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TRANSMISSION AND INTERFACE CRACK PROBLEMS OF THERMOELASTICITY FOR HEMITROPIC SOLIDS

Abstract. The purpose of this paper is to investigate basic transmission and interface crack problems for the differential equations of the theory of elasticity of hemitropic materials with regard to thermal effects. We consider the so called pseudo-oscillation equations corresponding to the time harmonic dependent case. Applying the potential method and the theory of pseudodifferential equations first we prove uniqueness and existence theorems of solutions to the Dirichlet and Neumann type transmission-boundary value problems for piecewise homogeneous hemitropic composite bodies. Afterwards we investigate the interface crack problems and study regularity properties of solution.

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რეზიუმე. სტატიის მიზანია ჰემიტროპული სხეულების დრეკადობის თეორიის დიფერენციალური განტოლებებისათვის ძირითადი საკონტაქტო და ბზარის ტიპის ამოცანების გამოკვლევა თერმული ეფექტების გათვალისწინებით. განხილულია ე. წ. ფსევდო-რხევის განტოლებები, რომლებიც შეესაბამება დროზე ჰარმონიულად დამოკიდებულ შემთხვევას. პოტენციალთა მეთოდისა და ფსევდოდიფერენციალურ განტოლებათა თეორიის გამოყენებით ჯერ დამტკიცებულია დირიხლესა და ნეიმანის ტიპის სასაზღვრო-საკონტაქტო ამოცანების ამონახსნების არსებობისა და ერთადერთობის თეორემები უბნობრივ ერთგვაროვანი ჰემიტროპული სხეულებისათვის, ხოლო შემდეგ გამოკვლეულია ბზარის ტიპის ამოცანა, როდესაც ბზარი მდებარეობს საკონტაქტო ზედაპირზე, და შესწავლილია ამონახსნის რეგულარობა.

1. Introduction

Technological and industrial developments, and also recent important progress in biological and medical sciences require the use of more general and refined models for elastic bodies. In a generalized solid continuum, the usual displacement field has to be supplemented by a microrotation field. Such materials are called micropolar or Cosserat solids. They model continua with a complex inner structure whose material particles have 6 degree of freedom (3 displacement components and 3 microrotation components). Recall that the classical elasticity theory allows only 3 degrees of freedom (3 displacement components).

Experiments have shown that micropolar materials possess quite different properties in comparison with the classical elastic materials (see, e.g., [3], [4], [7], [15], [23], [25], [26], and the references therein). For example, in non-centrosymmetric micropolar materials the propagation of left-handed and right-handed elastic waves is observed. Moreover, the twisting behaviour under an axial stress is a purely hemitropic (chiral) phenomenon and has no counterpart in classical elasticity. Such solids are called *hemitropic non-centrosymmetric, acentric*, or *chiral*. Throughout the paper we use the term *hemitropic*.

Hemitropic solids are not isotropic with respect to inversion, i.e., they are isotropic with respect to all proper orthogonal transformations but not with respect to mirror reflections.

Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules - DNA, as well as on a large scale, as in composites with helical or screw–shaped inclusions, certain types of nanotubes, fabricated structures such as foams, chiral sculptured thin films and twisted fibers. For more details see the references [3], [4], [14], [15], [20], [23], [24], [26]–[30], [34], [35], [46]–[50], [53], [56], [57].

Mathematical models describing the chiral properties of elastic hemitropic materials have been proposed by Aéro and Kuvshinski [3], [4] (for historical notes see also [14], [15], [46], and the references therein).

In the present paper we deal with the model of micropolar elasticity for hemitropic solids when the thermal effects are taken into consideration.

In the mathematical theory of hemitropic thermoelasticity there are introduced the asymmetric force stress tensor and couple stress tensor, which are kinematically related with the asymmetric strain tensor, torsion (curvature) tensor and the temperature function via the constitutive equations. All these quantities along with the heat flux vector are expressed in terms of the components of the displacement and microrotation vectors, and the temperature function. In turn, the displacement and microrotation vectors, and the temperature satisfy a coupled complex system of second order partial differential equations of dynamics. When the mechanical and thermal characteristics (displacements, microrotations, temperature, body force, body couple vectors, and heat source) do not depend on the time variable t we have the differential equations of statics. If time dependence is harmonic (i.e., the pertinent fields are represented as the product of the time dependent exponential function $\exp\{-i\sigma t\}$ and a function of the spatial variable $x \in \mathbb{R}^3$) then we have the steady state oscillation equations. Here σ is a real frequency parameter. Note that if $\sigma = 0$, then we obtain the equations of statics. If $\sigma = \sigma_1 + i\sigma_2$ is a complex parameter, then we have the so called *pseudo-oscillation equations* (which are related to the dynamical equations via the Laplace transform). All the above equations generate a strongly elliptic, formally non-self-adjoint 7×7 matrix differential operator.

The Dirichlet, Neumann and mixed type boundary value problems (BVP) corresponding to this model are well investigated for homogeneous bodies of arbitrary shape and the uniqueness and existence theorems are proved, and regularity results for solutions are established by the potential method, as well as by variational methods (see [39]–[43] and the references therein).

The main goal of our investigation is to study the Dirichlet and Neumann type transmission and interface crack problems of the theory of elasticity with regard to thermal effects for piecewise homogeneous hemitropic composite bodies of arbitrary geometrical shape. We develop the boundary integral equations method to obtain the existence and uniqueness results in various Hölder ($C^{k,\alpha}$), Sobolev–Slobodetski (W_p^s) and Besov ($B_{p,q}^s$) functional spaces. We study regularity properties of solutions at the crack edges and characterize the corresponding stress singularity exponents.

2. Field Equations

2.1. **Constitutive relations and basic differential equations.** Denote by $\mathbb{R}^3$ the three-dimensional Euclidean space and let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with a boundary $S := \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$. Further, let $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. We assume that $\overline{\Omega} \in \{\overline{\Omega^+}, \overline{\Omega^-}\}$ is filled with an elastic material possessing the hemitropic properties.

Denote by $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ the displacement vector and the microrotation vector, respectively. By ϑ we denote the temperature increment – temperature distribution function. Here and in what follows the symbol $(\cdot)^\top$ denotes transposition. Note that the microrotation vector in the hemitropic elasticity theory is kinematically distinct from the macrorotation vector $\frac{1}{2}$ curl u.

Throughout the paper the central dot denotes the real scalar product, i.e., $a \cdot b := \sum\limits_{k=1}^{N} a_k b_k$ for complex-valued N-dimensional vectors $a, b \in \mathbb{C}^N$.

The force stress $\{\tau_{pq}\}$ and the couple stress $\{\mu_{pq}\}$ tensors in the linear theory of hemitropic thermoelasticity read as follows (the constitutive equations)

$$\tau_{pq} = \tau_{pq}(U) := (\mu + \alpha)\partial_p u_q + (\mu - \alpha)\partial_q u_p + \lambda\delta_{pq} \operatorname{div} u + \delta\delta_{pq} \operatorname{div} \omega +$$

$$+(\varkappa+\nu)\partial_p\omega_q+(\varkappa-\nu)\partial_q\omega_p-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k-\delta_{pq}\eta\vartheta, \tag{2.1}$$

$$\mu_{pq}=\mu_{pq}(U):=\delta\delta_{pq}\,\mathrm{div}\,u+(\varkappa+\nu)\Big[\partial_p u_q-\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k\Big]+\beta\delta_{pq}\,\mathrm{div}\,\omega+$$

$$+(\varkappa-\nu)\Big[\partial_q u_p-\sum_{k=1}^{3}\varepsilon_{qpk}\omega_k\Big]+(\gamma+\varepsilon)\partial_p\omega_q+(\gamma-\varepsilon)\partial_q\omega_p-\delta_{pq}\zeta\vartheta, \tag{2.2}$$

where $U=(u,\omega,\vartheta)^\top$, δ_{pq} is the Kronecker delta, ε_{pqk} is the permutation (Levi–Civitá) symbol, and α, β, γ, δ, λ, μ, ν, $\varkappa$, and ε are the material constants, while $\eta>0$ and $\zeta>0$ are constants describing the coupling of mechanical and thermal fields (see [3], [14]), $\partial=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$, $j=1,2,3$.

The linear equations of dynamics of the thermoelasticity theory of hemitropic materials have the form (see, e.g., [14])

$$\sum_{p=1}^{3}\partial_p\tau_{pq}(x,t)+\varrho F_q(x,t)=\varrho\partial_{tt}^2u_q(x,t),\quad q=1,2,3,$$

$$\sum_{p=1}^{3}\partial_p\mu_{pq}(x,t)+\sum_{l,r=1}^{3}\varepsilon_{qlr}\tau_{lr}(x,t)+\varrho G_q(x,t)=\mathcal{I}\partial_{tt}^2\omega_q(x,t),\quad q=1,2,3,$$

$$\kappa'\Delta\vartheta(x,t)-\eta\partial_t\,\mathrm{div}\,u(x,t)-\zeta\partial_t\,\mathrm{div}\,\omega(x,t)-\kappa''\partial_t\vartheta(x,t)+Q(x,t)=0,$$

where t is the time variable, $\partial_t=\partial/\partial t$, $F=(F_1,F_2,F_3)^\top$ and $G=(G_1,G_2,G_3)^\top$ are the body force and body couple vectors per unit volume, Q is the heat source density, ϱ is the mass density of the elastic material, and $\mathcal{I}$ is a constant characterizing the so called spin torque corresponding to the microrotations (i.e., the moment of inertia per unit volume); here $\kappa'=\frac{\lambda_0}{T_0}$ and $\kappa''=\frac{c_0}{T_0}$, where $\lambda_0>0$ is the heat conduction coefficient, $T_0>0$ is an initial natural state temperature and $c_0>0$ is the specific heat coefficient.

Using the relations (2.1)–(2.2) we can rewrite the above dynamic equations as

$$(\mu+\alpha)\Delta u(x,t)+(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,u(x,t)+(\varkappa+\nu)\Delta\omega(x,t)+$$
$$+(\delta+\varkappa-\nu)\,\mathrm{grad}\,\mathrm{div}\,\omega(x,t)+2\alpha\,\mathrm{curl}\,\omega(x,t)-$$
$$-\eta\,\mathrm{grad}\,\vartheta(x,t)+\varrho F(x,t)=\varrho\partial_{tt}^2u(x,t),$$
$$(\varkappa+\nu)\Delta u(x,t)+(\delta+\varkappa-\nu)\,\mathrm{grad}\,\mathrm{div}\,u(x,t)+2\alpha\,\mathrm{curl}\,u(x,t)+$$
$$+(\gamma+\varepsilon)\Delta\omega(x,t)+(\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\omega(x,t)+4\nu\,\mathrm{curl}\,\omega(x,t)-$$
$$-4\alpha\omega(x,t)-\zeta\,\mathrm{grad}\,\vartheta(x,t)+\varrho G(x,t)=\mathcal{I}\partial_{tt}^2\omega(x,t),$$
$$\kappa'\Delta\vartheta(x,t)-\eta\partial_t\,\mathrm{div}\,u(x,t)-\zeta\partial_t\,\mathrm{div}\,\omega(x,t)-\kappa''\partial_t\vartheta(x,t)+Q(x,t)=0,$$

where Δ is the Laplace operator.

If all the quantities involved in these equations are harmonic time dependent, i.e., $u(x,t)=u(x)e^{-it\sigma}$, $\omega(x,t)=\omega(x)e^{-it\sigma}$, $\vartheta(x,t)=\vartheta(x)e^{-it\sigma}$, $F(x,t)=F(x)e^{-it\sigma}$, $G(x,t)=G(x)e^{-it\sigma}$ and $Q(x,t)=Q(x)e^{-it\sigma}$ with $\sigma\in\mathbb{R}$ and $i=\sqrt{-1}$, we obtain the *steady state oscillation equations* of the hemitropic theory of thermoelasticity:

$$\begin{gathered}
(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}u(x)+\varrho\sigma^2u(x)+\\
+(\varkappa+\nu)\Delta\omega(x)+(\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}\omega(x)+2\alpha\operatorname{curl}\omega(x)-\\
-\eta\operatorname{grad}\vartheta(x)=-\varrho F(x),\\
(\varkappa+\nu)\Delta u(x)+(\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}u(x)+2\alpha\operatorname{curl}u(x)+\\
+(\gamma+\varepsilon)\Delta\omega(x)+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x)+4\nu\operatorname{curl}\omega(x)-\\
-\zeta\operatorname{grad}\vartheta(x)+(\mathcal{I}\sigma^2-4\alpha)\omega(x)=-\varrho G(x),\\
(\kappa'\Delta+i\sigma\kappa'')\vartheta(x)+i\eta\sigma\operatorname{div}u(x)+i\zeta\sigma\operatorname{div}\omega(x)=-Q(x),
\end{gathered}\tag{2.3}$$

here u, ω, F, and G are complex-valued vector functions, while ϑ and Q are complex-valued scalar functions, and σ is a frequency parameter.

If $\sigma=\sigma_1+i\sigma_2$ is a complex parameter with $\sigma_2\neq0$, then the above equations are called the *pseudo–oscillation equations*, while for $\sigma=0$ they represent the *equilibrium equations of statics*.

Let us introduce the block wise 7×7 matrix differential operator corresponding to the system (2.3):

$$L(\partial,\sigma):=\begin{bmatrix}L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma) & L^{(5)}(\partial,\sigma)\\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma) & L^{(6)}(\partial,\sigma)\\ L^{(7)}(\partial,\sigma) & L^{(8)}(\partial,\sigma) & L^{(9)}(\partial,\sigma)\end{bmatrix}_{7\times7},\tag{2.4}$$

where

$$\begin{gathered}
L^{(1)}(\partial,\sigma):=\big[(\mu+\alpha)\Delta+\varrho\sigma^2\big]I_3+(\lambda+\mu-\alpha)Q(\partial),\\
L^{(2)}(\partial,\sigma)=L^{(3)}(\partial,\sigma):=(\varkappa+\nu)\Delta I_3+(\delta+\varkappa-\nu)Q(\partial)+2\alpha R(\partial),\\
L^{(4)}(\partial,\sigma):=[(\gamma+\varepsilon)\Delta+(\mathcal{I}\sigma^2-4\alpha)]I_3+(\beta+\gamma-\varepsilon)Q(\partial)+4\nu R(\partial),\\
L^{(5)}(\partial,\sigma):=-\eta\nabla^\top,\quad L^{(6)}(\partial,\sigma):=-\zeta\nabla^\top,\quad L^{(7)}(\partial,\sigma):=i\eta\sigma\nabla,\\
L^{(8)}(\partial,\sigma):=i\zeta\sigma\nabla,\quad L^{(9)}(\partial,\sigma):=\kappa'\Delta+i\sigma\kappa''.
\end{gathered}$$

Here and in the sequel I_k stands for the $k\times k$ unit matrix and

$$R(\partial):=[-\varepsilon_{kjl}\partial_l]_{3\times3},\ \ Q(\partial):=[\partial_k\partial_j]_{3\times3},\ \ \nabla:=[\partial_1,\partial_2,\partial_3].\tag{2.5}$$

Throughout the paper summation over repeated indexes is meant from one to three if not otherwise stated. It is easy to see that for $v=(v_1,v_2,v_3)^\top$

$$\begin{gathered}
R(\partial)v=\operatorname{curl}v,\quad Q(\partial)v=\operatorname{grad}\operatorname{div}v,\\
R(-\partial)=-R(\partial)=[R(\partial)]^\top,\quad Q(\partial)R(\partial)=R(\partial)Q(\partial)=0,\\
Q(\partial)=[Q(\partial)]^\top,\quad [R(\partial)]^2=Q(\partial)-\Delta I_3,\quad [Q(\partial)]^2=Q(\partial)\Delta.
\end{gathered}\tag{2.6}$$

Due to the above notation, the system (2.3) can be rewritten in matrix form as

$$L(\partial,\sigma)U(x)=\Phi(x),\quad U=(u,\omega,\vartheta)^\top,\;\;\Phi=(-\varrho F,-\varrho G,-Q)^\top.$$

Note that $L(\partial,\sigma)$ is not formally self-adjoint. Further, let us remark that the differential operator

$$L(\partial):=L(\partial,0) \tag{2.7}$$

corresponds to the static equilibrium case, while the formally self-adjoint differential operator

$$L_0(\partial):=\begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial) & [0]_{3\times 1}\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial) & [0]_{3\times 1}\\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\Delta \end{bmatrix}_{7\times 7} \tag{2.8}$$

with

$$\begin{aligned} L_0^{(1)}(\partial)&:=(\mu+\alpha)\Delta I_3+(\lambda+\mu-\alpha)Q(\partial),\\ L_0^{(2)}(\partial)=L_0^{(3)}(\partial)&:=(\varkappa+\nu)\Delta I_3+(\delta+\varkappa-\nu)Q(\partial),\\ L_0^{(4)}(\partial)&:=(\gamma+\varepsilon)\Delta I_3+(\beta+\gamma-\varepsilon)Q(\partial), \end{aligned}$$

represents the principal homogeneous part of the operators (2.4) and (2.7). Denote

$$\begin{aligned} \widetilde{L}(\partial,\sigma)&:=\begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(2)}(\partial,\sigma)\\ L^{(3)}(\partial,\sigma) & L^{(4)}(\partial,\sigma)\end{bmatrix}_{6\times 6},\\ \widetilde{L}_0(\partial)&:=\begin{bmatrix} L_0^{(1)}(\partial) & L_0^{(2)}(\partial)\\ L_0^{(3)}(\partial) & L_0^{(4)}(\partial)\end{bmatrix}_{6\times 6}. \end{aligned} \tag{2.9}$$

The operators (2.9) correspond to the equations of hemitropic elasticity when thermal effects are not taken into consideration ([40]). It is clear that the operator $L_0(\partial)$, $\widetilde{L}(\partial,\sigma)$ and $\widetilde{L}_0(\partial)$ are formally self-adjoint.

