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SUFFICIENCY CONDITIONS FOR ASYMPTOTIC STABILITY OF SOLUTIONS OF A LINEAR HOMOGENEOUS NONAUTONOMOUS DIFFERENTIAL EQUATION OF SECOND ORDER

Abstract. The problem on the stability of second order linear homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0$$

is investigated in the case where the roots $\lambda_i(t)$ $(i = 1, 2)$ of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

are such that

$$\lambda_i(t) < 0 \ \text{ for } \ t \geq t_0, \qquad \int\limits_{t_0}^{+\infty} \lambda_i(t)\, dt = -\infty \ \ (i = 1, 2)$$

and there exist finite or infinite limits $\lim\limits_{t\to+\infty} \lambda_i(t)$ $(i = 1, 2)$.

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მდგრადობის საკითხი იმ შემთხვევაში, როცა მახასიათებელი განტოლების

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ფესვებს $\lambda_i(t)$ $(i = 1, 2)$ გააჩნიათ სასრული ან უსასრულო ზღვრები, როცა $t \to +\infty$, ამასთან

$$\lambda_i(t) < 0, \ \text{ როცა } \ t \geq t_0, \qquad \int\limits_{t_0}^{+\infty} \lambda_i(t)\, dt = -\infty \ \ (i = 1, 2).$$

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1. Introduction

In the theory of stability of linear homogeneous on-line systems (LHS) of ordinary differential equations

$$\frac{dY}{dt} = P(t)Y, \ \ t \in [t_0; +\infty) = I,$$

where $P(t)$ is, in general, complex matrix, the interest is focused on the investigation of stability of LHS depending on the roots $\lambda_i(t)$ $(i = \overline{1, n})$ of the characteristic equation

$$\det(P(t) - \lambda E) = 0.$$

L. Cesàro [1] considered a system of n-th order differential equations

$$\frac{dY}{dt} = [A + B(t) + C(t)]Y,$$

where A is a constant matrix, the roots λ_i $(i = \overline{1, n})$ of characteristic equation are different and satisfy the condition

$$\operatorname{Re}\lambda_i \le 0 \ \ (i = \overline{1, n});$$

$$B(t) \to 0 \ \text{ as } \ t \to +\infty, \qquad \int\limits_{t_0}^{+\infty} \Big\| \frac{dB(t)}{dt} \Big\| \, dt < +\infty;$$

$$\int\limits_{t_0}^{+\infty} \|C(t)\| \, dt < +\infty;$$

the roots of characteristic equation of the matrix $A + B(t)$ have nonpositive real parts.

C. P. Persidsky's article [2] deals with the case, where elements of the matrix $P(t)$ are the functions with weak variation, that is, each such function can be represented as

$$f(t) = f_1(t) + f_2(t),$$

where $f_1(t) \in C_I$ and there exists $\lim\limits_{t \to +\infty} f_1(t) \in \mathbb{R}$, but $f_2(t)$ is such that

$$\sup_{t \in I} |f_2(t)| < +\infty, \qquad \lim_{t \to +\infty} f_2'(t) = 0,$$

and the condition $\operatorname{Re}\lambda_i(t) \le a \in \mathbb{R}_-$ $(i = \overline{1, n})$ is fulfilled.

N. Y. Lyashchenko [3] considered a case $\operatorname{Re}\lambda_i(t) < a \in \mathbb{R}_-$ $(i = \overline{1, n})$, $t \in I$,

$$\sup_{t \in I} \|A'(t)\| \le \varepsilon.$$

The case $n = 2$ is thoroughly studied by N. I. Izobov.

I. K. Hale [4] studied the asymptotic behavior of LHS comparing the roots of the characteristic equation with exponential functions

$$\operatorname{Re}\lambda_i(t) \le -gt^{\beta}, \ \ g > 0, \ \ \beta > -1 \ \ (i = \overline{1, n}).$$

Then there exist the constants $K > 0$ and $0 < \rho < 1$ such that for solving the system

$$\frac{dy}{dt} = A(t)y$$

the estimate

$$\|y(t)\| \le K e^{-\frac{\rho g}{1+\beta} t^{1+\beta}} \|y(0)\|$$

is fulfilled.

The present paper considers the problem of stability of a second order real linear homogeneous differential equation (LHDE)

$$y'' + p(t)y' + q(t)y = 0 \quad (t \in I) \tag{1}$$

provided that the roots $\lambda_i(t)$ $(i = 1, 2)$ of the characteristic equation

$$\lambda^2 + p(t)\lambda + q(t) = 0$$

are such that

$$\lambda_i(t) < 0 \quad (t \in I), \qquad \int\limits_{t_0}^{+\infty} \lambda_i(t)\, dt = -\infty \quad (i = 1, 2) \tag{2}$$

and there are finite or infinite limits $\lim\limits_{t\to+\infty} \lambda_i(t)$ $(i = 1, 2)$. We have not encountered with such a statement of the problem even in the well-known works. The case where at least one of the roots satisfies the condition

$$0 < \int\limits_{t_0}^{+\infty} |\lambda_i(t)|\, dt < +\infty \quad (i = 1, 2)$$

should be considered separately.

