\ \text{ when }\ t\uparrow\omega.\end{gathered}$$

That is why for each $k\in\{1,\dots,m\}\setminus\{s\}$,

$$\begin{gathered}\ln\left[\frac{p_k(t)\prod\limits_{j=0}^{n-1}\varphi_{kj}(y^{(j)}(t))}{p_s(t)\prod\limits_{j=0}^{n-1}\varphi_{sj}(y^{(j)}(t))}\right]=\ln\frac{p_k(t)}{p_s(t)}+\sum_{j=0}^{n-1}\Big[\ln\varphi_{kj}(y^{j)}(t)-\ln\varphi_{sj}(y^{(j)}(t)\Big]=\\ =\ln\frac{p_k(t)}{p_s(t)}+\ln|\pi_\omega(t)|\sum_{j=0}^{n-1}\big[(\sigma_{kj}-\sigma_{sj})(i-j-1)+o(1)\big]=\\ =\beta\ln|\pi_\omega(t)|\left[\frac{\ln p_k(t)-\ln p_s(t)}{\beta\ln|\pi_\omega(t)|}+\beta\sum_{\substack{j=0\\ j\neq i-1}}^{n-1}(\sigma_{kj}-\sigma_{sj})(i-j-1)+o(1)\right]\\ \text{as}\quad t\uparrow\omega.\end{gathered}$$

Since the expression, appearing on the right of this correlation, by virtue of (2.6_i) and the type of the function π_ω from (1.5), tends to $-\infty$ when $t\uparrow\omega$, therefore

$$\lim_{t\uparrow\omega}\frac{p_k(t)\prod\limits_{j=0}^{n-1}\varphi_{kj}(y^{(j)}(t))}{p_s(t)\prod\limits_{j=0}^{n-1}\varphi_{sj}(y^{(j)}(t))}=0 \ \text{ at all } \ k\in\{1,\dots,m\}\setminus\{s\}. \tag{3.4}$$

Then from (1.1) it follows that this solution implies asymptotic relation

$$y^{(n)}(t)=\alpha_s p_s(t)[1+o(1)]\prod_{j=0}^{n-1}\varphi_{sj}(y^{(j)}(t)) \ \text{ when } \ t\uparrow\omega. \tag{3.5}$$

Here, for all $j\in\{0,\dots,n-1\}\setminus\{i-1\}$, the functions L_{sj} in the representations (2.1) of functions φ_{sj} satisfy the condition S_0. Therefore, by virtue of (3.1_i) and Remark 2.2, for them we have

$$L_{sj}(y^{(j)}(t))=L_{sj}\big(\nu_j|\pi_\omega(t)|^{i-j-1}\big)[1+o(1)] \ \text{ when } \ t\uparrow\omega.$$

Taking into account (2.1) and the above representations, we can rewrite (3.5) in the form

$$\begin{aligned}y^{(n)}(t)=\alpha_s p_s(t)y^{(i-1)}(t)|^{\sigma_{si-1}}L_{si-1}(y^{(i-1)}(t))\times\\ \times\bigg(\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}|y^{(j)}(t)|^{\sigma_{sj}}L_{sj}\big(\nu_j|\pi_\omega(t)|^{i-j-1}\big)\bigg)[1+o(1)] \ \text{ at } \ t\uparrow\omega.\end{aligned}$$

Hence, using (1.7), (1.8) and bearing in mind the fact that according to (3.1_i),

$$\begin{aligned}y^{(n)}(t)=\frac{y^{(n)}(t)}{y^{(n-1)}(t)}\cdots\frac{y^{(i+2)}(t)}{y^{(i+1)}(t)}y^{(i+1)}(t)\sim\\ \sim\frac{(-1)^{n-i-1}(n-i)!}{\pi_\omega^{n-i-1}(t)}y^{(i+1)}(t) \ \text{ at } \ t\uparrow\omega,\end{aligned}$$

and the notation introduced before formulation of theorems, we get the following relation:

$$\begin{aligned}\frac{y^{(i+1)}(t)|y^{(i)}(t)|^{\gamma_{si}-1}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{si-1}(y^{(i-1)}(t))}=\\ =\alpha_s(-1)^{n-i-1}\big(\operatorname{sign}[\pi_\omega(t)]^{n-i-1}\big)C_{si}p(t)|\pi_\omega(t)|^{\mu_{si}}\times\\ \times\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}L_{sj}\big(\nu_j|\pi_\omega(t)|^{i-j-1}\big)[1+o(1)] \ \text{ at } \ t\uparrow\omega.\end{aligned} \tag{3.6}$$

By virtue of property M_2 of slowly varying functions, there is a continuously differentiated function $L_{0si-1}:\Delta_{Y_{i-1}}\to]0,+\infty[$ satisfying the conditions (2.3) for $k = s$ and $j = i-1$. Using these conditions and (3.1_i), we find that

$$\Big(\frac{|y^{(i)}(t)|^{\gamma_{si}}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}\Big)' = \frac{\nu_i y^{(i+1)}(t)|y^{(i)}(t)|^{\gamma_{si}-1}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}\times$$
$$\times\Bigg(\gamma_{si} - (\gamma_s-\gamma_{si})\frac{y^{(i)}(t)}{y^{(i+1)}(t)}\cdot\frac{y^{(i)}(t)}{y^{(i-1)}(t)} -$$
$$-\frac{y^{(i)}(t)}{y^{(i+1)}(t)}\cdot\frac{y^{(i)}(t)}{y^{(i-1)}(t)}\cdot\frac{y^{(i-1)}(t)L'_{0si-1}(y^{(i-1)}(t))}{L_{0si-1}(y^{(i-1)}(t))}\Bigg) =$$
$$= \frac{y^{(i+1)}(t)|y^{(i)}(t)|^{\gamma_{si}-1}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}\big[\nu_i\gamma_{si}+o(1)\big] \ \text{ at } \ t\uparrow\omega.$$

Therefore (3.6) can be rewritten in the form

$$\Big(\frac{|y^{(i)}(t)|^{\gamma_{si}}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))}\Big)' =$$
$$= \nu_i\alpha_s(-1)^{n-i-1}\gamma_{si}\big(\operatorname{sign}[\pi_\omega(t)]^{n-i-1}\big)C_{si}p(t)|\pi_\omega(t)|^{\mu_{si}}\times$$
$$\times\prod_{\substack{j=0\\ j\neq i-1}}^{n-1} L_{sj}\big(\nu_j|\pi_\omega(t)|^{i-j-1}\big)[1+o(1)] \ \text{ at } \ t\uparrow\omega.$$

