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ON THE WEIGHTED INITIAL PROBLEM FOR SINGULAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. For singular functional differential systems, sufficient conditions for solvability and well-posedness of the weighted initial problem are established.

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In a finite interval $]a,b[$ we consider the functional differential system

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1}$$

with the weighted initial condition

$$\limsup_{t\to a} \big\|\phi^{-1}(t)x(t)\big\| < +\infty. \tag{2}$$

Here, $f : C([a,b];\mathbb{R}^n) \to L_{loc}(]a,b];\mathbb{R}^n)$ is a singular operator satisfying the local Carathéorory conditions, $\phi(t) = \operatorname{diag}\big(\varphi_1(t),\dots,\varphi_n(t)\big)$, and $\varphi_i : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$ are continuous non-decreasing functions such that $\varphi_i(a) = 0$, $\varphi_i(t) > 0$ for $a < t \le b$ $(i=1,\dots,n)$.

The initial problem for the singular system (1) has been thoroughly investigated in the cases, in which f is either the Nemytski's operator [1]–[6], or the evolutionary operator [7]–[9]. The weighted initial problem for higher order singular functional differential equations is studied in [11]–[14]. As for the weighted singular problem (1), (2), it is not studied well enough. In the present paper unimprovable in a certain sense conditions are given which, respectively, guarantee solvability and well-posedness of this problem.

Throughout the paper, the use will be made of the following notation.

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$.

$\mathbb{R}^n$ is the space of n-dimensional real column-vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i|.$$

If $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, then

$$[x]_+ = \Big(\frac{x_i + |x_i|}{2}\Big)_{i=1}^n.$$

$r(X)$ is the spectral radius of the $n \times n$ matrix X, and X^{-1} is the inverse to X matrix.

$\mathrm{diag}(x_1, \dots, x_n)$ is the diagonal $n \times n$-matrix with diagonal elements $x_1, \dots, x_n$.

If $X = \mathrm{diag}(x_1, \dots, x_n)$, then $\mathrm{Sgn}(X) = \big(\mathrm{sgn}(x_1), \dots, \mathrm{sgn}(x_n)\big)$.

$\mathbb{R}_+^n$ and $\mathbb{R}_+^{n\times n}$ are the sets of n-dimensional vectors and $n \times n$-matrices with nonnegative components.

$C([a,b];\mathbb{R}^n)$ is the space of continuous vector functions $x : [a,b] \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\Big\{\|x(t)\| : \ a \le t \le b\Big\}.$$

$C_\phi([a,b];\mathbb{R}^n)$ is the space of continuous vector functions $x : [a,b] \to \mathbb{R}^n$, satisfying the condition (2), with the norm

$$\|x\|_{C_\phi} = \sup\Big\{\big\|\phi^{-1}(t)x(t)\big\| : \ a < t \le b\Big\}.$$

If $x = (x_i)_{i=1}^n \in C_\phi([a,b];\mathbb{R}^n)$, then

$$|x|_{C_\phi} = \big(\|x_i\|_{C_{\varphi_i}}\big)_{i=1}^n.$$

$L([a,b];\mathbb{R}^n)$ is the space of vector functions with Lebesgue integrable on $[a,b]$ components.

$L_{loc}(]a,b];\mathbb{R}^n)$ is the space of vector functions whose components are Lebesgue integrable on $[a+\varepsilon, b]$ for an arbitrarily small $\varepsilon > 0$.

$K_{loc}(]a,b] \times \mathbb{R}^k;\mathbb{R}^m)$ and $K_{loc}(C([a,b];\mathbb{R}^k); L_{loc}(]a,b];\mathbb{R}^m))$ are the sets of vector functions $g :]a,b] \times \mathbb{R}^k \to \mathbb{R}^m$ and operators $f : C([a,b];\mathbb{R}^k) \to L_{loc}(]a,b];\mathbb{R}^m)$, satisfying the local Carathéodory conditions (see [15]).

