Memoirs on Differential Equations and Mathematical Physics
Volume 51, 2010, 119–154

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ON FUNDAMENTAL SOLUTION OF STEADY STATE OSCILLATION EQUATIONS

Abstract. The system of differential equations of steady state oscillations of anisotropic elasticity are considered. By the generalized Fourier transform technique and with the help of the limiting absorbtion principle, we construct maximally decaying at infinity matrices of fundamental solutions explicitly. Their expressions contain surface integral over a certain semi-sphere and a line integral along the edge boundary of the semi-sphere. We investigate near field and far field properties of the fundamental matrices and show that they satisfy the generalized Sommerfeld–Kupradze type radiation conditions at infinity.

2010 Mathematics Subject Classification. 35J15, 74B05, 74J05.
Key words and phrases: Elliptic systems, fundamental solution, steady state oscillations.

რეზიუმე. განიხილება ანიზოტროპული დრეკადობის თეორიის მდგრადი დრეკადი რხევების განტოლებათა სისტემა. განზოგადებული ფურიეს გარდაქმნისა და ზღვრული ქრობის პრინციპის გამოყენებით ცხადი სახით აგებულია ფუნდამენტურ ამონახსნთა მატრიცა, რომლის ელემენტების წარმოდგენაში შედის ზედაპირული ინტეგრალი ერთეულოვან ნახევარსფეროზე და მრუდწირული ინტეგრალი ამ ნახევარსფეროს საზღვარზე, ხოლო საინტეგრო ფუნქციები ჩაწერილია ცხადი სახით ელემენტარული ფუნქციებით, რომლებიც დაკავშირებულია სიმბოლურ მატრიცასთან.

დადგენილია ფუნდამენტურ ამონახსნთა ასიმპტოტიკა. ნაჩვენებია, რომ ფუნდამენტური ამონახსნები აკმაყოფილებს ზომერფელდ-კუპრაძის განზოგადებული სახის გამოსხივების პირობებს.

1. Introduction

Fundamental solutions play an important role in investigation of boundary value problems for partial differential equations.

For isotropic bodies the matrix of fundamental solutions of steady state oscillation equations satisfying the so-called Sommerfeld–Kupradze radiation conditions at infinity is constructed in [5], where it is written explicitly in terms of standard functions.

In the paper, using the generalized Fourier transform method and the limiting absorbtion principle (see [1]), we represent the fundamental solution of steady state oscillation equations of anisotropic elasticity under the assumption that the characteristic surfaces satisfy some specific restrictions.

The fundamental solution is constructed by means of surface and curvilinear integrals. In the surface integral the integration manifold is a hemisphere, while in the curvilinear integral the integration line is a unit circumference. On the basis of these representations we define the generalized Sommerfeld–Kupradze radiation conditions in anisotropic elasticity. Similar results can be found in the references [2], [3], [6]–[9].

2. Representation of the Fundamental Solution

2.1. **Equations.** The homogeneous system of differential equations of steady state oscillations of anisotropic elasticity reads as follows (see, e.g., [6], [7])

$$\mathbb{C}(\partial,\omega)u := C(\partial)u + \omega^2 u = c_{kjpq}\partial_j\partial_q u_p + \omega^2 u = 0, \tag{2.1}$$

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector (amplitude), $\omega > 0$ is the oscillation (frequency) parameter,

$$\mathbb{C}(\partial,\omega) := C(\partial) + \omega^2 I_3 = \left[c_{kjpq}\partial_j\partial_q + \delta_{kp}\omega^2\right]_{3\times 3},$$
$$C(\partial) = [c_{kjpq}\partial_j\partial_q]_{3\times 3}.$$

Here $\partial_j = \frac{\partial}{\partial x_j}$, I_3 stands for the unit 3×3 matrix, δ_{kp} is the Kroneker delta, the superscript $(\cdot)^\top$ denotes transposition, c_{kjpq} are elastic constants

$$c_{kjpq} = c_{jkpq} = c_{pqkj}, \quad k,j,p,q = 1,2,3.$$

Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwartz space $S'(\mathbb{R}^3)$) which for regular summable functions f and g read as follows

$$\mathcal{F}_{x\to\xi}[f] = \int\limits_{\mathbb{R}^3} f(x)e^{ix\cdot\xi}dx, \quad \mathcal{F}^{-1}_{\xi\to x}[g] = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} g(\xi)e^{-ix\cdot\xi}d\xi,$$

where $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$ and $x\cdot\xi = x_k\xi_k$. Note that for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $f \in S'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f] = (-i\xi)^\alpha\mathcal{F}[f], \quad \mathcal{F}^{-1}[\xi^\alpha g] = (i\partial)^\alpha\mathcal{F}^{-1}[g].$$

Denote by $\Psi(x.\omega)$ the matrix of fundamental solutions of the operator $\mathbb{C}(\partial,\omega)$

$$\mathbb{C}(\partial,\omega)\Psi(x,\omega)=I_3\delta(x).$$

Here $\delta(\cdot)$ is the Dirac's delta distribution. By standard arguments we can show that

$$\Psi(x,\omega)=\mathcal{F}^{-1}[\mathbb{C}^{-1}(-i\xi,\omega)]=\mathcal{F}^{-1}\left[\frac{\mathbb{C}^*(-i\xi,\omega)}{H(\xi,\omega)}\right]=$$

$$=N(\partial_x,\omega)\mathcal{F}^{-1}\left[\frac{1}{H(\xi,\omega)}\right]=N(\partial_x,\omega)\Gamma(x,\omega), \tag{2.2}$$

where $\mathbb{C}^{-1}(-i\xi,\omega)$ is the inverse to the symbol matrix $\mathbb{C}(-i\xi,\omega)$, $\mathbb{C}^*(-i\xi,\omega)$ is the corresponding matrix of cofactors, $H(\xi,\omega) := \det\mathbb{C}(-i\xi,\omega)$, $N(\partial_x,\omega)=[N_{kj}(\partial_x,\omega)]_{3\times3}$ is the formally adjoint matrix to the matrix $\mathbb{C}(\partial,\omega)$, i.e.,

$$N(\partial_x,\omega)\mathbb{C}(\partial,\omega)=\mathbb{C}(\partial,\omega)N(\partial_x,\omega)=H(x,\omega)I_3.$$

It is clear that N_{kj} is a nonhomogeneous differential operator of order 4 containing 0th, 2nd and 4th order differential operators.

Assume that for any $\eta\in\Sigma_1$, where

$$\Sigma_1=\{\eta\in\mathbb{R}^3 \mid |\eta|=1\},$$

the equation $H(\xi,\omega)=0$ written in spherical coordinates

$$\xi_1=\rho\cos\varphi\sin\theta,$$
$$\xi_2=\rho\sin\varphi\sin\theta,$$
$$\xi_3=\rho\cos\theta,\quad 0\le\theta\le\pi,\quad 0\le\varphi<2\pi,\quad \rho=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}=|\xi|,$$

has three different roots t_1, t_2, t_3 with respect to $t=\frac{\rho^2}{\omega^2}$, so

$$H(\xi,\omega)=-a(\eta)\prod_{j=1}^{3}(\rho^2-\omega^2\mu_j^2(\eta)),$$

where $t_j=\mu_j^2(\eta)$, $j=1,2,3$, and

$$a(\eta)=[\mu_1^2(\eta)\mu_2^2(\eta)\mu_3^2(\eta)]^{-1},\quad \eta\in\Sigma_1;\quad \mu_j(-\eta)=\mu_j(\eta),\quad a(-\eta)=a(\eta).$$

It is clear that

$$\mathbb{C}(-i\xi,\omega)=-C(\xi)+I_3\omega^2,$$

where $C(\xi)=[c_{kjpq}\xi_k\xi_j]_{3\times3}$ and $C(\xi)$ is a positive definite matrix, which means that there exists $\delta>0$ such that

$$C(\xi)a\cdot\overline{a}\ge\delta|a|^2|\xi|^2\quad\text{for all }\ a\in\mathbb{C}^3.$$

Note that $a(\eta)=\det C(\eta)\ge\delta_1>0$, $\eta\in\Sigma_1$, and $H(-\xi,\omega)=H(\xi,\omega)$.

Lemma 2.1. *Let $\tau=\omega+i\varepsilon$ with $\varepsilon\ne0$ and $\omega>0$. Then*

$$H(\xi,\tau)=\det(-i\xi,\tau)\ne0\quad\textit{for all }\ \xi\in\mathbb{R}^3\setminus\{0\}.$$

Proof. Assume that $H(\xi,\tau)=0$ for some $\xi\in\mathbb{R}^3\setminus\{0\}$ and a complex τ. There exists $a_0\in\mathbb{C}^3$, $a_0\neq 0$, such that

$$\mathbb{C}(-i\xi,\tau)a_0=-C(\xi)a_0+\tau^2a_0=0.$$

Multiplying the last equation by $\overline{a}_0$ (in scalar sense) we have

$$\tau^2|a_0|^2=C(\xi)a_0\cdot\overline{a}_0,$$

or

$$\tau^2=\frac{1}{|a_0|^2}\,C(\xi)a_0\cdot\overline{a}_0>0$$

due to the positive definiteness of $C(\xi)$. But τ is a complex number. This contradiction completes the proof. □

2.2. **Fundamental solution of pseudooscillation.** First we consider the situation of complex $\tau=\omega+i\varepsilon$, $\varepsilon\neq 0$ instead of $\omega>0$ and construct the fundamental solution of the corresponding system of pseudooscillation.

Theorem 2.2. *The fundamental solution of* (2.1) *for a complex* $\tau=\omega+i\varepsilon$ *have the following form:*

$$\underset{(\varepsilon>0)}{\Psi(x,\tau)}=N(\partial_x,\tau)\Bigg[-\frac{i}{16\pi^2\tau^3}\int\limits_{\Sigma_1}\Bigg\{\sum_{q=1}^{3}\frac{e^{i|(x\cdot\eta)|\tau\mu_q}\mu_q}{\prod\limits_{j=1,\,j\neq q}^{3}(\mu_q^2-\mu_j^2)}\Bigg\}\frac{d\Sigma_1}{a(\eta)}\Bigg],\tag{2.3}$$

or

$$\underset{(\varepsilon<0)}{\Psi(x,\tau)}=N(\partial_x,\tau)\Bigg[\frac{i}{16\pi^2\tau^3}\int\limits_{\Sigma_1}\Bigg\{\sum_{q=1}^{3}\frac{e^{-i|(x\cdot\eta)|\tau\mu_q}\mu_q}{\prod\limits_{j=1,\,j\neq q}^{3}(\mu_q^2-\mu_j^2)}\Bigg\}\frac{d\Sigma_1}{a(\eta)}\Bigg].\tag{2.4}$$

Proof. Taking a complex $\tau=\omega+i\varepsilon$, $\varepsilon\neq 0$, we have $H(\xi,\tau)\neq 0$ due to Lemma 2.1 and

$$\Gamma(x,\tau)=\mathcal{F}^{-1}[H^{-1}(\xi,\tau)]=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{e^{-ix\cdot\xi}}{H(\xi,\tau)}\,d\xi\quad\text{(cf. (2.2))}.$$

It is easy to check that

$$\Gamma(x,\tau)=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{e^{-ix\cdot\xi}}{H(\xi,\tau)}\,d\xi=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{e^{ix\cdot\xi}}{H(-\xi,\tau)}\,d\xi=$$
$$=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{e^{ix\cdot\xi}}{H(\xi,\tau)}\,d\xi.$$

Taking into account that $x\cdot\xi=|x|\cdot\rho\cos\gamma=(x\cdot\eta)\rho$, $\cos\gamma=\big(\frac{x}{|x|}\cdot\eta\big)=\big(\frac{x}{|x|}\cdot\frac{\xi}{|\xi|}\big)$, we have

$$\Gamma(x,\tau)=(2\pi)^{-3}\int\limits_{\Sigma_1}\int\limits_{0}^{\infty}\Bigg\{\frac{e^{-i|x|\rho\cos\gamma}\rho^2d\rho\,d\Sigma_1}{-a(\eta)\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\Bigg\}=$$

$$= (2\pi)^{-3}\int\limits_{\Sigma_1}\int\limits_0^{\infty}\left\{\frac{e^{i|x|\rho\cos\gamma}\rho^2 d\rho\, d\Sigma_1}{-a(\eta)\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\right\} =$$

$$= -(2\pi)^{-3}\int\limits_{\Sigma_1}\frac{d\Sigma_1}{a(\eta)}\left\{\int\limits_0^{\infty}\frac{e^{\pm i|x|\rho\cos\gamma}}{\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\,\rho^2\,d\rho\right\}. \quad (2.5)$$