Integrating this relation on the interval between t_1 and t and taking into account that the fraction under the derivative sign due to the condition $\gamma_{si}\neq 0$ tends either to zero, or to $\pm\infty$ as $t\uparrow\omega$, we get

$$\frac{|y^{(i)}(t)|^{\gamma_{si}}}{|y^{(i-1)}(t)|^{\gamma_{si}-\gamma_s}L_{0si-1}(y^{(i-1)}(t))} =$$
$$= \nu_i\alpha_s(-1)^{n-i-1}\gamma_{si}\big(\operatorname{sign}[\pi_\omega(t)]^{n-i-1}\big)C_{si}J_{si}(t)[1+o(1)] \ \text{ at } \ t\uparrow\omega.$$

From here first of all follows that the inequality (2.9_i) is fulfilled. Moreover, from this and (3.6), due to the equivalence of functions L_{si-1} and L_{0si-1} as $y^{(i-1)}\to Y_{i-1}$, we have

$$\frac{y^{(i+1)}(t)}{y^{(i)}(t)} = \frac{J'_{si}(t)}{\gamma_{si}J_{si}(t)}\,[1+o(1)] \ \text{ at } \ t\uparrow\omega,$$

whence, according to (3.1_i) for $j = i+1$, it follows that the first condition of (2.11_i) is valid.

From the obtained relation we also have

$$\frac{y^{(i)}(t)}{|y^{(i-1)}(t)|^{\frac{\gamma_{si}-\gamma_s}{\gamma_{si}}}L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))} =$$
$$= \nu_i\big|C_{si}\gamma_{si}J_{si}(t)\big|^{\frac{1}{\gamma_{si}}}[1+o(1)] \ \text{ at } \ t\uparrow\omega. \tag{3.7}$$

By virtue of the fact that

$$\left(\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}\right)' =$$

$$= \frac{\nu_{i-1}y^{(i)}(t)|y^{(i-1)}(t)|^{\frac{\gamma_s-\gamma_{si}}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}\left[\frac{\gamma_s}{\gamma_{si}} - \frac{1}{\gamma_{si}}\frac{y^{(i-1)}(t)L'_{0si}(y^{(i-1)}(t))}{L_{0si}(y^{(i-1)}(t))}\right] =$$

$$= \frac{\nu_{i-1}y^{(i)}(t)|y^{(i-1)}(t)|^{\frac{\gamma_s-\gamma_{si}}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}\left[\frac{\gamma_s}{\gamma_{si}} + o(1)\right] \text{ at } t \uparrow \omega,$$

from (3.7) it follows

$$\left(\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}\right)' = \frac{\nu_i\nu_{i-1}\gamma_s}{\gamma_{si}}\left|C_{si}\gamma_{si}J_{si}(t)\right|^{\frac{1}{\gamma_{si}}}[1+o(1)] \text{ when } t \uparrow \omega.$$

Here the fraction appearing under the derivative sign tends either to zero or to $\pm\infty$ as $t \uparrow \omega$, since by virtue of (1.4) and properties of slowly varying functions (see (3.3)),

$$\ln\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))} = \ln|y^{(i-1)}(t)|\left[\frac{\gamma_s}{\gamma_{si}} - \frac{1}{\gamma_{si}}\frac{\ln L_{0si-1}(y^{(i-1)}(t))}{\ln|y^{(i-1)}(t)|}\right] =$$

$$= \ln|y^{(i-1)}(t)|\left[\frac{\gamma_s}{\gamma_{si}} + o(1)\right] \to \pm\infty \text{ at } t \uparrow \omega.$$

That is why, by integrating this correlation on the interval from t_1 to t, we get

$$\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))} = \frac{\nu_i\nu_{i-1}\gamma_s}{\gamma_{si}}|\gamma_{si}C_{si}|^{\frac{1}{\gamma_{si}}}J_{sii}(t)[1+o(1)] \text{ at } t \uparrow \omega. \tag{3.8}$$

From here it follows the validity of the second inequality of (2.8_i) and also, in view of the equivalence of functions L_{si-1} and L_{0si-1} as $y^{(i-1)} \to Y_{i-1}$, the validity of the asymptotic representation (2.14_i). Besides, (3.7) and (3.8) yield

$$\frac{y^{(i)}(t)}{y^{(i-1)}(t)} = \frac{\gamma_{si}J'_{sii}(t)}{\gamma_s J_{sii}(t)}[1+o(1)] \text{ at } t \uparrow \omega. \tag{3.9_i}$$

By virtue of the last relation and Lemma 1.1, the second conditions of (2.10_i) and (2.11_i) are fulfilled, and asymptotic representations (2.12_i) and (2.13_i) hold.

Sufficiency. Let the conditions (2.8_i)–(2.11_i) be satisfied, and the algebraic equation (2.7) have no roots with zero real part. Let us show that in this case the equation (1.1) has solutions admitting asymptotic relations (2.12_i)–(2.14_i) as $t \uparrow \omega$.

Towards this end, we consider first the relation

$$\frac{|Y|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(Y)} = |\gamma_{si} C_{si}|^{\frac{1}{\gamma_{si}}} \left|\frac{\gamma_s}{\gamma_{si}} J_{sii}(t)\right| [1+v_n], \tag{3.10}$$

where $L_{0si} : \Delta_{Y_i} \to]0, +\infty[$ are continuously differentiated slowly varying as $Y \to Y_{i-1}$ functions, satisfying the conditions (2.3) (for $k = s$ and $j = i-1$) and existing due to the property M_2 of slowly varying functions.