2.2. **Generalized stress operators.** The components of the force stress vector $\tau^{(n)}$ and the couple stress vector $\mu^{(n)}$, acting on a surface element with a unite normal vector $n=(n_1,n_2,n_3)$, read as

$$\tau^{(n)}=\big(\tau_1^{(n)},\tau_2^{(n)},\tau_3^{(n)}\big)^\top,\quad \mu^{(n)}=\big(\mu_1^{(n)},\mu_2^{(n)},\mu_3^{(n)}\big)^\top,$$

where

$$\tau_q^{(n)}=\sum_{p=1}^3\tau_{pq}n_p,\quad \mu_q^{(n)}=\sum_{p=1}^3\mu_{pq}n_p,\;\; q=1,2,3.$$

It is also well known that the normal component of the heat flux vector across a surface element with a normal vector $n=(n_1,n_2,n_3)$ is expressed with the help of the normal derivative of the temperature function

$$\kappa' n\cdot\nabla\vartheta=\kappa'\sum_{p=1}^3 n_p\partial_p\vartheta=\kappa'\partial_n\vartheta,$$

where $\partial_n = \partial/\partial n$ denotes the usual normal derivative.

Throughout the paper we will refer the six vector $(\tau^{(n)}, \mu^{(n)})^\top$ as the *mechanical thermo-stress vector*, while the seven vector $(\tau^{(n)}, \mu^{(n)}, \kappa'\partial_n\vartheta)^\top$ as *the generalized thermo-stress vector*.

Let us introduce the generalized thermo-stress operators

$$\mathcal{T}(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta n^\top \end{bmatrix}_{6\times 7}, \tag{2.10}$$

$$\mathcal{P}(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -\zeta n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\partial_n \end{bmatrix}_{7\times 7}, \tag{2.11}$$

where

$$T^{(j)} = [T^{(j)}_{pq}]_{3\times 3}, \quad j = \overline{1,4}, \quad n^\top = (n_1,n_2,n_3)^\top,$$
$$T^{(1)}_{pq}(\partial,n) = (\mu+\alpha)\delta_{pq}\partial_n + (\mu-\alpha)n_q\partial_p + \lambda n_p\partial_q,$$
$$T^{(2)}_{pq}(\partial,n) = (\varkappa+\nu)\delta_{pq}\partial_n + (\varkappa-\nu)n_q\partial_p + \delta n_p\partial_q - 2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,$$
$$T^{(3)}_{pq}(\partial,n) = (\varkappa+\nu)\delta_{pq}\partial_n + (\varkappa-\nu)n_q\partial_p + \delta n_p\partial_q,$$
$$T^{(4)}_{pq}(\partial,n) = (\gamma+\varepsilon)\delta_{pq}\partial_n + (\gamma-\varepsilon)n_q\partial_p + \beta n_p\partial_q - 2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.$$

One can easily check that for an arbitrary vector $U = (u,\omega,\vartheta)^\top$,

$$\mathcal{T}(\partial,n)U = \big(\tau^{(n)}, \mu^{(n)}\big)^\top, \quad \mathcal{P}(\partial,n)U = \big(\tau^{(n)}, \mu^{(n)}, \kappa'\partial_n\vartheta\big)^\top,$$

i.e., the six vector $\mathcal{T}(\partial,n)U$ corresponds to the mechanical thermo-stress vector and the seven vector $\mathcal{P}(\partial,n)U$ corresponds to the generalized thermo-stress vector.

Further, let us introduce the boundary differential operators which occur in Green's formulas and are associated with the adjoint differential operator $L^*(\partial,\sigma) := L^\top(-\partial,\sigma)$:

$$\begin{gathered}\mathcal{T}^*(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -i\sigma\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -i\sigma\zeta n^\top \end{bmatrix}_{6\times 7}, \\ \mathcal{P}^*(\partial,n) = \begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) & -i\sigma\eta n^\top \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) & -i\sigma\zeta n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa'\partial_n \end{bmatrix}_{7\times 7}.\end{gathered} \tag{2.12}$$

It is easy to see that the principal homogeneous parts of the operators $\mathcal{T}(\partial,n)$ and $\mathcal{T}^*(\partial,n)$ are the same, as well as the principal homogeneous parts of the operators $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$.

Note that when the thermal effects are not taken into consideration the hemitropic stress operator reads as [40]

$$T(\partial,n)=\begin{bmatrix}T^{(1)}(\partial,n) & T^{(2)}(\partial,n)\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n)\end{bmatrix}_{6\times 6}. \tag{2.13}$$

Evidently, for $U=(u,\omega,0)^\top$ and $\widetilde{U}=(u,\omega)^\top$ we have $\mathcal{T}(\partial,n)U=T(\partial,n)\widetilde{U}$ in view of (2.10) and (2.13).

2.3. **Green's identities.** For vector functions

$$\widetilde{U}=(u,\omega)^\top,\widetilde{U}'=(u',\omega')^\top\in[C^2(\overline{\Omega^+})]^6,$$

we have the following Green formula [40]

$$\int\limits_{\Omega^+}\Big[\widetilde{U}'\cdot\widetilde{L}(\partial,0)\widetilde{U}+E(\widetilde{U}',\widetilde{U})\Big]\,dx=\int\limits_{\partial\Omega^+}\{\widetilde{U}'\}^+\cdot\{T(\partial,n)\widetilde{U}\}^+\,dS, \tag{2.14}$$

where the operators $\widetilde{L}(\partial,0)$ and $T(\partial,n)$ are given by (2.9) and (2.13) respectively, $\partial\Omega^+$ is a piecewise smooth manifold, n is the outward unit normal vector to $\partial\Omega^+$, the symbols $\{\,\cdot\,\}^\pm$ denote the limiting values on $\partial\Omega^\pm$ from $\Omega^\pm$ respectively, $E(\cdot\,,\,\cdot)$ is the so called *energy bilinear form*,

$$\begin{aligned}E(\widetilde{U}',\widetilde{U})=E(\widetilde{U},\widetilde{U}')=\sum_{p,q=1}^{3}\Big\{&(\mu+\alpha)u'_{pq}u_{pq}+(\mu-\alpha)u'_{pq}u_{qp}+\\&+(\varkappa+\nu)(u'_{pq}\omega_{pq}+\omega'_{pq}u_{pq})+(\varkappa-\nu)(u'_{pq}\omega_{qp}+\omega'_{pq}u_{qp})+(\gamma+\varepsilon)\omega'_{pq}\omega_{pq}+\\&+(\gamma-\varepsilon)\omega'_{pq}\omega_{qp}+\delta(u'_{pp}\omega_{qq}+\omega'_{qq}u_{pp})+\lambda u'_{pp}u_{qq}+\beta\omega'_{pp}\omega_{qq}\Big\}\end{aligned} \tag{2.15}$$

with

$$u_{pq}=\partial_p u_q-\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k,\quad \omega_{pq}=\partial_p\omega_q,\quad p,q=1,2,3. \tag{2.16}$$

In what follows the over bar denotes complex conjugation. The necessary and sufficient conditions for the quadratic form $E(\widetilde{U},\overline{\widetilde{U}})$ to be positive definite with respect to the variables u_{pq} and ω_{pq}, read as (see [4], [14], [18])

$$\begin{gathered}\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ \lambda+2\mu>0,\ \ \mu\gamma-\varkappa^2>0,\ \ \alpha\varepsilon-\nu^2>0,\\ (\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2>0,\ \ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0,\\ (\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\ \ (\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0,\\ \mu\big[(\lambda+\mu)(\beta+\gamma)-(\delta+\varkappa)^2\big]+(\lambda+\mu)(\mu\gamma-\varkappa^2)>0,\\ \mu\big[(3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2\big]+(3\lambda+2\mu)(\mu\gamma-\varkappa^2)>0.\end{gathered}$$

Let us note that, if the condition $3\lambda+2\mu>0$ is fulfilled, which is very natural in the classical elasticity, then the above conditions are equivalent to the following simultaneous inequalities

$$\begin{gathered}\mu>0,\ \ \alpha>0,\ \ \gamma>0,\ \ \varepsilon>0,\ \ 3\lambda+2\mu>0,\ \ \mu\gamma-\varkappa^2>0,\\ \alpha\varepsilon-\nu^2>0,\ \ (\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\\ (3\lambda+2\mu)(3\beta+2\gamma)-(3\delta+2\varkappa)^2>0.\end{gathered}\tag{2.17}$$

For simplicity in what follows we assume that $3\lambda+2\mu>0$ and therefore the conditions (2.17) imply positive definiteness of the energy quadratic form $E(\widetilde{U},\overline{\widetilde{U}})$ defined by (2.15). From (2.17) it follows that

$$\begin{aligned}&\gamma>0,\quad \varepsilon>0,\quad \lambda+\mu>0,\quad \beta+\gamma>0,\\ &d_1:=(\mu+\alpha)(\gamma+\varepsilon)-(\varkappa+\nu)^2>0,\\ &d_2:=(\lambda+2\mu)(\beta+2\gamma)-(\delta+2\varkappa)^2>0.\end{aligned}$$

Formula (2.15) can be rewritten as

$$\begin{aligned}E(\widetilde{U},\widetilde{U}')=&\ \frac{3\lambda+2\mu}{3}\Big(\operatorname{div}u+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega\Big)\Big(\operatorname{div}u'+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega'\Big)+\\ &+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\varkappa)^2}{3\lambda+2\mu}\Big)(\operatorname{div}\omega)(\operatorname{div}\omega')+\\ &+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)\operatorname{curl}\omega\cdot\operatorname{curl}\omega'+\\ &+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big]\times\\ &\qquad\times\Big[\frac{\partial u_k'}{\partial x_j}+\frac{\partial u_j'}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k'}{\partial x_j}+\frac{\partial\omega_j'}{\partial x_k}\Big)\Big]+\\ &+\frac{\mu}{3}\sum_{k,j=1}^{3}\Big[\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big]\times\\ &\qquad\times\Big[\frac{\partial u_k'}{\partial x_k}-\frac{\partial u_j'}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial\omega_k'}{\partial x_k}-\frac{\partial\omega_j'}{\partial x_j}\Big)\Big]+\\ &+\Big(\gamma-\frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,\,k\neq j}^{3}\Big[\frac{1}{2}\Big(\frac{\partial\omega_k}{\partial x_j}+\frac{\partial\omega_j}{\partial x_k}\Big)\Big(\frac{\partial\omega_k'}{\partial x_j}+\frac{\partial\omega_j'}{\partial x_k}\Big)+\\ &\qquad+\frac{1}{3}\Big(\frac{\partial\omega_k}{\partial x_k}-\frac{\partial\omega_j}{\partial x_j}\Big)\Big(\frac{\partial\omega_k'}{\partial x_k}-\frac{\partial\omega_j'}{\partial x_j}\Big)\Big]+\\ &+\alpha\Big(\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\Big)\cdot\Big(\operatorname{curl}u'+\frac{\nu}{\alpha}\operatorname{curl}\omega'-2\omega'\Big).\end{aligned}$$

In particular,

$$\begin{aligned}E(\widetilde{U},\overline{\widetilde{U}})=&\ \frac{3\lambda+2\mu}{3}\Big|\operatorname{div}u+\frac{3\delta+2\varkappa}{3\lambda+2\mu}\operatorname{div}\omega\Big|^2+\\ &+\frac{1}{3}\Big(3\beta+2\gamma-\frac{(3\delta+2\varkappa)^2}{3\lambda+2\mu}\Big)|\operatorname{div}\omega|^2+\end{aligned}$$

$$+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\left|\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}+\frac{\varkappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\Big)\right|^2+$$
$$+\frac{\mu}{3}\sum_{k,j=1}^{3}\left|\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}+\frac{\varkappa}{\mu}\Big(\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\Big)\right|^2+$$
$$+\Big(\gamma-\frac{\varkappa^2}{\mu}\Big)\sum_{k,j=1,\,k\neq j}^{3}\left[\frac{1}{2}\left|\frac{\partial \omega_k}{\partial x_j}+\frac{\partial \omega_j}{\partial x_k}\right|^2+\frac{1}{3}\left|\frac{\partial \omega_k}{\partial x_k}-\frac{\partial \omega_j}{\partial x_j}\right|^2\right]+$$
$$+\Big(\varepsilon-\frac{\nu^2}{\alpha}\Big)|\operatorname{curl}\omega|^2+\alpha\Big|\operatorname{curl}u+\frac{\nu}{\alpha}\operatorname{curl}\omega-2\omega\Big|^2.$$

We formulate here the following technical lemma.

Lemma 2.1. *Let* $\widetilde{U}=(u,\omega)^\top\in[C^1(\Omega^+)]^6$ *and* $E(\widetilde{U},\overline{\widetilde{U}})=0$ *in* Ω^+. *Then*

$$u(x)=[a\times x]+b,\quad \omega(x)=a,\ \ x\in\Omega^+, \tag{2.18}$$

where a *and* b *are arbitrary three-dimensional constant complex vectors.*

Moreover,

(i) *for an arbitrary vector* $\widetilde{U}=(u,\omega)^\top$ *defined by formulas* (2.18) *and an arbitrary unit vector* $n=(n_1,n_2,n_3)$ *the generalized hemitropic stress vector* $T(\partial,n)\widetilde{U}$ *vanishes identically, i.e.,* $T(\partial,n)\widetilde{U}(x)=0$ *for all* $x\in\Omega^+$.

(ii) *for an arbitrary vector* $U:=(\widetilde{U},0)^\top=(u,\omega,0)^\top$, *where* u *and* ω *are given by formulas* (2.18), *and for an arbitrary unit vector* $n=(n_1,n_2,n_3)$ *the generalized hemitropic thermo-stress vector* $\mathcal{P}(\partial,n)U$ *vanishes identically, i.e.,* $\mathcal{P}(\partial,n)U(x)=0$ *for all* $x\in\Omega^+$.

Proof. The first part of the lemma is shown in [40]. The second part easily follows from the first part and from the formulas (2.10), (2.11), (2.13). □

Throughout the paper L_p, W_p^s, H_p^s, and $B_{p,q}^s$ (with $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$) denote the well–known Lebesgue, Sobolev–Slobodetski, Bessel potential, and Besov spaces, respectively (see, e.g., [54], [55], [31]). We recall that $H_2^s=W_2^s=B_{2,2}^s$, $W_p^t=B_{p,p}^t$, and $H_p^k=W_p^k$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Further, let $\mathcal{M}_0$ be a Lipschitz surface without boundary. For a Lipschitz sub-manifold $\mathcal{M}\subset\mathcal{M}_0$ we denote by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ the subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\widetilde{H}_p^s(\mathcal{M})=\Big\{g:\ g\in H_p^s(\mathcal{M}_0),\ \operatorname{supp}g\subset\overline{\mathcal{M}}\Big\},$$
$$\widetilde{B}_{p,q}^s(\mathcal{M})=\Big\{g:\ g\in B_{p,q}^s(\mathcal{M}_0),\ \operatorname{supp}g\subset\overline{\mathcal{M}}\Big\},$$

while $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ denote the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$H_p^s(\mathcal{M})=\big\{r_{\mathcal{M}}f:\ f\in H_p^s(\mathcal{M}_0)\big\},\quad B_{p,q}^s(\mathcal{M})=\big\{r_{\mathcal{M}}f\ :\ f\in B_{p,q}^s(\mathcal{M}_0)\big\}.$$

Here $r_{\mathcal{M}}$ is the restriction operator.

If $\widetilde{U}=\widetilde{U}^{(1)}+i\widetilde{U}^{(2)}$ is a complex–valued vector, where $\widetilde{U}^{(j)}=(u^{(j)},\omega^{(j)})^\top$ $(j=1,2)$ are real–valued vectors, then

$$E(\widetilde{U},\overline{\widetilde{U}})=E(\widetilde{U}^{(1)},\widetilde{U}^{(1)})+E(\widetilde{U}^{(2)},\widetilde{U}^{(2)}),$$

and, due to the positive definiteness of the energy form for real–valued vector functions, we have

$$E(\widetilde{U},\overline{\widetilde{U}})\geq c^*\sum_{p,q=1}^{3}\Big[(u_{pq}^{(1)})^2+(u_{pq}^{(2)})^2+(\omega_{pq}^{(1)})^2+(\omega_{pq}^{(2)})^2\Big],$$

where c^* is a positive constant depending only on the material constants, and $u_{pq}^{(j)}$ and $\omega_{pq}^{(j)}$ are defined by formulae (2.16) with $u^{(j)}$ and $\omega^{(j)}$ for u and ω.

