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R. Duduchava

A REVISED ASYMPTOTIC MODEL OF A SHELL

Dedicated to Professor N. Muskhelishvili
on his 120-th birthday anniversary

Abstract. Asymptotic model of a shell (Koiter, Sanchez-Palencia, Ciarlet etc.) is revised based on the calculus of tangent Gunter's derivatives, developed in the recent papers of the author with D. Mitrea and M. Mitrea [12]–[14], [16]. As a result the 2-dimensional shell equation on a middle surface $\mathscr{S}$ is written in terms of Gunter's derivatives, unit normal vector field and the Lamé constant, which coincides with the Lamé equation on the hypersurface $\mathscr{S}$, investigated in [12]–[14], [16].

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რეზიუმე. გარსის ასიმპტოტური მოდელის (კოიტერი, სანჩეს-პალენსია, სიარლე) წინამდებარე მოდიფიკაცია (გადახედვა) ეფუძნება გიუნტერის მხები დიფერენციალური ოპერატორების აღრიცხვას, რომელიც დამუშავდა ნაშრომის ავტორის დ. მიტრეა და მ. მიტრეას ბოლო ათწლეულის განმავლობაში გამოქვეყნებულ ნაშრომებში [12]–[14], [16]. ამის შედეგად გარსის 2-განზომილებიანი განტოლება ჩაწერილია გარსის შუა ზედაპირზე $\mathscr{S}$ გიუნტერის წარმოებულების, ერთეულოვანი ნორმალური ვექტორული ველისა და ლამეს კონსტანტების მეშვეობით და გარეგნულად ემთხვევა ლამეს განტოლებას დრეკადი ჰიპერზედაპირებისათვის $\mathscr{S}$, რომელიც იყო გამოყვანილი და შესწავლილი ნაშრომებში [12]–[14], [16].

Introduction

The purpose of the present investigation is to develop a proper mathematical tools for a two-dimensional theory of shells. There exist a number of approaches proposed for modeling linearly elastic shells. Started by the Cosserats pioneering work (1909), Goldenveiser (1961), Naghdi (1963), Vekua (1965), Novozhilov (1970), Koiter (1970) and many others contributed the development of the shell theories. Ellipticity of the corresponding partial differential equations was proved later by Roug'e (1969) for cylindrical shells, by Coutris (1973) for the shell model proposed by Naghdi, Gordeziani (1974) for the shell model proposed by Vekua, Shoikhet (1974) for the shell model proposed by Novozhilov, Ciarlet & Miara for the model proposed by Koiter (cf. [1]–[4], [10], [28]–[30] for surveys and further references).

Shell configuration consists of all points in the distance less or equal ε from a middle surface $\mathscr{S}$ given by a local immersion

$$\Theta : \omega \longrightarrow \mathscr{S}, \quad \omega \subset \mathbb{R}^{n-1}. \tag{1}$$

In particular,

$$\Omega^\varepsilon := \Big\{ \mathscr{X}_t \in \mathbb{R}^n : \ \mathscr{X}_t = \mathscr{X} + t\boldsymbol{\nu}(\mathscr{X}) = \Theta(x) + t\boldsymbol{\nu}\big(\Theta(x)\big), \\ x \in \omega, \ -\varepsilon < t < \varepsilon \Big\}, \tag{2}$$

where $\boldsymbol{\nu}(\mathscr{X}) = \boldsymbol{\nu}(\Theta(y))$ for $\mathscr{X} = \Theta(y) \in \mathscr{S}$, is the outer unit normal vector field. We look for the displacement vector field $\boldsymbol{U} = \sum\limits_{j=1}^{3} U_j \boldsymbol{e}^j$, represented in the natural basis $\boldsymbol{e}^1 = (1,0,0)^\top$, $\boldsymbol{e}^2 = (0,1,0)^\top$, $\boldsymbol{e}^3 = (0,0,1)^\top$ of the ambient Euclidean space $\mathbb{R}^3$. The approach applies the Gunter's tangential derivatives (cf. [17], [20], [11])

$$\mathscr{D}_j := \partial_j - \nu_j(\mathscr{X})\partial_{\boldsymbol{\nu}} = \partial_{\boldsymbol{d}^j}, \quad \boldsymbol{d}^j := \pi_{\mathscr{S}} \boldsymbol{e}^j, \ \ j = 1,2,3. \tag{3}$$

where $\partial_{\boldsymbol{\nu}} := \sum\limits_{j=1}^{n} \nu_j \partial_j$ denotes the normal derivative and

$$\pi_{\mathscr{S}} : \mathbb{R}^3 \longrightarrow \mathbb{T}\mathscr{S}, \quad \pi_{\mathscr{S}} \boldsymbol{U} := \boldsymbol{U} - \langle \nu, \boldsymbol{U} \rangle \nu, \ \ t \in \mathscr{S} \tag{4}$$

defines the canonical orthogonal projection onto the space of tangential vector fields $\mathbb{T}\mathscr{S}$ to the hypersurface. $\mathscr{D}_j^{\mathscr{S}} := \pi_{\mathscr{S}} \mathscr{D}_j$ denotes the covariant Gunter's derivatives.

The following form of the deformation (strain) tensor of the hypersurface $\mathscr{S}$ was identified in [16]

$$\operatorname{Def}_{\mathscr{S}}(\boldsymbol{U}) = \big[\mathfrak{D}_{jk}(\boldsymbol{U})\big]_{3\times 3}, \quad \boldsymbol{U} = \sum_{j=1}^{n} U_j^0 \boldsymbol{d}^j \in \mathbb{T}\mathscr{S}, \tag{5}$$

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \frac{1}{2}\left[(\mathscr{D}_j^{\mathscr{S}} \boldsymbol{U})_k^0 + (\mathscr{D}_k^{\mathscr{S}} \boldsymbol{U})_j^0 \right], \ \ j,k = 1,2,3,$$

where $(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0 := \langle \mathscr{D}_j^{\mathscr{S}}\boldsymbol{U}, \boldsymbol{e}^k\rangle$. It coincides with the linearized change of metric tensor on the surface $\mathscr{S}$ (see Theorem 5.5 below).

Considering the thickness as a "small" parameter, we apply a standard asymptotic analysis (see [1], [5], [18], [31], [32]) and derive two dimensional equation of a shell. The unknown is the displacement vector field

$$\boldsymbol{U}(\mathscr{X},t) = \sum_{j=1}^{3} U_j^0(\mathscr{X},t)\boldsymbol{e}^j = \sum_{j=1}^{3} U_j^0(\mathscr{X},t)\boldsymbol{d}^j(\mathscr{X}) + U_4^0(\mathscr{X},t)\boldsymbol{d}_4(\mathscr{X}),$$
$$\boldsymbol{d}^4(\mathscr{X}) := \boldsymbol{\nu}(\mathscr{X}), \quad U_4^0(\mathscr{X},t) := \langle \boldsymbol{U}(\mathscr{X},t), \boldsymbol{\nu}(\mathscr{X})\rangle, \ \ (\mathscr{X},t) \in \Omega^\varepsilon$$

of the elastic media Ω^ε, written on the local variables $\mathscr{X} \in \mathscr{S}$, $-\varepsilon < t < \varepsilon$. We start from the linearized equation in the variational form, governing the deformation of the shell

$$\begin{gathered}\int\limits_{\Omega^\varepsilon} \mathscr{E}^{jklm,\varepsilon}\boldsymbol{D}_{lm}(\boldsymbol{U}^\varepsilon)\boldsymbol{D}_{jk}(\boldsymbol{V}^\varepsilon)\, d\mathscr{X} = \\ = \int\limits_{\Omega^\varepsilon} \langle \boldsymbol{F}^\varepsilon, \boldsymbol{V}^\varepsilon\rangle\, d\mathscr{X} + \int\limits_{\mathscr{S}_-^\varepsilon\cup\mathscr{S}_+^\varepsilon} \langle \boldsymbol{H}^\varepsilon, \boldsymbol{V}^\varepsilon\rangle\, d\sigma, \\ \boldsymbol{U}^\varepsilon,\ \boldsymbol{V}^\varepsilon \in \mathbb{W}^1(\Omega^\varepsilon,\Gamma_0^\varepsilon) := \big\{\boldsymbol{X} \in \mathbb{W}^1(\Omega^\varepsilon):\ \boldsymbol{X}(\mathscr{X}) = 0 \text{ on } \Gamma_0^\varepsilon\big\}.\end{gathered} \tag{6}$$

Here $\boldsymbol{D}_{jk}(\boldsymbol{X}) = \frac{1}{2}(\partial_j X_k + \partial_k X_j)$ are the components of classical deformation tensor, $\mathscr{S}_\pm^\varepsilon = \mathscr{S} \times \{\pm\varepsilon\}$ are the upper and lower surfaces of the domain Ω^ε and $\Gamma_0^\varepsilon := \Gamma_0 \times (-\varepsilon,\varepsilon)$ is the part of the lateral surface $\Gamma^\varepsilon := \Gamma \times (-\varepsilon,\varepsilon)$, $\Gamma_0 \subset \Gamma := \partial\mathscr{S}$, $\operatorname{mes}\Gamma_0 \neq 0$. The corresponding elasticity tensor $\mathscr{E}^\varepsilon := \big[\mathscr{E}^{jklm,\varepsilon}\big]_{4\times4\times4\times4}$ is the standard one for an isotropic case (see (108)) and only depend on a couple of Lamé constants λ^ε and μ^ε:

$$\mathscr{E}^{jklm,\varepsilon} = \lambda^\varepsilon\delta_{jk}\delta_{lm} + \mu^\varepsilon\big[\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}\big]. \tag{7}$$

$\boldsymbol{F}^\varepsilon$ and $\boldsymbol{H}^\varepsilon$ are the volume and surface forces applied to the shell.

After the standard scaling $\Omega^\varepsilon \ni (\mathscr{X},t) \to (\mathscr{X},\varepsilon t) \in \Omega^1 := \mathscr{S} \times (-1,1)$, the following is assumed: the scaled forces and the scaled displacement vector field have the following asymptotic expansions:

$$\begin{aligned}\boldsymbol{F}(\varepsilon)(\mathscr{X},t) &= \frac{1}{\varepsilon^2}\boldsymbol{F}_{-2}(\mathscr{X},t) + \frac{1}{\varepsilon}\boldsymbol{F}_{-1}(\mathscr{X},t) + \boldsymbol{F}_0(\mathscr{X},t) + \\ &\qquad + \varepsilon\boldsymbol{F}_1(\mathscr{X},t) + \mathscr{O}(\varepsilon^2), \\ \boldsymbol{H}(\varepsilon)(\mathscr{X},\pm1) &= \frac{1}{\varepsilon}\boldsymbol{H}_{-1}(\mathscr{X},\pm1) + \boldsymbol{H}_0(\mathscr{X},\pm1) \\ &\qquad + \varepsilon\boldsymbol{H}_1(\mathscr{X},\pm1) + \mathscr{O}(\varepsilon^2), \\ \boldsymbol{U}(\varepsilon)(\mathscr{X},t) &= \boldsymbol{U}^0(\mathscr{X},t) + \varepsilon\boldsymbol{U}^1(\mathscr{X},t) + \cdots, \ \ (\mathscr{X},t) \in \Omega^1, \ \ \boldsymbol{U}^0 \neq 0\end{aligned} \tag{8}$$

where $\boldsymbol{F}_{-2}, \boldsymbol{F}_{-1}, \boldsymbol{F}_0, \ldots, \boldsymbol{H}_{-1}, \boldsymbol{H}_0, \boldsymbol{H}_1, \ldots$ and $\boldsymbol{U}^0, \boldsymbol{U}^1, \ldots$ are independent of ε. The asymptotic analysis, performed similar to that in [1], [5], [18], [31],

[32], leads to the following results:

$$\begin{gathered}\boldsymbol{F}_{-2}(\mathscr{X},t)=\boldsymbol{F}_{-1}(\mathscr{X},t)=\boldsymbol{H}_{-1}(\mathscr{X},\pm 1)=\\ =\boldsymbol{H}_0(\mathscr{X},\pm 1)=0,\quad \forall\,(\mathscr{X},t)\in\Omega^1,\end{gathered} \tag{9}$$

$$\boldsymbol{U}^0(\mathscr{X},t)=\boldsymbol{U}^0(\mathscr{X}),\;\; \boldsymbol{U}^0\in\mathbb{W}^1(\mathscr{S},\Gamma_0):=\left\{\boldsymbol{X}\in\mathbb{W}^1(\mathscr{S}):\ \boldsymbol{X}(\mathscr{X})=0 \text{ on } \Gamma_0\right\}$$

and the principal component $\boldsymbol{U}^0$ in the asymptotic expansion of the displacement vector field (cf. (8)) satisfies the variational equation

$$\sum_{j,k,l,m=1}^{3}\int_{\mathscr{S}}\mathfrak{G}^{jklm}\mathfrak{D}_{lm}(\boldsymbol{U}^0)\mathfrak{D}_{jk}(\boldsymbol{X})\,\boldsymbol{d}\sigma=\int_{\mathscr{S}}\langle\boldsymbol{P}_0,\boldsymbol{X}\rangle\,\boldsymbol{d}\sigma, \tag{10}$$

where $\boldsymbol{U}^0$, $\boldsymbol{X}\in\mathbb{W}^1(\mathscr{S},\Gamma_0)$ and

$$\begin{aligned}\mathfrak{G}^{jklm}&=2\mathscr{E}^{jklm}-2\,\frac{\mathscr{E}^{jk44}\mathscr{E}^{44lm}}{\mathscr{E}^{4444}}=\\ &=\frac{4\lambda\mu}{\lambda+2\mu}\,\delta_{jk}\delta_{lm}+2\mu\big[\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl}\big],\\ \boldsymbol{P}_0(\mathscr{X})&:=\frac{1}{2}\int_{-1}^{1}\boldsymbol{F}_0(\mathscr{X},t)\,dt+\frac{1}{2}\big[\boldsymbol{H}_1(\mathscr{X},-1)+\boldsymbol{H}_1(\mathscr{X},1)\big].\end{aligned} \tag{11}$$

The obtained two dimensional equation for a shell coincides with the Láme equation and operates with the deformation tensor of $\mathscr{S}$. It is remarkable that equation (9) has a unique solution $\boldsymbol{U}^0\in\mathbb{W}^1(\mathscr{S},\Gamma_0)$ for arbitrary resultant force $\boldsymbol{P}_0\in\mathbb{L}_2(\mathscr{S})$ (see Theorem 7.2 below). Moreover, if we assume the resultant forces are vanishing

$$\boldsymbol{P}_j(\mathscr{X}):=\frac{1}{2}\int_{-1}^{1}\boldsymbol{F}_j(\mathscr{X},t)\,dt+\frac{1}{2}\big[\boldsymbol{H}_{j+1}(\mathscr{X},-1)+\boldsymbol{H}_{j+1}(\mathscr{X},1)\big]=0 \tag{12}$$

for $j=0,1,\dots,q-1$, the entries of the asymptotic expansion of the displacement vector field in (8) vanish $\boldsymbol{U}^j(\mathscr{X},t)=0$ for $j=0,1,\dots,q-1$, the next component is independent of the transverse variable $\boldsymbol{U}^q(x,t)=\boldsymbol{U}^q(x)$ end is a solution to the variational problem, similar to (10):

$$\begin{gathered}\sum_{j,k,l,m=1}^{3}\int_{\mathscr{S}}\mathfrak{G}^{jklm}\mathfrak{D}_{lm}(\boldsymbol{U}^q)\mathfrak{D}_{jk}(\boldsymbol{X})\,\boldsymbol{d}\sigma=\int_{\mathscr{S}}\langle\boldsymbol{P}_q,\boldsymbol{X}\rangle\,\boldsymbol{d}\sigma,\\ \boldsymbol{U}^q,\ \boldsymbol{X}\in\mathbb{W}^1(\mathscr{S},\Gamma_0).\end{gathered} \tag{13}$$

The most significant is that the presented approach seems to be universal: If the "thickness" of a shell and the ratio d=(thickness of a shell=ε)/(minimal mean curvature of the middle surface of a shell) are related by the formulae $d=\varepsilon^s$, shells are usually divided in "membrane" $s<2$, "Novozhilov's" $s=2$ and "shallow" $s>2$ shell classes (see [10]). This is not the case in the present approach and the two-dimensional shell equation with the displacement vector filed of the middle surface as an unknown is uniquely solvable in all these cases.

The paper is organize as follows: After auxiliaries from classical differential geometry, exposed in § 1, in 2 we consider the deformation tensor, and the related Lamé equation on an elastic hypersurface $\mathscr{S}$. The calculus is based on the Gunter's tangential derivatives, developed in [11], [16]. The calculus turned out to be relevant for the shell problem.

In the next § 3 we describe configuration of a thin shell $\Omega^\varepsilon := \mathscr{S} \times (-\varepsilon, \varepsilon)$ (see (2)) and study properties of the normal vector field and surface measure on surfaces $\mathscr{S}_t$, which are equidistant from the middle surface $\mathscr{S} = \mathscr{S}_0$. These properties are crucial for the derivation of 2D shell equations later. In § 4 we rewrite some basic differential operators (the divergence, the gradient etc.) in curvilinear coordinates $(\mathscr{X}, t) \in \Omega^\varepsilon$, $\mathscr{X} \in \mathscr{S}$, $t \in (-\varepsilon, \varepsilon)$ of a thin shell. We also discuss the representation of vector fields in a full (but linearly dependent) system of tangent vectors $\{\boldsymbol{d}^j\}_{j=1}^n$, which represent the projections of the Cartesian base $\{\boldsymbol{e}^j\}_{j=1}^n$ to the hypersurface $\mathscr{S}$ (see (2)–(4)).

The most important result in § 5 is the form of deformation tensor and Lamé operator in the curvilinear coordinates $(\mathscr{X}, t) \in \Omega^\varepsilon$. We also prove coerciveness of the Lamé operator and show that the deformation tensor is nothing but the linearized change of the metric tensor on the surface. In § 6 we formulate rigorous variational problem for the 3D shell configuration and apply the scaling $t \to \varepsilon t$, which changes the integration in the transversal direction from the small interval $(-\varepsilon, \varepsilon)$ to the finite interval $(-1, 1)$.

Section 7 is the most important in the present paper: By using formal asymptotic expansion of solutions and applied forces (see (8)) and formal asymptotic analysis we establish properties of the displacement vector field (see (9)) and the 2D equation of the shell, written on the middle surface $\mathscr{S}$ (see (10), (11)). We also prove the unique solvability of the obtained equation of a shell.

In the concluding § 8 we describe, just for the readers convenience, two other asymptotic models of a shell, most relevant to the model derived in this paper: Koiter's and Sanchez-Palencia–Ciarlet models.

1. Auxiliaries from Differential Geometry

By a classical approach differential equations on surfaces are written with the help of covariant and contravariant frames, metric tensors and Christoffel symbols. To demonstrate a difference between a classical and the present approaches, let us consider an example. A surface $\mathscr{S}$ is given by a local immersion (1), which means that the derivatives $\{\boldsymbol{g}_k := \partial_k \Theta\}_{k=1}^{n-1}$ are linearly independent vector fields on the surface $\mathscr{S}$ and constitute a *covariant basis* in the space of tangential vector fields $\mathbb{T}\mathscr{S}$ to $\mathscr{S}$. Or, equivalently, the Gram matrix

$$G_{\mathscr{S}}(\mathscr{X}) = [g_{jk}(\mathscr{X})]_{n-1 \times n-1}, \quad g_{jk} := \langle \boldsymbol{g}_j, \boldsymbol{g}_k \rangle,$$

which is the *covariant metric tensor*, has the inverse $G^{-1}_{\mathscr{S}}(\mathscr{X}) = [g^{jk}(\mathscr{X})]_{n-1\times n-1}$. This inverse $G^{-1}_{\mathscr{S}}(\mathscr{X})$ is known as the *contravariant metric tensor*, represents the Gram matrix $g^{jk} := \langle \boldsymbol{g}^j, \boldsymbol{g}^k\rangle$ of the contravariant basis $\{\boldsymbol{g}^k\}_{k=1}^{n-1}$ and the latter is biorthogonal to the covariant basis

$$\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk}, \quad j,k = 1,\dots,n-1\,. \tag{14}$$

The Gram matrix $G_{\mathscr{S}}(\mathscr{X})$ is responsible for the *Riemannian metric* on $\mathscr{S}$.

Here and in what follows

$$\langle \boldsymbol{U}, \boldsymbol{V}\rangle := \sum_{j=1}^{n} U^0_j V^0_j, \quad \boldsymbol{U} = (U^0_1,\dots,U^0_n)^\top \in \mathbb{R}^n, \quad \boldsymbol{V} = (V^0_1,\dots,V^0_n)^\top \in \mathbb{R}^n,$$

denotes the scalar product in the Euclidean space $\mathbb{R}^n$.

The surface divergence and the surface gradient in classical differential geometry (i.e., in intrinsic parameters of the surface $\mathscr{S}$) are defined as follows:

$$\begin{gathered} \operatorname{div}_{\mathscr{S}}\boldsymbol{U} := \big[\det G_{\mathscr{S}}\big]^{-1/2}\sum_{j=1}^{n}\partial_j\Big\{\big[\det G_{\mathscr{S}}\big]^{1/2}U^j\Big\}, \\ \boldsymbol{\nabla}_{\mathscr{S}} f = \sum_{j,k=1}^{n-1}(g^{jk}\partial_j f)\partial_k, \quad \boldsymbol{U} = \sum_{j=1}^{n-1}U^j\boldsymbol{g}_j \end{gathered} \tag{15}$$

(see [33, Ch. 2, § 3]). The intrinsic parameters enable generalization to arbitrary manifolds, not necessarily immersed in the Euclidean space $\mathbb{R}^n$.