From (2.5) we can write that

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3}\int\limits_{\Sigma_1}\frac{d\Sigma_1}{a(\eta)}\left\{\int\limits_0^{\infty}\frac{e^{i(x\cdot\eta)\rho}}{\prod\limits_{j=1}^{3}[\rho^2-\tau^2\mu_j^2(\eta)]}\,\rho^2\,d\rho + \int\limits_0^{\infty}\frac{e^{-i(x\cdot\eta)\rho}}{\prod\limits_{j=1}^{3}[\rho^2-\tau^2\mu_j^2(\eta)]}\,\rho^2\,d\rho\right\}. \quad (2.6)$$

Taking into account

$$\int\limits_0^{\infty}\frac{e^{-i(x\cdot\eta)\rho}}{\prod\limits_{j=1}^{3}[\rho^2-\tau^2\mu_j^2(\eta)]}\,\rho^2\,d\rho = \begin{bmatrix}\rho=-r\\ d\rho=-dr\end{bmatrix} =$$

$$= \int\limits_0^{-\infty}\frac{e^{i(x\cdot\eta)r}}{\prod\limits_{j=1}^{3}[r^2-\tau^2\mu_j^2(\eta)]}\,r^2\,(-dr) = \int\limits_{-\infty}^{0}\frac{e^{i(x\cdot\eta)r}}{\prod\limits_{j=1}^{3}[r^2-\tau^2\mu_j^2(\eta)]}\,r^2\,dr,$$

(2.6) can be rewritten as

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3}\int\limits_{\Sigma_1}\frac{d\Sigma_1}{a(\eta)}\int\limits_{-\infty}^{\infty}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\,. \quad (2.7)$$

Assume that ρ is a complex variable $\rho=\rho'+i\rho''$, $\tau\mu_j(\eta)=\omega\mu_j(\eta)+i\varepsilon\mu_j(\eta)$, $\varepsilon\neq 0$, $j=1,2,3$.

In (2.7) the integrand is an analytic function with respect to ρ and (see Fig. 2.1)

$$\int\limits_{-\infty}^{\infty}\left\{\frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\right\}d\rho = \int\limits_{\ell_\varepsilon}\left\{\frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\right\}d\rho,$$

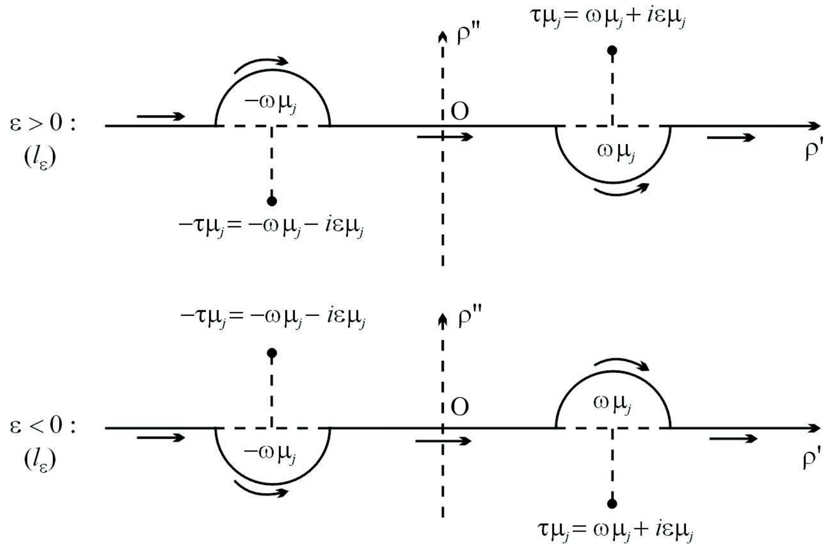


FIGURE 2.1.

or

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3}\int\limits_{\Sigma_1}\frac{d\Sigma_1}{a(\eta)}\int\limits_{\ell_\varepsilon}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}[\rho-\tau\mu_j(\eta)][\rho+\tau\mu_j(\eta)]}\,. \tag{2.8}$$

Let us denote by C_R^+ and C_R^- the upper and the lower half-part of the circumference with radius $R \gg 1$ on the plane $0\rho'\rho''$. If $(x\cdot\eta)\geq 0$, then $i(x\cdot\eta)\rho = i(x\cdot\eta)\rho' - (x\cdot\eta)\rho''$ and in this case $\operatorname{Re}\{i(x\cdot\eta)\rho\}\leq 0$.

Clearly, for $(x\cdot\eta)\geq 0$

$$\int\limits_{C_R^+}\frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}[\rho^2-\tau^2\mu_j^2(\eta)]}\,d\rho \to 0 \quad \text{as} \quad R\to+\infty,$$

because the integrand is $O(\rho^{-4})$.

Similarly, if $(x\cdot\eta)\leq 0$, then $\operatorname{Re}\big\{i(x\cdot\eta)\rho\big\}_{\rho\in C_R^-}\leq 0$ and

$$\int\limits_{C_R^-}\frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}[\rho^2-\tau^2\mu_j^2(\eta)]}\,d\rho \to 0 \quad \text{as} \quad R\to+\infty.$$

We have the following situations:

a) $(x \cdot \eta) \geq 0$, $\varepsilon > 0$;

$$\int\limits_{\ell_{3,R}} + \int\limits_{C_R^+} + \int\limits_{C_{j,\delta}} = 0.$$

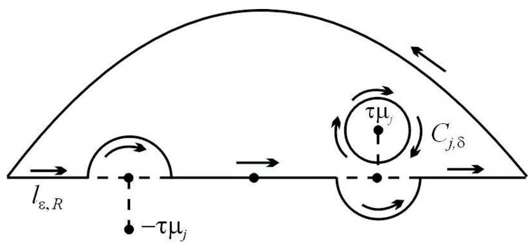


FIGURE 2.2.

Choosing $\delta > 0$ sufficiently small and taking limit as $R \to +\infty$, we get

$$\int\limits_{\ell_\varepsilon} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho = \sum_{q=1}^{3} \int\limits_{C_{q,\delta}(\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho. \tag{2.9}$$

b) $(x \cdot \eta) \leq 0$, $\varepsilon > 0$;

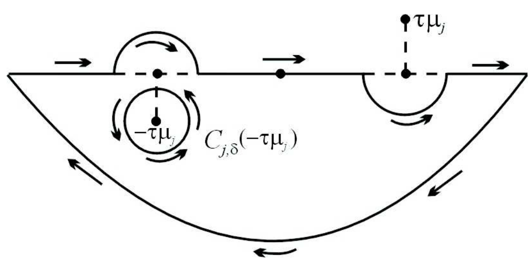


FIGURE 2.3.

$$\int\limits_{\ell_\varepsilon} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho = -\sum_{q=1}^{3} \int\limits_{C_{q,\delta}(-\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho. \tag{2.10}$$

c) $(x \cdot \eta) \geq 0$, $\varepsilon < 0$;

$$\int\limits_{\ell_\varepsilon} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho = \sum_{q=1}^{3} \int\limits_{C_{q,\delta}(-\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2 - \tau^2\mu_j^2)}\, d\rho. \tag{2.11}$$

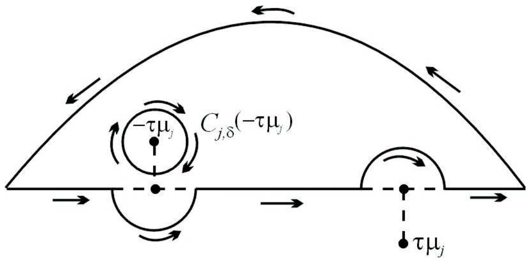


FIGURE 2.4.

d) $(x \cdot \eta) \le 0$, $\varepsilon < 0$;

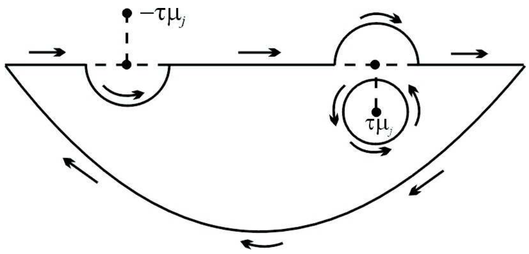


FIGURE 2.5.

$$\int\limits_{\ell_\varepsilon} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\,d\rho = -\sum_{q=1}^{3}\int\limits_{C_{q,\delta}(\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\,d\rho. \tag{2.12}$$

In what follows, we use the following notation (see Fig. 2.6)

$$\Sigma_x^+ = \{\eta \in \Sigma_1 \ : \ (x\cdot\eta) \ge 0\},$$
$$\Sigma_x^- = \{\eta \in \Sigma_1 \ : \ (x\cdot\eta) \le 0\}.$$

From the relations (2.9)–(2.12) and (2.8) we conclude that for $\varepsilon > 0$

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3}\Bigg[\int\limits_{\Sigma_x^+}\Bigg\{\sum_{q=1}^{3}\int\limits_{C_{q,\delta}(\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\Bigg\}\frac{d\Sigma_1}{a(\eta)} -$$
$$-\int\limits_{\Sigma_x^-}\Bigg\{\sum_{q=1}^{3}\int\limits_{C_{q,\delta}(-\tau\mu_q)} \frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\Bigg\}\frac{d\Sigma_1}{a(\eta)}\Bigg], \tag{2.13}$$

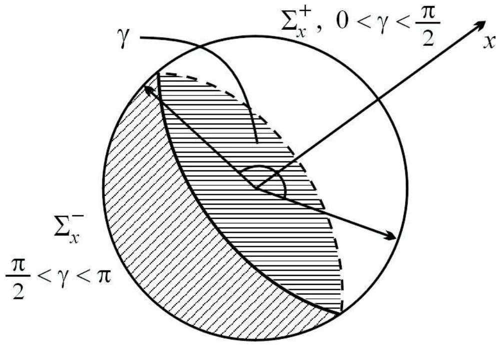


FIGURE 2.6.

and for $\varepsilon < 0$

$$\Gamma(x,\tau) = -\frac{1}{2(2\pi)^3}\left[\int\limits_{\Sigma_x^+}\left\{\sum_{q=1}^{3}\int\limits_{C_{q,\delta}(-\tau\mu_q)\circlearrowleft}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)}\right\}\frac{d\Sigma_1}{a(\eta)}\right]. \tag{2.14}$$

Using the Cauchy integral formula, we can write

$$\int\limits_{C_{q,\delta}(\tau\mu_q)\circlearrowleft}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)} = 2\pi i\frac{e^{i(x\cdot\eta)\tau\mu_q}\tau^2\,\mu_q^2}{\prod\limits_{j=1}^{3}[\tau\mu_q+\tau\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\tau\mu_q-\tau\mu_j]}\,;$$

$$\int\limits_{C_{q,\delta}(-\tau\mu_q)\circlearrowleft}\frac{e^{i(x\cdot\eta)\rho}\rho^2\,d\rho}{\prod\limits_{j=1}^{3}(\rho^2-\tau^2\mu_j^2)} = 2\pi i\frac{e^{-i(x\cdot\eta)\tau\mu_q}\tau^2\,\mu_q^2}{\prod\limits_{j=1}^{3}[-\tau\mu_q-\tau\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[-\tau\mu_q+\tau\mu_j]}\,.$$

Due to these relations, we can rewrite (2.13) and (2.14) as follows

$$\underset{\varepsilon>0}{\Gamma(x,\tau)} = -\frac{i}{8\pi^2}\left[\int\limits_{\Sigma_x^+}\left\{\sum_{q=1}^{3}\frac{e^{i(x\cdot\eta)\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[\mu_q+\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q-\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)}-\right.$$
$$\left.-\int\limits_{\Sigma_x^-}\left\{\sum_{q=1}^{3}\frac{e^{-i(x\cdot\eta)\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[-\mu_q-\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[-\mu_q+\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)}\right] \tag{2.15}$$

and

$$\underset{\varepsilon<0}{\Gamma(x,\tau)} = -\frac{i}{8\pi^2}\left[\int\limits_{\Sigma_x^+}\left\{\sum_{q=1}^{3}\frac{e^{-i(x\cdot\eta)\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[-\mu_q-\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[-\mu_q+\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)}-\right.$$

$$-\int\limits_{\Sigma_x^-}\left\{\sum_{q=1}^{3}\frac{e^{i(x\cdot\eta)\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[\mu_q+\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q-\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)}\Bigg]. \tag{2.16}$$

Clearly, (2.15) and (2.16) decay at infinity faster than any negative power of $|x|$.

