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ON THE ASYMPTOTICS OF SOLUTIONS OF NONLINEAR CYCLIC SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to the memory of Temur Chanturia

Abstract. The asymptotics for a class of solutions for cyclic nonlinear systems of ordinary differential equations of more general type than Emden–Fowler system are established.

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1. Statement of the Problem and Auxiliary Designations

We consider the system of differential equations

$$y_i' = \alpha_i p_i(t)\varphi_{i+1}(y_{i+1}) \ \ (i=\overline{1,n}),^{*} \tag{1.1}$$

where $\alpha_i \in \{-1,1\}$ $(i=\overline{1,n})$, $p_i : [a,\omega[\, \to]0,+\infty[$ $(i=\overline{1,n})$ are continuous functions, $-\infty < a < \omega \le +\infty$,[†] $\varphi_i : \Delta(Y_i^0) \to]0;+\infty[$ $(i=\overline{1,n})$ are continuously differentiable functions satisfying the conditions

$$\lim_{\substack{y_i\to Y_i^0\\ y_i\in\Delta(Y_i^0)}} \frac{y_i\varphi_i'(y_i)}{\varphi_i(y_i)} = \sigma_i \ \ (i=\overline{1,n}), \quad \prod_{i=1}^{n}\sigma_i \ne 1, \tag{1.2}$$

where Y_i^0 $(i \in \{1,\dots,n\})$ is equal either to 0, or to $\pm\infty$, $\Delta(Y_i^0)$ $(i \in \{1,\dots,n\})$ is a one-sided neighborhood of Y_i^0.

It follows from the conditions (1.2) that φ_i $(i=\overline{1,n})$ are regularly varying functions of orders σ_i as $y_i \to Y_i^0$, hence (see [1]) these functions admit the representation

$$\varphi_i(y_i) = |y_i|^{\sigma_i}\theta_i(y_i) \ \ (i=\overline{1,n}), \tag{1.3}$$

where θ_i $(i=\overline{1,n})$ are slowly varying functions as $y_i \to Y_i^0$. According to the definition and properties of slowly varying functions and also in view of (1.2),

$$\lim_{y_i\to Y_i^0}\frac{\theta_i(\lambda y_i)}{\theta_i(y_i)} = 1 \ \text{ for any } \ \lambda>0, \quad \lim_{y_i\to Y_i^0}\frac{y_i\theta_i'(y_i)}{\theta_i(y_i)} = 0 \ \ (i=\overline{1,n}), \tag{1.4}$$

and the first limits are uniform with respect to λ on any segment $[c,d] \in]0,+\infty[$.

If $\theta_i(y_i) \equiv 1$ $(i=\overline{1,n})$, then the system (1.1) is called an Emden–Fowler system. In case $n=2$, the asymptotic behavior of its nonoscillating solutions is thoroughly investigated in [2–6].

In the present paper (as distinct from [2–6]), the system (1.1) is considered in the case where the functions $\varphi_i(y_i)$ $(i=\overline{1,n})$ are close to the power functions in the neighborhoods of Y_i^0 in the sense of the definition of regularly varying functions.

In T. A. Chanturia's paper [7], for systems of differential equations that are close to (1.1) in a certain sense the criteria for the existence of A and B-properties are established.

A solution $(y_i)_{i=1}^n$ of the system (1.1) is called $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution, if it is defined on the interval $[t_0,\omega[\, \subset [a,\omega[$ and satisfies the following conditions:

$$\begin{gathered} y_i(t) \in \Delta(Y_i^0) \ \text{ while } \ t\in[t_0,\omega[\,, \quad \lim_{t\uparrow\omega} y_i(t) = Y_i^0, \\ \lim_{t\uparrow\omega}\frac{y_i(t)y_{i+1}'(t)}{y_i'(t)y_{i+1}(t)} = \Lambda_i \ \ (i=\overline{1,n-1}). \end{gathered} \tag{1.5}$$

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* Here and in the sequel, all functions and parameters with the index $n+1$ will be equivalent to the corresponding values with index 1.

† While $\omega=+\infty$ we consider $a>0$.

The aim of this work is to establish sufficient and necessary conditions for the existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions for the system (1.1), and also to provide the asymptotic representation (when $t\uparrow\omega$) for these solutions, when Λ_i $(i=\overline{1,n-1})$ are real numbers, including those equal to zero, and $\Lambda_{n-1}\sigma_n=1$.

Remark 1.1. The definition of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution does not give the direct connection between the first and the n-th components of the solution, which appear in the n-th equation of the system. To establish this connection, we define the following functions:

$$\lambda_i(t)=\frac{y_i(t)y'_{i+1}(t)}{y'_i(t)y_{i+1}(t)}\quad (i=\overline{1,n}). \tag{1.6}$$

We have

$$\lambda_n(t)=\frac{y_n(t)y'_1(t)}{y'_n(t)y_1(t)}=\frac{y_n(t)y'_{n-1}(t)}{y'_n(t)y_{n-1}(t)}\cdot\frac{y_{n-1}(t)y'_{n-2}(t)}{y'_{n-1}(t)y_{n-2}(t)}\cdots\frac{y_2(t)y'_1(t)}{y'_2(t)y_1(t)}=$$
$$=\frac{1}{\lambda_1(t)\dots\lambda_{n-1}(t)}\,. \tag{1.7}$$

It follows from (1.5) that $\lim\limits_{t\uparrow\omega}\lambda_i(t)=\Lambda_i$ $(i=\overline{1,n-1})$. Therefore, if there are zeroes among Λ_i $(i=\overline{1,n-1})$, taking into account (1.7), we obtain

$$\Lambda_n=\lim_{t\uparrow\omega}\lambda_n(t)=\pm\infty.$$

In particular, it is evident that the case in which among all Λ_i $(i=1,\dots,n-1)$ there is a single $\pm\infty$, while all others are real different from zero numbers, can be transformed into the case described in this work. This transformation is carried out by cyclic redesignation of variables, functions and constants. For instance, if $\Lambda_l=\pm\infty$ $(l\in\{1,\dots,n-1\})$, the indices are redesignated as follows:

$$l\to n,\ \ l+1\to 1,\dots,n\to n-l,\ \ 1\to n-l+1,\dots,l-1\to n-1.$$

It is obvious that $\Lambda_i=0$, if $i=n-l$.

Further, we introduce some auxiliary notation.
First, if

$$\mu_i=\begin{cases}1, & \text{as } Y_i^0=+\infty,\\ & \text{or } Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is right neighborhood of 0},\\ -1, & \text{as } Y_i^0=-\infty,\\ & \text{or } Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is left neighboorhood of 0},\end{cases}$$

it is obvious that μ_i $(i=\overline{1,n})$ determine the signs of the components of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution in some left neighborhood of ω.
Further, we denote the sets

$$\mathfrak{I}=\{i\in\{1,\dots,n-1\}:\ 1-\Lambda_i\sigma_{i+1}\neq 1\}\,,\quad \overline{\mathfrak{I}}=\{1,\dots,n-1\}\setminus\mathfrak{I}$$

and suppose that

$$r = \max \mathfrak{I} < n-1.$$

Taking into account the fact that $r < n-1$, we denote auxiliary functions I_i, Q_i $(i = 1, \dots, n)$ and none-zero constants β_i $(i = 1, \dots, n)$, supposing that

$$I_i(t) = \begin{cases} \int\limits_{A_i}^{t} p_i(\tau)\, d\tau & \text{for } i \in \mathfrak{I}, \\ \int\limits_{A_i}^{t} p_i(\tau) I_{i+1}(\tau)\, d\tau & \text{for } i \in \overline{\mathfrak{I}}, \\ \int\limits_{A_n}^{t} p_n(\tau) q_{r+1}(\tau)\, d\tau & \text{for } i = n, \end{cases} \qquad \beta_i = \begin{cases} 1 - \Lambda_i \sigma_{i+1}, & \text{for } i \in \mathfrak{I}, \\ \beta_{i+1}\Lambda_i, & \text{for } i \in \overline{\mathfrak{I}}, \\ 1 - \prod\limits_{k=1}^{n} \sigma_k & \text{for } i = n, \end{cases}$$

$$Q_i(t) = \begin{cases} \alpha_i \beta_i I_i(t) & \text{for } i \in \mathfrak{I} \cup \{n\}, \\ \dfrac{\alpha_i \beta_i I_i(t)}{I_{i+1}(t)} & \text{for } i \in \overline{\mathfrak{I}}, \end{cases}$$

where limits of integration $A_i \in \{\omega, a\}$ $(i \in \{1, \dots, n-1\})$, $A_n \in \{\omega, b\}$ $(b \in [a, \omega[)$ are chosen in such a way that the corresponding integral I_i tends either to zero, or to ∞ as $t \uparrow \omega$,

$$q_{r+1}(t) = \theta_1 \left(\mu_1 |I_1(t)|^{\frac{1}{\beta_1}} \right) |Q_r(t)|^{\prod\limits_{k=1}^{r} \sigma_k} \times \\ \times \prod_{k=1}^{r-1} \left| Q_k(t) \theta_{k+1} \left(\mu_{k+1} |I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}} \right) \right|^{\prod\limits_{i=1}^{k} \sigma_i}.$$

In addition, we introduce the numbers

$$\begin{aligned} A_i^* &= \begin{cases} 1, & \text{if } A_i = a, \\ -1, & \text{if } A_i = \omega \end{cases} \quad (i = 1, \dots, n-1), \\ A_n^* &= \begin{cases} 1, & \text{if } A_n = b, \\ -1, & \text{if } A_n = \omega. \end{cases} \end{aligned} \tag{1.8}$$

These numbers enable us to define the signs of the functions I_i $(i = 1, \dots, n-1)$ on the interval $]a, \omega[$ and the sign of the function I_n on the interval $]b, \omega[$.