Under the term "almost triangular LHS" we understand each LHS

$$\frac{dy_i(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t) y_k \quad (i = \overline{1, n}), \tag{3}$$

where $p_{ik}(t) \in C_I$ $(i, k = \overline{1, n})$, which differs little from a linear triangular system

$$\frac{dy_i^*(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t) y_k^* \quad (i = \overline{1, n}), \tag{4}$$

and the conditions of either Theorem 0.1 or Theorem 0.2 due to A. V. Kostin [5] are fulfilled.

Theorem 1. *Let the following conditions hold:*

1) *LHS* (4) *is stable for* $t \in I$*;*
2) *for a particular solution* $\sigma_i(t)$ $(i = \overline{1, n})$ *of a linear inhomogeneous triangular system*

$$\frac{d\sigma_i(t)}{dt} = \sum_{k=1}^{i-1} |p_{ik}(t)| + p_{ii}(t)\sigma_i(t) + \sum_{k=i+1}^{n} |p_{ik}(t)| \sigma_k(t) \quad (i = \overline{1, n}) \tag{5}$$

with the initial conditions $\sigma_i(t_0) = 0$ $(i = \overline{1,n})$ *the estimate of the form*

$$0 < \sigma_i(t) < 1 - \gamma \ \ (i = \overline{1,n}), \ \ \gamma = const, \ \ \gamma \in (0,1)$$

holds for all $t \in I$.

Then the zero solution of the system (3) *is a fortiori stable for* $t \in I$.

Theorem 2. *Suppose the system* (3) *satisfies all conditions of Theorem* 1 *and, moreover,*

1) *the triangular linear system* (4) *is asymptotically stable for* $t \in I$*;*
2) $\lim\limits_{t\to+\infty} \sigma_i(t) = 0$ $(i = \overline{1,n})$.

Then the zero solution of the system (3) *is asymptotically stable for* $t \in I$.

Theorem 3. *Suppose the system* (3) *satisfies all conditions of Theorem* 1 *and, moreover,*

1) *none of the functions*

$$\psi_i(t) = \sum_{k=1}^{i-1} |p_{ik}(t)| \ \ (i = \overline{2,n}) \not\equiv 0 \ \ for \ \ t \in I;$$

2) $\lim\limits_{t\to+\infty} \sigma_i(t) = 0$ $(i = \overline{1,n})$.

Then the zero solution of the system (3) *is stable for* $t \in I$.

We will also use the following lemma [5]:

Lemma 1. *If the functions* $p(t), q(t) \in C_I$, $p(t) < 0$, $t \in I$,

$$\int\limits_{t_0}^{+\infty} p(\tau)\,d\tau = -\infty, \qquad \lim_{t\to+\infty} \frac{q(t)}{\operatorname{Re} p(t)} = 0,$$

then

$$e^{\int\limits_{t_0}^{t} p(\tau)\,d\tau} \int\limits_{t_0}^{t} q(\tau) e^{-\int\limits_{\tau_0}^{\tau} p(\tau_1)\,d\tau_1}\,d\tau = o(1), \ \ t \to +\infty.$$

Further, all limits and symbols o, O are assumed to be considered when $t \to +\infty$.

2. The Main Results

2.1. **Reduction of equation (1) to the system of the type (5).** Consider the real second order LHDE (1)

$$y'' + p(t)y' + q(t)y = 0 \ \ (t \in I)$$

where $p(t), q(t) \in C^1_I$. Let $y = y_1$, $y' = y_2$. We reduce the equation to an equivalent system

$$\begin{cases} y_1' = 0 \cdot y_1 + 1 \cdot y_2, \\ y_2' = -q \cdot y_1 - p \cdot y_2. \end{cases} \tag{6}$$

Consider the characteristic equation of LHS (6):

$$\begin{vmatrix} 0-\lambda & 1 \\ -q & -p-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + p\lambda + q = 0, \tag{7}$$

and assume that $\frac{p^2}{2} - q > 0$ in I or $\frac{p^2}{2} - q \equiv 0$ in I. Then this equation has two roots: $\lambda_1(t)$ and $\lambda_2(t)$, $\lambda_i(t) \in C^1_I$ $(i = 1, 2)$, $\lambda_i(t)$ are real functions $(i = 1, 2)$.

There arises the question on the sufficient conditions for stability of a trivial solution of system (6).