An important particular case of the functional differential system (1) is the differential system with a deviating argument

$$\frac{dx(t)}{dt} = g\big(t, x(t), x(\tau(t))\big). \tag{3}$$

Along with the problem (1), (2), we consider the problem (3), (2). Everywhere below, when the question concerns these problems, it will be assumed that

$$f \in K_{loc}\big(C([a,b];\mathbb{R}^n); L_{loc}(]a,b];\mathbb{R}^n)\big), \quad g \in K_{loc}(]a,b] \times \mathbb{R}^{2n};\mathbb{R}^n),$$

and $\tau : [a,b] \to [a,b]$ is a measurable function.

We are mainly interested in the case, where the systems (1) and (3) are singular, i.e., in the case in which

$$\int_a^b f_\rho^*(t)\,dt = +\infty \ \text{ and } \ \int_a^b g_\rho^*(t)\,dt = +\infty \ \text{ for } \ \rho > 0,$$

where

$$f^*_\rho(t) = \sup\Big\{\big\|f(x)(t)\big\|:\ \|x\|_C \le \rho\Big\},$$
$$g^*_\rho(t) = \max\Big\{\big\|g(t,x,y)\big\|:\ \|x\| + \|y\| \le \rho\Big\}.$$

For an arbitrary positive number δ, we put

$$\chi(t,\delta,\lambda) = \begin{cases} 0 & \text{for } \ a \le t < a+\delta \\ \lambda & \text{for } \ t > a+\delta \end{cases},$$

and consider the auxiliary initial problem

$$\frac{dx(t)}{dt} = \chi(t,\delta,\lambda) f(x)(t), \tag{4}$$
$$x(a) = 0, \tag{5}$$

depending on the parameters $\lambda \in]0,1]$ and $\delta > 0$.

On the basis of Corollary 2 in [16], the following theorem can be proved.

Theorem 1. *Let there exist a positive number ρ_0 such that for arbitrary $\lambda \in]0,1]$ and $\delta > 0$ every solution x of the problem (4), (5) admits the estimate*

$$\|x\|_{C_\phi} \le \rho_0.$$

Then the problem (1), (2) has at least one solution.

This theorem allows one to get efficient sufficient conditions for the solvability of the problems (1), (2) and (3), (2). In particular, the following propositions are valid.

Theorem 2. *Let there exist a matrix $\mathcal{P} \in \mathbb{R}^{n\times n}_+$ and a vector function $q : \mathbb{R}_+ \to \mathbb{R}^n_+$ such that*

$$r(\mathcal{P}) < 1, \quad \lim_{\rho\to+\infty} \frac{\|q(\rho)\|}{\rho} = 0, \tag{6}$$

and for an arbitrary vector function $x \in C_\phi([a,b];\mathbb{R}^n)$ on the interval $[a,b]$ the inequality

$$\int_a^t \Big[\operatorname{sgn}(x(s)) f(x)(s)\Big]_+ ds \le \phi(t)\Big(\mathcal{P}|x|_{C_\phi} + q\big(\|x\|_{C_\phi}\big)\Big)$$

is fulfilled. Then the problem (1), (2) has at least one solution.

Corollary 1. *Let the functions φ_i $(i = 1,\dots,n)$ be absolutely continuous and let there exist a set of zero measure $I_0 \subset [a,b]$, matrices $\mathcal{P}_k \in \mathbb{R}^{n\times n}_+$ $(k = 1,2)$ and a vector function $q : \mathbb{R}_+ \to \mathbb{R}^n_+$ with non-decreasing components such that on the set $([a,b]\setminus I_0)\times\mathbb{R}^{2n}$ the inequality*

$$\operatorname{Sgn}(x) g(t,x,y) \le \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x| + \mathcal{P}_2\phi^{-1}(\tau(t))|y|\Big) + \\ + \phi'(t) q\Big(\big\|\phi^{-1}(t)|x| + \phi^{-1}(\tau(t))|y|\,\big\|\Big)$$

is fulfilled. If, moreover, the conditions (6) *are fulfilled, where* $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, *then the problem* (3), (2) *has at least one solution.*

Remark 1. In Theorem 2 and Corollary 1, the condition $r(\mathcal{P}) < 1$ is unimprovable and it cannot be replaced by the condition $r(\mathcal{P}) \le 1$. The validity of that fact follows directly from the theorem below.