An alternative form of these and other operators on a hypersurface $\mathscr{S} \subset \mathbb{R}^n$ is based on the calculus of Gunter's derivatives and applies the Cartesian coordinates of the ambient space $\mathbb{R}^n$ with the natural basis

$$\boldsymbol{e}^1 = (1,0,\dots,0)^\top, \dots, \boldsymbol{e}^n = (0,\dots,0,1)^\top. \tag{16}$$

The calculus itself operates with the field of unit normal vectors to the hypersurface $\mathscr{S}$

$$\boldsymbol{\nu}(\mathscr{X}) := \pm\frac{\boldsymbol{g}_1(\Theta^{-1}(\mathscr{X}))\wedge\cdots\wedge\boldsymbol{g}_{n-1}(\Theta^{-1}(\mathscr{X}))}{\big|\boldsymbol{g}_1(\Theta^{-1}(\mathscr{X}))\wedge\cdots\wedge\boldsymbol{g}_{n-1}(\Theta^{-1}(\mathscr{X}))\big|}, \quad \mathscr{X}\in\mathscr{S}. \tag{17}$$

where $\boldsymbol{U}^{(1)}\wedge\cdots\wedge\boldsymbol{U}^{(n-1)}$ (or also $\boldsymbol{U}^{(1)}\times\cdots\times\boldsymbol{U}^{(n-1)}$) denotes the vector product of vectors $\boldsymbol{U}^{(1)},\dots,\boldsymbol{U}^{(n-1)}\in\mathbb{R}^n$. If a hypersurface $\mathscr{S}$ in $\mathbb{R}^n$ is defined implicitly

$$\mathscr{S} = \big\{\mathscr{X}\in\omega : \ \Psi_{\mathscr{S}}(\mathscr{X}) = 0\big\}, \tag{18}$$

where $\Psi_{\mathscr{S}} : \omega \longrightarrow \mathbb{R}$ is a C^k-smooth, $k \geqslant 1$, or a Lipschitz continuous and is regular $\boldsymbol{\nabla}\Psi(\mathscr{X}) \neq 0$, the outer unit normal vector field coincides with the normalized gradient of the generating function

$$\boldsymbol{\nu}(\mathscr{X}) := \pm\frac{\boldsymbol{\nabla}\Psi_{\mathscr{S}}(\mathscr{X})}{\big|\boldsymbol{\nabla}\Psi_{\mathscr{S}}(\mathscr{X})\big|}, \quad \mathscr{X}\in\mathscr{S}. \tag{19}$$

The sign $\pm$ is chosen appropriately to make the vector field $\boldsymbol{\nu}(\mathscr{X})$ outer with respect to the compact domain bordered by $\mathscr{S}$, provided the surface $\mathscr{S}$ has no boundary. If a surface $\mathscr{S}$ is hypograph

$$\mathscr{S} = \Big\{x = (x', x_n)^\top,\ x' \in \omega \subset \mathbb{R}^{n-1}:\ x_n = \Phi_{\mathscr{S}}(x')\Big\}, \tag{20}$$

where Φ is either C^k-smooth, $k \geqslant 1$, or Lipschitz continuous, the outer unit normal vector is defined by the formula

$$\boldsymbol{\nu}(x') := \frac{(\boldsymbol{\nabla}\Phi_{\mathscr{S}}(x'), -1)^\top}{\sqrt{1 + \big[\boldsymbol{\nabla}\Psi_{\mathscr{S}}(\mathscr{X})\big]^2}}\,, \quad x' \in \omega. \tag{21}$$

The contravariant frame $\{\boldsymbol{g}^k\}_{k=1}^{n-1}$ (cf. (14)) can also be defined by the formulae:

$$\boldsymbol{g}^k = \frac{1}{G_{\mathscr{S}}}\,\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_{k-1} \wedge \boldsymbol{\nu} \wedge \boldsymbol{g}_{k+1} \wedge \cdots \wedge \boldsymbol{g}_{n-1}, \tag{22}$$
$$\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk}, \quad k = 1, \dots, n-1.$$

The collection of the tangential *Günter's derivatives* are defined as follows (cf. [17], [20], [22], [11], [16])

$$\mathscr{D}_j := \partial_j - \nu_j(\mathscr{X})\partial_{\boldsymbol{\nu}} = \partial_{\boldsymbol{d}^j}\,, \quad \boldsymbol{d}^j := \pi_{\mathscr{S}}\boldsymbol{e}^j. \tag{23}$$

Here $\partial_{\boldsymbol{\nu}} := \sum\limits_{j=1}^{n} \nu_j\partial_j$ is the standard normal derivative and

$$\pi_{\mathscr{S}} : \mathbb{R}^n \longrightarrow \mathbb{T}\mathscr{S}, \quad \pi_{\mathscr{S}}\boldsymbol{U} := \boldsymbol{U} - \nu\nu^\top\boldsymbol{U} = \boldsymbol{U} - \langle \nu, \boldsymbol{U}\rangle\nu, \ \ t \in \mathscr{S}, \tag{24}$$

denotes the canonical orthogonal projection $\pi^2_{\mathscr{S}} = \pi_{\mathscr{S}}$ onto the space $\mathbb{T}\mathscr{S}$ of tangential vector fields: $(\boldsymbol{\nu}, \pi_{\mathscr{S}}\boldsymbol{V}) = 0$ for all $\boldsymbol{V} \in \mathbb{R}^n$.

The collection $\{\mathscr{D}_j\}_{j=1}^n$ of the first-order derivatives represent directional derivatives along the tangential vector fields $\{\boldsymbol{d}_j\}_{j=1}^n$ to $\mathscr{S}$.

A tangential vector field $\boldsymbol{U} \in \mathbb{T}\mathscr{S}$ has representations

$$\boldsymbol{U} = \sum_{j=1}^{n} U_j^0\boldsymbol{e}^j = \sum_{j=1}^{n} U_j^0\boldsymbol{d}^j, \tag{25}$$

where the coefficients $U_1^0, \dots, U_n^0$ are the same. Written with the help of Gunter's derivatives, the *surface gradient* $\boldsymbol{\nabla}_{\mathscr{S}}\boldsymbol{U}$ and the *surface divergence* $\operatorname{div}_{\mathscr{S}}\boldsymbol{U}$ from (15) acquire the form

$$\boldsymbol{\nabla}_{\mathscr{S}}\boldsymbol{U} := (\mathscr{D}_1U^0, \dots, \mathscr{D}_nU^0)^\top, \quad \operatorname{div}_{\mathscr{S}}\boldsymbol{U}; = \sum_{j=1}^{n} \mathscr{D}_jU_j^0 \tag{26}$$

(cf. [16]), while the *derivative of a vector field* $\boldsymbol{V}$ along $\boldsymbol{U}$ and the corresponding *covariant derivative* have the form

$$\partial_{\boldsymbol{U}}\boldsymbol{V} = \sum_{j=1}^{n} U_j^0\mathscr{D}_j\boldsymbol{V}. \tag{27}$$

A derivative $\partial^{\mathscr{S}}_{\boldsymbol{U}} : C^1(\mathscr{S}) \longrightarrow C^0(\mathscr{S})$ along a vector field $\boldsymbol{U} \in C(\mathscr{S})$ is called *covariant* if it is a linear automorphism of the space of tangential vector fields

$$\partial^{\mathscr{S}}_{\boldsymbol{U}} : \mathbb{T}\mathscr{S} \longrightarrow \mathbb{T}\mathscr{S}. \tag{28}$$

The covariant derivative of a tangential vector field $\boldsymbol{V} = \sum\limits_{j=1}^{n} V^0_j \boldsymbol{d}^j \in \mathbb{T}\mathscr{S}$ along a tangential vector field $\boldsymbol{U} = \sum\limits_{j=1}^{n} U^0_j \boldsymbol{d}_j \in \mathbb{T}\mathscr{S}$ is defined by the formula

$$\partial^{\mathscr{S}}_{\boldsymbol{U}} \boldsymbol{V} := \pi_{\mathscr{S}} \partial_{\boldsymbol{U}} \boldsymbol{V} = \sum_{j=1}^{n} U^0_j \mathscr{D}^{\mathscr{S}}_j \boldsymbol{V} \tag{29}$$

(cf. [13], [14]). where the projection $\pi_{\mathscr{S}}$ is defined in (24) and $\mathscr{D}^{\mathscr{S}}_j : \mathbb{T}\mathscr{S} \longrightarrow \mathbb{T}\mathscr{S}$ is the *covariant Gunter's derivative*

$$\mathscr{D}^{\mathscr{S}}_j \boldsymbol{V} := \pi_{\mathscr{S}} \mathscr{D}_j \boldsymbol{V} = \mathscr{D}_j \boldsymbol{V} - \langle \boldsymbol{\nu}, \mathscr{D}_j \boldsymbol{V} \rangle \boldsymbol{\nu}, \quad j = 1, \dots, n. \tag{30}$$

In the classical differential geometry the derivatives of the covariant basis $\partial_j \boldsymbol{g}_k$, of the covariant basis $\partial_j \boldsymbol{g}^k$ and of the unit normal vector field are given by the following formulae

$$\partial_k \boldsymbol{g}_j(x) = \partial_k \partial_j \Theta(x) = \sum_{m=1}^{n-1} \Gamma^m_{jk}(x) \boldsymbol{g}^m(x) + b_{jk}(x) \boldsymbol{\nu}(x), \tag{31}$$

$$\partial_j \boldsymbol{g}^k(x) = -\sum_{m=1}^{n-1} \Gamma^k_{jm}(x) \boldsymbol{g}_m(x) + b^k_j(x) \boldsymbol{\nu}(x) \ \ \forall\, x \in \Omega, \tag{32}$$

$$\partial_j \boldsymbol{\nu} = -\sum_{k=1}^{n-1} b_{jk} \boldsymbol{g}^k = -\sum_{k=1}^{n-1} b^k_j \boldsymbol{g}_k, \quad j = 1, \dots, n-1, \tag{33}$$

where $\boldsymbol{\nu}(x) := \boldsymbol{\nu}(\Theta(x))$, $x \in \Omega$ and $\Gamma^m_{jk}(x)$ are the *Christoffel symbols*, defined by the equalities

$$\Gamma^m_{jk}(x) = \Gamma^m_{kj}(x) = \langle \partial_k \boldsymbol{g}_j(x), \boldsymbol{g}^m(x) \rangle. \tag{34}$$

The matrices

$$\begin{aligned} \mathscr{B}_{\mathscr{S}}(\mathscr{X}) &:= \big[b^k_j(\mathscr{X})\big]_{(n-1)\times(n-1)} = G^{-1}_{\mathscr{S}}(\mathscr{X}) B_{\mathscr{S}}(\mathscr{X}), \\ B_{\mathscr{S}}(\mathscr{X}) &:= \big[b_{jk}(\mathscr{X})\big]_{(n-1)\times(n-1)} = B^{\top}_{\mathscr{S}}(\mathscr{X}), \quad \mathscr{X} \in \mathscr{S}, \end{aligned} \tag{35}$$

compiled of projections of derivatives of the covariant and contravariant bases on the normal vector field (cf. (31))

$$\begin{aligned} b_{jk}(x) &:= \langle \partial_j \boldsymbol{g}_k(x), \boldsymbol{\nu}(x) \rangle = -\langle \boldsymbol{g}_k(x), \partial_j \boldsymbol{\nu}(x) \rangle \\ &= \langle \partial_k \boldsymbol{g}_j(x), \boldsymbol{\nu}(x) \rangle = b_{kj}(x), \end{aligned} \tag{36}$$

$$b^k_j(x) := \langle \partial_j \boldsymbol{g}^k(x), \boldsymbol{\nu}(x) \rangle = -\langle \boldsymbol{g}^k(x), \partial_j \boldsymbol{\nu}(x) \rangle, \quad \forall\, x \in \Omega. \tag{37}$$

represent the important *covariant curvature tensor* and the *mixed curvature tensor*. The covariant curvature tensor is symmetric $b_{jk} = b_{kj}$ (cf (36)), while the mixed curvature tensor is not: in general $b_j^k \neq b_k^j$. The coefficients b_{jk} and b_j^k are related as follows:

$$b_j^k = \sum_{m=1}^{n-1} g^{km} b_{mj}, \quad b_{jk} = \sum_{m=1}^{n-1} g_{km} b_j^m, \quad j,k = 1,\dots,n-1. \tag{38}$$

Within the classical theory the covariant derivative of a tangential vector field $\boldsymbol{V} \in \mathbb{T}\mathscr{S}$ along a vector field $\boldsymbol{U} \in \mathbb{T}\mathscr{S}$, where

$$\boldsymbol{V} = \sum_{j=1}^{n-1} V^j \boldsymbol{g}_j = \sum_{j=1}^{n-1} V_j \boldsymbol{g}^j = \sum_{j=1}^{n} V_j^0 \boldsymbol{d}^j,$$
$$\boldsymbol{U} = \sum_{j=1}^{n-1} U^j \boldsymbol{g}_j = \sum_{j=1}^{n-1} U_j \boldsymbol{g}^j = \sum_{j=1}^{n} V_j^0 \boldsymbol{e}^j,$$

are defined by the formulae

$$\partial_{\boldsymbol{U}}^{\mathscr{S}} \boldsymbol{V} := \pi_{\mathscr{S}} \partial_{\boldsymbol{U}} \boldsymbol{V} := \sum_{j,m=1}^{n-1} U^j V^m_{;j} \boldsymbol{g}_m, \quad V^m_{;j} = \partial_j V^m + \sum_{k=1}^{n-1} \Gamma^m_{jk} V^k, \tag{39}$$
$$:= \sum_{j,m=1}^{n-1} U_j V_{m;j} \boldsymbol{g}^m, \quad V_{m;j} := \partial_j V_m - \sum_{k=1}^{n-1} \Gamma^k_{jm} V_k \tag{40}$$

(see (27) for the third form of the covariant derivative).

2. The Deformation Tensor and Lamé Operator on a Hypersurface

The Lamé operator $\mathscr{L}_{\mathscr{S}}$ on $\mathscr{S}$ is the natural operator associated with the Euler-Lagrange equations for a variational integral. The starting point is the total free (elastic) energy

$$\mathscr{E}[\boldsymbol{U}] := \oint_{\mathscr{S}} E(y, \mathscr{D}^\sigma \boldsymbol{U}(y))\, dS, \quad \mathscr{D}^{\mathscr{S}} \boldsymbol{U} := \left[(\mathscr{D}_j^{\mathscr{S}} \boldsymbol{U})_k^0\right]_{n\times n}, \quad \boldsymbol{U} \in \mathbb{T}\mathscr{S}, \tag{41}$$

ignoring at the moment the displacement boundary conditions (Koiter's model). Equilibria states correspond to minimizers of the above variational integral (see [26, § 5.2]). The kernel $E = (\mathfrak{S}_{\mathscr{S}}, \mathrm{Def}_{\mathscr{S}})$ depends bi-linearly on the stress $\mathfrak{S}_{\mathscr{S}} = \left[\mathfrak{S}^{jk}\right]_{n\times n}$ and the deformation $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})$ tensors. The following form of the deformation (strain) tensor was identified in [16]

$$\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) = \left[\mathfrak{D}_{jk}(\boldsymbol{U})\right]_{n\times n}, \quad \boldsymbol{U} = \sum_{j=1}^{n} U_j^0 \boldsymbol{d}^j \in \mathbb{T}\mathscr{S}, \quad j,k = 1,\dots,n, \tag{42}$$

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \frac{1}{2}\left[(\mathscr{D}_j^{\mathscr{S}} \boldsymbol{U})_k^0 + (\mathscr{D}_k^{\mathscr{S}} \boldsymbol{U})_j^0\right] = \frac{1}{2}\left[\mathscr{D}_k U_j^0 + \mathscr{D}_j U_k^0 + \sum_{m=1}^{n} U_m^0 \mathscr{D}_m(\nu_j \nu_k)\right],$$

where $(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0 := \langle\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U},\boldsymbol{e}^k\rangle$. Hooke's law states that $\mathfrak{S}_{\mathscr{S}} = \mathfrak{C}\mathrm{Def}_{\mathscr{S}}$ for some linear fourth-order tensor $\mathfrak{C} := \big[c_{jk\ell m}\big]_{n\times n\times n\times n}$, which is positive definite:

$$\langle\mathfrak{C}\zeta,\zeta\rangle := \sum_{i,j,k,\ell=1}^{n} c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell} \geqslant C_0\sum_{i,j=1}^{n}|\zeta_{i,j}|^2 := C_0|\zeta|^2 \tag{43}$$

for all symmetric tensors $\zeta_{ij} = \zeta_{ji} \in \mathbb{C}$, $\zeta := \big[\zeta_{ij}\big]_{n\times n}$. Moreover, $\mathfrak{C}$ has the following symmetry properties:

$$c_{ijk\ell} = c_{ij\ell k} = c_{k\ell ij} \quad \forall\, i,j,k,\ell. \tag{44}$$

The following form of the Lamé operator for a linear anisotropic elastic medium was identified in [16]

$$\mathscr{L}_{\mathscr{S}} = \mathrm{Def}^*_{\mathscr{S}}\mathfrak{C}\,\mathrm{Def}_{\mathscr{S}} = \Big[\sum_{\ell m=1}^{n} c_{jk\ell m}\mathscr{D}_j^{\mathscr{S}}\mathscr{D}_\ell^{\mathscr{S}}\Big]_{n\times n}, \quad \boldsymbol{U}\in\mathbb{T}\mathscr{S}, \tag{45}$$

where the adjoint operator to the deformation tensor

$$\mathrm{Def}^*_{\mathscr{S}}\mathfrak{U} := \frac{1}{2}\sum_{j=1}^{n}\big\{(\mathscr{D}_j^{\mathscr{S}})^*\big[\mathfrak{U}^{jk}+\mathfrak{U}^{kj}\big]\big\}_{k=1}^{n} \;\text{ for }\; \mathfrak{U} = \|\mathfrak{U}^{jk}\|_{n\times n} \tag{46}$$

maps 2-tensor functions to vector functions.

For an isotropic medium, as usual, the number of distinct coefficients reduces from 21 to 2 and

$$c_{jklm} = \lambda\delta_{jk}\delta_{lm} + \mu\big[\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}\big], \tag{47}$$

where λ and μ are the Lamé constants. The corresponding Lamé operator acquires a simpler form

$$\begin{aligned}\mathscr{L}_{\mathscr{S}}\boldsymbol{U} &= -\lambda\boldsymbol{\nabla}_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U} + 2\mu\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U} = \\ &= -\mu\pi_{\mathscr{S}}\Delta_{\mathscr{S}}\boldsymbol{U} - (\lambda+\mu)\boldsymbol{\nabla}_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U} - \mu\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U}, \quad \boldsymbol{U}\in\mathbb{T}\mathscr{S}\end{aligned} \tag{48}$$

(cf. (24) for the projection $\pi_{\mathscr{S}}$). λ, $\mu\in\mathbb{R}$ are the Lamé coefficients, whereas

$$\begin{gathered}\mathscr{H}^0_{\mathscr{S}} = -\mathrm{div}_{\mathscr{S}}\boldsymbol{\nu} := -\sum_{j=1}^{n}\mathscr{D}_j\nu_j = \mathrm{Tr}\mathscr{W}_{\mathscr{S}} = \mathrm{Tr}\,\mathscr{B}_{\mathscr{S}} = \sum_{j=1}^{n-1}b_j^j, \\ \mathscr{W}_{\mathscr{S}} = -\big[\mathscr{D}_j\nu_k\big]_{n\times n}.\end{gathered} \tag{49}$$

Note, that $\mathscr{H}_{\mathscr{S}} := (n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}$ represents the *mean curvature* of the surface $\mathscr{S}$; $\mathscr{W}_{\mathscr{S}}$ is the *Weingarten curvature tensor* of $\mathscr{S}$; Eigenvalues of $\mathscr{W}_{\mathscr{S}}$, except one which is 0, represent all principal curvatures of the surface $\mathscr{S}$ and coincide with eigenvalues of the curvature tensors $\mathscr{B}_{\mathscr{S}}(\mathscr{X})$ and $B_{\mathscr{S}}(\mathscr{X})$ in (35).

Note, that Gunter's derivatives were already applied in [22] to minimal surfaces and in [17], [20] to the problems of 3D elasticity.

A vector field on $\mathbb{R}^n$ near a surface $\mathscr{S}$ can be represented in the extended covariant $\{\boldsymbol{g}_j\}_{j=1}^n$ and contravariant $\{\boldsymbol{g}^j\}_{j=1}^n$ frames, where $\boldsymbol{g}_n = \boldsymbol{g}^n := \boldsymbol{\nu}$ is the outer unit normal vector

$$\boldsymbol{U} = \sum_{j=1}^n U^j \boldsymbol{g}_j = \sum_{j=1}^n U_j \boldsymbol{g}^j, \quad U^n = U_n := \langle \boldsymbol{U}, \boldsymbol{\nu}\rangle. \tag{50}$$

Let us consider a natural extension of the curvature tensor:

$$\begin{gathered} B_{\mathscr{S}} = [b_{jk}]_{n\times n}, \quad b_{jk} = b_{kj} = \langle \partial_j \boldsymbol{g}_k, \boldsymbol{\nu}\rangle = -\langle \boldsymbol{g}_k, \partial_j \boldsymbol{\nu}\rangle, \\ b_{nn} = \langle \partial_{\boldsymbol{\nu}} \boldsymbol{\nu}, \boldsymbol{\nu}\rangle = 0, \\ b_{nj} = b_{jn} = \langle \partial_{\boldsymbol{\nu}} \boldsymbol{g}_j, \boldsymbol{\nu}\rangle = -\langle \boldsymbol{g}_j, \partial_{\boldsymbol{\nu}} \boldsymbol{\nu}\rangle = 0, \quad j,k = 1,\dots,n-1 \end{gathered} \tag{51}$$

(cf. (35) and (37)).

Similarly, we consider a natural extension of the deformation tensor

$$\widetilde{\mathfrak{D}}_{\mathscr{S}}(\boldsymbol{U}) = \big[\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U})\big]_{n\times n}, \quad \widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}) := \frac{1}{2}\left[U_{j;k} + U_{k;j}\right], \tag{52}$$

where the extended covariant derivatives $U_{j;k}$ are defined as

$$\begin{gathered} U_{j;k} := \partial_k U_j - \sum_{m=1}^n \Gamma_{kj}^m U_m, \quad \partial_j := \partial_{g_j}, \\ \Gamma_{kj}^m := \langle \partial_k \boldsymbol{g}_j, \boldsymbol{g}^m\rangle = -\langle \boldsymbol{g}_j, \partial_k \boldsymbol{g}^m\rangle, \quad j,k = 1,\dots,n \end{gathered} \tag{53}$$

(cf. (39) and (40) for $j = 1,\dots,n-1$). In particular, $\partial_n = \partial_{\boldsymbol{\nu}}$, $\boldsymbol{g}^n = \boldsymbol{g}_n = \boldsymbol{\nu}$ and the Christoffel symbols are extended by the curvature tensors (cf. (36)):

$$\begin{gathered} \Gamma_{jk}^n = \Gamma_{kj}^n := \langle \partial_k \boldsymbol{g}_j, \boldsymbol{\nu}\rangle = -\langle \boldsymbol{g}_j, \partial_k \boldsymbol{\nu}\rangle = b_{kj} = b_{jk}, \quad j,k = 1,\dots,n-1, \\ \Gamma_{jn}^k := \langle \boldsymbol{\nu}, \partial_j \boldsymbol{g}^k\rangle = -\langle \partial_j \boldsymbol{\nu}, \boldsymbol{g}^k\rangle = b_j^k, \quad j,k = 1,\dots,n-1, \\ \Gamma_{nj}^n = \Gamma_{jn}^n = \langle \partial_j \boldsymbol{\nu}, \boldsymbol{\nu}\rangle = 0, \quad \Gamma_{nn}^j = \langle \partial_{\boldsymbol{\nu}} \boldsymbol{\nu}, \boldsymbol{g}_j\rangle = 0, \quad j = 1,\dots,n. \end{gathered} \tag{54}$$

The notation (54) enables to write formulae (31) and (32) in an universal form

$$\partial_k \boldsymbol{g}_j = \sum_{m=1}^n \Gamma_{jk}^m \boldsymbol{g}^m, \quad \partial_j \boldsymbol{g}^k = -\sum_{m=1}^n \Gamma_{jm}^k \boldsymbol{g}_m \quad j,k = 1,\dots,n.$$

Theorem 2.1. *The linearized change of the extended metric tensor*

$$G_{\mathscr{S}} = \big[g_{jk}\big]_{n\times n}, \quad g_{jk} = \langle \boldsymbol{g}_j, \boldsymbol{g}_k\rangle, \quad g_{nj} = g_{jn} = \delta_{jn}, \quad j,k = 1,\dots,n \tag{55}$$

by a displacement vector field represented in the extended contravariant frame (50) *coincides with the extended deformation tensor:*

$$\mathfrak{g}_{jk} := \big\{g_{jk}(\boldsymbol{U}) - g_{jk}\big\}^{\rm lin} = \frac{1}{2}\big[U_{j;k} + U_{k;j}\big] = \widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}). \tag{56}$$

Here g_{jk} and $g_{jk}(\boldsymbol{U})$ denote the covariant metric tensors of the surfaces before $\mathscr{S}$ and after $\mathscr{S}(\boldsymbol{U}) = \mathscr{S} + \boldsymbol{U}$ the displacement $\boldsymbol{U}$ is applied, respectively; $U_{j;k}$ denotes the covariant derivative (*cf.* (39))*. The notation $\big\{\,\cdot\,\big\}^{\rm lin}$ indicates that we ignore all non-linear summands inside the bracket* (*for $\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U})$ see* (52)).