Having chosen an arbitrary number $d \in]0, |\frac{\gamma_{si}}{\gamma_s}|[$, let us show that for some $t_0 \in]a, \omega[$ the relation (3.10) defined uniquely, on the set $[t_0, \omega[\times \mathbb{R}_{\frac{1}{2}}$, where $\mathbb{R}_{\frac{1}{2}} = \{v \in \mathbb{R} : \ |v| \leq \frac{1}{2}\}$, a continuously differentiated implicit function $Y = Y(t, v_n)$ of the type

$$Y(t, v_n) = \nu_{i-1} |J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s} + z(t, v_n)}, \tag{3.11}$$

where z is the function such that

$$|z(t, v_n)| \leq d \ \text{ for } \ (t, v_n) \in [t_0, \omega[\times \mathbb{R}_{\frac{1}{2}} \ \text{ and } \ \lim_{t \uparrow \omega} z(t, v_n) = 0$$
$$\text{uniformly with respect to } \ v_n \in \mathbb{R}_{\frac{1}{2}}. \tag{3.12}$$

Assuming in (3.10)

$$Y = \nu_{i-1} |J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s} + z} \tag{3.13}$$

and then taking the logarithm of the obtained relation, after elementary manipulations, we find that

$$z = a(t) + b(t, v_n) + Z(t, z), \tag{3.14}$$

where

$$a(t) = \frac{\gamma_{si}}{\gamma_s} \cdot \frac{\ln|\frac{\gamma_s}{\gamma_{si}}| + \frac{1}{\gamma_{si}} \ln|\gamma_{si} C_{si}|}{\ln|J_{sii}(t)|}, \quad b(t, v_n) = \frac{\gamma_{si}}{\gamma_s} \cdot \frac{\ln[1+v_n]}{\ln|J_{sii}(t)|},$$
$$Z(t, z) = \frac{1}{\gamma_s} \cdot \frac{\ln L_{0si-1}(\nu_{i-1} |J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s} + z})}{\ln|J_{sii}(t)|}.$$

Here, by virtue of the second condition of (2.10_i), by the choice of the limit of integration in J_{sii} and by the property (3.3) of slowly varying functions,

$$\nu_{i-1} \lim_{t \uparrow \omega} |J_{sii}(t)|^{\frac{\gamma_{is}}{\gamma_s} + z} = Y_{i-1}$$
$$\text{uniformly with respect to } \ z \in [-d, d], \quad \lim_{t \uparrow \omega} a(t) = 0, \tag{3.15}$$

$$\begin{aligned} &\lim_{t \uparrow \omega} b(t, v_n) = 0 \quad \text{uniformly with respect to} \quad v_n \in \mathbb{R}_{\frac{1}{2}}, \\ &\lim_{t \uparrow \omega} Z(t, z) = 0 \quad \text{uniformly with respect to} \quad z \in [-d, d]. \end{aligned} \tag{3.16}$$

Since

$$\frac{\partial Z(t, z)}{\partial z} = \frac{1}{\gamma_s} \cdot \frac{\nu_{i-1} |J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s} + z} L'_{osi-1}(\nu_{i-1} |J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s} + z})}{L_{0si-1}(\nu_{i-1} |J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s} + z})},$$

by virtue of (2.3) and the first of the above-stated conditions, we likewise have

$$\lim_{t\uparrow\omega}\frac{\partial Z(t,z)}{\partial z}=0 \ \text{ uniformly with respect to } \ z\in[-d,d].$$

According to these conditions, there is a number $t_1\in[a,\omega[$ such that

$$\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{is}}{\gamma_s}+z}\in\Delta_{Y_{i-1}} \ \text{ at } \ (t,z)\in[t_1,\omega[\,\times\mathbb{R}_d,$$
$$\text{where } \ \mathbb{R}_d=\big\{z\in\mathbb{R}:\ |z|\le d\big\}, \tag{3.17}$$
$$\big|a(t)+b(t,v_1,v_2)+Z(t,z)\big|\le d \ \text{ at } \ (t,v_n,z)\in[t_1,\omega[\,\times\mathbb{R}_{\frac{1}{2}}\times\mathbb{R}_d$$

and

$$\big|Z(t,z_1)-Z(t,z_2)\big|\le\frac{1}{2}\,|z_1-z_2| \ \text{ at } \ t\in[t_1,\omega[\ \text{ and } \ z_1,z_2\in\mathbb{R}_d. \tag{3.18}$$

Having chosen in this way the number t_1, we denote by $\mathbf{B}$ the Banach space of continuous and bounded on set $\Omega=[t_1,\omega[\,\times\mathbb{R}_{\frac{1}{2}}$ functions $z:\Omega\to\mathbb{R}$ with the norm

$$\|z\|=\sup\big\{|z(t,v_n)|:\ (t,v_n)\in\Omega\big\}.$$

We distinguish from it the subspace $\mathbf{B}_0$ of those functions from $\mathbf{B}$, for which $\|z\|\le d$, and consider on $\mathbf{B}_0$, choosing a fortiori an arbitrary number $\nu\in(0,1)$, the operator

$$\Phi(z)(t,v_n)=z(t,v_n)-\nu\big[z(t,v_n)-a(t)-b(t,v_n)-Z(t,z(t,v_n))\big]. \tag{3.19}$$

By virtue of (3.17) and (3.18), for any $z\in\mathbf{B}_0$ and $z_1,z_2\in\mathbf{B}_0$, we have

$$|\Phi(z)(t,v_n)|\le(1-\nu)|z(t,v_n)|+\nu d\le d \ \text{ and } \ (t,v_n)\in\Omega$$

and

$$\begin{aligned}&|\Phi(z_1)(t,v_n)-\Phi(z_2)(t,v_n)|\le\\ &\quad\le(1-\nu)|z_1(t,v_n)-z_2(t,v_n)|+\nu|Z(t,z_1(t,v_n))-Z(t,z_2(t,v_n)|\le\\ &\qquad\le(1-\nu)|z_1(t,v_n)-z_2(t,v_n)|+\frac{\nu}{2}|z_1(t,v_n)-z_2(t,v_n)|\le\\ &\qquad\qquad\le\Big(1-\frac{\nu}{2}\Big)\|z_1-z_2\| \ \textit{at} \ (t,v_n)\in\Omega.\end{aligned}$$

This implies that $\Phi(\mathbf{B}_0)\subset\mathbf{B}_0$ and $\|\Phi(z_1)-\Phi(z_2)\|\le(1-\frac{\nu}{2})\|z_1-z_2\|$.