We write the system (6) in vector form

$$Y' = A(t)Y,$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

To reduce this system to almost triangular form, we use a linear transformation of the form

$$Y = B(t)Z,$$

$$B(t) = \begin{pmatrix} 1 & 0 \\ \lambda_1(t) & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

where $z_i(t)$ are new unknown functions $(i = 1, 2)$. We obtain

$$B'Z + BZ' = ABZ$$

or, after obvious transformations,

$$Z' = (B^{-1}AB - B^{-1}B')Z,$$

$$\det B(t) = 1, \quad B^{-1}(t) = \begin{pmatrix} 1 & 0 \\ -\lambda_1(t) & 1 \end{pmatrix},$$

$$B'(t) = \begin{pmatrix} 0 & 0 \\ \lambda_1'(t) & 0 \end{pmatrix}, \quad B^{-1}B' = \begin{pmatrix} 0 & 0 \\ \lambda_1'(t) & 0 \end{pmatrix},$$

$$B^{-1}AB = \begin{pmatrix} 1 & 0 \\ -\lambda_1(t) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda_1(t) & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1(t) & 1 \\ 0 & \lambda_2(t) \end{pmatrix}.$$

The system with respect to new unknowns $z_i(t)$ $(i = 1, 2)$ in scalar form looks as

$$\begin{cases} z_1'(t) = \lambda_1(t)z_1(t) + z_2(t), \\ z_2'(t) = -\lambda_1'(t)z_1(t) + \lambda_2(t)z_2(t). \end{cases} \tag{8}$$

In accordance with Theorem 1, let us write an auxiliary system of differential equations:

$$\begin{cases} \sigma_1'(t) = \lambda_1(t)\sigma_1(t) + \sigma_2(t), \\ \sigma_2'(t) = |\lambda_1'(t)| + \lambda_2(t)\sigma_2(t) \end{cases} \tag{9}$$

and consider its particular solution with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$. This solution has the form

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t} \lambda_2(\tau)\,d\tau} \displaystyle\int\limits_{t_0}^{t} |\lambda_1'(t)| e^{-\int\limits_{\tau_0}^{\tau} \lambda_2(\tau_1)\,d\tau_1}\, d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t} \lambda_1(\tau)\,d\tau} \displaystyle\int\limits_{t_0}^{t} \widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau} \lambda_1(\tau_1)\,d\tau_1}\, d\tau. \end{cases} \tag{10}$$

2.2. **Various cases of behavior of the roots $\lambda_i(t)$ $(i = 1, 2)$.** Consider the following cases of behavior of the roots of the characteristic equation, assuming that the condition (2) is satisfied:

1) $\lambda_i(+\infty) \in \mathbb{R}_-$ $(i = 1, 2)$;
2) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) = o(1)$;
3) $\lambda_i(t) = o(1)$ $(i = 1, 2)$;
4) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(+\infty) = -\infty$;
5) $\lambda_1(t) = o(1)$, $\lambda_2(+\infty) = -\infty$;
6) $\lambda_i(+\infty) = -\infty$ $(i = 1, 2)$.

Theorems 4–9 correspond to the above-indicated cases 1)–6).

Theorem 4. *In case* 1)*, a trivial solution of the equation* (1) *is asymptotically stable. Here it is sufficient to assume that $p(t), q(t) \in C_I$.*

This case is well-known. The validity of this theorem follows from the results obtained by A. M. Lyapunov.

Theorem 5. *Let the condition* (2) *for $i = 2$ and the conditions*

1) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) = o(1)$*;*
2) $\frac{\lambda_1'(t)}{\lambda_2(t)} = o(1)$

be fulfilled. Then a trivial solution of equation (1) *is asymptotically stable.*

Proof. We apply Theorem 3. Condition 1) of Theorem 3 is obviously satisfied: $\psi(t) = |\lambda_1'(t)| \not\equiv 0$ for $t \in I$. Therefore, it suffices to show that condition 2) of Theorem 3 also holds. By assumption 2)

$$\frac{|\lambda_1'(t)|}{\lambda_2(t)} = o(1).$$

Therefore, by virtue of Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. By condition 1) of this theorem,

$$\frac{\widetilde{\sigma}_2(t)}{\lambda_1(t)} = o(1), \quad \int\limits_{t_0}^{+\infty} \lambda_1(t)\,dt = -\infty,$$

and hence $\widetilde{\sigma}_1(t) = o(1)$ by Lemma 1. This implies that Theorem 5 is valid if we take into consideration that the transformation $B(t)$ is restricted in I. To obtain the estimate of solutions $y_i(t)$ $(i = 1, 2)$, we make in the system (8) the following change:

$$z_i(t) = e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\,d\tau} \eta_i(t) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then the system (8) takes the form

$$\begin{cases} \eta_1'(t) = (\lambda_1(t) - \delta\lambda_2(t))\eta_1(t) + \eta_2(t), \\ \eta_2'(t) = -\lambda_1'(t)\eta_1(t) + (1-\delta)\lambda_2(t)\eta_2(t) \end{cases}$$

and the system (9) takes the form

$$\begin{cases} \sigma_1'(t) = (\lambda_1(t) - \delta\lambda_2(t))\sigma_1(t) + \sigma_2(t), \\ \sigma_2'(t) = |\lambda_1'(t)| + (1-\delta)\lambda_2(t)\sigma_2(t). \end{cases}$$