Taking into account (2.15) and (2.16), we get

$$\underset{\varepsilon>0}{\Gamma(x,\tau)}=-\frac{i}{8\pi^2}\int\limits_{\Sigma_1}\left\{\sum_{q=1}^{3}\frac{e^{i|(x\cdot\eta)|\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[\mu_q+\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q-\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)} \tag{2.17}$$

and

$$\underset{\varepsilon<0}{\Gamma(x,\tau)}=\frac{i}{8\pi^2}\int\limits_{\Sigma_1}\left\{\sum_{q=1}^{3}\frac{e^{-i|(x\cdot\eta)|\tau\mu_q}(\tau^2\,\mu_q^2)}{\prod\limits_{j=1}^{3}[\mu_q+\mu_j]\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q-\mu_j]\tau^5}\right\}\frac{d\Sigma_1}{a(\eta)}. \tag{2.18}$$

Finally, from (2.17) and (2.18) we obtain (2.3) and (2.4). □

2.3. **Fundamental solution of steady state oscillation.** Using Theorem 2.2 and limiting procedure, we can prove

Theorem 2.3. *The fundamental solution of* (2.1) *has the following form*

$$\Psi(x,\omega,1)=N(\partial_x,\omega)\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1, \tag{2.19}$$

or

$$\Psi(x,\omega,2)=-N(\partial_x,\omega)\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)e^{-i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1, \tag{2.20}$$

where

$$F_q(\eta)=-\frac{i}{8\pi^2}\,\frac{\rho_q(\eta)}{\left\{\prod\limits_{j=1,\,j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]\right\}a(\eta)}. \tag{2.21}$$

Proof. Taking limit in (2.17) and (2.18) as $|\varepsilon|\to 0$, we get

$$\lim_{\varepsilon\to 0+}\Gamma(x,\tau)=-\frac{i}{16\pi^2\omega^3}\int\limits_{\Sigma_1}\sum_{q=1}^{3}\frac{e^{i|(x\cdot\eta)|\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\frac{d\Sigma_1}{a(\eta)}=$$
$$=:\Gamma(x,\omega,1); \tag{2.22}$$

$$\lim_{\varepsilon\to 0-}\Gamma(x,\tau)=-\frac{i}{16\pi^2\omega^3}\int\limits_{\Sigma_1}\sum_{q=1}^{3}\frac{e^{-i|(x\cdot\eta)|\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1,\,j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\frac{d\Sigma_1}{a(\eta)}=$$
$$=:\Gamma(x,\omega,2). \tag{2.23}$$

Clearly, $\Gamma(x,\omega,2)=\overline{\Gamma(x,\omega,1)}$.

(2.22) and (2.23) are the formulae similar to those in [4], but they are not identical. Another difference is that (2.22) and (2.23) satisfy the radiation conditions.

We can rewrite (2.22) as

$$\Gamma(x,\omega,1)=-\frac{i}{16\pi^2\omega^3}\Bigg\{\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}\frac{e^{i(x\cdot\eta)\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\, j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\,\frac{d\Sigma_1}{a(\eta)}+ \\ +\int\limits_{\Sigma_x^-}\sum_{q=1}^{3}\frac{e^{-i(x\cdot\eta)\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\, j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\,\frac{d\Sigma_1}{a(\eta)}\Bigg\}. \tag{2.24}$$

Using the substitution $\eta=-\widetilde{\eta}$ in the second integral of (2.24), we obtain $(\mu_q(-\eta)=\mu_q(\eta),\ d\Sigma_{1\eta}=d\Sigma_{1\widetilde{\eta}},\ \Sigma_x^-\to\Sigma_x^+,\ a(-\eta)=a(\eta))$

$$\Gamma(x,\omega,1)=-\frac{i}{8\pi^2\omega^3}\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}\frac{e^{i(x\cdot\eta)\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\, j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\,\frac{d\Sigma_1}{a(\eta)}\,.$$

$\Gamma(x,\omega,2)$ can be written in a similar form

$$\Gamma(x,\omega,2)=\frac{i}{8\pi^2\omega^3}\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}\frac{e^{-i(x\cdot\eta)\omega\mu_q(\eta)}\mu_q(\eta)}{\prod\limits_{j=1\, j\neq q}^{3}[\mu_q^2(\eta)-\mu_j^2(\eta)]}\,\frac{d\Sigma_1}{a(\eta)}\,.$$

Taking into account the notation (2.21) and the fact that $\rho_\eta(\eta)=\omega\mu_q(\eta)$, $q=1,2,3$, we get

$$\Gamma(x,\omega,1)=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)e^{i(x\cdot\eta)\rho_q}\,d\Sigma_1 \tag{2.25}$$

and

$$\Gamma(x,\omega,2)=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}(-F_q(\eta))e^{-i(x\cdot\eta)\rho_q}\,d\Sigma_1. \tag{2.26}$$

Evidently, (2.25) and (2.26) imply (2.19) and (2.20). □

Denote by S_q the characteristic surface given by the equation $\rho=\rho_q(\eta)$, $\eta\in\Sigma_1$ $(q=1,2,3)$. We assume that S_q is a star-shaped surface with respect to the origin and it is convex; it means that $\xi\cdot\eta(\xi)\geq 0$ for all $\xi\in S_q$, where $n(\xi)$ is the outward unit normal vector at $\xi\in S_q$.

Note that $\eta\rho_q(\eta)=\xi\in S_q$ and

$$\rho_q^2 d\Sigma_1=\left(\frac{\xi}{|\xi|}\cdot n(\xi)\right)dS_q=\frac{1}{\rho_q}\,(\xi\cdot n(\xi))dS_q.$$

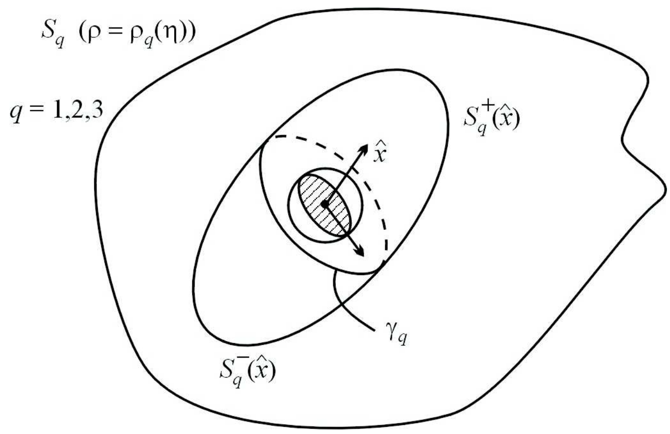


FIGURE 2.7.

Therefore we can rewrite (2.19) and (2.20) in the equivalent form

$$\Psi(x,\omega,1) = N(\partial_x,\omega)\sum_{q=1}^{3}\int\limits_{S_q^+(\hat{x})}\frac{F_q(\eta)e^{i(x\cdot\xi)}(\xi\cdot n(\xi))}{\rho_q^3(\eta)}\,dS_q;$$

$$\Psi(x,\omega,2) = -N(\partial_x,\omega)\sum_{q=1}^{3}\int\limits_{S_q^+(\hat{x})}\frac{F_q(\eta)e^{-i(x\cdot\xi)}(\xi\cdot n(\xi))}{\rho_q^3(\eta)}\,dS_q.$$

3. ASYMPTOTICS

3.1. **Singularity in Vicinity of the Origin.** Let S be a regular surface in $\mathbb{R}^3$. Then

$$\frac{\partial}{\partial S_k(\xi)} = \partial_k(n,\nabla_\xi) = [n\times\nabla_\xi]_k, \quad k=1,2,3,$$

i.e.,

$$\frac{\partial}{\partial S_1(\xi)} = \partial_1(n,\nabla_\xi) = n_2\,\frac{\partial}{\partial\xi_3} - n_3\,\frac{\partial}{\partial\xi_2}\,,$$
$$\frac{\partial}{\partial S_2(\xi)} = \partial_2(n,\nabla_\xi) = n_3\,\frac{\partial}{\partial\xi_1} - n_1\,\frac{\partial}{\partial\xi_3}\,,$$
$$\frac{\partial}{\partial S_3(\xi)} = \partial_3(n,\nabla_\xi) = n_1\,\frac{\partial}{\partial\xi_2} - n_2\,\frac{\partial}{\partial\xi_1}\,,$$

where $\nabla_\xi = \left(\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2},\frac{\partial}{\partial\xi_3}\right)$, $n(\xi)$ is the outward unit normal vector at $\xi\in S$ and $\times$ denotes the vector product.

If S is a closed regular surface and f, g are smooth functions, then by the Stokes theorem

$$\int\limits_S \left[\partial_k(n,\nabla_\xi)f(\xi)\right] f(\xi)\,dS = -\int\limits_S f(\xi)\left[\partial_k(n,\nabla_\xi)g(\xi)\right]\,dS.$$

Let us consider a special type of the function $\psi_*(\xi)=\psi\left(\frac{\xi}{r}\right)$, where $r=|\xi|$ and $\frac{\xi}{r}=\eta\in\Sigma_1$. We have

$$\begin{aligned}
[\nabla_\xi\psi_*(\xi)]_j &= \left[\nabla_\xi\psi\left(\frac{\xi}{r}\right)\right]_j = [\nabla_\xi\psi(\eta)]_j = \frac{\partial}{\partial\xi_j}\,\psi_*(\xi) = \frac{\partial}{\xi_j}\,\psi(\eta) = \\
&= \sum_{p=1}^{3}\frac{\partial\psi(\eta)}{\partial\eta_p}\cdot\frac{\partial\eta_p}{\partial\xi_j} = \sum_{p=1}^{3}\frac{\partial\psi(\eta)}{\partial\eta_p}\,\frac{\partial}{\partial\xi_j}\left[\frac{\xi_p}{r}\right] = \\
&= \sum_{p=1}^{3}\frac{\partial\psi(\eta)}{\partial\eta_p}\left[\frac{\delta_{jp}}{r}-\frac{\xi_p\xi_j}{r^3}\right] = \frac{1}{r}\left[\frac{\partial\psi(\eta)}{\partial\eta_j}-\eta_j(\eta\cdot\nabla_\eta\psi(\eta))\right],
\end{aligned}$$

i.e.,

$$\nabla_\xi\psi_*(\xi)=\nabla_\xi\psi(\eta)=\nabla_\xi\psi\left(\frac{\xi}{r}\right)=\frac{1}{r}\left[\nabla_\eta\psi(\eta)-\eta(\eta\cdot\nabla_\eta\psi(\eta))\right]. \tag{3.1}$$

It follows from (3.1) that for the case of Σ_1 $(\eta=n)$

$$\begin{aligned}
\partial_k(n,\nabla_\xi)\psi(\eta) &= \partial_k(n,\nabla_\xi)\psi\left(\frac{\xi}{r}\right) = \partial_k(\eta,\nabla_\xi)\psi(\eta) = \\
&= \frac{1}{r}\left[\eta\times\nabla_\eta\psi(\eta)\right]_k = \frac{1}{r}\,\partial_k(\eta,\nabla_\eta)\psi(\eta),
\end{aligned}$$

or

$$[\eta\times\nabla_\eta\psi(\eta)]_k=\partial_k(\eta,\nabla_\eta)\psi(\eta).$$

Hence

$$\partial_1(\eta,\nabla_\eta)\eta=\begin{bmatrix}0\\-\eta_3\\\eta_2\end{bmatrix},\quad \partial_2(\eta,\nabla_\eta)\eta=\begin{bmatrix}\eta_3\\0\\-\eta_1\end{bmatrix},\quad \partial_3(\eta,\nabla_\eta)\eta=\begin{bmatrix}-\eta_2\\\eta_1\\0\end{bmatrix}. \tag{3.2}$$

Let us consider $\psi(\eta)=e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}$ with $\frac{x}{|x|}=\widehat{x}\in\Sigma_1$, $\eta\in\Sigma_1$, $\lambda=const$ and $\rho(\eta)=\rho_k(\eta)$, $k=1,2,3$. We easily derive

$$\begin{aligned}
&\partial_k(\eta,\nabla_\eta)e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)} = \\
&\qquad = i\lambda e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}\left[(\widehat{x}\cdot\partial_k(\eta,\nabla_\eta)\eta)\rho(\eta)+(\widehat{x}\cdot\eta)\partial_k(\eta,\nabla_\eta)\rho(\eta)\right].
\end{aligned} \tag{3.3}$$