It means that the operator Φ maps the space $\mathbf{B}_0$ into itself and is a contractor operator on it. Then, by the contraction mapping principle, there is a unique function $z\in\mathbf{B}_0$ such that $z=\Phi(z)$. By virtue of (3.19), this continuous on set Ω function is a unique solution of the equation (3.14) satisfying the condition $\|z\|\le d$. From (3.14), with regard for (3.15), (3.16), it follows that the given solution tends to zero as $t\uparrow\omega$ uniformly with respect to $v_n\in\mathbb{R}_{\frac{1}{2}}$. Continuous differentiability of this solution on the set $[t_0,\omega[\,\times\mathbb{R}_{\frac{1}{2}}$, where t_0 is some number from $[t_1,\omega[$, follows directly from the well-known local theorem on the existence of an implicit function defined by the relation (3.14). In virtue of replacement (3.13), the obtained function z corresponds to a continuously differentiated on set $[t_0,\omega[\,\times\mathbb{R}_{\frac{1}{2}}$ function Y of type (3.11), where z possesses the properties (3.12) and which is a solution of the equation (3.10) and satisfies the conditions

$$\begin{gathered} Y(t,v_n)\in\Delta_{Y_{i-1}} \ \text{ for } \ (t,v_n)\in[t_0,\omega[\,\times\mathbb{R}_{\frac{1}{2}}, \\ \lim_{t\uparrow\omega} Y(t,v_n)=Y_{i-1} \ \text{ uniformly with respect to } \ v_n\in\mathbb{R}_{\frac{1}{2}}. \end{gathered} \tag{3.20}$$

Now, applying to differential equation (1.1) the transformation

$$\begin{gathered} y^{(j-1)}(t)=\frac{[\pi_\omega(t)]^{i-j}}{(i-j)!}\,y^{(i-1)}(t)[1+v_j(\tau)] \ \ (j=1,\dots,i-1), \\ y^{(j)}(t)=(-1)^{j-i}\frac{(j-i)!}{[\pi_\omega(t)]^{j-i}}\cdot\frac{\gamma_{si}J'_{sii}(t)}{\gamma_s J_{sii}(t)}\,y^{(i-1)}(t)[1+v_j(\tau)] \\ (j=i,\dots,n-1), \\ y^{(i-1)}(t)=Y(t,v_n(\tau)), \ \ \tau(t)=\beta\ln|\pi_\omega(t)|, \end{gathered} \tag{3.21$_i$}$$

where β is defined in (2.6_i), and bearing in mind that the function $y^{(i-1)}(t)=Y(t,v_n(\tau))$ for $t\in[t_0,\omega[$ and $v_n(\tau)\in\mathbb{R}^{\frac{1}{2}}$ satisfies equation

$$\frac{|y^{(i-1)}(t)|^{\frac{\gamma_s}{\gamma_{si}}}}{L_{0si-1}^{\frac{1}{\gamma_{si}}}(y^{(i-1)}(t))}=|\gamma_{si}C_{si}|^{\frac{1}{\gamma_{si}}}\Big|\frac{\gamma_s}{\gamma_{si}}\,J_{sii}(t)\Big|[1+v_n(\tau)],$$

with the use of sign conditions (2.8_i), (2.9_i), we get a system of differential equations of the form

$$\begin{cases} v'_j=\beta\Big[(i-j)(v_{j+1}-v_j)-\dfrac{\gamma_{si}}{\gamma_s}h_1(\tau)(1+v_j)(1+v_i)\Big] \ \ (j=1,\dots,i-2), \\ v'_{i-1}=\beta\Big[-v_{i-1}-\dfrac{\gamma_{si}}{\gamma_s}\,h_1(\tau)(1+v_{i-1})(1+v_i)\Big], \\ v'_j=\beta\Big[(j-i)(1+v_j)-(j+1-i)(1+v_{j+1})-\dfrac{1}{\gamma_{si}}\,h_2(\tau)(1+v_j)+ \\ \qquad +\dfrac{1}{\gamma_s}h_1(\tau)(1+v_j)(\gamma_s-\gamma_{si}-\gamma_{si}v_i)\Big] \ \ (j=i,\dots,n-2), \\ v'_{n-1}=\beta\Bigg[\dfrac{n-i}{\gamma_{si}}\,h_2(\tau)\,\dfrac{\prod\limits_{j=0}^{i-2}|1+v_{j+1}|^{\sigma_{sj}}\prod\limits_{j=i}^{n-1}|1+v_j|^{\sigma_{sj}}}{|1+v_n|^{\gamma_{si}}}\,G(\tau,v_1,\dots,v_n)+ \\ \qquad +(n-i-1)(1+v_{n-1})-\dfrac{1}{\gamma_{si}}\,h_2(\tau)(1+v_{n-1})+ \\ \qquad +\dfrac{1}{\gamma_s}\,h_1(\tau)(1+v_{n-1})(\gamma_s-\gamma_{si}-\gamma_{si}v_i)\Bigg], \\ v'_n=\beta h_1(\tau)\Big[(1+v_n)(1+v_i)-(1+v_n)-\dfrac{1}{\gamma_s}\,H(\tau,v_n)(1+v_n)(1+v_i)\Big], \end{cases}$$

in which

$$h_1(\tau)=h_1(\tau(t))=\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}\,, \quad h_2(\tau)=h_2(\tau(t))=\frac{\pi_\omega(t)J'_{si}(t)}{J_{si}(t)}\,,$$

$$G(\tau(t), v_1, \dots, v_n) = \frac{L_{si-1}(Y(t,v_n))}{L_{0si-1}(Y(t,v_n))} \cdot \frac{\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} L_{sj}(Y^{[j]}(t, v_j, v_{j+1}, v_n))}{\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} L_{sj}(\nu_j |\pi_\omega(t)|^{i-j-1})} \times$$

$$\times \frac{\sum\limits_{k=1}^{m} \alpha_k p_k(t) \varphi_{ki-1}(Y(t,v_n)) \prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} \varphi_{kj}(Y^{[j]}(t, v_j, v_{j+1}, v_n)}{\alpha_s p_s(t) \varphi_{si-1}(Y(t,v_n)) \prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} \varphi_{sj}(Y^{[j]}(t, v_j, v_{j+1}, v_n))},$$

$$H(\tau(t), v_n) = \frac{Y(t,v_n) L'_{0si-1}(Y(t,v_n))}{L_{0si-1}(Y(t,v_n))},$$

$$Y^{[j]}(t, v_j, v_{j+1}, v_n) =$$

$$= \begin{cases} \dfrac{\pi_\omega^{i-j-1}(t)}{(i-j-1)!}\, Y(t,v_n)(1+v_{j+1}) & \text{when } j = \overline{0, i-2}, \\ \dfrac{(j-i)!}{\pi_\omega^{j-i}(t)} \, \dfrac{\gamma_{si}}{\gamma_s} \, \dfrac{J'_{sii}(t)}{J_{sii}(t)}\, Y(t,v_n)(1+v_j) & \text{when } j = \overline{i, n-1}. \end{cases}$$