We will define that the function φ_k $(k \in \{1, \dots, n\})$ satisfies the condition **S**, if for any continuously differentiable function $l : \Delta(Y_k^0) \to]0, +\infty[$ with the property

$$\lim_{\substack{z \to Y_k^0 \\ z \in \Delta(Y_k^0)}} \frac{z\, l'(z)}{l(z)} = 0,$$

the function θ_k admits the asymptotic representation

$$\theta_k(zl(z)) = \theta(z)[1 + o(1)] \ \text{ as } \ z \to Y_k^0 \ \ (z \in \Delta(Y_k^0)). \tag{1.9}$$

For instance, the **S**-condition is, obviously, satisfied by the functions φ_k of the type

$$\varphi_k(y_k) = |y_k|^{\sigma_k} |\ln y_k|^{\gamma_1}, \quad \varphi_k(y_k) = |y_k|^{\sigma_k} |\ln y_k|^{\gamma_1} |\ln|\ln y_k||^{\gamma_2},$$

where $\gamma_1, \gamma_2 \neq 0$. The **S**-condition is also satisfied by the functions φ_k which include the functions θ_k that have the eventual limit as $y_k \to Y_k^0$. The **S**-condition is also satisfied by many other functions.

Remark 1.2. If φ_k $(k \in \{1, \dots, n\})$ satisfies the **S**-condition and $y_k : [t_0, \omega[\to \Delta(Y_k^0)$ is a continuously differentiable function with the property

$$\lim_{t\uparrow\omega} y_k(t) = Y_k^0, \quad \frac{y_k'(t)}{y_k(t)} = \frac{\xi'(t)}{\xi(t)}\,[r + o(1)] \ \text{ as } \ t \uparrow \omega,$$

where r is a non-zero real constant, ξ is a continuously differentiable in some left neighborhood of ω real function with $\xi'(t) \neq 0$, then

$$\theta_k(y_k(t)) = \theta_k\left(\mu_k|\xi(t)|^r\right)[1 + o(1)] \ \text{ as } \ t \uparrow \omega,$$

since in this case

$$y_k(t) = z(t)l(z(t)), \ \text{ where } \ z(t) = \mu_k|\xi(t)|^r,$$

and

$$\lim_{\substack{z\to Y_0\\ z\in\Delta_{Y_0}}} \frac{z\,l'(z)}{l(z)} = \lim_{t\uparrow\omega} \frac{z(t)\,l'(z(t))}{l(z(t))} =$$

$$= \lim_{t\uparrow\omega} \frac{z(t)\left(\frac{y_k(t)}{z(t)}\right)'}{\left(\frac{y_k(t)}{z(t)}\right) z'(t)} = \lim_{t\uparrow\omega}\left[\frac{\xi(t)y_k'(t)}{r\xi'(t)y_k(t)} - 1\right] = 0.$$

2. Main Results

Theorem 2.1. *Let* $\Lambda_i \in \mathbb{R}$ $(i = \overline{1, n-1})$ *include those equal to zero,* $m = \max\{i \in \Im :\ \Lambda_i = 0\}$ *and* $r = \max \Im < n-1$. *Let also the functions* φ_k $(k = \overline{1, r})$ *satisfy the* **S***-condition. Then for the existence of* $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$*-solutions of* (1.1) *it is necessary and, if the algebraic equation*

$$\Big(\prod_{j=1}^{n} \sigma_j - 1 - \lambda\Big) \prod_{j=m+1}^{n-1} (M_j + \lambda) =$$

$$= \Big(\prod_{j=1}^{n} \sigma_j\Big)\Big(\sum_{k=m}^{r} \prod_{j=m+1}^{k} (M_j + \lambda) \prod_{s=k+2}^{n-1} M_s\Big)\lambda,^{*} \tag{2.1}$$

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* Here and in what follows, we assume that $\prod\limits_{j=s}^{l} = 1$, $\sum\limits_{j=s}^{l} = 0$ if $l < s$.

where

$$M_j = \Big(\prod_{i=j}^{n-1} \Lambda_i\Big)^{-1} \quad (j = \overline{m+1, n-1}),$$

have no roots with a zero real part, it is also sufficient that

$$\lim_{t\uparrow\omega} \frac{I_i(t) I'_{i+1}(t)}{I'_i(t) I_{i+1}(t)} = \Lambda_i \frac{\beta_{i+1}}{\beta_i} \quad (i = \overline{1, n-1}) \tag{2.2}$$

and for each $i \in \{1, \dots, n\}$ the following conditions be satisfied:

$$A_i^* \beta_i > 0 \ \text{ if } \ Y_i^0 = \pm\infty, \quad A_i^* \beta_i < 0 \ \text{ if } \ Y_i^0 = 0, \tag{2.3}$$

$$\operatorname{sign}\,[\alpha_i A_i^* \beta_i] = \mu_i. \tag{2.4}$$

Moreover, the components of each solution of that type admit asymptotic representation when $t \uparrow \omega$,

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = Q_i(t)[1 + o(1)] \quad (i = \overline{1, n-1}), \tag{2.5}$$

$$\frac{y_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{i=1}^{r} \sigma_i}} = Q_n(t)[1 + o(1)], \tag{2.6}$$

and there exists the whole k-parametric family of these solutions if there are k positive roots among the solutions of the following algebraic equation:

$$\gamma_i = \begin{cases} \beta_i A_i^* & \text{if } i \in \Im \setminus \{m+1, \dots, n-1\}, \\ \beta_i A_i^* A_{i+1}^* & \text{if } i \in \overline{\Im} \setminus \{m+1, \dots, n-1\}, \\ A_n^* \Big(\prod\limits_{j=1}^{n-1} \sigma_j - 1\Big) \operatorname{Re} \lambda_{i-m}^0 & \text{if } i \in \{m+1, \dots, n\}, \end{cases} \tag{2.7}$$

where λ_j^0 $(j = \overline{1, n-m})$ are the roots of the algebraic equation (2.1) (*along with multiple*).

Remark 2.1. The algebraic equation (2.1) has, obviously, no roots with zero real part, if

$$\Big(\sum_{k=m+1}^{r+1} \prod_{j=k}^{n-1} |\Lambda_j|\Big) \prod_{k=1}^{n} |\sigma_k| < \Big|1 - \prod_{j=1}^{n} \sigma_j\Big|.$$

Proof of Theorem 2.1. *Necessity.* Let $y_i : [t_0, \omega[\to \Delta(Y_i^0)$ $(i = \overline{1, n})$ be an arbitrary $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution of (1.1). Then by virtue of (1.1), we obtain

$$\frac{y'_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i p_i(t) \quad (i = \overline{1, n}) \ \text{ as } \ t \in [t_0, \omega[\,. \tag{2.8}$$

When $i \in \Im$, integrating (2.8) over the interval from B_i to t, where $B_i = \omega$, if $A_i = \omega$, or $B_i = t_0$, if $A_i = a$, we get

$$\int\limits_{B_i}^{t} \frac{y_i'(\tau)}{\varphi_{i+1}(y_{i+1}(\tau))}\, d\tau = \alpha_i I_i(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{2.9}$$

In virtue of de L'Hospital's rule in the form of Stoltz, we get

$$\lim_{t\uparrow\omega} \frac{\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))}}{\int\limits_{B_i}^{t} \frac{y_i'(\tau)}{\varphi_{i+1}(y_{i+1}(\tau))}\, d\tau} = \lim_{t\uparrow\omega} \frac{\frac{y_i'(t)}{\varphi_{i+1}(y_{i+1}(t))} - \frac{y_i(t)\varphi_{i+1}'(y_{i+1}(t))y_{i+1}'(t)}{\varphi_{i+1}^2(y_{i+1}(t))}}{\frac{y_i'(t)}{\varphi_{i+1}(y_{i+1}(t))}} =$$

$$= 1 - \lim_{t\uparrow\omega} \frac{y_{i+1}(t)\varphi_{i+1}'(y_{i+1}(t))}{\varphi_{i+1}(y_{i+1}(t))} \lim_{t\uparrow\omega} \frac{y_i(t)y_{i+1}'(t)}{y_i'(t)y_{i+1}(t)} = 1 - \Lambda_i\sigma_{i+1} = \beta_i \neq 0.$$

Therefore, in view of (2.9), we have

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i\beta_i I_i(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{2.10}$$

Consequently, when $i \in \Im$, the asymptotic representation (2.5) is valid and, in virtue of (2.8) and (2.10),

$$\frac{y_i'(t)}{y_i(t)} = \frac{I_i'(t)}{\beta_i I_i(t)}\, [1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{2.11}$$

Further, taking into account that $r = \max \Im < n - 1$, we consider the relations (2.8) consistently starting with the maximum $i \in \overline{\Im}$, that is lower than r, since $i \in \overline{\Im} \setminus \{r+1, \dots, n-1\}$. We consider these relations taking into account that the relations (2.11) are valid for bigger values of $i \le r$. Multiplying (2.8) by $I_{i+1}(t)$ and integrating over the interval from B_i to t, where B_i are chosen in the above way, we get

$$\int\limits_{B_i}^{t} \frac{y_i'(\tau)I_{i+1}(\tau)}{\varphi_{i+1}(y_{i+1}(\tau))}\, d\tau = \alpha_i I_i(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega. \tag{2.12}$$