It is evident from (3.2) that

$$\widehat{x}\cdot\partial_1(\eta,\nabla_\eta)\eta=\begin{pmatrix}\widehat{x}_1\\\widehat{x}_2\\\widehat{x}_3\end{pmatrix}\cdot\begin{pmatrix}0\\-\eta_3\\\eta_2\end{pmatrix}=\widehat{x}_3\eta_2-\widehat{x}_2\eta_3=[\eta\times\widehat{x}]_1,$$

$$\widehat{x}\cdot\partial_2(\eta,\nabla_\eta)\eta=\begin{pmatrix}\widehat{x}_1\\ \widehat{x}_2\\ \widehat{x}_3\end{pmatrix}\cdot\begin{bmatrix}\eta_3\\ 0\\ -\eta_1\end{bmatrix}=\widehat{x}_1\eta_3-\widehat{x}_3\eta_1=[\eta\times\widehat{x}]_2,$$

$$\widehat{x}\cdot\partial_3(\eta,\nabla_\eta)\eta=\begin{pmatrix}\widehat{x}_1\\ \widehat{x}_2\\ \widehat{x}_3\end{pmatrix}\cdot\begin{bmatrix}-\eta_2\\ \eta_1\\ 0\end{bmatrix}=\widehat{x}_2\eta_1-\widehat{x}_1\eta_2=[\eta\times\widehat{x}]_3,$$

i.e.,

$$(\widehat{x}\cdot\partial_k(\eta,\nabla_\eta)\eta)=[\eta\times\widehat{x}]_k,\quad k=1,2,3. \tag{3.4}$$

Denoting

$$\Phi_k(\widehat{x},\eta)=[\eta\times\widehat{x}]_k\rho(\eta)+(\widehat{x}\cdot\eta)\partial_k(\eta,\nabla_\eta)\rho(\eta)= \\ =\eta\times(\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)),\quad k=1,2,3, \tag{3.5}$$

we can rewrite (3.3) as

$$\partial_k(\eta,\nabla_\eta)e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}=i\lambda e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}\Phi_k(\widehat{x},\eta),\quad k=1,2,3. \tag{3.6}$$

Lemma 3.1. *The following conditions are equivalent:*
i) $\Phi(\widehat{x},\eta)=\eta\times[\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)]\neq 0$*;*
ii) $\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)\nparallel\eta$*;*
iii) $\eta\times\Phi(\widehat{x},\eta)=-\widehat{x}\rho(\eta)-(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)\neq 0$.

Proof. Since $\sum\limits_{k=1}^{3}\eta_k\partial_k(\eta,\nabla_\eta)\equiv 0$, from (3.5) and (3.6) we obtain

$$\sum_{k=1}^{3}\eta_k\Phi_k(\widehat{x},\eta)\equiv 0,\quad \text{i.e.,}\quad \eta\cdot\Phi(\widehat{x},\eta)\equiv 0,\quad \eta\in\Sigma_1,$$

where $\Phi(\widehat{x},\eta)=(\Phi_1(\widehat{x},\eta),\Phi_2(\widehat{x},\eta),\Phi_1(\widehat{x},\eta))$ and $\eta=(\eta_1,\eta_2,\eta_3)$.

If $\Phi(\widehat{x},\eta)\neq 0$, then this condition is equivalent to $[\eta\times\Phi(\widehat{x},\eta)]\neq 0$.

On the other hand,

$$\Phi(\widehat{x},\eta)\neq 0\Longleftrightarrow\Phi(\widehat{x},\eta)=\eta\times(\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta))\neq 0,$$

i.e., the vector $\rho(\eta)\widehat{x}+(\widehat{x}\cdot\eta)\nabla_2\rho(\eta)$ is not parallel to η. Thus, i)$\Leftrightarrow$ii).

In the particular case under consideration it is clear that

$$\rho_q(t\eta)=\omega\mu_q(t\eta)=\frac{1}{t}\,\omega\mu_q(\eta)=\frac{1}{t}\,\rho_q(\eta),\quad t>0.$$

The functions $\rho_q(\eta)$, $q=1,2,3$, are homogeneous functions of order (-1) for $\eta\in\Sigma_1$. Ttherefore

$$(\eta\cdot\nabla_\eta\rho(\eta))=-\rho(\eta). \tag{3.7}$$

Taking into account (3.7) and the fact that for arbitrary vectors a,b and c, $a\times[b\times c]=b(a\cdot c)-c(a\cdot b)$, we have

$$\eta\times\Phi(\widehat{x},\eta)=\eta\times\{\eta\times(\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta))\}= \\ =\eta\{(\eta\cdot\widehat{x})\rho(\eta)+(\widehat{x}\cdot\eta)(\eta\cdot\nabla_\eta\rho(\eta))\}-(\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta))= \\ =(\eta\cdot\widehat{x})\{\rho(\eta)-\rho(\eta)\}\eta-\{\widehat{x}\rho(\eta)+(\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)\}=$$

$$= -\widehat{x}\rho(\eta) - (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta),$$

hence

$$\eta\Phi(\widehat{x},\eta) = -\widehat{x}\rho(\eta) - (\widehat{x},\eta)\nabla_\eta\rho(\eta).$$

Using (3.4), we conclude that i)⇔iii). □

Note that if $(\widehat{x}\cdot\eta)=0$, then $\widehat{x}\perp\eta$, $|\widehat{x}\times\eta|=1$ and

$$|\Phi(\widehat{x},\eta)| = |\eta\times\widehat{x}|\rho(\eta) = \rho(\eta) > 0,$$

i.e., if $(\widehat{x}\cdot\eta)=0$, then $\Phi(\widehat{x},\eta)\neq 0$.

From Lemma 3.1 we conclude that

$$\Phi(\widehat{x},\eta) = \eta\times[\widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)] = 0 \iff \\ \iff \widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta) = 0. \quad (3.8)$$

Since

$$\eta\cdot(\widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)) = (\eta\cdot\widehat{x})\rho(\eta) + (\widehat{x}\cdot\eta)(\eta\cdot\nabla_\eta\rho(\eta)) = \\ = (\eta\cdot\widehat{x})\rho(\eta) - (\widehat{x}\cdot\eta)\rho(\eta) = 0,$$

this means that $\{\widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)\}\perp\eta$ and

$$|\Phi(\widehat{x},\eta)| = |\widehat{x}\rho(\eta) + (\widehat{x}\cdot\eta)\nabla_\eta\rho(\eta)|.$$

The points $\eta\in\Sigma_1$ satisfying the equation (3.8) will be called critical points on Σ_1 corresponding to the direction $\widehat{x}$.

Denote by $\widetilde{S}$ the surface defined by the equation $\rho=\rho(\eta)$. Clearly,

$$\rho(\eta):\Sigma_1\to\widetilde{S}.$$

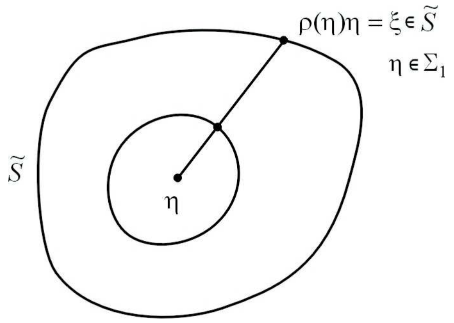


FIGURE 3.1.

Lemma 3.2. $\eta_0\in\Sigma_1$ *is a critical point corresponding to the direction* $\widehat{x}\in\Sigma_1$ *if and only if* $\eta(\xi_0)=\pm\widehat{x}$, *where* $\xi_0=\rho(\eta_0)\eta_0\in\widetilde{S}$.

Proof. Let us consider the function

$$F(\xi) = |\xi| - \rho\left(\frac{\xi}{|\xi|}\right), \quad \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\rho(\eta)$ is a positive function defined on Σ_1 as a function of η, is differentiable with respect to η and homogeneous of order -1.

It is evident that $F(\xi) = 0$ is an equation for $\widetilde{S}$, i.e., $\widetilde{S}$ is a level surface for the function F. Therefore $\nabla_\xi F(\xi)$ defines the field of outward normal directions on $\widetilde{S}$: $n(\xi) = \left.\dfrac{\nabla_\xi F(\xi)}{|\nabla_\xi F(\xi)|}\right|_{\xi\in\widetilde{S}}$ is the outward unit normal vector to $\widetilde{S}$ at the point $\xi \in \widetilde{S}$.

Elementary calculations show

$$\nabla_\xi F(\xi) = \frac{\xi}{|\xi|} - \nabla_\xi \rho\left(\frac{\xi}{|\xi|}\right) =$$
$$= \frac{\xi}{|\xi|} - \frac{1}{|\xi|}\left[\nabla_\xi \rho(\eta) - \eta(\eta\cdot\nabla_\eta\rho(\eta))\right] = \eta - \frac{1}{|\xi|}\left[\nabla_\eta\rho(\eta) + \eta\rho(\eta)\right].$$

Therefore

$$\nabla_\xi F(\xi)\big|_{\xi\in\widetilde{S}} = \eta - \frac{1}{\rho(\eta)}\left[\nabla_\eta\rho(\eta) + \eta\rho(\eta)\right] = -\frac{1}{\rho(\eta)}\nabla_\eta\rho(\eta).$$

Note that the surface $\widetilde{S} = S_q$, $q = 1, 2, 3$, are star shape with respect to the origin point 0, i.e., if $n(\xi)$ is the outward unit normal vector to $\widetilde{S}$ at $\xi \in \widetilde{S}$, then $(\eta \cdot n(\xi)) \geq 0$.

Since $(\eta \cdot n(\xi)) = -\dfrac{1}{\nabla_\eta\rho(\eta)}(\eta\cdot\nabla_\eta\rho(\eta)) = \dfrac{\rho(\eta)}{|\nabla_\eta\rho(\eta)|} > 0$, we conclude that

$$n(\xi) = -\frac{\nabla_\eta\rho(\eta)}{|\nabla_\eta\rho(\eta)|} \quad \text{for } \xi \in \widetilde{S} \tag{3.9}$$

defines the outward unit normal vector.

If $\eta_0 \in \Sigma_1$ is a critical point corresponding to $\widehat{x} \in \Sigma_1$, then using (3.8) and (3.9) we conclude that $\eta(\xi_0) = \pm\widehat{x}$, where $\xi_0 = \rho(\eta_0)\eta_0 \in \widetilde{S}$.

On the other hand, let $n(\xi_0)\,\|\,\widehat{x}$, i.e., $n(\xi_0) = \pm\widehat{x}$, or due to (3.9) $\widehat{x} = \pm\dfrac{\nabla_\eta\rho(\eta_0)}{|\nabla_\eta\rho(\eta_0)|}$.

Let us write (3.8) for η_0

$$\widehat{x}\rho(\eta_0) + (\widehat{x}\cdot\eta_0)\nabla_\eta\rho(\eta_0) =$$
$$= \pm\frac{1}{|\nabla_\eta\rho(\eta_0)|}\left\{(\nabla_\eta\rho(\eta_0))\rho(\eta_0) + (\nabla_\eta\rho(\eta_0)\cdot\eta_0)\nabla_\eta\rho(\eta_0)\right\} =$$
$$= \pm\frac{1}{|\nabla_\eta\rho(\eta_0)|}\left\{\rho(\eta_0)\nabla_\eta\rho(\eta_0) - \rho(\eta_0)\nabla_\eta\rho(\eta_0)\right\} = 0.$$

Therefore we get that $\Phi(\widehat{x}, \eta_0) = 0$, i.e., η_0 is a critical point. □

Remark 3.3. If the surface $\widetilde{S}$ does not contain a plane two-dimensional part (i.e., curvature of the surface $\widetilde{S}$ does not vanish on a subset of $\widetilde{S}$ of positive 2-dimensional measure), then the set of critical points consists of isolated points or lines on $\widetilde{S}$.

Using Lemmas 3.1 and 3.2, one can easily prove the following

Theorem 3.4. i) *If $\eta_0 \in \Sigma_1$ is not a critical point corresponding to the direction $\widehat{x} \in \Sigma_1$, then*

$$\Phi(\widehat{x}, \eta) = \eta \times [\rho(\eta)\widehat{x} + (\widehat{x} \cdot \eta)\nabla_\eta \rho(\eta)] \neq 0$$

and

$$|\Phi(\widehat{x}, \eta)| = |\rho(\eta)\widehat{x} + (\widehat{x}, \eta)\nabla_\eta \rho(\eta)| > 0.$$

ii) *If $(\widehat{x} \cdot \eta) = 0$, then $|\Phi(\widehat{x}, \eta)| = \rho(\eta) > 0$.*
iii) *$\Phi(\widehat{x}, \eta) = 0$ only at critical points.*

From ii) of Theorem 3.4 it follows

Corollary 3.5. *There exists a neighborhood $U(\delta, \partial\Sigma_x^{\pm})$ of the circumference $\partial\Sigma_x^{\pm}$ with $|\Phi(\widehat{x}, \eta)| \geq \delta > 0$ for $\eta \in U(\delta, \partial\Sigma_x^{\pm})$.*

Using the Stokes theorem for $f \in C^1(\Sigma_1)$, we can write

$$\int_{\Sigma^*} \partial_k(\eta, \nabla_\eta) f(\eta)\, d\Sigma_1 = \int_\gamma f(\eta)\ell_k(\eta)\, d\gamma, \tag{3.10}$$

where $\Sigma^* \subset \Sigma_1$, $\partial\Sigma^* = \gamma$, $n = \eta$ on Σ_1 and $\ell = (\ell_1, \ell_2, \ell_3)$ is the unit tangent vector to γ.