Next, consider a particular solution of this system with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t}(1-\delta)\lambda_2(\tau)\,d\tau} \displaystyle\int\limits_{t_0}^{t} |\lambda_1'(t)| e^{-\int\limits_{\tau_0}^{\tau}(1-\delta)\lambda_2(\tau_1)\,d\tau_1} d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t}(\lambda_1(\tau)-\delta\lambda_2(\tau))\,d\tau} \displaystyle\int\limits_{t_0}^{t} \widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau}(\lambda_1(\tau_1)-\delta\lambda_2(\tau_1))\,d\tau_1} d\tau. \end{cases}$$

In our case,

$$\lim_{t\to+\infty} \frac{|\lambda_1'(t)|}{(1-\delta)\lambda_2(t)} = 0.$$

Thus, by Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. Further,

$$\lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{\lambda_1(t) - \delta\lambda_2(t)} = \lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{\lambda_1(t)(1 - \delta\frac{\lambda_2(t)}{\lambda_1(t)})} = \lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{\lambda_1(t)} = 0$$

and hence $\widetilde{\sigma}_1(t) = o(1)$, by Lemma 1. Thus the validity of Theorem 5 is not violated. So,

$$z_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\,d\tau}\Big) \ \ (i = 1, 2).$$

Taking into account the transformation $B(t)$,

$$\begin{cases} y_1(t) = z_1(t), \\ y_2(t) = \lambda_1(t) z_1(t) + z_2(t); \end{cases}$$

$$\begin{cases} y_1(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big), \\ y_2(t) = o\Big(e^{\int\limits_{t_0}^{t} (\delta\lambda_2(\tau) + \frac{\lambda_1'(t)}{\lambda_1(t)})\, d\tau}\Big) + o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big); \end{cases}$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t} \lambda_2(\tau)(\delta + \frac{\lambda_1'(\tau)}{\lambda_1(\tau)\lambda_2(\tau)})\, d\tau}\Big) + o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big),$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t} \lambda_2(\tau)(\delta + o(1))\, d\tau}\Big) + o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big).$$

Therefore,

$$y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_2(\tau)\, d\tau}\Big) \ \ (i = 1, 2), \ \ \delta \in (0, 1). \qquad \square$$

Theorem 6. *Let the condition* (2) *and the conditions*

1) $\lambda_i(t) = o(1)$ $(i = 1, 2)$;

2) $\frac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$ *(or* $\frac{\lambda_2'(t)}{\lambda_2^2(t)} = o(1)$*),* $\frac{\lambda_1(t)}{\lambda_2(t)} = O(1)$

be fulfilled. Then a trivial solution of equation (1) *is asymptotically stable.*

Proof. We apply Theorems 3 and 2. We make in the system (8) the following change:

$$z_1(t) = \xi_1(t), \quad \frac{z_2(t)}{\lambda_1(t)} = \xi_2(t). \tag{11}$$

Then

$$z_1'(t) = \xi_1'(t), \quad z_2'(t) = \lambda_1'(t)\xi_2(t) + \lambda_1(t)\xi_2'(t).$$

Substituting these expressions into the system (8), we have

$$\begin{cases} \xi_1'(t) = \lambda_1(t)\xi_1(t) + \lambda_1(t)\xi_2(t), \\ \xi_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_1(t)}\xi_1(t) + \Big(\lambda_2(t) - \dfrac{\lambda_1'(t)}{\lambda_1(t)}\Big)\xi_2(t). \end{cases} \tag{12}$$

To obtain the estimate of solutions $y_i(t)$ $(i = 1, 2)$ we make in system (12) the following change:

$$\xi_i(t) = e^{\delta \int\limits_{t_0}^{t} \lambda_1(\tau)\, d\tau} \eta_i(t) \ \ (i = 1, 2), \ \ \delta \in (0, 1).$$

Then the system (12) takes the form

$$\begin{cases} \xi_1'(t) = (1 - \delta)\lambda_1(t)\xi_1(t) + \lambda_1(t)\xi_2(t), \\ \xi_2'(t) = -\dfrac{\lambda_1'(t)}{\lambda_1(t)}\, \xi_1(t) + \Big(\lambda_2(t) - \delta\lambda_1(t) - \dfrac{\lambda_1'(t)}{\lambda_1(t)}\Big)\xi_2(t). \end{cases}$$

Let us denote

$$\mu(t)=\frac{\lambda_1'(t)}{\lambda_1(t)}\,.$$

In accordance with Theorem 1, we write an auxiliary system of differential equations:

$$\begin{cases}\sigma_1'(t)=(1-\delta)\lambda_1(t)\sigma_1(t)+|\lambda_1(t)|\sigma_2(t),\\ \sigma_2'(t)=|\mu(t)|+\big(\lambda_2(t)-\delta\lambda_1(t)-\mu(t)\big)\sigma_2(t).\end{cases}\tag{13}$$