Here, the function $\tau(t) = \beta \ln |\pi_\omega(t)|$ possesses the properties

$$\tau'(t) > 0 \ \text{ at } \ t \in [t_0, \omega[, \quad \lim_{t \uparrow \omega} \tau(t) = +\infty$$

and that is why, according to conditions (2.11_i),

$$\begin{aligned} \lim_{\tau \to +\infty} h_1(\tau) &= \lim_{t\uparrow\omega} h_1(\tau(t)) = 0, \\ \lim_{\tau \to +\infty} h_2(\tau) &= \lim_{t\uparrow\omega} h_2(\tau(t)) = -\gamma_{si}. \end{aligned} \tag{3.22}$$

By virtue of (3.20) and (2.3) (for $k = s$ and $j = i-1$) the function H tends to zero as $\tau \to +\infty$ uniformly with respect to $v_n \in \mathbb{R}_{\frac{1}{2}}$, and first fraction in the representation of the function G tends to unity as $\tau \to +\infty$ uniformly with respect to $v_n \in \mathbb{R}_{\frac{1}{2}}$.

Let us show that the second and third fractions in the representation of function G likewise tends to unity as $\tau \to +\infty$ uniformly with respect to $(v_1, \dots, v_n) \in \mathbb{R}^n_{\frac{1}{2}}$.

By virtue of (2.11_i) and using the l'Hospital's rule, we have

$$\lim_{t\uparrow\omega} \frac{\ln |J_{sii}(t)|}{\ln |\pi_\omega(t)|} = \lim_{t\uparrow\omega} \frac{\pi_\omega(t) J'_{sii}(t)}{J_{sii}(t)} = 0,$$

$$\lim_{t\uparrow\omega} \frac{\ln |\frac{J'_{sii}(t)}{J_{sii}(t)}|}{\ln |\pi_\omega(t)|} = \lim_{t\uparrow\omega} \Big[\frac{\pi_\omega(t) J'_{si}(t)}{\gamma_{si} J_{si}(t)} - \frac{\pi_\omega(t) J'_{sii}(t)}{J_{sii}(t)} \Big] = -1.$$

Taking into account the type of functions Y and $Y^{[j]}$ ($j=\overline{0,n-1}$, $j\neq i-1$), we find

$$\lim_{t\uparrow\omega}\frac{\ln|Y(t,v_n)|}{\ln|\pi_\omega(t)|}=\lim_{t\uparrow\omega}\Big[\frac{\gamma_s}{\gamma_{si}}+z(t,v_n)\Big]\lim_{t\uparrow\omega}\frac{\ln|J_{sii}(t)|}{\ln|\pi_\omega(t)|}=0$$

uniformly with respect to $v_n\in\mathbb{R}_{\frac12}$,

$$\lim_{t\uparrow\omega}\frac{\ln|Y^{[j]}(t,v_j,v_{j+1},v_n)|}{\ln|\pi_\omega(t)|}=$$

$$=i-j-1+\lim_{t\uparrow\omega}\frac{\ln|Y(t,v_n)|}{\ln|\pi_\omega(t)|}+\lim_{t\uparrow\omega}\frac{\ln\frac{|1+v_{j+1}|}{(i-j-1)!}}{\ln|\pi_\omega(t)|}=i-j-1$$

uniformly with respect to $(v_{j+1},v_n)\in\mathbb{R}^2_{\frac12}$ for $j=\overline{0,i-2}$

and

$$\lim_{t\uparrow\omega}\frac{\ln|Y^{[j]}(t,v_j,v_{j+1},v_n)|}{\ln|\pi_\omega(t)|}=i-j+\lim_{t\uparrow\omega}\frac{\ln|Y(t,v_n)|}{\ln|\pi_\omega(t)|}+$$

$$+\lim_{t\uparrow\omega}\frac{\ln|\frac{J'_{sii}(t)}{J_{sii}(t)}|}{\ln|\pi_\omega(t)|}+\lim_{t\uparrow\omega}\frac{\ln\frac{(j-i)!|\gamma_{si}(1+v_j)|}{|\gamma_s|}}{\ln|\pi_\omega(t)|}=i-j-1$$

uniformly with respect to $(v_j,v_n)\in\mathbb{R}^2_{\frac12}$ for $j=\overline{i,n-1}$.

In view of these marginal ratios and using inequalities (2.6_i) we find, repeating the reasoning in proving the necessity, that for any $k\in\{1,\dots,m\}\setminus\{s\}$

$$\lim_{t\uparrow\omega}\frac{p_k(t)\varphi_{ki-1}(Y(t,v_n))\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\varphi_{kj}(Y^{[j]}(t,v_j,v_{j+1},v_n))}{p_s(t)\varphi_{si-1}(Y(t,v_n))\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\varphi_{sj}(Y^{[j]}(t,v_j,v_{j+1},v_n))}=0$$

uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac12}$.

Owing to these conditions, the last fraction in the representation of function G tends to unity as $\tau\to+\infty$ uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac12}$.

