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POSITIVE SOLUTIONS OF NONLOCAL PROBLEMS FOR NONLINEAR SINGULAR DIFFERENTIAL SYSTEMS

Abstract. For nonlinear differential systems with singularities with respect to phase variables, sufficient conditions for the existence of positive solutions of nonlocal problems are established.

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Let $-\infty < a < b < +\infty$, $\mathbb{R}^n_+$ be the set of n-dimensional real vectors $(x_i)_{i=1}^n$ with nonnegative components $x_1, \dots, x_n$,

$$\mathbb{R}^n_{0+} = \big\{(x_i)_{i=1}^n : \; x_1 > 0, \dots, x_n > 0\big\},$$

and let $C([a,b];\mathbb{R}^n_+)$ be the set of continuous vector functions $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}^n_+$. Consider the nonlocal problem

$$\frac{du_i}{dt} = f_i(t, u_1, \dots, u_n) \;\; (i = 1, \dots, n), \tag{1}$$

$$u_i(t_i) = \varphi_i(u_1, \dots, u_n) \;\; (i = 1, \dots, n), \tag{2}$$

where $f_i :]a,b[\, \times \mathbb{R}^n_{0+} \to \mathbb{R}$ are functions satisfying the local Carathéodory conditions, $a \le t_i \le b$ $(i = 1, \dots, n)$, and $\varphi_k : C([a,b];\mathbb{R}^n_+) \to \mathbb{R}_+$ $(k = 1, \dots, n)$ are continuous and bounded on every bounded subset of $C([a,b];\mathbb{R}^n_+)$ functionals.

In the case where the functions f_i $(i = 1, \dots, n)$ have no singularities with respect to phase variables, boundary value problems of the type (1), (2) have been studied in [1]–[4].

The present paper deals with the case not investigated yet, when f_i $(i = 1, \dots, n)$ have singularities with respect to the phase variables, that is the case, where

$$\lim_{x_k \to 0} \big|f_i(t, x_1, \dots, x_n)\big| = +\infty \;\; (i, k = 1, \dots, n).$$

Throughout the paper, along with the above-introduced we will use the following notations.

$(x_{ik})_{i,k=1}^n$ is the matrix with components x_{ik} $(i, k = 1, \dots, n)$.

$r(X)$ is the spectral radius of the $n \times n$ matrix X.

If $u : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\|u\|_C = \max\big\{\|u(t)\| : \ a \le t \le b\big\}.$$

If $\delta_k : [a,b] \to [0,+\infty[$ $(k = 1,\dots,n)$ are continuous functions satisfying the conditions

$$\delta_k(t) > 0 \ \text{ for almost all } \ t \in [a,b] \ \ (k = 1,\dots,n),$$

and $\rho > 0$, then

$$f^*(\delta_1,\dots,\delta_n,\rho)(t) = \sup\bigg\{ \sum_{i=1}^n \big|f_i(t,x_1,\dots,x_n)\big| : \\ \delta_1(t) < x_1 < \delta_1(t) + \rho, \dots, \delta_n(t) < x_n < \delta_n(t) + \rho \bigg\}.$$

Along with (1), (2), we consider the auxiliary problem

$$\frac{du_i}{dt} = \lambda f_i(t,u_1,\dots,u_n) + (1-\lambda)\delta_i(t) \ \ (i = 1,\dots,n), \tag{3}$$

$$u_i(t_i) = \lambda\varphi_i(u_1,\dots,u_n) \ \ (i = 1,\dots,n), \tag{4}$$

$$u_i(t) \ge \delta_i(t) \ \text{ for } \ a \le t \le b, \tag{5}$$

depending on the parameter $\lambda \in]0,1]$ and on absolutely continuous functions $\delta_i : [a,b] \to [0,+\infty[$ $(i = 1,\dots,n)$.

An absolutely continuous vector function $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}_+^n$ is said to be a positive solution of the system (1) (of the system (3)) if it almost everywhere on $[a,b]$ satisfies this system and

$$u_i(t) > 0 \ \text{ for almost all } \ t \in [a,b] \ \ (i = 1,\dots,n).$$

A positive solution $(u_i)_{i=1}^n$ of the system (1) (of the system (3)), satisfying the conditions (2) (the conditions (4) and (5)), is called a positive solution of the problem (1), (2) (a solution of the problem (3), (4), (5)).

The following theorem is valid.