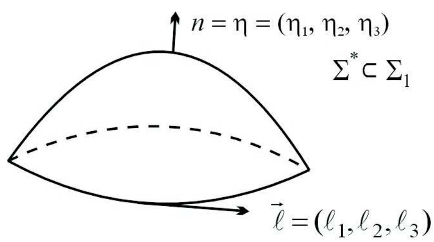


FIGURE 3.2.

As a result, from (3.10) we have

$$\int_{\Sigma^*} [\partial_k(\eta, \nabla_\eta) f(\eta)] g(\eta)\, d\Sigma_1 =$$
$$= -\int_{\Sigma^*} f(\eta)[\partial_k(\eta, \nabla_\eta) g(\eta)]\, d\Sigma_1 + \int_\gamma f(\eta) g(\eta) \ell_k(\eta)\, d\gamma. \tag{3.11}$$

If either $f|_\gamma = 0$ or $g|_\gamma = 0$, then

$$\int\limits_{\Sigma^*} [\partial_k(\eta, \nabla_\eta) f(\eta)] g(\eta)\, d\Sigma_1 = -\int\limits_{\Sigma^*} f(\eta)[\partial_k(\eta, \nabla_\eta) g(\eta)]\, d\Sigma_1.$$

Lemma 3.6. *If $\eta \in \Sigma_1$ is not a critical point corresponding to the direction $\widehat{x} \in \Sigma_1$, then*

$$e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)} = \frac{1}{i\lambda} \sum_{k=1}^{3} \frac{\Phi_k(\widehat{x},\eta)}{|\Phi_k(\widehat{x},\eta)|} \left[\partial_k(\eta, \nabla_\eta) e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}\right]. \tag{3.12}$$

Proof. Multiplying both sides of the formula (3.6) by $\Phi_k(\widehat{x},\eta)$ and summing, we obtain the equation

$$\sum_{k=1}^{3} \Phi_k(\widehat{x},\eta) \left[\partial_k(\eta, \nabla_\eta) e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)}\right] = i\lambda e^{i\lambda(\widehat{x}\cdot\eta)\rho(\eta)} |\Phi(\widehat{x},\eta)|^2 \tag{3.13}$$

(for $\rho(\eta) = \rho_q(\eta)$ we will use the notation $\Phi_k^{(q)}(\widehat{x},\eta)$ and $\Phi^{(q)}(\widehat{x},\eta)$).

If η is not a critical point, then $\Phi(\widehat{x},\eta) \neq 0$, and (3.13) can be rewritten in the form (3.12). □

In what follows, we essentially use the following

Lemma 3.7. *If $\Phi(x) = \int\limits_{\Sigma_x^+} \varphi(x,\eta) d_\eta\Sigma_1$ and $\varphi(\cdot,\eta) \in C^1(\mathbb{R}^3)$, $\Sigma_x^+ = \{\eta \in \Sigma_1 : (x\cdot\eta) \geq 0\}$, then*

$$\frac{\partial\Phi(x)}{\partial x_k} = \int\limits_{\Sigma_x^+} \frac{\partial\varphi(x,\eta)}{\partial x_k} d_\eta\Sigma_1 + \frac{1}{|x|} \int\limits_{\gamma_x} \varphi(x,\eta)\eta_k d_\eta\gamma_x, \tag{3.14}$$

where $\gamma_x = \partial\Sigma_x^+$.

Proof. First let us calculate the derivative of $\Phi(x)$ in the direction $e_0 = (e_{01}, e_{02}, e_{03})$, $|e_0| = 1$,

$$\frac{\partial\Phi(x)}{\partial e_0} = \lim_{t\to 0} \frac{\Phi(x+te_0) - \Phi(x)}{t}.$$

It is clear that

$$\Phi(x+te_0) - \Phi(x) = \int\limits_{\Sigma_{x+te_0}^+} \varphi(x+te_0,\eta) d_\eta\Sigma_1 - \int\limits_{\Sigma_x^+} \varphi(x,\eta) d_\eta\Sigma_1 =$$

$$= \int\limits_{\Sigma_x^+} [\varphi(x+te_0,\eta) - \varphi(x,\eta)] d_\eta\Sigma_1 + \int\limits_{\Sigma_{x+te_0}^+} \varphi(x+te_0,\eta) d_\eta\Sigma_1 -$$

$$- \int\limits_{\Sigma_x^+} \varphi(x+te_0,\eta) d_\eta\Sigma_1 = \int\limits_{\Sigma_x^+} [\varphi(x+te_0,\eta) - \varphi(x,\eta)] d_\eta\Sigma_1 +$$

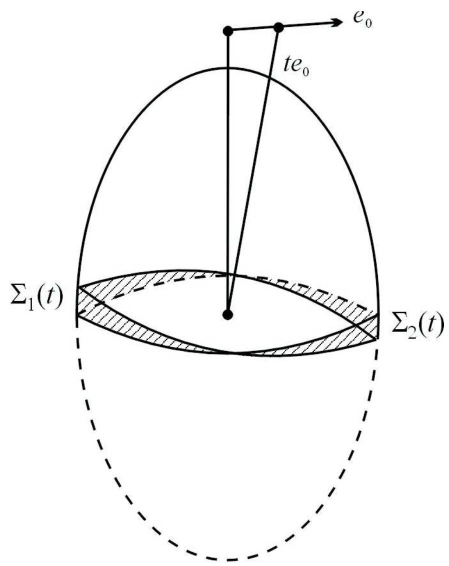


FIGURE 3.3.

$$+\int\limits_{\Sigma_2(t)}\varphi(x+te_0,\eta)d_\eta\Sigma_1-\int\limits_{\Sigma_1(t)}\varphi(x+te_0,\eta)d_\eta\Sigma_1. \quad (3.15)$$

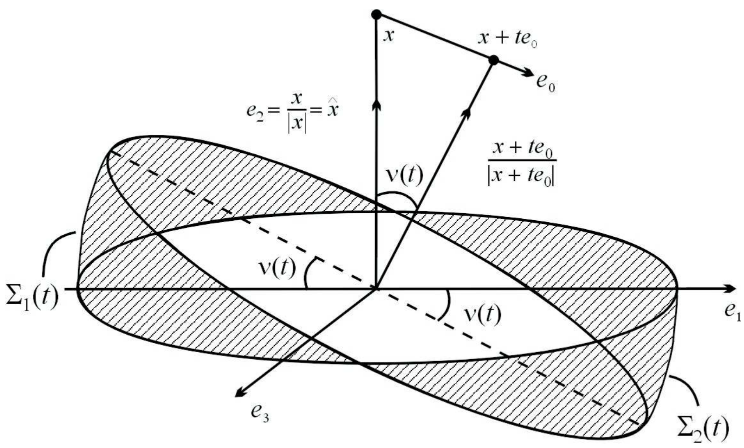


FIGURE 3.4.

$$e_3=-\frac{\widehat{x}\times e_0}{|\widehat{x}\times e_0|}=\frac{e_0\times\widehat{x}}{|\widehat{x}\times e_0|};\quad e_2=\widehat{x};\quad e_1=-e_3\times\widehat{x}=e_2\times e_3.$$

From (3.15) we get

$$\frac{1}{t}\left[\Phi(x+te_0)-\Phi(x)\right]=\int\limits_{\Sigma_x^+}\frac{\varphi(x+te_0,\eta)-\varphi(x,\eta)}{t}\,d_\eta\Sigma_1+$$
$$+\frac{1}{t}\int\limits_{\Sigma_2(t)}\varphi(x+te_0,\eta)\,d\Sigma_1-\frac{1}{t}\int\limits_{\Sigma_1(t)}\varphi(x+te_0,\eta)\,d\Sigma_1=$$
$$=\Phi_1(x,t)+\Phi_2(x,t)+\Phi_3(x,t), \quad (3.16)$$

where

$$\Phi_1(x,t)=\int\limits_{\Sigma_x^+}\frac{\varphi(x+te_0,\eta)-\varphi(x,t)}{t}\,d_\eta\Sigma_1,$$
$$\Phi_2(x,t)=\frac{1}{t}\int\limits_{\Sigma_2(t)}\varphi(x+te_0,\eta)\,d\Sigma_1,\quad \Phi_3(x,t)=-\frac{1}{t}\int\limits_{\Sigma_1(t)}\varphi(x+te_0,\eta)\,d\Sigma_1.$$

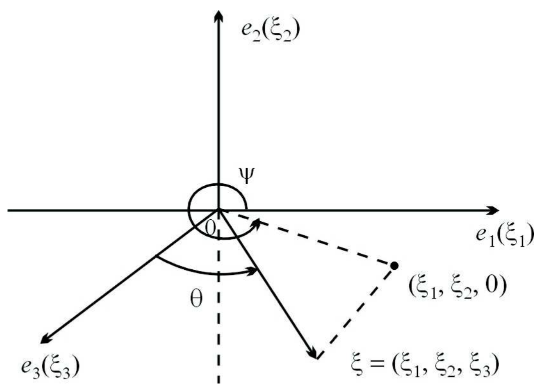
Evidently,

$$\lim_{t\to 0}\Phi_1(x,t)=\int\limits_{\Sigma_x^+}\frac{\partial\varphi(x,\eta)}{\partial e_0(x)}\,d\Sigma_1. \quad (3.17)$$

Let us make an orthogonal transform of the initial system such that $0\xi_1$ coincides with e_1, $0\xi_2$ with e_2 and $0\xi_3$ with e_3 (see Fig. 3.4). Denote by $B:=B(x,e)$ the orthogonal matrix of this transform ($B\xi=\eta$)

$$B=\begin{bmatrix} e_{11} & e_{21} & e_{31}\\ e_{12} & e_{22} & e_{32}\\ e_{13} & e_{23} & e_{33}\end{bmatrix},\quad e_k=\begin{bmatrix} e_{k1}\\ e_{k2}\\ e_{k3}\end{bmatrix},\quad k=1,2,3.$$

Using the spherical coordinates, we have



$$\xi_1=|\xi|\cos\psi\sin\theta,$$
$$\xi_2=|\xi|\sin\psi\sin\theta,$$
$$\xi_3=|\xi|\cos\theta,$$

$$0\le\theta\le\pi \quad \text{and}$$
$$0\le\psi<2\pi.$$

FIGURE 3.5

As it is seen from Fig. 3.4, $2\pi - \nu(t) < \psi < 2\pi$ for $\Sigma_2(t)$ and $\pi - \nu(t) < \psi < \pi$ for $\Sigma_1(t)$, i.e., for both surfaces $0 < \theta < \pi$. Let us estimate the angle $\nu(t)$ ($\nu(t) \geq 0$ is sufficiently small)

$$\cos\nu(t) = \widehat{x} \cdot \frac{x + te_0}{|x + te_0|} = \frac{x \cdot (x + te_0)}{|x|\,|x + te_0|} =$$
$$= \frac{|x|^2 + t(e_0 \cdot x)}{|x|\,|x + te_0|} = 1 + \frac{|x|^2 + t(e_0 \cdot x) - |x|\,|x + e_0 t|}{|x|\,|x + e_0 t|} =$$
$$= 1 + \frac{|x|^4 + 2t|x|^2(e_0 \cdot x)^2 + t^2(e_0 \cdot x)^2 - |x|^2[|x|^2 + 2t(e_0 \cdot x) + t^2]}{|x|\,|x + e_0 t|[|x|^2 + t(e_0 \cdot x) + |x|\,|x + e_0 t|]} =$$
$$= 1 - \frac{t^2[|x|^2 - (e_0 \cdot x)^2]}{|x|\,|x + e_0 t|[|x|^2 + t(e_0 \cdot x) + |x|\,|x + e_0 t|]}\,,$$

i.e.,

$$2\sin^2\frac{\nu(t)}{2} = t^2\,\frac{|x|^2 - (e_0 \cdot x)^2}{|x|\,|x + e_0 t|[|x|^2 + t(e_0 \cdot x) + |x|\,|x + e_0 t|]} =$$
$$= t^2\left\{\frac{|x|^2 - (e_0 \cdot x)^2}{2|x|^4}\right\} + O(t^3).$$