Let us consider its particular solution with the initial conditions $\sigma_i(t_0)=0$ $(i=1,2)$:

$$\begin{cases}\widetilde{\sigma}_2(t)=e^{\int_{t_0}^{t}(\lambda_2(\tau)-\delta\lambda_1(\tau)-\mu(\tau))\,d\tau}\displaystyle\int_{t_0}^{t}|\mu(\tau)|e^{-\int_{\tau_0}^{\tau}\lambda_2(\tau_1)-\delta\lambda_1(\tau_1)-\mu(\tau_1))\,d\tau_1}\,d\tau,\\ \widetilde{\sigma}_1(t)=e^{\int_{t_0}^{t}(1-\delta)\lambda_1(\tau)\,d\tau}\displaystyle\int_{t_0}^{t}|\lambda_1(\tau)|\widetilde{\sigma}_2(\tau)e^{-\int_{\tau_0}^{\tau}(1-\delta)\lambda_1(\tau_1)\,d\tau_1}\,d\tau.\end{cases}$$

Condition 1) of Theorem 3 is obviously satisfied:

$$\psi(t)=|\mu(t)|\not\equiv 0\ \text{ for }\ t\in I.$$

In our case,

$$\lim_{t\to+\infty}\frac{|\mu(t)|}{\lambda_2(t)-\delta\lambda_1(t)-\mu(t)}=-\lim_{t\to+\infty}\frac{\left|\frac{\lambda_1'(t)}{\lambda_1^2(t)}\right|}{\frac{\lambda_2(t)}{\lambda_1(t)}-\delta-\frac{\lambda_1'(t)}{\lambda_1^2(t)}}=0.$$

If

$$\frac{\lambda_1'(t)}{\lambda_1^2(t)}\neq o(1),$$

then interchanging the elements $\lambda_1(t)$ and $\lambda_2(t)$, we obtain

$$\lim_{t\to+\infty}\frac{\left|\frac{\lambda_2'(t)}{\lambda_2(t)}\right|}{\lambda_1(t)-\delta\lambda_2(t)-\frac{\lambda_2'(t)}{\lambda_2(t)}}=-\lim_{t\to+\infty}\frac{\left|\frac{\lambda_2'(t)}{\lambda_2^2(t)}\right|}{\frac{\lambda_1(t)}{\lambda_2(t)}-\delta-\frac{\lambda_2'(t)}{\lambda_2^2(t)}}=0.$$

Consequently, by Lemma 1, $\widetilde{\sigma}_2(t)=o(1)$. Then

$$\lim_{t\to+\infty}\frac{|\lambda_1(t)|}{(1-\delta)\lambda_1(t)}\widetilde{\sigma}_2(t)=-\lim_{t\to+\infty}\frac{\widetilde{\sigma}_2(t)}{1-\delta}=0.$$

Hence $\widetilde{\sigma}_1(t)=o(1)$, by Lemma 1. This implies that Theorem 6 is valid. Then, taking into account the change (11), we have

$$\begin{cases}z_1(t)=o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big),\\ z_2(t)=o\Big(e^{\int_{t_0}^{t}(\delta\lambda_1(\tau)+\mu(\tau))\,d\tau}\Big).\end{cases}$$

$$z_2(t) = o\Big(e^{\int\limits_{t_0}^{t} \lambda_1(\tau)(\delta + \frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\,d\tau}\Big) \Longrightarrow z_2(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big).$$

Then,

$$\begin{cases} y_1(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big), \\ y_2(t) = o\Big(e^{\int\limits_{t_0}^{t} (\delta\lambda_1(\tau) + \mu(\tau))\,d\tau}\Big) + o\Big(e^{\int\limits_{t_0}^{t} o(\lambda_1(\tau))\,d\tau}\Big); \end{cases}$$

$$y_2(t) = o\Big(e^{\int\limits_{t_0}^{t} \lambda_1(\tau)(\delta + \frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\,d\tau}\Big) \Longrightarrow y_2(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big).$$

Consequently,

$$y_i(t) = o\Big(e^{\delta \int\limits_{t_0}^{t} \lambda_1(\tau)\,d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1). \qquad \square$$

Theorem 7. *Let the condition* (2) *for* $i = 2$ *and the conditions*

1) $\lambda_1(+\infty) \in \mathbb{R}_-$, $\lambda_2(t) \to -\infty$, $\lambda_2(t) < 0$ *at* I*;*
2) $\lambda_1'(t)$ *is bounded at* $t \to +\infty$

be fulfilled. Then a trivial solution of the equation (1) *is asymptotically stable.*

Proof. In system (8) we make the following change:

$$z_i(t) = e^{\delta \int\limits_{t_0}^{t} \lambda_1(\tau)\,d\tau} \eta_i(t) \quad (i = 1, 2), \quad \delta \in (0, 1).$$

Then the system (8) takes the form

$$\begin{cases} \eta_1'(t) = (1 - \delta)\lambda_1(t)\eta_1(t) + \eta_2(t), \\ \eta_2'(t) = -\lambda_1'(t)\eta_1(t) + \big(\lambda_2(t) - \delta\lambda_1(t)\big)\eta_2(t). \end{cases}$$