In virtue of de L'Hospital's rule in the form of Stoltz, using (2.11) and the definition of $P_\omega(\Lambda_1, \dots, \Lambda_{n-1})$- solution, we obtain

$$\lim_{t\uparrow\omega} \frac{\frac{y_i(t)I_{i+1}(t)}{\varphi_{i+1}(y_{i+1}(t))}}{\int\limits_{A_i}^{t} \frac{y_i'(\tau)I_{i+1}(\tau)}{\varphi_{i+1}(y_{i+1}(\tau))}\, d\tau} =$$

$$= \lim_{t\uparrow\omega} \frac{\frac{y_i'(t)I_{i+1}(t)}{\varphi_{i+1}(y_{i+1}(t))} + \frac{y_i(t)I_{i+1}'(t)}{\varphi_{i+1}(y_{i+1}(t))} - \frac{y_i(t)I_{i+1}(t)\varphi_{i+1}'(y_{i+1}(t))y_{i+1}'(t)}{\varphi_{i+1}^2(y_{i+1}(t))}}{\frac{y_i'(t)I_{i+1}(t)}{\varphi_{i+1}(y_{i+1}(t))}} =$$

$$= 1 + \lim_{t\uparrow\omega} \frac{y_i(t)I_{i+1}'(t)}{y_i'(t)I_{i+1}(t)} - \lim_{t\uparrow\omega} \frac{y_{i+1}(t)\varphi_{i+1}'(y_{i+1}(t))}{\varphi_{i+1}(y_{i+1}(t))} \lim_{t\uparrow\omega} \frac{y_i(t)y_{i+1}'(t)}{y_i'(t)y_{i+1}(t)} =$$

$$= 1 - \Lambda_i \sigma_{i+1} + \beta_{i+1} \lim_{t \uparrow \omega} \frac{y_i(t) y'_{i+1}(t)}{y'_i(t) y_{i+1}(t)} =$$

$$= \beta_{i+1} \lim_{t \uparrow \omega} \left[\frac{y_i(t) y'_{i+1}(t)}{y'_i(t) y_{i+1}(t)} \right] = \beta_{i+1} \Lambda_i = \beta_i.$$

Hence, with regard for (2.12), we get

$$\frac{y_i(t)}{\varphi_i(y_{i+1}(t))} = \frac{\alpha_i \beta_i I_i(t)}{I_{i+1}(t)} \, [1 + o(1)] \ \text{ as } \ t \uparrow \omega. \tag{2.13}$$

Therefore, with regard for (2.8), the asymptotic formula (2.11) is valid. Consequently, the asymptotic representations (2.5) and (2.11) are admitted for all $i \in \overline{\mathfrak{I}} \setminus \{r+1, \dots, n-1\}$.

Taking into account that φ_i satisfy the **S**-condition for all $i \in \{1, \dots, r\}$ and asymptotic representations (2.11) are valid, in virtue of Remark 1.2, we get

$$\varphi_i(y_i(t)) = |y_i(t)|^{\sigma_i} \theta_i \left(\mu_i |I_i(t)|^{\frac{1}{\beta_i}} \right) [1 + o(1)] \ \ (i = \overline{1, r}) \ \text{ as } \ t \uparrow \omega.$$

According to these representations and the asymptotic representations (2.5) for $i = \overline{1, r}$, we have

$$\varphi_1(y_1(t)) = |y_1(t)|^{\sigma_1} \theta_1 \left(\mu_1 |I_1(t)|^{\frac{1}{\beta_1}} \right) [1 + o(1)] =$$

$$= \theta_1 \left(\mu_1 |I_1(t)|^{\frac{1}{\beta_1}} \right) \left| |y_2(t)|^{\sigma_2} Q_1(t) \theta_2 \left(\mu_2 |I_2(t)|^{\frac{1}{\beta_2}} \right) \right|^{\sigma_1} [1 + o(1)] =$$

$$= \theta_1 \left(\mu_1 |I_1(t)|^{\frac{1}{\beta_1}} \right) \left| Q_1(t) \theta_2 \left(\mu_2 |I_2(t)|^{\frac{1}{\beta_2}} \right) \right|^{\sigma_1} \times$$

$$\times \left| |y_3(t)|^{\sigma_3} Q_2(t) \theta_3 \left(\mu_3 |I_3(t)|^{\frac{1}{\beta_3}} \right) \right|^{\sigma_1 \sigma_2} [1 + o(1)] = \dots =$$

$$= q_{r+1}(t) \left[\varphi_{r+1}(y_{r+1}(t)) \right]^{\prod\limits_{i=1}^{r} \sigma_i} [1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

From this and the last formula in (2.8), we conclude that

$$\frac{y'_n(t)}{\left[\varphi_{r+1}(y_{r+1}(t)) \right]^{\prod\limits_{k=1}^{r} \sigma_k}} = \alpha_n p_n(t) q_{r+1}(t) [1 + o(1)] \ \text{ as } \ t \uparrow \omega. \tag{2.14}$$

Integrating (2.14) over the interval from B_n to t, where $B_n = \omega$, if $A_n = \omega$, and $B_n = t_0$, if $A_n = b$, we obtain

$$\int\limits_{B_n}^{t} \frac{y'_n(\tau)}{\left[\varphi_{r+1}(y_{r+1}(\tau)) \right]^{\prod\limits_{k=1}^{r} \sigma_k}} \, d\tau = \alpha_n I_n(t) [1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

Using de L'Hospital's rule, with regard for (1.2), (1.5) and the conditions $1 - \Lambda_j \sigma_{j+1} = 0$ as $j = \overline{r+1, n-1}$, we get

$$\lim_{t\uparrow\omega}\frac{\dfrac{y_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod_{k=1}^{r}\sigma_k}}}{\int\limits_{B_n}^{t}\dfrac{y_n'(\tau)}{[\varphi_{r+1}(y_{r+1}(\tau))]^{\prod_{k=1}^{r}\sigma_k}}\,d\tau}=$$

$$=\lim_{t\uparrow\omega}\frac{\dfrac{y_n'(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod_{k=1}^{r}\sigma_k}}\left[1-\Big(\prod_{k=1}^{r}\sigma_k\Big)\dfrac{y_{r+1}(t)\varphi_{r+1}'(y_{r+1}(t))}{\varphi_{r+1}(y_{r+1}(t))}\,\dfrac{y_{r+1}'(t)y_n(t)}{y_{r+1}(t)y_n'(t)}\right]}{\dfrac{y_n'(t)}{[\varphi_n(y_n(t))]^{\prod_{k=1}^{n-1}\sigma_k}}}=$$

$$=1-\Big(\prod_{j=1}^{r+1}\sigma_j\Big)\lim_{t\uparrow\omega}\frac{y_{r+1}'(t)y_{r+2}(t)}{y_{r+1}(t)y_{r+2}'(t)}\cdots\frac{y_{n-1}'(t)y_n(t)}{y_{n-1}(t)y_n'(t)}=$$

$$=1-\frac{\prod_{j=1}^{r+1}\sigma_j}{\Lambda_{r+1}\dots\Lambda_{n-1}}=1-\prod_{j=1}^{n}\sigma_j=\beta_n.$$

The previous asymptotic representation yields

$$\frac{y_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod_{k=1}^{r}\sigma_k}}=\alpha_n\beta_n I_n(t)[1+o(1)]\quad\text{as}\quad t\uparrow\omega.$$

Hence, the representation (2.6) is valid and, in virtue of (2.14), (2.11), it takes place when $i=n$.

Taking into account that the asymptotic representation (2.11) is valid for $i=n$, by the same reasoning (multiplying (2.8) by $I_{i+1}(t)$ and further integrating over the interval from B_i to t), we conclude that the asymptotic representations (2.5) and (2.11) are valid for all $i=\overline{r+1,n-1}$ starting with $i=\overline{r+1,n-1}$. The relations (2.11) are valid for $i=\overline{1,n}$ and the solution under consideration satisfies the last limiting condition from the definition of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution. Consequently, for all $i\in\{1,\dots,n-1\}$, the conditions (2.2) are valid. Moreover, from (2.11) it follows that

$$|y_i(t)|=|I_i(t)|^{\frac{1}{\beta_i}+o(1)}\quad(i=\overline{1,n})\ \text{ as }\ t\uparrow\omega.$$

On the basis of the above fact, from the condition $\lim\limits_{t\uparrow\omega}y_i(t)=Y_i^0$ in the definition of the $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution and from the definition of numbers A_i^*, there follow the sign conditions (2.3).

The validity of the sign conditions (2.4) follows immediately from (2.5), (2.6), if we consider the signs of the functions y_i and I_i $(i=\overline{1,n})$ over the interval $[t_0,\omega[$.

Sufficiency. Assume that the conditions (2.2)–(2.4) are satisfied and the algebraic equation (2.1) has no roots with zero real part. We will prove that the system (1.1) has at least one $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$- solution that admits the asymptotic representation (2.5), (2.6) as $t \uparrow \omega$. We will also study the question about the quantity of such solutions.