From the positive definiteness of the energy form $E(\cdot,\cdot)$ with respect to the variables (2.16) it follows that there exist positive constants c_1 and c_2 such that for an arbitrary real–valued vector $\widetilde{U}\in[C^1(\overline{\Omega^+})]^6$

$$\widetilde{\mathcal{B}}(\widetilde{U},\widetilde{U}):=\int\limits_{\Omega^+}E(\widetilde{U},\widetilde{U})\,dx\geq$$

$$\geq c_1\int\limits_{\Omega^+}\bigg\{\sum_{p,q=1}^{3}\big[(\partial_p u_q)^2+(\partial_p\omega_q)^2\big]+\sum_{p=1}^{3}[u_p^2+\omega_p^2]\bigg\}\,dx-$$

$$-c_2\int\limits_{\Omega^+}\sum_{p=1}^{3}[u_p^2+\omega_p^2]\,dx,$$

i.e., the following Korn's type inequality holds (cf. [17, Part I, § 12], [32, Ch. 10])

$$\widetilde{\mathcal{B}}(\widetilde{U},\widetilde{U})\geq c_1\|\widetilde{U}\|^2_{[H_2^1(\Omega^+)]^6}-c_2\|\widetilde{U}\|^2_{[H_2^0(\Omega^+)]^6},\tag{2.19}$$

where $\|\cdot\|_{[H_2^s(\Omega^+)]^6}$ denotes the norm in the Sobolev space $[H_2^s(\Omega^+)]^6$. Clearly, the counterpart of (2.19) holds for an arbitrary complex-valued vector $\widetilde{U}\in[H_2^1(\Omega^+)]^6$ as well,

$$\widetilde{\mathcal{B}}(\widetilde{U},\overline{\widetilde{U}})\geq c_1\|\widetilde{U}\|^2_{[H_2^1(\Omega^+)]^6}-c_2\|\widetilde{U}\|^2_{[H_2^0(\Omega^+)]^6}.\tag{2.20}$$

These results imply that the differential operators $\widetilde{L}(\partial,\sigma)$ and $\widetilde{L}_0(\partial)$ are *strongly elliptic* and the following inequality (*the accretivity condition*) holds (cf., e.g., [17, Part I, § 5], [32, Ch. 4, Lemma 4.5])

$$c_2'|\xi|^2|\eta|^2\geq\widetilde{L}_0(\xi)\eta\cdot\eta=\sum_{k,j=1}^{6}\widetilde{L}_{0kj}(\xi)\eta_j\overline{\eta_k}\geq c_1'|\xi|^2|\eta|^2\tag{2.21}$$

with some constants $c_k'>0$, $k=1,2$, for arbitrary $\xi\in\mathbb{R}^3$ and arbitrary complex vector $\eta\in\mathbb{C}^6$.

Consequently, in view of (2.8) and (2.21) the differential operator $L(\partial,\sigma)$ is strongly elliptic as well, since

$$C_2'|\xi|^2|\eta|^2 \geq L_0(\xi)\eta\cdot\overline{\eta} = \sum_{k,j=1}^{6} L_{0kj}(\xi)\eta_j\overline{\eta_k} \geq C_1'|\xi|^2|\eta|^2$$

with some constants $C_k' > 0$, $k=1,2$, for arbitrary $\xi\in\mathbb{R}^3$ and for arbitrary complex vector $\eta\in\mathbb{C}^7$.

Now let $U = (\widetilde{U},\vartheta)^\top = (u,\omega,\vartheta)^\top$ and $U' = (\widetilde{U}',\vartheta')^\top = (u',\omega',\vartheta')^\top$ be vector functions of the class $[C^2(\overline{\Omega^+})]^7$. With the help of relation (2.14) and standard manipulations we can show that the following Green's formulas hold

$$\int\limits_{\Omega^+} U'\cdot L(\partial,\sigma)U\,dx = \int\limits_{\partial\Omega^+} \{U'\}^+\cdot\big\{\mathcal{P}(\partial,n)U\big\}^+\,dS -$$
$$-\int\limits_{\Omega^+}\Big[E(\widetilde{U}',\widetilde{U}) - \varrho\sigma^2 u'\cdot u - \mathcal{I}\sigma^2\omega'\cdot\omega - \eta\vartheta\,\mathrm{div}\,u' - \zeta\vartheta\,\mathrm{div}\,\omega' -$$
$$-i\eta\sigma\vartheta'\,\mathrm{div}\,u - i\zeta\sigma\vartheta'\,\mathrm{div}\,\omega - i\sigma\kappa''\vartheta\vartheta' + \kappa'\,\mathrm{grad}\,\vartheta'\cdot\mathrm{grad}\,\vartheta\Big]\,dx, \qquad (2.22)$$

$$\int\limits_{\Omega^+}\Big[U'\cdot L(\partial,\sigma)U - L^*(\partial,\sigma)U'\cdot U\Big]\,dx =$$
$$= \int\limits_{\partial\Omega^+}\Big[\{U'\}^+\cdot\big\{\mathcal{P}(\partial,n)U\big\}^+ - \big\{\mathcal{P}^*(\partial,n)U'\big\}^+\cdot\{U\}^+\Big]\,dS, \qquad (2.23)$$

where $L^*(\partial,\sigma) = L^\top(-\partial,\sigma)$ is the operator formally adjoint to $L(\partial,\sigma)$, the differential operators $L(\partial,\sigma)$, $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$ are defined by (2.4), (2.11) and (2.12) respectively. The proof of (2.22) and (2.23) easily follows from (2.14) in view of the identity

$$U'\cdot L(\partial,\sigma)U - \widetilde{U}'\cdot\widetilde{L}(\partial,0)\widetilde{U} = \varrho\sigma^2 u'\cdot u - \eta\,\mathrm{grad}\,\vartheta\cdot u' + \mathcal{I}\sigma^2\omega'\cdot\omega -$$
$$-\zeta\,\mathrm{grad}\,\vartheta\cdot\omega' + \kappa'\vartheta'\Delta\vartheta + i\eta\sigma\vartheta'\,\mathrm{div}\,u + i\sigma\zeta\vartheta'\,\mathrm{div}\,\omega + i\sigma\kappa''\vartheta\vartheta'.$$

By the standard limiting approach, Green's formula (2.22) can be extended to Lipschitz domains (see, e.g., [45], [32]) and to the case of complex–valued vector functions $U\in[W_p^1(\Omega^+)]^7$ and $U'\in[W_{p'}^1(\Omega^+)]^7$ with $1/p+1/p'=1$, $1<p<\infty$, and $L(\partial,\sigma)U\in[L_p(\Omega^+)]^7$ (cf. [31], [10], [32])

$$\Big\langle\{U'\}^+,\big\{\mathcal{P}(\partial,n)U\big\}^+\Big\rangle_{\partial\Omega^+} = \int\limits_{\Omega^+} U'\cdot L(\partial,\sigma)U\,dx +$$
$$+\int\limits_{\Omega^+}\Big[E(\widetilde{U}',\widetilde{U}) - \varrho\sigma^2 u'\cdot u - \mathcal{I}\sigma^2\omega'\cdot\omega - \eta\vartheta\,\mathrm{div}\,u' - \zeta\vartheta\,\mathrm{div}\,\omega' -$$
$$-i\eta\sigma\vartheta'\,\mathrm{div}\,u - i\zeta\sigma\vartheta'\,\mathrm{div}\,\omega - i\sigma\kappa''\vartheta\vartheta' + \kappa'\,\mathrm{grad}\,\vartheta'\cdot\mathrm{grad}\,\vartheta\Big]\,dx, \qquad (2.24)$$

where $\langle \cdot , \cdot \rangle_{\partial\Omega^+}$ denotes the duality between the spaces $[B^{\frac{1}{p}}_{p,p}(\partial\Omega^+)]^7$ and $[B^{-\frac{1}{p}}_{p',p'}(\partial\Omega^+)]^7$, which extends the usual real L_2-scalar product, i.e., for $f, g \in [L_2(S)]^7$

$$\langle f, g\rangle_S = \sum_{k=1}^{7} \int\limits_S f_k g_k \, dS = (f, g)_{[L_2(S)]^7}.$$

Clearly, the generalized trace functional $\{\mathcal{P}(\partial, n)U\}^+ \in [B^{-\frac{1}{p}}_{p,p}(\partial\Omega^+)]^7$ is well defined by the relation (2.24).

Let us introduce the sesquilinear form related to the operator $L(\partial, \sigma)$

$$\mathcal{B}(U, U') := \int\limits_{\Omega^+} \Big[E(\widetilde{U}, \overline{\widetilde{U'}}) - \varrho\sigma^2 u \cdot \overline{u'} - \mathcal{I}\sigma^2 \omega \cdot \overline{\omega'} - \eta\vartheta \operatorname{div} \overline{u'} - \zeta\vartheta \operatorname{div} \overline{\omega'} - \\ - i\eta\sigma\overline{\vartheta'} \operatorname{div} u - i\zeta\sigma\overline{\vartheta'} \operatorname{div} \omega - i\sigma\kappa''\vartheta\overline{\vartheta'} + \kappa' \operatorname{grad} \vartheta \cdot \operatorname{grad} \overline{\vartheta'} \Big] \, dx. \quad (2.25)$$

With the help of (2.20) and (2.25) we derive the inequality

$$\operatorname{Re} \mathcal{B}(U, U) \geq C_1 \|U\|^2_{[H^1_2(\Omega^+)]^7} - C_2 \|U\|^2_{[H^0_2(\Omega^+)]^7}, \quad (2.26)$$

with some positive constants C_1 and C_2. This inequality plays a crucial role in the study of boundary value problems of the micropolar elasticity theory for hemitropic continua by means of the variational methods based on the well known Lax–Milgram theorem.

3. Formulation of Transmission Problems and Uniqueness Theorems

Let Ω be a bounded region in $\mathbb{R}^3$ with the smooth connected boundary $\partial\Omega = S_0$. Let $\overline{\Omega}_1 \subset \Omega$ be a sub-domain of Ω with a smooth simply connected boundary $\partial\Omega_1 = S_1 \subset \Omega$. Put $\Omega_0 := \Omega \setminus \overline{\Omega}_1$. In what follows, by $n(z)$, $z \in S_0 \cup S_1$, we denote the outward unit normal vector with respect to the domains Ω_1 and Ω, at the point z. We assume that $S_\ell \in C^{2,\gamma'}$, $0 < \gamma' \leq 1$, $\ell = 0, 1$, if not otherwise stated. Let the domains Ω_ℓ be filled up by elastic continua heaving different hemitropic material constants, $\alpha^{(\ell)}$, $\beta^{(\ell)}$, $\gamma^{(\ell)}$, $\delta^{(\ell)}$, $\lambda^{(\ell)}$, $\mu^{(\ell)}$, $\nu^{(\ell)}$, $\varkappa^{(\ell)}$ and $\varepsilon^{(\ell)}$, $\ell = 0, 1$; $\eta^{(\ell)} > 0$ and $\zeta^{(\ell)} > 0$, $\ell = 0, 1$, are constants describing the coupling of mechanical and thermal fields in Ω_ℓ (see [3], [14]), $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$.

Analogously, for the mechanical characteristics, e.g., the displacement and microrotation vectors, the force stress and couple stress vectors, and also for the differential operators, fundamental matrices and potentials related to the hemitropic material occupying the domain Ω_ℓ, $\ell = 0, 1$, we also employ the superscript (ℓ). In particular, $u^{(\ell)} = (u^{(\ell)}_1, u^{(\ell)}_2, u^{(\ell)}_3)^T$, $\omega^{(\ell)} = (\omega^{(\ell)}_1, \omega^{(\ell)}_2, \omega^{(\ell)}_3)^T$ and $\vartheta^{(\ell)}$ denote the displacement and microrotation vectors and temperature function in the domain Ω_ℓ; $E^{(\ell)}(U^{(\ell)}, U^{(\ell)})$ designates the appropriate potential energy density, $L^{(\ell)}(\partial, \sigma)$, $L^{(\ell)}(\partial)$, $L^{(\ell)}_0(\partial)$,

$\mathcal{P}^{(\ell)}(\partial, n)$ and $\mathcal{P}_0^{(\ell)}(\partial, n)$ are the corresponding differential operators given by the formulae (2.4), (2.7), (2.8), (2.5) and (2.6).

In what follows we treat transmission problems for the differential equations of pseudo-oscillations, i.e., we assume that

$$\sigma = \sigma_1 + i\sigma_2 \ \text{ with } \ \sigma_2 > 0. \tag{3.1}$$

It is clear that the nonhomogeneous differential equation $L^{(\ell)}(\partial, \sigma)U^{(\ell)} = \Psi^{(\ell)}$ in Ω_ℓ we can reduce to the homogeneous one, $L^{(\ell)}(\partial, \sigma)V^{(\ell)} = 0$, with the help of the volume Newtonian potential $N_{\Omega_\ell}(\Psi^{(\ell)})$ (see Appendix A). Therefore, without loss of generality we can assume that the body force and body couple vectors absent.

We will study the following boundary-transmission problems:

Find regular complex-valued vector-functions $U^{(\ell)} \in [C^1(\overline{\Omega}_\ell)]^7 \cap [C^2(\Omega_\ell)]^7$, $\ell = 0, 1$, satisfying the differential equations

$$L^{(\ell)}(\partial, \sigma)U^{(\ell)}(x) = 0 \ \text{ in } \ \Omega_\ell, \ \ \ell = 0, 1, \tag{3.2}$$

the transmission conditions on S_1

$$\{U^{(1)}(z)\}^+ - \{U^{(0)}(z)\}^- = f(z) \ \text{ on } \ S_1, \tag{3.3}$$

$$\big\{\mathcal{P}^{(1)}(\partial, n)U^{(1)}(z)\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial, n)U^{(0)}(z)\big\}^- = F(z) \ \text{ o } \ S_1, \tag{3.4}$$

and either the Dirichlet boundary condition on S_0

$$\{U^{(0)}(z)\}^+ = f^{(D)}(z) \ \text{ n } \ S_0, \tag{3.5}$$

or the Neumann boundary condition on S_0

$$\big\{\mathcal{P}^{(0)}(\partial, n)U^{(0)}(z)\big\}^+ = F^{(N)}(z) \ \text{ on } \ S_0. \tag{3.6}$$

We assume that the given transmission and boundary data are complex-valued vectors and

$$f \in [C^{1,\beta'}(S_0)]^7, \quad F \in [C^{0,\beta'}(S_0)]^7,$$
$$f^{(D)} \in [C^{1,\beta'}(S_1)]^7, \quad F^{(N)} \in [C^{0,\beta'}(S_1)]^7,$$

with $0 < \beta' < \gamma' \leq 1$. We refer to the boundary-transmission problem (3.2)–(3.5) as Problem (TD) and the boundary-transmission problem (3.2)–(3.4) and (3.6) as Problem (TN).

The above problem setting is a *classical* one in the space of continuously differentiable vector-functions.

In the case of a *weak setting* of the problems we look for a solution pair $(U^{(0)}, U^{(1)})$ in the Sobolev spaces, $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$, $\ell = 0, 1$, with $L^{(\ell)}(\partial, \sigma)U^{(\ell)} \in [L_p(\Omega_\ell)]^7$. Therefore, equations (3.2) are understood in the distributional sense. However, we remark that solutions to these homogeneous equations actually are analytical vector-functions of the real spatial variable x in the open domains Ω_0 and Ω_1, since the differential operators $L^{(\ell)}(\partial, \sigma)$ are strongly elliptic.

The Dirichlet type boundary and transmission conditions are understood in the usual trace sense, while the Neumann type conditions are understood in the generalized trace sense defined by Green's identity (2.24) (for details see [37], [42]).

We start with the study of uniqueness of solutions to these problems.