In accordance with Theorem 1, we write an auxiliary system of differential equations

$$\begin{cases} \sigma_1'(t) = (1 - \delta)\lambda_1(t)\sigma_1(t) + \sigma_2(t), \\ \sigma_2'(t) = |\lambda_1'(t)| + \big(\lambda_2(t) - \delta\lambda_1(t)\big)\sigma_2(t). \end{cases}$$

Its particular solution with the initial conditions $\sigma_i(t_0) = 0 \quad (i = 1, 2)$ has the form

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int\limits_{t_0}^{t} (\lambda_2(\tau) - \delta\lambda_1(\tau))\,d\tau} \displaystyle\int_{t_0}^{t} |\lambda_1'(t)| e^{-\int\limits_{\tau_0}^{\tau} (\lambda_2(\tau_1) - \delta\lambda_1(\tau_1))\,d\tau_1}\,d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int\limits_{t_0}^{t} (1-\delta)\lambda_1(\tau)\,d\tau} \displaystyle\int_{t_0}^{t} \widetilde{\sigma}_2(\tau) e^{-\int\limits_{\tau_0}^{\tau} (1-\delta)\lambda_1(\tau_1)\,d\tau_1}\,d\tau. \end{cases}$$

Since

$$\lim_{t\to+\infty}\frac{|\lambda_1'(t)|}{\lambda_2(t)-\delta\lambda_1(t)}=\lim_{t\to+\infty}\frac{|\lambda_1'(t)|}{\lambda_2(t)(1-\delta\frac{\lambda_1(t)}{\lambda_2(t)})}\lim_{t\to+\infty}\frac{|\lambda_1'(t)|}{\lambda_2(t)}=0,$$

by Lemma 1, $\widetilde{\sigma}_2(t)=o(1)$. As

$$\frac{\widetilde{\sigma}_2(t)}{(1-\delta)\lambda_1(t)}=o(1),$$

it is obvious that $\widetilde{\sigma}_1(t)=o(1)$. This implies that Theorem 7 is valid. Thus, $\eta_i(t)=o(1)$ $(i=1,2)$ and

$$z_i(t)=o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big)\quad(i=1,2),\quad \delta\in(0,1).$$

Moreover,

$$\begin{cases} y_1(t)=o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big),\\ y_2(t)=o\Big(e^{\int_{t_0}^{t}(\delta\lambda_1(\tau)+\mu(\tau))\,d\tau}\Big)+o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big). \end{cases}$$

and hence

$$\begin{cases} y_1(t)=o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big),\\ y_2(t)=o\Big(e^{\int_{t_0}^{t}\lambda_1(\tau)\delta+\frac{\lambda_1'(t)}{\lambda_1^2(t)})\,d\tau}\Big)+o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big), \end{cases}\qquad \delta\in(0,1).\qquad\square$$

Theorem 8. *Let the condition* (2) *for* $i=1$ *and the conditions*

1) $\lambda_1(t)=o(1)$, $\lambda_2(+\infty)=-\infty$;
2) $\frac{\lambda_1'(t)}{\lambda_1^2(t)}=o(1)$

be fulfilled. Then a trivial solution of the equation (1) *is asymptotically stable.*

Proof. In system (8) we make the following change:

$$\lambda_1(t)z_1(t)=\xi_1(t),\quad z_2(t)=\xi_2(t).\tag{14}$$

Then

$$z_1'(t)=\frac{\xi_1'(t)\lambda_1(t)-\xi_1(t)\lambda_1'(t)}{\lambda_1^2(t)}=\frac{1}{\lambda_1(t)}\,\xi_1'(t)-\frac{\lambda_1'(t)}{\lambda_1^2(t)}\,\xi_1(t),$$
$$z_2'(t)=\xi_2'(t).$$

After such a change, system (8) takes the form

$$\begin{cases}\xi_1'(t)=(\lambda_1(t)+\mu(t))\xi_1(t)+\lambda_1(t)\xi_2(t),\\ \xi_2'(t)=-\mu(t)\xi_1(t)+\lambda_2(t)\xi_2(t).\end{cases}\tag{15}$$

Now we make change in the system (15):

$$\xi_i(t) = e^{\delta \int_{t_0}^{t} \lambda_1(\tau)\, d\tau} \eta_i(t) \;\; (i = 1, 2), \quad \delta \in (0, 1).$$

After that the system (15) takes the form

$$\begin{cases} \eta_1'(t) = \Big((1-\delta)\lambda_1(t) + \dfrac{\lambda_1'(t)}{\lambda_1(t)}\Big)\eta_1(t) + \lambda_1(t)\eta_2(t), \\ \eta_2'(t) = -\mu(t)\eta_1(t) + (\lambda_2(t) - \delta\lambda_1(t))\eta_2(t). \end{cases}$$

According to Theorem 1, for the obtained system we write an auxiliary system of differential equations

$$\begin{cases} \sigma_1'(t) = \big((1-\delta)\lambda_1(t) + \mu(t)\big)\sigma_1(t) + |\lambda_1(t)|\sigma_2(t), \\ \sigma_2'(t) = |\mu(t)| + (\lambda_2(t) - \delta\lambda_1(t))\sigma_2(t). \end{cases}$$