First, consider the system of the following relations:

$$\begin{cases} \dfrac{y_i}{\varphi_{i+1}(y_{i+1})} = Q_i(t)(1+v_i) & (i=\overline{1,n-1}), \\ \dfrac{y_n}{[\varphi_{r+1}(y_{r+1})]^{\prod\limits_{k=1}^{r}\sigma_k}} = Q_n(t)(1+v_n). \end{cases} \tag{2.15}$$

We will establish that this system on the sets $D=[t_0,\omega[\times\mathbb{R}^n_{\frac12}$, where $t_0\in[a,\omega[$ and $\mathbb{R}^n_{\frac12}=\{\overline{x}\equiv(x_i,\dots,x_n)\in\mathbb{R}^n:\ |x_k|\le 1/2\ (k=\overline{1,n})\}$, defines uniquely continuously differential functions $y_i=Y_i(t,\overline{v})$ $(i=\overline{1,n})$ of the type

$$Y_i(t,\overline{v})=\mu_i|I_i(t)|^{\frac{1}{\beta_i}[1+z_i(t,\overline{v})]}\ \ (i=\overline{1,n}), \tag{2.16}$$

where z_i $(i=\overline{1,n})$ are the following functions

$$|z_i(t,\overline{v})|\le\frac12\ \text{ as }\ (t,\overline{v})\in D$$

and

$$\lim_{t\uparrow\omega} z_i(t,\overline{v})=0\ \text{ uniformly over }\ \overline{v}\in\mathbb{R}^n_{\frac12}.$$

Setting in (2.15)

$$y_i=\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\ \ (i=\overline{1,n}), \tag{2.17}$$

and taking into account (1.3), we obtain the following system of relations:

$$\begin{cases} \dfrac{|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}}{|I_{i+1}(t)|^{\frac{\sigma_{i+1}}{\beta_{i+1}}(1+z_{i+1})}} = \\ \quad = \mu_iQ_i(t)\theta_{i+1}\left(\mu_{i+1}|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}(1+z_{i+1})}\right)(1+v_i)\ (i=\overline{1,n-1}), \\ \dfrac{|I_n(t)|^{\frac{1}{\beta_n}(1+z_n)}}{|I_{r+1}(t)|^{\frac{\prod_{k=1}^{r+1}\sigma_k}{\beta_{r+1}}(1+z_{r+1})}} = \\ \quad = \mu_nQ_n(t)\left[\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_r)}\right)\right]^{\prod\limits_{k=1}^{r}\sigma_k}(1+v_n), \end{cases}$$

With regard to sign conditions (2.3), (2.4), the system is defined for all $|v_i|\le\frac12$, $|z_i|\le\frac12$ $(i=\overline{1,n})$ and t from some left neighborhood of ω.

Hence, taking the logarithm, we get

$$\begin{cases} \dfrac{1}{\beta_i}\,(1+z_i)\ln|I_i(t)| - \dfrac{\sigma_{i+1}}{\beta_{i+1}}\,(1+z_{i+1})\ln|I_{i+1}(t)| = \\ \qquad = \ln|Q_i(t)| + \ln\theta_{i+1}\left(\mu_{i+1}|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}(1+z_{i+1})}\right) + \ln|1+v_i| \\ \qquad\qquad\qquad (i=\overline{1,n-1}), \\ \dfrac{1}{\beta_n}\,(1+z_n)\ln|I_n(t)| - \dfrac{\prod\limits_{k=1}^{r+1}\sigma_k}{\beta_{r+1}}\,(1+z_{r+1})\ln|I_{r+1}(t)| = \\ \qquad = \ln|Q_n(t)| + \Big(\prod\limits_{k=1}^{r}\sigma_k\Big)\ln\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_{r+1})}\right) + \\ \qquad + \ln|1+v_n|. \end{cases}$$

Therefore

$$\begin{cases} 1+z_i - \dfrac{\beta_i\sigma_{i+1}}{\beta_{i+1}}\,\dfrac{\ln|I_{i+1}(t)|}{\ln|I_i(t)|}\,(1+z_{i+1}) = \\ \qquad = \dfrac{\beta_i\ln|Q_i(t)|}{\ln|I_i(t)|} + \dfrac{\beta_i\ln\theta_{i+1}\left(\mu_{i+1}|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}(1+z_{i+1})}\right)}{\ln|I_i(t)|} + \\ \qquad + \dfrac{\beta_i\ln|1+v_i|}{\ln|I_i(t)|} \quad (i=\overline{1,n-1}), \\ 1+z_n - \dfrac{\beta_n\prod\limits_{k=1}^{r+1}\sigma_k}{\beta_{r+1}}\,\dfrac{\ln|I_{r+1}(t)|}{\ln|I_n(t)|}(1+z_{r+1}) = \\ \qquad = \dfrac{\beta_n\ln|Q_n(t)|}{\ln|I_n(t)|} + \\ \qquad + \dfrac{\beta_n\Big(\prod\limits_{k=1}^{r}\sigma_k\Big)\ln\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_{r+1})}\right)}{\ln|I_n(t)|} + \\ \qquad + \dfrac{\beta_n\ln|1+v_n|}{\ln|I_n(t)|}\,. \end{cases}$$

Solving partly this system (as a system of nonhomogeneous linear equations with variables $1+z_i$ $(i=\overline{1,n})$), we obtain

$$z_i = a_i(t) + b_i(t,\overline{v}) + Z_i(t,\overline{z}) \quad (i=\overline{1,n}), \tag{2.18}$$

where the functions a_i, b_i, Z_i $(i=\overline{1,n})$ are defined by the following recurrent relations:

$$a_{r+1}(t) = -1 + \left[1 - \Big(\prod_{k=1}^{r+1}\sigma_k\Big)\frac{\beta_n\ln|I_{r+1}(t)|}{\beta_{r+1}\ln|I_n(t)|}\prod_{k=r}^{n-1}\frac{\sigma_{k+1}\beta_k\ln|I_{k+1}(t)|}{\beta_{k+1}\ln|I_k(t)|}\right]^{-1}\times$$

$$\times \sum_{k=r+1}^{n} \frac{\beta_k \ln|Q_k(t)|}{\ln|I_k(t)|} \prod_{j=r+1}^{k-1} \frac{\beta_j \sigma_{j+1} \ln|I_{j+1}(t)|}{\beta_{j+1} \ln|I_j(t)|},$$

$$b_{r+1}(t,\overline{v}) = \left[1 - \Big(\prod_{k=1}^{r+1} \sigma_k\Big) \frac{\beta_n \ln|I_{r+1}(t)|}{\beta_{r+1} \ln|I_n(t)|} \prod_{k=r}^{n-1} \frac{\sigma_{k+1}\beta_k \ln|I_{k+1}(t)|}{\beta_{k+1} \ln|I_k(t)|}\right]^{-1} \times$$

$$\times \sum_{k=r+1}^{n} \frac{\beta_k \ln|1+v_k|}{\ln|I_k(t)|} \prod_{j=r+1}^{k-1} \frac{\beta_j \sigma_{j+1} \ln|I_{j+1}(t)|}{\beta_{j+1} \ln|I_j(t)|},$$

$$Z_{r+1}(t,\overline{z}) = \left[1 - \Big(\prod_{k=1}^{r+1} \sigma_k\Big) \frac{\beta_n \ln|I_{r+1}(t)|}{\beta_{r+1} \ln|I_n(t)|} \prod_{k=r}^{n-1} \frac{\sigma_{k+1}\beta_k \ln|I_{k+1}(t)|}{\beta_{k+1} \ln|I_k(t)|}\right]^{-1} \times$$

$$\times \left[\sum_{k=r+1}^{n-1} \frac{\beta_k \ln\theta_{k+1}\Big(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}(1+z_{k+1})}\Big)}{\ln|I_k(t)|} \prod_{j=r+1}^{k-1} \frac{\beta_j \sigma_{j+1} \ln|I_{j+1}(t)|}{\beta_{j+1} \ln|I_j(t)|} + \right.$$

$$+ \frac{\beta_n\Big(\prod\limits_{k=1}^{r} \sigma_k\Big) \ln\theta_{r+1}\Big(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_{r+1})}\Big)}{\ln|I_n(t)|} \times$$

$$\left. \times \prod_{j=r+1}^{n-1} \frac{\beta_j \sigma_{j+1} \ln|I_{j+1}(t)|}{\beta_{j+1} \ln|I_j(t)|}\right],$$

$$a_n(t) = -1 + \frac{\beta_n \prod\limits_{k=1}^{r+1} \sigma_k}{\beta_{r+1}} \frac{\ln|I_{r+1}(t)|}{\ln|I_n(t)|} [1 + a_{r+1}(t)] + \frac{\beta_n \ln|Q_n(t)|}{\ln|I_n(t)|},$$

$$b_n(t,\overline{v}) = \frac{\beta_n \prod\limits_{k=1}^{r+1} \sigma_k}{\beta_{r+1}} \frac{\ln|I_{r+1}(t)|}{\ln|I_n(t)|} b_{r+1}(t,\overline{v}) + \frac{\beta_n \ln|1+v_n|}{\ln|I_n(t)|},$$

$$Z_n(t,\overline{z}) = \frac{\beta_n \prod\limits_{k=1}^{r+1} \sigma_k}{\beta_{r+1}} \frac{\ln|I_{r+1}(t)|}{\ln|I_n(t)|} Z_{r+1}(t,\overline{z}) +$$

$$+ \frac{\beta_n\Big(\prod\limits_{k=1}^{r} \sigma_k\Big) \ln\theta_{r+1}\Big(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}(1+z_{r+1})}\Big)}{\ln|I_n(t)|},$$

$$a_i(t) = -1 + \frac{\beta_i \sigma_{i+1}}{\beta_{i+1}} \frac{\ln|I_{i+1}(t)|}{\ln|I_i(t)|} [1 + a_{i+1}(t)] + \frac{\beta_i \ln|Q_i(t)|}{\ln|I_i(t)|} \quad \text{if } i \in \{1,\dots,n-1\} \setminus \{r+1\},$$

$$b_i(t,\overline{v}) = \frac{\beta_i \sigma_{i+1}}{\beta_{i+1}} \frac{\ln|I_{i+1}(t)|}{\ln|I_i(t)|} b_{i+1}(t,\overline{v}) + \frac{\beta_i \ln|1+v_i|}{\ln|I_i(t)|} \quad \text{if } i \in \{1,\dots,n-1\} \setminus \{r+1\},$$

$$Z_i(t,\overline{z}) = \frac{\beta_i\sigma_{i+1}}{\beta_{i+1}}\frac{\ln|I_{i+1}(t)|}{\ln|I_i(t)|}Z_{i+1}(t,\overline{z}) + \\ + \frac{\beta_i \ln\theta_{i+1}\Big(\mu_{i+1}|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}(1+z_{i+1})}\Big)}{\ln|I_i(t)|} \\ \text{as } \ i\in\{1,\dots,n-1\}\setminus\{r+1\}.$$