Moreover, taking into account marginal ratios stated above, we obtain the following representations:

$$Y^{[j]}(t,v_j,v_{j+1},v_n)=\nu_j e^{\ln|Y^{[j]}(t,v_j,v_{j+1},v_n)|}=$$

$$=\nu_j e^{[1+r_j(t,v_j,v_{j+1},v_n)]\ln|\pi_\omega(t)|^{i-j-1}}\ \text{ as }\ j\in\{0,\dots,n-1\}\setminus\{i-1\},$$

where

$$\lim_{t\uparrow\omega}r_j(t,v_j,v_{j+1},v_n)=0\ \text{ uniformly with respect to }\ (v_j,v_{j+1},v_n)\in\mathbb{R}^3_{\frac12}$$

$$\text{for all }\ j\in\{0,\dots,n-1\}\setminus\{i-1\}.$$

Since the functions L_{sj} ($j=\overline{1,n-1}$, $j\neq i-1$) satisfy the condition S_0, by Remark 2.3, it follows that

$$\lim_{t\uparrow\omega}\frac{\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} L_{sj}(Y^{[j]}(t,v_j,v_{j+1},v_n))}{\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} L_{sj}(\nu_j|\pi_\omega(t)|^{i-j-1})}=1$$

$$\text{uniformly with respect to } (v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}.$$

Therefore, the second fraction in the representation of function G tends to unity as $\tau\to+\infty$ uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}$.

Due to above stated, the obtained system of differential equations can be written in form

$$\begin{cases} v_j' = \beta\Big[f_i(\tau,v_1,\dots,v_n)+\sum\limits_{k=1}^{n} p_{jk}v_k\Big] \ \ (k=\overline{1,n-2}),\\ v_{n-1}'=\beta\Big[f_{n-1}(\tau,v_1,\dots,v_n)+\sum\limits_{k=1}^{n} p_{n-1k}v_k+V_{n-1}(v_1,\dots,v_n)\Big],\\ v_n' = \beta h_1(\tau)\Big[f_n(\tau,v_1,\dots,v_n)+\sum\limits_{k=1}^{n} p_{nk}v_k+V_n(v_1,\dots,v_n)\Big], \end{cases} \tag{3.23$_i$}$$

where the functions f_i $(i=\overline{1,n})$ are continuous on a set $[\tau_1,+\infty[\,\times\mathbb{R}^n_{\frac{1}{2}}$ for some $\tau_1\geq\beta\ln|\pi_\omega(t_0)|$ and are such that

$$\begin{gathered} \lim_{\tau\to+\infty} f_i(\tau,v_1,\dots,v_n)=0 \ \ (i=\overline{1,n})\\ \text{uniformly with respect to } (v_1,\dots,v_n)\in\mathbb{R}^n_{\frac{1}{2}}, \qquad (3.24)\\ p_{jj}=j-i, \quad p_{jj+1}=i-j,\\ {}_{jk}=0 \ \text{ at } \ k\in\{1,\dots,n\}\setminus\{j,j+1\} \ \ (j=\overline{1,i-2}),^{*}\\ p_{i-1i-1}=-1, \quad p_{i-1k}=0 \ \text{ at } \ k\in\{1,\dots,n\}\setminus\{i-1\},\\ p_{jj}=j-i+1, \quad p_{jj+1}=i-j-1,\\ p_{jk}=0 \ \text{ at } \ k\in\{1,\dots,n\}\setminus\{j,j+1\} \ \ (j=\overline{i,n-2}),\\ p_{n-1k}=-(n-i)\sigma_{sk-1} \ \ (k=\overline{1,i-1}),\\ p_{n-1k}=-(n-i)\sigma_{sk} \ \ (k=\overline{i,n-2}), \quad p_{n-1n-1}=(n-i)(1-\sigma_{sn-1}),\\ p_{n-1n}=(n-i)\gamma_{si}, \quad p_{ni}=i, \quad p_{nk}=0 \ \text{ at } \ k\in\{1,\dots,n\}\setminus\{i\}, \end{gathered}$$

$$V_n(v_1,\dots,v_n)=v_iv_n,$$

$$V_{n-1}(v_1,\dots,v_n)=(i-n)\frac{\prod\limits_{j=0}^{i-2}|1+v_{j+1}|^{\sigma_{sj}}\prod\limits_{j=i}^{n-1}|1+v_j|^{\sigma_{sj}}}{|1+v_n|^{\gamma_{si}}}+$$

$$+(n-i)\Big[1+\sum_{k=1}^{i-1}\sigma_{sk-1}v_k+\sum_{k=i}^{n-1}\sigma_{sk}v_k-\gamma_{si}v_n\Big].$$

Since conditions (3.24) are satisfied and

$$\lim_{|v_1|+\dots+|v_n|\to 0} \frac{\partial V_j(v_1,\dots,v_n)}{\partial v_k} = 0 \quad (j=n-1,n; \quad k=\overline{1,n}),$$

this system belongs to the class of systems of differential equations, for which the criteria for the existence of vanishing at infinity solutions were obtained in [24]. Let us show that for this system the conditions of Theorem 2.6 are fulfilled (based on this paper).

First of all, taking into account the conditions (3.22) and the type of integral $J_{sii}(t)$, we notice that the function h_1 possesses the properties

$$\lim_{\tau\to+\infty} h_1(\tau)=0, \qquad \int\limits_{\tau_1}^{+\infty} h_1(\tau)\,d\tau = \beta\int\limits_{t_1}^{\omega} \frac{J'_{sii}(t)}{J_{sii}(t)}\,dt =$$
$$= \beta\ln|J_{sii}(t)|\big|_{t_1}^{\omega} = \pm\infty \quad (\tau_1=\beta\ln|\pi_\omega(t_1)|),$$
$$\lim_{\tau\to+\infty}\frac{h_1'(\tau)}{h_1(\tau)} = \lim_{t\uparrow\omega}\frac{(h_1(\tau(t)))'_t}{\tau'(t)h_1(\tau(t))} =$$
$$=\beta\lim_{t\uparrow\omega}\left[\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}+\frac{1}{\gamma_{si}}\,\frac{\pi_\omega(t)J'_{si}(t)}{J_{si}(t)}\,\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}-\Big(\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}\Big)^2\right]=0.$$

Next, consider the matrices $P_n=(p_{jk})_{j,k=1}^{n}$ and $P_{n-1}=(p_{jk})_{j,k=1}^{n-1}$, for which we have

$$\det P_{n-1} = (-1)^{i-1}(i-1)!(n-i)!\gamma_{si}, \quad \det P_n = (-1)^{i}(i-1)!(n-i)!\gamma_{si},$$
$$\det\big[P_{n-1}-\rho E_{n-1}\big] = (-1)^{i-1}\prod_{k=1}^{i-1}(k+\rho)\Bigg[\prod_{m=1}^{n-i}(m-\rho)-$$
$$-(n-i)!\sum_{j=i+1}^{n-1}\frac{\sigma_{sj}}{(j-i)!}\prod_{m=1}^{j-i}(m-\rho)-(n-i)!\sigma_{si}\Bigg],$$

where E_{n-1} is the unit matrix of dimension $(n-1)\times(n-1)$.