Proof. By applying (32), (33), (39), (53) and (54) we get:

$$\partial_j\Big[\sum_{r=1}^n U_r\boldsymbol{g}^r\Big]=\sum_{r=1}^n(\partial_j U_r)\boldsymbol{g}^r+\sum_{r=1}^n U_r\partial_j\boldsymbol{g}^r=\sum_{r=1}^n\Big[\partial_j U_r-\sum_{m=1}^n\Gamma^m_{jr}U_m\Big]\boldsymbol{g}^r=$$
$$=\sum_{r=1}^n U_{r;j}\boldsymbol{g}^r,\quad j=1,\dots,n. \tag{57}$$

We recall that the vectors are biorthogonal $\langle\boldsymbol{g}_j,\boldsymbol{g}^k\rangle=\delta_{jk}$, $\langle\boldsymbol{\nu},\boldsymbol{g}^k\rangle=0$, $j,k=1,\dots,n-1$, apply the obtained equalities, ignore non-linear terms like $U_{r;j}U_{m;k}$ and get:

$$\mathfrak{g}_{jk}=\big\{g_{jk}(\boldsymbol{U})-g_{jk}\big\}^{\rm lin}=\frac12\Big\{\langle\partial_j\Theta+\partial_j\boldsymbol{U},\partial_k\Theta+\partial_k\boldsymbol{U}\rangle-\langle\partial_j\Theta,\partial_k\Theta\rangle\Big\}^{\rm lin}=$$
$$=\frac12\bigg\{\Big\langle\boldsymbol{g}_j+\partial_j\Big[\sum_{r=1}^n U_r\boldsymbol{g}^r\Big],\boldsymbol{g}_k+\partial_k\Big[\sum_{m=1}^n U_m\boldsymbol{g}^m\Big]\Big\rangle-\langle\boldsymbol{g}_j,\boldsymbol{g}_k\rangle\bigg\}^{\rm lin}=$$
$$=\frac12\big[U_{j;k}+U_{k;j}\big]=\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}).$$ □

3. Configuration of a thin Shell

We endeavor to study elastic shells, whose reference configuration Ω^ε is described by equalities (1) and (2). We will also use the notation $\boldsymbol{\nu}(y):=\boldsymbol{\nu}(\Theta(y))$ for brevity unless this does not leads to a confusion. The variable t in (2) will be referred to as the *transverse variable.*

Ω^ε is referred to as a *tubular domain* of the middle surface $\mathscr{S}$.

Lemma 3.1. *If the hypersurface $\mathscr{S}$ is, at least, C^1-smooth, the mapping*

$$\begin{aligned}&\Theta^\varepsilon:\omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon)\longrightarrow\Omega^\varepsilon,\quad\omega^\varepsilon\subset\mathbb{R}^n,\\&\Theta^\varepsilon(y,t):=\Theta(y)+t\boldsymbol{\nu}(y),\quad(y,t)\in\omega^\varepsilon,\end{aligned}\tag{58}$$

is a diffeomorphism, i.e.,

$$\det\boldsymbol{\nabla}_{(y,t)}\Theta^\varepsilon(y,t)\neq0\ \text{ for all }\ (y,t)\in\omega^\varepsilon,\tag{59}$$

provided that ε is sufficiently small.

Proof. Recall that the vectors $\boldsymbol{g}_j(y):=\partial_j\Theta(y)$, $j=1,\dots,n-1$ are linearly independent (Θ is an immersion), which implies that the corresponding Gram determinant does not vanishes

$$\mathscr{G}\big[\boldsymbol{g}_1(y),\dots,\boldsymbol{g}_{n-1}(y)\big]\neq0\ \ \forall\,y\in\omega.$$

Then the perturbed system

$$\big\{\boldsymbol{g}^\varepsilon_j(y)\big\}_{j=1}^n,\ \ \boldsymbol{g}^\varepsilon_j(y):=\partial_j\Theta^\varepsilon(y,t)=\boldsymbol{g}_j(y)+t\partial_j\boldsymbol{\nu}(y),\ \ j=1,\dots,n-1,$$
$$\boldsymbol{g}^\varepsilon_n(y):=\partial_t\Theta^\varepsilon(y,t)=\boldsymbol{\nu}(y)=\boldsymbol{g}_n(y)$$

remains linearly independent for $|t|\leqslant\varepsilon$ when ε is sufficiently small and (59) follows. □

Going into detail one finds easily that $1/\varepsilon$ is more than the maximum of modules of all principal curvatures of the surface $\mathscr{S}$, i.e.

$$\frac{1}{\varepsilon} > \max_{\substack{j=1,\dots,n\\ \mathscr{X}\in\mathscr{S}}} |\lambda_1(\mathscr{X})|,$$

where $\lambda_1(\mathscr{X}),\dots,\lambda_{n-1}(\mathscr{X}),\lambda_n(\mathscr{X})\equiv 0$ are all eigenvalues of the Weingarten matrix $\mathscr{W}_{\mathscr{S}}(\mathscr{X})$, $\mathscr{X}\in\mathscr{S}$, then the mapping Θ^ε in (58) is a diffeomorphism.

Let us use the notation

$$\mathscr{S}_t := \Big\{\mathscr{X}_t\in\Omega^\varepsilon:\ \mathscr{X}_t=\mathscr{X}+t\boldsymbol{\nu}(\mathscr{X}),\ \mathscr{X}\in\mathscr{S}\Big\} = \\ = \big\{z\in\Omega^\varepsilon:\ \operatorname{dist}(z,\mathscr{S})=t\big\},\quad -\varepsilon<t<\varepsilon, \qquad (60)$$

for the surface on the distance t from the middle surface $\mathscr{S}$. If t is fixed and sufficiently small, $\mathscr{S}_t$ is a surface, defined by the immersion $\Theta^\varepsilon(y,t)$ in (58). Let σ_t denote the surface measure on $\mathscr{S}_t$ and use $d\sigma=d\sigma_0$ for the measure on the middle surface $\mathscr{S}=\mathscr{S}_0$.

Lemma 3.2. *The unit normal vector field $\boldsymbol{\nu}_t(\mathscr{X}_t)$, $\mathscr{X}_t\in\mathscr{S}_t$, is independent of the transversal variable $t\in[-\varepsilon,\varepsilon]$:*

$$\boldsymbol{\nu}_t(\mathscr{X}_t)=\boldsymbol{\nu}(\mathscr{X}),\quad \mathscr{X}=\Theta(x)\in\mathscr{S},\quad \mathscr{X}_t=\mathscr{X}+t\boldsymbol{\nu}(\mathscr{X})=\Theta^\varepsilon(x,t)\in\mathscr{S}_t. \qquad (61)$$

Proof. The covariant bases on the surfaces $\mathscr{S}$ and $\mathscr{S}_t$, defined by the difseomorphisms $\Theta(x)$ in (1) and $\Theta^\varepsilon(x,t)$ in (58) are, respectively,

$$\boldsymbol{g}_j(x):=\partial_j\Theta(x)\ \text{ and }\ \boldsymbol{g}_{j,t}(x):=\partial_j\Theta^\varepsilon(t,x),\ \ (x,t)\in\omega^\varepsilon,\ j=1,\dots,n-1. \qquad (62)$$

First recall that

$$\boldsymbol{g}_{j_1}\wedge\cdots\wedge\boldsymbol{g}_{j_{n-1}}=\sigma(j_1,\dots,j_{n-1})\boldsymbol{g}_1\wedge\cdots\wedge\boldsymbol{g}_{n-1}, \qquad (63)$$

where $\sigma(j_1,\dots,j_{n-1})=0,\pm1$ denotes the permutation sign:

$$\sigma(j_1,\dots,j_k)= \\ =\begin{cases}+1 & \text{if } (j_1,\dots,j_k) \text{ is an even permutation of } (1,\dots,n-1),\\ 0 & \text{if } j_r=j_s \text{ for some } r,s=1,\dots,n-1 \text{ and } r\neq s,\\ -1 & \text{if } (j_1,\dots,j_k) \text{ is an odd permutation of } (1,\dots,n-1).\end{cases} \qquad (64)$$

Also recall, that

$$\partial_j\boldsymbol{\nu}=-\sum_{k=1}^{n-1}b_j^k\boldsymbol{g}_k, \qquad (65)$$

(see (33)) and

$$\mathscr{H}_{\mathscr{S}}=\frac{1}{n-1}\operatorname{Tr}\mathscr{B}_{\mathscr{S}}=\frac{1}{n-1}\sum_{j=1}^{n-1}b_j^j,\quad \mathscr{K}_{\mathscr{S}}=\det\mathscr{B}_{\mathscr{S}} \qquad (66)$$

are, respectively, the mean curvature (cf. (49)) and the Gaussian curvature of the hypersurface $\mathscr{S}$.

Using natural identifications

$$\boldsymbol{\nu}(\mathscr{X}) = \boldsymbol{\nu}(\Theta(x)) \ \text{ for } \ \mathscr{X} = \Theta(x) \in \mathscr{S},$$
$$\boldsymbol{\nu}(\mathscr{X}_t) = \boldsymbol{\nu}(\Theta_t(x)) \ \text{ for } \ \mathscr{X}_t = \Theta_t(x) \in \mathscr{S}_t$$

and by invoking equalities (65)–(66), we obtain the following

$$\begin{aligned} &\boldsymbol{g}_{1,t}(x) \cdots \wedge \boldsymbol{g}_{j_{n-1},t}(x) = \partial_1[\Theta(x) + t\boldsymbol{\nu}(x)] \wedge \cdots \wedge \partial_{n-1}[\Theta(x) + t\boldsymbol{\nu}(x)] = \\ &= \big[1 - (n-1)t\mathscr{H}_{\mathscr{S}}(x) + \cdots + (-t)^{n-1}\mathscr{K}_{\mathscr{S}}(x)\big]\boldsymbol{g}_1(x) \wedge \cdots \wedge \boldsymbol{g}_{n-1}(x) \end{aligned} \tag{67}$$

and the function $1 - (n-1)t\mathscr{H}_{\mathscr{S}}(x) + \cdots + (-t)^{n-1}\mathscr{K}_{\mathscr{S}}(x)$ is positive, provided t is sufficiently small. Then, in view of the established dependence (67) and by the definition of the unit normal vector fields,

$$\begin{aligned} \boldsymbol{\nu}_t(\mathscr{X}_t) &= \pm\frac{\boldsymbol{g}_{1,t}(x) \wedge \cdots \wedge \boldsymbol{g}_{j_{n-1},t}(x)}{|\boldsymbol{g}_{1,t}(x) \wedge \cdots \wedge \boldsymbol{g}_{j_{n-1},t}(x)|} = \\ &= \pm\frac{\boldsymbol{g}_1(x) \wedge \cdots \wedge \boldsymbol{g}_{j_{n-1}}(x)}{|\boldsymbol{g}_1(x) \wedge \cdots \wedge \boldsymbol{g}_{j_{n-1}}(x)|} = \boldsymbol{\nu}(\mathscr{X}), \end{aligned} \tag{68}$$
$$\mathscr{X} = \Theta(x) \in \mathscr{S}, \quad \mathscr{X}_t = \Theta^\varepsilon(x,t) = \mathscr{X} + t\boldsymbol{\nu}(\mathscr{X}) \in \Omega^\varepsilon, \quad \forall\, x \in \omega, \quad t \in [-\varepsilon, \varepsilon],$$

which proves the asserted equality. □

Remark 3.3. The foregoing Lemma 3.2 justifies that the extension of the unit normal vector field $\boldsymbol{\nu}(\mathscr{X})$ on the hypersurface $\mathscr{S}$ as a constant in the transversal direction is natural.

The next Corollary 3.4 is proved in [27, Theorem 2.5.18] for $n = 3$ and represents a consequence of equality (67).

Corollary 3.4. *Under the conditions of the foregoing Lemma 3.2, the surface measures $d\sigma_t$ on $\mathscr{S}_t$ and $d\sigma$ on $\mathscr{S}$ are related as follows*

$$\begin{aligned} d\sigma_t(\mathscr{X}_t) := \big[1 - (n-1)t\mathscr{H}_{\mathscr{S}}(\mathscr{X}) + \cdots + (-t)^{n-1}\mathscr{K}_{\mathscr{S}}(\mathscr{X})\big]\, d\sigma, \\ \mathscr{X}_t = \mathscr{X} + t\boldsymbol{\nu}(\mathscr{X}) \in \mathscr{S}_t, \quad \mathscr{X} \in \mathscr{S}, \end{aligned} \tag{69}$$

provided ε is small enough that

$$\inf_{-\varepsilon<t<\varepsilon} \big[1 - (n-1)t\mathscr{H}_{\mathscr{S}}(\mathscr{X}) + \cdots + (-t)^{n-1}\mathscr{K}_{\mathscr{S}}(\mathscr{X})\big] > 0. \tag{70}$$

In particular, for $n = 3$, the following equality is valid:

$$d\sigma_t := \big[1 - 2t\mathscr{H}_{\mathscr{S}}(\mathscr{X}) + t^2\mathscr{K}_{\mathscr{S}}(\mathscr{X})\big]\, d\sigma. \tag{71}$$

Proof. The proof is a direct consequence of the well known formulae for the surface measures

$$d\sigma = |\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_{n-1}|, \quad d\sigma_t = |\boldsymbol{g}_{1,t} \wedge \cdots \wedge \boldsymbol{g}_{n-1,t}|$$

and the equality (67) proved above. □

Remark 3.5. The estimate for the determinant $\mathscr{G}_{\mathscr{S}} = \det G_{\mathscr{S}}$

$$0 < C_0 \leqslant \mathscr{G}_{\mathscr{S}}(x) \leqslant C_1 < \infty \tag{72}$$

of the covariant metric tensor $G_{\mathscr{S}} := [\langle \boldsymbol{g}_j, \boldsymbol{g}_k \rangle]_{(n-1)\times(n-1)}$, is a consequence of the definition of the surface.

The similar estimate for the determinant $\mathscr{G}_{\mathscr{S},t} = \det G_{\mathscr{S},t}$

$$0 < D_0 \leqslant \mathscr{G}_{\mathscr{S},t}(x,t) \leqslant D_1 < \infty \tag{73}$$

of the covariant metric tensors $G_{\mathscr{S},t} := \big[\langle \boldsymbol{g}_{j,t}, \boldsymbol{g}_{k,t} \rangle\big]_{(n-1)\times(n-1)}$ of the equidistant surfaces $\mathscr{S}_t$, provided ε is so small that (70) holds, is a consequence of equality

$$d\sigma_t = \sqrt{\mathscr{G}_{\mathscr{S},t}(x,t)}dx, \quad d\sigma = \sqrt{\mathscr{G}_{\mathscr{S}}(x)}\,dx, \quad (x,t) \in \omega^\varepsilon = \omega \times [-\varepsilon, \varepsilon]. \tag{74}$$

and of (69), (72).

Moreover, (69) and (74) imply that

$$\mathscr{G}_{\mathscr{S},t}(x,t) := \big[1 - (n-1)t\mathscr{H}_{\mathscr{S}}(\mathscr{x}) + \cdots + (-t)^{n-1}\mathscr{K}_{\mathscr{S}}(\mathscr{x})\big]^2 \mathscr{G}_{\mathscr{S}}(x), \quad (x,t) \in \omega^\varepsilon = \omega \times [-\varepsilon, \varepsilon]. \tag{75}$$

In particular, for $n = 3$, the following equality is valid:

$$\mathscr{G}_{\mathscr{S},t}(x,t) := \big[1 - 2t\mathscr{H}_{\mathscr{S}}(\mathscr{x}) + t^2\mathscr{K}_{\mathscr{S}}(\mathscr{x})\big]^2 \mathscr{G}_{\mathscr{S}}(x). \tag{76}$$

Hereafter we will tacitly assume that ε is sufficiently small that the mapping Θ in (1) is a diffeomorhism and the estimates (70), (73) are both valid.

A vector field $\mathscr{N}(x)$, defined in the tubular neighborhood Ω^ε of the hypersurface $\mathscr{S}$, is called a *proper extension* of the unit normal vector field $\boldsymbol{\nu}$ on $\mathscr{S}$, if the conditions

$$\partial_k \mathscr{N}_j(x) - \partial_j \mathscr{N}_k(x) = 0 \quad \partial_{\mathscr{N}} \mathscr{N}_j(x) = 0 \quad \text{for all} \quad x \in \Omega^\varepsilon, \quad j,k = 1,\dots,n, \tag{77}$$

hold.

Easy to check, that the extensions given, for example, by equalities (19) and (21), are both proper. A simplest proper extension of $\boldsymbol{\nu}(x)$ is to make the extended field independent of the transversal variable

$$\mathscr{N}(x) = \mathscr{N}(\mathscr{x}, t) := \boldsymbol{\nu}(\mathscr{x}), \quad x = (\mathscr{x}, t) \in \Omega^\varepsilon. \tag{78}$$

Lemma 3.6. *The extended Günter's derivatives* $\mathscr{D}_j := \partial_j - \mathscr{N}_j \partial_{\mathscr{N}}$, $j = 1,\dots,n$ *(cf.* (23)) *for a properly extended unit normal vector field* $\mathscr{N}(x)$, *have the following properties:*

i. $\sum\limits_{j=1}^n \mathscr{N}_j \mathscr{D}_j = 0$;
ii. $\partial_j \mathscr{N}_k = \mathscr{D}_j \mathscr{N}_k = \mathscr{D}_k \mathscr{N}_j$;
iii. $[\mathscr{D}_j, \partial_{\mathscr{N}}] = \langle \mathscr{D}_j \mathscr{N}, \boldsymbol{\nabla} \rangle = \sum\limits_{r=1}^n (\mathscr{D}_j \mathscr{N}_r) \partial_r$;
iv. *The adjoint operator to the Günter's derivative is* $\mathscr{D}_j^* = -\mathscr{D}_j - (n-1)\mathscr{N}_j \mathscr{H}_{\mathscr{S}}$, *where* $\mathscr{H}_{\mathscr{S}}(\mathscr{x})$ *is the mean curvature of* $\mathscr{S}$;

v. *The adjoint operator to the normal derivative is* $\partial^*_{\mathscr{N}} = -\partial_{\mathscr{N}} + (n-1)\mathscr{H}_{\mathscr{S}}$.

Proof. The asserted properties are easy to check (see [12], [13], [14], [16] for these and other similar equalities). □

4. Basic Differential Operators in Curvilinear Coordinates

The n-tuple $\boldsymbol{g}_1 := \partial_1\Theta, \ldots, \boldsymbol{g}_{n-1} := \partial_{n-1}\Theta, \boldsymbol{g}_n := \mathscr{N}$, where $\mathscr{N}$ is the proper extension of $\boldsymbol{\nu}$ in the neighborhood Ω^ε (cf. (77)), is a basis in Ω^ε and arbitrary vector field $\boldsymbol{U} = \sum\limits_{j=1}^{n} U_j^0 \boldsymbol{e}^j \in \mathscr{V}(\Omega^\varepsilon)$ is represented with this basis in "curvilinear coordinates".

Let us consider the system of $(n+1)$-vectors

$$\boldsymbol{d}^j := \boldsymbol{e}^j - \mathscr{N}_j\mathscr{N}, \quad j = 1, \ldots, n \quad \text{and} \quad \boldsymbol{d}^{n+1} := \mathscr{N}, \tag{79}$$

where $\boldsymbol{e}^1, \ldots, \boldsymbol{e}^n$ is the Cartesian frame in $\mathbb{R}^n$ (cf. (16)); the first n vectors $\boldsymbol{d}^1, \ldots, \boldsymbol{d}^n$ are tangential to the surface $\mathscr{S}$ (cf. (23)), while the last one $\boldsymbol{d}^{n+1} = \mathscr{N}$ is orthogonal to it and, thus, to $\boldsymbol{d}^1, \ldots, \boldsymbol{d}^n$. This system is, obviously, linearly dependent, but full and any vector field $\boldsymbol{U} \in \mathscr{V}(\Omega^\varepsilon)$ is written in the following forms:

$$\boldsymbol{U} = \sum_{j=1}^{n} U_j^0 \boldsymbol{e}^j = \sum_{j=1}^{n+1} U_j^0 \boldsymbol{d}^j, \quad \text{where} \quad U_{n+1}^0 := \langle \mathscr{N}, \boldsymbol{U} \rangle = \sum_{j=1}^{n} \mathscr{N}_j U_j^0. \tag{80}$$

Note for a later use, that due to the conventions (79) and (80)

$$\mathscr{N}_{n+1} := \langle \mathscr{N}, \mathscr{N} \rangle = 1, \tag{81}$$

$$\partial_{\boldsymbol{U}} = \sum_{j=1}^{n} U_j^0 \partial_j = \sum_{j=1}^{n+1} U_j^0 \mathscr{D}_j. \tag{82}$$

Since the system $\{\boldsymbol{d}^j\}_{j=1}^{n+1}$ is linearly dependent, the representation of a vector is not unique. Next Lemma 4.1 addresses such representation because it is crucial in the present investigation.

Lemma 4.1. *Vector fields*

$$\boldsymbol{U} = \sum_{j=1}^{n+1} U_j^0 \boldsymbol{d}^j \quad \text{and} \quad \boldsymbol{V} = \sum_{j=1}^{n+1} V_j^0 \boldsymbol{d}^j. \tag{83}$$

coincide $\boldsymbol{U}(\mathscr{X}) = \boldsymbol{V}(\mathscr{X})$ *if and only if*

$$V_{n+1}^0 = U_{n+1}^0 = \langle \mathscr{N}, \boldsymbol{U} \rangle \quad \text{and} \quad V_j^0 = U_j^0 + c\mathscr{N}_j \quad \text{for} \quad j = 1, \ldots, n \tag{84}$$

for arbitrary function $c \in \mathbb{L}_\infty(\mathscr{S})$.