Here $\lim\limits_{t\uparrow\omega} I_i(t)$ $(i=\overline{1,n})$ is equal either to zero, or to $\pm\infty$. Moreover, by de L'Hospital's rule, (2.2), (1.4) and by the above-introduced notation β_i $(i=\overline{1,n})$, we get

$$\lim_{t\uparrow\omega}\frac{\beta_i\ln|I_{i+1}(t)|}{\beta_{i+1}\ln|I_i(t)|} = \lim_{t\uparrow\omega}\frac{\beta_i I_i(t)I'_{i+1}(t)}{\beta_{i+1}I'_i(t)I_{i+1}(t)} = \Lambda_i \ \ (i=\overline{1,n-1}),$$

$$\lim_{t\uparrow\omega}\frac{\beta_n\ln|I_{r+1}(t)|}{\beta_{r+1}\ln|I_n(t)|} = \lim_{t\uparrow\omega}\frac{\beta_n I'_{r+1}(t)I_n(t)}{\beta_{r+1}I_{r+1}(t)I'_n(t)} = (\Lambda_{r+1}\cdots\Lambda_{n-1})^{-1} = \prod_{k=r+2}^{n}\sigma_k,$$

$$\lim_{t\uparrow\omega}\frac{\beta_i\ln|Q_i(t)|}{\ln|I_i(t)|} = \begin{cases}\beta_i = 1-\Lambda_i\sigma_{i+1} & \text{if } \ i\in\mathfrak{I},\\ \beta_n = 1-\prod\limits_{k=1}^{n}\sigma_k & \text{if } \ i=n,\end{cases}$$

$$\lim_{t\uparrow\omega}\frac{\beta_i\ln|Q_i(t)|}{\ln|I_i(t)|} = \beta_i\lim_{t\uparrow\omega}\bigg(1-\frac{I_i(t)I'_{i+1}(t)}{I'_i(t)I_{i+1}(t)}\bigg) = \beta_i\big(1-\frac{\beta_{i+1}}{\beta_i}\Lambda_i\big) = 0 \ \text{ if } \ i\in\overline{\mathfrak{I}},$$

$$\lim_{t\uparrow\omega}\frac{\ln\theta_i\Big(\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\Big)}{\ln|I_i(t)|} = \frac{1}{\beta_i}\,(1+z_i)\lim_{t\uparrow\omega}\frac{\ln\theta_i\Big(\mu_i|I_i(t)|^{\frac{1}{\beta_i}(1+z_i)}\Big)}{\ln\big|\mu_i|I_i(t)|\big|^{\frac{1}{\beta_i}(1+z_i)}} =$$

$$= \frac{1}{\beta_i}\,(1+z_i)\lim_{y\to Y_i^0}\frac{\ln\theta_i(y)}{\ln|y|} =$$

$$= \frac{1}{\beta_i}\,(1+z_i)\lim_{y\to Y_i^0}\frac{y\theta'_i(y)}{\theta_i(y)} = 0 \ \text{ uniformly over } \ |z_i|\le\frac{1}{2}\,.$$

From these limiting relations, starting with $i=r+1$, and further for $i=r,r-1,\dots,1$ and $i=r+2,\dots,n$, we obtain

$$\lim_{t\uparrow\omega} a_i(t) = 0 \ \ (i=\overline{1,n}), \tag{2.19}$$

$$\lim_{t\uparrow\omega} b_i(t,\overline{v}) = 0 \ \ (i=\overline{1,n}) \ \text{ uniformly over } \ \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}\,, \tag{2.20}$$

$$\lim_{t\uparrow\omega} Z_i(t,\overline{z}) = 0 \ \ (i=\overline{1,n}) \ \text{ uniformly over } \ d\overline{z}\in\mathbb{R}^n_{\frac{1}{2}}\,. \tag{2.21}$$

Moreover, for each $i \in \{1, \dots, n\}$

$$\frac{1}{\ln|I_i(t)|} \frac{\partial \left[\ln \theta_i \left(\mu_i |I_i(t)|^{\frac{1}{\beta_i}(1+z_i)} \right) \right]}{\partial z_i} =$$

$$= \frac{1}{\beta_i} \frac{\mu_i |I_i(t)|^{\frac{1}{\beta_i}(1+z_i)} \theta_i' \left(\mu_i |I_i(t)|^{\frac{1}{\beta_i}(1+z_i)} \right)}{\theta_i \left(\mu_i |I_i(t)|^{\frac{1}{\beta_i}(1+z_i)} \right)}$$

and, therefore, this relation because of (1.4), tends to zero as $t \uparrow \omega$ uniformly over $|z_i| \le \frac{1}{2}$. Taking this fact into account, starting with $i = r+1$ (by the same method), we obtain

$$\lim_{t \uparrow \omega} \frac{\partial Z_i(t, \overline{z})}{\partial z_k} = 0 \ \ (i, k = \overline{1, n}) \ \text{ uniformly over } \ \overline{z} \in \mathbb{R}^n_{\frac{1}{2}}\, . \tag{2.22}$$

By conditions (2.19)–(2.22), there exists a number $t_0 \in [a, \omega[$ such that the following inequalities are valid:

$$\big| a_i(t) + b_i(t, \overline{v}) + Z_i(t, \overline{z}) \big| \le \frac{1}{2n} \ \ (i = \overline{1, n}) \quad \text{as } \ (t, \overline{v}, \overline{z}) \in [t_0, \omega[\, \times \mathbb{R}^n_{\frac{1}{2}} \times \mathbb{R}^n_{\frac{1}{2}} \tag{2.23}$$

and Lipschitz conditions are valid

$$\big| Z_i(t, \overline{z}^1) - Z_i(t, \overline{z}^2) \big| \le \frac{1}{n+1} \sum_{k=1}^{n} |z_k^1 - z_k^2| \ \ (i = \overline{1, n}) \tag{2.24}$$

as $(t, \overline{z}^1), (t, \overline{z}^2) \in [t_0, \omega[\, \times \mathbb{R}^n_{\frac{1}{2}}$.

Choosing the number t_0 by this method, let $\mathbf{B}$ denote the Banach space of vector-functions $z = (z_i)_{i=1}^n$; each its component, z_i $(i \in \{1, \dots, n\})$, is defined, continuous and bounded on the set $D = [t_0, \omega[\, \times \mathbb{R}^n_{\frac{1}{2}}$, with the norm

$$\|z\| = \sup \left\{ \sum_{i=1}^{n} |z_i(t, \overline{v})| : \ (t, \overline{v}) \in D \ (i = \overline{1, n}) \right\}.$$

Let us select from this space the subspace $\mathbf{B}_0$ of the functions from $\mathbf{B}$ with the property $\|z\| \le \frac{1}{2}$, and consider its elements, arbitrarily choosing the number $\nu \in (0,1)$ and the operator $\Phi = (\Phi_i)_{i=1}^n$, defined by the relations

$$\Phi_i(z)(t, \overline{v}) = z_i(t, \overline{v}) -$$

$$- \nu \left[z_i(t, \overline{v}) - a_i(t) - b_i(t, \overline{v}) - Z_i(t, z_1(t, \overline{v}), \dots z_n(t, \overline{v})) \right] \ \ (i = \overline{1, n}), \tag{2.25}$$

For each $z \in \mathbf{B}_0$, by the conditions (2.23) we get

$$|\Phi_i(z)(t, \overline{v})| \le (1 - \nu)|z_i(t, \overline{v})| + \frac{\nu}{2n} \ \ (i = \overline{1, n}) \ \text{ as } \ (t, \overline{v}) \in D.$$

Therefore, if $(t, \overline{v}) \in D$,

$$\sum_{i=1}^{n} |\Phi_i(z)(t,\overline{v})| \le (1-\nu)\sum_{i=1}^{n} |z_i(t,\overline{v})| + \frac{1}{2}\nu \le$$
$$\le (1-\nu)\|z\| + \frac{1}{2}\,\nu \le (1-\nu)\,\frac{1}{2} + \nu\,\frac{1}{2} = \frac{1}{2}\,.$$

This yields that $\|\Phi(z)\| \le \frac{1}{2}$, i.e., $\Phi(\mathbf{B}_0) \subset \mathbf{B}_0$.