Theorem 3.1. *Problems* (TD) *and* (TN) *may have at most one solution in the space of regular vector-functions.*

Proof. Due to linearity of the problems under consideration, it suffices to show that the corresponding homogeneous problems have only the trivial solutions. Let a pair of regular vectors

$$(U^{(0)},U^{(1)})\in\big([C^1(\overline{\Omega}_0)]^7\cap[C^2(\Omega_0)]^7\big)\times\big([C^1(\overline{\Omega}_1)]^7\cap[C^2(\Omega_1)]^7\big)$$

be a solution of either the homogeneous Problem (TD) or Problem (TN). Using Green's formulae for the vector-functions $U^{(0)}$ and $U^{(1)}$ and taking into account the chosen direction of the normal vector on the boundaries S_0 and S_1, we get

$$\int\limits_{\Omega_1}\Big[-E^{(1)}\big(\widetilde{U}^{(1)},\overline{\widetilde{U}^{(1)}}\big)+\varrho_1\sigma^2|u^{(1)}|^2+\mathcal{I}_1\sigma^2|\omega^{(1)}|^2-C_0\kappa_1'|\nabla\vartheta^{(1)}|^2-\kappa_1''|\vartheta^{(1)}|^2\Big]dx+$$
$$+\int\limits_{S_1}\Big\{\mathcal{T}^{(1)}(\partial,n)U^{(1)}\cdot\overline{\widetilde{U}^{(1)}}+C_0\kappa_1'\vartheta^{(1)}\partial_n\overline{\vartheta^{(1)}}\Big\}^+dS=0, \tag{3.7}$$

$$\int\limits_{\Omega_0}\Big[-E^{(0)}\big(\widetilde{U}^{(0)},\overline{\widetilde{U}^{(0)}}\big)+\varrho_0\sigma^2|u^{(0)}|^2+\mathcal{I}_0\sigma^2|\omega^{(0)}|^2-C_0\kappa_0'|\nabla\vartheta^{(0)}|^2-\kappa_0''|\vartheta^{(0)}|^2\Big]dx+$$
$$+\int\limits_{S_0}\Big\{\mathcal{T}^{(0)}(\partial,n)U^{(0)}\cdot\overline{\widetilde{U}^{(0)}}+C_0\kappa_0'\vartheta^{(0)}\partial_n\overline{\vartheta^{(0)}}\Big\}^+dS-$$
$$-\int\limits_{S_1}\Big\{\mathcal{T}^{(0)}(\partial,n)U^{(0)}\cdot\overline{\widetilde{U}^{(0)}}+C_0\kappa_0'\vartheta^{(0)}\partial_n\overline{\vartheta^{(0)}}\Big\}^-dS=0, \tag{3.8}$$

where

$$C_0=-\frac{i}{\sigma}\,,\;\;\kappa_\ell'=\frac{\lambda_0^{(\ell)}}{T_0^{(\ell)}},\;\;\kappa_\ell''=\frac{c_0^{(\ell)}}{T_0^{(\ell)}},\;\;\widetilde{U}^{(\ell)}=(u^{(\ell)},\;\;\omega^{(\ell)})^\top,\;\;\ell=0,1.$$

The homogeneous boundary and transmission conditions, $f^{(\ell)}=F^{(\ell)}=0$, yield

$$\sum_{\ell=0}^{1}\int\limits_{\Omega_\ell}\Big[E^{(\ell)}\big(\widetilde{U}^{(\ell)},\overline{\widetilde{U}^{(\ell)}}\big)-\varrho_\ell\sigma^2|u^{(\ell)}|^2-\mathcal{I}_\ell\sigma^2|\omega^{(\ell)}|^2+$$
$$+C_0\kappa_\ell'|\nabla\vartheta^{(\ell)}|^2+\kappa_\ell''|\vartheta^{(\ell)}|^2\Big]dx=0. \tag{3.9}$$

Separating the imaginary part leads to the relation

$$\sigma_1 \sum_{\ell=0}^{1} \int_{\Omega_\ell} \Big[2\sigma_2 \varrho_\ell |u^{(\ell)}|^2 + 2\sigma_2 \mathcal{I}_\ell |\omega^{(\ell)}|^2 + \frac{\kappa'_\ell}{|\sigma|^2} |\nabla \vartheta^{(\ell)}|^2 \Big]\, dx = 0.$$

If $\sigma_1 \neq 0$, we then conclude $u^{(\ell)} = 0$, $\omega^{(\ell)} = 0$, $\vartheta^{(\ell)} = const$. But from (3.9) we have $\vartheta^{(\ell)} = 0$ and consequently $U^{(\ell)} = 0$ in Ω_ℓ. If $\sigma_1 = 0$, then from (3.9) we have

$$E^{(\ell)}(\widetilde{U}^{(\ell)}, \overline{\widetilde{U}^{(\ell)}}) + \sigma_2^2 \varrho_\ell |u^{(\ell)}|^2 + \sigma_2^2 \mathcal{I}_\ell |\omega^{(\ell)}|^2 + \frac{\kappa'_\ell}{\sigma_2} |\nabla \vartheta^{(\ell)}|^2 + \kappa''_\ell |\vartheta^{(\ell)}|^2 = 0$$

for $\ell = 0, 1$, whence $u^{(\ell)} = 0$, $\omega^{(\ell)} = 0$, $\vartheta^{(\ell)} = 0$ in Ω_ℓ follow. □

By the quite similar arguments one can prove the following uniqueness theorem for the same transmission problems in the weak formulation.

Theorem 3.2. *Problems* (TD) *and* (TN) *may have at most one solution in the space* $(U^{(0)}, U^{(1)}) \in [W_2^1(\Omega_0)]^7 \times [W_2^1(\Omega_1)]^7$.

4. Existence Results for Problem (TD)

Here we develop the so called indirect boundary integral equations approach. We look for a solution pair of Problem (TD) in the form of single layer potentials, see Appendix A,

$$U^{(1)}(x) = V_{S_1}^{(1)}\big([\mathcal{H}_{S_1}^{(1)}]^{-1}\varphi\big)(x) \equiv$$

$$\equiv \int_{S_1} \Gamma^{(1)}(x-y,\sigma)\big([\mathcal{H}_{S_1}^{(1)}]^{-1}\varphi\big)(y)\, dS_y, \quad x \in \Omega_1, \tag{4.1}$$

$$U^{(0)}(x) = V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\psi\big)(x) + V_{S_1}^{(0)}\big([\mathcal{H}_{S_1}^{(0)}]^{-1}\chi\big)(x) \equiv$$

$$\equiv \int_{S_0} \Gamma^{(0)}(x-y,\sigma)\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\psi\big)(y)\, dS_y +$$

$$+ \int_{S_1} \Gamma^{(0)}(x-y,\sigma)\big([\mathcal{H}_{S_1}^{(0)}]^{-1}\chi\big)(y)\, dS_y, \quad x \in \Omega_0, \tag{4.2}$$

where $\varphi = (\varphi_1, \dots, \varphi_7)^\top$, $\psi = (\psi_1, \dots, \psi_7)^\top$ and $\chi = (\chi_1, \dots, \chi_7)^\top$ are unknown densities; $\Gamma^{(\ell)}(x-y,\sigma)$ is the fundamental matrix of the operator $L^{(\ell)}(\partial,\sigma)$, $\ell = 0,1$; $[\mathcal{H}_{S_j}^{(\ell)}]^{-1}$ stands for the operator inverse to $\mathcal{H}_{S_j}^{(\ell)}$, $\ell, j = 0, 1$, which is well defined due to Theorems A.5 and A.6 in Appendix A.

Recall that for the potentials and the boundary operators generated by them, the superscript (ℓ) shows the correspondence to the type of hemitropic material in Ω_ℓ.

Taking into consideration the transmission and boundary conditions of Problem (TD) and using the properties of the single-layer potentials we arrive at the system of boundary integral (pseudodifferential) equations:

$$\begin{gathered}\varphi(z)-\chi(z)-\int\limits_{S_0}\Gamma^{(0)}(z-y,\sigma)\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)(y)\,dS_y=f(z),\;\;z\in S_1,\\ \Big[(-2^{-1}I_7+\mathcal{K}^{(1)}_{S_1})[\mathcal{H}^{(1)}_{S_1}]^{-1}\varphi\Big](z)-\Big[(2^{-1}I_7+\mathcal{K}^{(0)}_{S_1})[\mathcal{H}^{(0)}_{S_1}]^{-1}\chi\Big](z)-\\ -\int\limits_{S_0}\mathcal{P}^{(0)}(\partial_z,n(z))\Gamma^{(0)}(z-y,\sigma)\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)(y)\,dS_y=F(z),\;\;z\in S_1,\\ \int\limits_{S_1}\Gamma^{(0)}(z-y,\sigma)\big([\mathcal{H}^{(0)}_{S_1}]^{-1}\chi\big)(y)\,dS_y+\psi(z)=f^{(D)}(z),\;\;z\in S_0.\end{gathered}\tag{4.3}$$

The operators $\mathcal{K}^{(\ell)}_{S_\ell}$, $\ell=0,1$, are defined in Appendix A (see Theorem A.1).

Introduce the so called Steklov–Poincaré type operators

$$\mathcal{A}^{(0)}_{S_1}:=(2^{-1}I_7+\mathcal{K}^{(0)}_{S_1})[\mathcal{H}^{(0)}_{S_1}]^{-1},\quad \mathcal{A}^{(1)}_{S_1}:=(-2^{-1}I_7+\mathcal{K}^{(1)}_{S_1})[\mathcal{H}^{(1)}_{S_1}]^{-1},\tag{4.4}$$

and rewrite system (4.3) as

$$\begin{aligned}\varphi-\chi-V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)&=f\;\text{ on }\;S_1,\\ \mathcal{A}^{(1)}_{S_1}\varphi-\mathcal{A}^{(0)}_{S_1}\chi-\mathcal{P}^{(0)}(\partial_z,n)V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)&=F\;\text{ on }\;S_1,\\ V^{(0)}_{S_1}([\mathcal{H}^{(0)}_{S_1}]^{-1}\chi)+\psi&=f^{(D)}\;\text{ on }\;S_0.\end{aligned}\tag{4.5}$$

Denote by r_Σ the restriction operator onto Σ. Clearly, the operators $r_{S_1}V^{(0)}_{S_0}$, $r_{S_1}\mathcal{P}^{(0)}V^{(0)}_{S_0}$ and $r_{S_0}V^{(0)}_{S_1}$, involved in the above equations are smoothing operators, since the surfaces S_1 and S_0 are disjoint.

Denote the operator generated by the left hand side expressions in (4.5) by $\mathcal{D}$ which acts on the triplet of the sought for vectors $(\varphi,\chi,\psi)^\top$,

$$\mathcal{D}:=\begin{bmatrix} I_7 & -I_7 & -r_{S_1}V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\big)\\ \mathcal{A}^{(1)}_{S_1} & -\mathcal{A}^{(0)}_{S_1} & -r_{S_1}\mathcal{P}^{(0)}V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\big)\\ 0 & r_{S_0}V^{(0)}_{S_1}([\mathcal{H}^{(0)}_{S_1}]^{-1}) & I_7\end{bmatrix}_{21\times 21}.$$

Set

$$\Psi=(\varphi,\chi,\psi)^\top,\quad Q=(f,F,f^{(D)})^\top,$$

and rewrite (4.5) in matrix form

$$\mathcal{D}\Psi=Q.$$

Let us introduce the function spaces:

$$\begin{gathered}\boldsymbol{X}^{k,\beta'}:=[C^{k,\beta'}(S_1)]^7\times[C^{k,\beta'}(S_1)]^7\times[C^{k,\beta'}(S_0)]^7,\\ \boldsymbol{Y}^{k,\beta'}:=[C^{k,\beta'}(S_1)]^7\times[C^{k-1,\beta'}(S_1)]^7\times[C^{k,\beta'}(S_0)]^7,\\ S_0,S_1\in C^{k+1,\gamma'},\;\;k\geq 1,\;\;0<\beta'<\gamma'\leq 1,\end{gathered}\tag{4.6}$$

$$\begin{aligned} \boldsymbol{X}_p^s &:= [H_p^s(S_1)]^7 \times [H_p^s(S_1)]^7 \times [H_p^s(S_0)]^7, \\ \boldsymbol{Y}_p^s &:= [H_p^s(S_1)]^7 \times [H_p^{s-1}(S_1)]^7 \times [H_p^s(S_0)]^7, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \boldsymbol{X}_{p,t}^s &:= [B_{p,t}^s(S_1)]^7 \times [B_{p,t}^s(S_1)]^7 \times [B_{p,t}^s(S_0)]^7, \\ \boldsymbol{Y}_{p,t}^s &:= [B_{p,t}^s(S_1)]^7 \times [B_{p,t}^{s-1}(S_1)]^7 \times [B_{p,t}^s(S_0)]^7, \\ & s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ 1 \le t \le \infty, \ \ S_0, S_1 \in C^\infty. \end{aligned} \tag{4.8}$$

The results collected in Appendix A yield the following mapping properties:

$$\begin{gathered} \mathcal{D} : \boldsymbol{X}^{k,\beta'} \longrightarrow \boldsymbol{Y}^{k,\beta'}, \ \ S_0, S_1 \in C^{k+1,\gamma'}, \ \ k \ge 1, \ \ 0 < \beta' < \gamma' \le 1, \\ \mathcal{D} : \boldsymbol{X}_p^s \longrightarrow \boldsymbol{Y}_p^s, \ \ s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ S_0, S_1 \in C^\infty, \\ \mathcal{D} : \boldsymbol{X}_{p,t}^s \longrightarrow \boldsymbol{Y}_{p,t}^s, \ \ s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ 1 \le t \le \infty \ \ S_0, S_1 \in C^\infty. \end{gathered}$$

Further, let us introduce the operator

$$\widetilde{\mathcal{D}} := \begin{bmatrix} I_7 & -I_7 & 0 \\ \mathcal{A}_{S_1}^{(1)} & -\mathcal{A}_{S_1}^{(0)} & 0 \\ 0 & 0 & I_7 \end{bmatrix}_{21\times 21}.$$

It is clear that $\widetilde{\mathcal{D}}$ has the same mapping properties as the operator $\mathcal{D}$ and the operator $\mathcal{D} - \widetilde{\mathcal{D}}$ with the same domain and range spaces is a compact operator. To establish the Fredholm properties of the operator $\mathcal{D}$ first we study the operator $\widetilde{\mathcal{D}}$.

Lemma 4.1. *The operators*

$$\widetilde{\mathcal{D}} : \boldsymbol{X}^{k,\beta'} \longrightarrow \boldsymbol{Y}^{k,\beta'}, \ \ k \ge 1, \ \ 0 < \beta' < \gamma' \le 1, \ \ S_0, S_1 \in C^{k+1,\gamma'}, \tag{4.9}$$

$$\widetilde{\mathcal{D}} : \boldsymbol{X}_p^s \longrightarrow \boldsymbol{Y}_p^s, \ \ s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ S_0, S_1 \in C^\infty, \tag{4.10}$$

$$\widetilde{\mathcal{D}} : \boldsymbol{X}_{p,t}^s \longrightarrow \boldsymbol{Y}_{p,t}^s, \ \ s \in \mathbb{R}, \ \ 1 < p < \infty, \ \ 1 \le t \le \infty, \ \ S_0, S_1 \in C^\infty \tag{4.11}$$

are invertible.

Proof. We prove the lemma into several steps.

Step 1. First we show that the null-space of the operator (4.9) is trivial. To this end, we have to prove that the simultaneous homogeneous equations

$$\begin{aligned} \varphi(z) - \chi(z) &= 0, \ \ z \in S_1, \\ [\mathcal{A}_{S_1}^{(1)}\varphi](z) - [\mathcal{A}_{S_1}^{(0)}\chi](z) &= 0, \ \ z \in S_1, \\ \psi(z) &= 0, \ \ z \in S_0, \end{aligned} \tag{4.12}$$

have only the trivial solution. Since $\psi = 0$ on S_0 it suffices to show that the first two equations imply $\varphi = \chi = 0$ on S_1. Indeed, let φ and χ solve the above homogeneous equations. Construct the single-layer potentials:

$$\begin{aligned} \widetilde{U}^{(1)}(x) &= V_{S_1}^{(1)}\big([\mathcal{H}_{S_1}^{(1)}]^{-1}\varphi\big)(x), \ \ x \in \Omega^+ := \Omega_1, \\ \widetilde{U}^{(0)}(x) &= V_{S_1}^{(0)}\big([\mathcal{H}_{S_1}^{(0)}]^{-1}\chi\big)(x), \ \ x \in \Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}_1. \end{aligned} \tag{4.13}$$

From the first two equations in (4.12) and the properties of the single-layer potentials it follows that the pair of vectors $(\widetilde{U}^{(0)}, \widetilde{U}^{(1)})$ solve the basic homogeneous transmission problem for the whole space with the interface S_1:

$$L^{(1)}(\partial,\sigma)\widetilde{U}^{(1)}(x)=0 \ \text{ in } \ \Omega^+, \quad L^{(0)}(\partial,\sigma)\widetilde{U}^{(0)}(x)=0 \ \text{ in } \ \Omega^-,$$
$$\{\widetilde{U}^{(1)}(z)\}^+ - \{\widetilde{U}^{(0)}(z)\}^- = 0 \ \text{ on } \ S_1,$$
$$\big\{\mathcal{P}^{(1)}(\partial,n)\widetilde{U}^{(1)}(z)\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial,n)\widetilde{U}^{(0)}(z)\big\}^- = 0 \ \text{ on } \ S_1.$$

Note that, if $\varphi,\chi\in[C^{k,\beta'}(S_1)]^7$, then the corresponding single-layer potentials are regular vectors in the $\overline{\Omega^\pm}$, i.e., $\widetilde{U}^{(1)}\in[C^{k,\beta'}(\overline{\Omega^+})]^7\cap[C^\infty(\Omega^+)]^7$ and $\widetilde{U}^{(0)}\in[C^{k,\beta'}(\overline{\Omega^-})]^7\cap[C^\infty(\Omega^-)]^7$. We recall that the entries of the fundamental matrix $\Gamma^{(\ell)}(x,\sigma)$ decay exponentially at infinity (see [44]), and therefore the vector $\widetilde{U}^{(0)}$ and its partial derivatives decay exponentially as $|x|\to+\infty$. It is clear that for such vectors the corresponding Green's formulae hold in the unbounded domain Ω^- (cf. (3.7), (3.8)).