Theorem 3. *Let the functions* φ_i $(i = 1, \dots, n)$ *be absolutely continuous and let there exist a set of zero measure* $I_0 \subset [a, b]$, *matrices* $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2)$ *and a vector* $q_0 = (q_{0i})_{i=1}^n$ *with positive components* q_{0i} $(i = 1, \dots, n)$ *such that on the set* $([a, b] \setminus I_0) \times \mathbb{R}^{2n}$ *the inequality*

$$g(t, x, y) \ge \phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x| + \mathcal{P}_2\phi^{-1}(\tau(t))|y| + q_0\Big)$$

is fulfilled. If, moreover, $r(\mathcal{P}_1 + \mathcal{P}_2) \ge 1$, *then the problem* (3), (2) *has no solution.*

Along with the problem (1), (2), we consider the perturbed problem

$$\frac{dy(t)}{dt} = f(y)(t) + h(t), \tag{7}$$

$$\limsup_{t\to a} \left\|\phi^{-1}(t)y(t)\right\| < +\infty, \tag{8}$$

and introduce the following

Definition. The problem (1), (2) is called well-posed if there exists a positive number ρ such that for an arbitrary function $h \in L([a, b]; \mathbb{R}^n)$, satisfying the condition

$$\nu_\phi(h) = \sup\left\{\left\|\phi^{-1}(t)\int_a^t |h(s)|\,ds\right\| :\ a < t \le b\right\} < +\infty,$$

the problem (7), (8) is uniquely solvable and its solution admits the estimate

$$\|y - x\|_{C_\phi} \le \rho\nu_\phi(h),$$

where x is a solution of the problem (1), (2).

Theorem 4. *Let there exist a matrix* $\mathcal{P} \in \mathbb{R}_+^{n\times n}$ *such that* $r(\mathcal{P}) < 1$, *and for arbitrary vector functions* x *and* $y \in C_\phi([a, b]; \mathbb{R}^n)$ *in the interval* $[a, b]$ *the inequality*

$$\int_a^t \Big[\operatorname{sgn}(y(s))\big(f(x + y)(s) - f(x)(s)\big)\Big]_+ ds \le \phi(t)\mathcal{P}|y|_{C_\phi}$$

is fulfilled. If, moreover,

$$\sup\left\{\left\|\phi^{-1}(t)\int_a^t \big|f(0)(s)\big|\,ds\right\| :\ a < t \le b\right\} < +\infty,$$

then the problem (1), (2) *is well-posed.*

Corollary 2. *Let the functions* φ_i $(i=1,\dots,n)$ *be absolutely continuous and let there exist a set of zero measure* $I_0 \subset [a,b]$ *and matrices* $\mathcal{P}_k \in \mathbb{R}_+^{n\times n}$ $(k=1,2)$ *such that* $r(\mathcal{P}_1+\mathcal{P}_2)<1$, *and for any* $t\in[a,b]\setminus I_0$, x, $\overline{x}$, y *and* $\overline{y}\in\mathbb{R}^n$ *the inequality*

$$\operatorname{sgn}(\overline{x})\Big(g\big(t,x+\overline{x},y+\overline{y}\big)-g(t,x,y)\Big)\le\phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|\overline{x}|+\mathcal{P}_2\phi^{-1}(\tau(t))|\overline{y}|\Big)$$

is fulfilled. If, moreover,

$$\sup\left\{\left\|\phi^{-1}(t)\int\limits_a^t\big|g(s,0,0)\big|\,ds\right\|:\ a<t\le b\right\}<+\infty,$$

then the problem (3), (2) *is well-posed.*

From Theorem 3 and Corollary 2 it follows

Corollary 3. *Let the functions* φ_i $(i=1,\dots,n)$ *be absolutely continuous and*

$$g(t,x,y)=\phi'(t)\Big(\mathcal{P}_1\phi^{-1}(t)|x|+\mathcal{P}_2\phi^{-1}(\tau(t))|y|+q_0\Big),$$