Suppose $z, \widetilde{z} \in \mathbf{B}_0$. Then, from (2.24), if $(t, \overline{v}) \in D$,

$$\Big|\Phi_i(z)(t,\overline{v}) - \Phi_i(\widetilde{z})(t,\overline{v})\Big| \le (1-\nu)|z_i(t,\overline{v}) - \widetilde{z}_i(t,\overline{v})| +$$
$$+\,\nu\Big|Z_i(t, z_1(t,\overline{v}),\dots,z_n(t,\overline{v}) - Z_i(t,\widetilde{z}_1(t,\overline{v}),\dots,\widetilde{z}_n(t,\overline{v})\Big| \le$$
$$\le (1-\nu)|z_i(t,\overline{v}_i) - \widetilde{z}_i(t,\overline{v}_i)| + \frac{\nu}{n+1}\sum_{k=1}^{n} |z_k(t,\overline{v}) - \widetilde{z}_k(t,\overline{v})| \ \ (i=\overline{1,n}).$$

Thus, if $(t, \overline{v}) \in D$ $(i=\overline{1,n})$,

$$\sum_{i=1}^{n} |\Phi_i(z)(t,\overline{v}_i) - \Phi_i(\widetilde{z})(t,\overline{v}_i)| \le$$
$$\le \left(1 - \frac{\nu}{n+1}\right)\sum_{i=1}^{n} |z_i(t,\overline{v}) - \widetilde{z}_i(t,\overline{v})| \le \left(1 - \frac{\nu}{n+1}\right)\|z - \widetilde{z}\|,$$

consequently,

$$\|\Phi(z) - \Phi(\widetilde{z})\| \le \left(1 - \frac{\nu}{n+1}\right)\|z - \widetilde{z}\|.$$

Thus, the operator Φ maps the space $\mathbf{B}_0$ into itself and is a contraction operator on this space. Then, according to the contraction mapping principle, there exists a unique vector-function $z \in \mathbf{B}_0$ such that $z = \Phi(z)$. By (2.25), this vector-function with continuous components $z_i : D \to \mathbb{R}$ $(i = \overline{1,n})$ is the only solution of the system (2.18) that satisfies the conditions $\|z\| \le \frac{1}{2}$. From (2.18) together with the above condition, and from (2.19)–(2.21) it follows that the components $z_i(t,\overline{v})$ $(i = \overline{1,n})$ of this solution tend to zero when $t \uparrow \omega$ uniformly over $\overline{v} \in \mathbb{R}^n_{\frac{1}{2}}$. Continuous differentiability of these components on some set $[t_1, \omega[\times\mathbb{R}^n_{\frac{1}{2}}$, where $t_1 \in [t_0, \omega[$, follows immediately from the well-known local theorem about the existence of implicit functions defined by the system of relations. According to the transformation (2.17), the obtained vector-function $z = (z_i)_{i=1}^n$ corresponds to the continuously differentiable vector-function $(Y_i)_{i=1}^n : [t_1, \omega[\,\times\mathbb{R}^n_{\frac{1}{2}}$ with components of the type (2.16). This vector-function is a solution of the system (2.15). Moreover, according to (2.16) and the sign conditions (2.3), (2.4),

$$\lim_{t\uparrow\omega} Y_i(t,\overline{v}) = Y_i^0 \ \text{ uniformly over } \ \overline{v} \in \mathbb{R}^n_{\frac{1}{2}} \ \ (i=\overline{1,n}). \tag{2.26}$$

Moreover, from (2.15) it follows

$$\begin{cases} \dfrac{(Y_i(t,\overline{v}))'_t}{Y_i(t,\overline{v})} = \dfrac{O'_i(t)}{Q_i(t)} + \dfrac{Y_{i+1}(t,\overline{v})\varphi'_{i+1}(Y_{i+1}(t,\overline{v}))}{\varphi_{i+1}(Y_{i+1}(t,\overline{v}))}\,\dfrac{(Y_{i+1}(t,\overline{v}))'_t}{Y_{i+1}(t,\overline{v})} \\ \qquad\qquad\qquad\qquad\qquad\qquad (i=\overline{1,n-1}), \\ \dfrac{(Y_n(t,\overline{v}))'_t}{Y_n(t,\overline{v})} = \dfrac{O'_n(t)}{Q_n(t)} + \\ \quad +\Big(\prod\limits_{k=1}^{r}\sigma_k\Big)\dfrac{Y_{r+1}(t,\overline{v})\varphi'_{r+1}(Y_{r+1}(t,\overline{v}))}{\varphi_{r+1}(Y_{r+1}(t,\overline{v}))}\,\dfrac{(Y_{r+1}(t,\overline{v}))'_t}{Y_{r+1}(t,\overline{v})}\,. \end{cases} \tag{2.27}$$

Here by virtue of (2.26) and (1.2),

$$\lim_{t\uparrow\omega}\frac{Y_i(t,\overline{v}_i)\varphi'_i\left(Y_i(t,\overline{v}_i)\right)}{\varphi_i\left(Y_i(t,\overline{v}_i)\right)} = \sigma_i \ \ (i=\overline{1,n}) \ \text{ uniformly over } \ \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}, \tag{2.28}$$

and according to the form of the functions Q_i $(i=\overline{1,n})$,

$$\frac{Q'_i(t)}{Q_i(t)} = \begin{cases} \dfrac{I'_i(t)}{I_i(t)} & \text{as } \ i\in\mathfrak{I}\cup\{n\}, \\ \dfrac{I'_i(t)}{I_i(t)} - \dfrac{I'_{i+1}(t)}{I_{i+1}(t)} & \text{as } \ i\in\overline{\mathfrak{I}}. \end{cases} \tag{2.29}$$

First, from (2.27) we obtain

$$\frac{(Y_{r+1}(t,\overline{v}))'_t}{Y_{r+1}(t,\overline{v})} = \left[1-\Big(\prod_{k=1}^{r}\sigma_k\Big)\prod_{k=r+1}^{n}\frac{Y_k(t,\overline{v})\varphi'_k\left(Y_k(t,\overline{v})\right)}{\varphi_k\left(Y_k(t,\overline{v})\right)}\right]^{-1}\times$$
$$\times\Big(\sum_{k=r+1}^{n}\frac{Q'_k(t)}{Q_k(t)}\prod_{j=r+1}^{k-1}\frac{Y_{j+1}(t,\overline{v})\varphi'_{j+1}\left(Y_{j+1}(t,\overline{v})\right)}{\varphi_j\left(Y_j(t,\overline{v})\right)}\Big).$$

Hence, according to (2.28), (2.29) and (2.2), we get

$$\lim_{t\uparrow\omega}\frac{I_{r+1}(t)\left(Y_{r+1}(t,\overline{v})\right)'_t}{I'_{r+1}(t)Y_{r+1}(t,\overline{v})} = \frac{1}{\beta_{r+1}} \quad \text{uniformly over} \quad \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}.$$

Further, by virtue of this limiting condition, from (2.27), consistently, starting from $i=n$ to $i=r+2$, and then, starting from $i=r$ to $i=1$, we get, (using (2.28), (2.29), (2.2))

$$\lim_{t\uparrow\omega}\frac{I_i(t)\left(Y_i(t,\overline{v})\right)'_t}{I'_i(t)Y_i(t,\overline{v})} = \frac{1}{\beta_i} \ \text{ uniformly over } \ \overline{v}\in\mathbb{R}^n_{\frac{1}{2}}. \tag{2.30}$$

Applying now to the system of differential equations (1.1) the transformation

$$y_i(t) = Y_i(t,\overline{v}_i(t)) \ \ (i=\overline{1,n}) \tag{2.31}$$

and taking into consideration that the vector-function $(Y_i(t,\overline{v}(t)))_{i=1}^n$ with $t \in [t_1,\omega[$ and $\overline{v}(t) \in \mathbb{R}^n_{\frac{1}{2}}$ is a solution of the system

$$\begin{cases} \dfrac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = Q_i(t)[1+v_i(t)] & (i=\overline{1,n-1}), \\ \dfrac{y_n(t)}{[\varphi_{r+1}(y_{r+1}(t))]^{\prod\limits_{k=1}^{r}\sigma_k}} = Q_n(t)[1+v_n(t)], \end{cases} \tag{2.32}$$

we obtain the system of differential equations of the type

$$\begin{cases} v_i' = \dfrac{I_i'(t)}{\beta_i I_i(t)} - \dfrac{Q_i'(t)}{Q_i(t)}\,(1+v_i) - \\ \quad -\dfrac{I_{i+1}'(t)}{\beta_{i+1} I_{i+1}(t)} \cdot \dfrac{1+v_i}{1+v_{i+1}}\, H_{i+1}(t,\overline{v}) \ \ (i=\overline{1,n-2}), \\ v_{n-1}' = \dfrac{I_{n-1}'(t)}{\beta_{n-1} I_{n-1}(t)} - \dfrac{Q_{n-1}'(t)}{Q_{n-1}(t)}\,(1+v_{n-1}) - \\ \quad -\dfrac{1+v_{n-1}}{1+v_n}\, H_n(t,\overline{v})\, \dfrac{H(t,\overline{v})}{Q_n(t)}\,, \\ v_n' = \dfrac{H(t,\overline{v})}{Q_n(t)} - \dfrac{Q_n'(t)}{Q_n(t)}\,(1+v_n) - \\ \quad -\Big(\prod\limits_{k=1}^{r}\sigma_k\Big)\dfrac{1+v_n}{1+v_{r+1}}\, H_{r+1}(t,\overline{v})\, \dfrac{I_{r+1}'(t)}{\beta_{r+1} I_{r+1}(t)}\,, \end{cases} \tag{2.33}$$

where

$$H_i(t,\overline{v}) = \frac{Y_i(t,\overline{v})\varphi_i'(Y_i(t,\overline{v}))}{\varphi_i(Y_i(t,\overline{v}))} \ \ (i=\overline{1,n}),$$
$$H(t,\overline{v}_1) = \frac{\alpha_n p_n(t)\varphi_1(Y_1(t,\overline{v}))}{[\varphi_{r+1}(Y_{r+1}(t,\overline{v}))]^{\prod\limits_{k=1}^{r}\sigma_k}}\,.$$