The components $U_1^0, \ldots, U_n^0, U_{n+1}^0$ *in the representation* (83) *of a vector field* $\boldsymbol{U}$ *are defined in a unique way if*

$$U_{n+1}^0 = \langle \widehat{\boldsymbol{U}}, \mathscr{N} \rangle = \sum_{j=1}^{n} U_j^0 \mathscr{N}_j, \quad \widehat{\boldsymbol{U}} := (U_1^0, \ldots, U_n^0)^\top. \tag{85}$$

Proof. We leave the proof to the reader. □

Definition 4.2. For a function $\varphi \in \mathbb{W}^1(\Omega^\varepsilon)$ we define the extended gradient

$$\boldsymbol{\nabla}_{\Omega^\varepsilon}\varphi = \left\{\mathscr{D}_1\varphi, \dots, \mathscr{D}_n\varphi, \mathscr{D}_{n+1}\varphi\right\}^\top, \quad \mathscr{D}_{n+1}\varphi := \partial_{\mathscr{N}}\varphi \tag{86}$$

and for a smooth vector field $\boldsymbol{U} = \sum\limits_{j=1}^{n} U_j^0 \boldsymbol{e}^j = \sum\limits_{j=1}^{n+1} U_j^0 \boldsymbol{d}^j \in \mathscr{V}(\Omega^\varepsilon)$ we define the extended divergence

$$\operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U} := \sum_{j=1}^{n+1} \mathscr{D}_j U_j^0 = -\boldsymbol{\nabla}^*_{\Omega^\varepsilon}\boldsymbol{U}, \tag{87}$$

where (cf. equality (77))

$$\mathscr{D}_{n+1}U^0_{n+1} := \partial_{\mathscr{N}}U^0_{n+1} = \langle \mathscr{N}, \partial_{\mathscr{N}}\boldsymbol{U}\rangle = (\mathscr{D}_{n+1}\boldsymbol{U})^0_{n+1}. \tag{88}$$

Caution: While defining the divergence in (87) we should only use the representation in Cartesian coordinates $\boldsymbol{U} = \sum\limits_{j=1}^{n} U_j^0 \boldsymbol{e}^j \in \mathscr{V}(\Omega^\varepsilon)$, because for other representations, differing from this one by the vector $c\mathscr{N}$ with arbitrary function $c(\mathscr{X})$ (cf. (84)), the divergence will differ by the summand $\operatorname{div}_{\Omega^\varepsilon}\big(c(\mathscr{X})\mathscr{N}(\mathscr{X})\big) = \partial_{\mathscr{N}}c(\mathscr{X}) - (n-1)c(\mathscr{X})\mathscr{H}_{\mathscr{S}}(\mathscr{X})$, where $\mathscr{H}_{\mathscr{S}}$ is the mean curvature of $\mathscr{S}$ (see (66)).

Lemma 4.3. *The classical gradient* $\boldsymbol{\nabla}\varphi := \{\partial_1\varphi, \dots, \partial_n\varphi\}^\top$*, written in the full system of vectors* $\left\{\boldsymbol{d}^j\right\}_{j=1}^{n+1}$ *in* (79) *coincides with the extended gradient* $\boldsymbol{\nabla}_{\Omega^\varepsilon}\varphi$ *in* (86)*.*

Similarly: the classical divergence $\operatorname{div}\boldsymbol{U} := \sum\limits_{j=1}^{n}\partial_j U_j^0$ *of a vector field* $\boldsymbol{U} := \sum\limits_{j=1}^{n} U_j^0\boldsymbol{e}^j$*, written in the full system* (79)*, coincides with the extended divergence* $\operatorname{div}\boldsymbol{U} = \operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}$ *in* (87)*.*

The extended gradient and the negative extended divergence are dual $\boldsymbol{\nabla}^*_{\Omega^\varepsilon} = -\operatorname{div}_{\Omega^\varepsilon}$ *with respect to the scalar product in* $\mathbb{R}^n$*.*

The Laplace–Beltrami operator $\Delta_{\Omega^\varepsilon} := \operatorname{div}_{\Omega^\varepsilon}\boldsymbol{\nabla}_{\Omega^\varepsilon}\varphi = -\boldsymbol{\nabla}^*_{\Omega^\varepsilon}(\boldsymbol{\nabla}_{\Omega^\varepsilon}\varphi)$ *on* Ω^ε*, written in the full system* (79)*, acquires the following form*

$$\Delta_{\Omega^\varepsilon}\varphi = \sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi, \quad \varphi \in \mathbb{W}^2(\Omega^\varepsilon). \tag{89}$$

Proof. That the gradients coincide follows from the choice of the system (79):

$$\begin{aligned}\boldsymbol{\nabla}\varphi := \left\{\partial_1\varphi, \dots, \partial_n\varphi\right\}^\top &= \sum_{j=1}^{n}\boldsymbol{e}^j\partial_j\varphi = \sum_{j=1}^{n}\boldsymbol{e}^j(\mathscr{D}_j\varphi + \mathscr{N}_j\mathscr{D}_{n+1}\varphi) = \\ &= \sum_{j=1}^{n}\boldsymbol{d}^j\mathscr{D}_j\varphi + \mathscr{N}\mathscr{D}_{n+1}\varphi = \sum_{j=1}^{n+1}\boldsymbol{d}^j\mathscr{D}_j\varphi = \boldsymbol{\nabla}_{\Omega^\varepsilon}\varphi\end{aligned}$$

since $\partial_j = \mathscr{D}_j + \mathscr{N}_j\mathscr{N}$ and $\sum\limits_{j=1}^{n} \boldsymbol{e}^j\mathscr{D}_j\varphi = \sum\limits_{j=1}^{n} \boldsymbol{d}^j\mathscr{D}_j\varphi$.

By applying (77) for the divergence we get:

$$\operatorname{div}\boldsymbol{U} = \sum_{j=1}^{n}\partial_j U_j^0 = \sum_{j=1}^{n}\mathscr{D}_j U_j^0 + \sum_{j=1}^{n}\mathscr{N}_j\partial_{\mathscr{N}} U_j^0 = \sum_{j=1}^{n}\mathscr{D}_j U_j^0 + \sum_{j=1}^{n}\partial_{\mathscr{N}}\big(\mathscr{N}_j U_j^0\big) =$$
$$= \sum_{j=1}^{n}\mathscr{D}_j U_j + \partial_{\mathscr{N}} U_{n+1}^0 = \sum_{j=1}^{n+1}\mathscr{D}_j U_j = \operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}. \quad (90)$$

The foregoing assertions combined with the classical equality $\boldsymbol{\nabla}^* = -\operatorname{div}$, ensures the equality $\boldsymbol{\nabla}^*_{\Omega^\varepsilon} = -\operatorname{div}_{\Omega^\varepsilon}$.

Formula (89) for the Laplace–Beltrami operator is a direct consequence of equalities (86) and (87). □

Let us check the following equalities for a later use:

$$\boldsymbol{\nabla}_{\Omega^\varepsilon}\boldsymbol{U} = \big[\mathscr{D}_j U_k^0\big]_{n+1\times n+1} \text{ and, in particular:}$$
$$\boldsymbol{\nabla}_{\Omega^\varepsilon} x = I_{n+1\times n+1}, \ x\in\mathbb{R}^n. \quad (91)$$

where

$$\boldsymbol{U} := \sum_{m=1}^{n+1} U_m^0\boldsymbol{d}^m = \sum_{m=1}^{n} U_m^0\boldsymbol{e}^m,$$
$$U_{n+1}^0 = \sum_{m=1}^{n} U_m^0\nu_m, \quad I_{n+1\times n+1} = [\delta_{jk}]_{n+1\times n+1}.$$

In fact:

$$\mathscr{D}_j\boldsymbol{U} := \mathscr{D}_j\Big[\sum_{m=1}^{n+1} U_k^0\boldsymbol{d}^k\Big] = \mathscr{D}_j\Big[\sum_{k=1}^{n} U_k^0\boldsymbol{e}^k\Big] = \sum_{k=1}^{n}(\partial_j U_k^0)\boldsymbol{e}^k - \nu_j\sum_{k=1}^{n}(\partial_{\boldsymbol{\nu}} U_k^0)\boldsymbol{e}^k =$$
$$= \sum_{k=1}^{n+1}(\partial_j U_k^0)\boldsymbol{d}^k - \nu_j\sum_{k=1}^{n+1}(\partial_{\boldsymbol{\nu}} U_k^0)\boldsymbol{d}^k = \sum_{k=1}^{n+1}(\mathscr{D}_j U_k^0)\boldsymbol{d}^k, \quad j = 1,\dots,n,$$
$$\mathscr{D}_{n+1}\boldsymbol{U} = \partial_{\boldsymbol{\nu}}\boldsymbol{U} = \sum_{m=1}^{n}(\partial_{\boldsymbol{\nu}}\boldsymbol{U}_m^0)\boldsymbol{e}^m = \sum_{k=1}^{n+1}(\partial_{\boldsymbol{\nu}}\boldsymbol{U}_m^0)\boldsymbol{d}^m.$$

The second equality in (91) is a direct consequence of the first one for $\boldsymbol{U}(x) = x$.

Next we will rewrite the deformation tensor. For this we need to define the *extended curvature tensor*

$$\begin{aligned} &\mathfrak{B}_{\Omega^\varepsilon}(x) := \big[\mathfrak{b}_{jk}(x)\big]_{(n+1)\times(n+1)}, \\ &\mathfrak{b}_{jk}(x) := \langle\mathscr{D}_j\boldsymbol{d}^k(x), \mathscr{N}(x)\rangle = -\langle\boldsymbol{d}^k(x), \mathscr{D}_j\mathscr{N}(x)\rangle \end{aligned} \quad (92)$$

in a tubular domain Ω^ε (cf. (2)), where $\mathscr{N}$ is a proper extension of the unit normal vector field $\boldsymbol{\nu}$ on $\mathscr{S}$ (cf. (77)). The concluding equality in (92) is based on the equalities $\langle \boldsymbol{d}^k, \mathscr{N}\rangle = 0, \quad k = 1,\dots,n$, and $\langle \boldsymbol{d}^{n+1}, \mathscr{N}\rangle = \langle \mathscr{N}, \mathscr{N}\rangle = 1$:

$$\langle \mathscr{D}_j \boldsymbol{d}^k(x), \mathscr{N}(x)\rangle = \mathscr{D}_j \langle \boldsymbol{d}^k(x), \mathscr{N}(x)\rangle - \langle \boldsymbol{d}^k(x), \mathscr{D}_j \mathscr{N}(x)\rangle = \langle \boldsymbol{d}^k(x), \mathscr{D}_j \mathscr{N}(x)\rangle.$$

Lemma 4.4. *The following is an equivalent definition of the extended curvature tensor:*

$$\mathfrak{B}_{\Omega^\varepsilon} := -\boldsymbol{\nabla}_{\Omega^\varepsilon} \mathscr{N}. \tag{93}$$

The last column and the last row in the extended curvature tensor $\mathfrak{B}_{\Omega^\varepsilon}$ are trivial

$$\mathfrak{b}_{j(n+1)} = \mathfrak{b}_{(n+1)j} = 0, \ \ j = 1,\dots,n+1. \tag{94}$$

The remainder $\big[\mathfrak{b}_{jk}\big]_{n\times n}$ coincides with the extended Weingarten matrix

$$\mathfrak{b}_{jk} = -\mathscr{D}_j \mathscr{N}_k, \ \ j,k = 1,\dots,n+1, \tag{95}$$

and restricted to the middle surface $\mathscr{S}$, coincides with the Weingarten matrix

$$\mathscr{W}_{\Omega^\varepsilon}\big|_{\mathscr{S}} = \mathscr{W}_{\mathscr{S}} = -[\mathscr{D}_j \nu_k]_{n\times n}.$$

Proof. Equality (93) is a direct consequence of equalities (94) and (95) . Thus, we concentrate on the proof of the latter two equalities.

To prove (94) we invoke (92) and proceed as follows:

$$\begin{aligned}\mathfrak{b}_{j(n+1)} &= \langle \mathscr{D}_j \boldsymbol{d}^{n+1}, \mathscr{N}\rangle = \langle \mathscr{D}_j \mathscr{N}, \mathscr{N}\rangle = \frac{1}{2}\mathscr{D}_j \langle \mathscr{N}, \mathscr{N}\rangle = \frac{1}{2}\mathscr{D}_j 1 = 0,\\ \mathfrak{b}_{(n+1)j} &= \langle \partial_{\boldsymbol{d}^{n+1}} \boldsymbol{d}^j, \mathscr{N}\rangle = \langle \partial_{\mathscr{N}} \boldsymbol{d}^j, \mathscr{N}\rangle = -\langle \boldsymbol{d}^j, \partial_{\mathscr{N}} \mathscr{N}\rangle = 0.\end{aligned}$$

We have applied that $\langle \mathscr{N}, \mathscr{N}\rangle = 1$, $\partial_{\mathscr{N}} \mathscr{N} = 0$ and $\langle \boldsymbol{d}^j, \mathscr{N}\rangle = 0$ for all $j = 1,\dots,n$.

To prove (95) it suffices to note that

$$\begin{aligned}\mathfrak{b}_{jk} &= -\langle \boldsymbol{d}^k, \mathscr{D}_j \mathscr{N}\rangle = -\langle \boldsymbol{e}^k - \mathscr{N}_k \mathscr{N}, \mathscr{D}_j \mathscr{N}\rangle = \\ &= -\mathscr{D}_j \mathscr{N}_k + \mathscr{N}_k \langle \mathscr{N}, \mathscr{D}_j \mathscr{N}\rangle = -\mathscr{D}_j \mathscr{N}_k, \ \ \forall\, j,k = 1,\dots,n,\end{aligned}$$

since $\langle \mathscr{D}_j \mathscr{N}, \mathscr{N}\rangle = 0$ for all $j = 1,\dots,n$. □

Corollary 4.5. *Let Ω^ε be a tubular domain in the 3-dimensional Euclidean space $\mathbb{R}^3$ with the middle hypersurfaces $\mathscr{S} = \mathscr{S}_0$ and $\mathscr{S}_t$ be the equidistant hypersurfaces defined in* (60).

The Weingarten matrix, the mean and the Gaußian curvatures of the surfaces $\mathscr{S}_t$ are independent of the parameter t:

$$\begin{gathered}\mathscr{W}_{\mathscr{S}_t} \equiv \mathscr{W}_{\mathscr{S}} = -\big[\mathscr{D}_j \nu_k\big]_{n\times n},\\ \mathscr{H}_{\mathscr{S}_t} = \mathscr{H}_{\mathscr{S}} = \frac{1}{2}\,\mathrm{Tr}\mathscr{W}_{\mathscr{S}_t}, \ \ \mathscr{K}_{\mathscr{S}_t} = \mathscr{K}_{\mathscr{S}} = \lambda_1\lambda_2 \ \ \forall\, t \in (-\varepsilon, \varepsilon),\end{gathered} \tag{96}$$

where λ_1 and λ_2 are those eigenvalues of the Weingarten matrix $\mathscr{W}_{\mathscr{S}}$ which are left after removing two trivial eigenvalues $\lambda_3 = \lambda_4 = 0$.

Proof. The proof follows immediately from the foregoing Lemma 4.4 and form the independence of the vector field $\boldsymbol{\nu}_t(\mathscr{X}_t) = \boldsymbol{\nu}(\mathscr{X})$ from the transversal variable $t \in (-\varepsilon, \varepsilon)$ (see (61)). □

5. Elastic Deformation in Curvilinear Coordinates

Theorem 5.1. *The classical deformation tensor*

$$\mathrm{Def}_{\Omega^\varepsilon}(\boldsymbol{U}) := \left[\frac{1}{2}\left[\partial_j U_k + \partial_k U_j\right]\right]_{n\times n}, \quad \boldsymbol{U} \in \mathscr{V}(\Omega^\varepsilon),$$

written in the full system of vectors $\{\boldsymbol{d}^j\}_{j=1}^{n+1}$ *in* (79), *has the form*

$$\mathrm{Def}_{\Omega^\varepsilon}(\boldsymbol{U}) := \left[\mathfrak{D}_{jk}(\boldsymbol{U})\right]_{(n+1)\times(n+1)}, \tag{97}$$

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \frac{1}{2}\Big[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0 + (\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U})_j^0\Big], \quad j,k = 1,\dots,n+1, \tag{98}$$

$$(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0 :=$$

$$:= \begin{cases} \mathscr{D}_j U_k^0 - \langle \mathscr{N}, \mathscr{D}_j \boldsymbol{U}\rangle \mathscr{N}_k & \text{for } j = 1,\dots,n+1, \ \ k = 1,2,\dots,n, \\ \langle \mathscr{N}, \mathscr{D}_j \boldsymbol{U}\rangle = (\mathscr{D}_j\boldsymbol{U})_{n+1}^0 & \text{for } j = 1,\dots,n+1, \ \ k = n+1 \end{cases}$$

and, in particular (*cf.* (88)),

$$(\mathscr{D}_{n+1}^{\mathscr{S}}\boldsymbol{U})_{n+1}^0 = \langle \mathscr{N}, \partial_{\mathscr{N}}\boldsymbol{U}\rangle = \partial_{\mathscr{N}}\langle \mathscr{N}, \boldsymbol{U}\rangle = \partial_{\mathscr{N}} U_{n+1}^0. \tag{99}$$

Proof. To prove (97) let us consider the full systems

$$\begin{gathered} \left\{\boldsymbol{e}^{jk} := \boldsymbol{e}^j \otimes \boldsymbol{e}^k\right\}_{j=1}^{n+1}, \quad \left\{\boldsymbol{d}^{jk} := \boldsymbol{d}^j \otimes \boldsymbol{d}^k\right\}_{j=1}^{n+1}, \\ \boldsymbol{e}^j = \boldsymbol{d}^j + \mathscr{N}_j \boldsymbol{d}^{n+1}, \ \ j = 1,\dots,n, \ \ \boldsymbol{e}^{n+1} := \boldsymbol{d}^{n+1} := \mathscr{N}, \end{gathered} \tag{100}$$

apply the equalities

$$\begin{gathered} \partial_j = \mathscr{D}_j + \mathscr{N}_j \mathscr{D}_{n+1}, \quad \partial_j \mathscr{N}_k = \partial_k \mathscr{N}_j, \quad \mathscr{D}_{n+1}\mathscr{N}_k = \partial_{\mathscr{N}}\mathscr{N}_k = 0, \\ \sum_{j=1}^n \mathscr{N}_j \mathscr{D}_j = 0, \quad U_{n+1}^0 = \sum_{j=1}^n \mathscr{N}_j U_j^0, \quad \sum_{j=1}^n \mathscr{N}_j \boldsymbol{d}^{jk} = \sum_{k=1}^n \mathscr{N}_k \boldsymbol{d}^{jk} = 0 \end{gathered} \tag{101}$$

(cf. (77)) and derive

$$\mathrm{Def}(\boldsymbol{U}) = \frac{1}{2}\sum_{j,k=1}^n (\partial_j U_k^0 + \partial_k U_j^0)\boldsymbol{e}^{jk} =$$

$$= \frac{1}{2}\sum_{j,k=1}^n \big[\partial_j U_k^0 + \partial_k U_j^0\big](\boldsymbol{d}^{jk} + \mathscr{N}_j \boldsymbol{d}^{n+1,k} + \mathscr{N}_k \boldsymbol{d}^{j,n+1} + \mathscr{N}_j\mathscr{N}_k \boldsymbol{d}^{n+1,n+1}) =$$

$$= \frac{1}{2}\sum_{j,k=1}^n \big[\partial_j U_k^0 + \partial_k U_j^0\big]\boldsymbol{d}^{jk} + \frac{1}{2}\sum_{j=1}^n \Big[\mathscr{D}_{n+1}U_j^0 + \sum_{k=1}^n \mathscr{N}_k \partial_j U_k^0\Big]\boldsymbol{d}^{j,n+1} +$$

$$+ \frac{1}{2}\sum_{k=1}^n \Big[\mathscr{D}_{n+1}U_k^0 + \sum_{j=1}^n \mathscr{N}_j \partial_k U_j^0\Big]\boldsymbol{d}^{n+1,k} +$$

$$+\sum_{j=1}^{n}\mathscr{N}_j(\mathscr{D}_{n+1}U_j^0)\boldsymbol{d}^{n+1,n+1}=$$

$$=\frac{1}{2}\sum_{j,k=1}^{n}\Big[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0+(\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U})_j^0+\langle\mathscr{D}_j\boldsymbol{U},\mathscr{N}\rangle\mathscr{N}_k+\langle\mathscr{D}_k\boldsymbol{U},\mathscr{N}\rangle\mathscr{N}_j\Big]\boldsymbol{d}^{jk}+$$

$$+\frac{1}{2}\sum_{j=1}^{n}\Bigg[(\mathscr{D}_{n+1}^{\mathscr{S}}\boldsymbol{U})_j^0+\langle\mathscr{D}_{n+1}\boldsymbol{U},\mathscr{N}\rangle\mathscr{N}_j+$$

$$+\sum_{k=1}^{n}\big[\mathscr{N}_k\mathscr{D}_jU_k^0+\mathscr{N}_j\mathscr{N}_k\mathscr{D}_{n+1}U_k^0\big]\Bigg]\boldsymbol{d}^{j,n+1}+$$

$$+\frac{1}{2}\sum_{k=1}^{n}\Bigg[(\mathscr{D}_{n+1}^{\mathscr{S}}\boldsymbol{U})_k^0+\langle\mathscr{D}_{n+1}\boldsymbol{U},\mathscr{N}\rangle\mathscr{N}_k+$$

$$+\sum_{j=1}^{n}\big[\mathscr{N}_j\mathscr{D}_kU_j^0+\mathscr{N}_k\mathscr{N}_j\mathscr{D}_{n+1}U_j^0\big]\Bigg]\boldsymbol{d}^{n+1,k}+$$

$$+\sum_{j=1}^{n}\mathscr{D}_{n+1}(\mathscr{N}_jU_j^0)\boldsymbol{d}^{n+1,n+1}=$$

$$=\frac{1}{2}\sum_{j,k=1}^{n}\Big[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0+(\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U})_j^0]\Big]\boldsymbol{d}^{jk}+$$

$$+\frac{1}{2}\sum_{j=1}^{n}\big[(\mathscr{D}_{n+1}^{\mathscr{S}}\boldsymbol{U})_j^0+(\mathscr{D}_j\boldsymbol{U})_{n+1}^0\big]\boldsymbol{d}^{j,n+1}+$$

$$+\frac{1}{2}\sum_{k=1}^{n}\big[(\mathscr{D}_{n+1}^{\mathscr{S}}\boldsymbol{U})_k^0+(\mathscr{D}_k\boldsymbol{U})_{n+1}^0\big]\boldsymbol{d}^{n+1,k}+\mathscr{D}_{n+1}U_{n+1}^0\boldsymbol{d}^{n+1,n+1}=$$

$$=\sum_{j,k=1}^{n+1}\mathfrak{D}_{jk}(\boldsymbol{U})\boldsymbol{d}^{jk}.\qquad\square$$

Now we are prepared to write the Lamé operator in curvilinear coordinates, namely in the full system (79). The medium is assumed to be isotropic.

Let $\boldsymbol{U}\in\mathbb{W}^1(\Omega^\varepsilon)$ be a displacement field of the body subject to deformation by surface and volume forces. We follow Koiter's model as in [16]: depart from the total free elastic energy

$$\mathscr{E}[\boldsymbol{U}]:=\int_{\Omega^\varepsilon}E(x,\boldsymbol{\nabla}\boldsymbol{U}(x))\,dx\quad\boldsymbol{\nabla}\boldsymbol{U}:=\big[(\partial_jU_k\big]_{n\times n},\tag{102}$$

$$\boldsymbol{U}=(U_1,\ldots,U_n)^\top\in\mathbb{W}^1(\Omega^\varepsilon)$$

and ignore at the moment the displacement boundary conditions. Equilibria states correspond to minimizers of the above variational integral (see [26, § 5.2]). At the moment the most important is to identify a correct form of the stored energy density $E(x, \boldsymbol{\nabla}\boldsymbol{U}(x))$. It turns out that the case of linear elasticity the energy density depends bi-linearly on the stress tensor $\mathfrak{S} = \left[\mathfrak{S}^{jk}\right]_{n\times n}$ and the deformation (strain) tensor $\mathrm{Def}_{\,\Omega^\varepsilon}$ (see (98)): $E(x, \boldsymbol{\nabla}\boldsymbol{U}(x)) = E(\mathfrak{S}, \mathrm{Def}_{\,\Omega^\varepsilon}\boldsymbol{U})$. The result formulated in the next Theorem 5.2 is well known (see [16] for the proof in the above described scenario).