where $\mathcal{P}_k\in\mathbb{R}_+^{n\times n}$ $(k=1,2)$, *and* $q_0\in\mathbb{R}_+^n$ *is the vector with positive components. Then the problem* (3), (2) *is well-posed if and only if*

$$r(\mathcal{P}_1+\mathcal{P}_2)<1.$$

Remark 2. According to Corollary 3, the inequality $r(\mathcal{P})<1$ $(r(\mathcal{P}_1+\mathcal{P}_2)<1)$ in Theorem 4 (in Corollary 2) is unimprovable and it cannot be replaced by the inequality $r(\mathcal{P})\le 1$ $(r(\mathcal{P}_1+\mathcal{P}_2)\le 1)$.

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References

1. V. A. Chechik, Investigation of systems of ordinary differential equations with a singularity. (Russian) *Tr. Mosk. Mat. Obs.*, **8** (1959), *GIFML, Moscow*, 155–198.
2. I. T. Kiguradze, On the Cauchy problem for singular systems of ordinary differential equations. (Russian) *Differentsial'nye Uravneniya* **1** (1965), No. 10, 1271–1291; English transl.: *Differ. Equations* **1** (1965), 995–1011.
3. I. T. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Izdat. Tbilis. Univ., Tbilisi*, 1975.
4. I. T. Kiguradze, On the singular Cauchy problem for systems of linear ordinary differential equations. (Russian) *Differentsial'nye Uravneniya* 32 (1996), No. 2, 215–223; English transl.: *Differ. Equations* **32** (1996), No. 2, 173–180.
5. I. Kiguradze, Initial and boundary value problems for systems of ordinary differential equations, I. (Russian) *Metsniereba, Tbilisi*, 1997.
6. T. I. Kiguradze, Estimates for the Cauchy function of linear singular differential equations and some applications. *Differ. Uravn.* **46** (2010), No. 1, 29–46; Engish transl.: *Differ. Equ.* **46** (2010), No. 1, 30–47.

7. I. T. Kiguradze and Z. P. Sokhadze, On the Cauchy problem for evolution singular functional-differential equations. (Russian) *Differentsial'nye Uravneniya* **33** (1997), No. 1, 48–59, 142; English transl.: *Differential Equations* **33** (1997), No. 1, 47–58.
8. I. Kiguradze and Z. Sokhadze, On the global solvability of the Cauchy problem for singular functional differential equations. *Georgian Math. J.* **4** (1997), No. 4, 355–372.
9. I. Kiguradze and Z. Sokhadze, On the structure of the set of solutions of the weighted Cauchy problem for evolution singular functional-differential equations. *Fasc. Math.* No. 28 (1998), 71–92.
10. Z. Sokhadze, The Cauchy problem for singular functional-differential equations. (Russian) *Kutaisskii Gosudarstvennyi Universitet, Kutaisi*, 2005.
11. Z. Sokhadze, On the solvability of the weighted initial value problem for high order evolution singular functional differential equations. *Mem. Differential Equations Math. Phys.* **15** (1998), 145–149.
12. Z. Sokhadze, The weighted Cauchy problem for linear functional differential equations with strong singularities, *Georgian Math. J.* **18** (2011), No. 3, 577–586.
13. B. Půža and Z. P. Sokhadze, The weighted Cauchy problem for nonlinear singular differential equations with deviating arguments. (Russian) *Differ. Uravn.* **48** (2012).
14. Z. Sokhadze, Kneser type theorems on a structure of sets of solutions of the weighted Cauchy problem for nonlinear singular delayed differential equations. *Georgian Math. J.* **19** (2012), No. 4.
15. B. Půža and Z. Sokhadze, Optimal solvability conditions of the Cauchy–Nicoletti problem for singular functional differential systems. *Mem. Differential Equations Math. Phys.* **54** (2011), 147–154.
16. I. Kiguradze and B. Půža On boundary value problems for functional differential equations. *Mem. Differential Equations Math. Phys.* **12** (1997), 106–113.

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