Since the conditions (2.28), (2.30) are valid and the functions φ_i $(i = 1,\dots,r)$ satisfy the **S** - condition, by virtue of Remark (2.2), we obtain

$$H_i(t,\overline{v}) = \sigma_i + R_i(t,\overline{v}) \ \ (i=\overline{1,n}),$$
$$H(t,\overline{v}) = \alpha_n p_n(t) q_{r+1}(t) \prod_{k=1}^{r} |1+v_k|^{\prod_{j=1}^{k}\sigma_j}[1+R(t,\overline{v})],$$

where

$$\lim_{t\uparrow\omega} R_i(t,\overline{v}) = 0 \ \text{ uniformly over } \ \overline{v} \in \mathbb{R}^n_{\frac{1}{2}} \ \ (i=\overline{1,n}),$$
$$\lim_{t\uparrow\omega} R(t,\overline{v}) = 0 \ \text{ uniformly over } \ \overline{v} \in \mathbb{R}^n_{\frac{1}{2}}.$$

By virtue of these representations and the conditions (2.2), the system (2.33) can be rewritten in the form

$$\begin{cases} v_i' = h_i(t)\,[f_i(t,\overline{v}) - v_i + \Lambda_i\sigma_{i+1}v_{i+1} + V_i(\overline{v})] \quad (i=\overline{1,n-2}), \\ v_{n-1}' = h_{n-1}(t)\left[f_{n-1}(t,\overline{v}) - \sum\limits_{k=1}^{r} a_{0k}v_k - v_{n-1} + v_n + V_{n-1}(\overline{v})\right], \\ v_n' = h_n(t)\left[f_n(t,\overline{v}) + \sum\limits_{k=1}^{r} a_{0k}v_k + a_{0n}v_{r+1} - v_n + V_n(\overline{v})\right], \end{cases} \tag{2.34}$$

where

$$h_i(t) = \frac{I_i'(t)}{\beta_i I_i(t)} \quad (i=\overline{1,n}),$$

$$a_{0k} = \prod_{j=1}^{k} \sigma_j \quad (k=\overline{1,n}),$$

$$\lim_{t\uparrow\omega} f_i(t,\overline{v}) = 0 \text{ uniformly over } \overline{v}\in\mathbb{R}^n_{\frac{1}{2}} \quad (i=\overline{1,n}),$$

$$V_i(\overline{v}) = -\Lambda_i\sigma_{i+1}\left[\frac{1+v_i}{1+v_{i+1}} - 1 - v_i + v_{i+1}\right] \quad (i=\overline{1,n-2}),$$

$$V_{n-1}(\overline{v}) = -\left[\frac{1+v_{n-1}}{1+v_n}\prod_{k=1}^{r}|1+v_k|^{a_{0k}} - 1 - \sum_{k=1}^{r} a_{0k}v_k - v_{n-1} + v_n\right],$$

$$V_n(\overline{v}) = \prod_{k=1}^{r}|1+v_k|^{a_{0k}} - a_{0n}\frac{1+v_n}{1+v_{r+1}} - 1 + a_{0n} - \\ - \sum_{k=1}^{r} a_{0k}v_k - a_{on}v_{r+1} + a_{0n}v_n.$$

Here

$$\lim_{|v_1|+\cdots+|v_n|\to 0} \frac{\partial V_i(\overline{v})}{\partial v_k} = 0 \quad (i,k=\overline{1,n}),$$

and, taking into consideration that $\lim\limits_{t\uparrow\omega} I_i(t)$ $(i=\overline{1,n})$ is equal either to zero, or to $\pm\infty$, the following conditions are satisfied:

$$\int_{t_1}^{\omega} h_i(t)\,dt = \pm\infty \quad (i=\overline{1,n}).$$

Since $m = \max\{i \in \Im : \ \Lambda_i = 0\} < n-1$ and the conditions (2.2) are valid, when $i = \overline{m+1, n-1}$, we have

$$h_i(t) = h_n(t)\frac{h_i(t)}{h_n(t)} = h_n(t)\frac{\beta_n I_n(t) I_i'(t)}{\beta_i I_i(t) I_n'(t)} =$$
$$= h_n(t)\frac{\beta_{i+1} I_i'(t) I_{i+1}(t)}{\beta_i I_i(t) I_{i+1}'(t)}\,\frac{\beta_{i+2} I_{i+1}'(t) I_{i+2}(t)}{\beta_{i+1} I_{i+1}(t) I_{i+2}'(t)}\cdots\frac{\beta_n I_{n-1}'(t) I_n(t)}{\beta_{n-1} I_{n-1}(t) I_n'(t)} =$$
$$= \frac{h_n(t)[1+o(1)]}{\Lambda_i \Lambda_{i+1}\dots\Lambda_{n-1}} \ \text{ as } \ t \uparrow \omega.$$

Therefore, the system (2.34) can be rewritten in the form

$$\begin{cases} v_i' = h_i(t)\Big[f_i(t,\overline{v}) - v_i + \Lambda_i \sigma_{i+1} v_{i+1} + V_i(\overline{v})\Big] \ \ (i = \overline{1, m-1}), \\ v_m' = h_m(t)\big[f_m(t,\overline{v}) - v_m\big], \\ v_i' = h_n(t)\Bigg[\widetilde{f}_i(t,\overline{v}) - \dfrac{v_i}{\Lambda_i\cdots\Lambda_{n-1}} + \\ \quad + \dfrac{\sigma_{i+1}}{\Lambda_{i+1}\cdots\Lambda_{n-1}}\, v_{i+1} + \dfrac{V_i(\overline{v})}{\Lambda_i\cdots\Lambda_{n-1}}\Bigg] \ \ (i = \overline{m+1, n-2}), \\ v_{n-1}' = h_n(t)\Bigg[\overline{f}_{n-1}(t,\overline{v}) - \sigma_n \displaystyle\sum_{k=1}^{r} a_{0k} v_k - \sigma_n v_{n-1} + \\ \qquad\qquad + \sigma_n v_n + \sigma_n V_{n-1}(\overline{v})\Bigg], \\ v_n' = h_n(t)\left[f_n(t,\overline{v}) + \displaystyle\sum_{k=1}^{r} a_{0k} v_k + a_{0n} v_{r+1} - v_n + V_n(\overline{v})\right], \end{cases} \tag{2.35}$$

where the functions $\overline{f}_i$ $(i = \overline{m+1, n-1})$ have the same properties as the functions f_i $(i = \overline{m+1, n-1})$ in the system (2.34).

The important peculiarity of the system is that the coefficient at v_{m+1} is equal to zero.

Suppose that B_{m+1} is a constant matrix of order $(n-m) \times (n-m)$. This matrix consists of the coefficients at $v_{m+1}, \dots, v_n$ in the last standing in brakets $n-m$ equations of the system (2.35). Its characteristic equation is $\det[B_{m+1} - \lambda E_{n-m}] = 0$, where E_{n-m} is the unit matrix of order $(n-m) \times (n-m)$ and is represented by (2.1). Taking into consideration the conditions of the theorem, it is evident that this equation has no roots with zero real part. Therefore, using the proof of Theorem 2.1 in [8], we conclude that there exists a nonsingular constant matrix D_{m+1} of order $(n-m) \times (n-m)$ and there exists a nonsingular continuously differentiable and bounded (together with its inverse matrix) on the interval $[t_0, \omega[$ matrix

$L_{m+1}(t)$ such that

$$L_{m+1}^{-1}(t)D_{m+1}^{-1}B_{m+1}D_{m+1}L_{m+1}(t)-\frac{1}{h_n(t)}\,L^{-1}(t)L'(t)=C_{m+1},$$

where C_{m+1} is the upper triangular matrix of the form

$$C_{m+1}=\begin{pmatrix} \operatorname{Re}\lambda_1^0 & c_{m+1m+2} & \dots & c_{m+1n-1} & c_{m+1n} \\ 0 & \operatorname{Re}\lambda_2^0 & \dots & c_{m+2n-1} & c_{m+2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \operatorname{Re}\lambda_{n-m-1}^0 & c_{n-1n} \\ 0 & 0 & \dots & 0 & \operatorname{Re}\lambda_{n-m}^0 \end{pmatrix},$$

where λ_i^0 $(i=\overline{1,n-m})$ are all roots (with multipliciting) of the algebraic equation (2.1), all c_{ik} $(k=\overline{i+1,n})$ as $i\in\{m+1,\dots,n\}$ are equal to zero, except for a single one that equals 1.