Theorem 5.2. *A vector field* $\boldsymbol{U} \in \mathscr{V}(\Omega^\varepsilon)$ *minimizes the free elastic energy* (102) *modeling a homogeneous, linear, isotropic, elastic medium, if and only if it is a solution to the following Lamé equation*

$$\mathscr{L}_{\Omega^\varepsilon}\boldsymbol{U} = 2\mu \mathrm{Def}^{\,*}_{\,\Omega^\varepsilon}\mathrm{Def}_{\,\Omega^\varepsilon}\boldsymbol{U} - \lambda\boldsymbol{\nabla}_{\Omega^\varepsilon}\mathrm{div}_{\,\Omega^\varepsilon}\boldsymbol{U} = 0, \tag{103}$$

where λ *and* μ *are the Lamé constants.*

Since $\mathrm{Def}_{\,\Omega^\varepsilon}\boldsymbol{U}$ is the classical deformation tensor, just represented in a different (full) system of vectors, solutions to the system of partial differential equations

$$\mathrm{Def}_{\,\Omega^\varepsilon}\boldsymbol{U} = 0, \quad \boldsymbol{U} \in \mathbb{W}^1(\Omega^\varepsilon)$$

coincides with the space of *rigid motions* $\mathscr{R}(\Omega^\varepsilon)$, which consists of linear vector-functions

$$\boldsymbol{V}(x) = a + \mathscr{B}x, \quad \mathscr{B} = \left[b_{jk}\right]_{n\times n}, \quad a \in \mathbb{R}^n, \quad x \in \Omega^\varepsilon, \tag{104}$$

restricted to the domain $\Omega\varepsilon$. The matrix $\mathscr{B}$ in (104) is skew symmetric

$$\mathscr{B} := \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1(n-2)} & b_{1(n-1)} \\ -b_{12} & 0 & b_{21} & \cdots & b_{1(n-3)} & b_{2(n-2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{1(n-2)} & -b_{2(n-3)} & -b_{3(n-4)} & \cdots & 0 & b_{(n-1)1} \\ -b_{1(n-1)} & -b_{2(n-2)} & -b_{3(n-3)} & \cdots & -b_{(n-1)1} & 0 \end{bmatrix} = \\ = -\mathscr{B}^\top \tag{105}$$

with real valued entries $b_{jk} \in \mathbb{R}$. For $n = 3, 4, \ldots$ the space $\mathscr{R}(\Omega^\varepsilon)$ is finite dimensional and $\dim \mathscr{R}(\Omega^\varepsilon) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Note that for $n = 3$ the vector field $\boldsymbol{V} \in \mathscr{R}(\Omega^\varepsilon)$ is the classical rigid displacement

$$\begin{array}{c} \boldsymbol{V}(x) = a + \mathscr{B}x = a + b \wedge x, \\ b := (b_1, b_2, b_3)^\top \in \mathbb{R}^3, \quad x \in \Omega^\varepsilon, \end{array} \qquad \mathscr{B} := \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}. \tag{106}$$

Moreover, Korn's inequality

$$\left\|\boldsymbol{U}\middle|\mathbb{W}^1(\Omega^\varepsilon)\right\| \leqslant M\left[\left\|\boldsymbol{U}\middle|\mathbb{L}_2(\Omega^\varepsilon)\right\|^2 + \left\|\mathrm{Def}_{\,\Omega^\varepsilon}\boldsymbol{U}\middle|\mathbb{L}_2(\Omega^\varepsilon)\right\|^2\right]^{1/2} \tag{107}$$

holds with some constant $M > 0$. For the classical deformation tensor in Cartesian coordinates $\mathrm{Def}\,(\boldsymbol{U})$ the inequality (107) is well known (see e.g., [2]).

Theorem 5.3. *The Lamé operator* $\mathscr{L}_{\Omega^\varepsilon}$ *in* (103) *is a formally self-adjoint differential operator of second order and, written in the full system* (79), *acquires the form*

$$\mathscr{L}_{\Omega^\varepsilon}\boldsymbol{U} = -\mu\boldsymbol{\Delta}_{\Omega^\varepsilon}\boldsymbol{U} - (\lambda+\mu)\boldsymbol{\nabla}_{\Omega^\varepsilon}\mathrm{div}\,_{\Omega^\varepsilon}\boldsymbol{U} = \\ = -\left[\mu\delta_{jk}\boldsymbol{\Delta}_{\Omega^\varepsilon} + (\lambda+\mu)\mathscr{D}_j\mathscr{D}_k\right]_{n+1\times n+1}\boldsymbol{U}. \tag{108}$$

The operator $\mathscr{L}_{\Omega^\varepsilon}$ *is positive definite modulo rigid motions* (*see* (104) *and* (106))*:*

$$\big(\mathscr{L}_{\Omega^\varepsilon}\boldsymbol{U},\boldsymbol{U}\big)_{\mathscr{S}} \geqslant 2\mu\big\|\mathrm{Def}\,_{\Omega^\varepsilon}\boldsymbol{U}\big|\mathbb{L}_2(\Omega^\varepsilon)\big\|^2. \tag{109}$$

Moreover, the operator $\mathscr{L}_{\Omega^\varepsilon}$ *satisfies Gårding's inequality*

$$\big(\mathscr{L}_{\Omega^\varepsilon}\boldsymbol{U},\boldsymbol{U}\big)_{\mathscr{S}} \geqslant \frac{2\mu}{M}\big\|\boldsymbol{U}\big|\mathbb{W}^1(\Omega^\varepsilon)\big\|^2 - \big\|\boldsymbol{U}\big|\mathbb{L}_2(\Omega^\varepsilon)\big\|^2, \tag{110}$$

where M *is the constant from* (107) *for* $p=2$ *and* μ *is the constant from* (109).

Proof. To prove (108), we depart from the representation of the deformation tensor

$$\mathrm{Def}\,(\boldsymbol{U}) = \frac{1}{2}\sum_{j,k=1}^{n}(\partial_j U_k^0 + \partial_k U_j^0)\boldsymbol{e}^{jk}$$

in the Cartesian frame, and get

$$\big(\mathrm{Def}\,^*\mathrm{Def}\,(\boldsymbol{U}),\boldsymbol{V}\big) = \big(\mathrm{Def}\,(\boldsymbol{U}),\mathrm{Def}\,\boldsymbol{V}\big) = \\ = \frac{1}{4}\sum_{j,k=1}^{n}\int_{\Omega^\varepsilon}(\partial_j U_k + \partial_k U_j)(\partial_j V_k + \partial_k V_j)\,dy = \\ = \frac{1}{2}\sum_{j,k=1}^{n}\int_{\Omega^\varepsilon}\partial_j^*(\partial_j U_k + \partial_k U_j)V_k\,dy,$$

Then

$$\mathrm{Def}\,^*\mathrm{Def}\,(\boldsymbol{U}) = \frac{1}{2}\left\{\sum_{k=1}^{n}\partial_k^*(\partial_k+\partial_j)U_j^0\right\}_{j=1}^{n} = \\ = \frac{1}{2}\left\{\sum_{k=1}^{n}(-\partial_k^2 - \partial_j\partial_k)U_j^0\right\}_{j=1}^{n} = \\ = -\frac{1}{2}\,\boldsymbol{\Delta}\boldsymbol{U} - \frac{1}{2}\,\boldsymbol{\nabla}\mathrm{div}\,\boldsymbol{U},\quad \boldsymbol{U}\in\mathscr{V}(\Omega^\varepsilon) \tag{111}$$

which gives the well-known representation of the Lamé operator in the same (canonical) basis

$$\mathscr{L}_{\Omega^\varepsilon}\boldsymbol{U} = 2\mu\mathrm{Def}\,^*\mathrm{Def}\,\boldsymbol{U} - \lambda\boldsymbol{\nabla}\mathrm{div}\,\boldsymbol{U} = \\ = -\big[\mu\delta_{jk}\boldsymbol{\Delta}_{\Omega^\varepsilon}U_k^0 + (\lambda+\mu)\mathscr{D}_j\mathscr{D}_k U_k^0\big]_{n\times n} = \\ = -\mu\boldsymbol{\Delta}\boldsymbol{U} - (\lambda+\mu)\boldsymbol{\nabla}\mathrm{div}\,\boldsymbol{U} = \\ = -\mu\boldsymbol{\Delta}_{\Omega^\varepsilon}\boldsymbol{U} - (\lambda+\mu)\boldsymbol{\nabla}_{\Omega^\varepsilon}\mathrm{div}\,_{\Omega^\varepsilon}\boldsymbol{U} \tag{112}$$

(cf. e.g. [2], [21]). The claimed equality (108) follows from (112) if we insert the representation of the operators $\boldsymbol{\nabla} = \boldsymbol{\nabla}_{\Omega^\varepsilon}$, div $= \operatorname{div}_{\Omega^\varepsilon}$ and $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\Omega^\varepsilon}$ in the system $\{\boldsymbol{d}^j\}_{j=1}^{n+1}$ from Lemma 4.3.

The inequality (109) is a simple consequence of (103)

$$(\mathscr{L}_{\Omega^\varepsilon}\boldsymbol{U},\boldsymbol{U})_{\Omega^\varepsilon} = 2\mu\big\|\operatorname{Def}_{\Omega^\varepsilon}(\boldsymbol{U})\big|\mathbb{L}^2(\Omega^\varepsilon)\big\|^2 + \lambda\big\|\operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}\big|\mathbb{L}^2(\Omega^\varepsilon)\big\|^2 \geqslant \\ \geqslant 2\mu\big\|\operatorname{Def}_{\Omega^\varepsilon}\boldsymbol{U}\big|\mathbb{L}^2(\Omega^\varepsilon)\big\|^2,$$

while Gårding's inequality (109) follows from (109) and from the classical Korn's inequality (107):

$$(\mathscr{L}_{\Omega^\varepsilon}\boldsymbol{U},\boldsymbol{U})_{\Omega^\varepsilon} \geqslant 2\mu\big\|\operatorname{Def}_{\Omega^\varepsilon}\boldsymbol{U}\big|\mathbb{L}^2(\Omega^\varepsilon)\big\|^2 \geqslant \\ \geqslant \frac{2\mu}{M}\big\|\boldsymbol{\nabla}_{\Omega^\varepsilon}\boldsymbol{U}\big|\mathbb{W}^1(\Omega^\varepsilon)\big\|^2 - \big\|\boldsymbol{\nabla}_{\Omega^\varepsilon}\boldsymbol{U}\big|\mathbb{L}_2(\Omega^\varepsilon)\big\|^2. \qquad \square$$

Let us reveal the geometric content of the extended deformation tensor. For this we consider the linearized change of the extended metric tensor on a surface.

If vector fields $\boldsymbol{U}$ and $\boldsymbol{V}$ are represented in the extended Cartesian system

$$\boldsymbol{U} = \sum_{j=1}^{n} U_j^0\boldsymbol{e}^j = \sum_{j=1}^{n+1} U_j^0\boldsymbol{d}^j, \quad \boldsymbol{V} = \sum_{j=1}^{n} V_j^0\boldsymbol{e}^j = \sum_{j=1}^{n+1} V_j^0\boldsymbol{d}^j, \tag{113}$$
$$\boldsymbol{d}^{n+1} = \boldsymbol{\nu}, \;\; U_{n+1}^0 = \langle\boldsymbol{U},\boldsymbol{\nu}\rangle, \;\; V_{n+1}^0 = \langle\boldsymbol{V},\boldsymbol{\nu}\rangle,$$

The linearized change of the metric tensor becomes a matrix of order $n+1$.

Corollary 5.4. *The linearized change*

$$\mathfrak{D}_{\mathscr{S}}(\boldsymbol{U}) = \big[\mathfrak{d}_{jk}(\boldsymbol{U})\big]_{(n+1)\times(n+1)} := \big[\{g_{jk}(\boldsymbol{U}) - g_{jk}\}^{\rm lin}\big]_{(n+1)\times(n+1)} \tag{114}$$

of the extended metric tensor by a displacement vector field $\boldsymbol{U}$ *in* (113) *coincides with the extended deformation tensor*

$$\mathfrak{d}_{jk}(\boldsymbol{U}) = \frac{1}{2}\left[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0 + (\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U})_j^0\right] = \mathfrak{D}_{jk}(\boldsymbol{U}), \;\; j,k = 1,\dots,n+1, \tag{115}$$

defined in (97), (98).

Proof. Changing the extended contravariant frame (see (50)) to the extended Cartesian full system (see (79)), due to Theorem 5.1, the deformation tensor $\widetilde{\mathfrak{D}}_{\mathscr{S}}(\boldsymbol{U}) = [\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U})]_{n\times n}$ changes to the deformation tensor $\mathfrak{D}_{\mathscr{S}}(\boldsymbol{U}) = [\mathfrak{D}_{jk}(\boldsymbol{U})]_{(n+1)\times(n+1)}$. Then (115) follows from (56). $\square$

Theorem 5.5. *The linearized change of metric the tensor on a surface*

$$G_{\mathscr{S}} = \big[g_{jk}\big]_{(n-1)\times(n-1)}, \quad g_{jk} = \langle\boldsymbol{g}_j,\boldsymbol{g}_k\rangle, \;\; j,k = 1,\dots,n-1, \tag{116}$$

by a non-tangential displacement vector field represented in the contravariant frame (50) *is given by the formulae:*

$$\mathfrak{g}_{jk} := \{g_{jk}(\boldsymbol{U}) - g_{jk}\}^{\rm lin} = \frac{1}{2}\left[U_{j;k} + U_{k;j}\right] - b_{jk}U_n = \\ = \widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}) - b_{jk}U_n, \quad j,k = 1,\dots,n-1. \tag{117}$$

Here g_{jk} and $g_{jk}(\boldsymbol{U})$ denote the covariant metric tensors of the surfaces $\mathscr{S}$ before and after a displacement $\boldsymbol{U}$ is applied $\mathscr{S}(\boldsymbol{U}) = \mathscr{S} + \boldsymbol{U}$, respectively; $U_{j;k}$ denotes the covariant derivative (*cf.* (39)). *The notation $\{\,\cdot\,\}^{\rm lin}$ indicates that we ignore all non-linear summands inside the bracket.*

Proof. Similarly to (57), by applying (32), (33) and (40) we get:

$$\partial_j\Big[\sum_{r=1}^{n-1} U_r\boldsymbol{g}^r + U_n\boldsymbol{\nu}\Big] = \sum_{r=1}^{n-1}(\partial_j U_r)\boldsymbol{g}^r + (\partial_j U_n)\boldsymbol{\nu} + \sum_{r=1}^{n-1} U_r\partial_j\boldsymbol{g}^r + U_n\partial_j\boldsymbol{\nu} = \\ = \sum_{r=1}^{n-1}\Big[\partial_j U_r - \sum_{m=1}^{n-1}\Gamma_{jr}^m U_m\Big] = \\ = \sum_{r=1}^{n-1} U_{r;j}\boldsymbol{g}^r + (\partial_j U_n)\boldsymbol{\nu} - U_n\sum_{r=1}^{n-1} b_{jr}\boldsymbol{g}^r, \quad j = 1,\dots,n-1. \tag{118}$$

Again, we apply that $\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk}$, $\langle \boldsymbol{\nu}, \boldsymbol{g}^k\rangle = 0$, $j,k = 1,\dots,n-1$, ignore non-linear terms like $U_{r;j}U_{m;k}$, $U_{r;j}\partial_j U_n$, $U_{r;j}U_n$, and get:

$$\mathfrak{g}_{jk} = \big\{g_{jk}(\boldsymbol{U}) - g_{jk}\big\}^{\rm lin} = \frac{1}{2}\Big\{\langle\partial_j\Theta + \partial_j\boldsymbol{U}, \partial_k\Theta + \partial_k\boldsymbol{U}\rangle - \langle\partial_j\Theta, \partial_k\Theta\rangle\Big\}^{\rm lin} = \\ = \frac{1}{2}\left\{\Big\langle \boldsymbol{g}_j + \partial_j\Big[\sum_{r=1}^{n} U_r\boldsymbol{g}^r\Big], \boldsymbol{g}_k + \partial_k\Big[\sum_{m=1}^{n} U_m\boldsymbol{g}^m\Big]\Big\rangle - \langle\boldsymbol{g}_j, \boldsymbol{g}_k\rangle\right\}^{\rm lin} = \\ = \frac{1}{2}\left[U_{j;k} - b_{jk}U_n + U_{k;j} - b_{kj}U_n\right] = \widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}) - b_{jk}U_n. \qquad \square$$

If vector fields $\boldsymbol{U}$ and $\boldsymbol{V}$ are represented in the extended Cartesian system (113), the linearized change of the metric tensor on a surface becomes a matrix of order n while the matrix $\big[b_{jk}\big]_{(n-1)\times(n-1)}$ transforms into the Weingarten matrix $\big[\partial_j\nu_k\big]_{n\times n}$ (cf. [15]) and we prove the following.

Corollary 5.6. *The linearized change*

$$\mathfrak{D}_{\mathscr{S}}(\boldsymbol{U}) = \big[\mathfrak{d}_{jk}(\boldsymbol{U})\big]_{n\times n} := \big[\{d^{jk}(\boldsymbol{U}) - d^{jk}\}^{\rm lin}\big]_{n\times n} \tag{119}$$

of metric tensor on a surface by a non-tangential displacement vector field $\boldsymbol{U}$ in (113) *coincides with the extended deformation tensor*

$$\mathfrak{d}_{jk}(\boldsymbol{U}) := \big\{d^{jk}(\boldsymbol{U}) - d^{jk}\big\}^{\rm lin} = \frac{1}{2}\left[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0 + (\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U})_j^0\right] - (\partial_j\nu_k)U_{n+1}^0 = \\ = \mathfrak{D}_{jk}(\boldsymbol{U}) - (\partial_j\nu_k)\langle\boldsymbol{U}, \boldsymbol{\nu}\rangle, \quad j,k = 1,\dots,n, \tag{120}$$

defined in (97), (98).

6. The Three Dimensional Equations Over a Domain Independent of ε

In the the present section we describe the basic preliminaries of the asymptotic analysis of a linearly elastic shell, based on a similar analysis by Sanchez-Palencia [31], [32], Miara & Sanchez-Palencia [23], Ciarlet & Lods [6]–[8], Ciarlet, Lods & Miara [9] and exposed in details in Ciarlet [3], [5].

In the present section we accept the common Einstein's convention and drop the sum, interpreting repeated indices as a summation. In particular, The vector filed $\boldsymbol{U}$ in a tubular domain of the three dimensional Euclidean space $\Omega^\varepsilon \subset \mathbb{R}^3$ with a middle surface $\mathscr{S} \subset \Omega^\varepsilon$ (cf. (1), (2)) will be written as

$$\begin{aligned} &\boldsymbol{U}^\varepsilon(x) = U_j^\varepsilon \boldsymbol{d}^j, \ \text{ which means } \ \boldsymbol{U}^\varepsilon(x) = \sum_{j=1}^{n+1} U_j^\varepsilon \boldsymbol{d}^j \\ &\text{or } \ \boldsymbol{U}^\varepsilon(x) = U_j^\varepsilon \boldsymbol{e}^j, \ \text{ which means } \ \boldsymbol{U}^\varepsilon(x) = \sum_{j=1}^{n} U_j^\varepsilon \boldsymbol{e}^j \end{aligned} \tag{121}$$

where the system of unit (linearly dependent) vectors $\boldsymbol{d}^1, \dots, \boldsymbol{d}^n, \boldsymbol{d}^{n+1}$ is defined in (79), while $\boldsymbol{e}^1, \dots, \boldsymbol{e}^n$ is the natural Cartesian basis, defined in (16). The sum will appear if the summation is not full, e.g.,

$$\sum_{j=1}^{n} U_j \boldsymbol{d}^j \ \text{ or } \ \sum_{j=1}^{n-1} U_j \boldsymbol{e}^j.$$

Along with the domains $\omega \subset \mathbb{R}^{n-1}$, $\omega^\varepsilon \subset \mathbb{R}^n$ (see (2)) and the tubular domain $\Omega^\varepsilon \subset \mathbb{R}^n$ (see (2)) we define the sets

$$\begin{gathered} \Gamma = \partial\mathscr{S}, \quad \omega_\pm^\varepsilon := \omega \times \{\pm\varepsilon\}, \\ \mathscr{S}_\pm^\varepsilon = \Theta(\omega_\pm^\varepsilon) := \big\{x \in \mathbb{R}^n : \ x = \Theta(y) : \ y \in \omega_\pm^\varepsilon\big\} = \mathscr{S} \times \{\pm\varepsilon\}. \end{gathered} \tag{122}$$

Let $\Gamma_0 \subset \Gamma := \partial\mathscr{S}$ be a measurable subset of the boundary of hypersurface $\mathscr{S}$. Along with a "thin" domain $\Omega^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$ we consider the scaled domain $\Omega^1 := \Omega \times (-1, 1)$, its lateral surface Γ^1, the subsurface Γ_0^1 and the upper and the lower hypersurfaces $\mathscr{S}_\pm$:

$$\begin{gathered} \operatorname{mes}\Gamma_0 \neq 0, \ \ \Gamma^1 := \Gamma \times [-1, 1], \ \ \Gamma_0^1 := \Gamma_0 \times [-1, 1], \\ \mathscr{S}_\pm = \mathscr{S}_\pm^1 := \mathscr{S} \times \{\pm 1\}. \end{gathered} \tag{123}$$

Similarly, we consider the "scaled" initial domain $\omega^1 := \omega \times [-1, 1]$ and its two "faces" $\omega_\pm := \omega_\pm^1 = \omega \times \{\pm 1\}$.

We establish the isomorphism

$$\pi^\varepsilon : \Omega^1 \longrightarrow \Omega^\varepsilon, \quad \pi^\varepsilon \mathscr{X} = \mathscr{X}^\varepsilon, \ \ \forall\, \mathscr{X} = (\mathscr{X}_1, \dots, \mathscr{X}_n, t)^\top \in \Omega^1, \tag{124}$$

$$\mathscr{X}^\varepsilon = (\mathscr{X}_1, \dots, \mathscr{X}_n, \varepsilon t)^\top \in \Omega^\varepsilon$$

and introduce the differentiation with respect to these variables

$$\mathscr{D}_j^\varepsilon := \mathscr{D}_j = \partial_{\boldsymbol{d}^j}, \ \ j = 1, , \dots, n, \ \ \mathscr{D}_{n+1}^\varepsilon := \partial_{\boldsymbol{d}_{n+1}^\varepsilon} = \frac{1}{\varepsilon}\, \mathscr{D}_{n+1} = \frac{1}{\varepsilon}\, \partial_{\mathscr{N}}. \tag{125}$$

The variables $\mathscr{X}^\varepsilon_{n+1} = \varepsilon t$ and t will be referred to as the "scaled transverse variable" and the "transverse variable", respectively.

Remark 6.1. Note that, due to the equality $\boldsymbol{\nu}_t(\mathscr{X}_t) = \boldsymbol{\nu}(\mathscr{X})$, $\mathscr{X}_t = \mathscr{X} + it\boldsymbol{\nu}(\mathscr{X}) \in \mathscr{S}_t$ (cf. Lemma 3.2) for the unit normal vector field to the equidistant (level) surface $\mathscr{S}_t$, $-\varepsilon < t < \varepsilon$ (cf. (60)), the Gunter's derivatives $\mathscr{D}_j = \partial_{\boldsymbol{d}_j} = \partial_j - \nu_j\partial_{\boldsymbol{\nu}}$, $j = 1, \dots, n$, are independent of the transverse variable $t \in (-\varepsilon, \varepsilon)$.