Condition 1) of Theorem 3 is obviously satisfied: $\psi(t) = |\mu(t)| \not\equiv 0$ for $t \in I$. Consider a particular solution of that system with the initial conditions $\sigma_i(t_0) = 0$ $(i = 1, 2)$:

$$\begin{cases} \widetilde{\sigma}_2(t) = e^{\int_{t_0}^{t} (\lambda_2(\tau) - \delta\lambda_1(\tau))\, d\tau} \displaystyle\int_{t_0}^{t} |\mu(\tau)| e^{-\int_{\tau_0}^{\tau} (\lambda_2(\tau_1) - \delta\lambda_1(\tau_1))\, d\tau_1}\, d\tau, \\ \widetilde{\sigma}_1(t) = e^{\int_{t_0}^{t} ((1-\delta)\lambda_1(\tau) + \mu(\tau))\, d\tau} \displaystyle\int_{t_0}^{t} |\lambda_1(\tau)|\widetilde{\sigma}_2(\tau) e^{-\int_{\tau_0}^{\tau} ((1-\delta)\lambda_1(\tau_1) + \mu(\tau_1))\, d\tau_1}\, d\tau. \end{cases}$$

According to condition 2) of the above theorem,

$$\mu(t) = o(\lambda_1(t)),$$

and, all the more,

$$\mu(t) = o(\lambda_2(t)).$$

Then

$$\lim_{t \to +\infty} \frac{|\mu(t)|}{\lambda_2(t) - \delta\lambda_1(t)} = \lim_{t \to +\infty} \frac{|\mu(t)|\, \frac{1}{\lambda_2(t)}}{1 - \delta\, \frac{\lambda_1(t)}{\lambda_2(t)}} = o(1).$$

Consequently, by Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. Further, we have

$$\lim_{t \to +\infty} \frac{|\lambda_1(t)|\widetilde{\sigma}_2(t)}{(1-\delta)\lambda_1(t) + \mu(t)} = \lim_{t \to +\infty} \frac{\widetilde{\sigma}_2(t)}{\delta - 1 - \frac{\lambda_1'(t)}{\lambda_1^2(t)}} = 0$$

and thus, $\widetilde{\sigma}_1(t) = o(1)$.

Then

$$\lim_{t\to+\infty} z_1(t) = \lim_{t\to+\infty} \frac{\xi_1(t)}{\lambda_1(t)} = \lim_{t\to+\infty} \frac{e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\eta_1(t)}{\lambda_1(t)} =$$

$$= \lim_{t\to+\infty} e^{\int_{t_0}^{t}(\delta\lambda_1(\tau)-\mu(\tau))\,d\tau}\eta_1(t) = \lim_{t\to+\infty} e^{\int_{t_0}^{t}\lambda_1(\tau)(\delta-\frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\,d\tau}\eta_1(t) =$$

$$= \lim_{t\to+\infty} e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\eta_1(t) = \lim_{t\to+\infty} \xi_1(t)\eta_1(t) = 0.$$

This implies that Theorem 8 is valid. Moreover,

$$\begin{cases} z_1(t) = \dfrac{\xi_1(t)}{\lambda_1(t)}, \\ z_2(t) = \xi_2(t) \end{cases} \Longrightarrow \begin{cases} z_1(t) = o\Big(e^{\int_{t_0}^{t}(\delta\lambda_1(\tau)-\mu(\tau))\,d\tau}\Big), \\ z_2(t) = o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big) \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} z_1(t) = o\Big(e^{\int_{t_0}^{t}\lambda_1(\tau)(\delta-\frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\,d\tau}\Big), \\ z_2(t) = o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big) \end{cases} \Longrightarrow$$

$$\Longrightarrow z_i(t) = o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big) \quad (i = 1, 2);$$

$$\begin{cases} y_1(t) = z_1(t), \\ y_2(t) = \lambda_1(t)z_1(t) + z_2(t) \end{cases} \Longrightarrow \begin{cases} y_1(t) = z_1(t), \\ y_2(t) = \xi_1(t) + z_2(t) \end{cases} \Longrightarrow$$

$$\Longrightarrow y_i(t) = o\Big(e^{\delta\int_{t_0}^{t}\lambda_1(\tau)\,d\tau}\Big) \quad (i = 1, 2), \quad \delta \in (0, 1). \quad \square$$

Theorem 9. *Let the conditions*

1) $\lambda_i(+\infty) = -\infty$ $(i = 1, 2)$*;*

2) $\frac{\lambda_1'(t)}{\lambda_1^2(t)} = o(1)$ *(or* $\frac{\lambda_2'(t)}{\lambda_2^2(t)} = o(1)$*),* $\frac{\lambda_1(t)}{\lambda_2(t)} = O(1)$

be fulfilled. Then a trivial solution of the equation (1) *is asymptotically stable.*