Therefore, by virtue of the homogeneous transmission conditions, as in the proof of Theorem 3.1, we arrive at the equalities $\widetilde{U}^{(1)}=0$ in Ω^+ and $\widetilde{U}^{(0)}=0$ in Ω^-, which in view of (4.13) proves that $\ker\widetilde{\mathcal{D}}$ is trivial.

Step 2. Let us consider the vectors

$$\begin{aligned} U^{(1)}(x)&=V^{(1)}_{S_1}\big([\mathcal{H}^{(1)}_{S_1}]^{-1}\chi\big)(x), \quad x\in\Omega^+,\\ U^{(0)}(x)&=V^{(0)}_{S_1}\big([\mathcal{H}^{(0)}_{S_1}]^{-1}\chi\big)(x), \quad x\in\Omega^-, \end{aligned} \tag{4.14}$$

then we have

$$\{U^{(1)}\}^+=\{U^{(0)}\}^-=\chi, \tag{4.15}$$
$$\big\{\mathcal{P}^{(1)}(\partial,n)U^{(1)}\big\}^+=\mathcal{A}^{(1)}_{S_1}\chi, \quad \big\{\mathcal{P}^{(0)}(\partial,n)\{U^{(0)}\big\}^-=\mathcal{A}^{(0)}_{S_1}\chi \ \text{ on } \ S_1.$$

With the help of formulae (2.24), for vectors $U'=\overline{U^{(1)}}$ and $U=U^{(1)}$ we have

$$\Big\langle\overline{\chi},\mathcal{A}^{(1)}_{S_1}\chi\Big\rangle_{S_1}=\mathcal{B}^{(1)}(U^{(1)},U^{(1)}), \tag{4.16}$$

where

$$\begin{aligned} \mathcal{B}^{(1)}(U^{(1)},U^{(1)})=\int\limits_{\Omega^+}\Big[&E^{(1)}(\overline{\widetilde{U}^{(1)}},\widetilde{U}^{(1)})+\kappa_1'|\nabla\vartheta^{(1)}|^2-\\ &-\varrho^{(1)}\sigma^2|u^{(1)}|^2-\mathcal{I}^{(1)}\sigma^2|\omega^{(1)}|^2-i\sigma\kappa_1''|\vartheta^{(1)}|^2-\\ -\vartheta^{(1)}\operatorname{div}\big(\eta^{(1)}\overline{u^{(1)}}+\zeta^{(1)}\overline{\omega^{(1)}}\big)&-i\sigma\overline{\vartheta^{(1)}}\operatorname{div}(\eta^{(1)}u^{(1)}+\zeta^{(1)}\omega^{(1)})\Big]\,dx. \end{aligned}$$

Quite similarly from (2.24) for vectors $U'=\overline{U^{(0)}}$ and $U=U^{(0)}$ we derive

$$-\Big\langle\overline{\chi},\mathcal{A}^{(0)}_{S_1}\chi\Big\rangle_{S_1}=\mathcal{B}^{(0)}(U^{(0)},U^{(0)}), \tag{4.17}$$

where

$$\mathcal{B}^{(0)}(U^{(0)},U^{(0)}) = \int\limits_{\Omega^-} \Big[E^{(0)}(\overline{\widetilde{U}^{(0)}},\widetilde{U}^{(0)}) + \kappa_0'|\nabla\vartheta^{(0)}|^2 - \\ - \varrho^{(0)}\sigma^2|u^{(0)}|^2 - \mathcal{I}^{(0)}\sigma^2|\omega^{(0)}|^2 - i\sigma\kappa_0''|\vartheta^{(0)}|^2 - \\ - \vartheta^{(0)}\operatorname{div}\big(\eta^{(0)}\overline{u^{(0)}} + \zeta^{(0)}\overline{\omega^{(0)}}\big) - i\sigma\overline{\vartheta^{(0)}}\operatorname{div}\big(\eta^{(0)}u^{(0)} + \zeta^{(0)}\omega^{(0)}\big)\Big]\,dx.$$

Now from (4.16) and (4.17) we have

$$\Big\langle(\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)})\chi,\overline{\chi}\Big\rangle_{S_1} = \mathcal{B}^{(1)}(U^{(1)},U^{(1)}) + \mathcal{B}^{(0)}(U^{(0)},U^{(0)}).$$

Let

$$U := \begin{cases} U^{(1)} & \text{in } \ \Omega^+, \\ U^{(0)} & \text{in } \ \Omega^-. \end{cases}$$

Since $U^{(1)} \in [H_2^1(\Omega^+)]^7$ and $U^{(0)} \in [H_2^1(\Omega^-)]^7$, by relation (4.15) we easily conclude that $U \in [H_2^1(\mathbb{R}^3)]^7$. Taking into consideration the coercivity relation (2.26), we have

$$\operatorname{Re}\Big\langle(\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)})\chi,\overline{\chi}\Big\rangle_{S_1} \geq C_1\|U\|^2_{[H_2^1(\mathbb{R}^3)]^7} - C_2\|U\|^2_{[H_2^0(\mathbb{R}^3)]^7}, \tag{4.18}$$

where C_1 and C_2 are some positive constants. Note that, by the trace theorem from (4.18) we derive

$$\operatorname{Re}\Big\langle(\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)})\chi,\overline{\chi}\Big\rangle_{S_1} \geq C_1'\|\{U\}^{\pm}\|^2_{[H_2^{\frac{1}{2}}(S_1)]^7} - C_2\|U\|^2_{[H_2^0(\mathbb{R}^3)]^7} \geq \\ \geq C_1'\|\chi\|^2_{[H_2^{\frac{1}{2}}(S_1)]^7} - C_2'\|\chi\|^2_{[H_2^{-\frac{1}{2}}(S_1)]^7}, \tag{4.19}$$

since by Theorem A.4 we have the estimate

$$\|U\|_{[H_2^0(\mathbb{R}^3)]^7} \leq C_2^*\|\chi\|_{[H_2^{-\frac{1}{2}}(S_1)]^7}.$$

In turn, the inequality (4.19) implies that the operator

$$\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)} : [H_2^{\frac{1}{2}}(S_1)]^7 \longrightarrow [H_2^{-\frac{1}{2}}(S_1)]^7 \tag{4.20}$$

is Fredholm with zero index (see, e.g., [32]).

Let us show that the null space of the operator (4.20) is trivial. Indeed, if $\chi \in [H_2^{\frac{1}{2}}(S_1)]^7$ is a solution of the homogeneous equation $(\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(0)})\chi = 0$ on S_1, then it follows that the vectors $U^{(1)}$ and $U^{(0)}$ defined by (4.14) solve the homogeneous transmission problem:

$$\begin{aligned} L^{(1)}(\partial,\sigma)U^{(1)}(x) &= 0 \ \text{ in } \ \Omega^+, \\ L^{(0)}(\partial,\sigma)U^{(0)}(x) &= 0 \ \text{ in } \ \Omega^-, \\ \{U^{(1)}(z)\}^+ - \{U^{(0)}(z)\}^- &= 0 \ \text{ on } \ S_1, \\ \big\{\mathcal{P}^{(1)}(\partial,n)U^{(1)}(z)\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial,n)U^{(0)}(z)\big\}^- &= 0 \ \text{ on } \ S_1. \end{aligned}$$

By the arguments applied in the proof of Theorem 3.1, we conclude that $U^{(1)} = 0$ in Ω^+ and $U^{(0)} = 0$ in Ω^-, implying $\chi = 0$ on S_1. Consequently, the null space of the operator (4.20) is trivial. Thus the operator (4.20) is invertible. Then from the general theory of pseudodifferential operators on manifolds without boundary it follows that

$$\begin{aligned}\mathcal{A}^{(1)}_{S_1}-\mathcal{A}^{(0)}_{S_1} &: [H^s_p(S_1)]^7 \longrightarrow [H^{s-1}_p(S_1)]^7,\\ &: [B^s_{p,t}(S_1)]^7 \longrightarrow [B^{s-1}_{p,t}(S_1)]^7\end{aligned}$$

are also invertible operators for arbitrary $s\in\mathbb{R}$, $1<p<\infty$, $1\le t\le\infty$ (see, e.g., [1], [2], [19], [51], [52]).

Step 3. In turn, this yields that the operator (4.10) is invertible for $s=1/2$ and $p=2$, i.e., the system of equations for the triplet $(\varphi,\chi,\psi)\in\boldsymbol{X}_2^{\frac12}$,

$$\begin{aligned}\varphi-\chi&=f \ \text{ on } \ S_1,\\ \mathcal{A}^{(1)}_{S_1}\varphi-\mathcal{A}^{(0)}_{S_1}\chi&=F \ \text{ on } \ S_1,\\ \psi&=f^{(D)} \ \text{ on } \ S_0,\end{aligned}$$

is uniquely solvable for arbitrary $(f,F,f^{(D)})\in\boldsymbol{Y}_2^{\frac12}$.

Applying again the results from the general theory of pseudodifferential operators on manifolds without boundary we conclude that all the operators in (4.9)–(4.11) are invertible. □

Now we are in a position to prove the following invertibility results.

Theorem 4.2. *The operators*

$$\mathcal{D}:\boldsymbol{X}^{k,\beta'}\longrightarrow\boldsymbol{Y}^{k,\beta'},\ \ k\ge1,\ \ 0<\beta'<\gamma'\le1,\ \ S_0,S_1\in C^{k+1,\gamma'},\qquad(4.21)$$

$$:\boldsymbol{X}^s_p\longrightarrow\boldsymbol{Y}^s_p,\ \ s\in\mathbb{R},\ \ 1<p<\infty,\ \ S_0,S_1\in C^\infty,\qquad(4.22)$$

$$:\boldsymbol{X}^s_{p,t}\longrightarrow\boldsymbol{Y}^s_{p,t},\ \ s\in\mathbb{R},\ \ 1<p<\infty,\ \ 1\le t\le\infty,\ \ S_0,S_1\in C^\infty,\qquad(4.23)$$

are invertible.

Proof. First let us note that by Lemma 4.1 the operators (4.21)–(4.23) are Fredholm with zero index, since they are compact perturbations of the invertible operators, due to the compactness of the difference $\mathcal{D}-\widetilde{\mathcal{D}}$ in the corresponding function spaces. Thus, for invertibility we need only to show that their null-spaces are trivial. Let the triplet $\Psi=(\varphi,\chi,\psi)^\top$ belonging to one of the spaces $\boldsymbol{X}^{k,\beta'}$ or $\boldsymbol{X}^s_p$ or $\boldsymbol{X}^s_{p,t}$ be a solution of the homogeneous equation $\mathcal{D}\Psi=0$, i.e., the homogeneous equation (4.5). Due to the regularity theorem for solutions to the elliptic pseudodifferential equations on manifolds without boundary we conclude that actually $\Psi\in\boldsymbol{X}^{k,\beta'}$. Further, with the help of the solution triplet (φ,χ,ψ) we construct the vectors $U^{(0)}$ and $U^{(1)}$ by formulae (4.1)–(4.2). Clearly, the pair $(U^{(0)},U^{(1)})$ is a regular solution to the homogeneous Problem (TD). Consequently, by the uniqueness Theorem 3.1 we have $U^{(1)}=0$ in Ω_1 and $U^{(0)}=0$ in Ω_0. Since $[U^{(1)}]^+=\varphi$ on S_1 we get $\varphi=0$.

The vector $U^{(0)}$ defined by formula (4.2) solves the homogeneous differential equation $L^{(0)}(\partial,\sigma)U^{(0)}=0$ in $R^3\setminus[S_0\cup S_1]$ and is identical zero in Ω_0.

Since the single layer potentials are continuous in $\mathbb{R}^3$ we have that $[U^{(0)}]^- = [U^{(0)}]^+ = 0$ on S_1 and $[U^{(0)}]^+ = [U^{(0)}]^- = 0$ on S_0. So $U^{(0)}$ solves the homogeneous Dirichlet problems for the operator $L^{(0)}(\partial, \sigma)$ in the domain Ω_1 and in the unbounded domain $\mathbb{R}^3 \setminus [\overline{\Omega}_0 \cup \overline{\Omega}_1]$. Moreover, $U^{(0)}$ decays exponentially at infinity. By the uniqueness theorem for the Dirichlet interior and exterior problems, which can be easily proved with the help of Green's formulae (2.22), we establish that $U^{(0)}$ vanishes in $\mathbb{R}^3$. Now, the jump relations for the singlelayer potential imply $[\mathcal{P}^{(0)}U^{(0)}]^- - [\mathcal{P}^{(0)}U^{(0)}]^+ = \chi = 0$ on S_1 and $[\mathcal{P}^{(0)}U^{(0)}]^- - [\mathcal{P}^{(0)}U^{(0)}]^+ = \psi = 0$ on S_0, which completes the proof. □

These invertibility properties for the operator $\mathcal{D}$ lead to the following existence results for Problem (TD).

Theorem 4.3. *Let*

$$S_0, S_1 \in C^{2,\gamma'}, \;\; f \in [C^{1,\beta'}(S_1)]^7, \;\; F \in [C^{0,\beta'}(S_1)]^7,$$
$$f^{(D)} \in [C^{1,\beta'}(S_0)]^7, \;\; 0 < \beta' < \gamma' \le 1.$$

Then the problem (3.2)–(3.5) *has a unique solution in the class of regular vector functions which can be represented by the single layer potentials* (4.1)–(4.2)*, where the triplet*

$$(\varphi, \chi, \psi)^\top \in [C^{1,\beta'}(S_1)]^7 \times [C^{1,\beta'}(S_1)]^7 \times [C^{1,\beta'}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (4.3).

Theorem 4.4. *Let* $p > 1$*,* $s \ge 1$*, and*

$$S_0, S_1 \in C^\infty, \;\; f \in \big[B^{s-\frac{1}{p}}_{p,p}(S_1)\big]^7,$$
$$F \in \big[B^{s-1-\frac{1}{p}}_{p,p}(S_1)\big]^7, \;\; f^{(D)} \in \big[B^{s-\frac{1}{p}}_{p,p}(S_0)\big]^7.$$

Then the problem (3.2)–(3.5) *has a unique solution*

$$(U^{(0)}, U^{(1)}) \in [W^s_p(\Omega_0)]^7 \times [W^s_p(\Omega_1)]^7$$

which can be represented by the single layer potentials (4.1)–(4.2)*, where the triplet*

$$(\varphi, \chi, \psi)^\top \in [B^{s-\frac{1}{p}}_{p,p}(S_1)]^7 \times [B^{s-\frac{1}{p}}_{p,p}(S_1)]^7 \times [B^{s-\frac{1}{p}}_{p,p}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (4.3).

Proof. Existence of solutions directly follows from the representations (4.1)–(4.2) and invertibility of the operator (4.23). Uniqueness for $p = 2$ follows from Theorem 3.1. It remains to show uniqueness of solutions for arbitrary $p > 1$ and $s = 1$.

First we prove that any solution $U^{(\ell)}\in[W^1_p(\Omega_\ell)]^7$ of the homogeneous equation

$$L^{(\ell)}(\partial,\sigma)U^{(\ell)}=0 \ \text{ in } \ \Omega_\ell, \ \ \ell=0,1,$$

can be represented by the single layer potentials:

$$U^{(1)}(x)=V^{(1)}_{S_1}(\varphi^*)(x), \ \ x\in\Omega_1, \tag{4.24}$$

$$U^{(0)}(x)=V^{(0)}_{S_0}(\psi^*)(x)+V^{(0)}_{S_1}(\chi^*)(x), \ \ x\in\Omega_0, \tag{4.25}$$

where $\varphi^*,\chi^*\in[B^{-\frac{1}{p}}_{p,p}(S_1)]^7$ and $\psi^*\in[B^{-\frac{1}{p}}_{p,p}(S_0)]^7$.