Formulation of 3D variational problem $\mathscr{P}(\Omega^\varepsilon)$ of a linearly elastic shell in curvilinear coordinates: We depart, as usual, from the following equation of linearized elasticity in variational formulation in curvilinear coordinates in a domain $\Omega^\varepsilon \subset \mathbb{R}^3$ (see (97) for the deformation tensors $\mathfrak{D}_{jk}$):

$$\begin{aligned}
&\int\limits_{\Omega^\varepsilon} \mathscr{E}^{jklm,\varepsilon}\mathfrak{D}_{lm}(\boldsymbol{U}^\varepsilon)\mathfrak{D}_{jk}(\boldsymbol{V}^\varepsilon)\, d\mathscr{X} = \\
&= \int\limits_{\Omega^\varepsilon} \langle \boldsymbol{F}^\varepsilon, \boldsymbol{V}^\varepsilon\rangle\, d\mathscr{X} + \int\limits_{\mathscr{S}^\varepsilon_-} \langle \boldsymbol{H}^\varepsilon, \boldsymbol{V}^\varepsilon\rangle\, d\sigma_{-\varepsilon} + \int\limits_{\mathscr{S}^\varepsilon_+} \langle \boldsymbol{H}^\varepsilon, \boldsymbol{V}^\varepsilon\rangle\, d\sigma_{+\varepsilon}, \qquad (126)\\
&\boldsymbol{U}^\varepsilon,\ \boldsymbol{V}^\varepsilon \in \mathbb{W}^1(\Omega^\varepsilon, \Gamma_0) := \big\{\boldsymbol{X} \in \mathbb{W}^1(\Omega^\varepsilon) :\ \boldsymbol{X}(\mathscr{X}) = 0 \text{ on } \Gamma_0\big\}.
\end{aligned}$$

Due to (74) we can rewrite (6) into the form

$$\begin{aligned}
&\int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{\mathscr{S}} \mathscr{E}^{jklm,\varepsilon}\mathfrak{D}_{lm}(\boldsymbol{U}^\varepsilon)\mathfrak{D}_{jk}(\boldsymbol{V}^\varepsilon)\, d\sigma_t\, dt = \\
&= \int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{\mathscr{S}} \langle \boldsymbol{F}^\varepsilon, \boldsymbol{V}^\varepsilon\rangle\, d\sigma_t\, dt + \int\limits_{\mathscr{S}_-} \langle \boldsymbol{H}^\varepsilon, \boldsymbol{V}^\varepsilon\rangle\, d\sigma_{-\varepsilon} + \int\limits_{\mathscr{S}_+} \langle \boldsymbol{H}^\varepsilon, \boldsymbol{V}^\varepsilon\rangle\, d\sigma_{+\varepsilon}, \quad (127)
\end{aligned}$$

where $d\sigma_t$ is the suraface measure on $\mathscr{S}_t$ (cf. (60)), An endeavor will be achieved by studying $\boldsymbol{U}^\varepsilon$ as $\varepsilon \to 0$. The main objective is the displacement field $\boldsymbol{U}^\varepsilon(\mathscr{X}) = U^\varepsilon_j(\mathscr{X})\boldsymbol{d}^j(\mathscr{X})$ in the tubular domain Ω^ε, and its behavior as $\varepsilon \to 0$. The vector fields $\boldsymbol{F}^\varepsilon$ on the tubular domain Ω^ε and $\boldsymbol{H}^\varepsilon$ on the boundary $\partial\Omega^\varepsilon$ are prescribed, while the vector field $\boldsymbol{V}^\varepsilon$ is arbitrary (the appropriate spaces for the participating vector fields are specified later).

$\mathscr{E}^\varepsilon := \big[\mathscr{E}^{jklm,\varepsilon}\big]_{4\times4\times4\times4}$ is the elasticity tensor for an isotropic case (see (108)) and only depend on a couple of Lamé constants λ^ε and μ^ε:

$$\mathscr{E}^{jklm,\varepsilon} = \lambda^\varepsilon\delta_{jk}\delta_{lm} + \mu^\varepsilon\big[\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}\big]. \qquad (128)$$

The bilinear form associated with the elasticity tensor on a space of symmetric matrices $\boldsymbol{\zeta} = \big[\zeta_{jk}\big]_{4\times4} = \boldsymbol{\zeta}^\top$ is positive definite

$$\langle \mathscr{E}^\varepsilon\boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle := \mathscr{E}^{jklm,\varepsilon}\zeta_{jk}\overline{\zeta}_{lm} \geqslant C_0|\zeta_{lm}|^2 \ \text{ for all } \ \zeta_{jk} = \zeta_{kj} \in \mathbb{C}, \qquad (129)$$

where the constant C_0 is independent of the parameter ε.

Note, that in a similar treatment of elasticity problems in curvilinear coordinates in [3], [4], [18], [24], [25], [34], the elasticity tensor in the variational problem (6), is not sparse (see § 8 below).

In order to carry out an asymptotic treatment we make the following assumptions:

A. For the unknown $\boldsymbol{U}^\varepsilon = \left\{U_j^\varepsilon\right\}_{j=1}^4 : \Omega^\varepsilon \to \mathbb{R}^4$ and the "test" $\boldsymbol{V}^\varepsilon = \left\{V_j^\varepsilon\right\}_{j=1}^4 : \Omega^\varepsilon \to \mathbb{R}^4$ vector fields we assume that

$$U_j^\varepsilon(\mathscr{x}^\varepsilon) = U_j(\varepsilon)(\mathscr{x}) \text{ and } V_j^\varepsilon(\mathscr{x}^\varepsilon) = V_j(\mathscr{x}) \ \forall\, \mathscr{x}^\varepsilon = \pi^\varepsilon \mathscr{x} \in \Omega^\varepsilon, \tag{130}$$

where the components of the scaled unknown $U_1^\varepsilon, \dots, U_4^\varepsilon$ are called the **scaled displacements**.

Moreover, it is supposed that the scaled displacement vector $\boldsymbol{U}^\varepsilon$ is of order zero with respect to ε. This means that if $\varepsilon \to 0$, the scaled displacement vector $\boldsymbol{U}^\varepsilon$ neither "blows up" nor converges to 0, provided the applied forces have the right orders.

B. The elasticity tensor

$$\mathscr{E}^{jklm,\varepsilon} = \mathscr{E}^{jklm}(\varepsilon), \ \ j,k,l,m = 1,2,3,4, \tag{131}$$

in general and Lamé constants $\lambda^\varepsilon = \lambda(\varepsilon)$ and $\mu^\varepsilon = \mu(\varepsilon)$ in particular, depend on the small parameter ε continuously and $\mathscr{E}^{jklm}(\varepsilon) \to \mathscr{E}^{jklm}(0) = \mathscr{E}^{jklm}(0)$ (similarly, $\lambda(\varepsilon) \to \lambda(0) = \lambda$, $\mu(\varepsilon) \to \mu(0) = \mu$) as $\varepsilon \to 0$.

Note that due to the independence of the constant C_0 from ε, the inequality (129) holds also for the limiting value $\varepsilon = 0$.

C. For the external force $\boldsymbol{F}$ and boundary data $\boldsymbol{H}$

$$\begin{aligned} &\boldsymbol{F}^\varepsilon(\mathscr{x}^\varepsilon) = \varepsilon^p \boldsymbol{F}(\mathscr{x}) \text{ for all } \mathscr{x}^\varepsilon = \pi^\varepsilon \mathscr{x} \in \Omega^\varepsilon, \\ &\boldsymbol{H}^\varepsilon(\mathscr{x}^\varepsilon) = \varepsilon^{p+1} \boldsymbol{H}(\mathscr{x}) \text{ for all } \mathscr{x}^\varepsilon = \pi^\varepsilon \mathscr{x} \in \partial\Omega^\varepsilon. \end{aligned} \tag{132}$$

where the exponent p will be specified later.

Theorem 6.2. *Consider a shell Ω^ε whose reference configuration coincides with the tubular domain Ω^ε (cf. (2)) around a middle surface given by a local immersion Θ in (1).*

With any vector field $\boldsymbol{U} \in \mathbb{W}^1(\Omega)$ we associate the scaled deformation tensors *defined by (cf. (98))*

$$\operatorname{Def}_{\Omega^\varepsilon}(\boldsymbol{V};\varepsilon) := \left[\mathfrak{D}_{jk}(\boldsymbol{V};\varepsilon)\right]_{4\times 4}, \tag{133}$$

$$\mathfrak{D}_{jk}(\boldsymbol{V};\varepsilon) := \frac{1}{2}\Big[(\mathscr{D}_j^{\mathscr{S},\varepsilon}\boldsymbol{V})_k^0 + (\mathscr{D}_k^{\mathscr{S},\varepsilon}\boldsymbol{V})_j^0\Big], \ \ j,k = 1,2,3,4, \tag{134}$$

$$\begin{aligned} &(\mathscr{D}_j^{\mathscr{S},\varepsilon}\boldsymbol{V})_k^0 = (\mathscr{D}_j^{\mathscr{S}}\boldsymbol{V})_k^0, \ \ (\mathscr{D}_4^{\mathscr{S},\varepsilon}\boldsymbol{V})_k^0 = \frac{1}{\varepsilon}\,(\mathscr{D}_4^{\mathscr{S}}\boldsymbol{V})_k^0, \ \ j,k = 1,2,3, \\ &(\mathscr{D}_j^{\mathscr{S},\varepsilon}\boldsymbol{V})_4^0 = (\mathscr{D}_j\boldsymbol{V})_4^0, \ \ (\mathscr{D}_4^{\mathscr{S},\varepsilon}\boldsymbol{V})_4^0 = \frac{1}{\varepsilon}\,(\mathscr{D}_4\boldsymbol{V})_4^0, \ \ j = 1,2,3. \end{aligned}$$

Let the assumptions on the data be as in (130)–(132). Then, rescaling the variable $\mathscr{x}^\varepsilon = (\mathscr{x}, t) \in \Omega^\varepsilon$ to $\mathscr{x} = (\mathscr{x}, \varepsilon t) \in \Omega^1$ (see (124)), due to (74), *the variational problem* $\mathscr{P}(\Omega^\varepsilon)$ *of a linearly elastic shell* (127) *for the scaled unknown vector* $\boldsymbol{U}(\varepsilon)$ *acquires the form:*

$$\int_{-1}^{1}\int_{\mathscr{S}} \mathscr{E}^{jklm}(\varepsilon)\mathfrak{D}_{lm}(\boldsymbol{U}(\varepsilon);\varepsilon)\mathfrak{D}_{jk}(\boldsymbol{V};\varepsilon)\,d\sigma_{\varepsilon t}\,dt =$$

$$= \varepsilon^p\int_{-1}^{1}\int_{\mathscr{S}}\langle\boldsymbol{F}(\varepsilon),\boldsymbol{V}\rangle\,d\sigma_{\varepsilon t}\,dt + \varepsilon^p\int_{\mathscr{S}_-}\langle\boldsymbol{H}(\varepsilon),\boldsymbol{V}\rangle\,d\sigma_{-ve}+$$

$$+\varepsilon^p\int_{\mathscr{S}_+}\langle\boldsymbol{H}(\varepsilon),\boldsymbol{V}\rangle\,d\sigma_{+\varepsilon}, \tag{135}$$

$$\boldsymbol{U}(\varepsilon),\ \boldsymbol{V}\in\mathbb{W}^1(\Omega^1,\Gamma_0^1) := \big\{\boldsymbol{X}\in\mathbb{W}^1(\Omega^1):\ \boldsymbol{X}(\mathscr{X})=0 \text{ on } \Gamma_0^1\big\},$$

where $\Omega^1 := \omega\times[-1,1]$, $\mathscr{S}_\pm = \mathscr{S}\times\{\pm1\}$, $\Gamma_0^1 = \Gamma_0\times[-1,1]$ (*see* (122), (123)).

7. Equations of Linearly Elastic Shells Derived by a Formal Asymptotic Analysis

Consider the 3D shell problem $\mathscr{P}(\Omega^\varepsilon)$ as stated in (6) for $\varepsilon>0$. To the assumptions A, B and C in (130)–(132) we add the smoothness requirement on the function Θ in (1) which defines the middle surface $\mathscr{S}$ of a shell configuration Ω^ε:

$$\Theta\in C^3(\overline{\omega},\mathbb{R}^3). \tag{136}$$

For the unknown $\boldsymbol{U}^\varepsilon = \big\{U_j^\varepsilon\big\}_{j=1}^4 : \Omega^\varepsilon\to\mathbb{R}^4$ and the "test" $\boldsymbol{V}^\varepsilon = \big\{V_j^\varepsilon\big\}_{j=1}^4 : \Omega^\varepsilon\to\mathbb{R}^4$ vector fields we assume the condition (130) holds. On the external force $\boldsymbol{F}$ and the boundary data $\boldsymbol{H}$ we impose less restrictions than (132)

$$\begin{aligned}\boldsymbol{F}^\varepsilon(\mathscr{X}^\varepsilon) &= \boldsymbol{F}(\varepsilon)(\mathscr{X}) \ \text{ for all } \ \mathscr{X}^\varepsilon = \pi^\varepsilon\mathscr{X}\in\Omega^\varepsilon,\\ \boldsymbol{H}^\varepsilon(\mathscr{X}^\varepsilon) &= \boldsymbol{H}(\varepsilon)(\mathscr{X}) \ \text{ for all } \ \mathscr{X}^\varepsilon = \pi^\varepsilon\mathscr{X}\in\partial\Omega^\varepsilon\end{aligned} \tag{137}$$

and will show later that the constraints (132) are natural (see (148), (151), (155) and (171) below).

For the volume measure $d\sigma_t dt$, where $d\sigma_t$ is the suraface measure on $\mathscr{S}_t$, holds the following equality (cf. (71)):

$$d\sigma_t dt = \big[1-2\mathscr{H}_{\mathscr{S}}(\mathscr{X})t + \mathscr{K}_{\mathscr{S}}(\mathscr{X})t^2\big]\,d\sigma\,dt \ \ \forall\,(\mathscr{X},t)\in\Omega^\varepsilon. \tag{138}$$

Under these constraints the variational problem $\mathscr{P}(\Omega^\varepsilon)$ in (127) reformulates into the following problem $\mathscr{P}^*(\Omega^\varepsilon)$:

$$\int_{\Omega^1}\mathscr{E}^{jklm}(\varepsilon)\mathfrak{D}_{lm}(\boldsymbol{U}(\varepsilon);\varepsilon)\mathfrak{D}_{jk}(\boldsymbol{V};\varepsilon)\,d\sigma_{\varepsilon t}\,dt =$$

$$= \int_{\Omega^1}\langle\boldsymbol{F}(\varepsilon),\boldsymbol{V}\rangle\,d\sigma_{\varepsilon t}\,dt + \frac{1}{\varepsilon}\int_{\mathscr{S}_-}\langle\boldsymbol{H}(\varepsilon),\boldsymbol{V}\rangle\,d\sigma_{-\varepsilon t}\,dt+$$

$$+\frac{1}{\varepsilon}\int\limits_{\mathscr{S}_+}\langle \boldsymbol{H}(\varepsilon),\boldsymbol{V}\rangle\,d\sigma_{\varepsilon t}\,dt, \tag{139}$$

$$\boldsymbol{U}(\varepsilon),\ \boldsymbol{V}\in\mathbb{W}^1(\Omega^1,\Gamma_0^1):=\left\{\boldsymbol{X}\in\mathbb{W}^1(\Omega^1):\ \boldsymbol{X}(x)=0 \text{ on } \Gamma_0^1\right\},$$

where $\Omega^1:=\mathscr{S}\times[-1,1]$, $\mathscr{S}_\pm=\mathscr{S}\times\{\pm1\}$, $\Gamma_0^1:=\Gamma_0\times[-1,1]$.

The scaled unknown $\boldsymbol{U}^\varepsilon=\left\{U_j^\varepsilon\right\}_{j=1}^4$, the scaled functions $\mathscr{E}^{jklm}(\varepsilon)$ and the scaled deformation tensor $\mathfrak{D}_{jk}(\boldsymbol{V};\varepsilon)$ are defined in (130), (88), (133), (134) and appeared in the problem $\mathscr{P}(\Omega^\varepsilon)$ (cf. (134)). Difference between problems $\mathscr{P}^*(\Omega^\varepsilon)$ and $\mathscr{P}(\Omega^\varepsilon)$ lies in the right-hand sides, where we have performed different scaling.

Let us assume that the scaled unknown vector has a formal asymptotic expansion:

$$\boldsymbol{U}(\varepsilon)(\mathscr{x},t)=\boldsymbol{U}^0(\mathscr{x},t)+\varepsilon\boldsymbol{U}^1(\mathscr{x},t)+\cdots,\quad(\mathscr{x},t)\in\Omega^1,\quad \boldsymbol{U}^0\neq0. \tag{140}$$

Recall, that $\mathscr{D}^\varepsilon_{n+1}=\partial_{\varepsilon\boldsymbol{\nu}}=\frac{1}{\varepsilon}\mathscr{D}_{n+1}$ (cf. (125)) while, in contrast to this, Gunter's first n derivatives are independent of the transverse variable t nd, therefore, from the small parameter ε: $\mathscr{D}^\varepsilon_j=\mathscr{D}_j$ for all $j=,\dots,n$ (cf. Remark 6.1), Then the formal asymptotic expansion of the scaled deformation tensor reads

$$\mathfrak{D}_{jk}(\boldsymbol{U};\varepsilon)=\frac{1}{\varepsilon}\,\mathfrak{D}^{-1}_{jk}(\boldsymbol{U})+\mathfrak{D}^0_{jk}(\boldsymbol{U})+\varepsilon\mathfrak{D}^1_{jk}(\boldsymbol{U})+\varepsilon^2\mathfrak{D}^2_{jk}(\boldsymbol{U})+\cdots, \tag{141}$$

$$\mathfrak{D}^{-1}_{jk}(\boldsymbol{U})=\begin{cases}0 & \text{if } j,k=1,2,3,\\ \frac{1}{2}\,(\mathscr{D}^{\mathscr{S}}_4\boldsymbol{U}^0)^0_j, & \text{if } j=1,2,3,\ \ k=4,\\ \frac{1}{2}(\mathscr{D}^{\mathscr{S}}_4\boldsymbol{U}^0)^0_k & \text{if } j=4,\ \ k=1,2,3,\\ (\mathscr{D}_4\boldsymbol{U}^0)^0_4, & \text{if } j=k=4,\end{cases} \tag{142}$$

$$\mathfrak{D}^m_{jk}(\boldsymbol{U})=\begin{cases}\frac{1}{2}\left[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U}^m)^0_k+(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U}^m)^0_j\right] & \text{if } j,k=1,2,3,\\ \frac{1}{2}\left[(\mathscr{D}_j\boldsymbol{U}^m)^0_4+(\mathscr{D}^{\mathscr{S}}_4\boldsymbol{U}^{m+1})^0_j\right] & \text{if } j=1,2,3,\ \ k=4\\ \frac{1}{2}\left[(\mathscr{D}^{\mathscr{S}}_4\boldsymbol{U}^{m+1})^0_k+(\mathscr{D}_k\boldsymbol{U}^m)^0_4\right] & \text{if } j=4,\ \ k=1,2,3,\\ (\mathscr{D}_4\boldsymbol{U}^{m+1})^0_4, & \text{if } j=k=4.\end{cases} \tag{143}$$

for $m=0,1,2,\dots$ (cf. (99)).

The deformations $\mathfrak{D}_{jk}(\boldsymbol{V};\varepsilon)$ have much simpler finite expansions, because $\boldsymbol{V}$ is independent of the parameter ε:

$$\mathfrak{D}_{jk}(\boldsymbol{V};\varepsilon)=\frac{1}{\varepsilon}\,\mathfrak{D}^{-1}_{jk}(\boldsymbol{V})+\mathfrak{D}^0_{jk}(\boldsymbol{V}), \tag{144}$$

$$\mathfrak{D}^{-1}_{jk}(\boldsymbol{V}) = \begin{cases} 0 & \text{if } \ j,k=1,2,3, \\ \dfrac{1}{2}(\mathscr{D}^{\mathscr{S}}_4\boldsymbol{V})^0_k & \text{if } \ j=4, \ \ k=1,2,3, \\ \dfrac{1}{2}(\mathscr{D}^{\mathscr{S}}_4\boldsymbol{V})^0_j, & \text{if } \ k=4, \ \ j=1,2,3, \\ (\mathscr{D}_4\boldsymbol{V})^0_4, & \text{if } \ j=k=4, \end{cases} \tag{145}$$

$$\mathfrak{D}^{0}_{jk}(\boldsymbol{V}) = \begin{cases} \dfrac{1}{2}\Big[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{V})^0_k + (\mathscr{D}^{\mathscr{S}}_k\boldsymbol{V})^0_j\Big] & \text{if } \ j,k=1,2,3, \\ \dfrac{1}{2}(\mathscr{D}_j\boldsymbol{V})^0_4 & \text{if } \ j=1,2,3, \ \ k=4, \\ \dfrac{1}{2}(\mathscr{D}_k\boldsymbol{V})^0_4 & \text{if } \ j=4, \ \ k=1,2,3, \\ 0 & \text{if } \ j=k=4. \end{cases} \tag{146}$$

For time being and the sake of simplicity we accept stronger convention than (131): The elasticity tensor $\mathscr{E}^{\varepsilon} = \big[\mathscr{E}^{jklm,\varepsilon}\big]_{4\times4\times4\times4}$ in general and Lamé constants λ^{ε}, μ^{ε} in particular, are independent of the small parameter ε:

$$\mathscr{E}^{jklm,\varepsilon} = \mathscr{E}^{jklm}, \ \ \forall\, j,k,l,m=1,2,3,4 \ \ (\lambda^{\varepsilon}=\lambda, \ \ \mu^{\varepsilon}=\mu). \tag{147}$$

As a concluding part of preparation let us prove the following.

Lemma 7.1 (see [3, § 3.4]). *Let $\mathscr{S}\subset\mathbb{R}^{n-1}$ be a Lipschitz hypersurface with the surface measure $d\sigma$, $\gamma:=\partial\mathscr{S}$ be its boundary and let $w\in\mathbb{L}_p(\Omega^1)$, $1<p<\infty$, $\Omega^1:=\mathscr{S}\times(-1,1)$. If*

$$\int\limits_{\Omega^1} w(\mathscr{x},t)\partial_t v(\mathscr{x},t)\,d\sigma\,dt = 0 \ \ \forall\, v\in \mathrm{Lip}(\Omega^1),$$

where $\mathrm{Lip}(\Omega^1)$ *is the space of Lipschitz continuous functions on Ω^1, then $\varphi=0$.*

Proof. Since $\partial_t:\mathrm{Lip}(\Omega^1)\longrightarrow\mathbb{L}_\infty(\Omega^1)$ is a surjection with an obvious right inverse

$$\partial_t^{-1}u(\mathscr{x},t) := \int\limits_{-1}^{t} u(\mathscr{x},\tau)\,d\tau,$$

the condition of the lemma reads

$$\int\limits_{\Omega^1} w(\mathscr{x},t)u(\mathscr{x},t)\,d\sigma\,dt = 0 \ \ \forall\, u\in\mathbb{L}_\infty(\Omega^1).$$

Now the result follows, since $\mathbb{L}_\infty(\Omega^1)$ is a dense linear subset of the space $\mathbb{L}_{p'}(\Omega^1)$, $p'=p/(p-1)$, which is dual to the space $\mathbb{L}_p(\Omega^1)$. □

Step 1 : Since the lowest power of ε in the equation (7) is -2 (see (141) and (144)), we shall suppose that

$$\begin{aligned}\boldsymbol{F}(\varepsilon)(\mathscr{x},t) = \frac{1}{\varepsilon^2}\boldsymbol{F}_{-2}(\mathscr{x},t) + \frac{1}{\varepsilon}\boldsymbol{F}_{-1}(\mathscr{x},t) + \boldsymbol{F}_0(\mathscr{x},t)+\\ + \varepsilon\boldsymbol{F}_1(\mathscr{x},t) + \mathscr{O}(\varepsilon^2),\\ \boldsymbol{H}(\varepsilon)(\mathscr{x},\pm1) = \frac{1}{\varepsilon}\,\boldsymbol{H}_{-1}(\mathscr{x},\pm1)+\boldsymbol{H}_0(\mathscr{x},\pm1)+\varepsilon\boldsymbol{H}_1(\mathscr{x},\pm1)+\mathscr{O}(\varepsilon^2)\end{aligned} \tag{148}$$

for the functions in (137), where $\boldsymbol{F}_{-2}, \boldsymbol{F}_{-1}, \boldsymbol{F}_0, \boldsymbol{F}_1, \ldots, \boldsymbol{H}_{-1}, \boldsymbol{H}_0, \boldsymbol{H}_1, \ldots$ are independent of ε.