In virtue of this fact, by means of the transformation

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}=\begin{pmatrix} E_m & O_1 \\ O_2 & D_{m+1}L_{m+1}(t) \end{pmatrix}\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \tag{2.36}$$

where O_1, O_2 are zero-matrices of orders $m\times(n-m)$ and $(n-m)\times m$ (respectively), E_m is the unit matrix of order $m\times m$, the system of differential equations (2.35) takes the form

$$\begin{cases} w_i'=h_i(t)\,[f_{1i}(t,\overline{w})-w_i+\Lambda_i\sigma_{i+1}w_{i+1}+f_{2i}(\overline{w})] \quad (i=\overline{1,m-1}), \\ w_m'=h_m(t)[f_{1m}(t,\overline{w})-w_m], \\ w_i'=h_n(t)\Big[f_{1i}(t,\overline{w})+\sum\limits_{k=1}^{m}c_{ik}(t)w_k+\big(\operatorname{Re}\lambda_{i-m}^0\big)\,w_i+ \\ \qquad\qquad +\sum\limits_{k=i+1}^{n}c_{ik}w_k+f_{2i}(t,\overline{w})\Big] \quad (i=\overline{m+1,n-1}), \\ w_n'=h_n(t)\Big[f_{1n}(t,\overline{w})+\sum\limits_{k=1}^{m}c_{nk}(t)w_k+\big(\operatorname{Re}\lambda_{n-m}^0\big)\,w_n+f_{2n}(t,\overline{w})\Big], \end{cases} \tag{2.37}$$

where the functions c_{ik} $(i=\overline{m+1,n},\ k\in\{1,\dots,m\})$ are continuous and bounded on the interval $[t_1,\omega[$, the functions $f_{1i}:[t_1,\omega[\,\times\mathbb{R}_\delta^n\to\mathbb{R}$ $(i=\overline{1,n})$, $f_{2i}:\mathbb{R}_\delta^n\to\mathbb{R}$ $(i=\overline{1,m-1})$, the functions $f_{2i}:[t_1,\omega[\,\times\mathbb{R}_\delta^n\to\mathbb{R}$ $(i=\overline{m+1,n})$ are continuous, where $\mathbb{R}_\delta^k=\{(x_1,\dots,x_k)\in\mathbb{R}^k:\ |x_j|\le\delta\}$,

$\delta > 0$ is sufficiently small and satisfy the conditions

$$\lim_{t\uparrow\omega} f_{1i}(t,\overline{w}) = 0 \ \ (i=\overline{1,m}) \ \text{ uniformly over } \ \overline{w}\in\mathbb{R}^n_\delta,$$

$$\lim_{|w_1|+\cdots+|w_{i+1}|\to 0} \frac{f_{2i}(\overline{w})}{|w_1‘+\cdots+|w_n|} = 0 \ \ (i=\overline{1,m-1}),$$

$$\lim_{|w_1|+\cdots+|w_n|\to 0} \frac{f_{2i}(t,\overline{w})}{|w_1|+\cdots+|w_n|} = 0 \ \ (i=\overline{m+1,n})$$

$$\text{uniformly over } \ t\in[t_1,\omega[\,.$$

Since the functions c_{ik} $(i=\overline{m+1,n},\ k\in\{1,\dots,m\})$ are bounded on the interval $[t_1,\omega[$, there exists a number $\varepsilon>0$ such that the constants B_i^0 $(i=\overline{m+1,n})$, defined (starting with $i=n$) by the recurrent relations

$$B_n^0 = \frac{\varepsilon}{|\mathrm{Re}\lambda^0_{n-m}|}\sum_{k=1}^m c^0_{nk},$$

$$B_i^0 = \frac{1}{|\mathrm{Re}\,\lambda^0_{i-m}|}\left(\varepsilon\sum_{k=1}^m c^0_{ik} + \sum_{i+1}^n |c_{ik}|B_k^0\right) \ \ (i=\overline{m+1,n-1}),$$

where

$$c^0_{ik} = \limsup_{t\uparrow\omega}|c_{ik}(t)| \ \ (i=\overline{m+1,n}, \ \ k\in\{1,\dots,m\}),$$

satisfy the inequalities $B_i^0<1$ $(i=\overline{m+1,n})$.

With this choice of the constant $\varepsilon>0$, the system (2.37) by means of the transformation

$$w_i = \varepsilon z_i \ \ (i=\overline{1,m}), \quad w_i = z_i \ \ (i=\overline{m+1,n}) \tag{2.38}$$

is reduced to a system of differential equations that satisfies all the conditions of Theorem 1.2 in [7]. According to this theorem, this system has at least one solution $(z_i)_{i=1}^n : [t_2,\omega[\,\mathbb{R}^n$ $(t_2\in[t_1,\omega[)$, which tends to zero when $t\uparrow\omega$. Moreover, there exists the whole k-parametric family of solutions, if there are k positive numbers among the numbers (2.7). In virtue of the transformations (2.38), (2.36) and (2.31), each of these solutions corresponds to the solution of the system (1.1), satisfying (as $t\uparrow\omega$) the asymptotic representations (2.5), (2.6). Furthermore, taking into consideration the form of functions (2.31) and conditions (2.2)–(2.4), it is easy to see that all these solutions are the $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions of the system (1.1). Thus the theorem is proved. □

Consider now the conditions that give an opportunity to rewrite the asymptotic representations (2.5), (2.6) in an explicit form.

Theorem 2.2. *Let* $\Lambda_i\in\mathbb{R}$ $(i=\overline{1,n-1})$ *include those equal zero,* $m=\max\{i\in\mathfrak{I}:\ \Lambda_i=0\}$ *and* $r=\max\mathfrak{I}<n-1$. *Moreover, let all the functions* φ_k $(k=\overline{1,n})$ *satisfy the* **S**-*condition. Then each* $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-*solution (in case it exists) of the system* (1.1) *admits for* $t\uparrow\omega$ *asymptotic*

representations

$$y_{r+1}(t) = \mu_{r+1} \prod_{k=r+1}^{n-1} \left| q_k(t)\theta_{k+1}\left(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}}\right)\right|^{\frac{\prod\limits_{j=r+2}^{k}\sigma_j}{1-\prod\limits_{j=1}^{n}\sigma_j}} \times$$

$$\times \left| Q_n(t)\left[\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}}\right)\right]^{\prod\limits_{j=1}^{r}\sigma_j}\right|^{\frac{\prod\limits_{j=r+2}^{n}\sigma_j}{1-\prod\limits_{j=1}^{n}\sigma_j}} [1+o(1)],$$

$$y_i(t) = \mu_i \prod_{k=i}^{r} \left| Q_k(t)\theta_{k+1}\left(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}}\right)\right|^{\prod\limits_{j=i+1}^{k}\sigma_j} \times$$

$$\times |y_{r+1}(t)|^{\prod\limits_{j=i+1}^{r+1}\sigma_j} [1+o(1)] \quad (i=\overline{1,r}), \tag{2.39}$$

$$y_i(t) = \mu_i \prod_{k=i}^{n-1} \left| Q_k(t)\theta_{k+1}\left(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}}\right)\right|^{\prod\limits_{j=i+1}^{k}\sigma_j} \times$$

$$\times \left| Q_n(t)\left[\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}}\right)\right]^{\prod\limits_{j=1}^{r}\sigma_j}\right|^{\prod\limits_{j=i+1}^{n}\sigma_j} \times$$

$$\times |y_{r+1}(t)|^{\prod\limits_{j=1}^{r+1}\sigma_j \prod\limits_{j=i+1}^{n}\sigma_j} [1+o(1)] \quad (i=\overline{r+2,n}).$$

Proof. In Theorem 2.1, it is proved, that for the existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions in (1.1), it is necessary that the conditions (2.2)–(2.4) valid, and each solution of that type admit for $t \uparrow \omega$ the asymptotic representations (2.5), (2.6). Moreover, the asymptotic representation (2.11) for these solutions was obtained. Since all functions φ_i $(i=\overline{1,n})$ satisfy the **S**-condition, in virtue of (2.11) and Remark 1.2, we get

$$\theta_i(y_i(t)) = \theta_i\left(\mu_i\,|I_i(t)|^{\frac{1}{\beta_i}}\right)[1+o(1)] \quad (i=\overline{1,n}) \ \text{ as } \ t\uparrow\omega.$$

That is why the asymptotic representations (2.5), (2.6) can be rewritten in the form

$$\frac{y_i(t)}{|y_{i+1}(t)|^{\sigma_{i+1}}} =$$

$$= Q_i(t)\theta_{i+1}\left(\mu_{i+1}\,|I_{i+1}(t)|^{\frac{1}{\beta_{i+1}}}\right)[1+o(1)] \quad (i=\overline{1,n-1}) \ \text{ as } \ t\uparrow\omega,$$

$$\frac{y_n(t)}{|y_{r+1}(t)|^{\prod\limits_{j=1}^{r+1}\sigma_j}} =$$

$$= Q_n(t)\left[\theta_{r+1}\left(\mu_{r+1}|I_{r+1}(t)|^{\frac{1}{\beta_{r+1}}}\right)\right]^{\prod\limits_{j=1}^{r}\sigma_j}[1+o(1)] \text{ as } t\uparrow\omega.$$

Hence, consistently, starting with $i = n$, we obtain the asymptotic representations (2.39). The theorem is proved. □

3. Conclusions

In this paper, for cyclic system (1.1) with regularly varying non-linearities, the class of the so-called $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions is introduced and the question of the existence of such solutions in special case (when $\Lambda_i \in \mathbb{R}$ $(i=\overline{1,n-1})$ include zeroes) is discovered. Peculiarity of this case demands both the validity of the additional **S** -condition for all nonlinearities of the system, except one, and the assumption that $\Lambda_{n-1} \in \mathfrak{I}$. As a result, the necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions for (1.1) are obtained. Implicit asymptotic formulas for components of these solutions (when $t\uparrow\omega$ $(\omega\le+\infty)$) are established. Explicit asymptotic formulas for components of these solutions are established, provided all nonlinearities satisfy the **S**-condition.

The results may be used, for instance, to establish the asymptotics of solutions for sufficiently nonlinear differential equations of the type

$$y'' = p(t)\varphi_1(y)\varphi_2(y') \text{ and } y^{(n)} = p(t)\varphi(y),$$

where $p : [a,\omega[\,\to \mathbb{R}\setminus\{0\}$ is a continuous function and $\varphi,\varphi_1 : \Delta(Y_0^0)\to\,]0,+\infty[$, $\varphi_2 : \Delta(Y_1^0)\to\,]0,+\infty[$, $\Delta(Y_i^0)$ is a one-sided neighborhood, Y_i^0 are continuously differentiable and regularly varying functions of certain orders (when $y\to Y_0^0$ and $y'\to Y_1^0$).

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