Since algebraic equation (2.7), according to the conditions of Theorem, has no roots with zero real part, the characteristic equation of the matrix P_{n-1} has likewise no such roots, and the given characteristic equation has $i-1$ roots (if $i>1$) of the type $\rho_k=-k$ $(k=\overline{1,i-1})$.

Thus, for the system (3.23_i), all the conditions of Theorem 2.6 of [24] are satisfied. According to this theorem, the system (3.23_i) has at least one solution $(v_j)_{j=1}^n : [\tau_2,+\infty[\to\mathbb{R}^n$ $(\tau_2\ge\tau_1)$ tending to zero as $\tau\to+\infty$.

Moreover, if l is a number of roots of the equation (2.7) with negative real part, and r is a number of roots with positive real part, then according to the same Theorem, in case $\beta=1$, this system has $i+1$ - parametric family of such solutions if the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si}>0$ is fulfilled, and has $i-1+l$- parametric family if the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si}<0$ is fulfilled, whereas, in case $\beta=-1$, there is $r+1$ - parameter family of such solutions if there is the inequality $\nu_i \nu_{i-1} \gamma_s \gamma_{si} > 0$ and r - parametric family if there is the inequality $\nu_i \nu_{i-1} \gamma_s \gamma_{si} < 0$.

To every such solution of the system (3.23_i) there corresponds, due to the replacements (3.21_i) and the first condition of (2.3), the solution $y : [t_2, \omega[\to \mathbb{R}$ $(t_2 \in [a, \omega[)$ of the equation (1.1) admitting as $t \uparrow \omega$ asymptotic representations (2.12_i)–(2.14_i). Using these representations and conditions (2.6_i), (2.8_i)–(2.11_i), it can be easily seen that it is a $P_\omega(Y_0, \dots, Y_{n-1}, \frac{n-i-1}{n-i})$-solution. □

Proof of Theorem 2.2. Let the equation (1.1) have $P_\omega(Y_0, \dots, Y_{n-1}, \frac{n-i-1}{n-i})$-solution $y : [t_0, \omega[\to \Delta_{Y_0}$. Then, according to Theorem 2.1, the conditions (2.8_i)–(2.11_i) are satisfied and for this solution the asymptotic representations (2.12_i)–(2.14_i) hold as $t \uparrow \omega$. Furthermore, from the proof of necessity of that theorem it is clear that the condition (3.9_i) is satisfied. Since the functions L_{si-1} satisfy the condition S_0, by virtue of (3.9_i) and Remark 2.2,

$$L_{si-1}(y^{(i-1)}(t)) = L_{sj}\big(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}\big)[1 + o(1)] \ \text{ at } \ t \uparrow \omega.$$

Therefore it follows from (2.14_i) that

$$|y^{(i-1)}(t)|^{\gamma_s} =$$
$$= |\gamma_{si} C_{si}| L_{si-1}\big(\nu_{i-1}|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}\big) \Big|\frac{\gamma_s}{\gamma_{si}} J_{sii}(t)\Big|^{\gamma_{si}} [1 + o(1)] \quad \text{at} \quad t \uparrow \omega,$$

which results in the presentation (2.15_i). □

4. Example of Equation with Regularly Varying as $t \uparrow \omega$ Coefficients

Suppose that in the differential equation (1.1), the continuous functions $p_k : [a, \omega[\to]0, +\infty[$ $(k = \overline{1, m})$ are regularly varying, as $t \uparrow \omega$, of orders ϱ_k $(k = \overline{1, m})$, and, moreover, the conditions of Theorem 2.1 as $i \in \{1, \dots, n-2\}$ are satisfied. In this case

$$\lim_{t \uparrow \omega} \frac{\ln p_k(t)}{\ln |\pi_\omega(t)|} = \varrho_k \tag{4.1}$$

and the conditions (2.6_i) take the form

$$\beta(\varrho_k - \varrho_s) < \beta \sum_{\substack{j=0 \\ j \neq i-1}}^{n-1} (\sigma_{sj} - \sigma_{kj})(i - j - 1)$$
$$\text{at all } \ k \in \{1, \dots, m\} \setminus \{s\}. \tag{4.2_i}$$

Since as $t \uparrow \omega$ the functions $L_{sj}(\nu_j |\pi_\omega(t)|^{i-j-1})$ $(j \in \{0, \dots, n-1\} \setminus \{i-1\})$ are slowly varying, and the function p_s is regularly varying of order ϱ_s, therefore the function J_{si} is regularly varying of order $1 + \varrho_s + \mu_{si}$, and the function $|J_{sii}(t)|$ is regularly varying of order $1+\frac{1}{\gamma_{si}}(1+\varrho_s+\mu_{si})$ as $t\uparrow\omega$. This implies that

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{si}(t)}{J_{si}(t)}=1+\varrho_s+\mu_{si},$$
$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'_{sii}(t)}{J_{sii}(t)}=1+\frac{1}{\gamma_{si}}\,(1+\varrho_s+\mu_{si}).$$

Therefore the conditions (2.11_i) will be of the following form:

$$1+\varrho_s+\gamma_{si}+\mu_{si}=0. \tag{4.3$_i$}$$

Taking into account this condition, the function $J_{sii}(t)$ should be slowly varying as $t\uparrow\omega$. In order to get asymptotic representation for this integral we have to know the type of a slowly varying component of the integrand equation.