We show it for the vector $U^{(0)}\in[W^1_p(\Omega_0)]^7$. By the general integral representation formula we have (see [44, corollary 3.6, formulae (3.77)])

$$\begin{aligned}U^{(0)}=W^{(0)}_{S_0}([U^{(0)}]^+)-V^{(0)}_{S_0}\big([\mathcal{P}^{(0)}U^{(0)}]^+\big)-\\ -W^{(0)}_{S_1}([U^{(0)}]^-)+V^{(0)}_{S_1}\big([\mathcal{P}^{(0)}U^{(0)}]^-\big) \ \text{ in } \ \Omega_0.\end{aligned} \tag{4.26}$$

Furthermore, we establish that the double-layer potentials $W^{(0)}_{S_0}([U^{(0)}]^+)$ and $W^{(0)}_{S_1}([U^{(0)}]^-)$ involved in (4.26) can be represented by the single layer potentials in the interior of S_0 (i.e., in Ω) and in the exterior of S_1 (i.e., in $\mathbb{R}^3\setminus\overline{\Omega}_1$), respectively. Indeed, denote $\widetilde{U}:=W^{(0)}_{S_0}([U^{(0)}]^+)$ in Ω, and consider the vector $U^*:=\widetilde{U}-V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}[\widetilde{U}]^+\big)\in[W^1_p(\Omega)]^7$. Clearly, $L^{(1)}(\partial,\sigma)U^*=0$ in Ω and $[U^*]^+=0$ on S_0. Therefore, applying again the general integral representation formula in Ω, we derive

$$U^*=-V^{(0)}_{S_0}\big([\mathcal{P}^{(0)}U^*]^+\big)\in[W^1_p(\Omega)]^7.$$

Whence it follows that

$$\widetilde{U}=V^{(0)}_{S_0}\Big([\mathcal{H}^{(0)}_{S_0}]^{-1}[\widetilde{U}]^+-[\mathcal{P}^{(0)}U^*]^+\Big) \ \text{ in } \ \Omega.$$

Quite analogously we can show that $W^{(0)}_{S_1}([U^{(0)}]^-)$ is representable by a single layer potential in $\mathbb{R}^3\setminus\overline{\Omega}_1$. Finally, from (4.26) we conclude that $U^{(0)}$ can be represented in the form (4.25). Similarly we derive the representation (4.24).

Due to invertibility of the operators $\mathcal{H}^{(\ell)}_{S_j}$, $\ell,j=0,1$, we conclude that any solution pair $(U^{(0)},U^{(1)})\in[W^1_p(\Omega_0)]^7\times[W^1_p(\Omega_1)]^7$ of the homogeneous Problem (TD) can be represented by formulae (4.1) and (4.2). This implies that the homogeneous problem (TD) with $p>1$ possesses only the trivial solution since the operator $\mathcal{D}$ is invertible by Theorem 4.2. □

Corollary 4.5. *Let*

$$S_0,S_1\in C^\infty, \ \ f\in[H^{\frac{1}{2}}_2(S_1)]^7, \ \ F\in[H^{-\frac{1}{2}}_2(S_1)]^7, \ \ f^{(D)}\in[H^{\frac{1}{2}}_2(S_0)]^7.$$

Then the problem (3.2)–(3.5) *has a unique solution*

$$(U^{(0)},U^{(1)})\in[W^1_2(\Omega_0)]^7\times[W^1_2(\Omega_1)]^7$$

which can be represented by the single layer potentials (4.1)–(4.2)*, were the triplet*

$$(\varphi,\chi,\psi)^\top\in[H_2^{\frac12}(S_1)]^7\times[H_2^{\frac12}(S_1)]^7\times[H_2^{\frac12}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (4.3)*.*

Remark 4.6. Applying the results in the references [8] and [42] (see also [32]) concerning the properties of the potentials on Lipschitz domains one can prove that the inequality (4.18) remains valid when S_1 is a Lipschitz surface and the operator (4.20) is invertible. This implies that Corollary 4.5 holds true when S_0 and S_1 are Lipschitz surfaces.

5. Existence Results for Problem (TN)

We look for a solution pair $(U^{(0)},U^{(1)})$ of Problem (TN) again in the form (4.1)–(4.2). Taking into consideration the transmission and boundary conditions of Problem (TN) and using the properties of the single layer potentials we arrive at the system of boundary pseudodifferential equations with respect to the triplet of unknown densities (φ,χ,ψ):

$$\begin{aligned}\varphi-\chi-r_{S_1}V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\psi\big)&=f\ \text{ on } S_1,\\ \mathcal{A}_{S_1}^{(1)}\varphi-\mathcal{A}_{S_1}^{(0)}\chi-r_{S_1}\mathcal{P}^{(0)}(\partial,n)V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\psi\big)&=F\ \text{ on } S_1,\\ r_{S_0}\mathcal{P}^{(0)}(\partial,n)V_{S_1}^{(0)}\big([\mathcal{H}_{S_1}^{(0)}]^{-1}\chi\big)+\mathcal{A}_{S_0}^{(0)}\psi&=F^{(N)}\ \text{ on } S_0,\end{aligned}\tag{5.1}$$

where $\mathcal{A}_{S_1}^{(1)}$ and $\mathcal{A}_{S_1}^{(0)}$ are the Steklov–Poincaré operators given by (4.4), and

$$\mathcal{A}_{S_0}^{(0)}:=(-2^{-1}I_7+\mathcal{K}_{S_0}^{(0)})[\mathcal{H}_{S_0}^{(0)}]^{-1}.$$

Denote by $\mathcal{N}$ the matrix integral operator generated by the left hand side expressions in (5.1)

$$\mathcal{N}=[\mathcal{N}_{kj}]_{21\times21}:=$$

$$:=\begin{bmatrix} I_7 & -I_7 & -r_{S_1}V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\big)\\ \mathcal{A}_{S_1}^{(1)} & -\mathcal{A}_{S_1}^{(0)} & -r_{S_1}\mathcal{P}^{(0)}V_{S_0}^{(0)}\big([\mathcal{H}_{S_0}^{(0)}]^{-1}\big)\\ 0 & r_{S_0}\mathcal{P}^{(0)}V_{S_1}^{(0)}([\mathcal{H}_{S_1}^{(0)}]^{-1}) & \mathcal{A}_{S_0}^{(0)}\end{bmatrix}_{21\times21}.\tag{5.2}$$

Set

$$\Psi=(\varphi,\chi,\psi)^\top,\quad Q=(f,F,F^{(N)})^\top,$$

and rewrite (5.1) in matrix form

$$\mathcal{N}\Psi=Q.$$

Further, let us introduce the function spaces

$$\begin{gathered}\boldsymbol{Z}^{k,\beta'}:=[C^{k,\beta'}(S_1)]^7\times[C^{k-1,\beta'}(S_1)]^7\times[C^{k-1,\beta'}(S_0)]^7,\\ S_0,S_1\in C^{k+1,\gamma'},\ \ k\geq1,\ \ 0<\beta'<\gamma'\leq1,\\ \boldsymbol{Z}_p^s:=[H_p^s(S_1)]^7\times[H_p^{s-1}(S_1)]^7\times[H_p^{s-1}(S_0)]^7,\end{gathered}$$

$$\boldsymbol{Z}^s_{p,t} := [B^s_{p,t}(S_1)]^7 \times [B^{s-1}_{p,t}(S_1)]^7 \times [B^{s-1}_{p,t}(S_0)]^7,$$
$$s \in \mathbb{R}, \;\; 1 < p < \infty, \;\; 1 \leq t \leq \infty, \;\; S_0, S_1 \in C^\infty.$$

The operator $\mathcal{N}$ possesses the mapping properties

$$\begin{aligned} \mathcal{N} : \boldsymbol{X}^{k,\beta'} &\longrightarrow \boldsymbol{Z}^{k,\beta'}, \\ : \boldsymbol{X}^s_p &\longrightarrow \boldsymbol{Z}^s_p, \\ : \boldsymbol{X}^s_{p,t} &\longrightarrow \boldsymbol{Z}^s_{p,t}, \end{aligned}$$

where the spaces $\boldsymbol{X}^{k,\beta'}$, $\boldsymbol{X}^s_p$, and $\boldsymbol{X}^s_{p,t}$ are defined in (4.6)–(4.8) respectively. To establish Fredholm properties of these operators let us consider the principal part $\widetilde{\mathcal{N}}$ of the operator (5.2)

$$\widetilde{\mathcal{N}} := \begin{bmatrix} I_7 & -I_7 & 0 \\ \mathcal{A}^{(1)}_{S_1} & -\mathcal{A}^{(0)}_{S_1} & 0 \\ 0 & 0 & \mathcal{A}^{(0)}_{S_0} \end{bmatrix}_{21\times 21}.$$

It is evident that $\widetilde{\mathcal{N}}$ has the same mapping properties as $\mathcal{N}$ and that the difference $\mathcal{N} - \widetilde{\mathcal{N}}$ is a compact operator in the corresponding spaces.

As we have shown in Section 4, the upper 14×14 principal block of the matrix operator $\widetilde{\mathcal{N}}$ and the elliptic pseudodifferential operator $\mathcal{A}^{(0)}_{S_0}$ are invertible in the appropriate function spaces. Consequently, $\widetilde{\mathcal{N}}$ is an invertible operator. Then it follows that the operator $\mathcal{N}$ is Fredholm with zero index. Now let us show that the operator $\mathcal{N}$ has a trivial kernel which implies its invertibility. Indeed, let $\Psi = (\varphi, \chi, \psi)^\top$ be a solution of the homogeneous equation

$$\mathcal{N}\Psi = 0.$$

Construct the single layer potentials:

$$U^{(1)}(x) = V^{(1)}_{S_1}\big([\mathcal{H}^{(1)}_{S_1}]^{-1}\varphi\big)(x), \;\; x \in \Omega_1,$$
$$U^{(0)}(x) = V^{(0)}_{S_0}\big([\mathcal{H}^{(0)}_{S_0}]^{-1}\psi\big)(x) + V^{(0)}_{S_1}\big([\mathcal{H}^{(0)}_{S_1}]^{-1}\chi\big)(x), \;\; x \in \Omega_0.$$

It is easy to verify that the pair $(U^{(0)}, U^{(1)})$ solves the homogeneous Problem (TN) and, consequently, by the uniqueness Theorem 3.2 we conclude that

$$U^{(1)}(x) = 0, \;\; x \in \Omega_1, \quad U^{(0)}(x) = 0, \;\; x \in \Omega_0. \tag{5.3}$$

As in the proof of Theorem 4.2 one can easily show that the relations (5.3) implies $\Psi = 0$.

Now we can formulate the following existence results for Problem (TN).

Theorem 5.1.

(i) *Let*

$$S_0, S_1 \in C^{2,\gamma'}, \;\; f \in [C^{1,\beta'}(S_1)]^7, \;\; F \in [C^{0,\beta'}(S_1)]^7,$$
$$F^{(N)} \in [C^{0,\beta'}(S_0)]^7, \;\; 0 < \beta' < \gamma' \leq 1.$$

Then the problem (3.2)–(3.4), (3.6) *possesses a unique solution in the class of regular vector functions which can be represented by single layer potentials* (4.1)–(4.2), *where the triplet*

$$(\varphi, \chi, \psi)^\top \in [C^{1,\beta'}(S_1)]^7 \times [C^{1,\beta'}(S_1)]^7 \times [C^{1,\beta'}(S_0)]^7$$

is uniquely defined by the system of boundary pseudodifferential equations (5.1).

(ii) *Let*

$$S_0, S_1 \in C^\infty, \quad f \in \big[B_{p,p}^{1-\frac{1}{p}}(S_1)\big]^7, \quad F \in \big[B_{p,p}^{-\frac{1}{p}}(S_1)\big]^7,$$
$$F^{(N)} \in \big[B_{p,p}^{-\frac{1}{p}}(S_0)\big]^7, \quad p > 1.$$

Then the problem (3.2)–(3.4), (3.6) *possesses a unique solution*

$$(U^{(0)}, U^{(1)}) \in [W_p^1(\Omega_0)]^7 \times [W_p^1(\Omega_1)]^7$$

which can be represented by the single layer potentials (4.1)–(4.2), *where the triplet*

$$(\varphi, \chi, \psi) \in [B_{p,p}^{1-\frac{1}{p}}(S_1)]^7 \times [B_{p,p}^{1-\frac{1}{p}}(S_1)]^7 \times [B_{p,p}^{1-\frac{1}{p}}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (5.1).

From this theorem, as a particular case, we have the following

Corollary 5.2. *Let*

$$S_0, S_1 \in C^\infty, \quad f \in [H_2^{\frac{1}{2}}(S_1)]^7, \quad F \in [H_2^{-\frac{1}{2}}(S_1)]^7, \quad F^{(N)} \in [H_2^{-\frac{1}{2}}(S_0)]^7.$$

Then the problem (3.2)–(3.4), (3.6) *has a solution*

$$(U^{(0)}, U^{(1)}) \in [W_2^1(\Omega_0)]^7 \times [W_2^1(\Omega_1)]^7$$

which can be represented by the singlelayer potentials (4.1)–(4.2), *where the triplet*

$$(\varphi, \chi, \psi) \in [H_2^{\frac{1}{2}}(S_1)]^7 \times [H_2^{\frac{1}{2}}(S_1)]^7 \times [H_2^{\frac{1}{2}}(S_0)]^7$$

is a unique solution of the system of boundary pseudodifferential equations (5.1).

Remark 5.3. Applying again the results in the references [8], [42], and [32]) concerning the properties of the potentials on Lipschitz domains one can prove that Corollary 5.2 holds true when S_0 and S_1 are Lipschitz surfaces.

6. Interface Crack Problem (ICP)

6.1. **Formulation of the problem.** Throughout this section, let $\Omega_1 = \Omega^+$ be a bounded region in $\mathbb{R}^3$ with a simply connected boundary $S = \partial\Omega_1 \in C^\infty$ and let $\Omega_0 = \Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}_1$. As in Section 3, we assume that the domains Ω_ℓ are filled with elastic hemitropic materials having different material constants, $\alpha^{(\ell)}$, $\beta^{(\ell)}$, $\gamma^{(\ell)}$, $\delta^{(\ell)}$, $\lambda^{(\ell)}$, $\mu^{(\ell)}$, $\nu^{(\ell)}$, $\varkappa^{(\ell)}$ and $\varepsilon^{(\ell)}$, $\ell = 0, 1$. We preserve the notation employed in Section 3 for differential and integral operators. In what follows, $n(z)$ stands for the outward unit normal vector with respect to the bounded domain Ω_1 at the point $z \in S$. Further, let the interface surface S be divided into two disjoint, simply connected parts S_T (where the transmission conditions are given) and S_C (where the crack conditions are given): $S = \overline{S}_T \cup \overline{S}_C$. We assume that $\partial S_T = \partial S_C$ is a simple, C^∞-smooth curve. We identify S_C as an interface crack surface with smooth boundary ∂S_C.

We will study the following interface crack type mixed transmission Problem (ICP):

Find vector-functions

$$U^{(1)} \in [W_p^1(\Omega_1)]^7, \;\; U^{(0)} \in [W_{p,loc}^1(\Omega_0)]^7, \;\; 1 < p < \infty,$$

satisfying the differential equations,

$$L^{(\ell)}(\partial, \sigma) U^{(\ell)} = 0 \;\text{ in }\; \Omega_\ell, \;\; \ell = 0, 1, \tag{6.1}$$

the transmission conditions on S_T,

$$\{U^{(1)}\}^+ - \{U^{(0)}\}^- = \widetilde{f}, \tag{6.2}$$

$$\big\{\mathcal{P}^{(1)}(\partial, n) U^{(1)}\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial, n) U^{(0)}\big\}^- = \widetilde{F} \;\text{ on }\; S_T, \tag{6.3}$$

and the interface crack conditions on S_C,

$$\big\{\mathcal{P}^{(1)}(\partial, n) U^{(1)}\big\}^+ = F^{(1)}, \;\; \big\{\mathcal{P}^{(0)}(\partial, n) U^{(0)}\big\}^- = F^{(0)} \;\text{ on }\; S_C. \tag{6.4}$$

Moreover, we assume that $U^{(0)}$ is bounded at infinity, whence in view of (3.1) it follows that actually $U^{(0)}$ decays exponentially at infinity and $U^{(0)} \in [W_p^1(\Omega_0)]^7 \cap [C^\infty(\Omega_0)]^7$ (for details see [44]).

In our analysis we replace the conditions (6.4) by the equivalent ones:

$$\big\{\mathcal{P}^{(1)}(\partial, n) U^{(1)}\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial, n) U^{(0)}\big\}^- = F^{(1)} - F^{(0)} \;\text{ on }\; S_C, \tag{6.5}$$

$$\big\{\mathcal{P}^{(1)}(\partial, n) U^{(1)}\big\}^+ + \big\{\mathcal{P}^{(0)}(\partial, n) U^{(0)}\big\}^- = F^{(1)} + F^{(0)} \;\text{ on }\; S_C. \tag{6.6}$$

The boundary data involved in the above formulation belong to the natural spaces:

$$\widetilde{f} \in \big[B_{p,p}^{1-\frac{1}{p}}(S_T)\big]^7, \;\; \widetilde{F} \in [B_{p,p}^{-\frac{1}{p}}(S_T)]^7, \;\; F^{(1)}, F^{(0)} \in [B_{p,p}^{-\frac{1}{p}}(S_C)]^7. \tag{6.7}$$

Denote

$$F := \begin{cases} \widetilde{F} & \text{on } S_T, \\ F^{(1)} - F^{(0)} & \text{on } S_C. \end{cases} \tag{6.8}$$

Clearly, F represents the difference of generalized traces of the stress vectors,

$$F = \big\{\mathcal{P}^{(1)}(\partial,n)U^{(1)}\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial,n)U^{(0)}\big\}^- \;\text{ on }\; S.$$

Therefore the imbedding

$$F \in [B_{p,p}^{-\frac{1}{p}}(S)]^7 \tag{6.9}$$

is the necessary condition for the interface crack problem (ICP) to be solvable in the space $[W_p^1(\Omega_0)]^7 \times [W_p^1(\Omega_1)]^7$.