Theorem 1 (The Principle of a Priori Boundedness)**.** *Let for any $i \in \{1,\dots,n\}$ on the set*

$$\Big\{(t,x_1,\dots,x_n) : \ t \in [a,b] \setminus I_0, \ x_k > \delta_k(t) \ \text{for}\ k \ne i, \ x_i = \delta_i(t)\Big\}$$

the inequality

$$\big[f_i(t,x_1,\dots,x_n) - \delta_i'(t)\big] \operatorname{sgn}(t - t_i) \ge 0$$

hold, where I_0 is a set of zero measure, and $\delta_k : [a,b] \to [0,+\infty[$ $(k = 1,\dots,n)$ are absolutely continuous functions such that

$$\delta_i(t) > 0 \ \text{ for } \ t \in [a,b] \setminus I_0 \ \ (i = 1,\dots,n),$$

$$\varphi_i(u_1,\dots,u_n) \ge \delta_i(t_i) \ \text{ for } \ (u_k)_{k=1}^n \in C([a,b];\mathbb{R}_+^n) \ \ (i = 1,\dots,n).$$

Let, moreover,

$$\int_a^b f^*(\delta_1,\dots,\delta_n;\rho)(t)\,dt < +\infty \ \ for \ \ \rho > 0$$

and there exist a positive constant ρ_0 such that for any $\lambda \in]0,1]$ every solution of the problem (3), (4), (5) *admits the estimate*

$$\sum_{i=1}^n \|u_i\|_C \le \rho_0.$$

Then the problem (1), (2) *has at least one positive solution.*

The operator $(\varphi_{0i})_{i=1}^n : C([a,b];\mathbb{R}_+^n) \to \mathbb{R}_+^n$ is said to be positively homogeneous if for any $i \in \{1,\dots,n\}$, $\lambda > 0$ and $(u_k)_{k=1}^n \in C([a,b];\mathbb{R}_+^n)$ the equality

$$\varphi_{0i}(\lambda u_1,\dots,\lambda u_n) = \lambda\varphi_{0i}(u_1,\dots,u_n)$$

is satisfied.

Following [1], we introduce

Definition 1. We say that the pair $((p_{ik})_{i,k=1}^n;(\varphi_{0i})_{i=1}^n)$, consisting of the matrix function $(p_{ik})_{i,k=1}^n$ with the Lebesgue integrable components $p_{ik} : [a,b] \to \mathbb{R}_+$ $(i,k=1,\dots,n)$ and the positively homogeneous nondecreasing operator $(\varphi_{0i})_{i=1}^n : C([a,b];\mathbb{R}_+^n) \to \mathbb{R}_+^n$ belongs to the set $\mathcal{U}(t_1,\dots,t_n)$ if the problem

$$u_i'(t)\operatorname{sgn}(t-t_i) \le \sum_{k=1}^n p_{ik}(t)u_k(t) \ \ (i=1,\dots,n),$$
$$u_i(t_i) \le \varphi_{0i}(u_1,\dots,u_n) \ \ (i=1,\dots,n)$$

has no a nonzero, nonnegative solution.

On the basis of Theorem 1, the following theorem can be proved.

Theorem 2. *Let*

$$\varphi_i(u_1,\dots,u_n) \le \varphi_{0i}\big(u_1,\dots,u_n\big) + \gamma \ \ for \ \ (u_k)_{k=1}^n \in C([a,b];\mathbb{R}_+^n) \quad (i=1,\dots,n)$$

and

$$0 \le \big(f_i(t,x_1,\dots,x_n) - p_i(t)x_i^{\lambda_i}\big)\operatorname{sgn}(t-t_i) \le \\ \le \sum_{k=1}^n p_{ik}(t)x_k \ \ for \ \ t \in [a,b]\setminus I_0, \ \ (x_k)_{k=1}^n \in \mathbb{R}_{0+}^n \ \ (i=1,\dots,n), \quad (6)$$

where I_0 is a set of zero measure, γ is a nonnegative constant, $\lambda_i < 1$ $(i=1,\dots,n)$, $p_i : [a,b] \to \mathbb{R}_{0+}$ $(i=1,\dots,n)$ are the Lebesgue integrable functions and

$$\big((p_{ik})_{i,k=1}^n;(\varphi_{0i})_{i=1}^n\big) \in \mathcal{U}(t_1,\dots,t_n).$$

Then the problem (1), (2) *has at least one positive solution.*

The above Theorem 2 and Lemma 5.4 of [1] result in

Corollary 1. *Let*

$$\varphi_i(u_1,\dots,u_n) \le \sum_{k=1}^{n} \ell_{ik}\|u_k\|_C + \gamma \ \ \textit{for} \ \ (u_k)_{k=1}^n \in C([a,b];\mathbb{R}_+) \quad (i=1,\dots,n),$$

and the inequalities (6) *be fulfilled, where* I_0 *is a set of zero measure,* ℓ_{ik} $(i,k=1,\dots,n)$ *and* γ *are nonnegative constants,* $\lambda_i < 1$ $(i=1,\dots,n)$, $p_i : [a,b] \to \mathbb{R}_{0+}$ *and* $p_{ik} : [a,b] \to \mathbb{R}_+$ $(i=1,\dots,n)$ *are the Lebesgue integrable functions. If, moreover,*

$$r(\Lambda) < 1, \ \ \textit{where} \ \ \Lambda = \Big(\ell_{ik} + \int_a^b p_{ik}(t)\,dt\Big)_{i,k=1}^n,$$

then the problem (1), (2) *has at least one positive solution.*

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