Hence

$$\lim_{t \to 0}\frac{\nu(t)}{t} = \sqrt{\frac{|x|^2 - (e_0 \cdot x)^2}{|x|^4}}\,. \tag{3.18}$$

If $B\Sigma_2(t) = \widetilde{\Sigma}_2(t)$, then

$$\Phi_2(x,t) = \frac{1}{t}\int\limits_{\Sigma_2(t)} \varphi(x + te_0, \eta)\,d\Sigma_1 = \frac{1}{t}\int\limits_{\widetilde{\Sigma}_2(t)} \varphi(x + te_0, B\xi)\,d\Sigma_1 =$$
$$= \frac{1}{t}\int\limits_{2\pi - \nu(t)}^{2\pi} d\psi \int\limits_0^{\pi} \varphi(x + te_0, B\xi)\sin\theta\,d\theta. \tag{3.19}$$

Using the mean value theorem in (3.19), we obtain

$$\Phi_2(x,t) = \frac{1}{t}\,\nu(t)\int\limits_0^{\pi} \varphi(x + te_0, B\xi')\sin\theta\,d\theta, \tag{3.20}$$

where

$$\begin{aligned} \xi' = (\xi_1', \xi_2', \xi_3') \ : \ & \xi_1' = \cos\psi'\sin\theta, \\ & \xi_2' = \sin\psi'\sin\theta, \\ & \xi_3' = \cos\theta \quad \text{and} \\ & 2\pi - \nu(t) \leq \psi' < 2\pi. \end{aligned} \tag{3.21}$$

Similarly, if $B\Sigma_1(t) = \widetilde{\Sigma}_1(t)$, then

$$\Phi_1(x,t) = -\frac{1}{t}\int\limits_{\Sigma_1(t)} \varphi(x+te_0,\eta)\,d\Sigma_1 = -\frac{1}{t}\int\limits_{\widetilde{\Sigma}_1(t)} \varphi(x+te_0,B\xi)\,d\Sigma_1 =$$

$$= -\frac{1}{t}\int\limits_{\pi-\nu(t)}^{\pi} d\psi \int\limits_0^{\pi} \varphi(x+te_0,B\xi)\sin\theta\,d\theta =$$

$$= -\frac{\nu(t)}{t}\int\limits_0^{\pi} \varphi(x+te_0,B\xi'')\sin\theta\,d\theta, \tag{3.22}$$

where

$$\xi'' = (\xi_1'',\xi_2'',\xi_3'') \ : \ \begin{aligned} &\xi_1'' = \cos\psi''\sin\theta,\\ &\xi_2'' = \sin\psi''\sin\theta,\\ &\xi_3'' = \cos\theta,\\ &2\pi-\nu(t) \le \psi'' < 2\pi. \end{aligned} \tag{3.23}$$

Due to (3.18)–(3.23) we find

$$\lim_{t\to 0}\Phi_2(x,t) = \frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int\limits_0^{\pi}\varphi(x,B\xi_0')\sin\theta\,d\theta, \tag{3.24}$$

where $\xi_0' = \begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix}$, and

$$\lim_{t\to 0}\Phi_3(x,t) = -\frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int\limits_0^{\pi}\varphi(x,B\xi_0'')\sin\theta\,d\theta, \tag{3.25}$$

where $\xi_0'' = \begin{bmatrix}-\sin\theta\\ 0\\ \cos\theta\end{bmatrix}$.

The substitution $\theta = \pi - \widetilde{\theta}$ in (3.25) leads to

$$\int\limits_0^{\pi}\varphi(x,B\xi_0'')\sin\theta\,d\theta = -\int\limits_{\pi}^{0}\varphi(x,B(-\xi_0'))\sin\theta\,d\theta = \int\limits_0^{\pi}\varphi(x,B\xi_0')\sin\theta\,d\theta.$$

If $\theta = \widetilde{\theta}-\pi$, then $\sin\theta = \sin(\widetilde{\theta}-\pi) = -\sin\widetilde{\theta}$, $\cos\theta = \cos(\widetilde{\theta}-\pi) = -\cos\widetilde{\theta}$, $0 \le \theta \le \pi$, $\pi \le \widetilde{\theta} \le 2\pi$,

$$-B\xi_0' = -B\begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix} = B\begin{bmatrix}\sin\overline{\theta}\\ 0\\ \cos\overline{\theta}\end{bmatrix} = B\overline{\xi}_0'$$

and

$$\int_0^\pi \varphi(x,-B\xi_0')\sin\theta\,d\theta = \int_\pi^{2\pi} \varphi(x,-B\overline{\xi}_0')\sin\overline{\theta}\,d\overline{\theta} = \int_\pi^{2\pi} \varphi(x,B\xi_0')\sin\theta\,d\theta.$$

Hence from (3.24) and (3.25)

$$\lim_{t\to 0}\Phi_2(x,t) = \frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int_0^\pi \varphi\left(x, B\begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix}\right)\sin\theta\,d\theta, \tag{3.26}$$

$$\lim_{t\to 0}\Phi_3(x,t) = -\frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int_0^\pi \varphi\left(x, -B\begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix}\right)\sin\theta\,d\theta =$$

$$= \frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int_\pi^{2\pi} \varphi\left(x, B\begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix}\right)\sin\theta\,d\theta. \tag{3.27}$$

Note that

$$B\begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix} = e_1\sin\theta + e_3\cos\theta = \begin{pmatrix}e_{11}\\ e_{12}\\ e_{13}\end{pmatrix}\sin\theta + \begin{pmatrix}e_{31}\\ e_{32}\\ e_{33}\end{pmatrix}\cos\theta = \zeta,$$

i.e., $\zeta = e_1\sin\theta + e_3\cos\theta$.

Clearly, $\zeta\in\gamma_x = \partial\Sigma_x^\pm$, and when θ varies from 0 to 2π, then ζ moves on γ_x in positive direction. Moreover,

$$\sin\theta = (e_1\cdot\zeta). \tag{3.28}$$

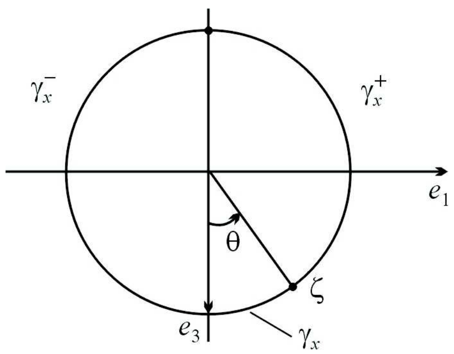


FIGURE 3.6.

Taking into account (3.26)–(3.28), we get

$$\lim_{t\to 0}\Phi_2(x,t) = \frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int_0^\pi \varphi(x,\zeta)(e_1\cdot\zeta)\,d\theta, \tag{3.29}$$

$$\lim_{t\to 0}\Phi_3(x,t)=\frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int\limits_{\pi}^{2\pi}\varphi(x,\zeta)(e_1\cdot\zeta)\,d\theta. \tag{3.30}$$

$d\theta = d\gamma_x$ on $\gamma_x=\partial\Sigma_x^{\pm}$ (see Fig. 3.6), and hence

$$\int\limits_{0}^{\pi}\varphi(x,\zeta)(e_1\cdot\zeta)\,d\theta=\int\limits_{\gamma_x^+}\varphi(x,\zeta)(e_1\cdot\zeta)\,d_\zeta\gamma_x \tag{3.31}$$

$$\int\limits_{\pi}^{2\pi}\varphi(x,\zeta)(e_1\cdot\zeta)\,d\theta=\int\limits_{\gamma_x^-}\varphi(x,\zeta)(e_1\cdot\zeta)\,d_\zeta\gamma_x. \tag{3.32}$$

Using (3.16), (3.17) and (3.29)–(3.32), we get

$$\frac{\partial\Phi}{\partial e_0}=\lim_{t\to 0}\frac{1}{t}[\Phi(x+te_0)-\Phi(x)]=\int\limits_{\Sigma_x^+}\frac{\partial\varphi(x,\eta)}{\partial e_0}\,d_\eta\Sigma_1+$$
$$+\frac{\sqrt{|x|^2-(x\cdot e_0)^2}}{|x|^2}\int\limits_{\gamma_x^+}\varphi(x,\zeta)(e_1\cdot\zeta)\,d_\zeta\gamma_x. \tag{3.33}$$

Note that the vector $e_0=(\delta_{1k},\delta_{2k},\delta_{3k})$ corresponds to $\frac{\partial}{\partial x_k}$:

I) If $e_0=(1,0,0)\sim\frac{\partial}{\partial x_1}$, then

$$[e_0\times\widehat{x}]=\begin{vmatrix} i & j & k\\ 1 & 0 & 0\\ \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3\end{vmatrix}=(0,-\widehat{x}_3,\widehat{x}_2)^\top,\quad r_1=\sqrt{\widehat{x}_2^2+\widehat{x}_3^2},$$

$$e_2^{(1)}=(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3),\quad e_3^{(1)}=\frac{e_0\times\widehat{x}}{|e_0\times\widehat{x}|}=e_3^{(1)}=\frac{1}{r_1}(0,-\widehat{x}_3,\widehat{x}_2),$$

$$e_1^{(1)}=[e_2^{(i)}\times e_3^{(1)}]=\begin{vmatrix} i & j & k\\ \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3\\ 0 & -\frac{\widehat{x}_3}{r_1} & -\frac{\widehat{x}_2}{r_1}\end{vmatrix}=\left(r_1,-\frac{\widehat{x}_1\widehat{x}_2}{r_1},-\frac{\widehat{x}_1\widehat{x}_3}{r_1}\right)=$$
$$=\frac{1}{r_1}\,(\widehat{x}_2^2+\widehat{x}_3^2+x_1^2-\widehat{x}_1^2;-\widehat{x}_1\widehat{x}_2,-\widehat{x}_1\widehat{x}_3)^\top=\frac{1}{r_1}\{(1,0,0)^\top-\widehat{x}_1\widehat{x}\};$$

II) If $e_0=(0,1,0)\sim\frac{\partial}{\partial x_2}$, then

$$[e_0\times\widehat{x}]=\begin{vmatrix} i & j & k\\ 0 & 1 & 0\\ \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3\end{vmatrix}=(\widehat{x}_3,0,-\widehat{x}_2)^\top,\quad r_2=\sqrt{\widehat{x}_1^2+\widehat{x}_3^2},$$

$$e_3^{(2)}=\frac{1}{r_2}(\widehat{x}_3,0,-\widehat{x}_2)^\top,\ \ e_2^{(2)}=\widehat{x}\ \text{ and }\ e_1^{(2)}=\frac{1}{r_2}\{(0,1,0)^\top-\widehat{x}_2\widehat{x}\};$$

III) If $e_0 = (0,0,1) \sim \frac{\partial}{\partial x_3}$, then

$$[e_0 \times \widehat{x}] = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3 \end{vmatrix} = (-\widehat{x}_2, \widehat{x}_1, 0)^\top, \quad r_3 = \sqrt{\widehat{x}_1^2 + \widehat{x}_2^2},$$

$$e_3^{(3)} = \frac{1}{r_3}(-\widehat{x}_2, \widehat{x}_1, 0)^\top, \;\; e_2^{(3)} = \widehat{x} \;\text{ and }\; e_1^{(3)} = \frac{1}{r_3}\{(0,0,1)^\top - \widehat{x}_3\widehat{x}\}.$$

The parametric equation of $\gamma_x = \partial\Sigma_x^\pm$ is

$$\zeta = e_1^{(k)} \sin\theta + e_3^{(k)} \cos\theta, \quad k = 1,2,3.$$

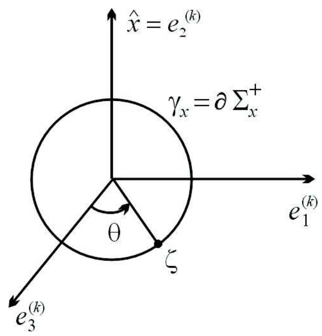


FIGURE 3.7.

Here the coordinates of $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ correspond to the initial system.