The cancelation of the variable ε^{-2} in the equation (7), together with the assumptions (138), (140)–(148), leads to the following equality:

$$\begin{aligned}\int\limits_{\Omega^1} \mathscr{E}^{jklm}\mathfrak{D}^{-1}_{lm}(\boldsymbol{U})\mathfrak{D}^{-1}_{jk}(\boldsymbol{V})\,d\sigma\,dt =\\ = \int\limits_{\Omega^1}\langle \boldsymbol{F}_{-2},\boldsymbol{V}\rangle\,d\sigma\,dt + \int\limits_{\mathscr{S}_-}\langle \boldsymbol{H}_{-1},\boldsymbol{V}\rangle\,d\sigma + \int\limits_{\mathscr{S}_+}\langle \boldsymbol{H}_{-1},\boldsymbol{V}\rangle\,d\sigma.\end{aligned} \tag{149}$$

for all $\boldsymbol{V} \in \mathbb{W}^1(\Omega^1,\Gamma^1_0)$. By taking $\boldsymbol{V}(\mathscr{x},t) = \boldsymbol{V}(x)$ independent of the transverse variable $t \in [-1,1]$, we find that the left-hand side in the equality vanishes

$$\begin{aligned}0 = \int\limits_{\Omega^1}\langle \boldsymbol{F}_{-2},\boldsymbol{V}\rangle\,\boldsymbol{d}\sigma\,dt + \int\limits_{\mathscr{S}_-}\langle \boldsymbol{H}_{-1},\boldsymbol{V}\rangle\,\boldsymbol{d}\sigma + \int\limits_{\mathscr{S}_+}\langle \boldsymbol{H}_{-1},\boldsymbol{V}\rangle\,\boldsymbol{d}\sigma =\\ = \int\limits_{\mathscr{S}_+}\langle \boldsymbol{P}_{-1}(x),\boldsymbol{V}\rangle\,\boldsymbol{d}\sigma.\end{aligned}$$

Here the first resultant force

$$\boldsymbol{P}_{-1}(\mathscr{x}) := \frac{1}{2}\int\limits_{-1}^{1}\boldsymbol{F}_{-2}(\mathscr{x},t)\,dt + \frac{1}{2}\left[\boldsymbol{H}_{-1}(\mathscr{x},-1) + \boldsymbol{H}_{-1}(\mathscr{x},+1)\right] \tag{150}$$

emerges after integration of the body force across the thickness $(-1,1)$. Consequently, if we let (cf. e.g., [3, Page 167])

$$\boldsymbol{F}_{-2}(\mathscr{x},t) = \boldsymbol{H}_{-1}(\mathscr{x},\pm1) = 0\ \ \forall\,\mathscr{x}\in\mathscr{S},\ \ \forall\,t\in[-1,1], \tag{151}$$

we get $\boldsymbol{P}_{-1}(\mathscr{x}) = 0$ and back to (149) this gives (see (142) and (145)):

$$\begin{aligned}\sum_{j,k=1}^{3}\int\limits_{\Omega^1}\mathscr{E}^{j4k4}\mathfrak{D}^{-1}_{k4}(\boldsymbol{U})\mathfrak{D}^{-1}_{j4}(\boldsymbol{V})\,\boldsymbol{d}\sigma\,dt =\\ = \frac{1}{4}\sum_{j,k=1}^{3}\int\limits_{\Omega^1}\mathscr{E}^{j4k4}(\partial_{\boldsymbol{\nu}}\boldsymbol{U}^0)^0_k(\partial_{\boldsymbol{\nu}}\boldsymbol{V})^0_j\,\boldsymbol{d}\sigma\,dt = 0.\end{aligned}$$

The tensor $\mathscr{E} = [\mathscr{E}^{jklm}]_{4\times4\times4\times4}$ is positive definite (see (129)); then its part, the matrix $\mathscr{E} = \left[\mathscr{E}^{j4k4}\right]_{4\times4}$, is positive definite as well (to check this it suffices to introduce $\zeta_{jk} = 0$ for $k \neq 4$ in (129)) and implies that, by introducing $\boldsymbol{V} = \boldsymbol{U}^0$, that

$$\int\limits_{\Omega^1} \left|(\partial_{\boldsymbol{\nu}}\boldsymbol{U}^0)^0_j\right|^2 \boldsymbol{d}\sigma\, dt = 0 \;\; \forall\, j = 1,2,3.$$

The obtained equality implies $(\partial_{\boldsymbol{\nu}}\boldsymbol{U}^0)^0_1 = (\partial_{\boldsymbol{\nu}}\boldsymbol{U}^0)^0_2 = (\partial_{\boldsymbol{\nu}}\boldsymbol{U}^0)^0_3 = 0$ and also

$$(\partial_{\boldsymbol{\nu}}\boldsymbol{U}^0)^0_4 = \partial_{\boldsymbol{\nu}}\langle \boldsymbol{U}^0, \boldsymbol{\nu}\rangle = 0.$$

Then $\partial_{\boldsymbol{\nu}}\boldsymbol{U}^0(\mathscr{x}, t) = \partial_t\boldsymbol{U}^0(\mathscr{x}, t) = 0$ and, consequently, the leading term of the asymptotic expansion (140) is independent of the transverse variable $t = \mathscr{x}_4$, parallel to $\boldsymbol{\nu}$.

Thus, the outcome of the Step 1 is:

$$\begin{cases} \boldsymbol{U}^0(\mathscr{x}, t) = \boldsymbol{U}^0(\mathscr{x}), \;\; \boldsymbol{U}^0 \in \mathbb{W}^1(\mathscr{S}, \Gamma_0), \\ \mathfrak{D}^{-1}_{jk}(\boldsymbol{U}) = 0 \;\text{ in }\; \Omega^1. \end{cases} \tag{152}$$

Step 2 : With the assumption (148) and the equality $\mathfrak{D}^{-1}_{jk}(\boldsymbol{U}) = 0$ (cf. (152)) at hand the cancelation of the variable ε^{-1} in the equation (7), together with the assumptions (140)–(147), lead to the following equality:

$$\int\limits_{\Omega^1} \mathscr{E}^{jklm}\mathfrak{D}^0_{lm}(\boldsymbol{U})\mathfrak{D}^{-1}_{jk}(\boldsymbol{V})\, \boldsymbol{d}\sigma\, dt =$$
$$= \int\limits_{\Omega^1} \langle \boldsymbol{F}_{-1}, \boldsymbol{V}\rangle\, \boldsymbol{d}\sigma\, dt + \int\limits_{\mathscr{S}_-} \langle \boldsymbol{H}_0, \boldsymbol{V}\rangle\, \boldsymbol{d}\sigma + \int\limits_{\mathscr{S}_+} \langle \boldsymbol{H}_0, \boldsymbol{V}\rangle\, \boldsymbol{d}\sigma \tag{153}$$

for all $\boldsymbol{V} \in \mathbb{W}^1(\Omega^1, \Gamma^1_0)$. And again, by taking $\boldsymbol{V}(x,t) = \boldsymbol{V}(x)$ independent of the transverse variable $t \in [-1, 1]$, we find that the left-hand side in the latter equality vanishes

$$0 = \int\limits_{\Omega^1} \langle \boldsymbol{F}_{-1}, \boldsymbol{V}\rangle\, \boldsymbol{d}\sigma\;\, dt + \int\limits_{\mathscr{S}_-} \langle \boldsymbol{H}_0, \boldsymbol{V}\rangle\, \boldsymbol{d}\sigma + \int\limits_{\mathscr{S}_+} \langle \boldsymbol{H}_0, \boldsymbol{V}\rangle\, \boldsymbol{d}\sigma =$$
$$= \int\limits_{\mathscr{S}_+} \langle \boldsymbol{P}_0(\mathscr{x}), \boldsymbol{V}\rangle\, \boldsymbol{d}\sigma,$$

where

$$\boldsymbol{P}_{-1}(\mathscr{x}) := \frac{1}{2}\int\limits_{-1}^{1} \boldsymbol{F}_{-1}(\mathscr{x}, t)\, dt + \frac{1}{2}\left[\boldsymbol{H}_0(\mathscr{x}, -1) + \boldsymbol{H}_0(\mathscr{x}, 1)\right] \tag{154}$$

is another resultant force. Consequently, if we let (cf. e.g., [3, Page 168])

$$\boldsymbol{F}_{-1}(\mathscr{x}, t) = \boldsymbol{H}_0(\mathscr{x}, \pm1) = 0 \;\; \forall\, \mathscr{x} \in \mathscr{S}, \;\; \forall\, t \in [-1, 1], \tag{155}$$

Introducing this into (153) we get

$$0 = 2\int\limits_{\Omega^1}\Big[\sum_{j,k=1}^{3}\mathscr{E}^{j4k4}\mathfrak{D}^0_{j4}(\boldsymbol{U})\mathfrak{D}^{-1}_{k4}(\boldsymbol{V}) + \sum_{j,k=1}^{4}\mathscr{E}^{jk44}\mathfrak{D}^0_{jk}(\boldsymbol{U})\mathfrak{D}^{-1}_{44}(\boldsymbol{V})\Big]\,\boldsymbol{d}\sigma\,dt =$$

$$= \int\limits_{\Omega^1}\Big[\sum_{j,k=1}^{3}\mathscr{E}^{j4k4}\mathfrak{D}^0_{j4}(\boldsymbol{U})\mathfrak{D}^{-1}_{k4}(\boldsymbol{V}) +$$

$$+ \Big(\sum_{j,k=1}^{3}\mathscr{E}^{jk44}\mathfrak{D}^0_{j4}(\boldsymbol{U}) + \mathscr{E}^{4444}\mathfrak{D}^0_{44}(\boldsymbol{U})\Big)\mathfrak{D}^{-1}_{44}(\boldsymbol{V})\Big]\,\boldsymbol{d}\sigma\,dt.$$

Invoking the positive definiteness of $\big[\mathscr{E}^{j4k4}\big]_{3\times3}$ (see inequality (129)), Lemma 7.1 and choosing $\mathfrak{D}^{-1}_{k4}(\boldsymbol{V}) = \mathfrak{D}^0_{k4}(\boldsymbol{U}) \in \mathbb{L}_2(\Omega^1, \Gamma^1_0)$ for $k = 1, 2, 3$, $\mathfrak{D}^{-1}_{44}(\boldsymbol{V}) \in \mathbb{L}_2(\Omega^1, \Gamma^1_0)$ arbitrary, we easily derive the following outcome of the Step 2:

$$\begin{cases}\mathfrak{D}^0_{j4}(\boldsymbol{U}) = \mathfrak{D}^0_{4j}(\boldsymbol{U}) = \dfrac{1}{2}\Big[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U}^0)^0_4 + (\partial_{\boldsymbol{\nu}}\boldsymbol{U}^1)^0_j\Big] = 0, \quad j = 1, 2, 3,\\ \mathfrak{D}^0_{44}(\boldsymbol{U}) = (\partial_{\boldsymbol{\nu}}\boldsymbol{U}^1)^0_4 = -\displaystyle\sum_{j,k=1}^{3}\frac{\mathscr{E}^{jk44}}{\mathscr{E}^{4444}}\mathfrak{D}^0_{jk}(\boldsymbol{U}) \qquad \text{in } \Omega^1.\end{cases} \tag{156}$$

Step 3 : With the assumption (148) and the equalities $\mathfrak{D}^{-1}_{jk}(\boldsymbol{U}) = 0$ (cf. (152)) and $\boldsymbol{F}_{-2} = \boldsymbol{F}_{-1} = \boldsymbol{H}_{-1} = \boldsymbol{H}_0 = 0$ (cf. (151) and (155)) at hand the cancelation of the variable $\varepsilon^0 = 1$ in the equation (7), together with the assumptions (140)–(147), leads to the following equality

$$\int\limits_{\Omega^1}\mathscr{E}^{jklm}\Big\{\big[\mathfrak{D}^0_{lm}(\boldsymbol{U})\mathfrak{D}^0_{jk}(\boldsymbol{V}) + \mathfrak{D}^1_{lm}(\boldsymbol{U})\mathfrak{D}^{-1}_{jk}(\boldsymbol{V})\big] + B^1\mathfrak{D}^0_{lm}(\boldsymbol{U})\mathfrak{D}^{-1}_{jk}(\boldsymbol{V})\Big\}d\sigma dt =$$

$$= \int\limits_{\mathscr{S}}\langle\boldsymbol{P}_0, \boldsymbol{V}\rangle\,\boldsymbol{d}\sigma \tag{157}$$

for all $\boldsymbol{V} \in \mathbb{W}^1(\Omega^1, \Gamma^1_0)$ and $B_1(\mathscr{X}) := -2\mathscr{H}_{\mathscr{S}}(\mathscr{X})t$ (see (138)) and with the resultant force

$$\boldsymbol{P}_0(\mathscr{X}) := \frac{1}{2}\int\limits_{-1}^{1}\boldsymbol{F}_0(\mathscr{X}, t)\,dt + \frac{1}{2}\big[\boldsymbol{H}_1(\mathscr{X}, -1) + \boldsymbol{H}_1(\mathscr{X}, 1)\big]. \tag{158}$$

Let $\boldsymbol{V} = \boldsymbol{X} \in \mathbb{W}^1(\mathscr{S}, \Gamma_0)$ be independent of the transverse variable $t \in [-1, 1]$. Then $\mathfrak{D}^{-1}_{jk}(\boldsymbol{X}) = 0$ (cf. (145)) and two summands in the obtained expression eliminate. By invoking the equalities (156) obtained for $\mathfrak{D}^0_{j4}(\boldsymbol{U})$, in Step 2 we proceed as follows:

$$\int\limits_{\Omega^1}\mathscr{E}^{jklm}\mathfrak{D}^0_{lm}(\boldsymbol{U})\mathfrak{D}^0_{jk}(\boldsymbol{X})\,\boldsymbol{d}\sigma\,dt =$$

$$= 2\sum_{j,k=1}^{3}\int_{\mathscr{S}}\Big[\sum_{l,m=1}^{3}\mathscr{E}^{jklm}\mathfrak{D}^0_{lm}(\boldsymbol{U})+\mathscr{E}^{jk44}\mathfrak{D}^0_{44}(\boldsymbol{U})\Big]\mathfrak{D}^0_{jk}(\boldsymbol{X})\,\boldsymbol{d}\sigma =$$

$$= 2\sum_{j,k,l,m=1}^{3}\int_{\mathscr{S}}\Big[\mathscr{E}^{jklm}-\frac{\mathscr{E}^{jk44}\mathscr{E}^{44lm}}{\mathscr{E}^{4444}}\Big]\mathfrak{D}^0_{lm}(\boldsymbol{U})\mathfrak{D}^0_{jk}(\boldsymbol{X})\,\boldsymbol{d}\sigma =$$

$$= \int_{\mathscr{S}}\langle \boldsymbol{P}_0,\boldsymbol{X}\rangle\,\boldsymbol{d}\sigma. \tag{159}$$

Due to formula (128) we get the following values for the *two-dimensional elasticity tensor for an isotropic shell*:

$$\mathfrak{S}^{jklm} = 2\mathscr{E}^{jklm}-2\frac{\mathscr{E}^{jk44}\mathscr{E}^{44lm}}{\mathscr{E}^{4444}} =$$

$$= 2\lambda\delta_{jk}\delta_{lm}+2\mu\big[\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl}\big]-2\,\frac{\lambda^2}{\lambda+2\mu}\,\delta_{jk}\delta_{lm} =$$

$$= \frac{4\lambda\mu}{\lambda+2\mu}\,\delta_{jk}\delta_{lm}+2\mu\big[\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl}\big]. \tag{160}$$

Step 4 : Consider the space

$$\mathbb{W}^1(\mathscr{S},\Gamma_0) := \big\{\boldsymbol{X}\in\mathbb{W}^1(\mathscr{S}):\ \boldsymbol{X}(x)=0 \text{ on } \Gamma_0\big\} \tag{161}$$

and note that

$$\begin{aligned}\mathfrak{D}^0_{lm}(\boldsymbol{U}) &= \mathfrak{D}_{lm}(\boldsymbol{U}^0)=\frac{1}{2}\Big[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U}^0)^0_k+(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U}^0)^0_j\Big],\\ \mathfrak{D}^0_{lm}(\boldsymbol{X}) &= \mathfrak{D}_{lm}(\boldsymbol{X})=\frac{1}{2}\Big[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{X})^0_k+(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{X})^0_j\Big],\quad j,k=1,2,3,\end{aligned} \tag{162}$$

where $\mathfrak{D}_{lm}$ is the surface deformation tensor (see (143) and (146)). The obtained equation in (159) is interpreted as a variational problem with surface deformation tensor for the leading term $\boldsymbol{U}^0\in\mathbb{W}^1(\mathscr{S},\Gamma_0)$ in the asymptotic expansion (140) of the displacement $\boldsymbol{U}$, which only depends on the tangent variable $\boldsymbol{U}^0(\mathscr{x},t)=\boldsymbol{U}^0(\mathscr{x})$ (see (152)):

$$\sum_{j,k,l,m=1}^{3}\int_{\mathscr{S}}\mathfrak{S}^{jklm}\mathfrak{D}_{lm}(\boldsymbol{U}^0)\mathfrak{D}_{jk}(\boldsymbol{X})\,\boldsymbol{d}\sigma=\int_{\mathscr{S}}\langle\boldsymbol{P}_0,\boldsymbol{X}\rangle\,\boldsymbol{d}\sigma, \tag{163}$$

$$\mathfrak{S}^{jklm}=2\mathscr{E}^{jklm}-2\frac{\mathscr{E}^{jk44}\mathscr{E}^{44lm}}{\mathscr{E}^{4444}},\quad \boldsymbol{U}^0,\ \boldsymbol{X}\in\mathbb{W}^1(\mathscr{S},\Gamma_0).$$

The resultant force $\boldsymbol{P}_0(\mathscr{x})$ is defined in (158) and $\mathfrak{S}^{jklm}$ in (163) is the *two-dimensional elasticity tensor for a shell.*

Note, that the factor 2 in the coefficients $\mathfrak{S}^{jklm}$ appeared due to the integration over the interval $[-1,1]$, since the integrand is independent of the transverse variable t.

Step 5 : Now we assume the resultant force is vanishing $\boldsymbol{P}_0=0$ and trace what happens with the displacement $\boldsymbol{U}(\varepsilon)(\mathscr{x},t)=\boldsymbol{U}^0(\mathscr{x},t)+\varepsilon\boldsymbol{U}^1(\mathscr{x},t)+$

$\cdots$ (cf. (140)). Due to forthcoming Theorem 7.2 the variational problem (163) then has a trivial solution

$$\boldsymbol{U}^0 = \mathfrak{D}^0_{jk}(\boldsymbol{U}) = \mathfrak{D}_{jk}(\boldsymbol{U}^0) = 0 \tag{164}$$

for all $j, k = 1, 2, 3$ and from (156) follows

$$\begin{aligned} (\partial_{\boldsymbol{\nu}} \boldsymbol{U}^1)^0_j &= 2\mathfrak{D}^0_{4j}(\boldsymbol{U}) - (\mathscr{D}^{\mathscr{S}}_j \boldsymbol{U}^0)^0_4 = 0, \quad j = 1, 2, 3, \\ (\partial_{\boldsymbol{\nu}} \boldsymbol{U}^1)^0_4 &= \mathfrak{D}^0_{44}(\boldsymbol{U}) = -\sum_{j,k=1}^{3} \frac{\mathscr{E}^{jk44}}{\mathscr{E}^{4444}} \mathfrak{D}^0_{jk}(\boldsymbol{U}) = 0. \end{aligned} \tag{165}$$

The latter equalities imply (compare with a similar equality for $\boldsymbol{U}^0$ in (152))

$$\boldsymbol{U}^1(\mathscr{X}, t) = \boldsymbol{U}^1(\mathscr{X}), \quad \boldsymbol{U}^1 \in \mathbb{W}^1(\mathscr{S}, \Gamma_0). \tag{166}$$

Next by introducing the equalities $\mathfrak{D}_{jk}(\boldsymbol{U}^0) = 0$, $\boldsymbol{P}_0 = 0$ into equation (7) we get

$$\begin{aligned} 0 &= \int_{\Omega^1} \mathscr{E}^{jkm4} \mathfrak{D}^1_{jk}(\boldsymbol{U}) \mathfrak{D}^{-1}_{m4}(\boldsymbol{V})\, \boldsymbol{d}\sigma\, dt = \\ &= \sum_{j,k=1}^{3} 4 \int_{\Omega^1} \mathscr{E}^{j4k4} \mathfrak{D}^1_{j4}(\boldsymbol{U}) \mathfrak{D}^{-1}_{k4}(\boldsymbol{V})\, \boldsymbol{d}\sigma\, dt + \\ &\quad + \int_{\Omega^1} \Big[\sum_{j,k=1}^{3} \mathscr{E}^{jk44} \mathfrak{D}^1_{jk}(\boldsymbol{U}) + \mathscr{E}^{4444} \mathfrak{D}^0_{44}(\boldsymbol{U}) \Big] \mathfrak{D}^{-1}_{44}(\boldsymbol{V})\, d\sigma\, dt = \\ &= \int_{\Omega^1} \Big[2 \sum_{j,k=1}^{3} \mathscr{E}^{j4k4} \mathfrak{D}^1_{j4}(\boldsymbol{U}) (\partial_{\boldsymbol{\nu}} \boldsymbol{V})^1_k + \\ &\quad + \Big(\sum_{j,k=1}^{3} \mathscr{E}^{jk44} \mathfrak{D}^1_{jk}(\boldsymbol{U}) + \mathscr{E}^{4444} (\partial_{\boldsymbol{\nu}} \boldsymbol{U}^1)^1_4 \Big) (\partial_{\boldsymbol{\nu}} \boldsymbol{V})^1_4 \Big]\, \boldsymbol{d}\sigma\, dt. \end{aligned}$$

From the obtained equality, similarly to (156) follows that

$$\begin{aligned} \mathfrak{D}^1_{j4}(\boldsymbol{U}) &= \mathfrak{D}^1_{4j}(\boldsymbol{U}) = \frac{1}{2} \left[(\mathscr{D}^{\mathscr{S}}_j \boldsymbol{U}^1)^0_4 + (\partial_{\boldsymbol{\nu}} \boldsymbol{U}^2)^0_j \right] = 0, \quad j = 1, 2, 3, \\ \mathfrak{D}^1_{44}(\boldsymbol{U}) &= (\partial_{\boldsymbol{\nu}} \boldsymbol{U}^1)^1_4 = -\sum_{j,k=1}^{3} \frac{\mathscr{E}^{jk44}}{\mathscr{E}^{4444}} \mathfrak{D}^1_{jk}(\boldsymbol{U}) \ \text{ in } \Omega^1. \end{aligned} \tag{167}$$