Now we reformulate the problem (ICP) (6.1)–(6.7) in the following form:

Find vector-functions $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$, $\ell = 0,1$, $1 < p < \infty$, satisfying the conditions

$$L^{(\ell)}(\partial,\sigma)U^{(\ell)} = 0 \;\text{ in }\; \Omega_\ell,\;\; \ell = 0,1, \tag{6.10}$$

$$\{U^{(1)}\}^+ - \{U^{(0)}\}^- = \widetilde{f} \;\text{ on }\; S_T, \tag{6.11}$$

$$\big\{\mathcal{P}^{(1)}(\partial,n)U^{(1)}\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial,n)U^{(0)}\big\}^- = F \;\text{ on }\; S, \tag{6.12}$$

$$\big\{\mathcal{P}^{(1)}(\partial,n)U^{(1)}\big\}^+ + \big\{\mathcal{P}^{(0)}(\partial,n)U^{(0)}\big\}^- = F^{(1)} + F^{(0)} \;\text{ on }\; S_C. \tag{6.13}$$

One can easily prove the following particular uniqueness result using Green's identities for domains Ω_1 and Ω_0 (see the proof of Theorem 3.1).

Theorem 6.1. *The interface crack problem* (6.10)–(6.13) *with* $p = 2$ *may have at most onesolution.*

6.2. **Auxiliary problem.** Let us consider the following basic transmission problem (BTP):

Find vector-functions $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$, $\ell = 0,1$, $1 < p < \infty$, satisfying the conditions $U^{(\ell)} \in [W_p^1(\Omega_\ell)]^7$:

$$L^{(\ell)}(\partial,\sigma)U^{(\ell)} = 0 \;\text{ in }\; \Omega_\ell,\;\; \ell = 0,1, \tag{6.14}$$

$$\{U^{(1)}\}^+ - \{U^{(0)}\}^- = f \;\text{ on }\; S, \tag{6.15}$$

$$\big\{\mathcal{P}^{(1)}(\partial,n)U^{(1)}\big\}^+ - \big\{\mathcal{P}^{(0)}(\partial,n)U^{(0)}\big\}^- = F \;\text{ on }\; S, \tag{6.16}$$

where

$$f \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^7, \quad F \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^7, \;\; 1 < p < \infty. \tag{6.17}$$

Using Green's formulas it can easily be shown that this problem possesses at most one solution for $p = 2$.

Let us look for a solution pair $(U^{(1)}, U^{(2)})$ in the form of single layer potentials:

$$U^{(\ell)}(x) = V^{(\ell)}\big([\mathcal{H}^{(\ell)}]^{-1}g^{(\ell)}\big)(x),\;\; \ell = 0,1, \tag{6.18}$$

where $V^{(\ell)} = V_S^{(\ell)}$ and $g^{(\ell)} \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^7$ are unknown densities.

The transmission conditions (6.15)–(6.16) lead then to the relations

$$g^{(1)} - g^{(0)} = f \;\text{ on }\; S, \tag{6.19}$$

$$\mathcal{A}^{(1)}g^{(1)} - \mathcal{A}^{(0)}g^{(0)} = F \;\text{ on }\; S, \tag{6.20}$$

where $\mathcal{A}^{(\ell)}$, $\ell = 0, 1$, are the above introduced Steklov–Poincaré operators (see (4.4)):

$$\mathcal{A}^{(1)} = (-2^{-1}I_7 + \mathcal{K}^{(1)})[\mathcal{H}^{(1)}]^{-1}, \quad \mathcal{A}^{(0)} = (2^{-1}I_7 + \mathcal{K}^{(0)})[\mathcal{H}^{(0)}]^{-1}.$$

From (6.19)–(6.20) we get

$$g^{(1)} = f - g^{(0)} \text{ on } S, \tag{6.21}$$

$$\big(\mathcal{A}^{(1)} - \mathcal{A}^{(0)}\big)g^{(0)} = F - \mathcal{A}^{(1)}f \text{ on } S. \tag{6.22}$$

As we have shown in the proof of Lemma 4.1 (Step 2) the operator

$$\mathcal{A}^{(1)} - \mathcal{A}^{(0)} : [B_{p,p}^{-\frac{1}{p}}(S)]^7 \longrightarrow [B_{p,p}^{-\frac{1}{p}}(S)]^7$$

is invertible. Therefore we have from (6.22)

$$g^{(0)} = [\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(1)}f). \tag{6.23}$$

By (6.21) we then get

$$g^{(1)} = [\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}F - [\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}f. \tag{6.24}$$

Substituting (6.23) and (6.24) into (6.18) finally we get the following representation of the solution to the (BTP)

$$U^{(1)} = V^{(1)}\Big([\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(0)}f)\Big) \text{ in } \Omega_1, \tag{6.25}$$

$$U^{(0)} = V^{(0)}\Big([\mathcal{H}^{(0)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(1)}f)\Big) \text{ in } \Omega_0. \tag{6.26}$$

Theorem 6.2. *Let* $1 < p < \infty$ *and conditions* (6.17) *be satisfied. Then the basic transmission problem* (6.14)–(6.17) *is uniquely solvable in the space* $[W_p^1(\Omega_1)]^7 \times [W_p^1(\Omega_0)]^7$ *and the solution can be represented by formulas* (6.25)–(6.26).

Proof. It is word for word of the proof of Theorem 4.4. □

6.3. **Existence and regularity of solutions to the (ICP).** Let us now consider the (ICP) (6.10)–(6.13). Denote by f a fixed extension of the vector $\widetilde{f}$ from S_T onto the whole of S, preserving the space. Any extension of the same vector can be then represented as a sum $f + \varphi$ with $\varphi \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^7$.

We look for a solution pair $(U^{(1)}, U^{(0)})$ to the (ICP) (6.10)–(6.13) in the form

$$U^{(1)} = V^{(1)}\Big([\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(0)}(f + \varphi))\Big) \text{ in } \Omega_1, \tag{6.27}$$

$$U^{(0)} = V^{(0)}\Big([\mathcal{H}^{(0)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}(F - \mathcal{A}^{(1)}(f + \varphi))\Big) \text{ in } \Omega_0, \tag{6.28}$$

where F is a known vector-function given by (6.8), f is the fixed extension of the vector $\widetilde{f}$ and $\varphi \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^7$ is unknown.

One can easily verify that the differential equations (6.10) and the transmission conditions (6.11) and (6.12) are automatically satisfied, while the boundary condition (6.13) on the crack surface S_C leads to the pseudodifferential equation on S_C for the unknown vector-function φ:

$$r_{S_C}\Big\{\mathcal{A}^{(1)}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}+\mathcal{A}^{(0)}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(1)}\Big\}\,\varphi=\Phi \ \text{ on } \ S_C, \quad (6.29)$$

where

$$\Phi := F^{(1)}-F^{(0)}-r_{S_C}(-2^{-1}I_7+\mathcal{K}^{(1)})[\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}(F-\mathcal{A}^{(0)}f)- \\ - r_{S_C}(2^{-1}I_7+\mathcal{K}^{(0)})[\mathcal{H}^{(0)}]^{-1}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}(F-\mathcal{A}^{(1)}f).$$

Clearly,

$$\Phi \in [B_{p,p}^{-\frac{1}{p}}(S_C)]^7.$$

Denote the principal homogeneous symbol matrices of the pseudodifferential operators $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(0)}$ by $\mathfrak{S}_1=\mathfrak{S}_1(x,\xi_1,\xi_2)$ and $\mathfrak{S}_0=\mathfrak{S}_0(x,\xi_1,\xi_2)$ respectively with $x\in\overline{S}_C$ and $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

Note that, since the principal homogeneous parts of the differential operators $L^{(\ell)}(\partial,\sigma)$ are formally selfadjoint, from (4.19) one can conclude that the principal homogeneous symbol matrices $\mathfrak{S}_1$ and $-\mathfrak{S}_0$ of the operators $\mathcal{A}^{(1)}_{S_1}$ and $-\mathcal{A}^{(0)}_{S_1}$ are positive definite for all $x\in\overline{S}_C$ and $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

For the principal homogeneous symbol matrix of the operator

$$\boldsymbol{K} := -\mathcal{A}^{(1)}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}-\mathcal{A}^{(0)}[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(1)} \qquad (6.30)$$

we have

$$\mathfrak{S}_{\boldsymbol{K}} = -\mathfrak{S}_1(\mathfrak{S}_1-\mathfrak{S}_0)^{-1}\mathfrak{S}_0-\mathfrak{S}_0(\mathfrak{S}_1-\mathfrak{S}_0)^{-1}\mathfrak{S}_1 = 2(\mathfrak{S}_1^{-1}-\mathfrak{S}_0^{-1})^{-1}. \quad (6.31)$$

Whence it follows that $\mathfrak{S}_{\boldsymbol{K}}=\mathfrak{S}_{\boldsymbol{K}}(x,\xi_1,\xi_2)$ is positive definite for all $x\in\overline{S}_C$ and $(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$.

Rewrite equation (6.29) in the form

$$r_{S_C}(\boldsymbol{K}\varphi) = -\Phi \ \text{ on } \ S_C,$$

Due to the results in [52] (see also Appendix C in [44]), since $\boldsymbol{K}$ is an elliptic pseudo differential operator of order $+1$ with positive definite principal homogeneous symbol, we conclude that the operator

$$r_{S_C}\boldsymbol{K} : [\widetilde{B}^s_{p,t}(S_C)]^7 \longrightarrow [B^{s-1}_{p,t}(S_C)]^7 \qquad (6.32)$$

is Fredholm with zero index for arbitrary $t\in[1,\infty]$, if

$$\frac{1}{p}-1 < s-\frac{1}{2} < \frac{1}{p}\,. \qquad (6.33)$$

In particular, for $s=1-\frac{1}{p}$ and $t=p$ we get that the operator

$$r_{S_C}\boldsymbol{K} : \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_C)\big]^7 \longrightarrow \big[B^{-\frac{1}{p}}_{p,p}(S_C)\big]^7 \qquad (6.34)$$

is Fredholm, if

$$\frac{4}{3} < p < 4. \qquad (6.35)$$

Moreover, the null space of the operator (6.32) does not depend on t, p and s if (6.33) holds (see, e.g., [5, Theorem 3.5]).

Now we show that the null space of the operator (6.34) with $p = 2$,

$$r_{S_C}\boldsymbol{K} : \big[\widetilde{H}_2^{\frac{1}{2}}(S_C)\big]^7 \longrightarrow [H_2^{-\frac{1}{2}}(S_C)]^7 \tag{6.36}$$

is trivial.

Let $\psi \in [H_2^{\frac{1}{2}}(S_C)]^7$ be a solution of the homogeneous equation

$$r_{S_C}\boldsymbol{K}\psi = 0 \ \text{ on } \ S_C \tag{6.37}$$

and construct the vectors

$$\widetilde{U}^{(1)}(x) = -V^{(1)}\Big([\mathcal{H}^{(1)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}\psi\Big) \ \text{ in } \ \Omega_1,$$

$$\widetilde{U}^{(0)}(x) = -V^{(0)}\Big([\mathcal{H}^{(0)}]^{-1}[\mathcal{A}^{(1)} - \mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(1)}\psi\Big) \ \text{ in } \ \Omega_0.$$

It is easy to check that the pair $(\widetilde{U}^{(1)}, \widetilde{U}^{(0)})$ solve the homogeneous problem (ICP) (6.10)–(6.13). Due to the uniqueness Theorem 6.1 it follows that

$$\widetilde{U}^{(1)} = 0 \ \text{ in } \ \Omega_0 \ \text{ and } \ \widetilde{U}^{(1)} = 0 \ \text{ in } \ \Omega_1.$$

Whence

$$0=\{\widetilde{U}^{(1)}\}^{+}-\{\widetilde{U}^{(0)}\}^{-}=-[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(0)}\psi+[\mathcal{A}^{(1)}-\mathcal{A}^{(0)}]^{-1}\mathcal{A}^{(1)}\psi=\psi.$$

Thus, equation (6.37) possesses only the zero solution and consequently the null space of the operator (6.36) is trivial. Therefore it follows that the operator (6.32) with s and p satisfying the condition (6.33) is invertible.

The same holds true for the operator (6.34) with p satisfying the inequalities (6.35). The above results lead to the following existence and regularity theorems.

Theorem 6.3. *Let* $4/3 < p < 4$,

$$\widetilde{f} \in \big[B_{p,p}^{1-\frac{1}{p}}(S_T)\big]^7, \quad \widetilde{F} \in \big[B_{p,p}^{-\frac{1}{p}}(S_T)\big]^7, \quad F^{(0)}, F^{(1)} \in \big[B_{p,p}^{-\frac{1}{p}}(S_C)\big]^7,$$

and for F given by (6.8) *the inclusion* (6.9) *hold. Then the interface crack problem* (ICP) *possesses a unique solution pair*

$$(U^{(0)}, U^{(1)}) \in [W_p^1(\Omega_0)]^7 \times [W_p^1(\Omega_1)]^7,$$

which is representable in the form (6.27)–(6.28)*, where the unknown vector φ is a unique solution to the pseudodifferential equation* (6.29).

Proof. It is quite similar to the proof of Theorem 4.4. The existence of solution follows from the mapping properties of the layer potentials described in Theorems A.1–A.4 (see Appendix A), while the uniqueness of solution is a consequence of the invertibility of the operator (6.34) with p satisfying the inequality (6.35). □

Theorem 6.4. *Let*

$$1 < t < \infty, \ \ 1 \le r \le \infty, \ \ \frac{4}{3} < p < 4, \ \ \frac{1}{t} - \frac{1}{2} < s < \frac{1}{t} + \frac{1}{2}, \tag{6.38}$$

and let a pair $(U^{(0)}, U^{(1)}) \in [W_p^1(\Omega_0)]^7 \times [W_{p,loc}^1(\Omega_1)]^7$ *be a solution to Problem* (ICP).

(i) *If* $\widetilde{f} \in [B^s_{t,t}(S_T)]^7$, $\widetilde{F} \in [B^{s-1}_{t,t}(S_T)]^7$, $F^{(0)}, F^{(1)} \in [B^{s-1}_{t,t}(S_C)]^7$, *and* $F \in [B^{s-1}_{t,t}(S)]^7$, *where* F *is defined by* (6.8), *then*

$$(U^{(0)}, U^{(1)}) \in [H_t^{s+\frac{1}{t}}(\Omega_0)]^7 \times [H_t^{s+\frac{1}{t}}(\Omega_1)]^7;$$

(ii) *If* $\widetilde{f} \in [B^s_{t,r}(S_T)]^7$, $\widetilde{F} \in [B^{s-1}_{t,r}(S_T)]^7$, $F^{(0)}, F^{(1)} \in [B^{s-1}_{t,r}(S_C)]^7$, *and* $F \in [B^{s-1}_{t,r}(S)]^6$, *where* F *is defined by* (6.8),*then*

$$(U^{(0)}, U^{(1)}) \in [B_{t,r}^{s+\frac{1}{t}}(\Omega_0)]^7 \times [B_{t,r}^{s+\frac{1}{t}}(\Omega_1)]^7; \tag{6.39}$$

(iii) *If*

$$\begin{gathered} \widetilde{f} \in [C^{\beta'}(S_T)]^7, \;\; \widetilde{F} \in \big[B^{\beta'-1}_{\infty,\infty}(S_T)\big]^7, \;\; F \in \big[B^{\beta'-1}_{\infty,\infty}(S)\big]^7, \\ F^{(0)}, F^{(1)} \in \big[B^{\beta'-1}_{\infty,\infty}(S_C)\big]^7, \;\; \beta' > 0, \end{gathered} \tag{6.40}$$

where F *is defined by* (6.8), *then*

$$U^{(\ell)} \in \bigcap_{\sigma'<\nu'} \big[C^{\sigma'}(\overline{\Omega}_\ell)\big]^7, \;\; \ell = 0, 1,$$

where $\nu' = \min\{\beta', 1/2\}$.

Proof. Under the restrictions on the parameters r, t and s stated in the theorem we see that the operator (6.32) is invertible. Therefore the items (i) and (ii) immediately follow from the mapping properties of the single layer potentials and the boundary operators $\mathcal{A}^{(1)} - \mathcal{A}^{(0)}$ and $\mathcal{H}^{(\ell)}$, $\mathcal{A}^{(\ell)}$, $\ell = 0, 1$.