Suppose that the functions p_s and φ_{sj} $(j=\overline{0,n-1})$ are of the form

$$\begin{gathered} p_s(t)=|\pi_\omega(t)|^{\varrho_s}\big|\ln|\pi_\omega(t)|\big|^{r_s},\\ \varphi_{sj}(y^{(j)})=|y^{(j)}|^{\sigma_{sj}}\big|\ln|y^{(j)}|\big|^{\lambda_{sj}}\quad (j=\overline{0,n-1}). \end{gathered}\tag{4.4}$$

In this case, $L_{sj}(y^{(j)})=\big|\ln|y^{(j)}|\big|^{\lambda_{sj}}$ $(j=\overline{0,n-1})$ and hence all of them satisfy the conditions of S_0. Additionally, we get as $t\uparrow\omega$ the following asymptotic relations

$$J_{si}(t)\sim-\frac{\beta}{\gamma_{si}}\prod_{\substack{j=0\\ j\neq i-1}}^{n-1}|i-j-1|^{\lambda_{sj}}|\pi_\omega(t)|^{-\gamma_{si}}\big|\ln|\pi_\omega(t)|\big|^{r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}}, \tag{4.5$_i$}$$

$$J_{sii}(t)\sim\begin{cases}\dfrac{\gamma_{si}\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}|i-j-1|^{\lambda_{sj}}}{|\gamma_{si}|^{\frac{1}{\gamma_{si}}}\Big(r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}+\gamma_{si}\Big)}\big|\ln|\pi_\omega(t)|\big|^{1+\frac{1}{\gamma_{si}}\Big(r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}\Big)}, & \\ \qquad\qquad \text{if } r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}\neq-\gamma_{si}, & \\ \dfrac{\beta}{|\gamma_{si}|^{\frac{1}{\gamma_{si}}}}\prod\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}|j-i-1|^{\lambda_{sj}}\ln\big|\ln|\pi_\omega(t)|\big|, & \\ \qquad\qquad \text{if } r_s+\sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1}\lambda_{sj}=-\gamma_{si}, & \end{cases}\tag{4.6$_i$}$$

$$\frac{J'_{sii}(t)}{J_{sii}(t)} \sim \begin{cases} \dfrac{r_s + \sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj} + \gamma_{si}}{\gamma_{si}\pi_\omega(t)\ln|\pi_\omega(t)|}\,, & \text{if } r_s + \sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj} \neq -\gamma_{si}, \\ \dfrac{1}{\pi_\omega(t)\ln|\pi_\omega(t)|\ln|\ln|\pi_\omega(t)||}\,, & \text{if } r_s + \sum\limits_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj} = -\gamma_{si}. \end{cases} \tag{4.7$_i$}$$

From the above relations it, in particular, follows that the inequalities (2.8_i), (2.9_i) and the conditions (2.10_i) take the form

$$\begin{aligned} &\nu_j\nu_{j-1}(i-j)\pi_\omega(t) > 0 \quad \text{at all } j \in \{1,\dots,n-1\}\setminus\{i\}, \\ &\nu_i\alpha_s(-1)^{n-i}\pi_\omega^{n-i}(t) > 0, \end{aligned} \tag{4.8$_i$}$$

$$\nu_i\nu_{i-1}\gamma_s\gamma_{si} > 0 \ (<0), \ \text{ if } 1 + \frac{1}{\gamma_{si}}\Big(r_s + \sum_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj}\Big) \geq 0 \ (<0), \tag{4.9$_i$}$$

$$\nu_{j-1}\lim_{t\uparrow\omega}|\pi_\omega(t)|^{i-j} = Y_{j-1} \ \text{ at } \ j \in \{1,\dots,n\}\setminus\{i\}, \tag{4.10$_i$}$$

$$\nu_{i-1}Y_{i-1} = \infty \ (=0), \ \text{ if } \ \gamma_s\Big(r_s + \sum_{\substack{j=0\\ j\neq i-1}}^{n-1} \lambda_{sj} + \gamma_{si}\Big) \geq 0 \ (<0). \tag{4.11$_i$}$$

By virtue of above-said, from Theorem 2.2 follows the following statement.

Corollary 4.1. *Let in the equation* (1.1) $n > 2$, *the functions* p_k $(k = \overline{1,m})$ *be regularly varying of orders* ϱ_k *at* $t \uparrow \omega$, $i \in \{1,\dots,n-2\}$ *and for some* $s \in \{1,\dots,m\}$, *the inequalities* (4.2_i) *be fulfilled. Let, moreover, the equation* $\gamma_s\gamma_{si} \neq 0$ *be fulfilled and the representations* (4.4) *hold. Then for the equation* (1.1) *to have* $P_\omega(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i})$*-solutions, it is necessary, and if algebraic equation* (2.7) *has no roots with zero real part, then it is sufficient that the conditions* (4.3_i), (4.8_i)–(4.11_i) (*along with* (1.3)) *are satisfied. Moreover, for each such solution there exist, as* $t \uparrow \omega$, *the following asymptotic representations:*

$$y^{(j-1)}(t) = \frac{[\pi_\omega(t)]^{i-j}}{(i-j)!}\,y^{(i-1)}(t)[1+o(1)] \ \ (j = 1,\dots,i-1), \tag{4.12$_i$}$$

$$y^{(j)}(t) = (-1)^{j-i}\,\frac{(j-i)!}{[\pi_\omega(t)]^{j-i}}\cdot\frac{\gamma_{si}J'_{sii}(t)}{\gamma_s J_{sii}(t)}\,y^{(i-1)}(t)[1+o(1)] \tag{4.13$_i$}$$
$$(j = i,\dots,n-1),$$

$$y^{(i-1)}(t) = \nu_{i-1}\Big|\gamma_{si}C_{si}\Big|\frac{\gamma_{si}}{\gamma_s}\Big|^{\lambda_{si-1}-\gamma_{si}}\Big|^{\frac{1}{\gamma_s}} \times$$
$$\times\,|J_{sii}(t)|^{\frac{\gamma_{si}}{\gamma_s}}\,\big|\ln|J_{sii}(t)|\big|^{\frac{\lambda_{si-1}}{\gamma_s}}[1+o(1)], \tag{2.15$_i$}$$

where the functions $J_{sii}(t)$ and $\frac{J'_{sii}(t)}{J_{sii}(t)}$ are defined by (4.6_i) and (4.7_i), respectively, and for such solutions in case $\omega = +\infty$ there exists an $i+1$-parametric family if the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si} > 0$ is fulfilled, and an $i-1+l$-parameter family if there is the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si} < 0$, while in case $\omega < +\infty$ there exists an $r+1$-parametric family of such solutions if the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si} > 0$ is fulfilled, and an r- parametric family if there is the inequality $\nu_i\nu_{i-1}\gamma_s\gamma_{si} < 0$, where l is a number of roots of the equation (2.7) with negative real part and r is a number of its roots with positive real part.

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(Received December 4, 2014)

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