With the assumption (148) and the equalities $\mathfrak{D}^{-1}_{jk}(\boldsymbol{U}) = \mathfrak{D}^0_{jk}(\boldsymbol{U}) = 0$ (cf. (152), (164) and $\boldsymbol{F}_{-2} = \boldsymbol{F}_{-1} = \boldsymbol{H}_{-1} = \boldsymbol{H}_0 = 0$ (cf. (151) and (155)) at hand the cancelation of the variable ε in the equation (7), together with the assumptions (140)–(147), leads to the following equality

$$\int_{\Omega^1} \mathscr{E}^{jklm} \Big[\mathfrak{D}^1_{lm}(\boldsymbol{U}) \mathfrak{D}^0_{jk}(\boldsymbol{V}) + \mathfrak{D}^2_{lm}(\boldsymbol{U}) \mathfrak{D}^{-1}_{jk}(\boldsymbol{V}) + \mathfrak{D}^1_{lm}(\boldsymbol{U}) \mathfrak{D}^{-1}_{jk}(\boldsymbol{V}) B^1 \Big] d\sigma dt =$$

$$= \int\limits_{\Omega^1} \langle \boldsymbol{F}_1, \boldsymbol{V}\rangle \, \boldsymbol{d}\sigma \, dt + \int\limits_{\mathscr{S}_-} \langle \boldsymbol{H}_2, \boldsymbol{V}\rangle \, \boldsymbol{d}\sigma + \int\limits_{\mathscr{S}_+} \langle \boldsymbol{H}_2, \boldsymbol{V}\rangle \, \boldsymbol{d}\sigma +$$

$$+ \int\limits_{\Omega^1} \langle \boldsymbol{F}_0, \boldsymbol{V}\rangle B^1(x) \, dx \, dt + \int\limits_{\mathscr{S}_-} \langle \boldsymbol{H}_1, \boldsymbol{V}\rangle B^1(x) \, dx + \int\limits_{\mathscr{S}_+} \langle \boldsymbol{H}_1, \boldsymbol{V}\rangle B^1(x) \, dx$$

for all $\boldsymbol{V} \in \mathbb{W}^1(\Omega^1, \Gamma_0^1)$ and $B_1(\mathscr{X}) := -2\mathscr{H}_{\mathscr{S}}(\mathscr{X})t$ (see (138)). Let $\boldsymbol{V} = \boldsymbol{X} \in \mathbb{W}^1(\mathscr{S}, \Gamma_0)$ be independent of the transverse variable $t \in [-1, 1]$. Then $\mathfrak{D}^{-1}_{jk}(\boldsymbol{X}) = \mathfrak{D}^0_{44}(\boldsymbol{X}) = 0$ (cf. (145)) and the corresponding summands in the obtained expression eliminate. Using the equalities (167) obtained for $\mathfrak{D}^1_{j4}(\boldsymbol{U})$ we proceed as follows:

$$\int\limits_{\Omega^1} \mathscr{E}^{jklm} \mathfrak{D}^1_{lm}(\boldsymbol{U}) \mathfrak{D}^0_{jk}(\boldsymbol{X}) \, \boldsymbol{d}\sigma \, dt =$$

$$= 2 \sum_{j,k=1}^{3} \int\limits_{\mathscr{S}} \Big[\sum_{l,m=1}^{3} \mathscr{E}^{jklm} \mathfrak{D}^1_{lm}(\boldsymbol{U}) + \mathscr{E}^{jk44} \mathfrak{D}^1_{44}(\boldsymbol{U}) \Big] \mathfrak{D}^0_{jk}(\boldsymbol{X}) \, \boldsymbol{d}\sigma =$$

$$= 2 \sum_{j,k,l,m=1}^{3} \int\limits_{\mathscr{S}} \mathfrak{S}^{jklm} \mathfrak{D}^1_{lm}(\boldsymbol{U}) \mathfrak{D}^0_{jk}(\boldsymbol{X}) \, \boldsymbol{d}\sigma =$$

$$= 2 \int\limits_{\mathscr{S}} \langle \boldsymbol{P}_1, \boldsymbol{X}\rangle \, \boldsymbol{d}\sigma + 2 \int\limits_{\mathscr{S}} \langle \boldsymbol{P}_0, \boldsymbol{X}\rangle B^1(\mathscr{X}) \, d\sigma = 2 \int\limits_{\mathscr{S}} \langle \boldsymbol{P}_1, \boldsymbol{X}\rangle \, \boldsymbol{d}\sigma. \quad (168)$$

Here we have applied that $\boldsymbol{P}_0 = 0$ by our assumption and

$$\boldsymbol{P}_1(\mathscr{X}) := \frac{1}{2} \int\limits_{-1}^{1} \boldsymbol{F}_1(\mathscr{X}, t) \, dt + \frac{1}{2} \left[\boldsymbol{H}_2(\mathscr{X}, -1) + \boldsymbol{H}_2(\mathscr{X}, 1) \right]. \quad (169)$$

Since $\mathfrak{D}^1_{lm}(\boldsymbol{U}) = \mathfrak{D}_{lm}(\boldsymbol{U}^1)$ and $\mathfrak{D}^0_{lm}(\boldsymbol{X}) = 2\mathfrak{D}_{lm}(\boldsymbol{X})$ for $j, k = 1, 2, 3$ (see (162)), equation (168) writes as

$$\sum_{j,k,l,m=1}^{3} \int\limits_{\mathscr{S}} \mathfrak{S}^{jklm} \mathfrak{D}_{lm}(\boldsymbol{U}^1) \mathfrak{D}_{jk}(\boldsymbol{X}) \, \boldsymbol{d}\sigma = \int\limits_{\mathscr{S}} \langle \boldsymbol{P}_1, \boldsymbol{X}\rangle \, \boldsymbol{d}\sigma, \quad (170)$$

$$\boldsymbol{U}^1, \ \boldsymbol{X} \in \mathbb{W}^1(\mathscr{S}, \Gamma_0).$$

What we get is similar to equation (163), but with respect to the displacement $\boldsymbol{U}^1$ instead of $\boldsymbol{U}^0$ and the resultant force $\boldsymbol{P}_1$ instead of $\boldsymbol{P}_0$.
Step 6 : It is clear that the process can be iterated: if we assume the resultant forces

$$\boldsymbol{P}_j(\mathscr{X}) := \frac{1}{2} \int\limits_{-1}^{1} \boldsymbol{F}_j(\mathscr{X}, t) \, dt + \frac{1}{2} \left[\boldsymbol{H}_{j+1}(\mathscr{X}, -1) + \boldsymbol{H}_{j+1}(\mathscr{X}, 1) \right] \quad (171)$$

are vanishing for $j = 0, 1, \ldots, q-1$, the entries of the asymptotic expansion of the displacement vector field $\boldsymbol{U}(\mathscr{X}, t)$ in (140) are independent of the transverse variable $\boldsymbol{U}^k(\mathscr{X},t)=\boldsymbol{U}^k(\mathscr{X})$ for $k=0,1,\ldots,q$ and all of them, except the last one, vanish $\boldsymbol{U}^j(\mathscr{X})=0$ for $j=0,1,\ldots,q-1$. The vector field $\boldsymbol{U}^q(\mathscr{X})$ is a solution to the variational problem (see similar (163) and (170))

$$\sum_{j,k,l,m=1}^{3}\int_{\mathscr{S}}\mathfrak{S}^{jklm}\mathfrak{D}_{lm}(\boldsymbol{U}^q)\mathfrak{D}_{jk}(\boldsymbol{X})\,\boldsymbol{d}\sigma=\int_{\mathscr{S}}\langle\boldsymbol{P}_q,\boldsymbol{X}\rangle\,\boldsymbol{d}\sigma, \tag{172}$$
$$\boldsymbol{U}^q,\ \boldsymbol{X}\in\mathbb{W}^1(\mathscr{S},\Gamma_0).$$

Theorem 7.2. *The variational problem* (172) *has a unique solution* $\boldsymbol{U}^q\in\mathbb{W}^1(\mathscr{S},\Gamma_0)$ *for arbitrary resultant force* $\boldsymbol{P}_q\in\widetilde{\mathbb{W}}^{-1}(\mathscr{S},\Gamma_0)$ *for all* $q=0,1,\ldots$.

Proof. First let us check the uniqueness of a solution. For this let the matrix $\left[\gamma^{jk}\right]_{3\times3}$ with real valued entries be non-degenerated and symmetric:

$$\det[\gamma^{jk}]_{3\times3}\neq0,\quad \mathfrak{G}^\top=\mathfrak{G}. \tag{173}$$

Then for $\lambda\geqslant0$, $\mu>0$ the tensor

$$\mathscr{A}^{jklm}=\frac{4\lambda\mu}{\lambda+2\mu}\gamma^{jk}\gamma^{lm}+2\mu\left[\gamma^{jl}\gamma^{km}+\gamma^{jm}\gamma^{kl}\right], \tag{174}$$

is positive definite: There exists a constant $C_0>0$ such that

$$\mathscr{A}^{jklm}\zeta_{jk}\overline{\zeta}_{lm}\geqslant C_0|\zeta_{lm}|^2\ \text{ for all }\ \zeta_{jk}=\zeta_{kj}\in\mathbb{C}. \tag{175}$$

The asserted positive definiteness (175) is is proved, e.g., in [3, Theorem 3.3-2,a] and in [20, Ch. 1, § 7]. Here is a sketch of the proof: From (174) and (173) follows that the tensor $\mathscr{A}^{jkml}$ in (174) has the following symmetry

$$\mathscr{A}^{jklm}=\mathscr{A}^{kjlm}=\mathscr{A}^{lmjk}\ \ \forall\, j,k,l,m=1,2,3. \tag{176}$$

The sum $\eta:=\mathscr{A}^{jklm}\zeta_{jk}\overline{\zeta}_{lm}$ is real valued because $\eta=\overline{\eta}$ and the symmetry properties (176) apply.

Note that, due to symmetries (173) and (175),

$$\left[\gamma^{jl}\gamma^{km}+\gamma^{jm}\gamma^{kl}\right]\zeta_{jk}\overline{\zeta}_{lm}=2\gamma^{jk}\gamma^{lm}\zeta_{jk}\overline{\zeta}_{lm}=2\left|\gamma^{jk}\zeta_{jk}\right|^2\geqslant0. \tag{177}$$

Using the representation

$$\left[\gamma^{jl}\gamma^{km}+\gamma^{jm}\gamma^{kl}\right]\zeta_{jk}\overline{\zeta}_{lm}=\langle\mathscr{A}\zeta,\overline{\zeta}\rangle,\quad \zeta:=(\zeta_{11},\zeta_{12},\zeta_{13},\zeta_{22},\zeta_{23},\zeta_{33})^\top,$$

where the 6×6 matrix $\mathscr{A}$ is positive definite (all six principal minors of $\mathscr{A}$ are positive), we get

$$\left[\gamma^{jl}\gamma^{km}+\gamma^{jm}\gamma^{kl}\right]\zeta_{jk}\overline{\zeta}_{lm}\geqslant\frac{C_0}{2}|\zeta|^2=C_0|\zeta_{jk}|^2. \tag{178}$$

for some C_0 (actually $C_0/2$ coincides with the minimal eigenvalue of $\mathscr{A}$). The inequality in (175) is an immediate consequence of the inequalities (177) and (178).

In particular, the elasticity tensor for an isotropic shell

$$\mathfrak{S}:=\left[\mathfrak{S}^{jklm}\right]_{3\times3\times3\times3},$$

exposed in (160), is positive definite, since the corresponding 3×3 tensor $\mathfrak{G}=\left[\gamma^{jk}\right]_{3\times 3}$ (see (173)) is the identity matrix $\mathfrak{G}:=\left[\delta^{jk}\right]_{3\times 3}=I_{3\times 3}$.

If the resultant force is zero $\boldsymbol{P}_q=0$, from (163) and the positive definiteness of the tensor $\mathfrak{S}:=\left[\mathfrak{S}^{jklm}\right]_{3\times 3\times 3\times 3}$ follows $\mathfrak{D}_{jk}(\boldsymbol{U}^q)=0$ for all $j,k=1,2,3$, i.e. $\boldsymbol{U}^q$ is a Killing's vector field $\boldsymbol{U}^q\in\mathscr{R}(\mathscr{S})$.

On the other hand, due to the strong unique continuation property from the boundary (cf. [13, Lemma 3.12]) a Killing's vector field which vanishes on a part of the boundary is trivial, i.e.,

$$\mathscr{R}(\mathscr{S})\cap\mathbb{W}^1(\mathscr{S},\Gamma_0)=\{0\}\,. \tag{179}$$

The property (179) implies $\boldsymbol{U}^q=0$.

The existence of a solution follows from the celebrated Lax–Milgramm Lemma (cf., e.g., [3, § 6.3]) and can also be proved by means of the potential method. □

8. Appendix: About Koiter's and Asymptotic Linear Models of a Shell

In the present appendix we describe shortly, just for the readers convenience, the models of shell equations, which are most relevant to the model presented above.

The two-dimensional Koiter's equation for a linearly elastic shell, proposed by Koiter in 1970 (see [19]), is formulated in the following form: The covariant displacement vector field $\boldsymbol{U}^\varepsilon(x)=\sum\limits_{j=1}^{3}U_j(x)\boldsymbol{g}^j$, $\boldsymbol{g}^3=\boldsymbol{\nu}$, of the middle surface $\mathscr{S}$ of the shell Ω^ε satisfies

$$\boldsymbol{U}\in\boldsymbol{V}_{\Gamma_0}(\omega):=$$

$$:=\Big\{\boldsymbol{U}\in\mathbb{W}^1(\omega)\times\mathbb{W}^1(\omega)\times\mathbb{W}^2(\omega):\ \boldsymbol{U}(x)=\partial_{\boldsymbol{\nu}}\boldsymbol{U}_3(x)=0\ \forall\,x\in\Gamma_0\Big\}, \tag{180}$$

$$\sum_{j,k,l,m=1}^{2}\int\limits_{\omega}\Big[\varepsilon\mathfrak{a}^{jklm}\gamma_{lm}(\boldsymbol{U})\gamma_{jk}(\boldsymbol{X})+\frac{\varepsilon^3}{3}\mathfrak{a}^{jklm}\rho_{lm}(\boldsymbol{U})\rho_{jk}(\boldsymbol{X})\Big]\sqrt{\mathscr{G}(x)}\,dx=$$

$$=\int\limits_{\omega}\langle\boldsymbol{P}_0,\boldsymbol{X}\rangle\sqrt{\mathscr{G}(x)}\,dx, \tag{181}$$

$$\mathfrak{a}^{jklm}=\frac{4\lambda\mu}{\lambda+2\mu}g^{jk}g^{lm}+2\mu\big[g^{jl}g^{km}+g^{jm}g^{kl}\big].$$

Here

$$\gamma_{jk}(\boldsymbol{U}):=\big\{g_{jk}(\boldsymbol{U})-g_{jk}\big\}^{\rm lin}=\frac{1}{2}\left[U_{j;k}+U_{k;j}\right]-b_{jk}U_3=$$

$$=\frac{1}{2}\left[\partial_jU_k+\partial_kU_j\right]-\sum_{m=1}^{2}\Gamma^m_{jk}U_m-b_{jk}U_3,\quad j,k=1,2 \tag{182}$$

is the linearized change of the metric tensor, while

$$\rho_{\alpha\beta}(\boldsymbol{U}) := \frac{1}{2}\left\{b_{jk}(\boldsymbol{U}) - b_{jk}\right\}^{\rm lin} =$$
$$= \partial_{\alpha\beta}\boldsymbol{g}^3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\boldsymbol{g}^3 - b^\sigma_\alpha b_{\alpha\sigma}\boldsymbol{g}^3 + b^\sigma_\alpha\big(\partial_\beta\boldsymbol{g}^3 - \Gamma^r_{\beta\sigma}\boldsymbol{g}^r\big) +$$
$$+ b^\beta_r\big(\partial_\alpha\boldsymbol{g}^r - \Gamma^\sigma_{\alpha r}\boldsymbol{g}^r\big) + \big(\partial_\alpha b^r_\beta + \Gamma^r_{\alpha\sigma}b^\sigma_\beta b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^r_\sigma\big)\boldsymbol{g}^r \quad (183)$$

is the linearized change of curvature tensor. $g_{jk} = \langle \boldsymbol{g}_j, \boldsymbol{g}_k\rangle$, $j,k = 1,2$ by the displacement $\boldsymbol{U}$ of the surface $\mathscr{S}$. The vectors $\boldsymbol{g}_1 := \partial_1\Theta$, $\boldsymbol{g}_2 := \partial_1\Theta$, $\boldsymbol{g}_3 = \boldsymbol{\nu} := \boldsymbol{g}^1 \wedge \boldsymbol{g}^2$ constitute the covariant basis in $\mathbb{T}\mathscr{S}$, while $\{\boldsymbol{g}^j\}_{j=1}^3$ is the corresponding contravariant basis $\langle \boldsymbol{g}^j, \boldsymbol{g}_k\rangle = \delta_{jk}$. $\Gamma^m_{jk}(x) = \Gamma^m_{kj}(x) = \langle \partial_k\boldsymbol{g}_j(x), \boldsymbol{g}^m(x)\rangle$ are the Christoffel symbols (see (34)).

$\boldsymbol{P}_0(x)$ is the resultant force, defined by formula (163). $g^{lm} := \langle \boldsymbol{g}^j, \boldsymbol{g}^k\rangle$, $j,k = 1,2$ and it represents the contravariant metric tensor (the inverse to the covariant metric tensor).

Further asymptotic analysis of Koiter's equation leaded to the asymptotic models of a shell, developed by Sanchez-Palencia [31], [32], Miara & Sanchez-Palencia [23], Ciarlet & Lods [6]–[8], Ciarlet, Lods & Miara [9] (see [1], [5] for rigorous formulations and details).

The asymptotic models are derived similarly by assuming the expansions (140) and (148). Although the equation derived in in the above mentioned papers (see [1], [5]) have a similar to (163) form, there is an essential difference: one has to distinguish between a *"membrane shell"* and a *"flexural shell"*, depending on the space

$$\mathbb{W}^1_0(\omega,\Gamma_0) := \Big\{\boldsymbol{U} \in \mathbb{W}^1(\omega,\Gamma_0):\ \gamma_{jk}(\boldsymbol{U}) = 0\ \forall\, j,k = 1,2,3,\ \text{in}\ \omega\Big\} \quad (184)$$

(see e.g., [3, § 3.3]).

If $\mathbb{W}^1_0(\omega,\Gamma_0) = \{0\}$ is trivial we deal with the "membrane shell" case, while if the space $\mathbb{W}^1_0(\omega,\Gamma_0) \neq \{0\}$ is non-trivial, we deal with the"flexural shell", respectively.

In the case of "membrane shell" the equation is similar to (163), written with respect to the linearized change of metric tensor $\gamma_{jk}(\boldsymbol{U})$ and has the form

$$\sum_{j,k,l,m=1}^{3} \int_\omega \mathfrak{a}^{jklm}\gamma_{lm}(\boldsymbol{U}^0)\gamma_{jk}(\boldsymbol{X})\sqrt{\mathscr{G}(x)}\,dx = \int_\omega \langle \boldsymbol{P}_0, \boldsymbol{X}\rangle\sqrt{\mathscr{G}(x)}\,dx, \quad (185)$$
$$\mathfrak{a}^{jklm} = \frac{4\lambda\mu}{\lambda+2\mu}g^{jk}g^{lm} + 2\mu\big[g^{jl}g^{km} + g^{jm}g^{kl}\big].$$

In the case of "flexural shell" the equation is written with respect to the linearized change of curvature tensor

$$\sum_{j,k,l,m=1}^{3} \int_\omega \mathfrak{a}^{jklm}\rho_{lm}(\boldsymbol{U}^0)\rho_{jk}(\boldsymbol{X})\sqrt{\mathscr{G}(x)}\,dx = \int_\omega \langle \boldsymbol{P}_0, \boldsymbol{X}\rangle\sqrt{\mathscr{G}(x)}\,dx. \quad (186)$$

Note that the two-dimensional elasticity tensor $\mathfrak{A} := [\mathfrak{a}^{jklm}]_{3\times3}$ for a shell in (8), in (186) and in (185), is more complicated and compiled of the metric tensor of the middle surface $\mathscr{S}$. This tensor is, in general, a fully populated matrix depending on the surface variable $x \in \mathscr{S}$ (see e.g., [3, § 3.3] and [5]).

The difference between Koiter's model and the model suggested here is that we address the covariant deformation tensor $\mathfrak{D}^{\mathscr{S}}_{jk}(\boldsymbol{U})$, which is independent of the metric tensor $\big[g_{jk}(\varepsilon;x)\big]_{2\times 2}$, Chtistoffel symbols $\big\{\Gamma^k_{ij}(\varepsilon;\mathscr{X})\big\}^3_{i,j,k=1}$ and other quantities dependent on the thickness parameter ε. This independence simplifies the obtained equations considerably. For example, in Koiter's model the deformation tensor is given by the formula

$$\widetilde{\mathfrak{D}}_{jk}(\varepsilon;\boldsymbol{U}) =$$
$$= \begin{cases} \dfrac{1}{2}\big[\partial_k U_j + \partial_j U_k\big] - \displaystyle\sum_{m=1}^{3} \Gamma^m_{jk}(\varepsilon) U_m, & j,k = 1,2,\\ \dfrac{1}{2}\big[\partial_j U_3 + \dfrac{1}{\varepsilon}\,\partial_3 U_j\big] - \displaystyle\sum_{m=1}^{2} \Gamma^m_{j3}(\varepsilon) U_m, & j = 1,2, \quad k = 3,\\ \dfrac{1}{2}\big[\partial_k U_3 + \dfrac{1}{\varepsilon}\,\partial_3 U_k\big] - \displaystyle\sum_{m=1}^{2} \Gamma^m_{jk}(\varepsilon) U_m, & j = 3, \quad k = 1,2\\ \dfrac{1}{\varepsilon}\,\partial_3 U_3, & j = k = 3. \end{cases} \tag{187}$$

To derive the shell equation (185) one needs, besides the expansion of the displacement vector field (140), the asymptotic expansion of the Christoffel's symbols

$$\Gamma^m_{jk}(\varepsilon) =$$
$$= \begin{cases} \Gamma^m_{jk} - \varepsilon x_3 b^m_{k;j} + \mathscr{O}(\varepsilon^2), & j,k,m = 1,2,\\ b_{jk} - \displaystyle\sum_{r=1}^{2} \varepsilon x_3 b^r_j b_{rk}, & j,k = 1,2, \quad m = 3,\\ -b^m_j - \displaystyle\sum_{r=1}^{2} \varepsilon x_3 b^r_j b^m_r, + \mathscr{O}(\varepsilon^2), & j = 1,2, \quad k = 3, \quad m = 1,2,\\ 0, & \text{otherwise} \end{cases} \tag{188}$$

(cf. [3, § 3.3]). The expansion (8) contributes the summand $b_{jk}U_3$ in the deformation tensor and converts it into the linearized change of metric tensor (182).

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Author's address:

Andrea Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: dudu@rmi.ge; RolDud@gmail.com