Proof. The condition (2) is obviously fulfilled. In the system (8) we make the substitution (14) and obtain the system (15). Next, we make the substitution

$$\xi_i(t) = e^{\int_{t_0}^{t}\nu(\tau)\,d\tau}\eta_i(t), \quad \nu(t) = o(\lambda_i(t)) \quad (i = 1, 2).$$

Then the system (15) takes the form

$$\begin{cases} \eta_1'(t) = \big((\lambda_1(t) - \nu(t) + \mu(t)\big)\eta_1(t) + \lambda_1(t)\eta_2(t), \\ \eta_2'(t) = -\mu(t)\eta_1(t) + (\lambda_2(t) - \nu(t))\eta_2(t). \end{cases}$$

In accordance with Theorem 1, we write an auxiliary system

$$\begin{cases} \sigma_1'(t) = \big((\lambda_1(t) - \nu(t) + \mu(t)\big)\sigma_1(t) + \lambda_1(t)\sigma_2(t), \\ \sigma_2'(t) = |\mu(t)| + (\lambda_2(t) - \nu(t))\sigma_2(t). \end{cases}$$

According to conditions 1) and 2) of the given theorem,

$$\lim_{t\to+\infty} \frac{|\mu(t)|}{\lambda_2(t) - \nu(t)} = -\lim_{t\to+\infty} \frac{\big|\frac{\lambda_1'(t)}{\lambda_1^2(t)}\big|\,\frac{\lambda_1(t)}{\lambda_2(t)}}{1 - \frac{\nu(t)}{\lambda_2(t)}} = 0.$$

If $\frac{\lambda_1(t)}{\lambda_2(t)}$ is unbounded as $t \to +\infty$, then we interchange $\lambda_1(t)$ and $\lambda_2(t)$ and get

$$\lim_{t\to+\infty} \frac{\big|\frac{\lambda_2'(t)}{\lambda_2(t)}\big|}{\lambda_1(t) - \nu(t)} = -\lim_{t\to+\infty} \frac{\big|\frac{\lambda_2'(t)}{\lambda_2^2(t)}\big|\,\frac{\lambda_2(t)}{\lambda_1(t)}}{1 - \frac{\nu(t)}{\lambda_1(t)}} = 0.$$

Consequently, by Lemma 1, $\widetilde{\sigma}_2(t) = o(1)$. Further, we have

$$\lim_{t\to+\infty} \frac{|\lambda_1(t)|\widetilde{\sigma}_2(t)}{\lambda_1(t) - \nu(t) + \mu(t)} = -\lim_{t\to+\infty} \frac{\widetilde{\sigma}_2(t)}{1 - \frac{\nu(t)}{\lambda_1(t)} + \frac{\lambda_1'(t)}{\lambda_1^2(t)}} = 0.$$

Then $\widetilde{\sigma}_1(t) = o(1)$. This implies that Theorem 9 is valid. Moreover,

$$\begin{cases} z_1(t) = \dfrac{\xi_1(t)}{\lambda_1(t)} \\ z_2(t) = \xi_2(t) \end{cases} \Longrightarrow \begin{cases} z_1(t) = o\Big(e^{\int_{t_0}^{t} (\nu(\tau) - \frac{\lambda_1'(\tau)}{\lambda_1(\tau)})\, d\tau}\Big), \\ z_2(t) = o\Big(e^{\int_{t_0}^{t} \nu(\tau)\, d\tau}\Big); \end{cases}$$

$$z_1(t) = o\Big(e^{\int_{t_0}^{t} \lambda_1(\tau)(\frac{\nu(\tau)}{\lambda_1(\tau)} - \frac{\lambda_1'(\tau)}{\lambda_1^2(\tau)})\, d\tau}\Big) \Longrightarrow$$

$$\Longrightarrow z_i(t) = o\Big(e^{\int_{t_0}^{t} \nu(\tau)\, d\tau}\Big), \quad \nu(t) = o(\lambda_i(t)) \ \ (i = 1, 2);$$

$$\begin{cases} y_1(t) = z_1(t), \\ y_2(t) = \lambda_1(t) z_1(t) + z_2(t). \end{cases} \Longrightarrow$$

$$\Longrightarrow y_i(t) = o\Big(e^{\int_{t_0}^{t} \nu(\tau)\, d\tau}\Big), \quad \nu(t) = o(\lambda_i(t)) \ \ (i = 1, 2).$$

Note that the condition $\frac{\lambda'(t)}{\lambda^2(t)} = o(1)$ is fulfilled for a sufficiently wide class of functions for which $\int\limits_{t_0}^{+\infty} \lambda(t)\, dt = -\infty$. □

Conclusion

The paper reveals the sufficient conditions for asymptotic stability and gives evaluation of solutions of the homogeneous linear nonautonomous second order differential equation depending on the behavior of roots of the corresponding characteristic equation in the case of real roots. The results of the work allow one to proceed to considering higher order equations and the questions connected with a simple stability and instability. The case of complex-conjugate roots has been considered by us and will be published in a separate article.

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