Applying (3.33), we have

$$\frac{\partial\Phi}{\partial x_k} = \int\limits_{\Sigma_x^+} \frac{\partial\varphi(x,\eta)}{\partial x_k}\, d_\eta\Sigma_1 + \frac{\sqrt{|x|^2 - x_k^2}}{|x|^2} \int\limits_{\gamma_x} \varphi(x,\zeta)(e_1^{(k)} \cdot \zeta) d_\zeta\gamma_x.$$

Clearly,

$$\frac{\sqrt{|x|^2 - x_k^2}}{|x|^2} = \frac{\sqrt{1 - \widehat{x}^2}}{|x|} = \frac{r_k}{|x|}, \;\; \zeta \cdot \widehat{x} = 0, \;\text{ and }\; r_k(e_1^{(k)} \cdot \zeta) = \zeta_k,$$

so

$$\frac{\partial\Phi}{\partial x_k} = \int\limits_{\Sigma_x^+} \frac{\partial\varphi(x,\eta)}{\partial x_k}\, d_\eta\Sigma_1 + \frac{1}{|x|} \int\limits_{\gamma_x} \varphi(x,\zeta)\zeta_k d_\zeta\gamma_x. \tag{3.34}$$

We can write η and η_k instead of ζ and ζ_k in (3.34) to get (3.14). □

Now we can prove the following

Theorem 3.8. *The fundamental solution* $\Psi(x,\omega,1)$ *of the equation* (2.1) *is represented as*

$$\Psi(x,\omega,1) = \Psi^{(1)}(x) + \Psi^{(0)}(x), \tag{3.35}$$

where

$$\Psi^{(1)}(x) = \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) N(i\eta\rho_q, \omega) e^{i(x\cdot\eta)\rho_q} d\Sigma_1, \tag{3.36}$$

$$\Psi^{(0)}(x) = -\frac{1}{8\pi^2|x|} \int\limits_{\gamma_x} C^{-1}(\eta)\, d\gamma. \tag{3.37}$$

Here $C^{-1}(\eta)$*is the inverse matrix of* $C(\eta)$ *(see* (2.1)*),* $d\gamma = d_\eta\gamma_x$*,* $F_q(\eta)$ *is defined by* (2.21)*.*

Moreover, if $|x| \to 0$*, then*

$$\frac{\partial}{\partial x_k}[\Psi^{(1)}(x)] = O(1); \quad \frac{\partial^2}{\partial x_k \partial x_j}[\Psi^{(1)}] = O\left(\frac{1}{|x|}\right) \tag{3.38}$$

and

$$\lim_{t\to 0} \Psi^{(1)}(x) = \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) N(i\eta\rho_q, \omega)\, d\Sigma_1. \tag{3.39}$$

Proof. Note that $a(\eta) > 0$ and $\rho_q(\eta) > 0$, $\eta \in \Sigma_q$, are even functions. Therefore

$$F_q(-\eta) = F_q(\eta).$$

It is easy to check that

$$\sum_{q=1}^{3} F_q(\eta)\rho_q^{\pm 1}(\eta) = 0. \tag{3.40}$$

Due to Lemma 3.7, we have

$$\frac{\partial}{\partial x_k}\Gamma(x,\omega,1) = \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) e^{i(x\cdot\eta)\rho_q(\eta)} i\eta_k\rho_q(\eta)\, d\Sigma_1 + \\ + \frac{1}{|x|}\int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta) e^{i(x\cdot\eta)\rho_q(\eta)}\eta_k\, d\gamma. \tag{3.41}$$

We know that $F_q(\eta)$ is an even function and $(x\cdot\eta) = 0$ on γ_x. Therefore

$$\int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta) e^{i(x\cdot\eta)\rho_q(\eta)}\eta_k\, d\gamma = \int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta)\eta_k\, d\gamma = \\ = \int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(-\eta)(-\eta_k)\, d\gamma = -\int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta)\eta_k\, d\gamma = 0.$$

Now we can rewrite (3.41) as

$$\frac{\partial}{\partial x_k}\,\Gamma(x,\omega,1)=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i\eta_k\rho_q(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1. \tag{3.42}$$

With the help of (3.14) and (3.40) we have

$$\begin{aligned}\frac{\partial^2}{\partial x_k\partial x_j}\,\Gamma(x,\omega,1)&=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)(i\eta_k)(i\eta_j)\rho_q^2(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1+\\&\quad+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)i\eta_k\rho_q(\eta)\eta_j\,d\gamma=\\&=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^2\eta_k\eta_j\rho_q^2(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1.\end{aligned} \tag{3.43}$$

Similarly,

$$\begin{aligned}\frac{\partial^3}{\partial x_k\partial x_j\partial x_m}\,\Gamma(x,\omega,1)&=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^3\eta_k\eta_j\eta_m\rho_q^3(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1+\\&\quad+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)i^2\eta_k\eta_j\rho_q^2(\eta)\eta_m\,d\gamma.\end{aligned} \tag{3.44}$$

The curvilinear integral in (3.44) vanishes since the integrand is an odd function, i.e.

$$\frac{\partial^3}{\partial x_k\partial x_j\partial x_m}\,\Gamma(x,\omega,1)=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^3\eta_k\eta_j\eta_m\rho_q^3(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1. \tag{3.45}$$

Another use of (3.14) gives

$$\begin{aligned}\frac{\partial^4}{\partial x_k\partial x_j\partial x_m\partial x_p}\,\Gamma(x,\omega,1)&=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)i^4\eta_k\eta_j\eta_m\eta_p\rho_q^4(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1+\\&\quad+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)i^3\eta_k\eta_j\eta_m\rho_q^3(\eta)\eta_p\,d\gamma=\\&=i^4\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\eta_p e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1+\\&\quad+\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\rho_q^3(\eta)\,d\gamma,\end{aligned} \tag{3.46}$$

where the curvilinear integral does not vanish, it is a homogeneous function of order -1. Clearly, the first integral in (3.46) is bounded in a vicinity of the origin.

Using (3.42)–(3.46), we can write

$$\Psi(x,\omega,1) = N(\partial_x,\omega)\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 =$$

$$= \int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 +$$

$$+ \frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^3(\eta)N^0(\eta)\,d\gamma, \tag{3.47}$$

where $N^0(\eta)$ is the principle part of the matrix $N(\eta,\omega)$.

Let us calculate

$$\sum_{q=1}^{3}F_q(\eta)\rho_q^3(\eta) = -\frac{i}{8\pi^2a(\eta)}\bigg\{\frac{\rho_1^4}{(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)} +$$

$$+ \frac{\rho_2^4}{(\rho_2^2-\rho_1^2)(\rho_2^2-\rho_3^2)} + \frac{\rho_3^4}{(\rho_3^2-\rho_1^2)(\rho_3^2-\rho_2^2)}\bigg\} =$$

$$= -\frac{i}{8\pi^2a(\eta)}\bigg\{\frac{\rho_1^4(\rho_2^2-\rho_3^2)-\rho_2^4(\rho_1^2-\rho_3^2)-\rho_3^4(\rho_1^2-\rho_2^2)}{(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)(\rho_2^2-\rho_3^2)}\bigg\} = -\frac{i}{8\pi^2a(\eta)}\,.$$

Now we can rewrite (3.47) as

$$\Psi(x,\omega,1) = \int\limits_{\Sigma_x^+}F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 - \frac{1}{8\pi^2|x|}\int\limits_{\gamma_x}C^{-1}(\eta)\,d\gamma,$$

where $C^{-1}(\eta) = \dfrac{1}{a(\eta)}\,N^0(\eta)$ is the matrix inverse to $C(\eta)$.

Using the notation (3.36) and (3.37), we arrive to (3.35). Note that $\Psi^0(x)$ is the fundamental solution of the static equation ($\omega = 0$)

$$C(\partial)\Psi^{(0)}(x) = \delta(x)I_3.$$

We know that

$$N(\partial_x,\omega) = [N_{kj}(\partial_x,\omega)]_{3\times3} \quad \text{and}$$
$$N_{kj}(i\eta\rho_q,\omega) = N_{kj}^0(\eta)i^4\rho_q^4 - N_{kj}^1(\eta)\rho_q^2\omega^2 + \omega^4\delta_{kj},$$

where $N_{kj}^0(\eta)$ is a 4th order polynomial with respect to η, $N_{kl}^1(\eta)$ is a second order polynomial,

$$\Psi^{(1)}(x) = \int\limits_{\Sigma_x^+}\sum_{q=1}^{3}N^0(\eta)F_q(\eta)\rho_q^4(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 -$$

$$-\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}N^1(\eta)F_q(\eta)\rho_q^2(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}I_3F_q(\eta)\omega^4e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1=$$

$$=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}\frac{\rho_q^5(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]}N^0(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$-\int\limits_{\Sigma_x^+}\omega^2\sum_{q=1}^{3}\frac{\rho_q^3(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]}N^1(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\int\limits_{\Sigma_x^+}\omega^4\sum_{q=1}^{3}\frac{\rho_q(\eta)}{\prod\limits_{j=1\,j\neq q}^{3}[\rho_q^2(\eta)-\rho_j^2(\eta)]}I_3e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1$$

$$=O(\omega)\to 0\quad\text{as}\quad\omega\to 0,$$

i.e., for any $x\in\mathbb{R}^3\setminus\{0\}$

$$\lim_{\omega\to 0}\Psi(x,\omega,1)=\Psi^{(0)}(x)$$

uniformly for all $|x|>\delta>0$.

Clearly, $\Psi^0(x)=O(1)$ as $|x|\to 0$,

Using (3.14) and the fact that $F_q(-\eta)=F_q(\eta)$, we obtain

$$\frac{\partial\Psi^{(1)}(x)}{\partial x_k}=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[i\eta_k\rho_q(\eta)]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)\eta_k\,d\gamma=$$

$$=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[i\eta_k\rho_q(\eta)]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1=O(1)$$

and

$$\frac{\partial^2\Psi^{(1)}(x)}{\partial x_k\partial x_j}=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[i^2\eta_k\eta_j]\rho_q^2(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)i\eta_k\rho_q(\eta)\eta_j\,d\gamma=O\left(\frac{1}{|x|}\right).$$

Taking into account that $F_q(\eta)N(i\eta\rho_q,\omega)$ is an even function, we derive (3.39). □

3.2. **Asymptotics at infinity and the radiation conditions.** Using Lemma 3.7, we can prove

Theorem 3.9. *For* $|x| \to +\infty$

$$\frac{\partial^5}{\partial x_k \partial x_j \partial x_m \partial x_p \partial x_n}\Gamma(x,\omega,1) = \\ = i^5 \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta)\eta_k\eta_j\eta_m\eta_p\eta_n\rho_q^5(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 + O(|x|^{-2}), \tag{3.48}$$

$$\frac{\partial^4}{\partial x_k \partial x_j \partial x_m \partial x_p}\Gamma(x,\omega,1) = \\ = i^4 \int\limits_{\Sigma_x^+} \psi_1(\eta) \sum_{q=1}^{3} F_q(\eta)\eta_k\eta_j\eta_m\eta_p e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 + O(|x|^{-2}), \tag{3.49}$$

where $\psi_1 \in C^\infty(\Sigma_1)$, $\psi_1(\eta) = 0$ *for* $\eta \in \gamma_x$.

Proof. Due to (3.14) and (3.46), we get

$$\frac{\partial^5}{\partial x_k \partial x_j \partial x_m \partial x_p \partial x_n}\Gamma(x,\omega,1) = \\ = \int\limits_{\Sigma_x^+} \sum_{q=1}^{3} F_q(\eta) i^5 \eta_k\eta_j\eta_m\eta_p\eta_n\rho_q^5(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1 + \\ + i^4 \int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta)\rho_q^4(\eta)\eta_k\eta_j\eta_m\eta_p\eta_n d\gamma + \\ + \frac{\partial}{\partial x_n}\left[\frac{i^3}{|x|}\int\limits_{\gamma_x} \sum_{q=1}^{3} F_q(\eta)\eta_k\eta_j\eta_m\eta_p\rho_q^3(\eta)d\gamma\right]. \tag{3.50}$$

The second integral in(3.50) vanishes since the integrand is an odd function. The third integral in(3.50) is $O(|x|^{-2})$ as $|x| \to \infty$ (or $|x| \to 0$), more precisely, it is a homogeneous function of order -2. Hence we can rewrite (3.50) as (3.48).

Let us consider the function

$$\Phi(x) = \int\limits_{\Sigma_x^+} \varphi(\eta)e^{i(x\cdot\eta)\rho(\eta)}d\Sigma_1, \quad \varphi \in C^1(\Sigma_1). \tag{3.51}$$

Due to Theorem 3.4 and Corollary 3.3, there exists $\varepsilon > 0$ such that there is no critical point in $\Sigma_x^+(\varepsilon)$ (see Fig. 3.8).