To prove (iii) we use the following embeddings (see, e.g., [54], [55])

$$B^{\alpha'}_{\infty,\infty}(\mathcal{S}) \subset B^{\alpha'-\varepsilon'}_{\infty,1}(\mathcal{S}) \subset B^{\alpha'-\varepsilon'}_{\infty,r}(\mathcal{S}) \subset B^{\alpha'-\varepsilon'}_{t,r}(\mathcal{S}), \tag{6.41}$$

$$\begin{aligned} C^{\beta'}(\mathcal{S}) = B^{\beta'}_{\infty,\infty}(\mathcal{S}) \subset B^{\beta'-\varepsilon'}_{\infty,1}(\mathcal{S}) \subset B^{\beta'-\varepsilon'}_{\infty,r}(\mathcal{S}) \subset \\ \subset B^{\beta'-\varepsilon'}_{t,r}(\mathcal{S}) \subset C^{\beta'-\varepsilon'-\frac{k}{t}}(\mathcal{S}), \end{aligned} \tag{6.42}$$

where $\alpha' \in \mathbb{R}$, ε' is an arbitrary small positive number, $\mathcal{S} \subset \mathbb{R}^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with smooth boundary, $1 \le r \le \infty$, $1 < t < \infty$, $\beta' - \varepsilon' - k/t > 0$, β' and $\beta' - \varepsilon' - k/t$ are not integers. From (6.40) and the embeddings (6.41) the condition (6.39) follows with any $s \le \beta' - \varepsilon'$.

Bearing in mind the conditions (6.38) and taking t sufficiently large and ε' sufficiently small, we may put $s = \beta' - \varepsilon'$ if

$$\frac{1}{t} - \frac{1}{2} < \beta' - \varepsilon' < \frac{1}{t} + \frac{1}{2}\,, \tag{6.43}$$

and $s \in (1/t - 1/2, 1/t + 1/2)$ if

$$\frac{1}{t} + \frac{1}{2} < \beta' - \varepsilon'. \tag{6.44}$$

By the inclusion (6.39) the vector $U^{(\ell)}$ belongs then to $[B_{t,r}^{s+\frac{1}{t}}(\Omega_\ell)]^7$ with $s+1/t = \beta' - \varepsilon' + 1/t$ if (6.43) holds, and with $s+1/t \in (2/t-1/2, 2/t+1/2)$

if (6.44) holds. In the last case we can take $s + 1/t = 2/t + 1/2 - \varepsilon'$. Therefore, we have either $U^{(\ell)} \in [B_{t,r}^{\beta'-\varepsilon'+\frac{1}{t}}(\Omega_\ell)]^7$, or $U^{(\ell)} \in [B_{t,r}^{\frac{1}{2}+\frac{2}{t}-\varepsilon'}(\Omega_\ell)]^7$ in accordance with the inequalities (6.43) and (6.44). The last embedding in (6.42) (with $k=3$) yields that either $U^{(\ell)} \in [C^{\beta'-\varepsilon'-\frac{2}{t}}(\overline{\Omega}_\ell)]^7$, or $U^{(\ell)} \in [C^{\frac{1}{2}-\varepsilon'-\frac{1}{t}}(\overline{\Omega}_\ell)]^7$ which lead to the inclusion

$$U^{(\ell)} \in \big[C^{\nu'-\varepsilon'-\frac{2}{t}}(\overline{\Omega}_\ell)\big]^7, \quad \ell = 0, 1, \tag{6.45}$$

where $\nu' := \min\{\beta', 1/2\}$. Since t is sufficiently large and ε' is sufficiently small, the embedding (6.45) completes the proof. □

Remark 6.5. More detailed analysis based on the asymptotic expansions of solutions (see [6], [9]) shows that for sufficiently smooth boundary data (e.g., C^∞-smooth data say) the leading asymptotic terms of the solution vectors $U^{(0)}$ and $U^{(1)}$ near the interface crack edge, i.e., near the curve $\partial S_T = \partial S_C$ can be represented as a product of a "good" vector-function and a singular factor of the form $[\ln \varrho(x)]^{q_j}[\varrho(x)]^{\alpha_j + i\beta_j}$, $0 \le q_j \le m_j - 1$. Here $\varrho(x)$ is the distance from a reference point x to the curve $\partial S_T = \partial S_C$. Therefore, near the interface crack edge, the leading dominant singular terms of the corresponding generalized stress vectors $\mathcal{P}^{(\ell)}U^{(\ell)}$ are represented as a product of a "good" vector-function and the factors $[\ln \varrho(x)]^{q_j}[\varrho(x)]^{-1+\alpha_j+i\beta_j}$. Clearly when the numbers β_j are different from zero then we have the oscillating stress singularities.

The exponents $\alpha_j + i\beta_j$ are related to the eigenvalues $\lambda_j = \lambda_j(x)$, $j = \overline{1,7}$, of the matrix (see (6.30), (6.31))

$$\big[\mathfrak{S}_{\boldsymbol{K}}(x,0,+1)\big]^{-1}\mathfrak{S}_{\boldsymbol{K}}(x,0,-1)$$

for $x \in \partial S_T = \partial S_C$, and the following relations hold

$$\alpha_j = \frac{1}{2} + \frac{\arg \lambda_j}{2\pi}\,, \quad \beta_j = -\frac{\ln|\lambda_j|}{2\pi}\,, \quad j = \overline{1,7}.$$

In the above expressions the parameter m_j denotes the algebraic multiplicity of the eigenvalue λ_j.

Note that due to the positive definiteness of the matrix $\mathfrak{S}_{\boldsymbol{K}}(x,\xi_1,\xi_2)$ for all $x \in S_1$ and $(\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$ it is easy to show that all eigenvalues λ_j are positive which implies that $\alpha_j = \frac{1}{2}$, $j = \overline{1,7}$.

It is evident that when $|\lambda_j| \neq 1$, then the corresponding $\beta_j \neq 0$ and oscillating stress singularities arise near the interface crack edge. Moreover, the components of the generalized stress vector $\mathcal{P}^{(\ell)}U^{(\ell)}$ behave like $\mathcal{O}\big([\ln \varrho(x)]^{q_0-1}[\varrho(x)]^{-\frac{1}{2}}\big)$, where q_0 denotes the maximal algebraic multiplicity of the eigenvalues. This is a global singularity effect for the first order derivatives of the vectors $U^{(0)}$ and $U^{(1)}$. As we see, the stress singularity exponents for the interface crack problem in the case of hemitropic solids have the form $-\frac{1}{2} + i\beta_j$ where β_j depends on the material parameters of the constituent solids of the composite structure.

7. Appendix A

Here we collect some results concerning mapping and regularity properties of the single and double layer potentials and the boundary pseudodifferential operators generated by them in the Hölder ($C^{m,\kappa}$), Sobolev–Slobodetski (W_p^s), Bessel potential (H_p^s) and Besov ($B_{p,q}^s$) spaces. They can be found in [10], [11], [12], [13], [16], [21], [22], [32], [36], [37], [38], [40], [43], and [44].

We assume (if not otherwise stated) that $\Omega^+ \subset \mathbb{R}^3$ is a bounded domain with boundary $S = \partial\Omega^+$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}^+$,

$$\begin{gathered} S = \partial\Omega^\pm \in C^{m,\gamma'} \text{ with integer } m \geq 2 \text{ and } 0 < \gamma' \leq 1, \\ \sigma = \sigma_1 + i\sigma_2, \ \ \sigma_1 \in \mathbb{R}, \ \ \sigma_2 > 0. \end{gathered} \tag{A.1}$$

Introduce the single and double layer potentials

$$V(x) = V_S(x) := \int\limits_S \Gamma(x-y,\sigma) g(y)\, dS_y, \tag{A.2}$$

$$W(x) = W_S(x) := \int\limits_S \big[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(x-y,\sigma)\big]^\top g(y)\, dS_y, \tag{A.3}$$

where $x \in \mathbb{R}^3 \setminus S$, $\Gamma(x-y,\sigma)$ is the fundamental matrix of the operator $L(\partial,\sigma)$ which is explicitly constructed in [44]. The proofs of the following theorems can be found in [44].

Theorem A.1. *Let S, m, and γ' be as in* (A.1), $0 < \beta' < \gamma'$, *and let $k \leq m-1$ be a nonnegative integer. Then the operators*

$$\begin{aligned} V &: [C^{k,\beta'}(S)]^7 \longrightarrow \big[C^{k+1,\beta'}(\overline{\Omega^\pm})\big]^7, \\ W &: [C^{k,\beta'}(S)]^7 \longrightarrow \big[C^{k,\beta'}(\overline{\Omega^\pm})\big]^7 \end{aligned} \tag{A.4}$$

are continuous.

For any $g \in [C^{0,\beta'}(S)]^7$, $h \in [C^{1,\beta'}(S)]^7$, and for all $x \in S$

$$[V(g)(x)]^\pm = V(g)(x) = \mathcal{H}g(x), \tag{A.5}$$

$$\big[\mathcal{P}(\partial_x, n(x))V(g)(x)\big]^\pm = [\mp 2^{-1} I_7 + \mathcal{K}]g(x), \tag{A.6}$$

$$[W(g)(x)]^\pm = [\pm 2^{-1} I_7 + \mathcal{N}]g(x), \tag{A.7}$$

$$\big[\mathcal{P}(\partial_x, n(x))W(h)(x)\big]^+ = [\mathcal{P}(\partial_x, n(x))W(h)(x)]^- = \mathcal{L}h(x), \tag{A.8}$$

where

$$\mathcal{H}g(x) = \mathcal{H}_S g(x) := \int\limits_S \Gamma(x-y,\sigma) g(y)\, dS_y, \tag{A.9}$$

$$\mathcal{K}g(x) = \mathcal{K}_S g(x) := \int\limits_S \big[\mathcal{P}(\partial_x, n(x))\Gamma(x-y,\sigma)\big] g(y)\, dS_y, \tag{A.10}$$

$$\mathcal{N}g(x)=\mathcal{N}_S g(x):=\int\limits_S \big[\mathcal{P}^*(\partial_y,n(y))\Gamma^\top(x-y,\sigma)\big]^\top g(y)\,dS_y, \tag{A.11}$$

$$\mathcal{L}h(x)=\mathcal{L}_S h(x):= \tag{A.12}$$

$$:=\lim_{\Omega^\pm\ni z\to x\in S}\mathcal{P}(\partial_z,n(x))\int\limits_S\big[\mathcal{P}^*(\partial_y,n(y))\Gamma^\top(z-y,\sigma)\big]^\top h(y)\,dS_y. \tag{A.13}$$

Theorem A.2. *Let S be a Lipschitz surface. Then the operators* (A.4) *can be extended to the continuous mappings*

$$V:[H_2^{-\frac12}(S)]^7\longrightarrow[H_2^1(\Omega^\pm)]^7,\quad W:[H_2^{\frac12}(S)]^7\longrightarrow[H_2^1(\Omega^\pm)]^7.$$

The jump relations (A.5)–(A.8) *on S remain valid for the extended operators in the corresponding function spaces.*

Theorem A.3. *Let S, m, γ', β' and k be as in Theorem A.*1. *Then the operators*

$$\begin{aligned}\mathcal{H}:[C^{k,\beta'}(S)]^7&\longrightarrow[C^{k+1,\beta'}(S)]^7,\\ :[H_2^{-\frac12}(S)]^7&\longrightarrow[H_2^{\frac12}(S)]^7,\end{aligned} \tag{A.14}$$

$$\begin{aligned}\mathcal{K}:[C^{k,\beta'}(S)]^7&\longrightarrow[C^{k,\beta'}(S)]^7,\\ :\ [H_2^{-\frac12}(S)]^7&\longrightarrow[H_2^{-\frac12}(S)]^7,\end{aligned} \tag{A.15}$$

$$\begin{aligned}\mathcal{N}:[C^{k,\beta'}(S)]^7&\longrightarrow[C^{k,\beta'}(S)]^7,\\ :[H_2^{\frac12}(S)]^7&\longrightarrow[H_2^{\frac12}(S)]^7,\end{aligned} \tag{A.16}$$

$$\begin{aligned}\mathcal{L}:[C^{k,\beta'}(S)]^7&\longrightarrow[C^{k-1,\beta'}(S)]^7,\\ :[H_2^{\frac12}(S)]^7&\longrightarrow[H_2^{-\frac12}(S)]^7\end{aligned} \tag{A.17}$$

are continuous. Moreover,

(i) *the principal homogeneous symbol matrices of the singular integral operators $\pm2^{-1}I_7+\mathcal{K}$ and $\pm2^{-1}I_7+\mathcal{N}$ are non-degenerate, while the principal homogeneous symbol matrices of the pseudodifferential operators $-\mathcal{H}$ and $\mathcal{L}$ are positive definite;*

(ii) *the operators $\mathcal{H}$, $\pm2^{-1}I_7+\mathcal{K}$, $\pm2^{-1}I_7+\mathcal{N}$, and $\mathcal{L}$ are elliptic pseudodifferential operators* (*of order -1, 0, 0, and 1, respectively*) *with zero index;*

(iii) *the following equalities hold in appropriate function spaces:*

$$\mathcal{N}\mathcal{H}=\mathcal{H}\mathcal{K},\quad \mathcal{L}\mathcal{N}=\mathcal{K}\mathcal{L},$$
$$\mathcal{H}\mathcal{L}=-4^{-1}I_7+\mathcal{N}^2,\quad \mathcal{L}\mathcal{H}=-4^{-1}I_7+\mathcal{K}^2.$$

(iv) *The operators* (A.14), (A.15), (A.16), *and* (A.17) *are bounded if S is a Lipschitz surface.*

Theorem A.4. *Let V, W, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{N}$, and $\mathcal{L}$ be as in Theorems A.1 and A.3 and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The layer potential operators* (A.2), (A.3) *and the boundary integral (pseudodifferential) operators* (A.9)–(A.12) *can be extended to the following continuous operators*

$$V : [B^s_{p,p}(S)]^7 \longrightarrow [H_p^{s+1+\frac{1}{p}}(\Omega^\pm)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow \big[B_{p,q}^{s+1+\frac{1}{p}}(\Omega^\pm)\big]^7\Big),$$

$$W : [B^s_{p,p}(S)]^7 \longrightarrow [H_p^{s+\frac{1}{p}}(\Omega^\pm)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow \big[B_{p,q}^{s+\frac{1}{p}}(\Omega^\pm)\big]^7\Big),$$

$$\mathcal{H} : [H^s_p(S)]^7 \longrightarrow [H^{s+1}_p(S)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow [B^{s+1}_{p,q}(S)]^7\Big), \tag{A.18}$$

$$\mathcal{K} : [H^s_p(S)]^7 \longrightarrow [H^s_p(S)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow [B^s_{p,q}(S)]^7\Big), \tag{A.19}$$

$$\mathcal{N} : [H^s_p(S)]^7 \longrightarrow [H^s_p(S)]^7 \quad \Big([B^s_{p,q}(S)]^7 \longrightarrow [B^s_{p,q}(S)]^7\Big), \tag{A.20}$$

$$\mathcal{L} : [H^{s+1}_p(S)]^7 \longrightarrow [H^s_p(S)]^7 \quad \Big([B^{s+1}_{p,q}(S)]^7 \longrightarrow [B^s_{p,q}(S)]^7\Big). \tag{A.21}$$

The jump relations (A.5)–(A.8) *remain valid for arbitrary $g \in [B^s_{p,q}(S)]^7$ with $s \in \mathbb{R}$ if the limiting values (traces) on S are understood in the sense described in* [51].

The operators (A.18)–(A.21) *are elliptic pseudodifferential operators with zero index. The null-spaces of the operators* (A.18)–(A.21) *are invariant with respect to p, q, and s.*

Theorem A.5. *Let $S \in C^{2,\gamma'}$ and $0 < \beta' < \gamma' \le 1$. Then the operator*

$$\mathcal{H} : [C^{0,\beta'}(S)]^7 \longrightarrow [C^{1,\beta'}(S)]^7$$

is invertible.

Theorem A.6. *Let S be Lipschitz. Then the operator*

$$\mathcal{H} : [H_2^{-\frac{1}{2}}(S)]^7 \longrightarrow [H_2^{\frac{1}{2}}(S)]^7$$

is invertible.

Let us introduce the volume Newtonian potential

$$N_\Omega(\Psi)(x) := \int\limits_\Omega \Gamma(x-y,\sigma)\Psi(y)\,dx,$$

where $\Omega \subset \mathbb{R}^3$ is an arbitrary bounded domain and either $\Psi \in [L_2(\Omega)]^7$ or $\Psi \in [C^{0,\beta'}(\overline{\Omega})]^7$ with $0 < \beta' < 1$. There holds the following proposition (see, e.g., [33], [32]).

Theorem A.7. *Let $S \in C^{1,\gamma'}$ and $0 < \beta' < \gamma' \le 1$. Then operators*

$$\begin{aligned} N_\Omega &: [L_2(\Omega)]^7 \longrightarrow [W^2_2(\Omega)]^7, \\ &: [C^{0,\beta'}(\overline{\Omega})]^7 \longrightarrow [C^{2,\beta'}(\Omega)]^7 \cap [C^{1,\beta'}(\overline{\Omega})]^7, \end{aligned} \tag{A.22}$$

are bounded. The mapping property (A.22) *holds for Lipschitz domains as well. Moreover,*

$$L(\partial,\sigma)N_{\Omega}(\Psi)(x)=\Psi(x),\quad x\in\Omega,$$

for almost all x *in* Ω *if* $\Psi\in[L_2(\Omega)]^7$ *and for all* x *in* Ω *if* $\Psi\in[C^{0,\beta'}(\overline{\Omega})]^7$.

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