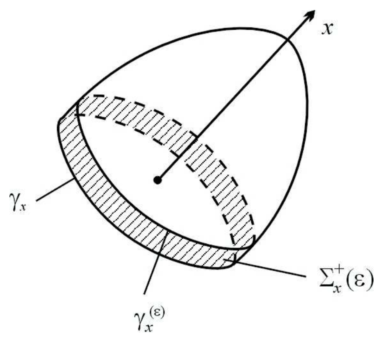


FIGURE 3.8.

Let us rewrite (3.51) as

$$\Phi(x) = \int\limits_{\Sigma_x^+} (\psi_0(\eta) + \psi_1(\eta))\varphi(\eta) e^{i(x\cdot\eta)\rho(\eta)} d\Sigma_1 = \Phi^*(x) + \Phi^{**}(x),$$

where

$$\Phi^*(x) = \int\limits_{\Sigma_x^+} \psi_0(\eta)\varphi(\eta) e^{i(x\cdot\eta)\rho(\eta)} d\Sigma_1,$$

$$\Phi^{**}(x) = \int\limits_{\Sigma_x^+} \psi_1(\eta)\varphi(\eta) e^{i(x\cdot\eta)\rho(\eta)} d\Sigma_1;$$

here $\psi_0(\eta) + \psi_1(\eta) = 0$, $\eta \in \Sigma_1$, $\psi_0, \psi_1 \in C^\infty(\Sigma_1)$, $\psi_0 \geq 0$, $\psi_0(\eta) = 0$ in vicinity of γ_x, $\operatorname{supp} \psi_0 \subset \Sigma_x^+(\varepsilon)$, $\psi_1(\eta) = 1 - \psi_0(\eta)$ vanishes on γ_x and in $\Sigma_x^+(\varepsilon)$.

Applying (3.11) and (3.12), we have

$$\Phi^*(x) = \int\limits_{\Sigma_x^+} \psi_0(\eta)\varphi(\eta) e^{i|x|(\widehat{x}\cdot\eta)\rho(\eta)} d\Sigma_1 =$$

$$= \int\limits_{\Sigma_x^+} \psi_0(\eta)\varphi(\eta) \bigg[\frac{1}{i|x|} \sum_{k=1}^3 \frac{\Phi_k(\widehat{x},\eta)}{|\Phi(\widehat{x},\eta)|^2}\, \partial_k(\eta,\nabla_\eta) e^{i|x|(\widehat{x}\cdot\eta)\rho(\eta)} \bigg] d\Sigma_1 =$$

$$= \frac{1}{i|x|} \Bigg\{ -\int\limits_{\Sigma_x^+} \bigg(\sum_{k=1}^3 \partial_k(\eta,\nabla_\eta) \bigg[\psi_0(\eta)\varphi(\eta)\, \frac{\Phi_k(\widehat{x},\eta)}{|\Phi(\widehat{x},\eta)|^2} \bigg] \bigg) e^{i|x|(\widehat{x}\cdot\eta)\rho(\eta)} d\Sigma_1 +$$

$$+ \int\limits_{\gamma_x} \psi_0(\eta)\varphi(\eta) \sum_{k=1}^3 \frac{\Phi_k(\widehat{x},\eta)}{|\Phi(\widehat{x},\eta)|^2}\, \ell_k(\eta) e^{i|x|(\widehat{x}\cdot\eta)\rho(\eta)} d\gamma \Bigg\}. \quad (3.52)$$

Applying the same procedure in the first integral of (3.52), we see that it is $O(|x|^{-2})$.

On the other hand, $(\widehat{x}\cdot\eta)=0$, $\eta\in\gamma_x$,

$$\Phi(\widehat{x},\eta)=[\eta\times\widehat{x}]\rho(\eta)=-\ell(\eta)\rho(\eta),\quad \Phi_k(\widehat{x},\eta)=-\rho(\eta)\ell_k(\eta)$$

(here $\ell(\eta)$ is the tangent vector to γ_x), so

$$\begin{aligned}\Phi^*(x)&=-\frac{1}{i|x|}\int\limits_{\gamma_x}\varphi(\eta)\sum_{k=1}^{3}\frac{\rho(\eta)\ell_k(\eta)}{\rho^2(\eta)}\,\ell_k\,d\gamma+O(|x|^{-2})=\\&=-\frac{1}{i|x|}\int\limits_{\gamma_x}\varphi(\eta)\frac{1}{\rho(\eta)}\,d\gamma+O(|x|^{-2}).\end{aligned}$$

For $x\gg 1$

$$\begin{aligned}\Phi(x)&=\int\limits_{\Sigma_x^+}\varphi(\eta)e^{i(\widehat{x}\cdot\eta)\rho(\eta)}d\Sigma_1=\\&=-\frac{1}{i|x|}\int\limits_{\gamma_x}\varphi(\eta)\frac{1}{\rho(\eta)}\,d\gamma+\Phi^{**}(x)+O(|x|^{-2}).\end{aligned}\tag{3.53}$$

Using (3.53) in (3.46), we can write

$$\begin{aligned}\frac{\partial^4}{\partial x_k\partial x_j\partial x_m\partial x_p}\Gamma(x,\omega,1)&=-\frac{i^4}{i|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^4(\eta)\eta_k\eta_j\eta_m\eta_p\,\frac{1}{\rho_q(\eta)}\,d\gamma+\\&+O(|x|^{-2})+\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^3(\eta)\eta_k\eta_j\eta_m\eta_p\,d\gamma+\\&+i^4\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)\eta_k\eta_j\eta_m\eta_p\rho_q^4(\eta)e^{i(x\cdot\eta)\rho_q(\eta)}\,d\Sigma_1.\end{aligned}$$

From this relation we obtain (3.49). □

Theorem 3.10. *For* $|x|\gg 1$

$$\Psi(x,\omega,1)=\sum_{q=1}^{3}\overset{(q)}{\Psi}(x,\omega,1),\quad \overset{(q)}{\Psi}(x,\omega,1)=O(|x|^{-1}),\tag{3.54}$$

$$\frac{\partial\overset{(q)}{\Psi}(x,\omega,1)}{\partial x_k}-i\xi_k^{(q)}\overset{(q)}{\Psi}(x,\omega,1)=O(|x|^{-2}),\tag{3.55}$$

where $\xi_k^{(q)}\in S_q$ *and* $\eta(\xi_k^{(q)})=\frac{x}{|x|}$.

These conditions are called the generalized Sommerfeld–Kupradze type radiation conditions.

Proof. Taking into account the form of N_{kj}, we can write

$$\Psi(x,\omega,1)=\int\limits_{\Sigma_x^+}(\psi_0(\eta)+\psi_1(\eta))\sum_{q=1}^{3}F_q(\eta)[N^0(\eta)i^4\rho_q^4(\eta)-$$

$$-N^1(\eta)\rho_q^2(\eta)\omega^2+\omega^2 I]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^3(\eta)N^0\,d\gamma=$$

$$=\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1-$$

$$-\frac{1}{i|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}\bigg[F_q(\eta)i^4\rho_q^4(\eta)N^0\,\frac{1}{\rho_1(\eta)}-$$

$$-F_q(\eta)N^1(\eta)\rho_q^2(\eta)\omega^2\,\frac{1}{\rho_q(\eta)}+F_q(\eta)\omega^2\,\frac{1}{\rho_q(\eta)}\bigg]d\gamma+$$

$$+\frac{i^3}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)\rho_q^3(\eta)N^0 d\gamma+O(|x|^{-2}). \tag{3.56}$$

Here ψ_0 and ψ_1 are the same as in the previous theorem.

Due to (3.40)

$$\Psi(x,\omega,1)=\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+O(|x|^{-2}),\quad |x|\gg 1. \tag{3.57}$$

Let us calculate $\dfrac{\partial\Psi(x,\omega,1)}{\partial x_k}$ with the help of (3.46) and (3.35)–(3.37)

$$\frac{\partial\Psi(x,\omega,1)}{\partial x_k}=\int\limits_{\Sigma_x^+}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[\eta_k\rho_q(\eta)]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+$$

$$+\frac{1}{|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)\eta_k\,d\gamma+\frac{\partial}{\partial x_k}\,\Psi^{(0)}(x). \tag{3.58}$$

The last term in (3.58) is $O(|x|^{-2})$. If we apply (3.53) in (3.58), we have

$$\frac{\partial\Psi(x,\omega,1)}{\partial x_k}=-\frac{1}{i|x|}\int\limits_{\gamma_x}\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[\eta_k\rho_q(\eta)]\,\frac{1}{\rho_q(\eta)}\,d\gamma+$$

$$+\int\limits_{\Sigma_x^+}\psi_1(\eta)\sum_{q=1}^{3}F_q(\eta)N(i\eta\rho_q,\omega)[\eta_k\rho_q(\eta)]e^{i(x\cdot\eta)\rho_q(\eta)}d\Sigma_1+O(|x|^{-2})=$$

$$= \int\limits_{\Sigma_x^+} \psi_1(\eta) \sum_{q=1}^{3} F_q(\eta) N(i\eta\rho_q, \omega)[\eta_k \rho_q(\eta)] e^{i(x\cdot\eta)\rho_q(\eta)} d\Sigma_1 + O(|x|^{-2}). \quad (3.59)$$

Note that

$$\rho_q^2 d\Sigma_1 = \cos\gamma \, dS_q = (\eta \cdot \eta(\xi)) \, dS_q = \left(\eta \cdot \frac{-\nabla_\eta \rho_q(\eta)}{|\nabla_\eta \rho_q(\eta)|}\right) dS_q = \frac{\rho_q(\eta)}{|\nabla_\eta \rho_q(\eta)|} \, dS_q,$$

so we can rewrite (3.57) as follows

$$\Psi(x,\omega,1) = \sum_{q=1}^{3} \int\limits_{S_q^+(x)} \psi_1(\eta) F_q(\eta) N(i\xi,\omega) e^{i(x\cdot\xi)} \frac{1}{\rho_q(\eta)|\nabla_\eta \rho_q(\eta)|} \, dS_q + $$

$$+ O(|x|^{-2}) =$$

$$= \sum_{q=1}^{3} \Bigg\{ -\frac{1}{8\pi^2} \int\limits_{S_q^+(x)} \psi_1(\eta) \frac{N(i\xi,\omega)}{a(\eta) \prod\limits_{j=1,\, j\neq q}^{3} [\rho_q^2(\eta) - \rho_j^2(\eta)]} \cdot \frac{e^{i(x\cdot\xi)}}{|\nabla_\eta \rho_q(\eta)|} \, dS_q \Bigg\} +$$

$$+ O(|x|^{-2});$$

here $\eta = \frac{\xi}{|\xi|} = \frac{\xi}{\rho_q(\eta)}$.

Now we can apply the results obtained in [8] and [9] to get

$$\Psi(x,\omega,1) = \sum_{q=1}^{3} \Bigg\{ -\frac{i}{8\pi^2} \frac{N(i\xi^{(q)},\omega) e^{-i\frac{\pi}{4}} \cdot 2\pi}{a(\eta^{(q)}) \prod\limits_{j=1,\, j\neq q}^{3} [\rho_q^2(\eta^{(q)}) - \rho_j^2(\eta^{(q)})]} \times$$

$$\times \frac{e^{i(x\cdot\xi^{(q)})}}{|\nabla_\eta \rho_q(\eta^{(q)})| \sqrt{K_q(\xi^{(q)})}} \Bigg\} + O(|x|^{-2}),$$

i.e.

$$\Psi(x,\omega,1) = \sum_{q=1}^{3} \Bigg\{ -\frac{1}{4\pi} \frac{N(i\xi^{(q)},\omega)}{a(\eta^{(q)}) \prod\limits_{j=1,\, j\neq q}^{3} [\rho_q^2(\eta^{(q)}) - \rho_j^2(\eta^{(q)})]} \times$$

$$\times \frac{e^{i(x\cdot\xi^{(q)})}}{|\nabla_\eta \rho_q(\eta^{(q)})| \sqrt{k_q(\xi^{(q)})}} \Bigg\} + O(|x|^{-2}), \quad (3.60)$$

where $\eta^{(q)} = \dfrac{\xi^{(q)}}{|\xi^{(q)}|}$, $\eta_k \rho_q(\eta^{(q)}) = \xi_k^{(q)}$ and k_q is the Gaussian curvature of S_q.

With the help of (3.60), (3.56) and (3.59) we obtain the radiation conditions (3.54), (3.55). □

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(Received 8.06.2009)

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